GAIT REGULATION FOR BIPEDAL LOCOMOTION

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DISSERTATION

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ABSTRACT

This work explores regulation of forward speed, step length, and slope walking for the passive-dynamic class of bipedal robots. Previously, an energy-shaping control for regulating forward speed has appeared in the literature; here we show that control to be a special case of a more general time-scaling control that allows for speed transitions in arbitrary time. As prior work has focused on potential energy shaping for fully actuated bipeds, we study in detail the shaping of kinetic energy for bipedal robots, giving special treatment to issues of underactuation. Drawing inspiration from features of human walking, an underactuated kinetic-shaping control is presented that provides efficient regulation of walking speed while adjusting step length. Previous results on energetic symmetries of bipedal walking are also extended, resulting in a control that allows regulation of speed and step length while walking on any slope. Finally we formalize the optimal gait regulation problem and propose a dynamic programming solution seeded with passive-dynamic limit cycles. Observations of the optimal solutions generated by this method reveal further similarities between passive dynamic walking and human locomotion and give insight into the structure of minimum-effort controls for walking.
For my family.
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have led to our present formalization of the optimal gait regulation problem and its solution based on the passive limit cycle of a biped.

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# TABLE OF CONTENTS

## CHAPTER 1 INTRODUCTION ................................................. 1
1.1 Background ......................................................... 1
1.1.1 Control of bipedal walking ................................. 1
1.2 Passive-Dynamic Locomotion ................................. 3
1.2.1 Analysis and control of passive-dynamic locomotion .... 5
1.2.2 Contributions .................................................. 7
1.3 Outline ............................................................. 8

## CHAPTER 2 MODELS AND ANALYSIS OF LOCOMOTION ............ 12
2.1 Biped Configuration and Dynamics .......................... 13
2.1.1 Dynamics of the single-support phase ............... 13
2.1.2 Dynamics of the double-support phase ............... 15
2.1.3 Hybrid dynamical model ................................. 16
2.2 Limit Cycles ....................................................... 17
2.2.1 Passive-dynamic walking ................................. 18
2.3 Finding Limit Cycles ............................................ 18
2.3.1 Poincaré maps ............................................... 19
2.3.2 Shooting method ............................................ 20
2.4 Stability of Limit Cycles ........................................ 21
2.4.1 The monodromy matrix .................................... 23
2.5 The Compass-Gait Biped ........................................ 25
2.5.1 Energy ......................................................... 25
2.5.2 Hybrid dynamics ............................................ 25
2.5.3 Passive limit cycle ......................................... 28
2.5.4 Stability of the limit cycle ............................... 29
2.6 The Biped with Knees ............................................ 30
2.6.1 Energy ......................................................... 30
2.6.2 Continuous-time dynamics of the single-support phase .. 32
2.6.3 Phases of walking ............................................ 33
2.6.4 Knee lock dynamics of the single-support phase ...... 34
2.6.5 Dynamics of the double-support phase ............... 35
2.6.6 Passive limit cycle ......................................... 36
2.6.7 Stability of the limit cycle ............................... 37
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.3.1</td>
<td>Faster transitions via total-energy shaping</td>
<td>97</td>
</tr>
<tr>
<td>6.3.2</td>
<td>Time derivative of total energy</td>
<td>100</td>
</tr>
<tr>
<td>6.3.3</td>
<td>Total energy-shaping control</td>
<td>101</td>
</tr>
<tr>
<td>7</td>
<td>Chapter 7 Regulating Walking on Slopes</td>
<td>105</td>
</tr>
<tr>
<td>7.1</td>
<td>Controlled Symmetries</td>
<td>106</td>
</tr>
<tr>
<td>7.2</td>
<td>Extending Controlled Symmetries</td>
<td>107</td>
</tr>
<tr>
<td>7.3</td>
<td>Controlled Symmetries with Kinetic Shaping</td>
<td>108</td>
</tr>
<tr>
<td>8</td>
<td>Chapter 8 Optimal Gait Regulation</td>
<td>111</td>
</tr>
<tr>
<td>8.1</td>
<td>Formalization of the Optimal Gait Regulation Problem</td>
<td>114</td>
</tr>
<tr>
<td>8.1.1</td>
<td>Simplification due to period-1 limit cycle requirement</td>
<td>115</td>
</tr>
<tr>
<td>8.1.2</td>
<td>Constraints on the continuous dynamics</td>
<td>117</td>
</tr>
<tr>
<td>8.1.3</td>
<td>Summary of the optimal gait regulation problem</td>
<td>120</td>
</tr>
<tr>
<td>8.2</td>
<td>Solution via Dynamic Programming</td>
<td>120</td>
</tr>
<tr>
<td>8.2.1</td>
<td>General solution</td>
<td>121</td>
</tr>
<tr>
<td>8.2.2</td>
<td>Summary</td>
<td>123</td>
</tr>
<tr>
<td>8.3</td>
<td>Fixed Terminal Velocities and Terminal Time</td>
<td>124</td>
</tr>
<tr>
<td>8.3.1</td>
<td>Solution for fixed terminal velocities and terminal time</td>
<td>126</td>
</tr>
<tr>
<td>8.4</td>
<td>Method of Approximating Solutions for the Optimal Gait Regulation Problem</td>
<td>128</td>
</tr>
<tr>
<td>8.4.1</td>
<td>Testing the method</td>
<td>129</td>
</tr>
<tr>
<td>8.4.2</td>
<td>Implementing the method</td>
<td>132</td>
</tr>
<tr>
<td>8.5</td>
<td>Results and Discussion</td>
<td>133</td>
</tr>
<tr>
<td>8.5.1</td>
<td>Approximate optimal terminal times and velocities</td>
<td>133</td>
</tr>
<tr>
<td>8.5.2</td>
<td>Forward speed and step length</td>
<td>135</td>
</tr>
<tr>
<td>8.5.3</td>
<td>Approximate optimal trajectories</td>
<td>136</td>
</tr>
<tr>
<td>8.5.4</td>
<td>Approximate optimal controls</td>
<td>138</td>
</tr>
<tr>
<td>8.5.5</td>
<td>Comparison with previous controls</td>
<td>142</td>
</tr>
<tr>
<td>9</td>
<td>Chapter 9 Conclusions</td>
<td>145</td>
</tr>
<tr>
<td>9.1</td>
<td>Future Work</td>
<td>146</td>
</tr>
<tr>
<td>A</td>
<td>Appendix A Optimal Control Approximation</td>
<td>148</td>
</tr>
<tr>
<td>A.1</td>
<td>Algorithm Overview</td>
<td>149</td>
</tr>
<tr>
<td>A.2</td>
<td>The Algorithm in Detail</td>
<td>150</td>
</tr>
<tr>
<td>A.2.1</td>
<td>Parameter values used in our simulation</td>
<td>154</td>
</tr>
<tr>
<td></td>
<td>REFERENCES</td>
<td>156</td>
</tr>
<tr>
<td></td>
<td>AUTHOR’S BIOGRAPHY</td>
<td>163</td>
</tr>
</tbody>
</table>
CHAPTER 1
INTRODUCTION

In this work we study gait regulation for passive-dynamic walking bipedal robots, a class of robots for which locomotion is a natural mode of its unforced dynamics. In this introductory chapter, we survey the literature and provide background for bipedal walking in general and for passive-dynamic walking in particular. We summarize the contributions we have made and outline the remainder of this document.

1.1 Background

Legged locomotion has long fascinated robotics researchers. While highly efficient, wheeled machines are limited to paved surfaces; legged devices have the capability of navigating irregular terrain. The study of bipedal walking is of particular interest due to its similarities to human locomotion. Research in bipedal locomotion holds potential both for the development of walking machines that can maneuver through nonsmooth environments and for the development of prosthetic and orthotic devices to assist disabled individuals.

1.1.1 Control of bipedal walking

Presently, the state-of-the-art in control of bipedal locomotion consists of preplanning joint trajectories while obeying various balance criteria and then commanding the legs of the biped to follow the trajectories using standard control techniques. Contemporary robots with trajectory planning-and-following control systems demon-
strate an impressive array of locomotion tasks. Honda’s famous ASIMO robot [1], whose legs follow preplanned trajectories while monitoring the location of its zero-moment point, is capable of walking on level ground, up and down stairs, turning, and (more recently) running.

The sophisticated collection of locomotion tasks available to bipeds using trajectory planning-and-following controls is impressive, but this type of control system suffers the prominent disadvantage of requiring massive amounts of energy. For example, it is estimated that ASIMO’s specific cost of transport

\[
\frac{\text{energy used}}{\text{weight} \times \text{distance traveled}}
\]

is some 32 times greater than that of a typical human [2]. As a consequence, ASIMO must carry a large, heavy battery backpack which it depletes in minutes.

An alternative strategy for controlling bipedal locomotion conceived by Grizzle, Westervelt, Chevallereau, and others in the 1990s is the method of hybrid zero dynamics [3,4]. This technique involves selecting a set of functions of the joint variables. The set of configurations corresponding to the functions being identically zero define a manifold, and trajectories forming closed orbits on the zero manifold correspond to walking cycles for the biped. Control is then applied to render the zero manifold invariant and to exponentially stabilize the closed orbits on the manifold.

The hybrid zero dynamics approach has been tested extensively on the RABBIT biped [5] and has drawn much interest in the control community on account of the elegant stability proofs it affords for bipedal locomotion, including underactuated walking. We note that the study of this control method has concentrated mainly on two-dimensional (2D) bipeds whose motion is restricted to the sagittal plane. Recently, however, Ames and Gregg [6] and Gregg and Spong [7] have combined hybrid zero dynamics methods with the classical technique of Routhian reduction, resulting in provably stable walking in three dimensions (3D).

Unfortunately, like the trajectory planning-and-following approach, the hybrid zero dynamics strategy suffers from energy inefficiency. Indeed the problem of en-
Energy efficiency is widespread in bipedal locomotion. Unlike the humans whose gait they approximate, most bipedal robots built to date are woefully inefficient. The usefulness of these bipeds is limited to applications that allow tethering with a power cable, frequent battery replacement, or high-output onboard power plants.

Recently, a handful of bipeds has been developed with remarkably high efficiency comparable to that of humans [2]. Control systems for these bipeds are based on the phenomenon of passive-dynamic walking, which we will explain in the following section. This relatively young area of research in biped control holds the promise of yielding a new generation of locomotion control systems with vastly improved efficiency.

1.2 Passive-Dynamic Locomotion

In 1984, McMahon [8] noted the similarities between human walking and a certain bipedal child’s toy shown in Figure 1.1(a). When set on a ramp and given a push, the toy waddled side to side and “walked” its way down the ramp. Remarkable about the toy was the absence of an external source of energy; its motion was driven entirely by conversion of potential energy into kinetic energy as it proceeded down the ramp. The toy’s simple, uncontrolled gait hinted that locomotion may be a natural mode for particular arrangements of links and joints and needs no external energy or planning to bring it about.

In 1990, McGeer [9] demonstrated unpowered bipedal locomotion with a simple planar robot similar to the one shown in Figure 1.1(b). When placed on a ramp and given a proper initial push, the biped walked down the ramp. With each step the walker gained energy from the change in potential on the decline with the ground impact at the end of each step dissipating the extra energy. With the right combination of initial conditions and ramp angle, each successive step exactly matched the previous step and a stable gait emerged.

This remarkable gait has been termed passive-dynamic walking. The term passive comes from the fact that no external source other than gravity provides the energy
Figure 1.1 Passive-dynamic walkers: (a) an unpowered toy discussed by McMahon [8], (b) McGeer’s 2D passive-dynamic walker without knees [9], (c) Ruina’s 2D passive-dynamic walker with knees [10], and (d) Collins’ 3D passive-dynamic walker with knees and arms [11]. These bipeds demonstrate stable, continuous walking on shallow downhill slopes.
necessary for locomotion. *Dynamic* is a term used to classify a type of stability associated with bipedal walking. It is perhaps defined best by first considering its counterpart, static stability. In contrast, consider quadrupedal walking, in which the ground projection of the center of mass always remains inside a support polygon defined by the contact points of the feet—three of which touch the ground at any given time. Locomotion for quadrupeds is known as *statically* stable walking; if at any moment the motion were to be halted, the quadruped would not fall over. In bipedal walking, however, the ground projection of the center of mass frequently exits the support region—defined by the one foot on the ground—and so the biped is constantly falling. Were motion to cease, the biped would topple over. Stability for such a biped requires that it be constantly in motion, continually interrupting the falling motion of the body with a successive step. This is called *dynamic* stability.

A variety of passive-dynamic walking mechanisms have been designed over the past two decades. McGeer, Garcia, and others have demonstrated passive walking for planar bipeds with knees [10,12] like the one pictured in Figure 1.1(c). More recently, Collins, Wisse, and Ruina [11] have demonstrated passive walking in three dimensions for the first time, using the biped with knees and arms shown in Figure 1.1(d).

Using the lessons learned from the passive walkers, researchers have built highly efficient biped robots capable of walking not only on downhill ramps, but on flat and uphill surfaces as well. A number of actuated robots based on passive-dynamic principles have been built at Cornell University, Delft Technical University, and the Massachusetts Institute of Technology [2,13], two of which are shown in Figure 1.2. These devices require actuators to compensate for the reduced contribution of gravity to level or uphill locomotion.

1.2.1 Analysis and control of passive-dynamic locomotion

McGeer’s seminal work in passive-dynamic walking [9] included analysis based on a linearized model of the planar (2D) walker. In the mid 1990s, Goswami et al. [14–16] reported results based on a full nonlinear 2D model, noting the sensi-
Figure 1.2 Actuated walkers based on passive-dynamic principles: (a) Wisse’s 2D walker with knees [13], and (b) Collin’s 3D walker with knees and arms [2]. These bipeds are capable of walking on level ground.

tivity of the walking limit cycle to changes in model parameters and slope of the ground and reporting the period-doubling bifurcations that occur when parameters are varied. Adolfsson et al. [17] performed similar analysis on a 3D passive-dynamic biped, identifying and classifying the bifurcations of the biped as model parameters are varied.

Goswami et al. [15] also proposed a control law allowing actuated bipeds to walk on different slopes. Spong [18] and Spong and Bullo [19] later showed that a general, $n$-degree-of-freedom passive-dynamic biped could be made to walk on any slope with a remarkably simple potential-shaping control law that enforces a particular symmetry in a biped’s potential energy-field. Further energy-shaping results were presented by Spong and Bhatia in [20] and Bhatia in [21], notably a total energy-shaping control law that broadens the basin of attraction and increases the rate of convergence to a biped’s limit cycle. In our earlier work, we reported a potential-shaping control that regulates a biped’s forward speed [22]. The work was followed a year later by
Licer, M’Sirdi, and Manamanni [23] who offered a rigorous proof for the use of potential shaping for speed regulation.

1.2.2 Contributions

In this dissertation, we present our recent work in gait regulation for passive-dynamic bipeds. Our contributions have diversified the set of locomotion tasks achievable for passive-dynamic bipeds, including regulation of walking speed, walking step length, and walking on slopes.

As mentioned above, our early work [22] focused on shaping the potential energy of passive-dynamic bipeds for regulation of forward speed. We have since discovered that this control law is a special case of a more general time-scaling control that allows for transitions between gaits of various speeds in arbitrary time [24, 25].

Collaboration with biomechanists and kinesiologists has uncovered numerous similarities between passive-dynamic walking and human locomotion. Noting these similarities prompted our own investigation of the role of passive dynamics in human gait and led to our report of a passive mechanical model that effectively duplicates the dynamics of the human ankle in level and downhill walking [26, 27]. These results and others in the biomechanics literature make a compelling case that passive dynamics accounts for a significant portion of human locomotion, suggesting biomimicry may be a useful tool when designing controls for passive-dynamic robots.

Noting several key features of human walking, we have designed a biomimetic energy-shaping control for regulating the speed of walking [28, 29]. This control exhibits marked improvements in energy efficiency over the previous time-scaling control, an advantage we attribute to its mimicry of efficient human gait. As previous work in energy-shaping for bipedal locomotion has focused on potential energy-shaping for fully actuated bipeds, we have designed our biomimetic control as a kinetic energy-shaping control for underactuated bipeds.

The achievable forms of a biped’s closed-loop energy are those that solve the matching condition, a particular nonlinear partial differential equation (PDE). We
have shown that this nonlinear PDE reduces to a single linear PDE in the case of potential energy-shaping and reduces to one algebraic equation and two linear PDEs in the case of kinetic energy-shaping [28, 29]. In this work we explicitly solve the matching condition for a particular passive-dynamic biped and report the energy forms achievable using potential-shaping or kinetic-shaping with actuation at the ankle alone.

Prior work on controlled symmetries has shown that the natural kinetic energy of a biped is invariant with respect to changes in the slope of the walking surface. Kinetic energy shaping, however, admits the possibility of closed-loop kinetic fields that vary with respect to slope. We have extended previous work to enforce a controlled symmetry in kinetic energy [29]. Here we show how this controlled symmetry allows bipeds with modified kinetic energy to walk on any given slope.

Finally, we have considered optimal gait regulation of passive-dynamic bipeds. We have formalized the optimal control problem, showing that the impact dynamics of a steady walking cycle for the hybrid dynamics of a bipedal robot may be recast as a constraint on a two-point boundary value problem for a continuous-time dynamical system. We propose a method for approximating the solutions to the optimal control problem using a numerical algorithm that is seeded with the passive limit cycle. Here we present the family of solutions generated by this method for various commanded step lengths. From the solutions we draw insight into the structure of the controls and gait trajectories that minimize actuator effort.

1.3 Outline

The remainder of this work is organized as follows.

- In Chapter 2, we present a general hybrid model for bipedal walking, combining a continuous differential equation for the step with a discrete map for the impact with the ground. Although the mathematical substance of continuous walking gait is nothing more than a loop in the state space of the biped, finding
and analyzing the stability of these loops, or *limit cycles*, requires many tools. The so-called *shooting method* for locating limit cycles is presented along with the use of linearized *Poincaré maps* to determine the stability of limit cycles. We demonstrate these techniques for a simple walking biped common in the literature, the so-called compass-gait biped, and also a more anthropomorphic biped with knees.

- Striking similarities between human and passive bipedal walking are noted in Chapter 3, including results from our own investigation of passive dynamics of the human ankle. Considering these results, we suggest a substantial connection exists between human and passive bipedal locomotion and hypothesize that mimicking features of human locomotion will prove useful when designing controls for passive-dynamic bipedal robots. We highlight two particular features of human walking we will duplicate when testing our hypothesis with a biomimetic control in Chapter 6.

- Chapter 4 lays the theoretical framework for energy-shaping control, which is the basis of part of the control design in Chapter 5 and all of the work in Chapter 6. In this chapter, we consider the *matching condition*, the nonlinear partial differential equation (PDE) whose solution defines the achievable forms of energy for fully actuated or underactuated bipeds. We show how the daunting task of solving this nonlinear PDE reduces to the solution of a single linear PDE in the case of potential energy shaping and to the solution of one algebraic equation and two linear PDEs in the case of kinetic energy shaping. As an example, we solve the matching condition for the compass-gait biped in the case of actuation at the ankle alone and contrast the achievable forms of the closed-loop energy if potential shaping is used with the possible forms if kinetic shaping is used.
• In Chapter 5 we review our previously reported potential-shaping control that regulates the forward speed of passive-dynamic bipeds. The fact that this control happens to hold step length constant suggests trajectory time-scaling is the mechanism underlying this speed regulating control. We develop a general time-scaling control for passive-dynamic bipeds and show that the potential-shaping control is a special case. We show how our more general time-scaling control enables transitions between speeds in arbitrary short time.

• We reconsider speed regulation in Chapter 6, noting that the behavior of the fully actuated speed-regulating control of the previous chapter differs from the behavior of humans when changing speed. Notably, the speed-regulating control of Chapter 5 holds step length constant while humans adjust step length for each walking velocity. Using the energy-shaping methods of Chapter 4, we design a new biomimetic control that regulates forward speed while altering step length, as do humans. We show that this biomimetic control requires significantly less actuator energy than previously reported speed controls.

• Walking on slopes is considered in Chapter 7, in which we examine the symmetry of bipedal energy with respect to the slope of the walking surface. The kinetic energy of a biped, while naturally invariant to changes in slope, may be altered in the closed-loop, destroying its natural symmetry with respect to the slope. We show how closed-loop kinetic energy may be rendered symmetric once again through extension of previous work on controlled symmetries. Equipped with this symmetry control, we demonstrate regulation of speed and step length while walking on any desired slope.

• Gait regulation is formulated as an optimal control problem in Chapter 8. We show how optimal control of the hybrid dynamics of walking may be recast as optimal control of a continuous dynamical system with particular boundary constraints. We present a method for approximating the solution to the optimal control problem that is seeded with the passive limit cycle. Analysis of
the optimal controls and trajectories generated by this method give structural insight for design of controls for bipedal walking that minimize actuator effort.

- In the final chapter, we make conclusions and note directions for future work.
CHAPTER 2
MODELS AND ANALYSIS OF LOCOMOTION

In this chapter, we present a mathematical framework for modeling walking of a general $n$-degree-of-freedom biped. The dynamics of bipedal locomotion comprise a *single-support* phase, in which the mass of the biped is born by one leg while the other leg is free to swing, and a *double-support* phase, in which both legs contact the walking surface and support is transferred from one leg to the other. A general hybrid dynamical model for walking is presented in which the single-support phase is described by a smooth differential equation and the double-support phase is governed by an instantaneous velocity map.

We explain how periodic orbits corresponding to a walking gait can be found using a so-called shooting method. How to determine the stability of these orbits using Poincaré sections is considered next. Finally, we present two planar bipeds which we will use throughout this thesis to explore controlled walking. The first is the compass-gait biped, a simple two-link model that appears frequently in the literature due to its simple dynamics and low dimensionality. The second is a three-link biped with knees, a more anthropomorphic and slightly more sophisticated walking model. The dynamics of both models give rise to passive walking limit cycles while descending shallow slopes.

Our treatment will be necessarily brief as these biped models have been studied by numerous other researchers, e.g., [3,6,14,22,30,31]. For a more detailed derivation of a general planar bipedal model see [4].
Figure 2.1 A general three-dimensional biped. Solid lines represent rigid links of the biped. Filled circles denote single degree of freedom, revolute joints. The ankle joint connecting the tip of the support leg to the walking surface is, in general, any three-dimensional rotation. We do not model foot segments on either leg.

2.1 Biped Configuration and Dynamics

Consider an \( n \)-link biped modeled as a kinematic chain, such as the biped illustrated in Figure 2.1. Assuming no slipping, the tip of the support leg has, in general, three degrees of freedom (DOF) relative to the walking surface. The configuration of the first link with respect to the surface may be described by a rotation in three dimensions, i.e., by an element of the group of three-dimensional rotations \( SO(3) \). The configuration of the remaining \( n - 1 \) links correspond to the angles of the joints of the biped, all of which are assumed to be revolute pin joints each allowing a single DOF of rotation. Each joint angle may be described by an element of \( S^1 \); hence the configuration space of all \( n - 1 \) remaining links is \( S \times S \times \cdots \times S = \mathbb{T}^{n-1} \), the \( n - 1 \) torus. Therefore the general configuration space of the entire biped is \( Q = SO(3) \times \mathbb{T}^{n-1} \). The tangent space is then \( TQ = so(3) \times \mathbb{R}^{n-1} \), where \( so(3) \) is the group of \( 3 \times 3 \) skew-symmetric matrices, i.e., the Lie algebra of \( SO(3) \).

2.1.1 Dynamics of the single-support phase

The dynamics of both legs of the biped during the single-support phase are governed by a differential equation. In this phase, the natural kinetic and potential energy vary smoothly in an interaction that swings the nonsupport leg ahead of the support leg. To write the equations of motion, it is common to introduce \( n \) local co-
ordinates for the configuration space $Q$. For example, one may choose \( \{q_1, q_2, \ldots, q_n\} \) as a coordinate chart, where $q_1, q_2, q_3$ are Euler angles that parameterize $SO(3)$ and $q_4, q_5, \ldots, q_n$ each specify one angle of $\mathbb{T}^{n-3}$.

Given a set of local coordinates, the kinetic energy $K : Q \times TQ \to \mathbb{R}$ of the biped may be expressed in the standard form quadratic in the velocities $\dot{q} \in TQ$ as follows:

$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

where $M \in \mathbb{R}^{n \times n}$ is the positive definite mass matrix of the biped. Writing the potential energy $V : Q \to \mathbb{R}$, we may now compose the Lagrangian $L : Q \times TQ \to \mathbb{R}$ of the biped in the usual way:

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q). \quad (2.1)$$

The Euler-Lagrange equations for the biped in the single-support phase are given by

$$\frac{d}{dt} \nabla^T_\dot{q} L(q, \dot{q}) - \nabla^T_q L(q, \dot{q}) = Bu \quad (2.2)$$

where $u \in \mathbb{R}^m$ is the vector of control input torques and the gradient $\nabla_x$ is the usual row vector of partial derivatives with respect to the components of vector $x$. The matrix $B \in \mathbb{R}^{n \times m}$ maps the control inputs $u$ to the various joints of the biped and is assumed to satisfy $\text{rank}(B) = m \leq n$, i.e., $B$ is of full column rank.

Substituting (2.1) into (2.2), the equations of motion may be factored into the form common in the robotics literature [32]:

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + G(q) = Bu \quad (2.3)$$

where $G = \nabla^T_q V$ is the $n$-vector of potential-dependent terms and $C \in \mathbb{R}^{n \times n}$ is the matrix of centrifugal and Coriolis terms (i.e., terms quadratic in the velocities $\dot{q}$) given by

$$C(q, \dot{q}) = D_q \left( M(q) \dot{q} \right) \dot{q} - \frac{1}{2} \nabla^T_q \left( \dot{q}^T M(q) \dot{q} \right) \quad (2.4)$$

where Jacobian $D_x(y)$ is the standard Jacobian matrix of partial derivatives of vector $y$ with respect to the components of vector $x$. 

14
Let \( x = [q, \dot{q}]^T \in Q \times TQ \) denote any point in the state space. We may rewrite (2.3) in affine form as
\[
\dot{x}(t) = f(x(t), u(t))
\]
where
\[
f(x(t), u(t)) = \begin{bmatrix} \dot{q}(t) \\ -M(q(t))^{-1}(C(x(t))\dot{q}(t) + G(q(t))) \end{bmatrix} + \begin{bmatrix} 0 \\ M(q(t))^{-1}B(q(t)) \end{bmatrix} u(t).
\]

### 2.1.2 Dynamics of the double-support phase

Support of the biped is transferred from one leg to the other in the double-support phase in an instantaneous dynamical event that results in a discontinuity in kinetic energy while potential energy is held constant. Let \( t_i \) denote the moment the tip of the nonsupport leg contacts the walking surface. Under standard assumptions, namely

1. the impact is perfectly plastic (no bounce occurs)
2. support is instantaneously transferred from one leg to the other
3. the legs do not slip along the ground during the impact

the angles of the legs \( q(t_i) \) are unchanged by the impact while the angular velocities \( \dot{q}(t_i) \) undergo an instantaneous discontinuity [30]. Static joint angles result in the potential energy \( V(q) \) being held constant through the impact, while the velocity discontinuity corresponds to an instantaneous change in the kinetic energy of the biped.

From the second assumption we assume the biped’s gait is “flat-footed”; i.e., we may define unambiguously a function \( H : Q \rightarrow \mathbb{R} \) specifying the height of the tip of the nonsupport leg above the walking surface. Therefore the impact occurs when \( H(q) = 0 \) and \( \frac{dH}{dt}(q) < 0 \), when the tip of the nonsupport leg contacts the surface in
the downward direction. This assumption of flat-footed walking avoids heel-strike—
double-support—toe-off scenarios which introduce subtleties beyond the scope of this
work.

Several methods exist for computing the discrete change in velocity (and, hence,
kinetic energy) that occurs during the double-support phase. These include the
method of [14] that involves equating angular momentum before and after the impact
as well as the method of [3, 33] that involves integrating the Lagrange dynamics
across the moment of impact. Both methods result in identical, linear mappings of
preimpact to postimpact velocities which may be written

$$\dot{q}(t^+_i) = h(q(t^-_i))\dot{q}(t^-_i)$$

(2.6)

for a suitable function $h : Q \to \mathbb{R}^{n \times n}$.

### 2.1.3 Hybrid dynamical model

We now join the smooth differential equation that governs the single-support
phase with the discrete velocity map for the double-support phase into a single hybrid
dynamical model for bipedal walking. In the state space of the biped $Q \times TQ$ the two
phases are distinguished by the *guard* $S_i \subset Q \times TQ$, a subset of the state space that
corresponds to angles and velocities that trigger the discrete impact map. In other
words, points in $S_i$ correspond to the tip of the nonsupport leg striking the walking
surface in a downward direction, i.e.,

$$S_i = \{(q, \dot{q})|H(q) = 0, \dot{H}(q) < 0\}. \quad (2.7)$$

Trajectories outside $S_i$ evolve smoothly according to the Lagrange differential equa-
tion (2.3) until entering $S_i$, at which point the discrete impact event (2.6) induces
a jump in the velocities. The impact maps the trajectory outside of $S_i$ where the
trajectory again evolves according to the differential equation until $S_i$ is encountered
once more. For the biped, this process of periodically encountering $S_i$ corresponds to
periodic behavior of bipedal walking in which the nonsupport leg strikes the walking
surface and support is transferred from one leg to the next.
Letting \( x = [q, \dot{q}]^T \) as before, the hybrid dynamics of the walking biped are summarized as follows:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u) \quad \text{for } x(t) \notin S_i \\
q(t_i^+) &= q(t_i^-) \\
\dot{q}(t_i^+) &= h(x(t_i^-))\dot{q}(t_i^-) \\
\end{align*}
\]  \hspace{1cm} (2.8)

for \( x(t_i) \in S_i \).

Some walking models incorporate additional discrete events during a single step. For instance, we will consider a biped with knees and assume the knee joint of the nonsupport leg is initially free at the start of a step but locks at the moment of full extension, as in [22, 31]. For such a biped, this knee-locking event is modeled under assumptions similar to those of Section 2.1.2 (i.e., instantaneous, slipless, and perfectly plastic) and results in an additional discrete map and a corresponding guard. The hybrid model (2.8) extends readily to encompass this additional event, as we will explain below.

2.2 Limit Cycles

We now define some terms common in the nonlinear control literature. The flow\( \phi(x_0, t) = [q(t), \dot{q}(t)]^T \in Q \times TQ \) denotes a solution trajectory of the dynamics (2.8) beginning with initial condition \( x_0 \) at time \( t = 0 \). If, for some initial condition \( x^* \) and some time \( T \),

\[ \phi(x^*, T) = x^* \]

we say the flow is a periodic orbit. Although difficult to visualize in higher dimensions, a periodic orbit in two or three dimensions would look like a closed loop. If the periodic orbit is isolated—that is, there is a neighborhood around it containing no other periodic solutions—then we call it a limit cycle. A limit cycle in the state space of a bipedal robot indicates a continuous walking pattern.

If all trajectories beginning in some neighborhood around a limit cycle converge to the limit cycle, we say it is stable. We give the name basin of attraction to the
neighborhood surrounding a stable limit cycle which contains all the initial conditions that will converge to the limit cycle.

2.2.1 Passive-dynamic walking

For a frictionless bipedal robot with no control input (i.e., $u(t) \equiv 0 \ \forall t$), no energy is lost during the single-support phase of walking. However, the velocity discontinuity during the double-support phase impact results in a loss of kinetic energy as described above. The existence of a stable walking cycle requires that the kinetic energy dissipated by the impacts be restored during the step.

One way this may be accomplished is by active feedback control (i.e., $u(t) \neq 0 \ \forall t$) that injects energy into the gait during the single-support phase and compensates for the loss of energy during the impact. Another way to replace the lost energy is to place the biped on a gentle slope and allow progression of the biped down the slope to convert the change in potential energy on the slope into additional kinetic energy. Careful selection of walking slope and the initial conditions of the biped will result in the kinetic energy dissipated by the impact being perfectly balanced by the addition of energy on the downhill slope, yielding a stable walking limit cycle [2,22,25]. This natural, uncontrolled mode of locomotion is known as passive-dynamic walking [9]. In the following sections we will consider two such passive-dynamic bipeds. We will refer to these bipeds throughout the remainder of the work as we develop various controls for regulating gait.

2.3 Finding Limit Cycles

For hybrid dynamical systems like our walking models, we find the use of Poincaré surfaces (also known as Poincaré sections) quite helpful in locating limit cycles. A Poincaré surface samples a near-periodic flow $\phi(t)$ once every period. If the intersections of the flow with the Poincaré surface converge to a single point over time, a stable limit cycle has been found. The effect is much like a strobe light illuminating
a walking figure at the same time during each step. If, over time, the strobed images become identical, the human has settled into a stable walking pattern.

In this section, we discuss the discovery of limit cycles using subsequent intersections of the flow with a Poincaré surface.

### 2.3.1 Poincaré maps

We begin by choosing the Poincaré surface, a hyperplane $\Gamma \subset Q \times TQ$ of dimension $2n - 1$ through which the flow passes periodically, as illustrated in Figure 2.2. In general, the Poincaré surface $\Gamma$ can be selected to be transverse to any point on the gait cycle. However, for analysis of bipedal walking it is convenient to choose $\Gamma$ to correspond to the boundary of guard $S_i$; i.e., choose the Poincaré surface to be the collection of points in the state space $Q \times TQ$ corresponding to the impacts that mark the double-support phase.

The Poincaré surface $\Gamma$ may be identified by the vector $n_\Gamma = \nabla_x \gamma(x)$ normal to it and any point $x_\Gamma$ on the plane itself. So $\gamma(x_\Gamma) = 0 \ \forall \ x_\Gamma \in \Gamma$. We may also define $\Gamma$ via an algebraic equation of the state variables $\gamma(x)$ such that

\[
\gamma(x) = 0 \ \forall \ x \in \Gamma \\
\gamma(x) \neq 0 \ \text{elsewhere}.
\]
Clearly, all trajectories $\phi(x_0, t)$ of (2.8) that intersect $\Gamma$ will satisfy $\gamma(\phi(t)) = 0$ for some $t$. For analysis of bipedal walking, we are interested only when the trajectory intersects $\Gamma$ in a certain direction, so we add the condition that $\phi(t)$ must intersect $\Gamma$ in the “positive” direction. Since $f(\phi(t))$ specifies the direction of the flow at time $t$, trajectories that intersect the Poincaré surface in the “positive” direction satisfy the following condition:

$$n_{\Gamma} \cdot f(\phi(t)) > 0$$

where $\cdot$ is the vector dot product. We make the following definition.

**Definition** The positive Poincaré map $p^+(x_0) : \Gamma \to \Gamma$ returns a point on the Poincaré surface $\Gamma$ corresponding to the first positive intersection of the flow $\phi(x_0, t)$ with $\Gamma$, as shown in Figure 2.2. That is, the point $y = p^+(x_0) = \phi(x_0, T)$ satisfies

$$\gamma(y) = 0$$

and

$$n_{\Gamma} \cdot f(y) > 0$$

for the smallest possible time $T > 0$.

The usefulness of the Poincaré map lies in the fact that finding limit cycles in the state space is tantamount to finding fixed points of the Poincaré map, i.e., finding all $x^* \in \Gamma$ that satisfy $p^+(x^*) = x^*$. We now consider a technique for finding these fixed points.

### 2.3.2 Shooting method

Let

$$r(x^i) = p^+(x^i) - x^i$$

where $x^i$ is an initial condition on the Poincaré surface $\Gamma$. Clearly, $r(x^*) = 0$. If we begin with a point $x^0 \in \Gamma$ near a fixed point $x^*$ of the Poincaré map, applying the following Newton-Raphson update [34] will move the initial condition $x^i$ toward the fixed point. We update the initial condition $x^i$ as follows:

$$x^{i+1} = x^i - [D_x r(x^i)]^{-1} r(x^i)$$

(2.10)
where the Jacobian of the Newton-Raphson equation $D_x r(x^i)$ is found by differentiating (2.9) with respect to the initial condition $x^i$

$$D_x r(x^i) = D_x p^+(x^i) - I.$$ 

To compute the linearization of the Poincaré map $D_x p^+$, we recall Theorem D.1 of [34], from which we have that

$$D_x p^+(x^i) = \left[ I - \frac{f(y)n_I^T}{n_I \cdot f(y)} \right] \Phi(x^i, T) \quad (2.11)$$

where $y = p^+(x^i)$, $T$ is the time required for the trajectory to travel from $x^i$ to $y$, and $\Phi(x^i, T)$ is the so-called monodromy matrix, which will be defined in the next section.

Beginning with a point $x^0$ on the Poincaré surface $\Gamma$, we iterate the update equation (2.10) until $|r(x^i)|$ is sufficiently close to zero, at which point we conclude $x^i \approx x^*$ is a fixed point of the Poincaré map. Consequently, the flow $\phi(x^*, t)$ is a limit cycle of the hybrid dynamics (2.8).

This iterative method is called the Poincaré shooting method. While this method will identify limit cycles, it does not give any information on whether the limit cycle is stable or unstable. In the next section, we turn our attention to classifying the stability of limit cycles.

### 2.4 Stability of Limit Cycles

In the previous section, we equated the task of finding limit cycles of the hybrid system (2.8) with the task of finding fixed points of the Poincaré map $p^+$. In this section, we will determine the stability of the limit cycle by establishing the stability of the fixed point of the Poincaré mapping of a point on the limit cycle.

The (local) stability of the Poincaré map at a fixed point $x^*$ is determined by the eigenvalues of the linearized Poincaré map $D_x p^+(x^*)$, which is computed using (2.11). Due to the fact that the Poincaré surface is dimension 1 less than the dimension of the state space (i.e. the state space $Q \times TQ$ is dimension $2n$, while the Poincaré surface...
Γ is dimension $2n - 1$), one of the eigenvalues of $D_xp^+(x^*)$ is always 0 [34]. The
eigenvector corresponding to the 0 eigenvalue is transverse to the Poincaré surface Γ and
points in the direction of the flow when it intersects Γ.

The remaining $2n - 1$ eigenvalues of the linearized Poincaré map $D_xp^+(x^*)$ are
called the characteristic multipliers [34], which we denote $m_i$ for $i = 1 \ldots 2n - 1$. Due
to the discrete nature of the Poincaré map, stability of the map is determined by the
locations of the characteristic multipliers with respect to the unit circle.

We classify the (local) stability of the Poincaré map, and hence the limit cycle, as follows:

1. $|m_i| \leq 1$ $\forall i$. The Poincaré map and limit cycle are stable. All trajectories in
some neighborhood of the limit cycle will remain nearby.

2. $|m_i| < 1$ $\forall i$. The Poincaré map and limit cycle are asymptotically stable. All
trajectories in some neighborhood of the limit cycle will converge to the limit
cycle as $t \to \infty$.

3. $|m_i| > 1$ $\forall i$. The Poincaré map and limit cycle are unstable. All trajectories
in some neighborhood of the limit cycle will diverge from the limit cycle as $t \to \infty$.

4. At least one $|m_i| < 1$ and at least one $|m_j| > 1$ for $i, j \in 1 \ldots 2n - 1, i \neq j$. In
this case, the Poincaré map and limit cycle are said to be nonstable [34]. Some
trajectories near the limit cycle will converge to the limit cycle while others will
diverge.

We note that computing the linearization of the Poincaré map (2.11) requires
knowledge of the monodromy matrix $\Phi(x^i, T)$. We next consider how to construct
this fundamental matrix.
2.4.1 The monodromy matrix

Recall the flow $\phi(x_0, t)$ is the solution of the hybrid system (2.8) with initial condition $x_0$ at time $t = 0$. We compute the flow by integrating the continuous-time dynamics

$$\dot{\phi}(x_0, t) = f(\phi(x_0, t))$$

(2.12)

while the flow remains outside the guard $S$ and applying the discrete impact map

$$q(t_i^+) = q(t_i^-)$$
$$\dot{q}(t_i^+) = h\left(\phi(x_0, t_i^-)\right)$$

(2.13)

when the flow enters the guard $S$. For trajectories beginning at a fixed point of the Poincaré map $x_0 = x^*$, the limit cycle completes its closed orbit at time $t = T$ and we have $\phi(x^*, T) = \phi(x^*, 0) = x^*$.

Define the trajectory sensitivity $\Phi(x_0, t)$ as the linearization of the flow $\phi(x_0, t)$, i.e., the Jacobian of the flow with respect to the initial conditions $x_0$

$$\Phi(x_0, t) = D_{x_0}\phi(x_0, t).$$

(2.14)

It follows from (2.14) that $\Phi(x_0, 0) = D_{x_0}x_0 = I$. Just as we compute the flow at time $t$ by integrating (2.12), we may compute the trajectory sensitivity at time $t$ by integrating the linearization of (2.12), i.e., by integrating

$$\dot{\Phi}(x_0, t) = D_x f\left(\phi(x_0, t)\right) D_{x_0} \phi(x_0, t)$$
$$= D_x f\left(\phi(x_0, t)\right) \Phi(x_0, t).$$

(2.15)

Equation (2.15) is called the variational equation, and we compute trajectory sensitivity $\Phi(x_0, t)$ by integrating this differential equation while the flow remains outside the guard $S$.

At time $t = t_i^-$, the flow enters guard $S$ and the discrete impact mapping (2.13) is applied to the flow. At the same time, the trajectory sensitivity undergoes an instantaneous discrete mapping. We recall the following result.
Proposition Subject to the conditions of Section 2.1.2, the trajectory sensitivity \( \Phi(x_0,t) \) experiences an instantaneous discrete change at time \( t = t_i \), when the flow enters guard \( S \). Define event function \( s(x) \) such that \( s(x) = 0 \) for all points \( x \in Q \times TQ \) on the boundary of guard \( S \) and \( s(x) \neq 0 \) elsewhere. Then the trajectory sensitivity the instant after the impact is given by

\[
\Phi(x_0,t_i^+) = D_x h(\phi^-)\Phi(x_0,t_i^-) - \left[ f(\phi^+) - D_x h(\phi^-)f(\phi^-) \right] \tau_{x_0} \quad (2.16)
\]

where

\[
\tau_{x_0} = -\nabla_x s(\phi^-)\Phi(x^*,t_i^-) \\
\phi^+ = \phi(x_0,t_i^+) \\
\phi^- = \phi(x_0,t_i^-).
\]

Proof This proposition for the sensitivity mappings of our class of bipedal robots is a specialization of a broader result that holds for general hybrid systems. Full development of the general case appears with proof in [35]. See [36] for the statement specialized to bipedal walking.

We combine (2.15) and (2.4.1) into the following hybrid system defining the trajectory sensitivity for all time \( t > 0 \):

\[
\dot{\Phi}(x_0,t) = D_x f\left(\phi(x_0,t)\right)\Phi(x_0,t) \quad \text{for } \phi(x_0,t) \notin S \\
\Phi(x_0,t_i^+) = D_x h(\phi^-)\Phi(x_0,t_i^-) - \left[ f(\phi^+) - D_x h(\phi^-)f(\phi^-) \right] \tau_{x_0} \in S \quad \text{for } \phi(x_0,t_i^-) \in S.
\]

If the flow is nearly periodic, we call the trajectory sensitivity matrix after one period \( T \) the monodromy matrix \( \Phi(x_0,T) \). The monodromy matrix is needed for computing the update equation of the Poincaré shooting method (2.10).

Once the shooting method has converged to a fixed point \( x^* \) of the Poincaré map, the monodromy matrix \( \Phi(x^*,T) \) may be used to determine the stability of the limit cycle. As with the linearized Poincaré map \( D_x p^+(x^*) \), the stability of the limit cycle is indicated by the location of \( 2n - 1 \) of the monodromy matrix’s \( 2n \) eigenvalues.
with respect to the unit circle. In fact, $2n - 1$ eigenvalues of the monodromy matrix exactly match the $2n - 1$ characteristic multipliers of the linearized Poincaré map. The remaining monodromy matrix eigenvalue is 1 and corresponds to an eigenvector pointing in the direction $f\left(\phi(x^*, T)\right)$, i.e., the direction the trajectory is heading when it intersects the Poincaré surface $\Gamma$.

We now present two bipedal walking models and analyze both using the methods presented thus far in this chapter.

### 2.5 The Compass-Gait Biped

The *compass-gait biped* [2, 9, 14] is shown in Figure 2.3. All motion of the biped is confined to the sagittal plane; therefore, the configuration of this two-link mechanism can be described by $q = [\theta_1, \theta_2]^T$, where $\theta_i \in S^1, i = 1, 2$ are the angles of the support leg and the nonsupport leg, hence $Q = \mathbb{T}^2$. In the bipedal walking literature it is common to define $\theta_1, \theta_2$ w.r.t. the vertical as shown in the figure.

#### 2.5.1 Energy

The kinetic and potential energy of the biped are given by

$$K(q, \dot{q}) = \frac{1}{2}\left[\left((a^2 + \ell^2) + \ell^2 m_H\right)\dot{\theta}_1^2 + b^2 m_2 \dot{\theta}_2^2\right]$$

$$-mb\ell \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2$$

$$V(q) = g\left(m(a + \ell) + m_H \ell\right) \cos(\theta_1) - gb m \cos(\theta_2)$$

where the parameters used in our simulation are specified in Table 2.1.

#### 2.5.2 Hybrid dynamics

Locomotion of the compass-gait biped is divided into two phases, as shown in Figure 2.4.
Table 2.1 Parameters for the compass gait biped used in our simulations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>5 kg</td>
<td>mass of each leg</td>
</tr>
<tr>
<td>$m_H$</td>
<td>10 kg</td>
<td>mass of hip</td>
</tr>
<tr>
<td>$a$</td>
<td>0.5 m</td>
<td>distance between tip of leg and leg mass</td>
</tr>
<tr>
<td>$b$</td>
<td>0.5 m</td>
<td>distance between leg mass and hip</td>
</tr>
<tr>
<td>$\ell = a + b$</td>
<td>1 m</td>
<td>total length of leg</td>
</tr>
<tr>
<td>$g$</td>
<td>9.8 m/s$^2$</td>
<td>acceleration due to gravity</td>
</tr>
</tbody>
</table>

Figure 2.4 Walking phases of the compass gait biped: (a) the single-support phase and (b) the double-support phase, i.e., the impact with the walking surface.
The differential equation of motion (2.3) that governs the step is straightforwardly derived from the energy and its components are given by

\[ M(q) = \begin{bmatrix} (m_H + m)\ell^2 + ma^2 & -mlb \cos(\theta_1 - \theta_2) \\ -mlb \cos(\theta_1 - \theta_2) & mb^2 \end{bmatrix} \]  

(2.19)

\[ G(q) = g \begin{bmatrix} - (m_H \ell + ma + m\ell) \sin(\theta_1) \\ mb \sin(\theta_2) \end{bmatrix} \]  

(2.20)

and \( C(q, \dot{q}) \) follows from (2.4).

The perpendicular distance from the walking surface to the tip of the nonsupport leg is given by

\[ H(q) = \ell \left( \cos(\theta_1 + \psi) - \cos(\theta_2 + \psi) \right) \]  

(2.21)

where \( \psi \) is the slope of the ground. Impacts occur when the tip of the swing leg contacts the walking surface in a downward direction. These two conditions define the guard \( S \) for the compass-gait biped as follows:

\[ S_i = \left\{ (q, \dot{q}) \left| H(q) = 0, \dot{H}(q) < 0 \right. \right\}. \]

Conservation of angular momentum provides an explicit solution for the impact map (2.6), resulting in the following linear mapping of pre- to postimpact velocities [14]:

\[ \dot{q}(t^+) = h \left( q(t^-) \right) \dot{q}(t^-) \]  

(2.22)

where

\[ h(q(t^-)) = \begin{bmatrix} h_{11}^+ & h_{12}^+ \\ h_{21}^+ & h_{22}^+ \end{bmatrix}^{-1} \begin{bmatrix} h_{11}^- & h_{12}^- \\ h_{21}^- & h_{22}^- \end{bmatrix} \]
and

\[
\begin{align*}
    h_{11}^+ &= m\ell(b - \ell \cos(\theta_1 - \theta_2^-)) + ma^2 + m_H \ell^2 \\
    h_{12}^+ &= mb(b - \ell \cos(\theta_1 - \theta_2^-)) \\
    h_{21}^+ &= -mbl \cos(\theta_1^- - \theta_2^-) \\
    h_{22}^+ &= mb^2 \\
    h_{11}^- &= -mab + (m_H \ell^2 + 2ma\ell) \cos(\theta_1^- - \theta_2^-) \\
    h_{12}^- &= h_{21}^- = -mab \\
    h_{22}^- &= 0.
\end{align*}
\]

2.5.3 Passive limit cycle

Due to its deeply nonlinear nature, there is no known closed-form solution to the dynamics of the compass-gait biped. Instead, we will perform numerical integration using the MATLAB differential equation solver \texttt{ode45} as we search for a limit cycle of the compass-gait biped.

We set the compass-gait robot on a slope \( \phi = 3^\circ \) and set the torque input \( u(t) \equiv 0 \) \( \forall t \). The sole source of energy for the robot is now the change in potential on the slope. We will now search for a passive limit cycle using the Poincaré shooting method. We find it useful to choose our Poincaré surface \( \Gamma \) to be the boundary of the guard \( S \) as suggested in \cite{36}, i.e.,

\[
\gamma(x) = s(x) = \theta_1 + \theta_2 - 2\psi.
\]

So \( \gamma(x) = s(x) = 0 \) when the tip of the nonsupport leg is touching the ground.

Starting from a near-periodic initial condition on the Poincaré surface \( \Gamma \), we iterate the shooting method of Section 2.3.1 until it converges to the following fixed point:

\[
x^* = \begin{bmatrix} 0.2187 & -0.3234 & -1.0918 & -0.3772 \end{bmatrix}^T.
\]

Using this \( x^* \) as an initial condition, we numerically solve the hybrid dynamics (2.8) of the compass-gait biped and find the limit cycle plotted in Figure 2.5. The
filled circles in the figure denote the initial conditions for the two legs and the arrows indicate the direction in which the trajectories evolve in forward time. At the ground impacts, the legs switch support roles instantaneously as shown by the vertical jumps from support to nonsupport trajectory and vice-versa.

For the compass-gait biped, period-1 limit cycles for the given set of parameters exist only on shallow slopes in approximate range $\psi \in (0^\circ, 4.5^\circ)$. Extensive parameter variation studies with several passive-dynamic bipeds [14, 17, 37] have reported no more than one period-1 limit cycle for any biped on any given slope. However, as parameters of a passive-dynamic biped or the ground slope are altered, the bipeds experience period-doubling bifurcations. As parameter values or slope are further changed, period-doubling bifurcations continue and the dynamic system descends into chaotic behavior.

2.5.4 Stability of the limit cycle

The eigenvalues of the compass-gait biped’s monodromy matrix $\Phi(x^*, T)$ and linearized Poincaré map $D_x p^+(x^*)$ are shown in Table 2.2. As expected, one eigenvalue
Table 2.2 Eigenvalues of the monodromy matrix and linearized Poincaré map for the passive limit cycle of the compass-gait biped on slope $\psi = 3^\circ$.

<table>
<thead>
<tr>
<th>\Phi(x^*, T) :</th>
<th>1.0000</th>
<th>-0.1997 + j0.5445</th>
<th>-0.1997 - j0.5445</th>
<th>0.1316</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_x p^+(x^*)$ :</td>
<td>0.0000</td>
<td>-0.1997 + j0.5445</td>
<td>-0.1997 - j0.5445</td>
<td>0.1316</td>
</tr>
</tbody>
</table>

of the monodromy matrix is 1 and one eigenvalue of the linearized Poincaré map is 0. The remaining eigenvalues—the so-called characteristic multipliers—match up. Since all the characteristic multipliers lie within the unit circle, we conclude this limit cycle is (locally) asymptotically stable.

## 2.6 The Biped with Knees

The more anthropomorphic biped with knees [22,31] is shown in Figure 2.6 and the parameters used in our simulations are given in Table 2.3. As with the compass-gait biped, all motion is confined to the sagittal plane; therefore, the configuration of this three-link mechanism can be described by $q = [\theta_1, \theta_2, \theta_3]^T$, where $\theta_i \in S^1, i = 1, 2, 3$ are the angles of the support (stance) leg, nonsupport (swing) thigh, and nonsupport (swing) shank, respectively, all measured with respect to the vertical. Hence $Q = T^3$.

### 2.6.1 Energy

The kinetic and potential energy of this biped are given by

$$K(q, \dot{q}) = \frac{1}{2} \left[ (a_1^2m_1 + (m_2 + m_3 + m_H)\ell_1^2) \dot{\theta}_1^2 + (b_2^2m_2 + m_3\ell_2^2) \dot{\theta}_2^2 + b_3^2m_3\dot{\theta}_3^2 \right]$$

$$-\ell_1(b_2m_2 + m_3\ell_2) \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 - b_3m_3\ell_1 \cos(\theta_1 - \theta_3)\dot{\theta}_1\dot{\theta}_3$$

$$+b_3m_3\ell_2 \cos(\theta_2 - \theta_3)\dot{\theta}_2\dot{\theta}_3$$ (2.23)

$$V(q) = g(a_1m_1 + (m_2 + m_3 + m_H)\ell_1) \cos(\theta_1) - g(b_2m_2 + m_3\ell_2) \cos(\theta_2)$$

$$-gb_3m_3 \cos(\theta_3).$$ (2.24)
Figure 2.6 The biped with knees.

Table 2.3 Parameters for the biped with knees.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>5 kg</td>
<td>mass of the support leg</td>
</tr>
<tr>
<td>$m_2$</td>
<td>3.5 kg</td>
<td>mass of the nonsupport leg thigh</td>
</tr>
<tr>
<td>$m_3$</td>
<td>1.5 kg</td>
<td>mass of the nonsupport leg shank</td>
</tr>
<tr>
<td>$m_h$</td>
<td>10 kg</td>
<td>mass of the hip</td>
</tr>
<tr>
<td>$a_1$</td>
<td>0.53 m</td>
<td>distance between the mass and tip of the support leg</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.47 m</td>
<td>distance between hip and support leg mass</td>
</tr>
<tr>
<td>$\ell_1$</td>
<td>$a_1 + b_1$</td>
<td>total length of support leg</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0.15 m</td>
<td>distance between knee and mass of nonsupport leg thigh</td>
</tr>
<tr>
<td>$b_2$</td>
<td>0.35 m</td>
<td>distance between hip and mass of nonsupport leg thigh</td>
</tr>
<tr>
<td>$\ell_2$</td>
<td>$a_2 + b_2$</td>
<td>total length of nonsupport leg thigh</td>
</tr>
<tr>
<td>$a_3$</td>
<td>0.25 m</td>
<td>distance between shank mass and tip of the nonsupport leg</td>
</tr>
<tr>
<td>$b_3$</td>
<td>0.25 m</td>
<td>distance between knee and mass of nonsupport leg shank</td>
</tr>
<tr>
<td>$\ell_3$</td>
<td>$a_3 + b_3$</td>
<td>total length of nonsupport leg shank</td>
</tr>
<tr>
<td>$g$</td>
<td>9.8 m/s$^2$</td>
<td>acceleration due to gravity</td>
</tr>
</tbody>
</table>
2.6.2 Continuous-time dynamics of the single-support phase

From the energy we may derive the Lagrange dynamics that govern the continuous-time portion of the step, given by

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = B(q)u - \tau$$  \tag{2.25}$$

where (2.25) includes an extra term $\tau$ which will be used to model the change in the dynamics that occurs when the knee locks. The term $\tau$ is a fictional torque that locks the knee after it has been fully extended. In practice, no torque is applied; locking of the knee joint is accomplished by a passive mechanism. In simulation, we will set $\tau = 0$ when the knee is unlocked; its value when the knee is locked will be described below.

The inertia matrix $M$ and gravity vector $G$ are given by

$$M(q) = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$m_{11} = m_{1} a_{1}^2 + (m_{h} + m_{2} + m_{3})\ell_{1}^2$$

$$m_{12} = m_{21} = -(m_{2} b_{2} \ell_{1} + m_{3} \ell_{1} \ell_{2}) \cos(\theta_{1} - \theta_{2})$$

$$m_{13} = m_{31} = -m_{3} b_{3} \ell_{1} \cos(\theta_{1} - \theta_{3})$$

$$m_{22} = m_{2} b_{2}^2 + m_{3} \ell_{2}^2$$

$$m_{23} = m_{32} = m_{3} b_{3} \ell_{2} \cos(\theta_{2} - \theta_{3})$$

$$m_{33} = m_{3} b_{3}^2$$

$$G(q) = g \begin{bmatrix} -(m_{1} a_{1} + m_{2} \ell_{1} + m_{3} \ell_{1} + m_{h} \ell_{1}) \sin(\theta_{1}) \\ (m_{2} b_{2} + m_{3} \ell_{2}) \sin(\theta_{2}) \\ m_{3} b_{3} \sin(\theta_{3}) \end{bmatrix}$$

and $C$ follows from (2.4). We may rewrite (2.25) in affine form as

$$\dot{x}(t) = f(x(t), u(t), \tau)$$
Figure 2.7 The walking cycle of the biped with knees occurs in two phases: the single-support phase (a-c) and the double-support phase (d). The single-support phase is divided into two subphases, (a) single-support with unlocked knee and (c) single-support with locked knee, separated by (b) the knee lock event.

2.6.3 Phases of walking

As with the compass-gait biped, the dynamics of the biped with knees comprise a single-support phase and a double-support phase. However, we further divide the dynamics of the single-support phase for the biped with knees into two subphases as shown in Figure 2.7. In the first single-support subphase, the knee is “unlocked” allowing the thigh ($\theta_2$) and shank ($\theta_3$) of the swing leg to move independently of each other. Once the knee is fully extended, i.e., when $\theta_2 = \theta_3$, a passive mechanism at the knee “locks” the knee joint. The second subphase of the single-support phase follows the knee locking event. During this subphase, the knee is “locked” (i.e., the passive mechanism at the knee enforces the constraint $\theta_2 = \theta_3$) and the swing leg moves as a single rigid link until the tip of the swing leg encounters the walking surface and begins the double-support phase.

Since the velocities and accelerations of the swing thigh and shank are, in general, different prior to the moment of knee lock, requiring the velocities and accelerations to
be identical immediately after the event results in an impulsive force on the biped and a corresponding discontinuity in the joint velocities. This discontinuity establishes a second discrete event in the hybrid dynamics of the biped with knees. As with the surface impact event of the compass-gait biped, we detect the knee lock event using a guard, which we will designate $S_k$, and upon entering the guard we apply an instantaneous velocity mapping
\[
\dot{q}(t_k^+) = h_k(q(t_k^-))\dot{q}(t_k^-)
\]
where the time $t_k$ denotes the moment of knee lock and the superscripts $-$ and $+$ denote the instants immediately before and after the lock.

We summarize the hybrid dynamics of the biped with knees as follows:
\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t), \tau) & \text{for } x(t) \notin S_i \cup S_k \\
q(t_i^+) &= q(t_i^-) & \text{for } x(t_i) \in S_i \\
\dot{q}(t_i^+) &= h(x(t_i^-))\dot{q}(t_i^-) & \text{for } x(t_i) \in S_i \\
q(t_k^+) &= q(t_k^-) & \text{for } x(t_k) \in S_k \\
\dot{q}(t_k^+) &= h_k(x(t_k^-))\dot{q}(t_k^-) & \text{for } x(t_k) \in S_k
\end{align*}
\]
where $S_i$ and $h$ denote the guard and velocity map corresponding to the surface impact, as with the compass-gait biped.

### 2.6.4 Knee lock dynamics of the single-support phase

The knee-locking event occurs when the knee is fully extended, i.e., when the angles of the nonsupport thigh and nonsupport shank match while the nonsupport shank is swinging forward. Therefore we define the guard $S_k \in Q \times TQ$ corresponding to the knee locking event as follows:
\[
S_k = \left\{ (q, \dot{q}) \mid \theta_2 - \theta_3 = 0, \dot{\theta}_2 - \dot{\theta}_3 < 0 \right\}.
\]

Following the knee locking event, the swing thigh and shank move together, i.e., $\theta_2 = \theta_3$. In practice, this constraint is enforced by a passive mechanism at the knee.
In simulation, we add a fictitious torque $\tau$ to the continuous dynamics to enforce the constraint.

Differentiating the constraint $\theta_2 - \theta_3 = 0$ twice, we have

\[
\ddot{\theta}_2 - \ddot{\theta}_3 = [0 \ 1 \ -1]\ddot{q} = J\ddot{q} = 0 \tag{2.28}
\]

where we have defined row vector $J = [0, 1, -1]$ as in [31]. Solving Equation (2.25) for $\ddot{q}$ we have

\[
\ddot{q} = M^{-1}(q)\left(B(q)u - \tau - C(q, \dot{q})\dot{q} - G(q)\right) \tag{2.29}
\]

Substituting (2.29) into (2.28) and solving for $\tau$ we find

\[
\tau = J^\dagger(q)JM^{-1}(q)\left(B(q)u - C(q, \dot{q})\dot{q} - G(q)\right) \tag{2.30}
\]

when the knee is locked, where $J^\dagger$ is the left pseudo-inverse of $JM^{-1}$, i.e.,

\[
J^\dagger(q) = J^T\left(JM^{-1}(q)J^T\right)^{-1}.
\]

When the knee is unlocked, the constraint is not enforced and $\tau = 0$.

Making the standard assumptions about the knee locking event, we may write the change in velocities induced by the event as in (2.26) where the mapping $h_k$ is given by [31]

\[
h_k(q(t_k^-)) = I - M(q(t_k^-))^{-1}J^\dagger(q(t_k^-))J.
\]

### 2.6.5 Dynamics of the double-support phase

After the knee locks, the biped consists of two rigid legs and is dynamically identical to the compass-gait biped. The configuration can be described by the vector $[\theta_1, \theta_2]^T$ where $\theta_2 = \theta_3$. When the nonsupport leg strikes the ground, the behavior is identical to that of the compass-gait biped. Therefore, the velocity mapping $h$ of pre- to post-impact velocities is identical to (2.22).
2.6.6 Passive limit cycle

A passive \((u(t) \equiv 0 \ \forall t)\) limit cycle exists for hybrid dynamics of the biped with knees on a slope of \(\psi = 0.052 \text{ rad (}3^\circ\)) with initial conditions

\[
x^* = \begin{bmatrix} 0.2210 & -0.3257 & -0.3257 & -1.0834 & -0.3589 & -0.3589 \end{bmatrix}^T
\]

and is shown in Figure 2.8. As before, the filled circles denote the initial conditions for the two legs (the swing thigh and shank begin from the same initial condition) and the arrows indicate the direction in which the trajectories evolve in forward time. Velocity discontinuities are visible at the two events: the knee lock and ground impact. \(\theta_2 = \theta_3\) at the moment of knee lock, and this equality is enforced by a constraint beyond the lock event. After the surface impact, the legs switch support roles as in the compass-gait biped.

These initial conditions above correspond to the start of a step. The continuous time dynamics (2.25) with \(\tau = 0\) are integrated until the trajectory enters guard \(S_k\) corresponding to knee locking conditions at time \(t_k = 0.4128 \text{ s, i.e., the moment the knee is fully extended. We apply the knee lock velocity map (2.26), implementing the dynamical effects of the knee locking impulse on the biped. Beyond the knee}
lock, we integrate (2.25) once more with $\tau$ given by (2.30) to enforce the knee lock constraint. Integration is halted when the trajectory enters guard $S_i$ corresponding to contact with the ground surface at time $t_i = 0.7339$ s. Applying the impact map (2.6) to the velocities just prior to impact $\dot{q}(t_i^-)$ and swapping $\theta_1$ and $\theta_2 = \theta_3$ due to the change of support roles after the double-support phase, the trajectory returns to the same initial conditions to begin the next step, completing the limit cycle.

### 2.6.7 Stability of the limit cycle

The shooting method presented in this chapter may be used to find the limit cycle of the biped with knees; however, the dynamics of the biped with knees preclude implementation of the variational equation to compute the monodromy matrix. The problem is due to the knee lock event, after which the dynamics are constrained by $\theta_2 = \theta_3$. This constraint effectively reduces the dimensionality of the dynamics by 2 and results in a singularity. Trajectories prior to the knee lock are transverse to the guard $S_k$, but trajectories following the knee lock are tangential to the guard. The lack of transversality following the knee lock event introduces a singularity into the trajectory sensitivity [38].

As an alternative to analytically computing the monodromy matrix using the variational equation, we may approximate the monodromy matrix as discussed in [22]

$$\tilde{\Phi}(x_0, T) \approx \begin{bmatrix}
\frac{\phi(x_0 + dx e_1, T_1) - \phi(x_0, T)}{dx} & \frac{\phi(x_0 + dx e_2, T_2) - \phi(x_0, T)}{dx} & \cdots & \frac{\phi(x_0 + dx e_6, T_6) - \phi(x_0, T)}{dx}
\end{bmatrix}
$$

(2.31)

where $dx \in \mathbb{R}$ is some scalar perturbation, $e_i$ is the unit vector for the $i$th coordinate of the state space $Q \times TQ$, and $T_i, i = 1...6$ are the times required for the perturbed flows to return to the Poincaré surface. Care must be taken when selecting the perturbation $dx$. If $dx$ is too large, flows originating from the perturbed initial conditions will not return to the Poincaré surface. If $dx$ is too small, the fractions in (2.31) will become very large.

While approximating the monodromy matrix in this fashion allows us to successfully implement the shooting method and find the limit cycle, we note the eigenvalues
Table 2.4 Approximate eigenvalues for the limit cycle of the biped with knees with $dx = 0.01$.

| $\Phi(x^*, T)$: | $0.5023 + j0.7114$ | $0.5023 - j0.7114$ | $0.0911$ | $0.0000$ | $0.0000$ | $0.0000$ |
| $D_x p^+(x^*)$: | $0.5023 + j0.7114$ | $0.5023 - j0.7114$ | $0.0911$ | $0.0000$ | $0.0000$ | $0.0000$ |

of the approximated monodromy matrix $\tilde{\Phi}(x_0, T)$ cannot be used to determine stability. Adjusting the perturbation value $dx$ will move the eigenvalues toward or away from the origin while preserving the arrangement of the eigenvalues about the origin. The eigenvalues of $\tilde{\Phi}(x_0, T)$ and the Poincaré map $p^+(x^*)$ for the biped with knees with $dx = 0.01$ are shown in Table 2.4. Two of the eigenvalues at zero are consequences of the knee strike. The third eigenvalue at zero corresponds to the vector transverse to the guard, as in the case of the compass-gait biped.
CHAPTER 3

BIOLOGICAL LOCOMOTION

The biomechanics literature is rich with data that may inspire useful biomimetic controls for walking bipeds. However, before using results from biology to inform control design, we discuss in this chapter relevant results from the biomechanics and robotics literature which indicate a connection between human walking and passive-dynamic robot locomotion.

Our starting point for comparing human and passive-dynamic locomotion is visual; the locomotion pattern of passive-dynamic robots appears visually similar to human walking to our eyes. However, the connection between human and passive-dynamic locomotion runs much deeper. Just as passive-dynamic bipeds have a particular gait for which energy input is minimized, humans exhibit minimum energy consumption when walking with “self-selected” gait, i.e., the gait that feels most comfortable or natural. Numerous observations of this phenomenon [39] have led to the widely held (though yet unproven) conjecture that humans subconsciously select walking patterns in ways that minimize energy consumption.

These and other results considered in this chapter reveal strong similarities in the energetics and dynamics of human and passive-dynamic robot locomotion. We hypothesize that the energy efficiency of self-selected human gait is largely due to underlying passive dynamics. Such a link between humans and machines suggests we may use observation of energy-efficient human walking to inform control decisions for passive-dynamic bipeds. We here discuss two such cues: the critical role of the
ankle in the energetics of human gait and the human pattern of altering locomotion speed and step length simultaneously. We will mimic these features when designing an efficient control to regulate forward speed in Chapter 6, and consider whether they are visible in the optimal gaits considered in Chapter 8.

This chapter concludes with results from our own biomechanics investigation [26, 27] which tests a part of our hypothesis that a connection exists between passive-dynamic walking and human locomotion. In particular, we have studied the behavior of the human ankle during downhill walking and compared it with that of a simple passive-dynamic ankle under identical conditions. Our results indicate the behavior of the human ankle in level and downhill walking may be effectively duplicated by this simple passive-dynamic mechanism: a pin joint coupled with a revolute spring and damper. Moreover, we report that this passive-dynamic model is most effective on shallow downhill slopes—inclinations similar to those on which both humans and passive-dynamic robots walk with minimal energy input.

3.1 Energetic Similarities between Human and Passive-Dynamic Locomotion

Compelling energetic similarities are evident between passive-dynamic robot locomotion and human walking. As mentioned in Chapter 1, bipedal robots based on the passive-dynamic walking phenomenon are the first locomoting robots to report specific cost of transport, i.e.,

\[
\frac{\text{energy used}}{\text{weight} \times \text{distance traveled}}
\]

roughly equivalent to that of humans [2]. The comparable energy efficiency of these bipeds suggest that some portions of human locomotion may be due to passive dynamics.

Further similarities are evident in the energy required for locomotion on various slopes. Studies of overall energetic cost of human walking [40, 41] demonstrate that the metabolic cost is minimized on a downhill slope of about 5.7°, as shown in
Figure 3.1 Overall metabolic energy requirement for human walking on various slopes [40, 41]. We note that gradient = 100 · tangent(ψ), where ψ is the angle measured between the horizontal and the walking surface.

Figure 3.1. Similarly, passive-dynamic robots require minimum (in fact, zero) energy input on similarly shallow slopes. A variety of controls may be employed to cause the biped to walk on slopes outside this range, all of which require nonzero energy input. The result in energy consumption profiles for robotic locomotion that are remarkably similar to those of humans, i.e., profiles that demonstrate local minimums in energy consumption on shallow downhill slopes. For example, the compass-gait biped requires zero actuator input for steady walking on downhill slopes of (0, 4.5] degrees, or (−0, −7.9] gradient; energy consumption on slopes outside this range is nonzero. The exact energy consumption versus slope profile for the robot will vary depending on what control is used; however, we know the minimum energy consumption will occur in the gradient range (−0, −7.9], similar to the profile shown in Figure 3.1.

Other results from biomechanics suggest that portions of human locomotion may be due to natural passive dynamics. Electromyography (EMG) involves placing electrodes over muscle groups and monitoring the electrical activity of the muscles during particular activities. A plot of the electrical activity observed in lower-body muscle groups during walking is given in Figure 3.2, taken from [42]. A cartoon of a walking
Figure 3.2 EMG activity of the major muscle groups involved in human walking [42].

human helps us identify what part of the walking cycle corresponds to the activity in various muscle groups.

While spikes of muscle activity are present during impact absorption (between heel contact and foot-flat) and during push-off (between heel-off and toe-off), most of the muscle groups are relatively quiet during the other portions of gait (a notable exception is the calf muscle group, which we will discuss in detail in the next section). The muscles of the lower body are particularly inactive during the nonsupport (swing) phase of walking from toe off to heel contact, indicating that passive dynamics may govern much of the behavior of the nonsupport leg during this portion of walking. These EMG results are corroborated by the work of [43].

3.1.1 Implications

Results considered in this section reveal that portions of biological locomotion with natural gait and speed involve little or no muscle activity and bear striking similarity to the energetics of passive-dynamic bipeds. While it is unclear to what degree human walking is due to passive dynamics (something that will be explored
further in the last section of this chapter), there is strong evidence indicating a
significant portion of human locomotion is due to passive dynamics.

The energetic similarities between human and passive-dynamic locomotion sug-
gest that passive dynamics underlies the efficiency of human walking. Consequently,
results from the biomechanics of human walking may inform intelligent control de-
cisions for passive-dynamics bipeds. We may take cues from human walking when
designing control laws for passive-dynamic robots and—just as biological gait ap-
ppears to be tuned to minimize the metabolic cost of walking [39]—we may expect to
reap results in energy efficiency from biomimicry.

3.2 Important Features of Human Locomotion

In this section, we discuss two features of human locomotion that may be mim-
icked in control of passive-dynamic walking bipeds. In particular, we consider the
adjustment of step length when changing walking speeds and the critical role of ankle
actuation in human walking. We will apply these features of biological locomotion
to the control of bipedal robots in Chapter 6.

3.2.1 Adjusting step length when changing speed

Studies of human and animal locomotion [44,45] indicate that walking organisms
adjust step length whenever they change speed. Grieve and Gear [45] reported the
following exponential relationship between step length and forward speed for human
walking:

\[
\text{step length} = \frac{1}{2} \text{ stride length} = \frac{30 h_{\text{body}}}{64.8} \left( \frac{\bar{s}}{h_{\text{body}}} \right)^{0.43}
\]  

(3.1)

where \( \bar{s} \) is the average forward speed during the step and \( h_{\text{body}} \) is the height of the
body from the bottom of the foot to the top of the head. Alexander [44] found that
a similar expression holds for a variety of locomoting mammals:

\[
\text{step length} = 1.15 h_{\text{hip}} \left( \frac{\bar{s}^2}{g \cdot h_{\text{hip}}} \right)^{0.3}
\]  

(3.2)
Figure 3.3 Relationship between step length and forward speed observed in humans [45] and locomoting mammals (humans, horses, dogs, cats) [44]. The range of values plotted reflects the range of walking speeds tested in each of the studies. The speeds and step lengths of the passive limit cycles of the compass-gait biped and the biped with knees walking on a $3^\circ$ slope are also shown for reference.

where $g$ is the acceleration of gravity and $h_{\text{hip}}$ is the height from the bottom of the feet to the hip. We note that on average the hip height and overall body height of humans are related by $h_{\text{hip}} = 0.530 h_{\text{body}}$ [46,47].

Step length vs. speed relationships (3.1) and (3.2) are shown in Figure 3.3, where the ranges of values plotted for each relationship reflect the ranges of speeds tested by each of the studies. We have used $h_{\text{hip}} = 1$ m, the hip height of both the compass-gait biped and biped with knees. The speeds and step lengths corresponding to the passive limit cycles of the compass-gait biped and the biped with knees are also shown for reference.

It is believed that walking organisms alter their step length when changing speed in this fashion in order to reduce effort. In Chapter 6, we design controls mimicking this feature of biological walking. The result is an energy savings when compared to controls that regulate speed while holding step length constant.
3.2.2 Critical role of the ankle during stance

Several researchers have reported the muscles of the ankle play a critical role in supplying the energy necessary for human walking. The calf muscle group—which consists of the ankle extensor muscles soleus and gastrocnemius and the ankle flexor muscle tibialis anterior—is notably active during the stance phase of human walking as seen from EMG data in Figure 3.2. The absence of EMG activity during swing indicates ankle muscles play little to no role during this portion of the human gait cycle. However, during the stance phase the muscles of the calf group are active from foot-flat to toe-off, a portion of gait during which the other muscle groups of the lower body are relatively quiet. Moreover, the ankle muscles exhibit a steady crescendo of activity throughout the stance phase.

By monitoring EMG activity while applying horizontal forces to the body, the particular importance of the ankle extensor gastrocnemius in supplying the energy that propels the body forward during walking has been demonstrated [48]. When applying assistive forces in the direction of walking, Gottschall and Kram reported simultaneous reduction of EMG activity in the gastrocnemius (up to 59%) and the overall metabolic cost (up to 53%) of walking while EMG activity in other muscles remained relatively constant. This finding led these researchers to conclude the ankle extensor gastrocnemius is responsible for supplying a significant proportion of the energy that propels the body forward during walking.

The results of Winter [49] corroborate the findings of Gottschall and Kram. Winter showed that the dynamics of the human ankle joint in stance phase contribute more mechanical energy to the forward motion of the body than do the knee and hip. The propulsion supplied by the ankle is provided primarily through a spike in mechanical power that occurs between heel-off and toe-off [49,50] which corresponds to the occurrence of peak ankle EMG activity, visible in Figure 3.2.

We note that important structural differences exist between human ankles and the ankles of the bipedal robots considered in this thesis. Specifically, our models...
have point feet; they do not include foot segments of nonzero length. The ankle of both the compass-gait biped and the biped with knees consists of a simple pin joint connecting the support leg to the walking surface. Actuation of the ankle is modeled as a torque applied at this pin joint to alter the angle between the leg and the surface.

The presence of a foot between the ankle and the walking surface in human walking introduces dynamical subtleties that are absent in our models, such as heel strike events prior to the foot being flat on the surface, noninstantaneous double-support phases, and heel-off events prior to the foot lifting away from the walking surface. These subtleties complicated analysis considerably, so we have opted to exclude them from our work. However, we note it is unclear if the dynamics of our simplified ankles is comparable to that of human ankles.

Due to the similarities in energetics and dynamics we have already observed—similarities that hold in spite of the fact that our bipeds have simplified ankles—we hypothesize that the ankle behavior of our biped models is indeed similar to that of human ankles. Consequently, from the critical role played by the human ankle muscles in the energetics of walking, we hypothesize that actuation at the ankle of our bipedal robots will play a similarly crucial role in the energetics of walking. By taking this cue from biology, we may be able to reduce energy requirements for steady robot locomotion using bipeds with revolute ankles by designing controls that supply energy primarily through the ankle. We will explore this biomimetic control choice in Chapter 6.

We will return to evaluate this hypothesis in Chapter 8, when we consider controls that explicitly minimize a measure of actuator effort. We will compare the optimal actuation profile of the ankle of the compass-gait biped with that of the hip joint over a variety of commanded gaits and observe whether the relative significance of each joint’s control input resembles that of humans.
3.3 Passive Dynamics of the Human Ankle

Though substantial evidence exists to suggest passive dynamics plays a significant role in human locomotion, we note there is a need for further investigation to verify this claim. This observation motivated our study [26, 27] of the passive behavior exhibited by the human ankle in level and downhill walking. In this section, we briefly summarize the methodology of our study and the results we found.

Hansen et al. [51] reported no net gain or loss of mechanical energy through the ankle during level walking, leading them to conjecture that the ankle is mechanically passive. Its behavior might be equivalent to that of a simple revolute spring and damper coupled to a pin joint. The activity observed in Figure 3.2 at the ankle may correspond to springy tissues being compressed during foot-fall and gradually being relaxed over the course of the step.

To date, two studies have attempted to quantify the passive mechanical dynamics of the ankle. Weiss et al. [52] modeled the dynamics of the ankles of reclined participants and reported spring and damper coefficients for mid-range motions. However, this dynamic model of the ankle, which holds when the ankle is unloaded, may not hold under walking conditions, during which the ankle bears the mass of the body. Indeed, Palmer [53] modeled the passive dynamics of the ankle during level walking and reported the spring and damper coefficients one order of magnitude greater than the values reported by Weiss et al. [52].

Palmer’s study also had limitations. He [53] modeled only the first two portions of the stance phase (from heel-strike to foot-flat and from flat-foot to heel-off) with passive mechanical components. The dynamics of each stage were treated separately resulting in parameter discontinuities between stages, and in the final investigation, we sought a passive model which could accurately duplicate the dynamics of the ankle throughout all of the stance phase, in both level and slope walking.
3.3.1 Methods

Eight healthy men ranging in age from 22 to 27, average mass 74 kg (± 4.7 standard deviation), and average height 178 cm (± 4.7 standard deviation), with no known gait impairments participated in this study. Participants were selected to minimize gait variation due to gender, age, mass, and stature. Each participant was informed of the experimental protocol and provided consent in compliance with the University of Illinois Institutional Review Board. Each participant wore rigid bicycling shoes (U.S. size 10.5, within one-half size of the participant’s normal shoe size) that restricted the motion of the foot to approximate a single rigid segment.

A variable slope apparatus illustrated in Figure 3.4 was constructed consisting of three walking surfaces joined by hinges. All walking surfaces were painted with nonslip paint. A force platform (Advanced Medical Technology Inc. model 02172) level with the walking surface was embedded in the center section. The starting section of the walkway was mounted on four hydraulic jacks (Pro-lift model F-2365). As the jacks were raised, hinges between the sections allowed the starting and ending platforms to remain horizontal while the center section assumed various angles. A number of structural reinforcements were used to minimize movement of the walking surfaces during data collection.

A six-camera optoelectric system (Vicon Motion Systems Ltd. model 460) was used to collect kinematic data. Kinematic and kinetic data were sampled at 100 Hz. Reflective markers were placed on the lower body as shown in Figure 3.5 and described in [26].

The participants were instructed to walk the length of the walkway at a self-selected pace for each of seven angles of the ramp: 0°, 2°, 3°, 4°, 5°, 6°, and 8°. The angles of the ramp used in this study were selected based on the range of angles tested in the previously mentioned studies of human [40, 41, 51, 54] and passive-dynamic biped [2, 9, 12, 14, 22] walking, with the steepest ramp angle not exceeding the legal maximum for a wheelchair ramp in the United States [United States Access Board,
Figure 3.4 Ramp apparatus used to test human participants.

Figure 3.5 Locations of the reflective markers and illustration of how $\theta$ was defined as the angle between the shank segment and the line perpendicular to the bottom of the foot segment.
The set of test angles was selected with the most resolution in a range between $2^\circ$ and $6^\circ$, in which humans and passive-dynamic robots exhibit minimum energy consumption.

No mention was made of the force platform, and participants were instructed to look straight ahead in order to prevent them from biasing their gait based on its location. A successful trial consisted of heel-strike and toe-off of either foot completely within the boundaries of the force plate without contact of the other foot. The participant’s starting position was adjusted until a successful trial was achieved. Participants completed two successful trials at each ramp angle and were provided breaks while the setting of the ramp angle was adjusted.

### 3.3.2 Data analysis

Kinematic data were conditioned using a second-order zero-phase Butterworth lowpass filter with 10-Hz cutoff using MATLAB. Analysis was computed in the sagittal plane which was deemed sufficient since 93% of the work done at the ankle during walking occurs in this plane [55]. The ankle angle was defined as shown in Figure 3.5, in agreement with [56]. The angular velocity $\dot{\theta}(t)$ was computed using a numeric approximation of the derivative, i.e. the change in angle divided by the time between frames. Note from Figure 3.5 that, unlike the bipedal robot models considered in this thesis, the model of human walking used in this study included a foot segment of nonzero length. The dynamical subtleties of human walking that result from a nontrivial foot—including heel-strike, noninstantaneous double-support phase, and heel-off—were considered in this investigation.

Measures of the two successful trials on each ramp angle for each participant were averaged. Exceptions were made in two instances in which successive trials for a given angle produced power and moment profiles starkly inconsistent due to technical errors. In these instances, only data from the trials without technical errors were considered. These two exceptions occurred in different participants and on
different angles, suggesting the errors were not correlated and were therefore not repeated.

The human ankle torque $\tau(t)$ was computed using the measured ground reaction force, center of pressure, and the motion data in a standard inverse dynamics procedure [47]. The following linear, passive model for the ankle torque was used:

$$\tau_{sd}(t) = k_p \theta(t) + k_d \dot{\theta}(t) + c$$

where $k_p$ and $k_d$ represent the unknown spring and damper coefficients, respectively, and $c$ is a constant accounting for the angle of spring relaxation as well as minor offsets in marker placement that affect the calculation of the ankle angle. The passive model parameters $k_p$, $k_d$, and $c$ were estimated by fitting the model ankle torque $\tau_{sd}(t)$ to the experimentally measured torque $\tau(t)$ using linear regression, i.e., by computing

$$\begin{bmatrix} k_p \\ k_d \\ c \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ 1 \end{bmatrix}^{-1} \begin{bmatrix} \tau(t) \end{bmatrix}$$

where $[\cdot]^{-1}$ denotes the Moore-Penrose pseudoinverse calculated using MATLAB.

To judge the quality of fit we computed the root-mean-square (RMS) error between the model-generated torque $\tau_{sd}(t)$ and the experimentally measured torque $\tau(t)$

$$e_{RMS} = \sqrt{\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} (\tau(t) - \tau_{sd}(t))^2 dt}$$

where time limits of integration $t_0$, $t_1$ were chosen by the boundaries of the stance phase, i.e. $t_0 =$heel contact to $t_1 =$toe-off.

A repeated measures analysis of variance was performed using SPSS (SPSS Inc., version 15.0) to evaluate the significance of slope variation on the measures of power, energy, and the passive ankle parameters.
3.3.3 Results

The ankle torque profiles were in general agreement with related studies [49, 50, 55, 56]. The passive mechanical model was fit to the experimentally measured ankle torque using the methods described above. A representative plot showing the fitting of the torque from the passive model to the experimental torque is shown in Figure 3.6.

The RMS error between the torque of the passive model and the experimentally measured torque remained low for all angles, bounded below 14 Nm and never surpassing 25% of the RMS torque value as shown in Figure 3.7(a). The relatively low error values suggest that the spring and damper mechanism effectively models level walking, confirming the prediction of Hansen et al. [51]. Furthermore, measures of error were minimized on slopes of $2^\circ - 4^\circ$ as shown in Figure 3.7(b), indicating the
Figure 3.7 RMS error of the passive-dynamic model $e_{RMS}$ (a) compared to RMS value of the experimentally measured torque $\tau(t)$, (b) plotted alone to emphasize that the error is minimized on slopes between $2^\circ - 4^\circ$.

The angle of the ramp had statistically significant effect on the RMS error of the model ($p < 0.0001$).

We emphasize that the passive-dynamic model is most accurate on the range of slopes that approximately corresponds to the range of slopes on which humans and passive-dynamic robots require minimum energy input, as discussed in Section 3.1.

Evolution of the passive parameters with variation in slope angle is shown in Figure 3.8, revealing nearly monotonic trends in all parameters. Ramp angle had a statistically significant effect on all model parameters ($p < 0.0001$).

The spring coefficient $k_p$ decreased as ramp angle increased while the damping coefficient $k_d$ and linear offset $c$ increased with increasing ramp angle. This implies that while a passive mechanism may effectively model the human ankle on a particular slope, the passive components must be adjusted for most accurate performance when walking on other slopes.

Values of the damping parameter $k_d$ were more than one order of magnitude smaller than values of the spring parameter $k_p$. This is consistent with the relative
Figure 3.8 Variation of the parameters of the passive model as ramp angle changed: (a) spring $k_p$, (b) damper $k_d$, (c) offset $c$. 
magnitudes of ankle spring and damper coefficients reported in related studies \cite{52,53}. We concluded that damping at the ankle is negligible on the range of slopes tested.

The offset parameter $c$ was nonzero on all slopes, suggesting the angle of relaxation for the spring is not the line perpendicular to the bottom of the foot shown in Figure 3.5. Moreover, the variation of the offset parameter as slope changed indicated the spring relaxation angle changes with walking slope.

### 3.3.4 Discussion

These results show that the behavior of the human ankle during walking can be effectively duplicated by a simple passive mechanism. This suggests that passive dynamics may account for much of the dynamics of the human ankle during walking, perhaps explaining the energy efficiency of human locomotion. In addition, these findings strengthen the existing body of work, indicating a connection between human walking and passive-dynamic locomotion in bipedal robots such as the compass-gait biped and the biped with knees considered in this dissertation.

While our human passive-dynamic model is not a perfect match to the human data, we note that the linear model we proposed and analyzed is only one of many possible passive models that could be tested. Indeed, there is an infinite set of possible nonlinear passive models that could also be studied which may better mimic human locomotion.
CHAPTER 4

ENERGY SHAPING

Our point of departure from previous work in control of bipedal locomotion is
the use of energy shaping for gait regulation. As we have seen in Section 2.1, the
phenomenon of passive-dynamic locomotion arises from the elegant interplay of po-
tential and kinetic energy during slope walking. However, passive dynamics alone
yields at most one period-1 walking cycle on any given slope, thereby furnishing a
single locomotion pattern with fixed gait characteristics. We hypothesize that careful
modification of a biped’s natural energy using feedback control will give rise to stable
walking limit cycles possessing desirable characteristics.

Shaping the energy of a system has long been a useful tool for control design-
ers. Over one hundred years ago Lord Kelvin proved that trajectories of dissipative
systems asymptotically approach the local minima of the potential energy (see proof
in [57]), a fact that has made potential shaping useful for stabilizing isolated equilib-
rium points [58]. Potential energy shaping has also proven effective in path-planning
applications, in which trajectories are planned through complex environments by
sculpting “walls” of high potential near obstacles and “valleys” of low potential in
free areas [59].

Recently there has been much activity in the area of total energy shaping (i.e.
shaping potential and kinetic energy simultaneously) for the purpose of stabilizing
equilibria. Total energy shaping for stabilization has evolved into two main ap-
proaches: the so-called methods of controlled Lagrangians [60,61] and interconnection
and damping assignment passivity-based control (IDA-PBC) [62,63]. Both methods shape the total energy of dynamical systems into forms with local minima at the desired equilibrium points and then inject damping to insure trajectories asymptotically descend to the equilibria. The closed-loop energy functions form natural Lyapunov function candidates, thereby lending nice stability proofs to these energy-shaping methods.

Energy shaping control has also proven useful for gait regulation of the passive-dynamic class of bipedal walking robots. Potential energy shaping in particular has received attention for its usefulness in regulating walking on slopes and in regulating the forward speed of a biped. Spong [18] and Spong and Bullo [19] have presented a potential-shaping control that renders a biped’s limit cycle invariant to changes in slope, thereby allowing the biped to walk on any incline. Several authors have shown that regulation of forward walking speed may be accomplished via another potential-shaping control [22–24]. In addition, passivity-based total energy shaping has been shown in [20,25] to enhance orbital stability by enlarging a biped’s basin of attraction, increasing rates of convergence to the walking limit cycle, and improving robustness to disturbances.

To date, much of the work in energy-shaping control for bipedal locomotion has focused on potential energy shaping alone. Moreover, almost all work reported in the literature has required full actuation. In this chapter, we consider the possible forms of the closed-loop energy for a bipedal robot and give special treatment to the case of underactuation, laying the framework for the energy-shaping controls of Chapters 5 an 6. The matching condition, a nonlinear partial differential equation (PDE) whose solutions define the achievable forms of the closed-loop energy is presented here. We show that the solution to the matching condition PDE is trivial in the case of full actuation while solving in the case of underactuation is a generally difficult task.

We consider and compare the special cases of shaping the potential energy of the biped alone and that of shaping the kinetic energy alone. We show that while it is relatively easy to solve for all possible closed-loop forms of a biped’s potential
energy, potential-shaping control suffers from a lack of velocity feedback information. We compare this with kinetic energy shaping, which affords feedback of both angular and velocity information in the control. We show how all feasible forms of the closed-loop kinetic energy may be found by solving a sequence of three equations: one algebraic equation and two linear PDEs.

The chapter concludes with an application of these energy-shaping results to the compass-gait biped. We consider a particular case of underactuation inspired by the biological results presented in Section 3.2.2. The matching condition PDE is solved explicitly for the compass-gait biped for the case of actuation at the ankle alone and all possible forms the closed-loop kinetic energy may assume are presented, a result that will be useful for the gait-regulating control presented in Chapter 6.

4.1 Shaping Closed-Loop Energy

Recall from (2.2) that the dynamics of the biped may be written

\[ \frac{d}{dt} \nabla_q^T L(q, \dot{q}) - \nabla_{\dot{q}}^T L(q, \dot{q}) = Bu \]  

(4.1)

where \( L : Q \times TQ \to \mathbb{R} \) is the Lagrangian of the open-loop system, \( u \in \mathbb{R}^m \) is the vector of input forces, and the gradient \( \nabla_x \) is the usual row vector of partial derivatives with respect to the components of vector \( x \). The matrix \( B \in \mathbb{R}^{n \times m} \) maps the control inputs \( u \) to the various joints of the biped and is assumed to satisfy \( \text{rank}(B) = m \leq n \), i.e., \( B \) is of full column rank. Substituting the Lagrangian of the biped (2.1)

\[ L(q, \dot{q}) = K(q, \dot{q}) - V(q) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q) \]

into (2.2), the equations of motion may be factored into the form common in the robotics literature (2.3)

\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Bu. \]  

(4.2)
Let \( \hat{L} : Q \times TQ \to \mathbb{R} \) denote the Lagrangian of the closed loop system defined with the same mechanical form as (2.1), i.e.,

\[
\hat{L}(q, \dot{q}) = \hat{K}(q, \dot{q}) - \hat{V}(q) = \frac{1}{2} \dot{q}^T \hat{M}(q) \dot{q} - \hat{V}(q)
\]

(4.3)

where \( \hat{K} : Q \times TQ \to \mathbb{R} \) and \( \hat{V} : Q \to \mathbb{R} \) are the kinetic and potential energy, respectively, of the closed-loop mechanical system and \( \hat{M} \in \mathbb{R}^{n \times n} \) is its positive definite mass matrix. The closed-loop Lagrangian dynamics may be written

\[
\hat{M}(q) \ddot{q} + \hat{C}(q, \dot{q}) \dot{q} + \hat{G}(q) = 0
\]

(4.4)

### 4.1.1 The energy-matching control

To impose the closed-loop dynamics (4.4) on the original biped dynamics (4.2) we must have

\[
Bu = \left( \frac{d}{dt} \nabla_\dot{q}^T L - \nabla_q^T L \right) - \left( \frac{d}{dt} \nabla_\dot{q}^T \hat{L} - \nabla_q^T \hat{L} \right) = \left( \hat{M}(q) \ddot{q} + \hat{C}(q, \dot{q}) \dot{q} + \hat{G}(q) \right) - \left( \hat{M}\ddot{q} + \hat{C}(q, \dot{q}) \dot{q} + \hat{G}(q) \right).
\]

(4.5)

Solving (4.4) for \( \ddot{q} \) we obtain an explicit expression for the acceleration

\[
\ddot{q} = -\hat{M}(q)^{-1} \left( \hat{C}(q, \dot{q}) \dot{q} + \hat{G}(q) \right).
\]

(4.6)

Substituting (4.6) into (4.5) and rearranging we have

\[
Bu = M \left[ M^{-1}(C\ddot{q} + G) - \hat{M}^{-1}(\hat{C}\ddot{q} + \hat{G}) \right].
\]

(4.7)

Multiplying (4.7) by the left pseudo-inverse of \( B \), i.e., \( B^\dagger = (B^T B)^{-1} B^T \), a simple computation shows the control that matches the open- and closed-loop dynamics is given by

\[
u = B^\dagger M \left[ M^{-1}(C\ddot{q} + G) - \hat{M}^{-1}(\hat{C}\ddot{q} + \hat{G}) \right].
\]

(4.8)

### 4.1.2 The matching condition

As in [64], let \( B^\perp \) be a rank \( m \) left-annihilator of \( B \) in the form of an orthogonal projection matrix, i.e., \( B^\perp B = 0 \), \( (B^\perp)^T = B^\perp \), and \( (B^\perp)^2 = B^\perp \). It follows from
Substituting (2.4) and recalling that \( G = \nabla_q V \), we may rewrite (4.9) as

\[
\begin{aligned}
B^\perp M \left[ M^{-1} (C\dot{q} + G) - \hat{M}^{-1} (\hat{C}\dot{q} + \hat{G}) \right] &= 0 \\
\hat{B} \left[ \hat{M}^{-1} \left( D_q (\hat{M}\dot{q})\dot{q} - \frac{1}{2} \nabla_q^T (\dot{q}^T \hat{M}\dot{q}) + \nabla_q V \right) - \hat{M}^{-1} \left( D_q (\hat{M}\dot{q})\dot{q} - \frac{1}{2} \nabla_q^T (\dot{q}^T \hat{M}\dot{q}) + \nabla_q \hat{V} \right) \right] &= 0.
\end{aligned}
\] (4.10)

Equation (4.10) is called the matching condition [60,64,65] and is a nonlinear PDE in the two unknowns: \( \hat{M}, \hat{V} \) which define the closed-loop kinetic and potential energy, respectively.

If \( m = n \) (the number of actuators matches the degrees of freedom of the robot, i.e., the biped is fully actuated), the left-annihilator of \( B \) is \( B^\perp = 0 \) and the matching condition (4.10) is trivially satisfied for all \( \hat{M} \) and \( \hat{V} \). Therefore, any closed-loop Lagrangian \( \hat{L} \) of the form (4.3) can be achieved in the case of full actuation.

However, if \( m < n \) (i.e., the biped is underactuated) not all forms of \( \hat{M} \) and \( \hat{V} \) can be achieved. Indeed, in the case of underactuation, \( B^\perp \neq 0 \) and the matching condition admits a restricted set of solutions for \( \hat{M} \) and \( \hat{V} \).

We next consider and compare two special cases of energy shaping: shaping potential energy alone, which has been employed by many authors for gait regulation [19,23–25], and shaping kinetic energy alone, which to the best of our knowledge has not been considered in the literature.

### 4.1.3 Potential energy shaping

Consider the case of shaping the closed-loop potential energy alone. That is, fix the closed-loop kinetic energy \( \hat{K} = K \) (equivalently, fix \( \hat{M} = M \)) and consider only variations in \( \hat{V} \). In this case, the matching condition (4.10) reduces to a single, linear PDE given by

\[
\begin{aligned}
B^\perp (\nabla_q V - \nabla_q \hat{V}) &= 0.
\end{aligned}
\] (4.11)
Finding the achievable forms of the closed-loop potential energy $\hat{V}$ amounts to solving for vectors in the null space of $B^\perp$, subtracting the open-loop potential energy $V$, and integrating to find the feasible $\hat{V}$.

Under potential energy shaping, the general energy-shaping control (4.8) reduces to

$$u(q) = B^\dagger \left( \nabla_q V(q) - \nabla_q \hat{V}(q) \right)$$

$$= B^\dagger \left( G(q) - \hat{G}(q) \right).$$

(4.12)

Note that potential-shaping control allows feedback based only on the joint angles $q$. The control law lacks any feedback of velocity information.

### 4.1.4 Kinetic energy shaping

Consider now the case of shaping the closed-loop kinetic energy alone. Therefore, let $\hat{V} = V$ in the matching condition (4.10) and consider only variations in $\hat{K}$ (or, equivalently, variations in $\hat{M}$).

The matching condition may be separated as in [64] into terms that are dependent on and independent of the velocities $\dot{q}$. Separation in this fashion yields the following two coupled matching “subconditions”:

$$B^\perp M \left( M^{-1} \nabla_q V - \hat{M}^{-1} \nabla_q V \right) = 0$$

(4.13)

$$B^\perp M \left[ M^{-1} \left( D_q(M\dot{q})\dot{q} - \frac{1}{2} \nabla^T_q (\dot{q}^T M \dot{q}) \right) - \hat{M}^{-1} \left( D_q(\hat{M}\dot{q})\dot{q} - \frac{1}{2} \nabla^T_q (\dot{q}^T \hat{M} \dot{q}) \right) \right] = 0.$$  

(4.14)

The first subcondition (4.13) is an algebraic equation that is easily solved for the unknown $\hat{M}$. However, the solution $\hat{M}$ of (4.13) is only defined with respect to the projection $B^\perp$. For a complete solution for $\hat{M}$, we must consider solutions $\hat{M}$ that simultaneously satisfy the nonlinear PDE (4.14).
Under kinetic energy shaping, the general energy-shaping control (4.8) becomes

\[ u(q, \dot{q}) = B^\dagger M \left[ M^{-1} \left( D_q (M \dot{q}) \dot{q} - \frac{1}{2} \nabla^T_q (q^T M \dot{q}) + \nabla_q V \right) \right. \]

\[ \left. - \hat{M}^{-1} \left( D_q (\hat{M} \dot{q}) \dot{q} - \frac{1}{2} \nabla^T_q (q^T \hat{M} \dot{q}) + \nabla_q V \right) \right] \]

\[ = B^\dagger M \left[ M^{-1} (C \dot{q} + G) - \hat{M}^{-1} (\hat{C} \dot{q} + G) \right] \]

where we have noted explicitly the kinetic-shaping control’s dependence on the joint angles \( q \) and velocities \( \dot{q} \).

An advantage of controls that shape kinetic energy over those that shape potential energy alone (4.12) is the allowance of velocity feedback. Conversely, the equations that must be solved to find all feasible closed-loop energy forms are considerably more complicated in the case of kinetic energy shaping than in the case of potential energy shaping. In the following section we show how Equations (4.13) and (4.14), whose solutions define the possible closed-loop kinetic energy, may be simplified by rewriting them as an algebraic equation and two linear PDEs to be solved sequentially.

**4.1.5 Solving the kinetic energy shaping matching condition**

To simplify the algebra in this section, define matrix \( \Lambda = \hat{M}^{-1} M \). Premultiplying (4.13) by \( M \), the algebraic condition may be rewritten

\[ MB^\perp G - (AB^\perp M)^T G = 0. \quad (4.15) \]

Equation (4.15) is an algebraic equation in the new unknown \( \Lambda \). Although it is possible to solve this equation for \( \Lambda \) directly, a complete solution for \( \Lambda \) will be obscured by the projection matrix \( B^\perp \). Instead, we will solve (4.15) for \( \Lambda B^\perp M \), which we will bring to bear when solving (4.14). Equation (4.15) is the first equation in a sequence of three equations we will solve to find all possible forms of the closed-loop kinetic energy.

Next we borrow from the general technique of Auckly et al. [66] specialized to mechanical systems in [64] to break down the second matching subcondition (4.14),
a nonlinear PDE, into two linear PDEs that are easier to solve. Here we present a brief overview of the necessary steps.

The kinetic energy inner product

\[ \langle \dot{x}, \dot{y} \rangle = \dot{x}^T M(q) \dot{y} \]  

(4.16)
defines a metric on the configuration manifold \( Q \) for all velocity pairs \( \dot{x}, \dot{y} \in TQ \) [67].

Recall the covariant derivative, \( \nabla_{\dot{x}} \dot{y} : TQ \times TQ \rightarrow TQ \), which associates with any two vectors \( \dot{x}, \dot{y} \) in the tangent space of \( Q \) another vector in \( TQ \) given by [68]

\[ \nabla_{\dot{x}} \dot{y} = \left( \dot{x}^i \dot{y}^j \Gamma^k_{ij} + \dot{x}^i \frac{\partial \dot{y}^k}{\partial \dot{x}^i} \right) e_k \]

where \( \Gamma^k_{ij} \) are Christoffel symbols of the second kind and \( e_k \) denotes the unit vector corresponding to coordinate \( q_k \in Q \). In our notation, the covariant derivative may be written as follows:

\[
2 \nabla_{\dot{x}} \dot{y} = 2 \left( \dot{x}^i \dot{y}^j \Gamma^k_{ij} + \dot{x}^i \frac{\partial \dot{y}^k}{\partial \dot{x}^i} \right) e_k \\
= 2 \left( \dot{x}^i \dot{y}^j \Gamma^k_{ij} \right) e_k + 2 D_q(\dot{y}) \dot{x} \\
= 2 \dot{x}^i \dot{y}^j \Gamma^k_{ij} e_k + D_q(\dot{y}) \dot{x} - D_q(\dot{x}) \dot{y} + D_q(\dot{y}) \dot{x} + D_q(\dot{x}) \dot{y} \\
= 2 \dot{x}^i \dot{y}^j \Gamma^k_{ij} e_k + [\dot{x}, \dot{y}] + D_q(\dot{y}) \dot{x} + D_q(\dot{x}) \dot{y} \\
= \dot{x}^i \dot{y}^j M^{-1}_{km} \left( \frac{\partial M_{mi}}{\partial q^j} + \frac{\partial M_{mj}}{\partial q^i} - \frac{\partial M_{ij}}{\partial q^m} \right) e_k \\
+ [\dot{x}, \dot{y}] + D_q(\dot{y}) \dot{x} + D_q(\dot{x}) \dot{y} \\
= \dot{x}^i \dot{y}^j M^{-1}_{km} \partial M_{mi} \partial q^j + \dot{x}^i \dot{y}^j M^{-1}_{km} \partial M_{mj} \partial q^i - \dot{x}^i \dot{y}^j M^{-1}_{km} \partial M_{ij} \partial q^m e_k \\
+ [\dot{x}, \dot{y}] + D_q(\dot{y}) \dot{x} + D_q(\dot{x}) \dot{y} \\
= M^{-1} D_q(M \dot{x}) \dot{y} + M^{-1} D_q(M \dot{y}) \dot{x} - M^{-1} \nabla^T_q (\dot{x}^T M \dot{y}) \\
+ [\dot{x}, \dot{y}] + D_q(\dot{y}) \dot{x} + D_q(\dot{x}) \dot{y} \\ 
(4.17)

where \([,]\) denotes the standard Lie bracket. (We note the typo in [64], which incorrectly reports (4.17) with the final two terms premultiplied by \( M^{-1} \).)

It may now be shown that the second subcondition (4.14) is the difference of covariant derivatives; i.e., (4.14) is equivalent to

\[
B^\perp M (\nabla_q \dot{q} - \tilde{\nabla}_q \dot{q}) = 0 
\]  

(4.18)
for all velocities $\dot{q} \in TQ$, where $\hat{\nabla}_z \hat{y}$ denotes the covariant derivative for the closed-loop system with mass matrix $\hat{M}$.

Polarizing Equation (4.18) as in [66] we have, for all $\dot{x}, \dot{y} \in TQ$,

$$
0 = B^\perp M \left( \nabla_{\dot{x}+\dot{y}} (\dot{x} + \dot{y}) - \hat{\nabla}_{\dot{x}+\dot{y}} (\dot{x} + \dot{y}) \right)
$$

$$
= B^\perp M \left( \nabla_{\dot{x}} \dot{x} + \nabla_{\dot{y}} \dot{y} + \nabla_{\dot{y}} \dot{x} + \nabla_{\dot{x}} \dot{y} - \hat{\nabla}_{\dot{x}} \dot{x} - \hat{\nabla}_{\dot{y}} \dot{y} \right)
$$

$$
= B^\perp M \left[ \left( \nabla_{\dot{x}} \dot{x} - \hat{\nabla}_{\dot{x}} \dot{x} \right) + \left( \nabla_{\dot{y}} \dot{y} - \hat{\nabla}_{\dot{y}} \dot{y} \right) + 2 \left( \nabla_{\dot{x}} \dot{y} - \hat{\nabla}_{\dot{x}} \dot{y} \right) \right]
$$

$$
= B^\perp M \left( \nabla_{\dot{x}} \dot{x} - \hat{\nabla}_{\dot{x}} \dot{x} \right) + B^\perp M \left( \nabla_{\dot{y}} \dot{y} - \hat{\nabla}_{\dot{y}} \dot{y} \right) + 2 B^\perp M \left( \nabla_{\dot{x}} \dot{y} - \hat{\nabla}_{\dot{x}} \dot{y} \right)
$$

$$
= B^\perp M \left( \nabla_{\dot{x}} \dot{y} - \hat{\nabla}_{\dot{y}} \dot{y} \right)
$$

(4.19)

where we have made use of the bilinearity of the covariant derivative and the fact that $\nabla_{\dot{x}} \dot{y} - \nabla_{\dot{y}} \dot{x} = [\dot{x}, \dot{y}] = \hat{\nabla}_{\dot{x}} \dot{y} - \hat{\nabla}_{\dot{y}} \dot{x}$ (see e.g. [67]).

Equation (4.19) defines a vector in $TQ$; therefore, it satisfies the metric (4.16) on $Q$. Choosing $\dot{x} = \Lambda B^\perp M \dot{z}$, where $\dot{z} \in TQ$, we may rewrite (4.19) as

$$
\left< \left( B^\perp M (\nabla_{AB^\perp M \dot{z}} - \hat{\nabla}_{AB^\perp M \dot{z}}) \right), \dot{z} \right> = 0
$$

(4.20)

which holds for all $\dot{y}, \dot{z} \in TQ$. A sequence of algebraic steps given in [66] allows elimination of $\hat{\nabla}$ from (4.20) resulting in an expression based solely on the covariant derivative in terms of the open-loop mass matrix $M$

$$
\left< B^\perp M \dot{z}, \nabla_{\dot{y}} \Lambda B^\perp M \dot{z} \right> - \left< \Lambda B^\perp M \dot{z}, \nabla_{\dot{y}} B^\perp M \dot{z} \right> = 0.
$$

(4.21)

Expanding the metric (4.16) and using the definition of the covariant derivative (4.17) we have, for all velocities $\dot{y}, \dot{z} \in TQ$,

$$(B^\perp M \dot{y})^T \left[ D_q(M \dot{z})(\Lambda B^\perp M) \dot{y} + D_q(M(\Lambda B^\perp M)) \dot{z} - \nabla_q^T \left( \dot{z}^T M (\Lambda B^\perp M) \dot{y} \right) \right] -$$

$$
(\Lambda B^\perp M) \dot{y})^T \left[ D_q(M \dot{z}) B^\perp M \dot{y} + D_q(M B^\perp M) \dot{y} \dot{z} - \nabla_q^T \left( \dot{z}^T M B^\perp M \dot{y} \right) \right] = 0.
$$

(4.22)

Both Equations (4.22) and (4.14) are PDEs that represent the second subcondition. However, while (4.14) is nonlinear in the unknown $\dot{M}$, (4.22) is linear in the unknown $\Lambda B^\perp M$. Equation (4.22) is the second equation we will solve en route to a solution for $\dot{M}$. As with (4.15), we solve (4.22) for $\Lambda B^\perp M$. 

64
Our ultimate goal is to find solutions for \( \hat{K} \) (equivalently, solutions for \( \hat{M} \)). Therefore, we rewrite the second subcondition once more in a form that relates \( \Lambda B^\perp M \) and \( \hat{M} \). Noting that \( M\hat{M}^{-1} = \Lambda^T \), it is easy to rewrite (4.14) as

\[
MB^\perp \left( D_q(M\dot{q})\dot{q} - \frac{1}{2} \nabla^T_q(\dot{q}^T M\dot{q}) \right) - \\
(\Lambda B^\perp M)^T \left( D_q(\hat{M}\dot{q})\dot{q} - \frac{1}{2} \nabla^T_q(\dot{q}^T \hat{M}\dot{q}) \right) = 0
\]  

(4.23)

which is the **third equation** of our method. Given \( \Lambda B^\perp M \) that simultaneously solves (4.15) and (4.22), (4.23) becomes a linear PDE in \( \hat{M} \). Solving (4.23) for \( \hat{M} \) defines all achievable forms of the closed-loop kinetic energy.

We summarize the equations to be solved for finding all possible forms of the closed-loop kinetic energy as follows:

1. Solve (4.15), obtaining a set of algebraic constraints on \( \Lambda B^\perp M \).
2. Solve (4.22) for \( \Lambda B^\perp M \) subject to the constraints found by solving (4.15).
3. Substitute the matrix \( \Lambda B^\perp M \) that simultaneously solves (4.15) and (4.22) into (4.23) and solve for \( \hat{M} \).

In the following section we present an example of this method using the compass-gait biped. We consider a case of underactuation and solve the necessary equations to show the forms of the closed-loop energy that are feasible with kinetic energy shaping.

### 4.2 Example: Possible Closed-Loop Energy for Ankle-Only Underactuation

In Section 3.2.2 we learned that ankle actuation plays a dominant role in human walking. This inspires our interest in studying passive-dynamic bipeds with actuation at the ankle joint alone. Consider the compass-gait biped with actuation only at the ankle joint. That is, consider the placement of a revolute actuator between the support leg and the walking surface and the absence of an actuator at the hip joint. In
this section we consider the possible forms of the closed-loop if kinetic energy shaping is employed and compare this with the possible forms under potential energy shaping.

For the compass-gait biped, actuation at the ankle alone implies $u \in \mathbb{R}^1$ and $B = [1, 0]^T$ in (4.2). Therefore, the orthogonal projection matrix that serves as a left-annihilator for $B$ is given by

$$B^\perp = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (4.24)

To simplify the solution of the matching condition, let

$$M(q) = \begin{bmatrix} a & b \cos(\theta_1 - \theta_2) \\ b \cos(\theta_1 - \theta_2) & c \end{bmatrix} \hspace{1cm} (4.25)$$

$$G(q) = \begin{bmatrix} d \sin(\theta_1) \\ e \sin(\theta_2) \end{bmatrix} \hspace{1cm} (4.26)$$

where $a, b, c, d, e$ are lumped constants defined by the parameters of (2.19)-(2.20).

4.2.1 Potential energy shaping

For controls that shape potential energy alone, fix the closed-loop kinetic energy $\hat{K} = \hat{K}$ and solve (4.11) for the possible forms of the closed-loop potential energy. Substituting (4.24) and (4.26) into (4.11) we have

$$\nabla_{\theta_2}\hat{V} = e \sin(\theta_2). \hspace{1cm} (4.27)$$

The implication of (4.27) is that variation of the closed-loop potential energy with respect to $\theta_2$, the angle of the unactuated nonsupport leg, is fixed, while variation with respect to $\theta_1$ is unspecified. Hence, potential energy shaping controls are free to shape the closed-loop energy with respect to the angle of the actuated support leg, $\theta_1$.

4.2.2 Kinetic energy shaping

Consider fixing the potential energy $\hat{V} = V$ and allowing the closed-loop kinetic energy $\hat{K}$ to vary. We will now solve the sequence of equations presented in the
previous section to find the possible forms of the closed-loop kinetic energy using energy-shaping control. Let
\[
\dot{\hat{M}}(q) = \begin{bmatrix}
\hat{m}_a(\theta_1, \theta_2) & \hat{m}_b(\theta_1, \theta_2) \\
\hat{m}_b(\theta_1, \theta_2) & \hat{m}_c(\theta_1, \theta_2)
\end{bmatrix}
\quad (4.28)
\]
\[
\Lambda \beta M = \begin{bmatrix}
\lambda_{11}(\theta_1, \theta_2) & \lambda_{12}(\theta_1, \theta_2) \\
\lambda_{21}(\theta_1, \theta_2) & \lambda_{22}(\theta_1, \theta_2)
\end{bmatrix}.
\quad (4.29)
\]

Note that, by definition (4.28), we have required mechanical mass matrix \(\hat{M}\) to be symmetric. The matching condition may now be solved by following the method given in Section 4.1.5.

1. **First Equation.** Solving (4.15) for \(\Lambda \beta M\) we get a set of algebraic constraints on the components of \(\Lambda \beta M\)
\[
d\sin(\theta_1) \begin{bmatrix}
\lambda_{11} \\
\lambda_{12}
\end{bmatrix} + e\sin(\theta_2) \begin{bmatrix}
\lambda_{21} \\
\lambda_{22}
\end{bmatrix} - \begin{bmatrix}
be \cos(\theta_1 - \theta_2) \sin(\theta_2) \\
ce \sin(\theta_2)
\end{bmatrix} = 0
\quad (4.30)
\]

from which we conclude
\[
\lambda_{21} = b \cos(\theta_1 - \theta_2) - \frac{d\sin(\theta_1)}{e \sin(\theta_2)} \lambda_{11}
\quad (4.31)
\]
\[
\lambda_{22} = c - \frac{d\sin(\theta_1)}{e \sin(\theta_2)} \lambda_{12}.
\quad (4.32)
\]

2. **Second Equation.** \(\Lambda \beta M\) must simultaneously satisfy PDE (4.22) as well as algebraic contraints (4.31) and (4.32). Substituting (4.24), (4.25), and (4.29) into (4.22) we get a scalar equation in terms of the components and partial derivatives of \(\Lambda \beta M\). We may factor this expression as follows
\[
\eta_1 \left( \begin{bmatrix}
\lambda_{11} \\
\nabla^T_q \lambda_{11}
\end{bmatrix} \dot{z}_1 + \begin{bmatrix}
\lambda_{12} \\
\nabla^T_q \lambda_{12}
\end{bmatrix} \dot{z}_2 \right) +
\eta_2 \left( \begin{bmatrix}
\lambda_{21} \\
\nabla^T_q \lambda_{21}
\end{bmatrix} \dot{z}_1 + \begin{bmatrix}
\lambda_{22} \\
\nabla^T_q \lambda_{22}
\end{bmatrix} \dot{z}_2 \right) = 0
\quad (4.33)
\]
which holds for all velocities \( \dot{y}, \dot{z} \in TQ \), where \( \nabla_q^T \lambda_i = \left[ \frac{\partial \lambda_i}{\partial \theta_1} \frac{\partial \lambda_i}{\partial \theta_2} \right]^T \) and the coefficients \( \eta_i(q, \dot{y}, \dot{z}) \) of the components and partial derivatives of \( AB^{-1}M \) are given by

\[
\eta_1^T = \left[ \begin{array}{c}
-b \sin(\theta_1 - \theta_2)((c\dot{y}_2 + b\cos(\theta_1 - \theta_2)(2\dot{y}_1 + 3\dot{y}_2))\dot{z}_1 + 2c(\dot{y}_1 + \dot{y}_2)\dot{z}_2) \\
b \cos(\theta_1 - \theta_2)\dot{y}_1(b \cos(\theta_1 - \theta_2)\dot{z}_1 + c\dot{z}_2) \\
a\dot{y}_1(b \cos(\theta_1 - \theta_2)\dot{z}_1 + c\dot{z}_2)
\end{array} \right],
\]

\[
\eta_2^T = \left[ \begin{array}{c}
b(c + b \cos(\theta_1 - \theta_2)) \sin(\theta_1 - \theta_2)\dot{y}_1\dot{z}_1 \\
c\dot{y}_1(b \cos(\theta_1 - \theta_2)\dot{z}_1 + c\dot{z}_2) \\
b \cos(\theta_1 - \theta_2)\dot{y}_1(b \cos(\theta_1 - \theta_2)\dot{z}_1 + c\dot{z}_2)
\end{array} \right].
\]

Note that all terms in (4.33) are cubic in the velocities \( \dot{y}_1, \dot{y}_2, \dot{z}_1, \dot{z}_2 \). Substituting (4.31)-(4.32) and factoring the PDE into common velocity terms, the PDE may be written

\[
\rho_1(\dot{z}_1^2\dot{y}_1) + \rho_2(\dot{z}_1^2\dot{y}_2) + \rho_3(\dot{z}_1\dot{z}_2\dot{y}_1) + \rho_4(\dot{z}_1\dot{z}_2\dot{y}_2) + \rho_5(\dot{z}_2^2\dot{y}_1) + \rho_6(\dot{z}_2^2\dot{y}_2) = 0 \quad (4.34)
\]

where

\[
\rho_1 = \frac{b}{2c} \left( \rho_{11} \lambda_{11} + \rho_{12} \frac{\partial \lambda_{11}}{\partial \theta_1} + \rho_{13} \frac{\partial \lambda_{11}}{\partial \theta_2} \right)
\]

\[
\rho_2 = -b(c + 3b \cos(\theta_1 - \theta_2)) \sin(\theta_1 - \theta_2)\lambda_{11}
\]

\[
\rho_3 = \frac{c}{e} \left( \rho_{31} \lambda_{11} + \rho_{32} \frac{\partial \lambda_{11}}{\partial \theta_1} + \rho_{33} \frac{\partial \lambda_{11}}{\partial \theta_2} \right) + \frac{b}{4e} \left( \rho_{34} \lambda_{12} + \rho_{35} \frac{\partial \lambda_{12}}{\partial \theta_1} + \rho_{36} \frac{\partial \lambda_{12}}{\partial \theta_2} \right)
\]

\[
\rho_4 = b \sin(\theta_1 - \theta_2)(-2c\lambda_{11} - (c + 3b \cos(\theta_1 - \theta_2))\lambda_{12})
\]

\[
\rho_5 = \frac{c}{e} \left( \rho_{51} \lambda_{12} + \rho_{52} \frac{\partial \lambda_{12}}{\partial \theta_1} + \rho_{53} \frac{\partial \lambda_{12}}{\partial \theta_2} \right)
\]

\[
\rho_6 = -2bc \sin(\theta_1 - \theta_2)\lambda_{12}
\]
\[\begin{align*}
\rho_{11} &= (2d \cot(\theta_2)(b + c \cos^2(\theta_1 - \theta_2) \csc(\theta_2) \sin(\theta_1)) \\
&\quad + b(2e + d \csc(\theta_2) \sin(\theta_1)) \sin(2(\theta_1 - \theta_2))) \\
\rho_{12} &= 2 \cos(\theta_1 - \theta_2)(be \cos(\theta_1 - \theta_2) - cd \csc(\theta_2) \sin(\theta_1)) \\
\rho_{13} &= 2 \cos(\theta_1 - \theta_2)(-ae + bd \cos(\theta_1 - \theta_2) \csc(\theta_2) \sin(\theta_1)) \\
\rho_{31} &= -d \csc(\theta_2)(c \cos(\theta_1) + b \cos(\theta_1 - \theta_2) \cot(\theta_2) \sin(\theta_1)) - 2eb \sin(\theta_1 - \theta_2) \\
\rho_{32} &= be \cos(\theta_1 - \theta_2) - d \csc(\theta_2) \sin(\theta_1) \\
\rho_{33} &= -ae + bd \cos(\theta_1 - \theta_2) \csc(\theta_2) \sin(\theta_1) \\
\rho_{34} &= -\csc^2(\theta_1)(-be \sin(2\theta_1) - bd \sin(\theta_1 - 3\theta_2) - be \sin(2\theta_1 - 2\theta_2)) \\
&\quad + bd \sin(\theta_1 - \theta_2) + 2be \sin(2(\theta_1 - \theta_2)) + bd \sin(3(\theta_1 - \theta_2)) \\
&\quad + 2cd \sin(2\theta_2) + bd \sin(\theta_1 + \theta_2)) \\
\rho_{35} &= 4 \cos(\theta_1 - \theta_2)(be \cos(\theta_1 - \theta_2) - cd \csc(\theta_1) \sin(\theta_1)) \\
\rho_{36} &= 4 \cos(\theta_1 - \theta_2)(-ae + bd \cos(\theta_1 - \theta_2) \csc(\theta_1) \sin(\theta_1)) \\
\rho_{51} &= -cd \cos(\theta_1) \csc(\theta_2) + bd \cos(\theta_1 - \theta_2) \cot(\theta_2) \csc(\theta_2) \sin(\theta_1) \\
&\quad + 2be \sin(\theta_1 - \theta_2) \\
\rho_{52} &= be \cos(\theta_1 - \theta_2) - cd \csc(\theta_2) \sin(\theta_1) \\
\rho_{53} &= -ae + bcd \cos(\theta_1 - \theta_2) \csc(\theta_2) \sin(\theta_1). \\
\end{align*}\]

In the equations above, sec and csc denote the secant and cosecant, respectively.

For (4.34) to hold for all \(\dot{y}, \dot{z} \in TQ\) we must have that each \(\rho_i = 0\). From \(\rho_2 = 0\) and \(\rho_6 = 0\) we conclude

\[\lambda_{11}(\theta_1, \theta_2) = \lambda_{12}(\theta_1, \theta_2) = 0.\]

From (4.31)-(4.32) we deduce

\[\Lambda B^+ M = \begin{bmatrix} \lambda_{11}(\theta_1, \theta_2) & \lambda_{12}(\theta_1, \theta_2) \\ \lambda_{21}(\theta_1, \theta_2) & \lambda_{22}(\theta_1, \theta_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b \cos(\theta_1 - \theta_2) & c \end{bmatrix}.\]
3. **Third Equation** (PDE). Finally, $\Lambda B^\perp M$ may be substituted into PDE (4.23) to find a solution for $\hat{M}$. We get a vector equation for partial derivatives of $\hat{M}$ in terms of the components of $\Lambda B^\perp M$

$$
\begin{bmatrix}
\lambda_1 & \lambda_{21} \\
\lambda_{12} & \lambda_2
\end{bmatrix}
\left(\chi_a \nabla_q^T \hat{m}_a + \chi_b \nabla_q^T \hat{m}_b + \chi_c \nabla_q^T \hat{m}_c\right) - 
\begin{bmatrix}
\frac{1}{2} b^2 \sin(2(\theta_1 - \theta_2)) \dot{q}_1^2 \\
bc \sin(\theta_1 - \theta_2) \dot{q}_1^2
\end{bmatrix} = 0
$$

(4.36)

which holds for all velocities $\dot{q} \in TQ$, where the coefficients $\chi_i(\dot{q})$ of the partial derivatives of $\hat{M}$ are given by

$$
\chi_a(\dot{q}) = \frac{1}{2} \begin{bmatrix}
-q_1^2 & -2\dot{q}_1 \dot{q}_2 \\
0 & \dot{q}_1^2 & 0
\end{bmatrix}
\chi_b(\dot{q}) = \begin{bmatrix}
0 & -\dot{q}_2^2 \\
-\dot{q}_1^2 & 0
\end{bmatrix}
$$

$$
\chi_c(\dot{q}) = \frac{1}{2} \begin{bmatrix}
\dot{q}_2^2 & 0 \\
-2\dot{q}_1 \dot{q}_2 & -\dot{q}_2^2
\end{bmatrix}.
$$

All terms in (4.36) are quadratic in the velocities $\dot{q}_1, \dot{q}_2$. Substituting (4.35) and factoring the PDE into common velocity terms, we have

$$
\chi_{11} \dot{q}_1^2 + \chi_{12} \dot{q}_1 \dot{q}_2 + \chi_{22} \dot{q}_2^2 = 0 \quad \forall \dot{q} \in TQ
$$

(4.37)

where

$$
\chi_{11} = \left[ b \cos(\theta_1 - \theta_2) \right] \left( \frac{1}{2} \frac{\partial \hat{m}_a}{\partial \theta_2} - \frac{\partial \hat{m}_b}{\partial \theta_1} \right) - \left[ \frac{1}{2} b^2 \sin(2(\theta_1 - \theta_2)) \right] 
\chi_{12} = - \left[ b \cos(\theta_1 - \theta_2) \right] \frac{\partial \hat{m}_c}{\partial \theta_1}
\chi_{22} = - \left[ b \cos(\theta_1 - \theta_2) \right] \frac{\partial \hat{m}_c}{\partial \theta_2}.
$$

For (4.37) to hold for all $\dot{q} \in TQ$, we must have that each $\chi_{ij} = 0$. From $\chi_{12} = 0$ and $\chi_{22} = 0$, we conclude that

$$
\hat{m}_c \text{ must be a constant.}
$$
Setting $\chi_{11} = 0$ we deduce

$$\frac{\partial \hat{m}_a}{\partial \theta_2} = 2b \sin(\theta_1 - \theta_2) + 2 \frac{\partial \hat{m}_b}{\partial \theta_1}. \quad (4.39)$$

The feasible forms of the closed-loop kinetic energy with ankle-only underactuation and kinetic energy shaping only are therefore given by

$$\hat{K}(q, \dot{q}) = \frac{1}{2} \dot{q}^T \hat{M}(q) \dot{q}$$

where $\hat{M}$ is given by (4.28) subject to constraints (4.38) and (4.39).

Comparing this result for kinetic energy shaping with the achievable closed-loop energy under potential energy shaping (4.27), we note that not only does kinetic energy shaping allow feedback of both joint angles and velocities but it also permits shaping the closed-loop energy \textit{with respect to the angle of the unactuated joint $\theta_2$}, albeit subject to constraint (4.39).
CHAPTER 5

REGULATING SPEED

In this chapter we study controls that regulate the forward speed of bipedal robots. We begin by establishing a definition of forward speed and review our previous energy-shaping work [22] that affords speed regulation by means of a simple potential-shaping control. An interesting consequence of this control was that it results in time-scaled system trajectories, i.e., trajectories that follow the same configuration path at various rates. While much work has been invested in the study of trajectory time scaling for continuous dynamical systems [69–72], comparatively little has been reported about time scaling for hybrid systems such as bipedal walking. In the case of bipedal walking, the presence of the velocity discontinuities due to impacts complicates the direct application of time-scaling results for continuous systems.

To this end, we have explored time scaling as a tool for regulating the speed of bipedal gait in our recent work [24, 25], and we discuss our findings here. We note that the velocity discontinuity due to impacts is a linear mapping of velocities before and after the impact. The linear nature of the impact map gives rise to a time-scaling control law with a great amount of flexibility; using this control law, we may transition between any forward speeds in arbitrary time. Whereas previous controls allowed only very gradual transitions between speeds, the control we present here allows for speed transitions in any length of time we desire.
We illustrate these controls for the compass-gait biped and the biped with knees, noting that they hold for the broader class of bipeds described in Section 2.1 (e.g., see [24] for application of these results to a biped with a torso).

5.1 Forward Speed of a Biped

We first define the forward speed of a biped. The instantaneous velocity of any point on the biped frame varies throughout the step. We choose the hip as a reference point and consider its velocity throughout the gait cycle. By “forward,” we refer to motion of the biped parallel to the walking surface in the direction of progression. Finally, since the velocity of any point on the biped frame varies throughout the gait cycle, we normalize by the gait period $T$. We summarize these features in the following definition.

**Definition** For a planar bipedal robot, the forward speed, denoted $\bar{s}$, is the average magnitude of the hip velocity parallel to the walking surface and in the direction of forward motion of the biped.

As an example, consider the geometry of the support leg of the compass-gait biped shown in Figure 5.1. With respect to the contact point of the support leg, the hip displacement parallel to the walking surface may be written

$$x_{\text{hip}}(t) = -\ell \sin \left( \theta_1(t) + \psi \right) \tag{5.1}$$

where the negative sign in (5.1) is present so that $x_{\text{hip}}$ is positive for values in the direction of forward motion of the biped. The time derivative of $x_{\text{hip}}$ is the instantaneous forward speed of the hip $s(t) = \frac{d}{dt}x_{\text{hip}}(t)$. We average $s(t)$ across the duration of a step to arrive at an explicit expression for the forward speed

$$\bar{s} = \frac{1}{T} \int_0^T s(t)dt \tag{5.2}$$

where $T$ is the period of the step. If we know the step length $d$, i.e., the distance between the surface contact points of the legs during the double-support phase, we
may write the forward speed alternatively as
\[ \bar{s} = \frac{d}{T}. \]

### 5.2 Speed Regulation by Potential Shaping

In our earlier work [22] we considered shaping the closed-loop potential energy of a fully actuated biped by multiplying the acceleration of gravity by a constant control gain. The open-loop potential energy of a biped may be written
\[ V(q) = gV(q) \]
where \( g \) is the acceleration of gravity and \( V(q) : Q \to \mathbb{R} \) is some scalar function of the configuration variables. We have considered closed-loop potential fields of the form
\[ \hat{V}(g) = pgV(q) = pV(q) \quad (5.3) \]
where \( p \in \mathbb{R} \) is a scalar gain of our choice. Choosing gain \( p \) different from 1 in (5.3) amounts to scaling the effect of gravity on the biped. Imposing this artificial potential field \( \hat{V} \) on the dynamics of a bipedal robot requires application of the energy-shaping control (4.8), which may be written
\[
    u = B(q)^{-1}(1 - p)\nabla_q V(q) \\
    = B(q)^{-1}(1 - p)G(q). \quad (5.4)
\]
Under feedback control (5.4), stable limit cycles exist for all $p \in \mathbb{R}$. Moreover, the forward speed of the biped varies with the value of gain $p$, related by the square of the ratio of the forward speeds. These results are summarized in the following proposition.

**Proposition** Consider a fully actuated bipedal robot with hybrid dynamics as in Section 2.1. Denote the forward speed of the passive (i.e., $u(t) \equiv 0 \ \forall t$) limit cycle $\bar{s}_0$. Under potential-shaping control (5.4), a stable limit cycle with desired forward speed $\bar{s}_d$ exists when the gravity gain is chosen as

$$p = \left( \frac{\bar{s}_d}{\bar{s}_0} \right)^2. \quad (5.5)$$

Furthermore, if $[q_0, \dot{q}_0]^T$ is a point on the passive limit cycle, then

$$\left[ q_0, \frac{\bar{s}_d}{\bar{s}_0} \dot{q}_0 \right]^T = [q_0, \sqrt{p} \dot{q}]^T$$

is a point on the limit cycle with desired forward speed $\bar{s}_d$.

**Proof** The quadratic relationship between forward speed and the acceleration of gravity was first noted by McMahon [8] and McGeer [9]. See Licer, M’Sirdi, and Manamanni [23] for a rigorous proof via reparameterization of the integral curves on the configuration manifold $Q$.

In [24, 25] we have shown that this potential-shaping control is a special case of a more general control that also affords speed transitions in arbitrary time. We will present this broader result in the next section.

### 5.2.1 Compass-gait biped

Limit cycles of the compass-gait biped under potential-shaping control (5.4) are shown in Figure 5.2(a) for various values of gain $p$. The forward speed of the biped as a function of the gain $p$ is shown in Figure 5.2(b), from which the purely quadratic relationship (5.5) between $p$ and the velocity ratio is apparent.

The tell-tale signs of trajectory time scaling are visible in the limit cycles of Figure 5.2(a); various values of gain $p$ result in the limit cycle being scaled with...
Figure 5.2 The compass-gait biped under potential-shaping control (5.4) with various settings of gain $p$. (a) Limit cycles are scaled with respect to the velocity axis. (b) Quadratic relationship between forward speed and gravity gain $p$.

respect to the velocity axis, while the limit cycle shape relative to the angular position axis remained constant. The time scaling effect can be seen more clearly in plots of the individual joint trajectories themselves, shown in Figure 5.3.

5.2.2 Biped with knees

Limit cycles for the biped with knees under potential-shaping control (5.4) are shown in Figure 5.4 for various values of gain $p$. As in Figure 5.2(a), we see that the limit cycle of the biped with knees is scaled with respect to the velocity axis, while the shape of the limit cycle with respect to the angular position axis remains constant. These are again the signs that $p \neq 1$ corresponds to the time-scaling of the biped’s trajectories.

As with the compass-gait biped, a purely quadratic relationship between gain $p$ and the speed ratio exists for the biped with knees and is given by (5.5).
Figure 5.3 Time-scaling effects of potential-shaping control (5.4) with various settings of gain $p$ on (a) trajectory of the support leg $\theta_1(t)$ and (b) trajectory of the nonsupport leg $\theta_2(t)$.

Figure 5.4 Limit cycles of the biped with knees under potential-shaping control (5.4).
5.3 Speed Regulation by Time Scaling

While control (5.4) is effective for purposes of regulating a constant forward speed of the biped, its usefulness is limited in transitioning from one forward speed to another. The basins of attraction of the limit cycles for each value of gain \( p \) are quite small; therefore, transitioning between speeds by varying \( p \) requires generous time to ensure the trajectories do not escape the basin of attraction [22]. Our desire for faster transitions, e.g., in just a few steps, motivates further work on speed regulation which we explore in this section.

In the previous section we have seen that the simple potential-shaping control (5.4) results in time scaling the trajectories of the compass-gait biped and the biped with knees. We now explore the general time-scaling control problem for the bipedal robots. We begin by reviewing the well-known theory of time scaling trajectories of continuous dynamical systems [69–72]. We show how the linearity of the velocity discontinuity at impact (2.6) allows extension of time-scaling theory to the hybrid dynamics of the bipeds considered in this work. In the case of time scaling by a constant, the general result reduces to the simple potential-shaping control (5.4) for regulating forward speed. In the case of variable time scaling, the more general result affords transitions between forward speeds in arbitrary time.

5.3.1 General time scaling for continuous dynamical systems

Consider the following continuous Lagrangian dynamics:

\[
M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + G(q(t)) = u(t)
\]

where \( M \) is the inertia matrix, \( C \) is the matrix of centrifugal and Coriolis terms, \( G \) is the gravity vector, and \( u(t) \) is the vector of generalized input forces. Suppose that control input \( u(t) = u_0(t) \) and initial conditions \([q_0(0), \dot{q}_0(0)]^T\) yield solution trajectory \( q(t) = q_0(t) \) for \( 0 \leq t \leq T \) for some \( T > 0 \).

Suppose now that the dynamics (5.6) follows reference trajectory \( q_0(t) \) w.r.t. scaled time \( t' = \phi(t) \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a monotonic map s.t. \( \frac{d}{dt}\phi(t) > 0 \)
for $0 \leq t \leq \phi^{-1}(T)$ and $\phi(0) = 0$. The scaled trajectory is

$$q_{sc}(t') = q_0(\phi(t)) = q_0(t')$$  (5.7)

where $0 \leq t' \leq \phi^{-1}(T)$. The scaled velocity and acceleration are given by

$$\dot{q}_{sc}(t') = \frac{\partial q_0(\phi)}{\partial \phi} \frac{d\phi}{dt} = \dot{q}_0(t')\dot{\phi}$$  (5.8)

$$\ddot{q}_{sc}(t') = \frac{\partial \dot{q}_0(\phi)}{\partial \phi} \frac{d\phi}{dt} \ddot{\phi} + \dot{q}_0(\phi) \frac{d\dot{\phi}}{dt} = \ddot{q}_0(t')\dot{\phi}^2 + \dot{q}_0(t')\ddot{\phi}$$  (5.9)

We rewrite dynamics (5.6) for the scaled trajectory $q_{sc}(t')$

$$M(q_{sc}(t'))\ddot{q}_{sc}(t') + C(q_{sc}(t'), \dot{q}_{sc}(t'))\dot{q}_{sc}(t') + G(q_{sc}(t')) = u_{sc}(t').$$  (5.10)

Substituting (5.7)-(5.9) into (5.10), we have

$$u_{sc}(t') = \ddot{\phi}^2 u_0(t') - G(q_0(t')) + M(q_0(t'))\ddot{q}_0(t')\dot{\phi} + G(q_0(t'))$$

where the second equality follows from the fact that $C(q, \star)\star$ is quadratic in $\star$, and the third equality follows from substitution of (5.6). Collecting common terms in the final expression above, we have

$$u_{sc}(t') = \ddot{\phi}^2 u_0(t') + (1 - \ddot{\phi}^2)G(q_0(t')) + M(q_0(t'))\ddot{q}_0(t')\dot{\phi}$$  (5.11)

which is the general control law for time-scaling systems with continuous dynamics given by (5.6).

### 5.3.2 Special cases of time scaling

We now consider two special cases of time scaling for continuous dynamical systems.
1. **Constant time scaling.** Consider the case of constant time-scaling for which the scaled time \( t' = \phi(t) \) is given by a constant multiple of normal time, i.e., \( \phi(t) = \lambda t \) for some constant \( \lambda \in \mathbb{R}^+ \). Under this restriction, \( \dot{\phi}(t) = \lambda \) and \( \ddot{\phi}(t) = 0 \ \forall t \geq 0 \) and the general time-scaling control law (5.11) reduces to

\[
\dot{q}_\text{sc}(t') = \lambda \dot{q}_0(t')
\]

\[
\ddot{q}_\text{sc}(t') = \lambda^2 \ddot{q}_0(t')
\]

for \( 0 \leq t \leq \phi^{-1}(T) = T/\lambda \) with initial condition \([q_\text{sc}(0), \dot{q}_\text{sc}(0)]^T = [q_0(0), \lambda \dot{q}_0(0)]^T\).

2. **Constant time scaling with passive reference trajectory.** If, in addition to constant time scaling, the reference trajectory is passive (i.e., \( u_0(t) = 0 \ \forall t \)) the time-scaling control law (5.11) reduces further to the simple potential energy-shaping control

\[
u_\text{sc}(t') = (1 - \lambda^2)G(q_0(t'))
\]

which effectively “cancels” the effect of normal gravity \( G \) on the system dynamics (5.6) and substitutes the effect of scaled gravity \( \lambda^2 G \). Note that (5.13) is identical to (5.4) with \( \lambda^2 = p \) and \( B = B^{-1} = I \).

### 5.3.3 Time scaling for hybrid dynamics of locomotion

The dynamics of bipedal walking comprise a continuous single-support phase and a discrete double-support phase as described in Section 2.1. The general time-scaling results for continuous-time systems presented above apply immediately to the continuous phase of bipedal locomotion. We now consider how the discrete dynamics of the double-support phase are affected by time scaling.
Recall the map relating pre- to postimpact velocities (2.6), i.e.,
\[ \dot{q}(t_i^+) = h(q(t_i^-))\dot{q}(t_i^-) \]  
(5.14)
where \( t_i \) is the moment of impact. We make two key observations. First, the map (5.14) is linear with respect to the velocities \( \dot{q} \). Second, the mapping function \( h(q(t_i^-)) \) is completely determined by the configuration at the end of the step \( q(t_i^-) \).

Consider choosing the reference trajectory for the general time-scaling control (5.11) such that the terminating configuration of the trajectory \( q_0(T) \) is the configuration of a biped the moment before impact, i.e., \( q_0(T) = q_0(t_i^-) \). Since the final configuration of a time-scaled trajectory \( q_{sc}(\phi^{-1}(T)) = q_0(T) \) is always the same, the function \( h \) remains the same regardless of time-scaling
\[ h\left(q_{sc}(\phi^{-1}(T))\right) = h(q_0(T)). \]

We now illustrate the application of time-scaling to our passive-dynamic bipeds. For both the compass-gait biped and the biped with knees we select as the reference trajectory the passive limit cycle of the biped. We let initial time \( t = 0 \) of the reference trajectory correspond to the configuration at the beginning of the single-support phase (the instant after impact) and let the terminal time \( t = T \) correspond to the configuration at the end of the single-support phase (the instant prior to impact), i.e.,
\[ t_i^+ = 0 \text{ and } t_i^- = T. \]  
(5.15)
Therefore, the postimpact velocity of the reference trajectory is the velocity at the start of the step
\[ \dot{q}_0(t_i^+) = \dot{q}_0(0). \]
From (5.14) we have that
\[ \dot{q}_0(0) = h(q_0(T))\dot{q}_0(T). \]  
(5.16)

### 5.3.4 Constant time scaling

We first consider the case of constant time scaling given by \( \phi(t) = \lambda t \) with \( \lambda > 0 \). Since for our bipeds we are making use of the passive limit cycles as reference
trajectories, the time-scaling control law reduces to (5.13), which is identical to the potential energy-shaping control (5.12) presented earlier in the chapter with \( p = \lambda^2 \).

Under constant time scaling, the reference trajectory (the passive limit cycle) is scaled linearly by the constant \( \lambda \), i.e.,

\[
\dot{q}_{sc}(t') = \lambda \dot{q}_0(t').
\]

Similarly, the linearity of the impact map results in the preimpact velocities emerging as post-impact velocities linearly scaled by the same factor \( \lambda \)

\[
\dot{q}_{sc}(\phi(t_i^+)) = h(q_{sc}(\phi(t_i^-))) \cdot \dot{q}_{sc}(\phi(t_i^-))
\]

\[
= h(q_0(T)) \cdot \lambda \dot{q}_0(T)
\]

\[
= \lambda \cdot h(q_0(T)) \dot{q}_0(T)
\]

\[
= \lambda \cdot \dot{q}_0(0)
\]

where the second equality follows from selection of reference times for the limit cycle given by (5.15) and the fourth equality follows from (5.16). The consequence is that the linear scaling of the reference trajectory during the continuous portion of the biped dynamics \( \dot{q}_{sc}(t') = \lambda \dot{q}_0(t') \) is unaffected by the discrete velocity map. If a velocity equal to the reference velocity multiplied by a constant enters the velocity map, a velocity equal to the reference velocity multiplied by the same constant emerges from the map. As a result, constant time-scaling of the passive limit cycle results in trajectories that form new closed orbits in the state space—new limit cycles. Furthermore, as constant time-scaling amounts to linear scaling of velocities, these new limit cycles appear to be stretched with respect to velocity while they remain constant with respect to angular position.

Figure 5.5 shows limit cycles of the compass-gait biped under constant time-scaling for various values of \( \lambda \). Comparison of these plots with those of Figure 5.2 confirms our earlier observation that the effect of the potential-shaping control (4.11) is constant time-scaling. In fact, the control for constant time-scaling of a passive reference trajectory (5.13) is identical to potential-shaping control (4.11) with \( p = \lambda^2 \).
Figure 5.5 Limit cycles of the compass-gait biped under constant time-scaling control for various values of parameter $\lambda$.

The dynamics of the biped with knees include not one but two velocity discontinuities: one due to impacts with the walking surface as in the compass-gait biped, the other due to the knee locking event that occurs when the nonsupport leg is fully extended. As the surface impact map is identical to that of the compass-gait biped, it accommodates the time-scaling as described above.

The velocity discontinuity due to the knee locking event (2.26), while different from the impact velocity discontinuity, possesses the same key properties noted in this section: it is linear with respect to the velocities $\dot{q}$ and the mapping function $h_k(q(t_k))$ is completely determined by the configuration at the end of the step $q(t_k)$. Analysis similar to what we have already shown for the velocity discontinuity at impact may be employed to show that the knee lock discontinuity is not affected by time-scaling. Constant scaling of velocities entering the velocity map is passed along to the velocities exiting the map.

Figure 5.6 shows limit cycles of the compass-gait biped under constant time-scaling for various values of $\lambda$. Comparison of these plots with those of Figure 5.4
Figure 5.6 Limit cycles of the biped with knees under constant time-scaling control for various values of parameter $\lambda$.

again shows that the effect the potential-shaping control (4.11) is constant time-scaling.

5.3.5 Variable time scaling

In the previous section, we showed that constant time scaling the passive limit cycles of our bipeds yields limit cycles with forward speeds of our choice. In this section, we consider nonconstant time-scaling of the passive limit cycles. Nonconstant time scaling allows for transitions between limit cycles with different forward speeds.

Previous work accomplished varying time scaling by gradually changing parameter $\lambda$ of the constant time-scaling control law (5.12) [22]. This approach is imprecise and time-consuming, requiring changes in $\lambda$ to be sufficiently slow to prevent trajectories from exiting the basin of attraction. In this work, we have formulated a general time-scaling control law (5.11) based on arbitrary time-scaling function $\phi(t)$; this formulation affords the flexibility of designing $\phi(t)$ to change the time scaling of the system in arbitrary time.
For the velocity discontinuity under nonuniform time scaling, a derivation similar to (5.17) holds, except that the scaling of the postimpact velocity \( \dot{q}_{sc}(\phi(t_i^+)) \) is not the same as the scaling of the initial velocity of the trajectory \( \dot{q}_{sc}(0) \) due to the fact that the time-scaling is nonuniform over the range \( t' \in [0, \phi(t_i^+)] \). Consequently, trajectories of the biped under nonconstant time-scaling do not form closed orbits. Instead, these trajectories scaled with nonlinear functions may be used to transition from one limit cycle to another. In this way, nonconstant time-scaling may be used for transitions in forward speeds.

Consider the task of transitioning from a limit cycle described by constant time-scaling function \( \phi_1(t) = \lambda_1 t \) and another limit cycle described by \( \phi_2(t) = \lambda_2 t \). Clearly, \( \dot{\phi}_i(t) = \lambda_i \) for each of these limit cycles. To move from one to the other, we construct a time-scaling function \( \phi_\Delta(t) \) whose derivative is equal to \( \lambda_1 \) at the beginning of the step and equal to \( \lambda_2 \) at the end of the step. That is,

\[
\begin{align*}
\phi_\Delta(0) &= 0 \\
\phi_\Delta(t_F) &= T \\
\dot{\phi}_\Delta(0) &= \lambda_1 \\
\dot{\phi}_\Delta(t_F) &= \lambda_2
\end{align*}
\]

where \( t_F = \phi^{-1}(T) \) is the desired time (in seconds) of the end ground impact, which we are free to choose provided \( \frac{d}{dt} \phi_\Delta(t) > 0 \) for \( 0 \leq t \leq t_F \). These four conditions are satisfied by the cubic polynomial given by

\[
\phi_\Delta(t) = \lambda_1 t + \left( \frac{3T}{t_F^2} - \frac{2\lambda_1 + \lambda_2}{t_F^2} \right) t^2 + \left( -\frac{2T}{t_F^2} + \frac{\lambda_1 + \lambda_2}{t_F^2} \right) t^3. \tag{5.18}
\]

Figure 5.7 is a phase portrait of the compass-gait biped under our time-scaling control transitioning between limit cycles with different forward speeds. For the first two steps, the biped is under constant time-scaling control (5.13) with \( \lambda = 1 \). For the third step, we implement the general time-scaling control (5.11) with time scaled according to the polynomial (5.18) with \( \lambda_1 = 1 \), \( \lambda_2 = 2 \), and \( t_F = 0.5 \) s. The result is nonlinear scaling of the limit cycle, beginning with an initial condition on the limit cycle corresponding to \( \lambda = 1 \) and concluding after 0.5 s with a terminal condition on the limit cycle corresponding to \( \lambda = 2 \). At this point, we switch back to the constant
Figure 5.7 Bipeds under varying time-scaling; here the bipeds transition from their passive limit cycle ($\lambda_1 = 1$) to time-scaled limit cycles described by $\lambda_2 = 2$ in 0.5 s: (a) compass-gait biped (b) biped with knees.

time-scaling control (5.13) with $\lambda = 2$, and the trajectory of the biped traces a limit cycle scaled with respect to velocities by 2. The result is that the biped maintains a steady speed for two steps, transitions from one speed to another in 0.5 s, and maintains the new speed for all steps thereafter.

A phase portrait of the biped with knees making a similar transition from a limit cycle defined by $\lambda = 1$ to a limit cycle defined by $\lambda = 2$ in 0.5 s is shown in Figure 5.7(b).
CHAPTER 6

REGULATING SPEED AND STEP LENGTH

We now consider biomimetic regulation of forward speed. In Chapter 3 we noted that biological locomotion is subconsciously tuned to minimize energy requirements. We hypothesize that designing controls for bipedal locomotion that mimic features of human walking will result in energy savings when compared to other control approaches.

In this chapter we focus on duplicating the two key features of human locomotion discussed in Section 3.2, i.e., simultaneously adjusting step length when changing forward speed and mimicking the dominant role played by the ankle of the support leg in supplying the energy for walking. This biomimetic approach is in contrast to that taken in the previous chapter, where we regulated forward speed in a manner that held step length constant and required actuation at all joints.

We mimic the first feature, the simultaneous variation of step length and speed, by designing a control such that step length is varied monotonically with respect to speed similar to the biological trends shown in Figure 3.3. We approximate the second feature, the dominant role of the support ankle in supplying energy for walking, by limiting control input to be solely from the ankle of the support leg; that is, we consider an underactuated control system with actuators positioned only at the ankles of the bipeds.

We have chosen to design the biomimetic control within the framework of energy-shaping subject to underactuation presented in Chapter 4. As potential energy-
shaping has been considered in previous work [22–24], we here explore shaping the kinetic energy of our bipeds into closed-loop forms that accomplish our goals.

The chapter begins with the formulation of a kinetic-shaping control that simultaneously alters forward speed and step length while requiring actuation only at the ankle of the support leg [28, 29]. We next compare this biomimetic control with that of the speed-regulating control presented in the previous chapter (which maintained constant step length across all speeds and required full actuation) and note the biologically inspired control exhibits multiple energetic advantages. The biomimetic control results in limit cycles whose total energy is significantly closer to that of the passive limit cycle than the previous speed-regulating control for the same commanded speeds. The implication is that substantially less energy is required to transition from the passive limit cycle to a biomimetic limit cycle with a desired forward speed. Moreover, the biomimetic control requires lower control effort per step than the previous speed-regulating control for the majority of realizable speeds.

The chapter concludes with the design of an energy-shaping control allowing rapid transitions between biomimetic limit cycles with different forward speeds and step lengths. A simple approach to gradually change the design parameter of our biomimetic control requires unacceptably long transition times due to the small basins of attraction of the biomimetic limit cycles. Therefore, we extend previous work on passivity-based total energy-shaping control [20, 22] to the present kinetic-shaping control, expanding the basins of attraction and thereby allowing for a significant reduction in transition times between limit cycles of different forward speeds and step lengths.

6.1 Biomimetic Kinetic Shaping

Recall from Section 3.2.1 that humans increase step length monotonically when increasing forward speed. Furthermore, recall the biomechanics results of Section 3.2.2 that show the critical role played by the human ankle in supplying the energy necessary for forward motion of the body during the stance phase of walking. In this
section, we seek to imitate these features of biological locomotion by designing a control law requiring ankle-only actuation that scales step length and forward speed simultaneously.

We search for a suitable shaping of the kinetic energy that realizes these features. As we are considering underactuated kinetic shaping, we recall the method of Section 4.1.4 for solving the matching condition in this case. For the compass-gait biped, the solution to the matching condition defining all possible forms of the closed-loop kinetic energy in the case of actuation at the ankle alone is provided in Section 4.2.

Using the results from Chapter 4, we will design a biomimetic control first for the compass-gait biped. This amounts to selecting one of the possible forms of the closed-loop kinetic energy that achieves our two goals: simultaneous variation of step length and forward speed and control input from the ankle alone. Our biomimetic control design for the compass-gait biped is complete once we successfully find a feasible kinetic form that meets these criteria.

Rather than repeat the entire process for the biped with knees, we note that the same kinetic form that accomplishes the biomimetic goals for the compass-gait biped satisfies the matching condition for the biped with knees. This indicates it is possible to realize the same kinetic form in the closed loop for the biped with knees with actuation at the ankle alone. Implementing the same kinetic form on the biped with knees, we discover that the biomimetic criteria are also satisfied for this biped as well.

### 6.1.1 Form of the kinetic energy modification

We now consider the possible forms of the closed-loop kinetic energy of the compass-gait biped subject to ankle-only actuation as presented in the solution to the underactuated matching condition in Section 4.2. Note that simple additions to the kinetic energy of the form $f(\theta_1)\dot{\theta}_1^2$ where $f : S \to \mathbb{R}$ are achievable with actuation at the ankle alone. That is, we note that kinetic energy of the form

$$\dot{K}(q, \dot{q}) = K(q, \dot{q}) + f(\theta_1)\dot{\theta}_1^2$$  \hspace{1cm} (6.1)
where $K$ is the natural kinetic energy given by (2.17) may be realized in the closed loop for the compass-gait biped. It follows from (6.1) that

$$
\hat{M}(q) = M(q) + \begin{bmatrix}
2f(\theta_1) & 0 \\
0 & 0
\end{bmatrix}.
$$

(6.2)

It is easy to verify that the component functions $\hat{m}_{a}$, $\hat{m}_{b}$, $\hat{m}_{c}$ of (6.2) satisfy both restrictions for ankle-only actuation presented in Section 4.2. We now restrict our attention to closed-loop kinetic energy of the form (6.1) as we search for a form that realizes our biomimetic criteria.

Previous work with energy shaping for bipedal walking [22] indicates that increasing the closed-loop energy function $\hat{E} = \hat{K} + \hat{V}$ of the biped near the beginning of the single-support phase (i.e., when $0 < \theta_1 < \pi$) and decreasing the closed-loop energy function near the end of the single-support phase ($-\pi < \theta_1 < 0$) is useful for increasing both the speed and step length of the limit cycle. Since we are performing kinetic energy shaping only (we have chosen $\hat{V} = V$), we search for kinetic energy modifications with this same profile. That is, we seek a closed-loop form of the kinetic energy that is higher than the natural kinetic energy in the early portion of each step and lower than the natural in the latter portion. Therefore, we experiment with simple functions $f(\theta_1)$ for which the following properties hold:

$$
\hat{K}(q, \dot{q}) - K(q, \dot{q}) = f(\theta_1)\dot{\theta}_1^2 > 0 \text{ when } 0 < \theta_1 < \pi
$$

$$
f(\theta_1)\dot{\theta}_1^2 = 0 \text{ when } \theta_1 = 0
$$

$$
f(\theta_1)\dot{\theta}_1^2 < 0 \text{ when } -\pi < \theta_1 < 0.
$$

(6.3)

Consider the following modification to the kinetic energy of the compass-gait biped:

$$
\hat{K}(q, \dot{q}) = K(q, \dot{q}) + k\sin(\theta_1)\dot{\theta}_1^2
$$

(6.4)

i.e., we have chosen $f(\theta_1) = k\sin(\theta_1)$, where $k \in \mathbb{R}$ is a free scalar parameter. It is clear that kinetic modification (6.4) satisfies the desired energetic profile (6.3).
Figure 6.1 (a) Limit cycles of the compass-gait biped with shaped kinetic energy (6.4) and various values of gain $k$. (b) The range of achievable speeds and step lengths as $k$ is varied.

6.1.2 Results

Phase portraits of the compass-gait biped under energy-shaping control (4.8) with closed-loop kinetic energy chosen as (6.4) are shown in Figure 6.1(a). For positive values of gain $k$, the limit cycle expands along both angular velocity and position axes, indicating an increase in both the speed and step length of the biped. For negative values of $k$, the limit cycle contracts with respect to both axes, indicating reduction in speed and step length. The relationship between the biped’s speed and step length at $k$ is varied as shown in Figure 6.1(b). Comparing the step length vs. speed profile of the biped in Figure 6.1 with that of humans and other locomoting mammals in Figure 3.3, we conclude we have accomplished our goal of biomimicry.

While it would be time-consuming to solve the underactuated matching condition for the biped with knees subject to ankle-only actuation and then repeat our search for a suitable closed-loop energetic form that meets our biomimetic goals, we note that it is relatively easy to show that kinetic energy modification (6.4) satisfies the matching condition for the biped with knees. That is, rather than solving for all solutions of the matching condition, it is simple to verify that a candidate solution
in fact solves the matching condition. A straightforward computation shows (6.4) satisfies the matching condition (4.10) when the biped with knees is restricted to actuation at the ankle alone. Therefore, we may accomplish the same kinetic energy modification on the closed-loop energy of the biped with knees.

Simulations of the biped with knees under energy-shaping control (4.8) with closed-loop kinetic energy (6.4) are shown in Figure 6.2, with results comparable to those of the compass-gait biped. Positive values of gain $k$ stretch in the limit cycle along both angular and velocity axes, corresponding to an increase in speed and step length; negative values of $k$ result in contractions of the limit cycle, corresponding to reduction in speed and step length. The overall effect is a biomimetic variation of speed and step length, as shown in Figure 6.2(b).

From phase portraits of the biped with knees shown in Figure 6.2(a), we notice a minute change in the behavior of the biped as $k$ is varied. For most values of $k$ in the range $[-12, 6]$, the nonsupport shank angle decreases momentarily at the start of the single-support phase before reversing direction and increasing angle up to the moment of knee lock; the effect is that the shank segment sweeps slightly backwards.
at the start of the step. For positive values of $k$ beyond some breakpoint, however, the shank angle increases uniformly from the start of the single-support phase up to the moment of knee lock, eliminating the initial backward sweep of the shank.

This behavioral artifact of the kinetic energy modification is analogous to that of the nonsupport leg of the compass-gait biped. From Figure 6.1(a) it is evident that, for most values of $k$ in the range $[-6, 6]$, the angle of the nonsupport leg initially decreases at the start of the single-support phase before reversing direction and increasing in value; this corresponds to the nonsupport leg kicking backwards and up very slightly at the start of the step. For positive values of $k$ beyond a particular breakpoint, however, the nonsupport angle demonstrates no negative motion in the early portion of the step.

### 6.2 Comparison of the Speed-Regulating Controls

In Chapter 3 it was observed that biological organisms subconsciously alter gait under various conditions in ways that seem to minimize the energy required for walking. It is believed this is the underlying reason why humans adjust step length when walking at various speeds and why the muscles of the ankle play such a prominent energetic role in walking. When designing our biomimetic control, we hypothesized that mimicking features of human locomotion will result in lower energy requirements than controls that do not duplicate these aspects of biological locomotion.

In this section, we compare the kinetic-shaping biomimetic speed-regulating control designed in this chapter against the potential-shaping speed-regulating control presented in the previous chapter. For our comparison, we use the following two energetic measures: total energy on the closed-loop limit cycles and actuator effort per step. The total energy on a controlled limit cycle may be written

$$\hat{E}(q, \dot{q}) = \hat{K}(q, \dot{q}) + \hat{V}(q)$$

for both controls. The total energy on the compass-gait limit cycle is constant for any speed, due to the facts that our bipeds are assumed to be frictionless and that
Figure 6.3 Comparison of the two different speed-regulating controls for the compass-gait biped: (a) total energy on the limit cycle and (b) actuator effort.

the kinetic energy dissipated in the impact between steps is exactly matched by the control input and the energy contribution from the change in potential energy on the walking slope. Our second energetic metric for comparing our controls is the following measure of actuator effort per step that is commonly found in the literature [73–78]:

$$\text{effort} = \frac{1}{2} \int_0^T u(t)^T u(t) dt$$

where $T$ is the period of the step.

In Figure 6.3 we compare the energetics of the compass-gait biped under our biomimetic control (i.e., (4.8) with closed-loop kinetic energy (6.4)) against the control of the previous chapter (4.11). From Figure 6.3(a) it is apparent the biomimetic control results in limit cycles with total energy far closer to the passive limit cycle than the previous control. This implies that, to change from the passive limit cycle to a biomimetic limit cycle with a nonpassive speed, the actuators of the compass-gait biped need to generate or dissipate vastly less energy than when changing to nonpassive limit cycles of the previous control. Further, from Figure 6.3(b) we see that the biomimetic control requires less effort per step for commanded forward speeds slower than 1.05 m/s.
Similar comparisons of the energetic requirements of the two controls implemented with the biped with knees are shown in Figure 6.4. The limit cycle of the biped with knees actually possesses two different total energy values—one at the start of the step when the knee is free and one following the knee lock. The knee lock event results in a small loss of kinetic (hence, total) energy. However, the loss in kinetic energy is so minute (about 0.5 J) that the difference in pre- and post-knee-lock total energy levels is indistinguishable in Figure 6.4(a). A zoom of Figure 6.4(a) is shown in Figure 6.4(b) revealing the two energy levels associated with each limit cycle for the biped with knees.

As with the compass-gait biped, comparison of the two controls in Figure 6.4(a) indicates the biomimetic control results in limit cycles with total energy far closer to the passive limit cycle than the previous control. Also, from Figure 6.4(b) we see that the biomimetic control requires less effort per step for the biped with knees for commanded forward speeds slower than 1.02 m/s.

6.3 Transitioning between Forward Speeds and Step Lengths

We now consider the task of transitioning between limit cycles with different speeds and step lengths. For our biomimetic control with closed-loop kinetic field (6.4), these limit cycles correspond to different values of the parameter $k$. We propose the following time function for $k(t)$ that transitions smoothly from one constant value $k_1$ to another $k_2$ in some set length of time $t_d > 0$

$$k(t) = \begin{cases} 
  k_1 & t < 0 \\
  \frac{1}{2} \left[ (k_2 + k_1) - (k_2 - k_1) \cos \left( \frac{\pi t}{t_d} \right) \right] & 0 \leq t \leq t_d \\
  k_2 & t_d < t.
\end{cases} \quad (6.5)$$

Using this smooth $k(t)$ transition along with our kinetic-shaping control, we can transition among limit cycles defined by different values of $k$ and hence dynamically change the speed and step length of a biped as it walks.
Figure 6.4 Comparison of the two different speed-regulating controls for the biped with knees: (a) and (b) total energy on the limit cycle, (c) actuator effort. (b) Shows a zoom of (a) revealing the two energy levels on the limit cycle of the biped with knees. The higher energy level corresponds to the total energy on the limit cycle prior to the knee lock; the lower energy level corresponds to the total energy after knee lock.
A stable transition between limit cycles requires that the system trajectory remain inside the basin of attraction. As the basin is quite small, transitions between limit cycles with different values of $k$ require conservative selections of transition time $t_d$. For example, consider the compass-gait biped transitioning between the limit cycles corresponding to $k_1 = 0$ and $k_2 = 6$, shown in Figure 6.5(a) and (c). In order to avoid the biped becoming unstable and tipping over, the minimum transition time for this change is about $t_d = 20$ s (roughly 30 steps).

The biped with knees is even more sensitive to $k$ transitions than the compass-gait biped. Figure 6.5(b) and (d) show the biped with knees transitioning from the limit cycles corresponding to $k_1 = 0$ to the one corresponding to $k_2 = 6$. The minimum stable transition time for this change is about $t_d = 27$ s (roughly 39 steps).

Since the transition time is limited by the size of the basins of attraction, expanding the basins would allow for reduced transition times. We next consider modifying our energy-shaping control to expand the basins of attraction.

### 6.3.1 Faster transitions via total-energy shaping

In [18,20], Spong et al. introduced a passivity-based control that shapes the total energy (i.e., both potential and kinetic energy) of the compass-gait biped. One effect of this total energy-shaping control is the expansion of the small basin of attraction of the biped. This method has been effectively incorporated into other controls for the compass-gait biped, such as potential-shaping controls [22]. In this section, we adapt this total energy-shaping control to our present biomimetic kinetic energy-shaping control to expand the basin of attraction of our controlled limit cycles. As a result, we are able to achieve faster transitions among limit cycles corresponding to different values of $k$.

As mentioned previously, the total energy $\hat{E}$ on the limit cycle of the compass-gait biped is constant for constant $k$. For the biped with knees, the total energy $\hat{E}$ assumes two values on the limit cycle, one prior to the knee lock event and one following the knee lock event. In the derivation that follows, we augment our existing biomimetic
Figure 6.5 Stable transitions using $k(t)$ planner (6.5) require conservative settings of the transition time $t_d$. (a) and (c) show the compass-gait biped transitioning between limit cycles corresponding to $k_1 = 0$ and $k_2 = 6$ in the minimum stable transition time $t_d = 20$ s (roughly 30 steps). (b) and (d) show the biped with knees transitioning between limit cycles corresponding to $k_1 = 0$ and $k_2 = 3$ in the minimum stable transition time $t_d = 27$ s (roughly 39 steps). (a) and (b) show the $k$ transition for each biped. (c) and (c) show the evolution of the bipeds’ trajectories as $k$ is varied.
kinetic energy-shaping control to include an additional total energy-shaping term which is based on the total energy $\hat{E}$ on the limit cycle. We show the derivation for the compass-gait biped, which has a single value of total energy on the limit cycle, while noting that the derivation extends easily to the biped with knees if we perform the identical derivations for the total energy before and after the knee lock.

Let $E_i$ denote the total energy on the compass-gait limit cycle corresponding to gain $k_i$ for $i = 1, 2$. For a transition between gains $k_1$ and $k_2$ as described in (6.5), consider varying a reference of the total energy on the limit cycle $\hat{E}_{ref}$ in a similar smooth form, i.e.,

$$\hat{E}_{ref}(t) = \begin{cases} 
E_1 & t < 0 \\
\frac{1}{2} \left[ (E_2 + E_1) - (E_2 - E_1) \cos \left( \frac{\pi}{t_d} t \right) \right] & 0 \leq t \leq t_d \\
E_2 & t_d < t.
\end{cases} \quad (6.6)$$

We now modify our biomimetic energy-shaping control. Consider adding a new control input $\hat{u}$ to the closed-loop dynamics (4.4), resulting in

$$\hat{M}(q) \ddot{q} + \hat{C}(q, \dot{q}) \dot{q} + G(q) = \hat{u}. \quad (6.7)$$

The additional control input $\hat{u}$ appears in the overall energy-shaping control law as follows:

$$u = M(q) \left[ M^{-1}(q) \left( \hat{C}(q, \dot{q}) \dot{q} + G(q) \right) - \hat{M}^{-1}(q) \left( \hat{C}(q, \dot{q}) \dot{q} + G(q) - \hat{u} \right) \right].$$

Adapting the approach of [20] as in [22], we propose the following Lyapunov function candidate, a so-called storage function:

$$S(q, \dot{q}, k) = \frac{1}{2} \left( \hat{E}(q, \dot{q}, k) - \hat{E}_{ref}(t) \right)^2.$$ 

Note that $S$ is positive semidefinite in some neighborhood of the limit cycle—it is identically zero on the limit cycle of the robot and positive in a region containing the limit cycle. Note also that the storage function $S$ depends on gain $k$, a feature that distinguishes this storage function from the function considered in [20].
differentiation of $S$ yields
\[ \frac{d}{dt} S(q, \dot{q}, k) = \left( \hat{E}(q, \dot{q}, k) - \hat{E}_{ref}(t) \right) \cdot \frac{d}{dt} \left( \hat{E}(q, \dot{q}, k) - \hat{E}_{ref}(t) \right). \] (6.8)

It is straightforward to compute the time derivative of $\hat{E}_{ref}(t)$, which is given by
\[
\frac{d}{dt} \hat{E}_{ref}(t) = \begin{cases} 
0 & t < 0 \\
\frac{\pi}{2t_d}(E_2 - E_1) \sin \left( \frac{\pi}{t_d} t \right) & 0 \leq t \leq t_d \\
0 & t_d < t.
\end{cases}
\] (6.9)

We now consider the time derivative of the total energy.

### 6.3.2 Time derivative of total energy

The total energy of the compass-gait robot under our biomimetic kinetic-shaping control is given by
\[
\hat{E}(q, \dot{q}, k) = \hat{K}(q, \dot{q}, k) + V(q) \\
= \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} + V(q) \\
= \frac{1}{2} \dot{q}^T \left( M(q) + \begin{bmatrix} 2k \sin(\theta_1) & 0 \\ 0 & 0 \end{bmatrix} \right) \dot{q} + V(q).
\]

Since the time-derivative of the closed-loop inertia matrix $\dot{M}$ may be written
\[
\frac{d}{dt} \dot{M} = \frac{d}{dt} M(q) + \frac{d}{dt} \begin{bmatrix} 2k \sin(\theta_1) & 0 \\ 0 & 0 \end{bmatrix} \\
= \dot{M}(q) + \begin{bmatrix} 2k \cos(\theta_1) \dot{\theta}_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2k \sin(\theta_1) & 0 \\ 0 & 0 \end{bmatrix},
\]
the time-derivative of the total energy may be written

\[
\frac{d}{dt} \dot{E}(q, \dot{q}, k) = \dot{q}^T \dot{M}(q) \dot{q} + \frac{1}{2} \dot{q}^T \left( \frac{d}{dt} \dot{M}(q) \right) \dot{q} + \frac{d}{dt} V(q)
\]

\[
= \dot{q}^T (\dot{M}(q) \dot{q} + G) + \frac{1}{2} \dot{q}^T \left( \frac{d}{dt} \dot{M}(q) \right) \dot{q}
\]

\[
= \dot{q}^T (\dot{u} - \dot{\hat{C}} \dot{q}) + \frac{1}{2} \dot{q}^T \left( \frac{d}{dt} \dot{M}(q) - 2 \dot{\hat{C}} \right) \dot{q}
\]

\[
= \dot{q}^T \dot{u} + \frac{1}{2} \dot{q}^T \begin{bmatrix} 2 \dot{k} \sin(\theta_1) & 0 \\ 0 & 0 \end{bmatrix} \dot{q}
\]

\[
= \dot{q}^T \dot{u} + \dot{k} \sin(\theta_1) \dot{\theta}_1^2
\]

(6.10)

where the second equality follows from the fact that \( \frac{d}{dt} V = \dot{q}^T G \) and the third equality follows from substitution of (6.7).

The fifth equality in the derivation above follows from the skew-symmetry property of rigid robots [32], i.e., the fact that \( \frac{d}{dt} \dot{M} - 2 \dot{\hat{C}} \) is skew-symmetric for a robot whose mass parameters are constant. For the compass-gait robot with shaped kinetic energy (6.4), the skew-symmetry property holds while gain \( k \) is constant; i.e., \( \frac{d}{dt} \dot{M} - 2 \dot{\hat{C}} \) is skew-symmetric when \( \dot{k} = 0 \). Indeed, consider (6.10) when \( \dot{k} = 0 \) and notice that any contribution due to \( \frac{d}{dt} \dot{M} - 2 \dot{\hat{C}} \) vanishes entirely. However, varying the mass parameter \( k \) with respect to time introduces a term to \( \frac{d}{dt} \dot{M} - 2 \dot{\hat{C}} \) that is not skew-symmetric, the component of \( \frac{d}{dt} \dot{M} \) that is nonzero for variable \( k \), i.e.,

\[
\begin{bmatrix} 2 \dot{k} \sin(\theta_1) & 0 \\ 0 & 0 \end{bmatrix}.
\]

### 6.3.3 Total energy-shaping control

Returning to (6.8), we may now write the time derivative of the storage function

\[
\frac{d}{dt} S(q, \dot{q}, k) = \left( \dot{E}(q, \dot{q}, k) - \dot{\hat{E}}_{\text{ref}}(t) \right) \cdot \frac{d}{dt} \left( \dot{E}(q, \dot{q}, k) - \dot{\hat{E}}_{\text{ref}}(t) \right)
\]

\[
= \left( \dot{E}(q, \dot{q}, k) - \dot{\hat{E}}_{\text{ref}}(t) \right) \cdot \left( \dot{q}^T \dot{u} + \dot{k} \sin(\theta_1) \dot{\theta}_1^2 - \frac{d}{dt} \dot{\hat{E}}_{\text{ref}}(t) \right)
\]
where $\frac{d}{dt}\hat{E}_{ref}(t)$ is given by (6.9).

We now select the additional control input $\hat{u}$ so that $\dot{S}$ is negative semidefinite in a neighborhood of the limit cycle. Hence, choose

$$\hat{u} = \frac{\dot{q}}{q^T\dot{q}} \left[ -\dot{\theta}_1 \sin(\theta_1) \dot{q}^2 + \frac{d}{dt}\hat{E}_{ref}(t) - \eta \left( \hat{E}(q, \dot{q}, k) - \hat{E}_{ref}(t) \right) \right] \quad (6.11)$$

where $\eta \in \mathbb{R}^+$, resulting in

$$\dot{S} = -\eta \left( \hat{E}(q, \dot{q}, k) - \hat{E}_{ref}(t) \right)^2.$$

Now, $S = 0$ and $\dot{S} = 0$ occur simultaneously when $\hat{E}(q, \dot{q}, k) = \hat{E}_{ref}(t)$ (i.e., when the trajectory is on the limit cycle corresponding to $k$), and we have that $S > 0$ and $\dot{S} < 0$ in some neighborhood of the limit cycle. This is sufficient to show that trajectories in the neighborhood where $S > 0$ and $\dot{S} < 0$ will asymptotically descend to toward the limit cycle during the continuous single-support phase of walking. Unfortunately, the discrete dynamics of the double-support phase and the time-varying dynamics during transitions prevent application of LaSalle's invariance principle across all of the dynamics of a step. For example, it is entirely possible that the double-support velocity discontinuity (2.6) may map some converging trajectories outside the basin of attraction for the control. However, as in [20, 22], we suspect this control will render the system locally asymptotically stable for trajectories in some neighborhood of the transitioning limit cycle. Indeed, we expect the basins of attraction under this control will be larger than the original basins of attraction.

This expectation is born out in simulations in which we have employed this total energy-shaping control augmentation to our biomimetic kinetic-shaping control, expanding the original basin of attraction and allowing for faster variation of $k$ while preserving the stability of the bipeds. Simulations of the compass-gait biped and the biped with knees under this control are shown in Figure 6.6, allowing more rapid transitions between limit cycles than with the biomimetic control alone.

Figure 6.6(a), (c), and (e) show the compass-gait biped transitioning from a limit cycle corresponding to $k_1 = 0$ to one corresponding to $k_2 = 6$ using total energy-shaping control (6.11) with gain $\eta = 10$ and transition time $t_d = 2$ s (roughly 3
Figure 6.6 Faster transitions between limit cycles with differing speeds and step lengths may occur under total energy-shaping control (6.11). (a), (c), and (e) show the compass-gait biped transitioning between limit cycles corresponding to $k_1 = 0$ and $k_2 = 6$ using control (6.11) with gain $\eta = 10$ and transition time $t_d = 2$ s (roughly 3 steps). (b), (d), and (f) show the biped with knees transitioning between limit cycles corresponding to $k_1 = 0$ and $k_2 = 3$ using control (6.11) with gain $\eta = 1$ and transition time $t_d = 5$ s (roughly 7 steps). (a) and (b) show the $k$ transition for each biped. (c) and (d) show the evolution of the bipeds’ trajectories as $k$ is varied. (e) and (f) show evolution of the reference energy $\hat{E}_{ref}$ and the actual energy $\hat{E}$ of the bipeds.
steps). Figure 6.6(b), (d), and (f) show the biped with knees transitioning from a limit cycle corresponding to $k_1 = 0$ to one corresponding to $k_2 = 3$ using the total energy-shaping control with gain $\eta = 1$ and transition time $t_d = 5$ s (roughly 7 steps). Implementation of the total energy-shaping control results in transitions requiring 25-35 fewer steps for the bipeds.

In Figure 6.6(e) and (f) the energy $\hat{E}$ of the bipeds is shown converging asymptotically to the reference energy $\hat{E}_{ref}$. The reference energy for the biped with knees is shown in Figure 6.6(f) as a square wave due to the fact that there are two reference energies per step for the biped with knees, as discussed in Section 6.2. For both bipeds, the smooth asymptotic convergence during the single-support phase is interrupted by the velocity discontinuities due to impacts with the walking surface (for both the compass-gait biped and the biped with knees) and knee lock (for the biped with knees only). Because the biped is in transition, these discontinuities sometimes map the transitioning trajectory away from the reference limit cycle. As a result, we see “spikes” in the evolution of the bipeds’ energy in Figure 6.6(e) and (f) at the times of the impacts and knee locks.
CHAPTER 7

REGULATING WALKING ON SLOPES

As we have mentioned earlier in this work, the passive-dynamic walking phenomenon gives rise to zero-input walking on downhill slopes in the range $0°$-$4.5°$. Walking on all other slopes requires active control input. Spong and Bullo [19] and Spong et al. [25] have shown that passive limit cycles exhibited by the class of bipeds described in Section 2.1 may be rendered invariant to the slope of the walking surface by applying a particular potential-shaping control law. That is, if an unforced ($u(t) \equiv 0 \forall t$) limit cycle is known to exist on one slope, applying a certain potential-shaping control will give rise to the limit cycle on any desired slope, thereby allowing a biped to walk on any incline. This is made possible by the enforcement of a particular energetic symmetry called a controlled symmetry.

In this short chapter, we present a corollary to the controlled symmetry law of [19] extending slope-walking flexibility to a larger set of limit cycles. We show how any forced ($u(t) \neq 0 \forall t$) limit cycle created by the energy-shaping methods described in Chapter 4 may be reproduced on any desired slope [29]. As an example we consider our kinetic energy-shaping biomimetic control of Chapter 6, which gives rise to limit cycles with a range of speeds and step lengths. Applying the extended controlled symmetry law, we reproduce these limit cycles on any slope of our choice, thereby allowing regulation of speed and step length on any incline. These results are demonstrated with the compass-gait biped and the biped with knees.
7.1 Controlled Symmetries

Spong and colleagues [18, 19, 25] have shown that passive limit cycles may be rendered invariant to slope by enforcing a controlled symmetry, a certain energetic symmetry with respect to slope. Creating this controlled symmetry amounts to the application of a particular potential energy-shaping control law. In this section, we present a rapid review of the mathematics behind controlled symmetries and present the original potential-shaping law. We then introduce a corollary extending the controlled symmetries law to handle cases where the reference limit cycle is not passive but rather is itself the creation of an energy-shaping control.

Changing the slope on which the biped walks amounts to the action of rotation group $SO(3)$ on the configuration manifold $Q$. This group action defines a mapping $\Phi : SO(3) \times Q \to Q$ which we denote as

$$\Phi(A, q) = \Phi_A(q) \quad \forall A \in SO(3), \forall q \in Q.$$ 

The group action $\Phi$ on the configuration manifold $Q$ induces corresponding maps on functions on $Q$ and its tangent space $TQ$, e.g., the kinetic and potential energy of the biped. The action tangent to the group action, called the lifted action, defines a mapping $Tq\Phi_A : TqQ \to T\Phi_A(q)Q$ of vectors in the tangent space at $q \in Q$ to vectors in the tangent space at $\Phi_A(q) \in Q$.

Using these facts, Spong and Bullo [19] have shown that, for all rotations $A \in SO(3)$, the following potential-shaping law

$$u = B(q)^{-1} \nabla_q^T \left( V(q) - V(\Phi_A(q)) \right)$$

$$= B(q)^{-1} \left( G(q) - G(\Phi_A(q)) \right)$$

renders the biped dynamics invariant under the slope-changing action $\Phi$; i.e., the closed-loop dynamics with input (7.1) obeys

$$\dot{L}(q, \dot{q}) = L\left( \Phi_A(q), Tq\Phi_A(\dot{q}) \right).$$

We note from the inversion of $B$ in control law (7.1) that this control requires full actuation.
The proof of this result relies on the fact (Proposition 3.1 of [19]) that the kinetic energy of the class of bipeds described in Section 2.1 is naturally invariant under the slope-changing action, i.e.,

\[ K(\Phi_A(q), T_q \Phi_A(\dot{q})) = K(q, \dot{q}). \tag{7.2} \]

Kinetic symmetry property (7.2) is a natural consequence of the dynamics of kinematic chains, the underlying structure of the class of bipedal robots studied in this work. The potential energy \( V(q) \) of a biped does not possess this symmetry naturally; therefore, we employ control (7.1) to shape the potential energy (and hence the closed-loop Lagrangian) into a form that is symmetric with respect to changes in slope. This gives rise to the name controlled symmetry.

### 7.2 Extending Controlled Symmetries

Consider now the application of our energy-shaping results of Chapters 4 and 6 to a walking biped. Among the feasible forms for the closed-loop kinetic energy \( \hat{K} \) are many that do not possess the symmetry property (7.2). We now extend the controlled symmetry results of [19, 25] to the case of “asymmetric” kinetic energy fields by introducing a corollary to Theorem 4.1 of [19].

**Corollary.** Let \( \gamma : [0, T] \to Q \) be a solution trajectory of the closed-loop dynamics (4.4) with energy-shaping control input \( u \) as in (4.8). Suppose the closed-loop kinetic energy \( \hat{K} \) lacks symmetry with respect to the slope-changing action \( \Phi \), i.e., \( \hat{K} \) does not satisfy (7.2). Let \( A \in SO(3) \) and redefine the energy-shaping control input as

\[ u = B^{-1}(q) M(q) \left[ M^{-1}(q) \left( C(q, \dot{q}) \dot{q} + G(q) \right) - \hat{M}^{-1}(q) \left( \hat{C}(q, \dot{q}) \dot{q} + \hat{G}(q) - \hat{u} \right) \right] \tag{7.3} \]

where \( \hat{M}, \hat{C} \) and \( \hat{G} \) are as before. Choose

\[
\dot{u} = \frac{d}{dt} \nabla_q^T \left( \hat{L}(q, \dot{q}) - \hat{L}(\Phi_A(q), T_q \Phi_A(\dot{q})) \right) \\
- \nabla_q^T \left( \hat{L}(q, \dot{q}) - \hat{L}(\Phi_A(q), T_q \Phi_A(\dot{q})) \right) \tag{7.4}
\]

\[
= \hat{M}(q) \left[ \hat{M}^{-1}(q) \left( \hat{C}(q, \dot{q}) \dot{q} + \hat{G}(q) \right) - \hat{M}^{-1}(\Phi_A(q)) \left( \hat{C}(\Phi_A(q), T_q \Phi_A(\dot{q})) \dot{q} + \hat{G}(\Phi_A(q)) \right) \right] \tag{7.5}
\]
where $\hat{L}$ is the closed-loop Lagrangian corresponding to the shaped energy $\hat{K}$ and $\hat{G}$.

Then $\Phi_A(\gamma) : [0, T] \rightarrow Q$ is a solution trajectory on the surface rotated by $A$ with control input $u$ as in (7.3).

**Proof:** The core of our extended symmetry control law appears in (7.4), where the asymmetric kinetic and potential energy defined in the original closed-loop Lagrangian $\hat{L}$ are replaced with the symmetric energy of $\tilde{L}$. Substituting $u$ of (7.3) into (4.2) we have

$$\hat{M}(q)\ddot{q} + \hat{C}(q, \dot{q})\dot{q} + \hat{G}(q) = \hat{u}. \tag{7.6}$$

Equation (7.6) looks like the closed-loop dynamics (4.4), but now additional term $\hat{u}$ appears on the right-hand side, affording further flexibility. Applying $\hat{u}$ as defined in (7.5) to (7.6), we have

$$\hat{M}(\Phi_A(q))\ddot{q} + \hat{C}(\Phi_A(q), T_q\Phi_A(\dot{q}))\dot{q} + \hat{G}(\Phi_A(q)) = 0. \tag{7.7}$$

Thus, if $\gamma$ is a solution (limit cycle) of the closed-loop dynamics (4.4) with control input $u(t)$ given by (4.8), then $\Phi_A(\gamma)$ is a solution (limit cycle) of (7.7) on the walking surface rotated by $A$.

We now apply this corollary to the compass-gait biped and biped with knees under the biomimetic kinetic energy from (6.4) discussed in the previous chapter.

### 7.3 Controlled Symmetries with Kinetic Shaping

In this section, we offer examples of controlled symmetries. We first illustrate the invariance of the natural kinetic energy (2.17) of the compass-gait biped to the slope of the walking surface. Next the biomimetic closed-loop kinetic energy form (6.4) of the previous chapter is considered and shown to lack to the natural invariance with respect to slope. We apply our extended controlled symmetry law to both the compass-gait biped and the biped with knees under the biomimetic control, illustrating how the shaped kinetic energy may be rendered invariant to the slope, allowing regulation of speed and step length while walking on any slope.
Consider the passive compass-gait biped and the case of a one-dimensional rotation of the slope in the sagittal plane; i.e., let \( A = R_{z,a} \in SO(3) \) where \( R_{z,a} \) denotes a rotation by angle \( a \in S^1 \) about the \( z \) axis \( z \), i.e. the axis perpendicular to the sagittal plane. The action of \( A \) on the compass-gait biped is equivalent to changing the inclination of the walking surface by angle \( a \). The action on the position variables can be written as \( \Phi_A(q) = q - a \), and the lifted action on the velocity variables can be written as \( T_q \Phi_A(\dot{q}) = \dot{q} \). Applying this action to the natural kinetic energy of the compass-gait biped (2.17), we have

\[
K\left(\Phi_A(q), T_q(\Phi_A(\dot{q}))\right) = K(q - a, \dot{q}) = K(q, \dot{q}).
\]

As expected, symmetry (7.2) holds for the natural kinetic energy of the compass-gait biped. A similar computation holds for the natural kinetic energy of the biped with knees (2.23).

Shaping the kinetic energy of the compass-gait biped as we have done in Chapter 6 with closed-loop form \( \hat{K}(q, \dot{q}) \) given by (6.4) destroys this symmetry, which is evident in the following calculation:

\[
\hat{K}\left(\Phi_A(q), T_q(\Phi_A(\dot{q}))\right) = \hat{K}(q - a, \dot{q}) \\
= \hat{K}(q - a, \dot{q}) + k \sin(\theta_1 - a)\dot{\theta}_1^2 \\
= \hat{K}(q, \dot{q}) + k \sin(\theta_1 - a)\dot{\theta}_1^2 \\
\ne \hat{K}(q, \dot{q}).
\]

A similar calculation shows the kinetic energy of the biped with knees, modified as in (6.4), also lacks symmetry with respect to slope.

We may compensate for this asymmetry by applying controlled symmetry law (7.3), thus rendering the closed-loop dynamics invariant to the slope-changing action. Simulations of the bipeds on various slopes with biomimetic kinetic form (6.4) applied using symmetry control (7.3) are shown in Figure 7.1 for the compass-gait biped and in Figure 7.2 for the biped with knees. As shown in the figures, the symmetry controller maps the limit cycles to the slopes of our choice, allowing the bipeds to walk on any incline.
Figure 7.1 Compass-gait biped with shaped kinetic energy (6.4) and controlled kinetic symmetry (7.3). The cluster of limit cycles on the left correspond to the biped walking on a 20° downhill slope; limit cycles on the right are for the biped walking on a 20° uphill slope.

Figure 7.2 Biped with knees with shaped kinetic energy (6.4) and controlled kinetic symmetry (7.3). The cluster of limit cycles on the left correspond to the biped walking on a 20° downhill slope; limit cycles on the right are for the biped walking on a 20° uphill slope.
Thus far we have considered two different controls for regulating the speed and step length of walking: the time-scaling control of Chapter 5 and the biomimetic kinetic energy-shaping control of Chapter 6. Comparison of the energetic costs of these controls in Section 6.2 indicates the biomimetic kinetic energy-shaping control is more efficient than the time-scaling control for most forward speeds. This discovery suggests deeper questions of optimality. What control minimizes the energetic cost of walking for a biped? How does optimal robotic gait compare with human locomotion? Are the key features of human walking noted in Chapter 3 evident in the optimal motion of a bipedal robot?

These questions motivate our investigation in this chapter of optimal gait regulation. We seek the control for a biped and the corresponding trajectory that minimize actuator effort. Our prediction is that the most efficient control for walking will exhibit the two key features of human walking discussed in Chapter 3. That is, we expect (1) that optimal trajectories will exhibit monotonic adjustment of step length and speed similar to that of humans and (2) that the ankle effort exerted by the optimal control will be significantly greater than the effort of the other actuators in the biped. We emphasize that our goal is to learn structural information about the form of the optimal control and corresponding optimal trajectory; as such, we will accept approximations to the true optimal control and trajectory.
Due to the present state-of-the-art in bipedal locomotion, which involves tracking of preplanned gait trajectories, considerable attention has been given to generation of biped trajectories that minimize actuator effort. Results in the literature may be divided into three broad categories. In the first, the trajectory is parameterized using polynomials, piecewise linear functions, or finite Fourier series [73–75]. Off-the-shelf optimization software such as DIRCOL [79] or MATLAB’s Optimization Toolbox is then employed to tune the trajectory to minimize the performance criterion, often involving sequential quadratic programming algorithms. (We note [74] is a special case of this technique, as it applies Bezier parameterization to the selection of certain functions that indirectly constrain the possible trajectories of a biped; this work fits nicely into the hybrid zero dynamics (HZD) approach to bipedal walking, summarized in Section 1.1.1.)

A second, related avenue of work applies various learning algorithms to tune the gait of physical laboratory robots [80–82]. For example, a genetic algorithm is implemented in [81] to improve the walking efficiency of the “PINO” bipedal robot.

In the final category of work, optimal gait problems are analytically solved via dynamic programming. Pontryagin’s maximum principle [83–85] gives a set of equations which the optimal control and trajectory must satisfy [77, 78]. Due to the nonlinear hybrid dynamics of bipedal locomotion, these equations typically contain a two-point boundary value problem (BVP) for which closed-form solutions are elusive. Various numerical techniques have been brought to bear to approximate the BVP solution, such as linearization of the equations of motion about a specific operating point [77] and implementation of a shooting method [78].

As many numerical algorithms are limited to providing approximations of locally optimal controls and trajectories, a critical component of the optimal gait regulation problem is the supply of a nominal control and trajectory sufficiently close to the optimal control and trajectory. As stated in [78], “the main difficulty to overcome in order for the algorithm to converge towards an optimal solution consists in finding a sufficiently accurate guess solution.”
The extant work in optimal gait regulation has been conducted on level walking surfaces and consequently has never considered passive limit cycles as reference optimal solutions. In this chapter, we consider the optimal gait regulation problem on a downhill slope, and we use the passive limit cycle to seed an algorithm that searches for optimal controls and trajectories. To the best of our knowledge, this is the first time that passive-dynamic walking has been applied to optimal gait regulation.

The nature of our study of optimal gait regulation is finite-horizon dynamic programming. This approach is attractive on account of the closed-form equations for the optimal control and trajectory it supplies. We limit our investigation to the relatively lower dimensional compass-gait biped (the dimension of the state space for the compass-gait biped is 4, whereas the biped with knees has a state space of dimension 6), as dynamic programming techniques become computationally prohibitive for high-dimensional state spaces, the unfortunate and well-known “curse of dimensionality” [85,86].

The chapter begins with a formulation of the optimal gait regulation problem (OGRP). We show how the generally complex task of optimization for the hybrid dynamics of locomotion may be simplified to optimization for a continuous dynamical system subject to boundary constraints. We present a complete dynamic programming solution to the OGRP using Pontryagin’s maximum principle [83–85], which furnishes us with a set of equations that the optimal control, optimal trajectory, and adjoint variables must solve. As is often the case in dynamic programming, these equations include a difficult two-point boundary value problem (BVP). However, we note that this BVP reduces to a simple BVP that is well-known in the optimal control literature, provided we presuppose the terminal time and terminal velocities of the optimal gait. This simpler problem may then be solved using many possible algorithms. We select the algorithm of Sakawa and colleagues [87,88], which has been employed in the past for finding energy-optimal controls and trajectories for systems with revolute joints [88,89].
We consider the problem of optimizing gait for various commanded step lengths. Our study begins with step lengths near that of the passive limit cycle, and we seed the search for optimal controls and trajectories under these conditions with the passive limit cycle control \( u(t) \equiv 0 \ \forall t \) and trajectory. For commanded step lengths further from that of the passive limit cycle, we seed the optimal search with the optimal controls and trajectories we have discovered for step lengths nearby.

The chapter concludes with presentation of the results of our search. Trends are clearly evident in the approximate optimal controls and trajectories as step length is varied, leading us to infer several structural conclusions about the optimal control and trajectory. The optimal control does indeed exhibit a simultaneous variation of speed and step length remarkably similar to the profile observed in human locomotion. However, the actuator effort required of the ankle by the optimal control effort does not dominate the effort of the hip. These findings lead us to accept part 1 of our hypothesis and reject part 2.

Furthermore, we consider the time profiles of the optimal controls, which demonstrate consistent trends as well. From these, we infer energy input/dissipation patterns that should be followed by high-efficiency bipedal walking controls.

### 8.1 Formalization of the Optimal Gait Regulation Problem

Our goal is to determine the control that minimizes the following measure of actuator effort [73–78]:

\[
J(u) = \frac{1}{2} \int_{t_0}^{t_f} u^T(t)u(t)dt \tag{8.1}
\]

for the hybrid dynamical system given in (2.8), i.e.,

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \quad \text{for } x(t) \notin S_i \\
q(t_i^+) &= q(t_i^-) \\
\dot{q}(t_i^+) &= h(x(t_i^-))\dot{q}(t_i^-) \quad \text{for } x(t_i) \in S_i.
\end{align*}
\tag{8.2}
\]
We note the finite terminal time $t_F$ appearing in the performance metric (8.1) is unspecified; optimization of this problem includes selection of the optimal terminal time, denoted $t^*_F$. The control which minimizes (8.1) is called the optimal control and is denoted $u^*(t)$, $t_0 \leq t \leq t^*_F$. The corresponding trajectory is called the optimal trajectory and is denoted $x^*(t)$.

Clearly, (8.1) is minimized (in fact, is identically zero) on the passive limit cycle of a biped. Therefore, $u^*(t) = 0 \forall t$ is the optimal control for (8.2) and the corresponding optimal trajectory is the passive limit cycle.

Passive-dynamic walking, however, affords only a single period-1 limit cycle with fixed gait characteristics. In this chapter, we are concerned with the minimum-effort controls that achieve period-1 walking cycles with a commanded step length $d$. We refer to the minimization of (8.1) for the hybrid system (8.2) subject to the requirement that the optimal trajectory be a period-1 walking cycle with step length $d$ as the optimal gait regulation problem, which we hereafter abbreviate as OGRP.

### 8.1.1 Simplification due to period-1 limit cycle requirement

Performing optimal control analysis for a hybrid dynamical system such as (8.2) is, in general, quite difficult. However, in this section we show that the OGRP requirement that the optimal trajectory be a period-1 walking cycle allows us to recast the OGRP as an optimization of the purely continuous dynamical system

$$\dot{x}(t) = f(x(t), u(t)) \text{ for } t \in [t_0, t_F] \quad (8.3)$$

where $f$ in (8.3) is identical to $f$ given in (8.2), subject to constraints on the initial and final states $x(t_0)$ and $x(t_F)$.

Let $t_0$ correspond to the instant after impact, i.e., the start of the single-support phase of the step. Similarly, let $t_F$ correspond to the instant prior to impact when the nonsupport leg is just striking the walking surface, i.e., the end of the single-support phase. Consider the state of the biped at the start and end of a period-1 walking cycle for bipedal walking, as illustrated in Figure 8.1.
Figure 8.1 Illustration of key components of the limit cycle. Selecting step length $d$ establishes the joint angles at the beginning and end of the step, i.e., $q(t_0) = [\theta_1(t_0), \theta_2(t_0)]^T$ and $q(t_F) = [\theta_1(t_F), \theta_2(t_F)]^T$. The impact mapping establishes a constraint on the initial and terminal joint velocities $\dot{q}(t_0) = [\dot{\theta}_1(t_0), \dot{\theta}_2(t_0)]^T$ and $\dot{q}(t_F) = [\dot{\theta}_1(t_F), \dot{\theta}_2(t_F)]^T$. The terminal time $t_F$ is free.

If the trajectory is a period-1 walking cycle, it must necessarily form a closed orbit in the state space $Q \times TQ$ after one step. Periodicity dictates the concluding state $x(t_F) = [q(t_F), \dot{q}(t_F)]^T$ of each successive single-support phase must be identical to the concluding state $x(t_F)$ of the previous single-support phase, and the same statement holds true for the initial state $x(t_0) = [q(t_0), \dot{q}(t_0)]^T$ of the single-support phase.

Since the angular positions of the legs are not changed during the double-support phase, we have

$$q(t_0) = q(t_F). \quad (8.4)$$

Further, as will be described in the following section, the angular positions at the start and end of each single-support phase may be computed directly from the commanded step length $d$ of the OGRP. Consequently, we know a priori half of the state information at the start and end of each step.

The discrete velocity change (2.6) that governs the double-support phase of locomotion maps the velocities at the end of each single-support phase to the velocities at
the start of each single-support phase. For a period-1 walking cycle, we may rewrite (2.6) as

\[ \dot{q}(t_0) = h(q(t_F))\dot{q}(t_F). \] (8.5)

Since (8.4) and (8.5) must be satisfied for any period-1 limit cycle, we may think of these equations as defining constraints on the initial and final states of the trajectory during the single-support phase. Consequently, solving the OGRP subject to the assumption that the optimal trajectories are period-1 walking cycles, we may exchange the problem of optimizing the full hybrid dynamical model (8.2) for that of optimizing the continuous dynamics of the single-support phase alone, subject to these constraints on the trajectory.

### 8.1.2 Constraints on the continuous dynamics

We now formalize the constraints on the OGRP. Two of these constraints are the boundary angular position and angular velocity constraints discussed above. The third constraint is simply a restatement of the dynamics of the single-support phase, a constraint common in dynamic programming problems [84,85].

- **Angular position constraints.** Step length \( d \) is the distance between the points of contact of the support and nonsupport legs with the walking surface during the double-support phase. Specifying \( d \) defines exactly the angular positions \( q(t_0) \) and \( q(t_F) \) of the legs at the start and end of each step. Consider the simple geometric relationship between step length \( d \) and between-leg angle \( \alpha \) for the compass-gait biped as illustrated in Figure 8.2.

  By the law of cosines, we have

  \[ d^2 = \ell^2 + \ell^2 - 2\ell^2 \cos(\alpha). \]

  We may compute the between-leg angle as a function of \( d \) as

  \[ \alpha(d) = \cos^{-1} \left( 1 - \frac{d^2}{2\ell^2} \right). \]
The between-leg angle is related to the angles of the each leg by
\[ \alpha(d) = \theta_2(t_F) - \theta_1(t_F) = \theta_1(t_0) - \theta_2(t_0). \]

Therefore, for a commanded step length \( d \), we compute the necessary angles of the legs at the end of each step as
\[
q(t_F, d) = \begin{bmatrix}
\theta_1(t_F, d) \\
\theta_2(t_F, d)
\end{bmatrix} = \begin{bmatrix}
-\left( \frac{1}{2} \alpha(d) + \psi \right) \\
\frac{1}{2} \alpha(d) - \psi
\end{bmatrix}
\]
where \( \psi \) is the slope of the walking surface. The configuration of the biped is not changed during the double-support phase, although the support roles of the legs are reversed. Therefore we write the necessary angles of the legs at the beginning of the step as a function of \( d \) as follows:
\[
q(t_0, d) = \begin{bmatrix}
\theta_1(t_0, d) \\
\theta_2(t_0, d)
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} q(t_F, d).
\]

We now define the following functions of the initial and terminal angular positions:
\[
P_0(x(t_0), d) = q(t_0) - q(t_0, d)
\]
\[
P_F(x(t_F), d) = q(t_F) - q(t_F, d).
\]

\( P_0 \) and \( P_F \) are referred to, respectively, as the initial and terminal angular position manifolds. Requiring that the initial and terminal configurations of the
optimal trajectory correspond to the commanded step length $d$ is tantamount to enforcing the following two constraints:

$$P_0(x(t_0), d) = 0 \quad (8.6)$$
$$P_F(x(t_F), d) = 0. \quad (8.7)$$

- **Angular velocity constraints.** A period-1 walking cycle must satisfy (8.5). Consequently, we may think of the double-support phase velocity map $h$ as defining a boundary velocity constraint on the single-support phase of walking. Noting as above that the terminal configuration $q(t_F)$ is a function of the step length $d$, we may rewrite the double-support phase velocity relationship (8.5) as

$$\dot{q}(t_0) = h(q(t_F))\dot{q}(t_F)$$
$$= h(d)\dot{q}(t_F). \quad (8.8)$$

We define the following functions of the initial and terminal velocities:

$$V_F(x(t_F), d) = h(d)\dot{q}(t_F)$$
$$V_0(x(t_0)) = \dot{q}(t_0)$$
$$V(x(t_F), x(t_0), d) = V_F(x(t_F), d) - V_0(x(t_0))$$

and refer to $V$ as the angular velocity manifold. Requiring that the optimal trajectory be a period-1 walking cycle requires enforcing the constraint

$$V(x(t_F), x(t_0), d) = 0. \quad (8.9)$$

Note that angular velocity constraint (8.9) allows some flexibility in the velocities at the boundaries of the single-support phase. In this sense, the velocity constraint is different from the angular position constraints (8.6) and (8.7), which completely specify both the initial and the terminal angular positions based on the commanded step length $d$. The commanded step length $d$ specifies the relationship between the initial and terminal velocities, but it does not specify the velocities themselves.
• **Dynamics constraints.** In dynamic programming for continuous dynamical systems, it is common to treat the dynamics itself as an additional constraint on the optimal control problem. Hence, we rewrite the continuous dynamics of the single-support phase (8.3) as a constraint as

\[ \dot{x}(t) - f(x, u, t) = 0. \]

• **Time constraints** (or lack thereof). Finally, while the OGRP is a finite-horizon optimal control problem (i.e., the optimal trajectory is finite in duration), there is no constraint on the terminal time of this optimal control problem. The length of time elapsed during an optimal step is unknown a priori; therefore terminal time \( t_F \) is unconstrained.

### 8.1.3 Summary of the optimal gait regulation problem

The optimal gait regulation problem (OGRP) may now be summarized as follows. Given commanded step length \( d \), we seek the control \( u^*(t) \) that minimizes

\[ J(u) = \frac{1}{2} \int_{t_0}^{t_F} u^T(t)u(t)dt \]

for the continuous dynamical system (8.3) subject to the following constraints:

\[
\begin{align*}
P_0(x(t_0), d) &= 0 \\
P_F(x(t_F), d) &= 0 \\
V(x(t_F), x(t_0), d) &= 0 \\
\dot{x}(t) - f(x, u, t) &= 0
\end{align*}
\]

where terminal time \( t_F \) is free. Equations (8.10) and (8.12) are constraints on the initial state \( x(t_0) \), and (8.11) and (8.12) constrain the terminal state \( x(t_F) \).

### 8.2 Solution via Dynamic Programming

In this section we consider solving the OGRP by standard dynamic programming techniques [83–85]. Our approach is to adjoin each of the constraints summarized in
8.2.1 General solution

We begin by adjoining the dynamics constraint (8.13) to the cost function (8.1) as follows

\[ J = \int_{t_0}^{t_F} \left[ \frac{1}{2} u(t)^T u(t) + \lambda(t)^T (f(x, u, t) - \dot{x}(t)) \right] dt \quad (8.14) \]

where \( \lambda(t) \) is a 4-vector of Lagrange multipliers. Next define the Hamiltonian

\[ H(x, u, p, t) = \frac{1}{2} u(t)^T u(t) + \lambda(t)^T f(x, u, t) \quad (8.15) \]

and rewrite (8.14) as

\[ J = \int_{t_0}^{t_F} \left[ H(x, u, p, t) - \lambda(t)^T \dot{x}(t) \right] dt. \quad (8.16) \]

Noting that \( \frac{d}{dt}(\lambda^T x) = \dot{\lambda}^T x + \lambda^T \dot{x} \), we may integrate (8.16) by parts to yield

\[ J = \int_{t_0}^{t_F} \left[ H(x, u, \lambda, t) + \dot{\lambda}(t)^T x(t) \right] dt - \left( \lambda(t_F)^T x(t_F) \right) \bigg|_{t_0}^{t_F} \quad (8.17) \]

We now adjoin the boundary constraints (8.10) through (8.12) to (8.17) to get

\[ J = v^T P_F(x(t_F), d) - \xi^T P_0(x(t_0), d) + \eta^T V(x(t_F), x(t_0), d) \]
\[ + \int_{t_0}^{t_F} \left[ H(x, u, \lambda, t) + \dot{\lambda}(t)^T x(t) \right] dt - \left( \lambda(t_F)^T x(t_F) \right) \bigg|_{t_0}^{t_F} \quad (8.18) \]

where \( v, \xi, \) and \( \eta \) are 2-vectors of Lagrange multipliers. Equation (8.18) may be rearranged to group the terms dependent on the state at time \( t_0 \), on the state at time \( t_F \), and on the trajectory between \( t_0 \) and \( t_F \) as follows:

\[ J = \left[ v^T P_F(x(t_F)) + \eta^T V_F(x(t_F)) - \lambda(t_F)^T x(t_F) \right] \]
\[ - \left[ \xi^T P_0(x(t_0)) + \eta^T V_0(x(t_0)) - \lambda(t_0)^T x(t_0) \right] \]
\[ + \int_{t_0}^{t_F} \left[ H(x, u, \lambda, t) + \dot{\lambda}(t)^T x(t) \right] dt. \quad (8.19) \]
Let \( u = u^* + \delta u \), \( t_F = t_F^* + \delta t_F \), and \( x = x^* + \delta x \), where \( u^* \) and \( t_F^* \) denote the optimal control and terminal time and \( x^* \) denotes the corresponding optimal trajectory. Consider variations of \( J \) away from the optimum, i.e.,

\[
\Delta J = J(x^* + \delta x, u^* + \delta u, t_F^* + \delta t_F) - J(x^*, u^*, t_F^*).
\]

We may write a Taylor series expansion of \( \Delta J \) about the optimal control \( u^* \), optimal terminal time \( t_F^* \), and optimal trajectory \( x^* \). The first variation of \( J \), written \( \delta J \), is defined as the components of the Taylor series expansion of \( \Delta J \) that are linear in \( \delta u \), \( \delta t_F \), and \( \delta x \). The first variation may be written as

\[
\delta J = \delta x^T(t_F^*) \left[ \frac{\partial P_F^T(t_F^*)}{\partial x} v + \frac{\partial V_F^T(t_F^*)}{\partial x} \eta - \lambda(t_F^*) \right] \\
- \delta x^T(t_0) \left[ \frac{\partial P_0^T(t_0)}{\partial x} \xi + \frac{\partial V_0^T(t_0)}{\partial x} \eta - \lambda(t_0) \right] \\
+ \delta t_F \left[ \dot{H}(x^*, u^*, \lambda, t_F^*) + \frac{\partial P_F(t_F^*)}{\partial t} v + \frac{\partial V_F(t_F^*)}{\partial t} \eta \right] \\
+ \int_{t_0}^{t_F} \delta x^T \left( \frac{\partial H(t)}{\partial x} + \dot{\lambda}(t) \right) + \partial u^T \frac{\partial H(t)}{\partial u} \right] dt.
\]

A necessary condition for \( J \) to be a minimum is that the first variation be zero [85]. Therefore, setting \( \delta J = 0 \), we must have

\[
\lambda(t_F^*) = \frac{\partial P_F^T(t_F^*)}{\partial x} v + \frac{\partial V_F^T(t_F^*)}{\partial x} \eta 
\]

(8.20)

\[
\lambda(t_0) = \frac{\partial P_0^T(t_0)}{\partial x} \xi + \frac{\partial V_0^T(t_0)}{\partial x} \eta 
\]

(8.21)

\[
H(x^*, u^*, \lambda, t_F^*) = - \left[ \frac{\partial P_F^T(t_F^*)}{\partial t} v + \frac{\partial V_F^T(t_F^*)}{\partial t} \eta \right] 
\]

(8.22)

\[
\dot{\lambda}(t) = - \frac{\partial H}{\partial x}(x^*, u^*, \lambda, t) 
\]

(8.23)

\[
\frac{\partial H}{\partial u}(x^*, u^*, \lambda, t) = 0 
\]

(8.24)

where (8.23) and (8.24) hold for all time \( t \in [0, t_F^*] \).
8.2.2 Summary

Denoting the components of the double-support phase impact map $h$ (2.6) as

$$ h = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}. $$

we compute the partial derivatives that appear in Equations (8.20)-(8.24). From (8.22), we find that the optimal terminal time $t^*_F$ must satisfy

$$ H(x^*, u^*, \lambda, t^*_F) = \left. \begin{bmatrix} \dot{\theta}_1 v_1 + \dot{\theta}_2 v_2 + \left( \frac{\partial h_{11}}{\partial \theta_1} \dot{\theta}_1 + h_{11} \ddot{\theta}_1 + \frac{\partial h_{12}}{\partial \theta_2} \dot{\theta}_2 + h_{12} \ddot{\theta}_2 \right) \eta_1 \\ \frac{\partial h_{21}}{\partial \theta_1} \dot{\theta}_1 + h_{21} \ddot{\theta}_1 + \frac{\partial h_{22}}{\partial \theta_2} \dot{\theta}_2 + h_{22} \ddot{\theta}_2 \right) \eta_2 \right|_{t=t^*_F}. \quad (8.25) $$

The optimal control $u^*(t)$ satisfies

$$ \frac{\partial H}{\partial u}(x^*, u^*, \lambda, t) = 0 \quad (8.26) $$

for $t \in [t_0, t^*_F]$ while the optimal trajectory and Lagrange multiplier pair $x^*(t), \lambda(t)$ satisfy

$$ \dot{x}^*(t) = f(x^*, u^*, t) \quad \text{and} \quad \dot{\lambda}(t) = -\frac{\partial H}{\partial x}(x^*, u^*, \lambda, t) \quad (8.27) $$

for $t \in [t_0, t^*_F]$, subject to initial conditions

$$ P_0(x(t_0)) = 0 \quad \text{and} \quad \lambda(t_0) = [\xi_1, \xi_2, \eta_1, \eta_2]^T \quad (8.28) $$

and terminal conditions

$$ P_0(x(t^*_F)) = 0 \quad (8.29) $$

and

$$ \lambda(t^*_F) = \begin{bmatrix} v_1 + \left( \frac{\partial h_{11}}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial h_{12}}{\partial \theta_2} \dot{\theta}_2 \right) \eta_1 + \left( \frac{\partial h_{21}}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial h_{22}}{\partial \theta_2} \dot{\theta}_2 \right) \eta_2 \\ v_2 + \left( \frac{\partial h_{11}}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial h_{12}}{\partial \theta_2} \dot{\theta}_2 \right) \eta_1 + \left( \frac{\partial h_{21}}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial h_{22}}{\partial \theta_2} \dot{\theta}_2 \right) \eta_2 \\ h_{11} \eta_1 + h_{21} \eta_2 \\ h_{12} \eta_1 + h_{22} \eta_2 \end{bmatrix}. $$
where $\xi_i, v_i, \eta_i, i = 1, 2$ are the components of the Lagrange multiplier 2-vectors $\xi, v,$ and $\eta$.

For some dynamical systems it is possible to explicitly solve Equations (8.25) through (8.29). For example, systems with dynamics that depend explicitly on time $t$ may allow direct solution of (8.25) for the optimal terminal time $t_F$. Unfortunately, the dynamics of our walking bipeds possess no such dependence on time, thereby making explicit solutions of (8.25) elusive.

Moreover, the trajectory and multiplier dynamics (8.27) subject to the initial and terminal conditions (8.28) and (8.29) define a two-point boundary value problem (BVP) whose solutions are remarkably difficult to obtain. The difficulty involved with directly solving this BVP for the explicit solution to the OGRP inspires us to consider an alternative method for approximating the solution.

### 8.3 Fixed Terminal Velocities and Terminal Time

For a given step length $d$, the boundary angular positions are immediately known from (8.10) and (8.11), while the terminal time is free and the boundary velocities are unspecified, albeit subject to (8.12). Consider now the OGRP with fixed boundary velocities and terminal time. That is, assume we presuppose the values of the terminal time $t_F$ and terminal velocities $\dot{q}(t_F)$ of the optimal trajectory. We may compute the necessary initial velocities $\dot{q}(t_0)$ using the velocity constraint (8.12). In such a case, we now have completely specified the boundary conditions $x(t_0) = x_0$ and $x(t_F) = x_F$ for our optimization problem, and we find the general OGRP now reduces to the following well-known optimal control problem.

Let us consider the solution to this new optimal control problem, that of finding the control that minimizes (8.1) in specified time $t_F$, subject to specified boundary conditions $x_0$ and $x_F$. That is, consider the problem of minimizing

$$J(u) = \frac{1}{2} \int_{t_0}^{t_F} u^T(t) u(t) dt$$
for the continuous dynamical system (8.3) subject to the constraint
\[ \dot{x}(t) - f(x, u, t) = 0 \]
and boundary conditions \( x(t_0) = x_0 \) and \( x(t_F) = x_F \), where terminal time \( t_F \) is fixed. We call this the optimal gait regulation problem subject to fixed velocities and terminal time and denote it OGRP-fixed. Posed in such a way, the OGRP-fixed already satisfies constraints (8.10), (8.11), and (8.12) of the OGRP.

The dynamic programming solution to the OGRP-fixed is as follows. We adjoin the dynamics constraint to the cost function (8.1) as before:
\[ J = \int_{t_0}^{t_F} \left[ \frac{1}{2} u(t)^T u(t) + \lambda(t)^T \left( f(x, u, t) - \dot{x}(t) \right) \right] dt \]  
(8.30)
where \( \lambda(t) \) is a 4-vector of Lagrange multipliers. Defining the Hamiltonian as in (8.15), we integrate (8.30) by parts to once again yield (8.17). Letting \( u = u^* + \delta u \) and \( x = x^* + \delta x \), we consider deviations from the optimum, i.e.,
\[ \Delta J = J(x^* + \delta x, u^* + \delta u) - J(x^*, u^*) \]
and compute the first variation of \( J \), given by
\[ \delta J = -\delta x^T(t_F)\lambda(t_F) + \delta x^T(t_0)\lambda(t_0) + \int_{t_0}^{t_F} \left[ \delta x^T \left( \frac{\partial H(t)}{\partial x} + \dot{\lambda}(t) \right) + \delta u^T \frac{\partial H(t)}{\partial u} \right] dt. \]
We note that any possible trajectory \( x(t) \) must satisfy the given boundary conditions \( x(t_0) = x_0 \) and \( x(t_F) = x_F \); therefore, we must have \( \delta x^T(t_0) = \delta x^T(t_F) = 0 \). Setting \( \delta J = 0 \) to zero to satisfy the necessary condition that the first variation be zero, we find the optimal control \( u^*(t) \) must satisfy
\[ \frac{\partial H}{\partial u}(x^*, u^*, \lambda, t) = 0 \]  
(8.31)
while the trajectory and Lagrange multiplier pair \( x^*(t) \) and \( \lambda(t) \) must satisfy
\[ \dot{x}^*(t) = f(x^*, u^*, t) \]  
(8.32)
\[ \dot{\lambda}(t) = -\frac{\partial H}{\partial x}(x^*, u^*, \lambda, t) \]  
(8.33)
subject to the presupposed boundary conditions \(x(t_0) = x_0\) and \(x(t_F) = x_F\).

Equations (8.31) through (8.33) that solve the OGRP-fixed are far simpler than Equations (8.25) through (8.29) which solve the more general OGRP. We note, however, that (8.32) subject to boundary conditions \(x_0\) and \(x_F\) amounts to another two-point BVP. This two-point BVP, however, is well-known in optimal control theory, and we may employ a standard trick to approximate its solution.

### 8.3.1 Solution for fixed terminal velocities and terminal time

It is common practice in optimal control to avoid the two-point BVP that arises in the OGRP-fixed by recasting the terminal boundary condition \(x(t_F) = x_F\) as a terminal penalty on the cost function (8.1). As in [88], define vector function

\[
\psi : Q \times TQ \rightarrow Q \times TQ
\]

\[
\psi(x) = x - x_F
\]

and scalar function \(\Theta : Q \times TQ \rightarrow \mathbb{R}\)

\[
\Theta(x) = \frac{1}{2} \psi(x)^T P \psi(x) \tag{8.34}
\]

where \(P \in \mathbb{R}^{n \times n}\) is a positive-definite, diagonal matrix with large values on its diagonal. Note that the terminal condition \(x(t_F) = x_F\) may be equivalently expressed as \(\psi(x(t_F)) = 0\) or \(\Theta(x(t_F)) = 0\).

We now compose a new cost function

\[
J_1(u) = \Theta(x(t_F)) + \frac{1}{2} \int_{t_0}^{t_F} u^T(t)u(t)dt. \tag{8.35}
\]

The new optimal control problem is to minimize (8.35) subject to the presupposed terminal time \(t_F\), and initial condition \(x(t_0) = x_0\). Choosing large positive values for diagonal gain matrix \(P\) in \(\Theta\) adds a steep penalty to the cost for trajectories terminating away from the desired state \(x_F\). Minimizing (8.35) with large \(P\) will result in an optimal trajectory that terminates near \(x_F\), thereby approximating the constraint that \(x(t_F) = x_F\). We call this the approximated optimal gait regulation problem and denote it OGRP-approximate.
As in [88], note that
\[ J_1 = \Theta(x(t_F)) + \frac{1}{2} \int_{t_0}^{t_F} u^T(t)u(t)dt \]
\[ = \Theta(x(t_0)) + \int_{t_0}^{t_F} \left[ \frac{1}{2} u^T(t)u(t) + \nabla_x \Theta f(x, u, t) \right] dt \]
\[ = \Theta(x(t_0)) + \int_{t_0}^{t_F} \left[ \frac{1}{2} u^T(t)u(t) + \psi^T(x)f(x, u, t) \right] dt. \]

Since \( x(t_0) = x_0 \) for all possible trajectories, \( \Theta(x(t_0)) \) is a constant. Therefore, minimizing \( J_1 \) is equivalent to minimizing
\[ J_2 = \int_{t_0}^{t_F} \left[ \frac{1}{2} u^T(t)u(t) + \psi^T(x)f(x, u, t) \right] dt. \]

For cost function \( J_2 \) we define the Hamiltonian as
\[ H(x, u, \lambda, t) = \frac{1}{2} u^T(t)u(t) + \left( \psi^T(x)P + \lambda(t)^T \right)f(x, u, t). \]

It is now a straightforward task to adjoin the dynamics \( \dot{x} = f(x, u, t) \) as a constraint to cost function \( J_2 \), integrate by parts, and compute the first variation \( \delta J_2 \). We note that for this optimal control problem with its approximated terminal constraint, all possible trajectories \( x(t) \) must satisfy \( x(t_0) = x_0 \), but not necessarily \( x(t_F) = x_F \). This implies \( \delta x^T(t_0) = 0 \) while leaving \( \delta x^T(t_F) \) free.

Setting \( \delta J_2 = 0 \), we find that the optimal control \( u^*(t) \) solving the OGRP-approximate must satisfy
\[ \nabla_u H(x^*, u^*, \lambda, t) = 0 \]
while the trajectory and Lagrange multiplier pair \( x^*(t), \lambda(t) \) satisfy
\[ \dot{x}^*(t) = f(x^*, u^*, t) \text{ and } \dot{\lambda}(t) = -\nabla_x H(x^*, u^*, \lambda, t) \]
subject to the initial condition
\[ x^*(t_0) = x_0 \]
and the terminal condition
\[ \lambda(t_F) = \nabla_x \Theta(t_F) = \psi(x(t_F))^Tf(x, u, t_F). \]
It is easy to compute a trajectory for $x(t)$ by solving the ODE $\dot{x} = f$ in forward time from the initial condition (8.37), then solving multiplier ODE $\dot{\lambda} = -\nabla_x H$ in reverse time, starting at the terminal condition (8.38). In this manner, we may approximate the solution to the OGRP-fixed while avoiding the two-point BVP that appears in the explicit solution of the OGRP-fixed.

Various algorithms for solving equations (8.36) through (8.38) simultaneously have been introduced in the optimal control literature. We have chosen the algorithm of Sakawa and colleagues [87,88] (summarized in Appendix A) as it has been used in the literature to approximate optimal controls for dynamical systems with revolute joints [88,89] similar to those of the bipeds considered in this thesis.

8.4 Method of Approximating Solutions for the Optimal Gait Regulation Problem

Our overall goal is to find solutions to the general OGRP, i.e., to find the control that minimizes the actuator effort metric (8.1) for a commanded step length $d$ while satisfying constraints (8.10)-(8.13). In order to avoid two-point BVPs, we developed in the previous section the OGRP-approximate, which approximates the controls that minimizes (8.1) for a commanded step length $d$ subject to presupposed values for the terminal time $t_F$ and the terminal velocities $\dot{\theta}(t_F)$. As our guesses of these values may be inaccurate, in this section we propose a method for solving the OGRP-approximate for a range of possible terminal times and terminal velocities. From the resulting grid of solutions to the OGRP-approximate, we select the control with the overall minimum actuator effort as the approximate solution of the general OGRP.

For a given step length $d$, we create a grid of guesses of terminal times and terminal velocities. For the two-link compass-gait biped, such an approach requires a three-dimensional grid of guesses as illustrated in Figure 8.3: one dimension of the grid for variations in the terminal time $t_F$, the other two dimensions for variations in the two terminal velocities $\dot{\theta}_1(t_F)$ and $\dot{\theta}_2(t_F)$. 128
Figure 8.3 Varying guesses of the terminal velocities $\dot{q}(t_F)$ and terminal time $t_F$ in fixed step sizes results in a three-dimensional grid for the compass-gait biped. The origin of the parameter variation grid correspond to the values of $\dot{q}(t_F)$ and $t_F$ for a known near-optimal reference trajectory.

The origin of the grid corresponds to the terminal time and terminal velocities of a reference trajectory known to be near-optimal. For example, for commanded step lengths $d$ near that of the passive limit cycle, the reference trajectory is the passive limit cycle itself, and the origin of the grid would correspond to the terminal time and terminal velocities of the passive limit cycle.

For each parameter combination in the grid of values, we propose solving the approximation to the OGRP-approximate and noting the cost (8.1) for the resulting control. Once approximate optimal solutions have been prepared for all vertices, we select the solution with the overall minimum cost as the closest approximation to the solution of the general OGRP.

8.4.1 Testing the method

We test this method by implementing it with the known passive limit cycle. Fixing $d = 0.5355$ m (the step length of the passive limit cycle) we establish a grid of guesses for the terminal time $t_F$ and terminal velocities $\dot{\theta}_1(t_F)$ and $\dot{\theta}_2(t_F)$ as in Figure 8.4(a).
The origin of our grid of guesses corresponds to the known values for the passive limit cycle, i.e., $t_F = 0.735$ s, $\dot{\theta}_1(t_F) = -1.4934 \text{ rad/s}$ and $\dot{\theta}_2(t_F) = -1.8048 \text{ m/s}$ at the origin. We vary the guesses in the grid with fixed step sizes $\Delta t_F = 0.005$ s, $\Delta \dot{\theta}_1(t_F) = 0.01 \text{ rad/s}$ and $\Delta \dot{\theta}_2(t_F) = 0.05 \text{ rad/s}$. These values were selected through experimentation using the algorithm of [87, 88] as these step sizes result in convergence of the algorithm in < 20 iterations.

The guesses of terminal time and terminal velocities are varied up to ±3 fixed step sizes on either side of the nominal values. That is, for the terminal time we consider the range of values

$$t_F = \{0.735, 0.735 \pm \Delta t_F, 0.735 \pm 2\Delta t_F, 0.735 \pm 3\Delta t_F\}$$

and we consider similar ranges for the terminal velocities $\dot{q}(t_F)$. There are 343 combinations of parameter values in our grid; we execute the optimal control approximation algorithm for each of these combinations, a process that requires about 10 hours on a dedicated Matlab workstation. We begin execution with the parameter combination at the origin of the grid and work in a radiant fashion from the center to the edges of the grid. Once the algorithm has converged for the parameter combination at the origin, we use the resultant optimal control as the nominal optimal control necessary for executing the algorithm for the adjacent parameter combinations in the grid.

We store the value of the actuator effort (8.1) for each combination

$$J\left(t_F, \dot{\theta}_1(t_F), \dot{\theta}_2(t_F)\right) = \frac{1}{2} \int_{t_0}^{t_F} u^T(t)u(t)dt.$$

Values of the actuator effort $J$ for the various parameter combinations when $d = 0.5355$ m are shown in Figure 8.4. It is difficult to graphically display the values for each of the vertices in the three-dimensional grid, so we instead present the values for several two-dimensional “slices” of the grid that pass through the origin.

While the concavity of the “slice” shown in Figure 8.4(c) is apparent, it is not so obvious for the “slices” shown in Figure 8.4(a) and (d). Nevertheless, zooms of these plots show the control effort is concave with a local minimum occurring at the origin.
Figure 8.4 Minimum values of the control effort (8.1) for the grid of guessed values of $t_F, \dot{\theta}_1(t_F),$ and $\dot{\theta}_2(t_F)$ when step length $d$ corresponds to that of the passive limit cycle. (a) Three-dimensional grid of values of Figure 8.3 reproduced here for reference; the three “slices” shown in (b),(c), and (d) correspond to the values of the control effort on each of the three planes shown in (a). (b) The “slice” of the three-dimensional values showing control effort when $\dot{\theta}_1(t_F)$ and $\dot{\theta}_2(t_F)$ are varied and $t_F$ is held to its nominal value. (c) The “slice” of the three-dimensional values showing control effort when $t_F$ and $\dot{\theta}_1(t_F)$ are varied and $\dot{\theta}_2(t_F)$ is held to its nominal value. (d) The “slice” of the three-dimensional values showing control effort when $t_F$ and $\dot{\theta}_2(t_F)$ are varied and $\dot{\theta}_1(t_F)$ is held to its nominal value.
Indeed, for the overall grid, there is a local minimum at the origin, i.e., the control effort is minimized when the values of the terminal time and terminal velocities correspond to the passive limit cycle. This indicates (as expected) that the optimal values of the terminal time and terminal velocities when step length \( d = 0.5355 \text{ m} \) are the values corresponding to the passive limit cycle.

### 8.4.2 Implementing the method

We implement the method to approximate solutions of the OGRP for the compass-gait biped over a range of commanded step lengths \( d \). We begin by constructing the grid for step length \( d = 0.5355 \text{ m} \), i.e. the step length known to correspond to the passive limit cycle. Once the method has converged on the approximate optimal control for this step length, we use this optimal control to seed the optimal control searches for step length slightly longer and slightly shorter than \( d = 0.5355 \text{ m} \). We choose fixed step size \( \Delta d = 0.002 \text{ m} \) and create grids of guesses for the terminal times and terminal velocities for \( d = 0.5355 \text{ m} \pm \Delta d \). The approximate optimal control that resulted from the grid for \( d = 0.5355 \text{ m} \) is used as the nominal control for solving the OGRP-approximate for the origin vertices of the \( d = 0.5355 \text{ m} + \Delta d \) and \( d = 0.5355 \text{ m} - \Delta d \) grids.

For each of the new \( d \) grids, we vary the terminal time guesses and terminal velocity guesses with the same fixed step sizes as before, up to \( \pm 3 \) fixed steps away from the origin of the grid along each parameter axis. We solve the OGRP-approximate using the terminal time and terminal velocity guesses corresponding to each parameter combination in the grid. The actuator effort (8.1) for each combination is recorded, and we select as the optimal control for step length \( d \) the control corresponding to the overall minimum actuator effort for that grid.

Once the method has converged to approximate optimal controls for the \( d = 0.5355 \text{ m} \pm \Delta d \) grids, we proceed to consider the grid for step lengths \( d = 0.5355 \text{ m} \pm 2\Delta d \) using the approximate optimal controls from the \( d = 0.5355 \text{ m} \pm \Delta d \) grids to seed the searches for the new grids.
Figure 8.5 Approximated optimal values of the terminal time and terminal velocities as commanded step length \( d \) is varied: (a) optimal terminal time \( t_F \) and (b) optimal terminal velocities \( \dot{\theta}_1(t_F) \) and \( \dot{\theta}_2(t_F) \). The dotted lines in these and subsequent figures indicate the step length of the passive limit cycle.

8.5 Results and Discussion

In this section, we discuss how the parameters, trajectories, and controls of the approximated OGRP vary over the range of commanded step lengths from \( d = 0.5005 \) m to \( d = 0.5955 \) m.

8.5.1 Approximate optimal terminal times and velocities

Values of the terminal time \( t_F \) and terminal velocities \( \dot{\theta}_1(t_F) \) and \( \dot{\theta}_2(t_F) \) corresponding to the approximate optimal control for a range of commanded step lengths \( d \) are shown in Figure 8.5. The dotted vertical line in these and subsequent figures indicates the parameter values corresponding to the passive limit cycle, i.e., the limit cycle with step length \( d = 0.5355 \) m. An immediate contrast is apparent between the terminal values of approximate optimal controls with step lengths longer and shorter than that of the passive limit cycle: terminal values remained constant for step lengths shorter than the passive limit cycle, while terminal values for longer step lengths demonstrate relatively steady upward or downward trends.
We note that the constant behavior of parameter values for step lengths less than that of the passive limit cycle does not appear to be a mere artifact of the optimization algorithm considering smaller step lengths than its seed trajectories. After completing the optimization for the range of step lengths \( d = 0.5005 \) m to \( d = 0.5955 \) m, we considered starting our approximation method beginning with a step length greater than that of the passive limit cycle. From this nonpassive starting point, we directed the algorithm to approximate the optimal parameters for increasingly shorter step lengths less than its initial value. The approximation method faithfully reproduced the behavior of the parameters shown in Figure 8.5, first regenerating the trends for the step lengths greater than that of the passive limit cycle, then fixing to constant parameter values at the step length of the passive limit cycle and maintaining these constant values for step lengths less than that of the passive limit cycle.

The starkly different behavior of the parameters for step lengths greater than or less than that of the passive limit cycle indicates a fundamental change in the optimal solution at the passive limit cycle. We will discuss this fundamental change in a subsequent section, after we have considered the behavior of the approximate optimal controls.

Trends are clearly visible in the terminal values for commanded step lengths greater than that of the passive limit cycle. From Figure 8.5(a), it appears that the approximate optimal terminal time peaks briefly for step lengths \( d \approx 0.545 \) m and then trends downward as step length increases. In Figure 8.5(b), we see that the terminal velocity for the support leg \( \dot{\theta}_1(t_F) \) trends downward (increasingly negative) as step length is increased. In contrast, the terminal velocity for the nonsupport leg \( \dot{\theta}_2(t_F) \) appears to reach a local minimum for step lengths \( d \approx 0.545 \) m and then trends upward (decreasingly negative values) as step length increases.

From Figure 8.5(b) we see that the approximate terminal velocity of the nonsupport leg \( \dot{\theta}_2(t_F) \) fluctuates much more widely than does the terminal velocity for the support leg as step length is increased. We expect this is due to the fact that changes in the support leg affect all the mass of the biped, whereas changes in the
nonsupport leg mainly affect only the mass of nonsupport leg itself. Consequently, changes in the support leg result in significant variation of the control, and therefore have a greater effect on the actuator effort (8.1). Since our algorithm is optimizing for minimum actuator effort, it follows that comparatively small variation would be observed in parameters related to the support leg.

We note a correlation in the behavior of the approximate terminal time in Figure 8.5(a) and the approximate terminal velocity of the nonsupport leg \( \dot{\theta}_2(t_F) \) in Figure 8.5(b). These parameters exhibit inflection points for step lengths of \( d \approx 0.545 \) m, a local maximum for the terminal time and a local minimum for the nonsupport terminal velocity.

### 8.5.2 Forward speed and step length

The forward speed

\[
\bar{s} = \frac{d}{t_F}
\]

of the biped under the approximated optimal control as step length \( d \) is varied is shown in Figure 8.6. In Figure 8.6(a) we see the speed trends roughly upward as step length increases. The perfectly linear trend in speed for step lengths shorter than that of the passive limit cycle is a direct consequence of the constant behavior of the terminal values seen in Figure 8.5. This speed-step length relationship is inverted in Figure 8.6(b) for comparison with the step length vs. speed relationship for humans that we reported in Chapter 3.

Comparing the step length vs. speed profile of the compass-gait biped under our approximate optimal control with that of humans and other locomoting mammals, we conclude optimal control of the compass-gait biped does indeed result in behavior similar to that of biological locomotion. This observation leads us to accept the first part of our hypothesis, i.e., that minimizing actuator effort results in biomimetic variation of step length and speed.
Figure 8.6 (a) Approximated optimal forward speed of the biped as step length $d$ is varied. This relationship is inverted in (b) for comparison with the speed-step length relationship of human locomotion in Figure 3.3.

### 8.5.3 Approximate optimal trajectories

Trajectories corresponding to the approximate optimal controls for various values of the step length $d$ are shown in Figure 8.7. From these figures we may draw conclusions about the optimal structure of a walking limit cycle. As step length is decreased, it appears from Figure 8.7(a) that the optimal limit cycle will contract along both joint angle and joint velocity axes, corresponding to the simultaneous decrease in step length and speed. As step length is increased, it appears from Figure 8.7(b) that the optimal limit cycle is expanded along both axes, although this expansion is more uniform throughout the step for the support leg than the nonsupport leg.

Once again, we attribute the lower variation in the optimized behavior of the support leg to the greater sensitivity of the cost function (8.1) to the motion of support leg, as changes to the support leg move all the mass of the biped. Motion of the nonsupport leg, on the other hand, affects only the mass of the swing leg and, hence, has a limited affect on the cost function.
Figure 8.7 Approximated optimal trajectories as step length $d$ is varied: (a) step lengths shorter than that of the passive limit cycle, (b) step lengths longer than that of the passive limit cycle.
The general pattern of dilating the optimal walking cycle along both joint angle and angular velocity axes as step length is increased indicates the optimal speed and step length increase as the commanded step length increases. This corresponds with the findings discussed in the previous section and displayed in Figure 8.6.

Also in Figure 8.7 we see a problematic effect of approximating the terminal boundary condition with a terminal penalty in our OGRP-approximate. In order for these trajectories to correspond to period-1 walking cycles, they must obey the requirement that the angular positions of the legs remain unchanged during the double-support phase. A simple visual way to verify this requirement is to check whether the trajectories for each leg terminate exactly below the initial conditions for the contralateral leg in the figures. Particularly in Figure 8.7(a) it is apparent that this requirement is not achieved precisely, indicating a weakness in our approach. We suggest experimentation with other algorithms for approximating the solution to the OGRP-fixed. For example, one might try the shooting method approach discussed in [78].

8.5.4 Approximate optimal controls

Components of the approximate optimal control as step length \( d \) is varied are shown in Figure 8.8, with control inputs for step lengths shorter than that of the passive limit cycle shown in Figure 8.8(a) and control inputs for longer step lengths shown in Figure 8.8(b). From these plots, we draw some conclusions about the structure of the optimal control as step length varies.

In Figure 8.8(a) we see that the approximate optimal control of both the support and nonsupport legs is positive throughout the step for commanded step lengths shorter than that of the passive limit cycle. Positive torque applied to both legs implies the actuators at the ankle and the hip are consistently applying torque in opposition to the motion of the legs throughout the step, dissipating energy throughout the step. The consistent dissipation of the actuators suggests that passive damping
elements may be introduced at each joint of the biped to achieve walking cycles with step lengths shorter than that of the original passive limit cycle.

In Figure 8.8(b) we see that, for commanded step lengths longer than that of the passive limit cycle, the control input to the support leg $u_1$ is positive early in the step and negative late in the step. This indicates the optimal ankle actuation regimen for increasing the step length of the compass-gait biped involves opposing the motion of the support leg during the early portion of the step and reinforcing the motion of the support leg during the later portion of the step. This behavior is similar to that of the human ankle during the stance phase of walking, shown in Figure 3.6 (note the ankle angle measured in our human experiments has the opposite sign of the ankle angle of the compass-gait biped in our simulations).

The approximate optimal control input of the hip actuator $u_2$ has a more complex evolution, displaying several inflections over the course of a step. One consistent structural property as step length is varied is the presence of local minima near the start and end of each step and the presence of a local maximum near the middle of each step.

We draw attention to the substantial difference in the behavior of the approximate optimal control for step lengths shorter than and greater than that of the passive limit cycle. We believe the consistent dissipative role of both control inputs for step lengths shorter than the passive limit cycle explains the constant terminal values seen in Figure 8.5. On the other hand, step lengths greater than that of the passive limit cycle result in control inputs with zero crossings, indicating the actuators switch roles—from opposing the motion of the biped to contributing to the motion—sometimes several times a step. Moreover, the times of these control input zero crossings vary as the step length is increased. We believe this relatively complex behavior of the optimal control for step lengths greater than that of the passive limit cycle gives rise to the nonconstant trends in terminal values seen in Figure 8.5.
Figure 8.8 Components of the approximate optimal control: (a) for commanded step lengths $d$ shorter than and forward speeds slower than that of the passive limit cycle; (b) for commanded step lengths $d$ longer than and forward speeds faster than that of the passive limit cycle.
We next test the second part of our hypothesis: that the optimal control would exhibit substantially greater ankle control effort than hip control effort. In Figure 8.9 we compare measures of the approximate optimal control effort of the ankle actuator and the hip actuator as the commanded step length is varied.

For step lengths shorter than that of the passive limit cycle, we find that the approximate optimal control exhibits roughly equivalent ankle and hip control effort. For step lengths longer than that of the passive limit cycle, the measure of ankle control effort climbs steadily as commanded step length \( d \) increases. The hip control effort, on the other hand, demonstrates greater variation, exhibiting values greater than that of the ankle control effort for step lengths \( d \approx 0.545 \) m then demonstrating less control effort than that of the ankle for step lengths greater than \( d \approx 0.565 \) m.

We find no clear dominance of the effort of one actuator over the other; therefore, we reject the second component of our hypothesis.

We do not interpret the rejection of the second part of our hypothesis to indicate a significant difference between human walking and passive-dynamic locomotion. Rather, we believe the dissimilarity in relative contributions of the ankle and other
joints is due to the difference in structure of the human ankle and the particular bipeds considered in this study. As noted in Chapter 3, feet are absent in our bipeds, a feature that simplifies analysis but leads to some dissimilarity when compared to the dynamics of human walking. We attribute the failure of the second part of our hypothesis to this dissimilarity in dynamical structure.

A possible avenue for future research would be the addition of nonzero length feet to the biped model. For a such a model, the present optimal control investigation could be repeated. It would be interesting to see how the ankle effort of the optimal controls of such a biped would compare with that of humans.

8.5.5 Comparison with previous controls

As the speed of the biped under the approximate optimal control trends positive as step length increases, we note that Figure 8.8(a) corresponds to speeds slower than that of the passive limit cycle and Figure 8.8(b) corresponds to faster speeds. We compare the approximated optimal control profiles of Figure 8.8 with the control profiles for our two speed-regulating controls shown in Figure 8.10 with various values of parameters $\lambda$ and $k$.

At high speeds, the control profile of our time-scaling control bears some gross resemblance to the approximate optimal control profile, namely positive control input $u_1$ early in the step and negative input late in the step, as seen in Figure 8.10(c) and (d). At low speeds, neither the time-scaling control nor the biomimetic control bears much resemblance to the approximate optimal control. However, we note that the biomimetic control maintains positive ankle input, as does the approximate optimal control. Perhaps the biomimetic control’s vague resemblance to the optimal control at low speeds and the time-scaling control’s gross resemblance to the optimal control at high speeds indicates the reason why the biomimetic control requires less actuator effort than the time-scaling control for low speeds while the time-scaling control performs better at high speeds as seen in Figure 6.3(b).
Figure 8.10 Comparison of the component control inputs of the compass-gait biped for the two speed-regulating controls: (a) and (c) for the time-scaling control of Chapter 5, (b) and (d) for the biomimetic control of Chapter 6. In (a) and (b) the biped is walking with commanded forward speed less than that of the passive limit cycle ($\lambda = 0.5$ in (a), $k = -6$ in (b)). In (c) and (d) the biped is walking with command forward speed greater than that of the passive limit cycle ($\lambda = 2$ in (c), $k = 6$ in (d)).
Figure 8.11 Control effort of the approximated optimal control compared with the control effort of time-scaling control of Chapter 5 and the biomimetic kinetic energy-shaping control of Chapter 6.

Our comparison of the control profiles in Figure 8.10 of the two speed-regulating controls presented in Chapter 5 and 6 against the profiles of the approximated optimal control in Figure 8.8 indicates our two speed-regulating controls are both far from optimal. It is therefore no surprise to find in Figure 8.11 that the control effort of the approximated optimal control is far lower than that of either of our two previous controls.
This work has furnished passive-dynamic bipeds with an expanded collection of useful walking controls. We have presented controls for regulation of forward speed, step length, and walking on slopes, and have shown gait regulation is possible with remarkably high efficiency and in cases of underactuation. We believe these findings hold great promise for the future of bipedal robotic locomotion. Our expectation is that continued work in passive dynamics-based control will revolutionize the state-of-the-art, resulting in bipeds capable of performing sophisticated locomotion tasks with low energetic costs.

A crucial motivator of our work has been the remarkable similarities in the energetics and dynamics of human and passive-dynamic robot locomotion. Awareness of these connections has resulted in some of the most compelling findings of this work. In Chapter 3, we found that the behavior of the human ankle could be effectively duplicated by a simple passive mechanism. In Chapter 6, we showed that mimicking key features of human walking led to highly efficient gait regulation. Further, our results on optimal control in Chapter 8 noted anthropomorphic behavior exhibited by the most efficient controls for robot locomotion. These results reinforce our conjecture that a substantial connection exists between human and passive-dynamic robotic locomotion and that further work along these lines will benefit not only robot locomotion, but human locomotion as well. We anticipate further developments may lead to more useful prosthetic and orthotic devices for assisting disabled individuals.
Our technical findings in this work have contributed results in the areas of time-scaling, energy-shaping, underactuated control, controlled symmetries, and optimal control. We have shown in Chapter 5 that discrete events of the type that appear in the hybrid dynamics of walking behave well under linear and nonlinear time-scaling. This result led to our development of a versatile speed-regulating control, which we note may be used for any system with similar dynamics. Moreover, Chapter 4 has drawn attention to kinetic energy shaping and underactuated energy shaping, two areas that have received little attention in the bipedal walking literature. From these results, we have demonstrated in Chapter 6 that highly efficient speed regulation is possible using underactuation at the ankle alone. In Chapter 7 we extended existing results on controlled symmetries to the case of shaped kinetic energy, a finding that has allowed for regulation of speed and step length on any walking slope. Finally, in Chapter 8, we have conducted what is to the best of our knowledge the first study of optimal control seeded with passive limit cycle trajectories. The results of this investigation have yielded structural information on how optimal locomotion controls evolve over a range of a commanded step lengths, providing insight for the development of future walking controls.

9.1 Future Work

As this work comes to its conclusion, many interesting areas for future work remain open. We believe the starting point of future work is further collaboration and sharing of ideas among experts in robot and human locomotion. We applaud efforts to gather experts in human walking, robotic locomotion, and lower-limb prosthetics, such as the recent Dynamic Walking workshops and conferences held at Delft University, Aland University, the University of Michigan, and Carnegie Mellon University. These and other opportunities to exchange insights into locomotion from biological and robotic perspectives are critical for the advancement of bipedal walking.

Along these lines, we recommend the consideration of more anthropomorphic mathematical models. In particular, we advise the development of models with feet
and articulated ankles. Although several models with feet have appeared in the literature, most we have seen have had rigid ankles. Addition of feet with articulated ankles will introduce substantial complexity to the analysis of bipedal walking, such as incorporating heel-contact to foot-flat and heel-off to toe-off dynamics to the double-support phase. However, it is imperative that such added complexity be adequately researched and understood, as feet with articulated ankles are necessary for applying ankle torque to a physical biped.

The ultimate goal of this work is the development useful locomoting devices. As such, we recommend construction of suitable bipedal prototypes for testing these controls. We anticipate that the physical insights gained from implementation will provide critical feedback for the refinement of these and future controls, in addition to informing the construction of future bipedal robots. As our research group at Illinois and others researchers [90] have learned the hard way, construction should be limited to bipeds that one is fully capable of simulating. We recommend fully developing a walking model of any biped one intends to build prior to beginning construction.
We here review the algorithm of Sakawa, Shindo, and Hashimoto [87, 88] for approximating the solution to the optimal control problem with fixed terminal time and terminal velocities stated in Section 8.3.1. The task is to minimize the cost function

\[ J = \int_{t_0}^{t_F} \frac{1}{2} u^T(t)u(t) + \psi^T(x)Pf(x, u, t)dt = \int_{t_0}^{t_F} L(x, u, t)dt \]

for the continuous dynamics of the single-support phase of bipedal locomotion, which may be written in affine form as

\[ \dot{x} = f(x, u, t) = A(x) + Bu. \]

We establish the Hamiltonian as follows:

\[ H(x, u, \lambda, t) = L(x, u, t) + \lambda(t)^T f(x, u, t) \]
\[ = \frac{1}{2} u^T(t)u(t) + \left( \psi^T(x)P + \lambda(t)^T \right) f(x, u, t). \]

It is relatively straightforward to apply Pontryagin’s maximum principle [84,85] to deduce the following:

1. There exists an optimal control satisfying

\[ u^*(t) = \min_u H(x, u, \lambda, t) \tag{A.1} \]

for \(0 \leq t \leq t_F\), where \( \dot{x}(t) = f(x, u, t) \).
2. The state and multiplier pair $x^*(t), \lambda(t)$ satisfies the coupled differential equations

\[
\begin{align*}
\dot{x}^*(t) &= \nabla_\lambda H(x, u, \lambda, t) = f(x, u) \\
\dot{\lambda}(t) &= -\nabla_x H(x, u, \lambda, t)
\end{align*}
\]  

(A.2)

subject to the initial condition

\[ x(0) = x_0 \]

and the terminal condition

\[ \lambda(t_F) = \nabla_x \Theta(x(t_F)) \]

where $\Theta$ is as defined in (8.34).

A simple computation based on (A.1) gives the optimal control as a function of the optimal trajectory

\[ u^* = -\left(\psi^T(x^*)P + \lambda^T\right)B(x^*) \]

The complexity lies in finding $x^*$. We consider the following numerical algorithm proposed by Sakawa and Shindo in [87] that has been applied to problems of optimal control for dynamical systems with revolute joints in [88, 89]. The algorithm requires a nominal control be supplied as an initial guess of the optimal control. An iterative scheme is then used to adjust the nominal control in the direction of the (local) optimal control.

A.1 Algorithm Overview

In this section, we give a general overview of the Sakawa and Shindo algorithm.

- Given a nominal control $u^0(t)$ and corresponding nominal trajectory $x^0(t)$, solve the costate differential equation

\[ \dot{\lambda}^0(t) = -\nabla_x H(x^0, u^0, \lambda^0, t) \]
in reverse-time, beginning with terminal condition \( \lambda^0(t_F) \) near \( \nabla_x \eta(x(t_F)) = P\psi(x^0(t_F)) \) (the true terminal condition should be \( \lambda(t_F) = \nabla_x \eta(x^*(t_F)) = P\psi(x^*(t_F)) \), but the exact value of \( x^*(t_F) \) is presently unknown).

• Simultaneously solve the state differential equation

\[
\dot{x}^i(t) = f(x^i, u^i, t)
\]

and the approximate optimal control

\[
u^i = -\left( \psi^T(x^i) P + (\lambda^{i-1})^T \right) B(x^i).
\]

Simultaneous solution of these two equations is technically impossible; therefore, an approximation given in [87] will be used. Also, it is likely that the approximated \( u^i \) will blow up, so Sakawa, Shindo, and Hashimoto incorporate a convergence control technique [91] to mitigate this problem.

• Compute the terminal penalty \( \Theta(t_F) \), i.e., determine how far the trajectory strayed from the commanded terminal state. Based on this error, adjust the costate terminal condition \( \lambda^i(t_F) \), recompute the costate trajectory

\[
\dot{\lambda}^i(t) = -\nabla_x H(x^i, u^i, \lambda^i, t)
\]

and repeat the previous step. Repeat until the terminal penalty has become sufficiently small.

### A.2 The Algorithm in Detail

The algorithm begins by defining the function

\[
K(x, u, \lambda; v, C) = L(x, u, t) + \lambda^T f(x, u, t) + \frac{1}{2} (u - v)^T C (u - v)
\]

where \( C \) is a nonnegative, diagonal matrix, called the convergence control parameter (CCP) matrix [91]. A subalgorithm of the following method will focus on minimizing \( K \) (instead of minimizing \( H \) directly) while appropriately choosing \( C \). The reasoning
Figure A.1 Flowchart of the numerical algorithm proposed in [88].

is as follows: choosing $C$ to be large reduces the variation in the approximate optimal control $u$ between successive iterations of the subalgorithm. This keeps the current approximate optimal control nearby, preventing the subalgorithm (and, hence, the overall method) from “blowing up.”

A graphical view of the algorithm is provided in Figure A.1.

- **Step 0: Initialization.** Given a nominal control $u^0(t)$, compute the nominal trajectory by solving

$$\dot{x}^0(t) = f(x^0(t), u^0(t))$$
subject to the initial condition \( x^0(0) = x_0 \). Also solve the co-state differential equation

\[
\dot{\lambda}^0(t) = -\nabla_x H(x^0(t), u^0(t), \lambda^0(t))
\]

subject to the terminal condition

\[
\lambda^0(t_F) = c P \psi(x^0(t_F))
\]

where \( c \) is a positive constant. Initialize the multiplier \( b \) to \( b^0 = 0 \) and the iteration count to \( i = 1 \). Choose nonnegative diagonal matrix \( C^1 \) to be somewhat large, as it will limit the variation in the control, i.e., \( \Delta u^i = u^i - u^{i-1} \). Compute the cost of the nominal control and trajectory

\[
J(u^0) = \int_0^{t_F} \frac{1}{2} (u^0)^T u^0 + \psi^T(x^0) P f(x^0, u^0) dt + \eta(x_0).
\]

Proceed to Step 1.

- **Step 1: Improve \( u \) and \( x \).** Find \( u^i \) that minimizes the criterion \( K \) in the following sense:

\[
K(x^i, u^i, \lambda^{i-1}; u^{i-1}, C^i) = \min_u K(x^i, u, \lambda^{i-1}; u^{i-1}, C^i)
\]

\[
= \min_u \left[ H(x^i, u, \lambda^{i-1}) + \frac{1}{2} (u - u^{i-1})^T C^i (u - u^{i-1}) \right]
\]

where \( C \) is the so-called convergence control parameter, while simultaneously solving

\[
\dot{x}^i(t) = f(x^i(t), u^i(t))
\]

subject to the initial condition \( x^i(0) = x_0 \). \( K \) is minimized by the following choice of \( u^i \):

\[
u^i = (I + C^i)^{-1} \left[ C u^{i-1} - B(x^i)^T (\lambda^{i-1} + P \psi(x^i)) \right].
\]

However, selection of this \( u^i \) is coupled to the solution \( x^i \) of the differential equation. Simultaneously computing \( u^i \) and \( x^i \) is, in itself, difficult to accomplish. Therefore consider the following approximation given in [87].
1. Choose a uniform step length $0 < \Delta << 1$. Let $x^i(0) = x_0$ and

$$u^i(0) = (I + C^i)^{-1}\left[ Cu^{i-1}(0) - B(x^i(0))\right] \left( \lambda^{i-1} + P\psi(x^i(0)) \right).$$

Set $k = 1$ and proceed to the next step.

2. Denote $x^i(k\Delta)$ as $x^i(k)$ and $u^i(k\Delta)$ as $u^i(k)$. Approximate $x^i(k)$ as

$$\hat{x}^i(k) = x^i(k-1) + \Delta \cdot f(x^i(k-1), u^i(k-1), (k-1)\Delta).$$

Compute $x^i(k\Delta)$ as

$$x^i(k) = x^i(k-1) + \frac{\Delta}{2} \left( f(x^i(k-1), u^i(k-1), (k-1)\Delta) + f(\hat{x}^i(k), u^i(k-1), k\Delta) \right).$$

3. Compute $u^i(k)$ using the explicit minimization of $K$, i.e.,

$$u^i(k) = (I + C^i)^{-1}\left[ Cu^{i-1}(k) - B(x^i(k))\right] \left( \lambda^{i-1} + P\psi(x^i(k)) \right).$$

4. Let $k = k + 1$. Return to the second step of this approximation routine.

Once $x^i$ and $u^i$ have been computed, proceed to Step 2.

- **Step 2: Compare Costs.** Compute the cost for the current control and trajectory, i.e.,

$$J(u^i) = \int_0^{t_f} \frac{1}{2} (u^i)^T u^i + \psi^T(x^i) P f(x^i, u^i) dt.$$ 

Compare the costs $J(u^i)$ and $J(u^{i-1})$.

If $J(u^i) > J(u^{i-1})$ AND $C^i < C_{max}$ then set $C^{i+1} = \min\{\alpha C^i, C_{max}\}$, where $\alpha > 1$ and return to Step 1.

If $J(u^i) > J(u^{i-1})$ AND $C^i \geq C_{max}$ then set $C^{i+1} = C_{max}$ and proceed to Step 3.

If $J(u^i) \leq J(u^{i-1})$ then set $C^{i+1} = \beta C^i$, where $0 < \beta < 1$, and proceed to Step 3.
• **Step 3: Test for Completion.** If

\[ \Theta(x(t_F)) = \frac{1}{2} \psi^T(x(t_F)) P \psi(x(t_F)) < \epsilon_1 \]

and

\[ \frac{1}{2} \int_0^{t_F} \| u^i(t) - u^{i-1}(t) \| \, dt < \epsilon_2 \]

then stop the algorithm; the approximate optimal trajectory is \( x^* \approx x^i \) and the approximate optimal control is \( u^* \approx u^i \). Otherwise, proceed to Step 4.

• **Step 4: Update Multiplier.** If \( J(u^i) > J(u^{i-1}) \) and \( \Theta(x^i(t_F)) > \Theta(x^{i-1}(t_F)) \) then set

\[ u^i = u^{i-1}, x^i = x^{i-1}, b^{i-1} = 0. \]

Update the multiplier \( b \) as follows:

\[ b^i = b^{i-1} + c P \psi(x^i(t_F)). \]

Proceed to Step 5.

• **Step 5: Improve \( \lambda \).** Compute the co-state trajectory \( \lambda^i \) by solving the differential equation

\[ \dot{\lambda}^i(t) = -\nabla_x H(x^i, u^i, \lambda^i, t) \]

subject to the terminal condition

\[ \lambda^i(t_F) = b^i. \]

Increment the iteration count \( i = i + 1 \) and return to Step 1.

### A.2.1 Parameter values used in our simulation

The several parameters used to adjust the convergence of this algorithm must be tuned to each dynamical system and particular optimal control problem. We have experimented with a variety of parameter values for our application and chosen a set that require few iterations for convergence while minimizing the terminal penalty \( \Theta(x(t_F)) \).
For our simulations, we set the terminal penalty gain matrix to $P = 1000I$. We initialize the convergence control parameter with the value to $C = 1000$ and set the parameter maximum to be $C_{\text{max}} = 200\,000$. We choose the convergence control parameter increment multiplier to be $\alpha = 2$ and the decrement multiplier to be $\beta = 0.75$. The fraction used to set and update the Lagrange multiplier terminal condition $\lambda(t_F)$ is chosen as $c = 1$. 
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