Final Report

Project Number 65-03G

The Analysis of the Uptake of Water by Plant Root Systems

A project supported by the Water Resources Center of the University of Illinois

with funds from the Office of Water Resources Research,

Department of Interior,

Washington, D.C.

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I. Introduction and Objectives

The flow equation for water in unsaturated soil may be written to include a source term which can be used to represent the uptake of water by roots of plants. The roots are considered to be distributed in the soil in a continuous manner and the use of water by roots is represented as a negative source. The objective of the research work described herein was to obtain solutions of the flow equation for soil water for various boundary conditions and to investigate the effect of the source term on the flow patterns. The choice of the source function was studied. Experiments were planned and started to obtain data for verifying the predictions made from the flow theory and to aid in the development of a model for the source function.

Drs. Richard J. Millington and Frank D. Whisler have contributed much to the work accomplished on this project. Dr. Millington gave counsel on the nature of the source function and contributed to the development of the experimental system. Dr. Whisler carried out the numerical computations with the IBM 7094 computer. In addition to the contributions of these two, Dr. D. B. Peters has contributed to the development of the experimental system.

II. Theory and Analysis

A. Flow theory

In the analysis to be described below it will be assumed that the modified Darcy equation for flow in an unsaturated soil is valid, viz.:

\[ v = -K(\theta) \nabla H \]  \hspace{1cm} (1)  

where \( v \) is the volume flux, i.e. the volume of water passing through unit cross sectional area of soil in unit time, \( \nabla H \) is the gradient of the hydraulic head \( H \), and \( K(\theta) \) is the conductivity of the soil to water as a function of water content (Gardner, 1960a). The hydraulic head is considered to be the sum of a gravitational head \( z \) and a pressure head \( h \), both expressed in length units.
The conservation of matter principle is also imposed on the flow by use of the equation of continuity:

\[ \frac{\partial \theta}{\partial t} = - \nabla \cdot \mathbf{v} + S \]  \hspace{1cm} (2)

In equation (2) \( \frac{\partial \theta}{\partial t} \) is the time rate of increase of the volumetric water content \( \theta \) which is considered to occur because of the excess of inflow over outflow and because of a source term \( S \), which represents the volume of water produced at a point in the flow system per unit volume of soil per unit time.

In the present analysis the source function \( S \) will be used to represent the uptake of water by plant roots in the soil. The roots are considered to be distributed in the soil in a continuous (but not necessarily uniform) manner. The volume of water extracted from the soil per unit volume of soil per unit time can be represented as a negative source. The present macroscopic approach to the uptake of water by plant roots stands in contrast to the microscopic approach in which flow to a single root is analyzed (Gardner, 1960b). The analysis is thus made on the macroscopic level; the level at which observations and measurements are made.

The flux may be eliminated as a variable by combining equations (1) and (2):

\[ \frac{\partial \theta}{\partial t} = \nabla \cdot (K(\theta) \nabla h) + S \] \hspace{1cm} (3)

If it is further assumed that a defined relation between the water content and the pressure head of the soil water exists, i.e. that \( \theta = \theta(h) \), then one may write equation (3) as:

\[ C(\theta) \frac{\partial h}{\partial t} = \nabla \cdot (K(\theta) \nabla h) + S \] \hspace{1cm} (4)

where the water capacity \( C(\theta) \) is defined as \( d\theta/dh \), the rate of change of water content with pressure head.

It is the objective of the present analysis to apply equation (4) to the problem of extraction of water by roots from a uniform vertical soil column.
with a water table at its lower end and with evaporation from the soil surface occurring at the upper end. For one-dimensional vertical flow equation (4) becomes:

\[ C(h) \frac{\partial h}{\partial t} = \frac{\partial}{\partial z} \left( K(h) \frac{\partial h}{\partial z} \right) + \frac{\partial K(h)}{\partial z} + S \]  

(5)

where we have regarded the water capacity and conductivity as functions of the pressure head. A coordinate system will be chosen so that \( z = 0 \) at the upper end of the soil column and \( z = -L \) at the lower end.

There are two types of situations that may be treated, viz., the steady state and unsteady state. We shall treat only the steady state flow here and leave the unsteady state case for future analysis. In the steady state we have:

\[ \frac{\partial h}{\partial t} = 0 \]

so that equation (5) becomes:

\[ 0 = \frac{\partial}{\partial z} \left( K(h) \frac{\partial h}{\partial z} \right) + \frac{\partial K(h)}{\partial z} + S \]  

(6)

The boundary conditions chosen for this analysis are:

\[ h(-L) = 0 \]  

(7)

\[ v_0 = - \left[ K(h) \left( \frac{\partial h}{\partial z} + 1 \right) \right]_{z=0} \]  

(8)

The first of these represents the presence of a water table at the lower end of the column and the second represents the flux of water \( v_0 \) at the upper end of the soil column. We shall regard \( v_0 \) as a positive constant, i.e., evaporation will be assumed to occur at a constant rate from the soil surface.

In principle, a particular solution of (6) is determined by (7) and (8) when the conductivity \( K(h) \) and source function \( S \) are specified. The conductivity function can be obtained by separate measurements on representative samples of the soil. These measurements, while difficult, are in principle quite clearly defined (Klute, 1965). The source function is not so well known and understood. One of the objectives of this analysis is to assess the effect of
various assumptions for the source function on the flow in the soil.

For purposes of analysis it will be convenient to use dimensionless variables. We define a dimensionless pressure head:

$$\omega = \frac{h}{L}$$ (9)

distance variable:

$$\zeta = \frac{Z}{L}$$ (10)

conductivity:

$$\kappa = \frac{K(h)}{K_s}$$ (11)

and a source term:

$$\Gamma = S \frac{L}{K_s}$$ (12)

where $\theta_s$ is the volumetric water content at saturation, and $K_s$ is the conductivity at saturation. Using these variables equation (6) becomes:

$$0 = \frac{\partial}{\partial \zeta} \left( \kappa \frac{\partial \omega}{\partial \zeta} \right) + \frac{\partial \kappa}{\partial \zeta} + \Gamma$$ (13)

and the boundary conditions become:

$$\omega(-1) = 0$$ (14)

$$\frac{\nu_0}{K_s} = -\left[ \kappa \left( \frac{\partial \omega}{\partial \zeta} + 1 \right) \right]_{\zeta=0}$$ (15)

B. The source function

The source function is at this point unspecified. This representation of root uptake of water has not been widely used and knowledge of its behavior is limited. The theory outlined above forms the basis for a considerable amount of experimentation to determine the factors that may affect the source function and establish their importance. At this time we can only investigate the effect of the source function on the flow by assuming various forms for the function and examining their effect on the flow. Several such models of the source function
have been examined. In terms of the dimensionless source function \( \Gamma \) these were:

\[
\Gamma = \Delta_0 e^{-A_1 \zeta} 
\]

\[
\Gamma = \kappa(\phi)A_0 e^{-A_1 \zeta} 
\]

\[
\Gamma = \kappa(\phi) A(\zeta) 
\]

\[
\Gamma = \alpha(\zeta) \kappa(\phi) (\phi_p - \phi) 
\]

In equation (19) \( \phi_p \) is the pressure head in the plant root tissue and \( \alpha(\zeta) \) is a root density function. In equations (16) and (17) \( A_0 \) and \( A_1 \) are arbitrary constants.

The development of the source function given in equation (19) proceeds as follows: It will be assumed that the source is a function of depth because of its dependence on (1) a root density function \( A(z) \), (2) the conductivity of the soil to water and (3) the difference between the pressure head of the water in the plant root tissue \( h_p \) and the pressure head of the water in the soil \( h \). Thus we define the source as follows:

\[
S(z) = A(z) K(h) (h_p - h) 
\]

The dimensionless form of this source function is given in equation (19). An alternate formulation of the source term is as follows:

\[
S(z) = a(z) K(h) \left( \frac{h_p - h}{L_e} \right) 
\]

where \( L_e \) is some effective length over which the pressure head difference \( h_p - h \) is presumed to act. Thus the product \( K(h) \left( \frac{h_p - h}{L_e} \right) \) is the flux of water to the root, viz., volume of water passing through unit area of absorbing surface of the roots per unit time. Viewed in this context the function \( a(z) \) has the dimensions of root absorbing surface per unit volume of soil. Since \( L_e \) is essentially unknown we combine it with \( a(z) \) to give a root density function \( A(z) \) and hence obtain equation (20). The \( A(z) \) function is used to represent the depth distribution of
roots. One possible physical interpretation of it is to regard it as the length of water absorbing roots per unit volume of soil. Questions arise immediately with regard to its measurement and interpretation. However, these will not be considered here. We shall adopt the attitude that \( A(z) \) is known and try to deduce the consequences of a given \( A(z) \) function.

The assumption of the source term model given by equation (19) introduces a new unknown, \( h_p \), the pressure head of the water in the plant root tissue at a given depth. In the numerical analysis to be described below the partial differential equation of flow and boundary conditions will be written in terms of algebraic finite difference equations. There will be a set of algebraic equations equal in number to the number of discrete values of \( h \). However, additional equations will be needed to determine the necessary values of \( h_p \).

This is handled as follows: We assume that the flow of water in the plant root tissue occurs against negligible impedance. Thus there will be only one value of \( h_p \) throughout the root system which is appropriate to a given set of external boundary conditions, and only one additional equation is needed. The additional equation is obtained via the integral relation:

\[
\int_{-L}^{0} S \, dz = I \tag{22}
\]

where \( I \) is the negative of the transpiration rate, i.e., the volume of water evaporated from the leaves of the plant, per unit area of soil surface per unit time.

We assume that the net energy available for evaporation of water is used (1) to evaporate water from the plant leaves and (2) to evaporate water from the soil surface and that the sum of these is a constant, i.e., there is a constant combined rate of evaporation from soil and plant surfaces. Denote this evapotranspiration rate by \( e \). The fraction of \( e \) evaporated from the soil
surface is defined as \( r \), and that lost from plant leaves is \((1-r)e\). Then we have

\[
\int_{-L}^{0} S \, dz = i = -(1-r)e
\]  

(23)

and

\[
\int_{-L}^{0} A(z) \, K(h) \, (h_p-h) \, dz = -(1-r)e
\]

\[
h_p \int_{-L}^{0} A(z) \cdot K(h) \cdot dz - \int_{-L}^{0} A(z) \, K(h) \, h \, dz = (1-r)e
\]

Solving for \( h_p \):

\[
h_p = \frac{-(1-r)e + \int_{-L}^{0} A(z) \, K(h) \, h \, dz}{\int_{-L}^{0} A(z) \, K(h) \, dz}
\]

(24)

In terms of the dimensionless variables this becomes:

\[
\varphi_p = \frac{-(1-r)E + \int_{-1}^{0} \alpha(\zeta) \, \kappa(\varphi) \, d\zeta}{\int_{-1}^{0} \alpha(\zeta) \, \kappa(\varphi) \, d\zeta}
\]

(25)

where \( E = \frac{\varphi}{K_n} \) is the dimensionless evapotranspiration rate, and \( \alpha(\zeta) = L^2 A(z) \).

Using the above model of the source function, the flow equation in dimensionless variables becomes:

\[
Q = \frac{\partial}{\partial \zeta} \left( \kappa \frac{\partial \varphi_p}{\partial \zeta} \right) + \frac{\partial \varphi_p}{\partial \zeta} + \alpha(\zeta) \, \kappa(\varphi_p) \, (\varphi_p - \varphi)
\]

(26)

which is to be solved subject to (14) and (15) and the auxiliary relation for \( \varphi_p \) equation (25).

C. Numerical procedures

A numerical solution procedure was used. The region \( 0 \leq \zeta \leq 1 \) was divided into \( N \) intervals and the terms in equation (26) were replaced by finite difference
representations. At a general point \( n \) in the region these representations were:

\[
\frac{\partial}{\partial \zeta} \left( \kappa \frac{\partial \phi}{\partial \zeta} \right) \approx \frac{1}{\Delta \zeta} \left[ \kappa_{n+\frac{1}{2}} \frac{\phi_{n+1} - \phi_p}{\Delta \zeta} - \kappa_{n-\frac{1}{2}} \frac{\phi_n - \phi_{n-1}}{\Delta \zeta} \right] \tag{27}
\]

\[
\frac{\partial \kappa}{\partial \zeta} \approx \frac{1}{\Delta \zeta} \left[ \kappa_{n+\frac{1}{2}} - \kappa_{n-\frac{1}{2}} \right] \tag{28}
\]

\[
\alpha(\zeta) \kappa(\phi) (\phi_p - \phi) \approx \alpha_n \kappa_n \phi_p^{(1-1)} - \phi_n^{(1)} \tag{29}
\]

In the above:

\[
\kappa_{n+\frac{1}{2}} = \frac{1}{2} \left[ \kappa_{n+1}^{(1-1)} (\phi) + \kappa_n^{(1-1)} (\phi) \right] \tag{30}
\]

with a similar definition for \( \kappa_{n-\frac{1}{2}} \). Also:

\[
\alpha_n = \alpha(\zeta_n) \tag{31}
\]

and

\[
\kappa_n = \kappa(\phi_n^{(1-1)}) \tag{32}
\]

The superscript \((i)\) on \( \phi \) denotes the iteration stage which will be explained below. Substitution of equations (27), (28) and (29) in (26) followed by algebraic rearrangement yields:

\[
A_n \phi_n^{(i)} - B_n \phi_n^{(i)} + C_n \phi_{n+1} = -H_n \tag{33}
\]

with:

\[
A_n = \kappa_{n-\frac{1}{2}} \tag{34}
\]

\[
B_n = \kappa_{n+\frac{1}{2}} + \kappa_{n-\frac{1}{2}} + \Delta \zeta^2 \alpha_n \kappa_n \tag{35}
\]

\[
C_n = \kappa_{n+\frac{1}{2}} = A_{n+1} \tag{36}
\]

\[
H_n = \Delta \zeta [C_n - A_n] + \Delta \zeta^2 \alpha_n \kappa_n \phi_p^{(1-1)} \tag{37}
\]

From the lower boundary condition equation (14) we have for \( n = 1 \), \( \phi_1 = 0 \) and \( \kappa_1 = 1 \). At the upper boundary condition, where \( n = N+1 \), the equation (15) becomes:

\[
rE = J(0) = \frac{v_0}{K_a} = -\kappa_{N+\frac{1}{2}} \left[ \frac{\phi_{n+1} - \phi_{n+1}^{(1)}}{\Delta \zeta} \right]^{(1)} \tag{38}
\]

Equation (25) for \( \phi_p \) was approximated by finite differences as follows:
\[
\varphi_p = \frac{e^{-1} - (1-r)E + \sum_{n=2}^{N} \left( \alpha_1 \varphi_n + 2 \alpha_0 \varphi_n^{(i-1)} + \alpha_{N+1} \varphi_n^{(i-1)} \right) \kappa_n + \alpha_{N+1} \varphi_n^{(i-1)} \kappa_n^{(i-1)}}{2 \sum_{n=2}^{N} \left( \alpha_1 \kappa_n + 2 \alpha_0 \kappa_n + \alpha_{N+1} \kappa_n^{(i-1)} \right)}
\]  

(39)

After selection of the parameters \(E, r\) and the functions \(\kappa(\varphi)\) and \(\alpha(\zeta)\), the following iterative procedure was used: An initial estimate of the \(\varphi_n\) values, denoted as \(\varphi_n^{(0)}\), was used in (39) to calculate \(\varphi_p^{(i-1)}\). The set of values \(\varphi_n^{(0)}\) was used to evaluate \(A_n, B_n, C_n\) and \(H_n\). Equations (33) and (38) and the boundary condition \(\varphi_1 = 0\) form a set of \(N+1\) equations in \(N+1\) values of \(\varphi_n^{(1)}\) which was then solved for an improved estimate of the \(\varphi_n\) values, viz., \(\varphi_n^{(1)}\). These were compared one by one with the corresponding values \(\varphi_n^{(0)}\). If they were not significantly different (\(\left| \varphi_n^{(1)} - \varphi_n^{(i-1)} \right| \leq 1 \times 10^{-4}\)) from \(\varphi_n^{(0)}\) the calculations were stopped, but if any of the values of \(\varphi_n^{(1)}\) differed from \(\varphi_n^{(0)}\) a new calculation of \(\varphi_p\) was made using the values of \(\varphi_n^{(1)}\) and a calculation of \(\varphi_n^{(2)}\) was made.

This iterative cycle was repeated until the values of \(\varphi_n\) at a given stage of iteration were not significantly different from those of the previous stage of iteration.

It should be noted that the set of equations represented by (33) and (38) would be non-linear if it were not for the fact that the coefficients \(A_n, B_n, C_n\) and \(H_n\) are evaluated using the \(\varphi\) values from the previous iteration stage.

There are two possibilities for calculation of the flux of water in the soil at any position. It may be calculated from the hydraulic gradient in which case in finite difference terms it would be given by:

\[
J_n + \frac{1}{2} = -\kappa_n + \frac{1}{2} \left( \varphi_{n+1}^{(i)} - \varphi_n^{(i)} \right) \frac{(i)}{J_0} + 1
\]  

(40)

This requires differentiation of a \(\varphi\) profile and we have found this to be generally less desirable than the following scheme of calculation. The flux in the bottom of the column below the rooting depth \(\zeta_c\) is equal to \(E\), the imposed evaporation-transpiration rate. Equation (40) was applied to the first interval of the column.
at \( \zeta = -1 \) and the resulting calculated flux compared with the imposed evapo-
transpiration rate \( E \) to see if there was a significant discrepancy between them. 
In all cases the flux at \( \zeta = -1 \) as calculated in this way was in very close 
agreement with the value of \( E \). The flux in the root zone, \( \zeta_c < \zeta < 0 \), was then 
calculated from a finite difference form of:

\[
E + \int_{\zeta_c}^{\zeta} T \, d\zeta = J(\zeta) \tag{41}
\]

D. Selection of functions and parameters

The analysis described in this report was carried out using conductivity-
pressure head data for Pachappa sandy loam as given by Gardner and Fireman (1958). 
The empirical function used to represent the data was of the form

\[
K = \frac{a}{(-h)^n + b} \tag{42}
\]

For Pachappa sandy loam \( n=3 \), a \( 32 \times 10^4 \) cm\(^4\) day\(^{-1}\) and \( b = 2.6 \times 10^4 \) cm\(^3\) when 
\( K \) is expressed in cm day\(^{-1}\). The conductivity at \( h=0 \), assumed to be saturation, 
is given by \( a/b \), i.e., \( K_s = 12.3 \) cm/day. Using the definitions of \( \kappa \) and \( \varphi \) the 
above relation can be written:

\[
\kappa = \frac{1}{\frac{L^n}{b} (-\varphi)^n + 1} \tag{43}
\]

This relation was used to construct \( \kappa(\varphi) \) functions based on the Pachappa sandy 
loam conductivity function for column lengths, \( L \), of 50, 100, 200 and 400 cms. 
These are shown in Figure 1. It should be noted that while the four \( \kappa \) functions 
used were based on a Pachappa conductivity function, any other combination of 
column length and conductivity function that happened to give the same \( \kappa \) function 
as one of these would behave in the same manner with regard to \( \varphi \) profiles, fluxes, 
etc. In Figure 1 the conductivity data are shown only for the range \( -1 < \varphi < 0 \), 
but in the numerical analysis values of \( \kappa \) were needed at values of \( \varphi < -1 \). All 
\( \kappa \) values were obtained by the use of equation (43).
In most cases the evapotranspiration rate $E$ was selected at 0.006 and 0.06. Further remarks about this parameter are made below.

In the results shown here the root distribution function $\alpha(z)$ was arbitrarily chosen as a constant down to a rooting depth $z_c$ and as zero below $z_c$. Other choices are certainly possible and a few have been partially investigated. However, the physical interpretation of this function is not entirely clear at this time, and the question of appropriate magnitude and shape of the function is still unanswered.

In view of the lack of knowledge of appropriate values for $A(z)$ we decided to use a range of constant values of $A_0$ for its dimensionless counterpart $\alpha(z)$. By trial and error $A_0$ values were selected that produced discernible effects on the $\phi$ profiles. It is not known whether these magnitudes are physically significant.

E. Results

An example of the calculated $\phi$ profile is given in Figure 2. The designations $\kappa_{50}$ and $\kappa_{200}$ indicate that the $\kappa$ functions labeled 50 and 200 in Figure 1 were used for the calculations. These $\phi$ profiles apply to any combination of $K(h)$ and $L$ giving these $\kappa(\phi)$ functions. There are two possible interpretations of these curves. They may be regarded as a comparison of the same soil in two column lengths, or of two soils in the same column length. If one takes the first of these interpretations it is seen that the value of $r$, i.e., the partitioning between evaporation at the soil surface and transpiration, has little influence on the $\phi$ profile in the short ($L = 50$ cm) column but has considerable influence in the longer ($L = 200$ cm) column. When $r = 0$ there is no soil surface flux and $\frac{\partial \phi}{\partial z}$ at the surface becomes -1. As $r$ becomes larger more and more flow across the soil surface occurs and $\frac{\partial \phi}{\partial z}$ becomes more negative and larger in magnitude in order to drive the imposed flux across the soil surface. At the same time the pressure head becomes more negative in the upper portion of
the profile. When \( r = 1 \), all the water is being evaporated from the soil surface, and there is no transpiration. When \( A_0 \neq 0 \) this might correspond to a root system without plant tops. The \( \phi \) curve labeled \( r = 1, 0, A_0 = 0 \) applies to a profile with no roots. Observe that the pressure heads were more negative in the "no root" case. The presence of the root system with its negligible impedance acted as a "short" to transfer water from the lower wetter portions of the profile to the upper drier part of the profile.

The second interpretation of these \( \phi \) profiles, viz., that of two soils in the same column length, will be discussed in a later section of the report.

The dimensionless flux profiles for the case corresponding to that in Figure 2 of \( \chi_{200} \) are shown in Figure 3. The flux below the rooting depth \( \zeta_c \) is equal to \( E \). Within the root zone the flux varies in accordance with equation (41). As long as the integral in this equation decreases monotonically as \( \zeta \) increases (toward zero), the flux will decrease monotonically with increasing \( \zeta \). This will be the case when \( \Gamma \) is a negative function at all \( \zeta \).

The flux was calculated from a finite difference form of equation (41). There are slight discrepancies at \( \zeta \) equal to zero between the flux as calculated in this manner and the value imposed, viz., \( rE \) but these are not considered serious.

The source function \( \Gamma \) is shown in Figure 4. For all values of \( r \) the source was larger (less negative) at the top than at the bottom of the rooting zone, i.e., the uptake of water by the roots was greatest at the bottom of the rooting zone. This occurs because the conductivity is largest there and because the quantity \( (\phi_p - \phi) \) has its largest negative value there. At \( r = 0.75 \) the source function became positive above \( \zeta \approx -0.07 \) indicating that the root system was giving up water to the soil in the upper part of the profile. Observe in connection with this that the flux profile for \( r = 0.75 \) has a minimum value at \( \zeta \approx -0.07 \). Whenever there is "shorting", i.e., whenever the source function
changes from negative to positive values, the flux profile will display a minimum at the point where the change of sign of the source term occurs. As $r$ is made larger with $E$ fixed the source function becomes generally less negative because less water is being transpired.

In soil columns without roots, i.e. without a source term, the theory of flow as used here predicts that there will be a finite limiting flux upward through the column as the pressure head at the top is made to approach a very large negative value (Gardner, 1958). For the conductivity function used in this analysis the limiting flux is given by:

$$v_{\text{lim}} \approx 1.76 \frac{a}{L}$$ \hspace{1cm} (44)

If we divide both sides of this expression by $\frac{a}{b}$, the saturated conductivity, we obtain the limiting flux in non-dimensional terms:

$$j_{\text{lim}} \approx 1.76 \frac{b}{L^3}$$ \hspace{1cm} (45)

Thus, the deeper the water table the lower the limiting flux.

In a soil column with roots it is more difficult to derive an expression for the limiting flux. However, some bounds can be set on its value. Consider a column with roots in a zone $\zeta_c < \zeta < 0$. If all the root uptake of water were concentrated near the top end at $\zeta = 0$ the limiting possible steady-state evapotranspiration rate $E$ would approach the limiting flux value one would calculate from equation (48) using the full column length for $L$. On the other hand, if all the root uptake of water were concentrated at the depth $\zeta_c$ it appears that the maximum possible steady-state evapotranspiration rate $E$ would approach that given by equation (45) using $L$ equal to the distance from the water table to the bottom of the root zone. This value of $E$ will be larger than the previous one.

In Figure 5 the solid line shows the limiting flux versus column length as given by equation (45) for the case when no roots are present. The dashed
lines labeled with values of \( C_C \) show the limiting flux that would be obtained when roots are present on the assumption that all the root uptake occurred at the depth \( C_C \). Thus, for example, the limiting flux \( E \) for a 100 cm column of Pachappa sandy loam would be approximately \( 0.45 \times 10^{-2} \). If roots were present down to 25 cm below the surface, \( C_C \) would be \( -0.25 \). The limiting flux through the lower 75 cm of the 100 cm column, assuming that all the water is removed by root activity at a depth of 25 cm, would be about \( 1.1 \times 10^{-2} \). The limiting flux through such a column with roots distributed in the upper 25 cm would then be somewhere in the range \( 0.45 \times 10^{-1} \) to \( 1.1 \times 10^{-1} \). The root distribution pattern and proportion of evaporation at the soil surface would influence the value of the limiting flux. The effect of insertion of a root system in a soil column is to increase the maximum possible steady state flux through the bottom of the column.

From a survey of the results of the calculations of \( \varphi \) profiles for various \( k \) functions (Figure 1) and evapotranspiration rates the following tentative conclusion can be reached: If the evapotranspiration rate is significantly less than the limiting flux discussed in the preceding paragraphs the source term will have a relatively minor effect on the \( \varphi \) profile. On the other hand, if the evapotranspiration rate approaches or exceeds the limiting flux possible when no roots are present, the presence of the source term and variation of parameters such as \( r \) and \( A_0 \) has a relatively larger and in some cases extremely important effect on the \( \varphi \) profiles.

In Figure 2 if we interpret the cases \( k_{50} \) and \( k_{200} \) as the same soil in two column lengths, the case denoted by \( k_{200} \), with \( E = 0.006 \), was operating at about the limiting flux rate (see Figure 5 for \( L = 200 \)). Varying \( r \), which varied the amount of transpiration between 0 and \( E \), had a very significant effect on the pressure head profile. In the case denoted by \( k_{50} \) in Figure 2, varying \( r \) had little effect on the \( \varphi \) profiles. In this instance an \( E \) of 0.006 is about 0.03 of the maximum possible flux through the column.
Referring again to Figure 2, we may interpret the curves in the sense of two soils in a given column length. In the case labeled \( \kappa_{200} \) in Figure 2 the appropriate \( \kappa \) function is denoted by 200 in Figure 1. All the \( \kappa \) functions in Figure 1 are of the form:

\[
\kappa = \frac{1}{C \phi^n + 1}
\]  

(46)

with \( C = \frac{L_n}{b} \), and \( n = 3 \). The conductivity curve labeled 50 was constructed from equation (45) with \( L = 50 \) cms and \( b = 2.6 \times 10^4 \). However, this \( \kappa \) function can also be assumed to apply to any column length subject to the restriction that \( \frac{L_n}{b} = C \) where \( C \) is the constant appropriate to the \( \kappa_{50} \) function. Thus, for example, if we want to interpret the curves in Figure 2 as applying to a column length of 200 cms the appropriate value of \( b \) for the \( \kappa_{50} \) curve to be applied to a 200 cm column can be calculated from:

\[
b = \frac{L_n}{C}
\]  

(47)

For the \( \kappa_{50} \) curve \( C = 50^3/2.6 \times 10^4 \) and hence:

\[
b_{50} = \frac{200^3}{50^3} \times 2.6 \times 10^4
\]

\[
b_{50} = 64 \times 2.6 \times 10^4
\]

where \( b_{50} \) is the value of \( b \) for the \( \kappa_{50} \) curve when applied to a 200 cm column.

For the \( \kappa_{200} \) curve the value of \( C \) is \( 200^3/2.6 \times 10^4 \) and hence if \( L = 200 \) cms, \( b_{200} \) is \( 2.6 \times 10^4 \) cm\(^3\). The limiting flux through the column is given by equation (45). The limiting flux through a 200 cm column of material characterized by the \( \kappa_{50} \) curve is:

\[
J_{lim\ 50} = \frac{1.76 \times 50 \times 2.6 \times 10^4}{200^3} = 3.7 \times 10^{-1}
\]

and the limiting flux through a 200 cm column of material characterized by the \( \kappa_{200} \) curve is:

\[
J_{lim\ 200} = \frac{1.76 \times 2.6 \times 10^4}{200^3} = 0.579 \times 10^{-2}
\]
The limiting flux for the $\kappa_{50}$ case is 64 times as large as that for the $\kappa_{200}$ case. Since $E = 0.006$ in both cases and since this value of $E$ is near the limiting value for the $\kappa_{200}$ case in Figure 2 it is seen that the $\kappa_{50}$ case is not operating anywhere near the limiting flux condition.

Additional examples that show the effect of the source term on the $\varphi$ profiles are given in Figure 6. When $E = 0.006$, which is much less than the limiting flux for the $\kappa_{100}$ function, the presence or absence of the source term had relatively little effect on the $\varphi$ profile (dashed lines). When $E = 0.06$, which is approximately of the same magnitude as the limiting flux for this case, the source term had a strong effect on the $\varphi$ profile (compare solid curve labeled $A_0 = 0$ and those labeled with $A_0 = 2.0$, $0.2$ and $0.20 \times 10^{-3}$).

An interesting phenomenon, with regard to the behavior of the source function, that has appeared in the results, is the "shorting" effect. Whenever $\varphi_p$, the pressure head of the water in the plant, is greater than the pressure head of the soil water, the sign of the source function becomes positive and water is transferred from the "root system" to the soil. This effect has already been mentioned above in the previous examples, and additional examples are shown in Figures 7, 8 and 9. In Figure 7 the source function $r$ is positive in the upper part of the root zone and negative in the lower part. Water is thus transferred via the root system from the lower part of the rooting zone to the upper. The effect of this shorting on the flux is quite dramatic as shown in Figure 8. Whenever shorting occurs, the flux will exhibit a minimum at some point in the rooting zone. In the example shown, there were three cases, viz., $r = 0$, $0.1$ and $0.25$, where there was sufficient shorting to cause a reversal of the direction of flow in the root zone as shown by the negative values of flux in part of the rooting zone. In the case $r = 0$ the flux curve should pass through zero flux at $\xi = 0$. The fact that it doesn't reflects some inaccuracies in calculation of the flux that have been referred to above. In this case ($r = 0$) a circulatory system is set up in the root zone with water flowing upward in the
root system and back down again in the soil. At the same time water is being lost from the plant by transpiration.

In Figure 9 the $\phi$ profiles are shown for the case depicted in Figures 7 and 8. Observe that as $r$ approaches zero the pressure heads become less negative and that when $r = 0$, $\phi$ at the top of the column is greater than $-1$. This is a result of the shorting within the column. The numbers in parentheses on the $\phi$ profiles are the values of $\frac{\phi}{r}$. Observe that in all cases they fall within the range of $\phi$ values displayed in the profile.

From an inspection of the calculated results obtained thus far it appears that the "shorting" phenomenon is most apt to occur when $E$ is small, and when $|\xi_c|$, $r$ and $A_0$ are large, i.e. under conditions of a well developed deep root system.

Transfer of water from wet soil to drier soil by plant roots has been observed by Breazeale (1930) and Hunter and Kelley (1946). The model of the source term used in this analysis at least qualitatively predicts the occurrence of such a phenomenon.

The effect of rooting depth on the flow can be examined in various ways. Several of these will now be described.

In Figures 10, 11 and 12 some of the effects of varying $\xi_c$ at constant integral root density and constant $E$ are shown. In these examples the parameter $A_0$ was chosen such that

$$\int_0^0 \alpha d\xi = -A_0 \xi_c = 2 \quad (48)$$

As $|\xi_c|$ increases the $\phi$ profile becomes less and less negative. As the rooting depth is increased the general level of flux in the soil in the upper part of the profile decreases, more of the water is removed from the profile at greater depths by the roots, and the gradient necessary to supply the flux at the soil surface becomes smaller. At the greater rooting depths some "shorting" is displayed (e.g. see $A_0 = 2.67$, $\xi_c = -0.75$, Figure 12). In this instance the flux
is downward in the region $-0.35 < \zeta < -0.62$. As the rooting depth increases
the source function tends to become less negative (see Figure 11, curves for
$\zeta_c = -0.1$ and $-0.25$). With further increase of rooting depth, the source function
approaches zero and in some cases becomes positive in the upper part of the pro-
file. When this happens the source function at the lower part of the rooting zone
is necessarily more negative because of the condition that:

$$\int_{\zeta_c}^{0} \Gamma d\zeta = -(1-r)E$$

(49)

where $r$ and $E$ are constants.

Another way to examine rooting depth effects is to fix the physical
depth of rooting and vary the column length. Some results of such a comparison
at constant $E$ are shown in Figure 13. The rooting depth was 25 cms and column
lengths of 50, 100, 200 and 400 cms were chosen. The root distribution function
was a constant and chosen such that the integral of the physical root distribu-
tion function, $\int_{z_c}^{0} A(z)dz$, equaled a constant. Since $a(\zeta) = A_0 = A(z)L^2$ this
meant that $-z_c A_c/L^2$ was a constant, which in this case was 0.5. The $\varphi$ profiles
are shown in Figure 13. The $\varphi$ profiles for the 50, 100 and 200 cm columns were
not affected by the source term because the evapotranspiration rate was much
less than the limiting value possible through these columns. However, in the
400 cm column the source term was effective in modifying the $\varphi$ profile as shown
by the effect of variation of $r$ which varies the proportion between transpiration
loss of water at the soil surface.

At higher values of $E$ steady state flow will become impossible in the
longer columns, and the effect of the source term on the flow will become more
evident in the shorter columns.

Still another way to examine rooting depth effects is to fix the
distance between the water table and the lower boundary of the root zone and
vary the column length. Some examples of this kind of comparison are shown in
Figures 14, 15, 16 and 17. Two distances between the water table and the lower
boundary of the root zone were chosen, viz., 25 and 75 cms. Column lengths chosen were 100, 200 and 400 cms. As in the previous comparison the integral \[ \int_{z_c}^{P} A(z)dz \] was kept constant. In the 100 and 200 cm columns the pressure head profiles were relatively straight-line and uninteresting, but in the 400 cm column the pressure head decreased quite drastically at the top of the column. As the column length increased the source function became less negative in the lower part of the root zone. When the roots were terminated 75 cms above the water table no shorting occurred, but when the roots were extended to 25 cms above the water table shorting occurred (see Figures 16 and 18).

The flux at the soil surface is given by \( rE \), which in this case was 0.0006. This flux is much less than the limiting flux through the 200 cm and shorter columns when no roots are present. The hydraulic gradient required to cause the flux \( \phi = 0.0006 \) was not very great in these columns because they were relatively wet at the top and the conductivity was consequently larger than it was in the case of the 400 cm column. Also the root system obtained the bulk of its demand for water from relatively wet soil near to the water table. These factors account for the relatively straight-line pressure head profiles in the columns of length 200 cms and shorter.

III. Development of an Experimental System

Considerable progress has been made in setting up an experimental system in which measurements of the soil water content, pressure head, evaporation, and transpiration rates can be made. The equipment and supplies for this system have largely been obtained from those on hand in the Agronomy Department and from other funds. At the present time we are in the midst of design and construction of three setups of the sort shown schematically in Figure 18. The soil column is approximately 1 meter high and 15 x 15 cm in cross section. Tensiometers are installed at intervals in the column and the column can be scanned with a gamma absorption system for the purpose of measurement of water content. Provision is made for separate measurement of the water loss from the
soil surface and from the leaves of the plant by means of condensation on cooling coils. The carbon dioxide and oxygen content of the air in the plant growth chamber can be monitored and controlled. At present, measurements are being made on the soil columns without plants to determine the conductivity function of the soil, to test the operation of the air conditioning equipment, etc. We intend to test the validity of the mathematical model of the root uptake of water as described in the first section of this report and to search for other possible ways of modeling the uptake of water by roots.

IV. Literature Cited


V. Summary, Conclusions and Future Work

While this is a final report on this particular project it is in fact a progress report on our work in the area of uptake of water by roots. The numerical analysis and experimental work will be continued if at all possible. Analyses using other conductivity functions should be made. There are a number of variations on the boundary conditions that should be examined, and the question of the most appropriate model of the source function must remain unanswered at
least until the analysis can be supplemented with experimental work designed to measure the variables involved in various models of the source term.

The work to the present has (1) demonstrated that solutions to the flow equation with a source term can be obtained by numerical means, (2) shown something of the theoretical influence of the parameters in the source term model on the flow in the soil, (3) qualitatively indicated a transfer of water from wet soil to dry soil via the root system, (4) indicated little effect of the source term on the pressure head profile unless the evapotranspiration rate is about the same order of magnitude as the theoretical limiting value, and (5) has formed a theoretical background for some experimental work which is being started. A paper is being prepared for publication in the Soil Science Society of America Proceedings and is being presented at the annual meeting of the Society in Stillwater, Oklahoma, August 21-26, 1966.
Figure 1. Dimensionless conductivity curves used in the analyses. The parameters on the curves are values of \( L \) used to generate the dimensionless curves from the physical data for Sachappa sandy loam.
Figure 2. Examples of the dimensionless pressure head.
Figure 3. Typical dimensionless flux profiles obtained in the analysis. $F_l = 0.006$

$A_0 = 8.0$

$S_{c} = -0.25$

$X_200$

Figure 4. Typical dimensionless source function profiles obtained in the analysis.

$F_l = 0.006$

$A_0 = 8.0$

$S_{c} = -0.25$

$X_200$
Figure 3. Typical dimensionless flux profiles obtained in the analysis.

Figure 4. Typical dimensionless source function profiles obtained in the analysis.
Figure 5: The limiting dimensionless flux as a function of column length for the conductivity functions used in the analysis.
Figure 9. Effect of the "shorting" phenomenon on the pressure head profile.
Figure 10. Effect of rooting depth $c_r$ on the pressure head profiles with $f_{ad}$ held constant.
Figure 11. Effect of rooting depth $z_c$ on the source function with $I_{ad}$ held constant.
Figure 12. Effect of rooting depth $S_c$ on the flux, with $A_0$ held constant.
Figure 13: Pressure head profiles for 50, 100, 200, and 400 cm columns with rooting depth fixed at 25 cm.
Figure 14. Pressure head profiles for 100, 200 and 75 cm above the water table.

400 cm columns with rooting depth to 25 cm.

Lower boundary of root zone 2.5 cm below water table.

Root zone 7.5 cm below water table.

Higher boundary of root zone 15 cm below water table.
Figure 12. Source Term, S/Kg, Protons for 100 to 200 and 400 cm columns with rooting depth to 25 and 75 cm above the water table.
Figure 18. Schematic diagram of the experimental arrangement for study of uptake of water by a root system.