PARABOLAS INFILTRATING THE FORD CIRCLES

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THESIS
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Abstract

After briefly discussing classical results of Farey fractions and Ford circles, we define and study a new family of parabolas in connection with Ford circles and discover some interesting properties that they have. Using these properties, we provide an application of such a family.
## Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Proof of theorem [1]</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>The set of parabolas associated to a set of Ford circles</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>Proof of theorem [2]</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>Theorem 2 on short intervals</td>
<td>13</td>
</tr>
<tr>
<td>6</td>
<td>Application to Kunik’s function</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>Bibliography</td>
<td>17</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

In studying the set of fractions in 1938, L.R. Ford discovered and defined a new geometric object known as a Ford circle. Such a circle, as defined by Ford [10], is tangent to the x-axis at a given point with rational coordinates \((a/q, 0)\), with \(a/q\) in lowest terms, and centered at \((a/q, 1/2q^2)\). For each such rational coordinate, a circle of such radius exists. Ford proved the following theorem in regards to these circles:

The representative circles of two distinct fractions are either tangent or wholly external to one another.

In the case that two circles are tangent to one another, Ford called the representative fractions adjacent. Several mathematicians, including Ford himself, have made connections between these circles and the set of Farey fractions. For some recent results on the distribution of Ford circles, the reader is referred to [1], [2], [6], and the references therein. For additional properties and results regarding Farey fractions refer to [3], [4], [5], [7], [8], [9], [11], and those references therein.

A natural continuation of study, and that which the authors will discuss in this paper, is the discussion of the relationship between the rational numbers and other geometric objects, and connections that might be made between these objects and Ford circles. In this spirit we define, at each rational \(a/q\) in reduced terms, a parabola in the following way:

\[
P_{a/q} : y = \frac{q^2}{2} \left( x - \frac{a}{q} \right)^2.
\]  

(1.0.1)

These parabolas, as we will see below, exhibit many interesting geometric properties, as well as a close relationship with Ford circles. First, we fix several families of objects which will be important in what follows. Namely, the family of all such parabolas as defined in (1.0.1), the family of Ford circles, and the family of centers of such Ford circles:

\[
P = \{P_{a/q} : a/q \in \mathbb{Q}\}
\]

\[
C = \{C_{a/q} : a/q \in \mathbb{Q}\}
\]

\[
O = \{O_{a/q} : a/q \in \mathbb{Q}\}.
\]
In Figure 1.1 these three families are shown for certain rational numbers in the interval $[0, 1]$.

In the following theorem, we collect some remarkable, yet easy to prove, geometric properties of these parabolas in connection with Ford circles.

**Theorem 1.** Given rational numbers $r_1 = a_1/q_1, r_2 = a_2/q_2$ in lowest terms, $C_{r_1}$ and $C_{r_2}$ of different radii, we have the following:

(i) The center $O_{r_2}$ lies on the parabola $P_{r_1}$ if and only if the circles $C_{r_1}$ and $C_{r_2}$ are tangent.

(ii) If the circles $C_{r_1}$ and $C_{r_2}$ are tangent, then the parabolas $P_{r_1}$ and $P_{r_2}$ intersect at exactly two points which are the centers of two Ford circles.

(iii) If $C_{r_3}$ and $C_{r_4}$ are such that $C_{r_1}$ is tangent to both, $C_{r_2}$ is tangent to both, and $C_{r_1}$ and $C_{r_2}$ are disjoint, then the parabolas $P_{r_1}$ and $P_{r_2}$ intersect at two centers of Ford circles, namely $O_{r_3}$ and $O_{r_4}$.

(iv) If $C_{r_3}$ is such that $C_{r_1}$ and $C_{r_2}$ are both tangent to it, and $C_{r_1}$ and $C_{r_2}$ are disjoint, then the parabolas $P_{r_1}$ and $P_{r_2}$ intersect at one center of a Ford circles, namely $O_{r_3}$.

(v) If there is no $C_{r_3}$ such that $C_{r_1}$ and $C_{r_2}$ are both tangent to it, then the parabolas $P_{r_1}$ and $P_{r_2}$ do not intersect at the center of any Ford circle.

(vi) **Reciprocity:** The center $O_{r_2}$ lies on the parabola $P_{r_1}$ if and only if $O_{r_1}$ lies on $P_{r_2}$.

It is clear from the theorem that there is a distinct connection between the tangency of Ford circles and the points which lie on a given parabola as defined. It may be interesting to study the points of intersection
which are not centers of circles to see what statements might be made about these points. We first prove
the above theorem, and then present an application of this family of parabolas.
Chapter 2

Proof of theorem 1

We first prove (i). Given two circles, $C_{r_1}$ and $C_{r_2}$, with $r_1 = a_1/q_1$ and $r_2 = a_2/q_2$, then the radii of these circles are $1/2q_1^2$ and $1/2q_2^2$, respectively. Therefore, the circles are tangent if and only if the distance between their centers equals the sum $1/2q_1^2 + 1/2q_2^2$. This is true if and only if

$$
\left(\frac{a_2}{q_2} - \frac{a_1}{q_1}\right)^2 + \left(\frac{1}{2q_2^2} - \frac{1}{2q_1^2}\right)^2 = \left(\frac{1}{2q_2^2} + \frac{1}{2q_1^2}\right)^2,
$$

which further simplifies to

$$
\left|\frac{a_2}{q_2} - \frac{a_1}{q_1}\right| = \frac{1}{q_1q_2}.
$$

Next, the center $O_{r_2} = (a_2/q_2, 1/2q_2^2)$ lies on the parabola $P_{r_1}$ if and only if

$$
\frac{1}{2q_2^2} = \frac{q_1^2}{2} \left(\frac{a_2}{q_2} - \frac{a_1}{q_1}\right)^2,
$$

which reduces to

$$
\frac{1}{q_1q_2} = \left|\frac{a_2}{q_2} - \frac{a_1}{q_1}\right|.
$$

Therefore the center $O_{r_2}$ lies on the parabola $P_{r_1}$ if and only if the circles $C_{r_1}$ and $C_{r_2}$, are tangent. This completes the proof of (i). Similarly, the center $O_{r_1}$ lies on the parabola $P_{r_2}$ if and only if $C_{r_1}$ and $C_{r_2}$ are tangent. Hence the center $O_{r_1}$ lies on $P_{r_2}$ if and only if $O_{r_2}$ lies on $P_{r_1}$. This proves (vi).

For (ii), we provide a geometric proof. Refer to Figure 2.1 for the proof. Given two Ford circles $C_{r_1}$ and $C_{r_2}$ of different radii, which are tangent to one another, by Ford’s theory we know that there are exactly two Ford circles, $C_{r_3}$ and $C_{r_4}$, which are tangent to $C_{r_1}$ and $C_{r_2}$. By (i), we know that $P_{r_1}$ and $P_{r_2}$ pass through both centers $O_{r_3}$ and $O_{r_4}$. Since $P_{r_1}$ and $P_{r_2}$ cannot intersect at more than two points, it follows that their intersection consists exactly of $O_{r_3}$ and $O_{r_4}$. This completes the proof of (ii).

The proof of (iii) follows from (i) in the same way as the proof of (ii) (refer to Figure 2.2). From (i), since $C_{r_3}$ and $C_{r_4}$ are tangent to $C_{r_1}$ and $C_{r_2}$, it follows that the parabolas $P_{r_1}$ and $P_{r_2}$ pass through the centers $O_{r_3}$ and $O_{r_4}$. Since $P_{r_1}$ and $P_{r_2}$ intersect at two points, this intersection consists of $O_{r_3}$ and $O_{r_4}$.
Figure 2.1: The parabolas $P_{r_1}$, $P_{r_2}$, centers $O_{r_3}$, $O_{r_4}$, and representative circles of the rational numbers $r_1$, $r_2$, $r_3$, and $r_4$ as described in Theorem 1 (ii).

This completes the proof of (iii). By the same logic, the proofs of (iv) and (v) follow also from (i). Refer to Figure 2.3. This completes the proof of Theorem 1.
Figure 2.2: The parabolas $P_{r_1}$, $P_{r_2}$, centers $O_{r_3}$, $O_{r_4}$, and representative circles of the rational numbers $r_1$, $r_2$, $r_3$, and $r_4$ as described in Theorem 1 (iii).

Figure 2.3: An example of parts (iv) and (v) of Theorem 1. The circle $C_{r_1}$ shares a tangent circle with both $C_{r_2}$ and $C_{r_3}$, thus the parabola $P_{r_1}$ intersects both $P_{r_2}$ and $P_{r_3}$ at the center of a Ford circle. The circles $C_{r_2}$ and $C_{r_3}$ do not share any tangent circles, so their parabolas $P_{r_2}$ and $P_{r_3}$ do not intersect at the center of any Ford circle.
Chapter 3

The set of parabolas associated to a set of Ford circles

A natural continuation of this study of parabolas is to consider the area given under the arcs of a certain set of parabolas. We then associate a set of parabolas with a finite set of Ford circles. To proceed, we fix a positive integer $Q$. Then we consider the finitely many Ford circles with centers on or above the horizontal line $y = 1/2Q^2$. Let $F_Q$ be the ordered set of Farey fractions of order $Q$ and let $N = N(Q)$ be its cardinality. One knows that

$$N(Q) = \frac{3}{\pi^2} Q^2 + O(Q \log Q).$$

We now define a real valued function $F_Q$ on the interval $[0, 1]$ as follows. Let $a/q$ and $a'/q'$ be consecutive fractions in $F_Q$. From basic properties of Farey fractions we know that the fraction $\theta = (a + a')/(q + q')$ does not belong to $F_Q$, and it is the first fraction that would be inserted between $a/q$ and $a'/q'$ if one would allow $Q$ to increase. By Ford’s theory, the circle $C_\theta$ is tangent to both $C_{a/q}$ and $C_{a'/q'}$. By its definition, $C_\theta$ is also tangent to the $x$-axis at the point $T_\theta = (\theta, 0)$. We define $F_Q$ on the interval $[a/q, a'/q']$ in such a way that its graph over this interval coincides with the arc of the parabola $P_\theta$ lying above the same interval $[a/q, a'/q']$. We know from Theorem 1 (i) that the end points of this arc are exactly the centers of the Ford circles $C_{a/q}$ and $C_{a'/q'}$. This shows that the function $F_Q$ is well defined and continuous on the entire interval $[0, 1]$.

On each subinterval of the form $[a/q, a'/q']$, $F_Q$ is given explicitly by

$$F_Q(t) = \frac{(q + q')^2}{2} \left( t - \frac{a + a'}{q + q'} \right)^2. \tag{3.0.1}$$

In Figure 3.1, we show the Ford circles and the parabolas described above in the case $Q = 5$. Thus in this case the graph of $F_Q$ is a union of parabolas joining those centers of Ford circles that lie on or above the line $y = 1/(2 \cdot 5^2) = 1/50$.

Our next goal is to estimate the area under these parabolas, that is,

$$\text{Area}(Q) = \int_0^1 F_Q(t) \, dt.$$
For example, in the particular case shown in Figure 3.1 the area under the parabolas equals Area(5) = 0.0675925. In Table 3.1 one sees how Area(Q) changes as Q increases.

Table 3.1: The size of Area(Q) and Q Area(Q) for selected values of Q.

<p>| | | | |</p>
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<thead>
<tr>
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<tbody>
<tr>
<td>Q</td>
<td>N(Q)</td>
<td>Area(Q)</td>
<td>Q · Area(Q)</td>
</tr>
<tr>
<td>---</td>
<td>------</td>
<td>---------</td>
<td>-------------</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.1250000000</td>
<td>0.2500000000</td>
</tr>
<tr>
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<td>0.0390560699</td>
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</tr>
<tr>
<td>100</td>
<td>3045</td>
<td>0.0044701813</td>
<td>0.4470181307</td>
</tr>
<tr>
<td>500</td>
<td>76117</td>
<td>0.0009078709</td>
<td>0.4539354502</td>
</tr>
<tr>
<td>1000</td>
<td>304193</td>
<td>0.0004549847</td>
<td>0.4549847094</td>
</tr>
<tr>
<td>5000</td>
<td>7600459</td>
<td>0.0000911752</td>
<td>0.4558762747</td>
</tr>
<tr>
<td>10000</td>
<td>30397487</td>
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<td>0.4560057781</td>
</tr>
<tr>
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<td>0.0000228035</td>
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</tr>
<tr>
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<td>0.4561061307</td>
</tr>
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<td>50000</td>
<td>759924265</td>
<td>0.0000091222</td>
<td>0.4561127077</td>
</tr>
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Since the areas appear to decay like 1/Q, in the last column we also included the quantity Q Area(Q). Based on the data from the table it is natural to conjecture that the sequence \( \{ Q \text{ Area}(Q) \} \) converges to a certain constant, 0.456... The following theorem shows that this is indeed true.
**Figure 3.2:** The behavior of the function $Q \to Q \text{Area}(Q)$.

**Theorem 2.** For all positive integers $Q$,

$$\text{Area}(Q) = \frac{\zeta(2)}{3\zeta(3)} \cdot \frac{1}{Q} + O\left(\frac{\log Q}{Q^2}\right),$$

where $\zeta(s)$ denotes the Riemann zeta function.

In particular, Theorem 2 confirms the above conjecture, as

$$\lim_{Q \to \infty} Q \text{Area}(Q) = \frac{\zeta(2)}{3\zeta(3)} = 0.45614 \ldots$$

It would be interesting to further study the oscillatory behavior of the function $Q \to Q \text{Area}(Q)$, shown in Figure 3.2.
Chapter 4

Proof of theorem 2

Recall that between two consecutive Farey fractions \( \alpha = a/q \) and \( \beta = a'/q' \), \( F_Q(t) \) has the form \( F_Q(t) = A(t - \theta)^2 \), where \( \theta = (a + a')/(q + q') \) is the point of tangency to the x-axis and \( A = (q + q')^2/2 \). Thus

\[
\int_{\alpha}^{\beta} F_Q(t) \, dt = \frac{A(t - \theta)^3}{3} \bigg|_{\alpha}^{\beta}.
\] (4.0.1)

To simplify the integral, notice that

\[
\beta - \theta = \frac{a'}{q'} - \frac{a + a'}{q + q'} = \frac{1}{q'(q + q')},
\]

\[
\alpha - \theta = \frac{a}{q} - \frac{a + a'}{q + q'} = -\frac{1}{q(q + q')},
\]

since by a fundamental property of Farey fractions, \( a'q - aq' = 1 \). We find that the integral in (4.0.1) reduces to

\[
\int_{\alpha}^{\beta} F(t) \, dt = \frac{1}{6} \left( \frac{1}{q'^3(q + q')} + \frac{1}{q^3(q + q')} \right).
\]

Taking into account the symmetry of the Farey sequence with respect to 1/2, or in other words taking into account that for each pair of consecutive Farey fractions \( a/q \) and \( a'/q' \) there is another pair having the same denominators \( q, q' \), in reverse order, namely \( 1 - a'/q' \) and \( 1 - a/q \), one sees that

\[
\text{Area}(Q) = \frac{1}{3} \sum \frac{1}{q^3(q + q')},
\] (4.0.2)

where the sum is over all pairs \( (q, q') \) of consecutive denominators of the sequence of Farey fractions of order \( Q \). From basic properties of Farey fractions we know that this set of pairs of consecutive denominators coincides with the set of pairs \( (q, q') \) of integer numbers for which \( 1 \leq q, q' \leq Q \), \( \gcd(q, q') = 1 \) and \( q + q' > Q \).
Thus we may rewrite \((4.0.2)\) as
\[
\text{Area}(Q) = \frac{1}{3} \sum_{1 \leq q, q' \leq Q \atop \gcd(q, q') = 1} \frac{1}{q^3(q + q')} := \frac{1}{3} S(Q) .
\]

For convenience, we introduce the function \(f(x, y) = \frac{1}{x^3(x+y)}\) and the triangular region \(\Omega = \Omega(Q)\), which is defined by the inequalities \(1 \leq x, y \leq Q\) and \(x + y > Q\). Then the sum in \((4.0.3)\) can be written as
\[
S(Q) = \sum_{(x,y) \in \Omega \atop \gcd(x,y) = 1} f(x, y) .
\]

By Möbius summation,
\[
S(Q) = \sum_{(x,y) \in \Omega} f(x, y) \sum_{d \mid \gcd(x,y)} \mu(d)
= \sum_{1 \leq d \leq Q} \mu(d) \sum_{(x,y) \in \Omega \atop d \mid x, d \mid y} f(x, y) .
\]

We continue the evaluation in \((4.0.4)\) by changing the variables \(x = dm\) and \(y = dn\), to obtain
\[
S(Q) = \sum_{1 \leq d \leq Q} \mu(d) \sum_{(m,n) \in \frac{1}{d} \Omega} f(dm, dn)
\]
\[
= \sum_{d \leq Q} \frac{\mu(d)}{d^4} \sum_{(m,n) \in \frac{1}{d} \Omega} \frac{1}{m^3(m + n)} := \sum_{d \leq Q} \frac{\mu(d)}{d^4} S_1(d, Q) .
\]

The inner sum \(S_1(d, Q)\) can be written as
\[
S_1(d, Q) = \sum_{1 \leq m \leq Q/d} \frac{1}{m^3} \sum_{Q/d - m < n \leq Q/d} \frac{1}{m + n}
= \sum_{1 \leq m \leq Q/d} \frac{1}{m^3} \sum_{Q/d < r \leq Q/d + m} \frac{1}{r} .
\]
Here we introduce a variable $s$ to take into account the size of $r$ near $Q/d$. We obtain

\[ S_1(d, Q) = \sum_{1 \leq m \leq Q/d} \frac{1}{m^3} \sum_{s=1}^{m} \frac{1}{\left\lfloor \frac{Q}{d} \right\rfloor + s} \]

\[ = \sum_{1 \leq m \leq Q/d} \frac{1}{m^3} \sum_{s=1}^{m} \frac{1}{\frac{Q}{d} - \left\{ \frac{Q}{d} \right\} + s} \]

\[ = \frac{d}{Q} \sum_{1 \leq m \leq Q/d} \frac{1}{m^3} \sum_{s=1}^{m} \left( 1 + O \left( \frac{sd}{Q} \right) \right) \]

\[ = \frac{d}{Q} \sum_{1 \leq m \leq Q/d} \frac{1}{m^2} + O \left( \frac{d^2 \log Q/d}{Q^2} \right). \]

The completion of the sum over all positive integers $m$, yields

\[ S_1(d, Q) = \frac{d}{Q} \sum_{m=1}^{\infty} \frac{1}{m^2} - \frac{d}{Q} \sum_{m>Q/d} \frac{1}{m^2} + O \left( \frac{d^2 \log Q/d}{Q^2} \right) \]

\[ = \frac{d}{Q} \zeta(2) + O \left( \frac{d^2 \log Q/d}{Q^2} \right). \tag{4.0.6} \]

Inserting this estimate into (4.0.5), we find that

\[ S(Q) = \sum_{d \leq Q} \frac{\mu(d)}{d^3} \left( \frac{d\zeta(2)}{Q} + O \left( \frac{d^2 \log Q/d}{Q^2} \right) \right) = \frac{\zeta(2)}{\zeta(3)} \frac{1}{Q} + O \left( \frac{\log Q}{Q^2} \right). \]

Hence, in view of (4.0.3), the proof of the theorem is completed.
Chapter 5

Theorem 2 on short intervals

The estimate given in Theorem 2 is satisfied for the full interval \((0, 1)\). From this, we deduce the following corollary regarding the given sum on a short interval \((\alpha, \beta) \subseteq (0, 1)\).

**Corollary 1.** For all positive integers \(Q\), and intervals \(I = (\alpha, \beta) \subseteq (0, 1)\),

\[
\text{Area}_I(Q) = |I| \cdot \frac{\zeta(2)}{3\zeta(3)} \cdot \frac{1}{Q} + \frac{C_1}{Q} + O\left(\frac{\log^2 Q}{Q^2}\right).
\]

The proof is as follows. Rewrite the sum given in (4.0.3) on a small interval \(I\) as follows:

\[
\text{Area}_I(Q) = \frac{1}{3} S_I(Q) = \frac{1}{3} \sum_{1 \leq q \leq Q} \sum_{Q - q \leq q' \leq Q} \frac{1}{q^3(q + q')} \leq \frac{1}{3} \sum_{1 \leq q \leq Q} \frac{1}{q^3Q}.
\] (5.0.1)

where \(q'\) is the inverse of \(q\). We see that this can be written as

\[
S_I(Q) \leq \sum_{1 \leq q \leq Q} \frac{1}{q^3Q} \sum_{Q - q \leq q' \leq Q} \frac{1}{q^3Q} = 1,
\]

which is just a counting function in the inner sum. Thus we can write:

\[
S_I(Q) \leq \sum_{1 \leq q \leq Q} \frac{1}{q^3Q} \# \{ Q - q \leq q' \leq Q; (q, q') = 1; \overline{q'} \in [q(1 - \beta), q(1 - \alpha)] \}.
\] (5.0.2)

Now we rewrite this counting function, denoted \(M\), in the following way:

\[
M = \# \{ m \in \mathbb{Z}; m \in [q(1 - \beta), q(1 - \alpha)]; (m, q) = 1 \}
\]
\[
= \sum_{q(1-\beta) \leq m \leq q(1-\alpha)} \sum_{d|m \atop d \mid q} \mu(d) .
\]

Reversing the order of summation,

\[
M = \sum_{d \mid q} \mu(d) \sum_{q(1-\beta) \leq m \leq q(1-\alpha)} 1
\]

\[
= \sum_{d \mid q} \mu(d) \left( \frac{q|I|}{d} + O(1) \right) .
\]

Simplifying, we see that

\[
M = q|I| \sum_{d \mid q} \frac{\mu(d)}{d} + O(q^3)
\]

\[
= |I| \varphi(q) + O(q^3) .
\]

Now we return to (5.0.2), and write

\[
S_I(Q) \leq \sum_{1 \leq \eta \leq Q} \frac{1}{q^3} \left( |I| \varphi(q) + A(q) \right) ,
\]

where \( A(q) \) is defined as

\[
A(q) = \# \{ m \in \mathbb{Z}; m \in [q(1-\beta), q(1-\alpha)]; (m,q) = 1 \} - |I| \varphi(q) . \tag{5.0.3}
\]

It follows from Theorem 2 that

\[
S_I(Q) \leq |I| \cdot \frac{\zeta(2)}{3\zeta(3)} + O \left( \frac{\log^2 Q}{Q^2} \right) + \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{A(q)}{q^3} .
\]

Lastly, we estimate the sum in this formula by some constant which depends on the interval I. That is,

\[
\frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{A(q)}{q^3} = C_1 ,
\]

where we define \( C_1 \) by

\[
C_1 := \sum_{q=1}^{\infty} \frac{A(q)}{q^3} . \tag{5.0.4}
\]

Then we have

\[
S_I(Q) \leq |I| \cdot \frac{\zeta(2)}{3\zeta(3)} \cdot \frac{1}{Q} + C_1 \left( \frac{1}{Q^2} \right) + O \left( \frac{\log^2 Q}{Q^2} \right) . \tag{5.0.5}
\]
We now find a lower bound for $S_I$. Notice that in (5.0.1) we can instead write

$$S_I(Q) \geq \sum_{1 \leq q \leq Q} \frac{1}{q^3(q + Q)} \sum_{\substack{q \leq q' \leq Q \\ \gcd(q,q') = 1}} \frac{1}{q'} .$$

That is,

$$S_I(Q) \geq \sum_{1 \leq q \leq Q} \frac{1}{q^3(q + Q)} \sum_{\substack{q \leq q' \leq Q \\ \gcd(q,q') = 1}} \frac{1}{q'} ,$$

which we see is the same counting function as defined above for the upper bound. Then we can say that

$$S_I(Q) \geq \sum_{1 \leq q \leq Q} \frac{1}{q^3(q + Q)} (|I|\varphi(q) + A(q)) ,$$

$A(q)$ defined as in (5.0.3). Following the exact same computations, it follows that

$$S_I(Q) \geq |I| \cdot \frac{\zeta(2)}{3\zeta(3)} \cdot \frac{1}{Q} + C_I \frac{1}{Q} + O \left( \frac{\log^2 Q}{Q^2} \right) ,$$

where $C_I$ is defined as in (5.0.4). Combining (5.0.5) and (5.0.7) we find the equality

$$S_I(Q) = |I| \cdot \frac{\zeta(2)}{3\zeta(3)} \cdot \frac{1}{Q} + C_I \frac{1}{Q} + O \left( \frac{\log^2 Q}{Q^2} \right) ,$$

which completes the proof of Corollary 1.

Notice that when $I = [0, 1]$, we get

$$S_{[0,1]} = \frac{\zeta(2)}{3\zeta(3)} \cdot \frac{1}{Q} + O \left( \frac{\log^2 Q}{Q^2} \right) ,$$

which is exactly our estimate for the full interval as determined in the previous section.
Chapter 6

Application to Kunik’s function

In a recent paper, Kunik [12] defines a function $t \mapsto \Psi^*_Q(t)$, for each positive integer $Q$, as follows. Between any two consecutive Farey fractions in $F_Q$, say $\frac{a_{j-1}}{q_{j-1}}$ and $\frac{a_j}{q_j}$, $j = 1, \ldots, N(Q) - 1$, the function $\Psi^*_Q(t)$ is defined by:

$$
\Psi^*_Q(t) := -\frac{q_j + q_{j-1}}{2} \left[ t - \frac{a_j + a_{j-1}}{q_j + q_{j-1}} \right], \quad \text{for } \frac{a_{j-1}}{q_{j-1}} \leq t < \frac{a_j}{q_j}.
$$

The function $\Psi^*_Q(t)$ is a step function, being linear between subsequent Farey fractions in $F_Q$ and having a jump of height $1/q_j$ at each Farey fraction $\frac{a_j}{q_j}$. Kunik established a number of interesting results, including the following $L^2$-estimate:

$$
||\Psi^*_Q||^2_2 = \int_0^1 \Psi^*_Q(t)^2 dt = O\left(\frac{\log Q}{Q}\right).
$$

We notice a connection between the above function $F_Q(t)$ and Kunik’s function $\Psi^*_Q(t)$. More precisely,

$$
F_Q(t) = 2\Psi^*_Q(t)^2, \quad \text{for all } t \in [0,1] \text{ and } Q \geq 1.
$$

Thus, by Theorem 2, one may replace the above estimate by an asymptotic formula:

$$
\int_0^1 \Psi^*_Q(t)^2 dt = \frac{1}{2} \text{Area}(Q) = \frac{\zeta(2)}{6\zeta(3)} \cdot \frac{1}{Q} + O\left(\frac{\log Q}{Q^2}\right).
$$
Bibliography


