WRC RESEARCH REPORT NO. 191

DIFFUSION AND ADVECTION IN TWO-DIMENSIONAL ROTATING FLOW

By Vahid Alavian and Sally M. Broeren
Department of Civil Engineering
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

The research on which this report is based was financed in part by the U.S. Department of the Interior, as authorized by the Water Research and Development Act of 1978. (P.L. 95-467).

Project No. G 840-02
Grant No. 14-08-0001-G 840

October 1984

Project Completion Report

U.S. Department of the Interior
Washington, D.C. 20240

ABSTRACT

This paper presents the results of an investigation of mass transport by diffusion and advection in two-dimensional steady, spatially varied confined flow. In practice, this type of flow occurs in recirculating regions of streams and rivers or behind hydraulic structures. A simple vorticity transport model is used to simulate shear-induced flow in a square region confined on three sides and open to a uniform flow on one side. The differential equation governing diffusion and advection of a known quantity of tracer mass introduced in such flow is solved. The solution scheme is based on Galerkin finite element approximation of the transport equation. Diffusion is represented by a second-order tensor, the components of which are related to the eddy diffusion coefficient, as well as the magnitude and direction of the mean local velocity. The unsteady transport term is approximated by implicit finite differencing in time. Simulation results are given for one and two-dimensional test cases and for shear induced rotating flow.

Alavian, Vahid and Broeren, Sally M.

DIFFUSION AND ADVECTION IN TWO-DIMENSIONAL ROTATING FLOW
Completion Report to Department of the Interior, September 1984, Washington, D.C., 34 pp.

KEYWORDS--*mass transport/*mixing/flow pattern/circulating flow/shear flow/*dispersion/*diffusion/*finite element/fluid mechanics

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LIST OF SYMBOLS

Symbols of secondary importance which appear only briefly in the text are not included in this list.

a	time dependent portion of local concentration
A	finite element area
B	global boundary matrix
BG	global matrix assembled from B matrix
c	local concentration
D_{ij}	second-order diffusion tensor (molecular and turbulent)
D	diffusion coefficient
D_L	streamwise diffusion coefficient
D_T	transverse diffusion coefficient
D_{xx}	diffusion coefficient along x-axis
D_{yy}	diffusion coefficient along y-axis
$D_{xy} = D_{yx}$	cross product of D_{xx} and D_{yy}
G	global matrix assembled from the sum of M, K, and H over all elements
H	coefficient matrix for diffusive approximation of concentration
i,j,k,l,m,n	summation indices
k	ratio of transverse to longitudinal diffusion coefficient
K	coefficient matrix for connective approximation of concentration
M	coefficient matrix for temporal approximation of concentration
MG	global matrix assembled from M matrix
R	residual error from approximation of concentration equation
R_e	characteristic Reynolds number
s	source or sink term

t	time
u_i	time-averaged velocity tensor
u	velocity component along x-axis
U	magnitude of local velocity vector
$\overline{U}$	velocity vector
v	velocity component along y-axis
x_i	cartesian coordinate tensor
x,y,z	coordinate axes
ε_L	longitudinal turbulent diffusion coefficient
ν	kinematic viscosity
ϕ	basis function for finite element approximation
$\bar{\omega}$	vorticity vector

1. INTRODUCTION

Knowledge of the mechanisms governing the mixing and transport of substances released, accidentally or by design, in the environment plays a significant role in maintenance and management of the quality of the environment. Transport phenomena associated with water resources, surface or subsurface, is generally quite complicated. This is, among other factors, due to the irregular geometry of natural flow systems and the complexity of motion in the transporting medium. One example of such complex situations is motion in streams and rivers with large circulating regions along their banks or behind hydraulic structures. Motion within these regions is significantly different from the main flow, though induced by it, and can affect the transport characteristics of the otherwise relatively straight flow. Investigation of mixing and transport within circulating regions is the motivation for the study reported herein. In the analysis, the flow situation is simplified somewhat by casting the problem in two dimensions in the horizontal plane and considering the simple flow geometry shown in Figure 1.

In general, a property (e.g., mass, momentum, energy) is transported as a result of the combined effects of the two mechanisms advection and diffusion. Advective transport is related to the flow pattern and velocity magnitude. The mean transport of a property may be characterized by the time averaged velocities in the medium. Following Fick's [1855] molecular diffusion theory, Taylor [1954] hypothesized that this portion of advective transport is also proportional to the property gradient. Diffusion has thus evolved to include the Fickian molecular diffusion and the diffusive portion of advection and is generally referred to as eddy diffusion. The coefficient of proportionality for molecular diffusion is a scalar and assumed constant. The coefficient characterizing eddy diffusion is, in general, a second-order

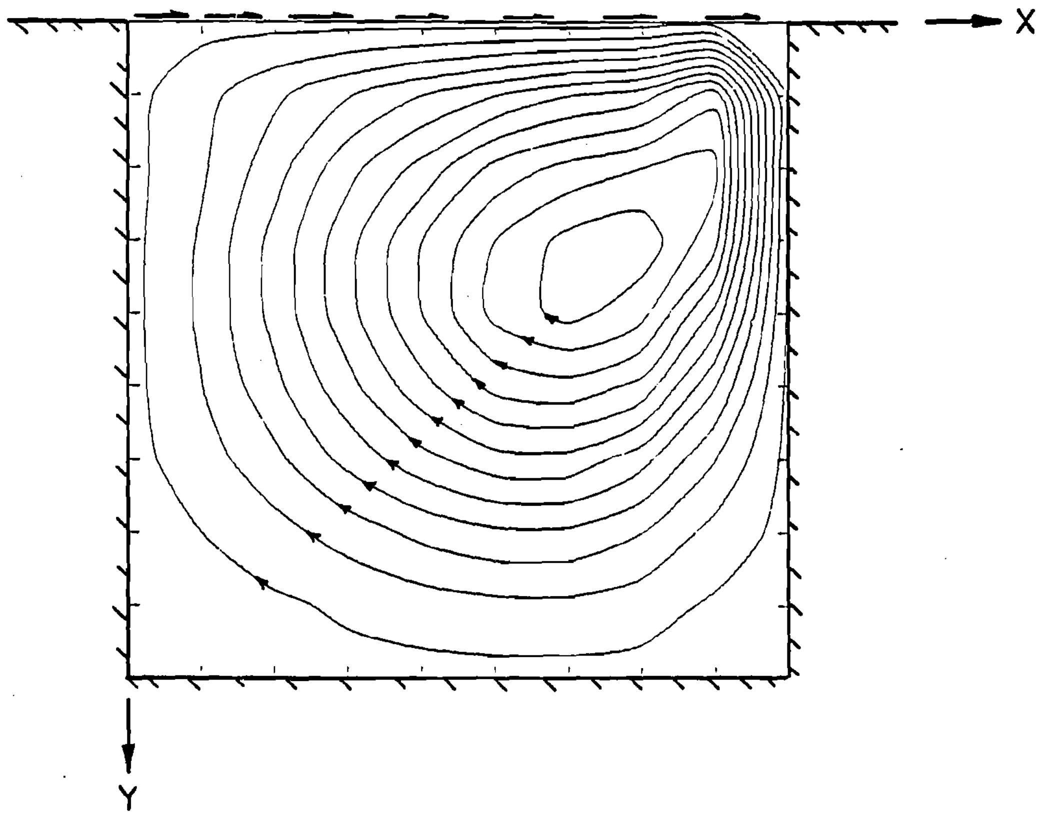


Fig. 1. Flow Pattern in Two-Dimensional Shear Driven Confined Flow

tensor, the components of which are functions of both position and time. In relatively simple flows, this tensor reduces to a scalar or at least a lower order tensor. In more complex situations, such as rotating flow considered here, the coefficient of proportionality (i.e., the diffusion coefficient) must be treated in its full tensor form. The role of the components of this eddy diffusion tensor in spreading mass is discussed in this paper for two flow patterns: solid body rotation, represented by linear velocity distribution with radial distance, and shear induced confined rotation. The latter flow pattern is a simplified representation of circulating (or secondary) flow regions along the stream bank or behind hydraulic structures.

2. RELATED WORK

Hydrologic transport has been a topic of interest to many researchers. Effort is continuously made to improve the performance of the existing models by better mathematical represention of the physical processes and/or by developing more sophisticated numerical solution schemes. Some of the previous work useful in investigating advection and diffusion of mass in spatially varying flow is discussed here.

McGuirk and Rodi [13] developed a depth-averaged model for calculation of mass exchange between a mainstream and a recirculating region. The turbulent concentration fluxes are represented as the ratio of eddy viscosity to a constant Schmidt number. Eddy viscosity and velocity distributions are obtained from solving a $k-\varepsilon$ turbulence model developed by Rastogi and Rodi [17]. They do not discuss variability of diffusion along and normal to the flow lines. Eddy viscosity (representing diffusion) is treated as a scalar at each computation node. Their model, even in two dimensions, is quite expensive computationally. Westrich [24] developed a one-dimensional model for predicting the mixing process between a mainflow and an adjacent circulating region. He found an exponential decrease of the tracer concentration with time under steady flow conditions. He assumed a well mixed circulation region with constant concentration and thereby did not address the transport phenomena within the region. This is, of course, not the case in practical situations. His work, however, is useful for comparison with two-dimensional analysis where diffusion and advection within a circulating region are taken into account.

Scheidegger [20] arrived at a general tensor form for dispersion in porous media. He showed that in isotropic media there are only two constants of dispersivity, one parallel to flow and one normal to it. This result was also arrived at by Nikolaevskii [14] who used a theorem of the statistical theory of turbulence and assumed an isotropic media. The simplest form of dispersion in porous media was given as the product of a fourth-order dispersivity tensor and a scalar velocity scale. The symmetrical properties of the dispersivity tensor reduces it to only two constants when aligned with the principal axis in one-dimensional flow. Bear [5] and Bachmat and Bear [4] arrived at a general form of the hydrodynamic dispersion equation in porous media. Most of the published research treating diffusion (or dispersion) as a tensor seems to be in the area of flow in porous media. This is partly due to the fact that the dispersive properties of a porous medium can be uncoupled from the flow properties; whereas in turbulent flow, diffusion is directly governed by the structure of turbulence and cannot be easily uncoupled from it. Subsequently, diffusion is most often treated as a scalar property in practical surface flow problems.

Smith et al. [22] performed a series of tests comparing the Galerkin finite element method to several optimized finite difference models. Finite element results for the two-dimensional convection-diffusion problem were found to be as conservative as Siemons [1970] iterative alternating direction implicit finite difference method. The authors compare the Galerkin finite element solution for a two-dimensional convection to the solution obtained from Robert and Weise's [18] fourth order finite difference model. The Galerkin finite element results for this problem were sensitive to the time step selected but had somewhat less distortion of contours than the finite difference model. The Galerkin form of the finite element method is reported by Grove [9] to be less subject to numerical dispersion than many standard finite difference schemes.

3. MASS TRANSPORT EQUATION

The second-order time averaged partial differential equation describing advection and diffusion of mass in turbulent flow is

$$\frac{\partial c}{\partial t} + u_i \frac{\partial c}{\partial x_j} = \frac{\partial}{\partial x_i} [D_{ij} \frac{\partial c}{\partial x_j}] + s \qquad (1)$$

where

- c concentration;
- t time;
- x_i cartesian corrdinate (x,y,z);
- u_i time-averaged velocity (u,v,w) along (x,y,z);
- D_{ij} second-order diffusion tensor (molecular and turbulent);
- s source or sink term.

The diffusion tensor in equation (1) has nine components which are a function of position and time. In most situations, the turbulent diffusion part of D_{ij} is orders of magnitude larger than the molecular diffusion part. The latter is, therefore, neglected unless one is concerned with diffusion close to boundaries where turbulence is damped. Kolmogorov [10,11] showed that equation (1) can be applied only after the diffusing substance has been in the flow long enough for sharp gradients to be smoothed out over the distance traveled. The time and distance over which the particle "forgets" its initial conditions, and begins to diffuse according to Fick's law, are known as Lagrangian scales. Equation (1), however, is written from the Eulerian viewpoint since a fixed (usually cartesian) frame of reference is used in most field observations and practical problems. With respect to its principal axis, $D_{ij} = 0$ for $i \neq j$ and reduces to three components. In the case of isotropic turbulence, a spherical symmetry exists and D_{ij} reduces to a scalar. However, since turbulence is often anisotropic and non-homogeneous, the diffusion tensor is written in a spatially varied form in equation (1) and will be treated as such throughout this paper.

The problem formulation is simplified by assuming that variations in the vertical direction are negligible. Thus, the expanded form of equation (1) for two-dimensional turbulent flow is

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = \frac{\partial}{\partial x}\left[D_{xx}\frac{\partial c}{\partial x} + D_{xy}\frac{\partial c}{\partial y}\right] + \frac{\partial}{\partial y}\left[D_{yy}\frac{\partial c}{\partial y} + D_{yx}\frac{\partial c}{\partial x}\right] \quad (2)$$

where c is now the time dependent concentration of a conservative substance (s = 0) released in the flow. Since equation (2) is valid in any coordinate system, the diffusion tensor is invariant under coordinate transformation. If one of the coordinate axes (e.g., x) coincides with the principal axis of flow, the cross-variant components (i.e., D_{xy} and D_{yx}) of the diffusion tensor become identically zero. The diffusion tensor thus reduces to two components: the longitudinal diffusion coefficient, D_L; and the transverse diffusion coefficient, D_T. Horizontal spreading of mass described by this formulation is different from that described in vertically averaged shear flows (e.g., Fischer [7]). Here, the diffusion term will not contain spatial averaging of any kind whereas in depth-averaged flows the effects of vertical shear distribution is represented by a dispersion (not diffusion) tensor. In situations where the average value of a substance over a cross-section is sufficient, the diffusion equation is written in one-dimension. In this case, a scalar bulk transport (dispersion) coefficient is defined to replace the spacially averaged diffusion term. The resulting one-dimensional dispersion equation is useful in the analysis of mass transport in streams and estuaries. A comprehensive discussion of one and two-dimensional dispersion in environmental flows without including circulating regions is given in Fischer et al. [8].

To analyze advection and diffusion in two-dimensional circulating or rotating flows, equation (2) must be solved in its complete form. Simplifications similar to those made in relatively straight flows cannot be made here because shear induced circulating flow is highly non-uniform. Averaging the flow properties spatially (i.e., one-dimensional analysis) provides even less information about the nature of convection and diffusion in such flows. Since defining a convenient coordinate axis along the flow lines is not possible, a fixed cartesian frame of reference is selected. Diffusion along and normal to the streamlines must now be mapped onto this fixed reference in order to define the components of the diffusion tensor shown in equation (2). This functional relationship must, of course, remain invarient under coordinate transformation.

Here, components of the diffusion tensor in equation (1) are expressed in terms of turbulent diffusion coefficients scaled by the components of the local velocity vector. With some manipulation, components of the diffusion tensor for two-dimensional flow, can be written as

$$D_{xx} = \varepsilon_L \left(\frac{u^2 + kv^2}{U^2} \right) \; ;$$

$$D_{yy} = \varepsilon_L \left(\frac{ku^2 + v^2}{U^2} \right) \; ; \qquad (3)$$

$$D_{xy} = D_{yx} = \varepsilon_L (1 - k) \frac{uv}{U^2} \; ;$$

where ε_L is the longitudinal turbulent diffusion coefficient (along the streamlines), $k \leq 1$ is the ratio of transverse to longitudinal turbulent diffusion coefficient, and U is the magnitude of the local velocity vector. Equations (3) calculate the diagonal components of the diffusion tensor as positive values with respect to the cartesian coordinate axes regardless of the local flow orientation. The off diagonal (cross-variant) terms are allowed to assume positive or negative values depending on direction of the local velocity vector.

Fischer et al. [8] note that experimental measurement of the longitudinal diffusion coefficient has been encumbered by difficulty in isolating the effects of turbulent diffusion from other mechanisms. Experiments conducted by Sayre and Cheng [19] indicated that the diffusion coefficient in the longitudinal or streamwise direction is greater than that in the transverse direction. The authors do not know of any theoretical or experimental investigation providing information on diffusion coefficient in rotating flow. Thus, values of longitudinal and transverse turbulent diffusion coefficients used in this paper are mainly for the purpose of illustrating the geometric characteristics of a diffusing cloud in a rotating flow field and the significance of the cross variance terms (usually ignored) in the diffusion tensor.

Equation (2), with diffusion terms given by equation (3), can be used to analyze mass transport in a variety of two-dimensional problems with known velocity field. The numerical scheme developed by Alavian et al. [1983, 1984] has been used to solve equation (2) for the flow situations presented in this paper. The solution method is based on the Galerkin finite element approximation of the transport equation and uses linear basis functions in the interpolation scheme. Fully implicit finite-differencing is used to approximate the unsteady term in equation (2). Spatial distribution of concentration, velocity, and diffusion are approximated viz:

$$c(x,y) \approx \sum_{j=1}^{3} a_j(t)\ \phi_j(x,y)\ ;$$

$$u(x,y) \approx \sum_{k=1}^{3} u_k\ \phi_k(x,y)\ ;$$

$$v(x,y) \approx \sum_{k=1}^{3} v_k\ \phi_k(x,y)\ ; \qquad (4)$$

$$D_{xx}(x,y) \approx \sum_{m=1}^{3} D_{xxm}\ \phi_m(x,y)\ ; \qquad + \text{ similar terms for } D_{yy} \text{ and } D_{xy} = D_{yx}\ ;$$

where ϕ is a linear basis function, which for triangular elements can be integrated explicitly as given by Pinder and Gray [15]; a, u, and v are the nodal values of concentration and velocity components, respectively; and D_{xxm}, D_{yym}, and $D_{xym} = D_{yxm}$ are evaluated from equation (3) for given values of the longitudinal diffusion coefficient and k.

Substituting equation (4) into equation (2) and setting the weighted residual error over each element area equal to zero, the approximated mass transport equation for each element becomes

$$\sum_{j=1}^{3} \int_A \left\{ \frac{\partial a_j}{\partial t}\ \phi_j\phi_i + [\sum_{k=1}^{3} u_k\phi_k]\ a_j\ \frac{\partial \phi_j}{\partial x}\ \phi_i + [\sum_{k=1}^{3} v_k\phi_k]\ a_j\ \frac{\partial \phi_j}{\partial y}\ \phi_i \right.$$
$$+ [\sum_{m=1}^{3} D_{xxm}\phi_m]\ a_j\ \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x} + [\sum_{m=1}^{3} D_{yym}\phi_m]\ a_j\ \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}$$
$$\left. + 2[\sum_{m=1}^{3} D_{xym}\phi_m]\ a_j\ \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y} \right\} dA = \int_A R(x,y)\ \phi_i\ dA = 0 \quad (i=1,3)\ . \qquad (5)$$

where R is the residual error and A is the element area. Equation (5) after integration over the element area and finite-differencing in time can be written in the general form

$$\left[\frac{1}{\Delta t} M + K + H \right] \sum_{j=1}^{3} a_j^{n+1} = [B] + \left[\frac{1}{\Delta t} M \right] \sum_{j=1}^{3} a_j^{n} \qquad (6)$$

where

- M — coefficient matrix for the unstready term in equation (5);
- K — $= K_x + K_y$, coefficient matrices for the convective terms in equation (5);
- H — $= H_{xx} + H_{yy} + H_{xy} + H_{yx}$, coefficient matrices for the diffusion terms in equation (5);
- B — residual matrix, includes the boundary conditions;
- Δt — finite-difference time step between n and n+1.

Since equation (6) holds for every element, a global system of equations can be derived by summing this equation over all the elements used to represent the flow geometry. Hence, matrices M, K, and H are assembled into a global matrix, G, with (k x k) components where k is the number of nodes in the descritized flow field. The element matrices M and B are similarly assembled into the global matrices MG(k x k) and BG(k x 1), respectively. Thus, the approximate form of the mass transport equation with diffusion formulated as a tensor becomes

$$[G] \sum_{j=1}^{k} a_j^{n+1} = [MG] \sum_{j=1}^{k} a_j^{n} + [BG] \ . \qquad (7)$$

The system of equations (7) can be solved once the flow geometry, velocity distribution, and turbulent diffusion coefficient are given.

4. ROTATING FLOW SIMULATION

Two patterns of rotating flow are considered here. One is a solid body rotation where velocity varies linearly with radial distance from a center of rotation. The other is shear induced rotating motion generated and maintained by a known uniform flow at the open boundary of a square (classical cavity flow problem). The latter pattern is selected because it, to some degree, resembles flow in circulating regions of stream and rivers, generated due to meanders, sudden changes in geometry and presence of engineering structures. There is growing evidence that circulating flows, depending on their size, geometry, and zone of influence can significantly affect mixing and transport of contaminants in water bodies.

Many models and schemes exist for numerical generation of circulating motion including those proposed by Kuipers and Vreugdenhil [12], Abbott and Rasmussen [1], McGuirk and Rodi [13], and Ponce and Yabusaki [1981]. Numerical generation of shear induced circulation is not very difficult, the real task is generating the correct circulation. Unfortunately, field or laboratory verification of the numerical models generating circulation is almost non-existent. Thus, a simple vorticity transport flow model is considered sufficient for simulating shear induced confined rotating flow. In practical situations mass and momentum are exchanged between the main flow and the recirculating flow through a free shear zone. This is ignored here since emphasis of the paper is on the advection and diffusion of mass in a known rotating flow without regard to the mechanisms causing the initial motion.

The dimensionless vorticity transport equation governing two-dimensional motion of an incompressible viscous fluid is

$$\frac{\partial \bar{\omega}}{\partial t} - \nabla \cdot (\bar{U}\bar{\omega}) = \frac{1}{R_e} \nabla^2 \bar{\omega} \qquad (8)$$

where

ω	=	$1/2(\partial v/\partial x - \partial u/\partial y)$, vorticity vector in two-dimensions;
U		velocity vector;
R_e	=	VL/ν, Reynolds number with V and L as characteristic velocity and length, respectively;
ν		viscosity

Equation (8) is well known and has been used to solve a variety of problems including shear driven flow in a square cavity. The boundary conditions are specified in terms of velocity or gradient of velocity (i.e., stream function). The no slip condition requires that both components of velocity be zero at the stationary walls. At the open boundary, one component of velocity has a non-zero value (usually unity).

There are many numerical schemes for solving the vorticity transport equation. A second-order upwind finite-differencing scheme with appropriate time step gave stable, steady state solution for a wide range of Reynolds numbers. Details of approximating equation (8), formulating the boundary conditions, and the numerical procedure for repeatedly solving this equation until an acceptable steady-state solution is obtained are given by Alavian et al. [2,3]. The steady flow pattern resulting from a Reynolds number of 10,000 is depicted on Figure 1. The Reynolds number is based on the scales of the free stream velocity and cavity opening. As the Reynolds number increases, the "eye" of the rotating flow moves off center towards the downstream wall.

5. SIMULATION RESULTS AND DISCUSSION

Numerical models are typically verified by comparing the simulation results with analytical solutions or experimental evidence. Neither exists for the case of mass transport by non-uniform shear driven rotating flow. In order to confirm the proper performance of the solution algorithm and to ascertain a relative measure of potential numerical problems, a series of simulations were conducted for transport of mass by simple flows. Simulation of two-dimensional diffusion from a stationary point source produced excellent agreement with the analytical solution. Pure advection of mass in a uniform flow was simulated next. Figure 2 shows streamwise cross-section of the computed concentration values depicted on the initial distribution, which was a piecewise linearized cosine hill. The model proved capable of advecting mass in a uniform flow with minimal distortion of the initial concentration pattern. Drop in the peak concentration in time and the oscillations around the edges are due to artificial diffusion common in fully implicit unsupressed schemes.

Mass transport in flow rotating as a solid body was simulated in the third phase of the model testing. The objective of the series of tests with solid body rotation flow was to develop a better understanding of the model performance under spatially changing, yet relatively simple flow pattern. Flow rotating as a solid body has a linear tangential velocity distribution and no radial velocity. Since shear is constant everywhere, mass is transported on a circular path without any differential advection. Any spreading is due to diffusion only. Similar to pure advection in uniform flow, a given distribution of mass must retain its initial form as it moves around the center of rotation. With isotropic physical diffusion at work, mass will spread uniformly in all directions while being advected. Numerical

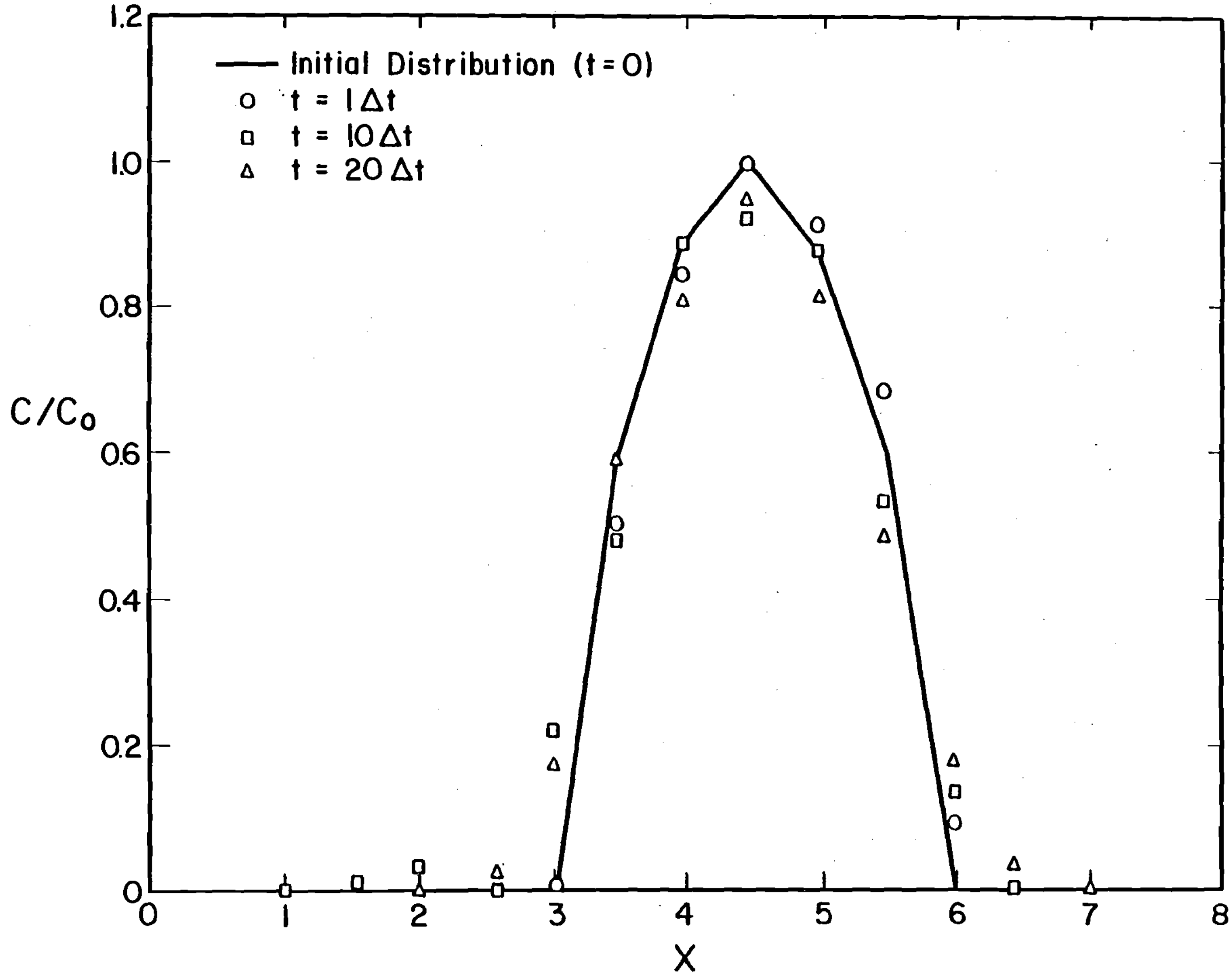


Fig. 2. Concentration Profiles for Pure Advection of Instantaneous Sources in a Uniform Flow. Profiles are centered on the Initial Distribution.

diffusion, on the other hand, causes the mass to spread irregularly thereby causing distortion and wiggles in the concentration pattern. Figure 3 is a three-dimensional plot of an initial cosine hill concentration distribution being transported by a flow rotating as a solid body. Progressive distortion in the concentration pattern at 1/4 and 1/2 turn of the center of mass is apparent in Figures 3b and 3c. Numerical problems became severe at about 3/4 turn and loss of mass was considerable. Using the same initial concentration distribution, physical isotropic diffusion was introduced by specification of a small scalar diffusion coefficient. The results after 1/4 and 1/2 turn of the center of mass are shown in Figure 3d and 3e. The so-called "wiggles" are no longer present and mass has diffused nearly uniformly in all directions.

Using the arithmetic mean velocity, $\bar{u}$, an overall Courant condition for non-uniform flow may be defined as $\bar{u}\Delta t/\Delta x$. Simulation results for pure advection in solid body rotation showed sensitivity to the Courant number thus defined. Simulations for overall Courant numbers, ranging between 1.3 and 0.13 were compared on the basis of peak concentration variation and mass conservation. Results showed progressively better solutions (i.e., less drop in peak concentration and smaller mass loss or gain) with decrease in Courant number. It should be noted that the overall Courant condition defined for highly spatially varied flow, such as those considered in this paper, is useful only as a general guide and not as an absolute measure of numerical stability.

The Peclet number ($P = \bar{u}\ \Delta x/D$) is generally used as an indicator of the interference of numerical diffusion with the physical diffusion. Usually, if $P \leq 1$, the flow is considered diffusion dominated and for $P > 2$ oscillations and wiggles may be present. A Peclet number calculated in the same overall

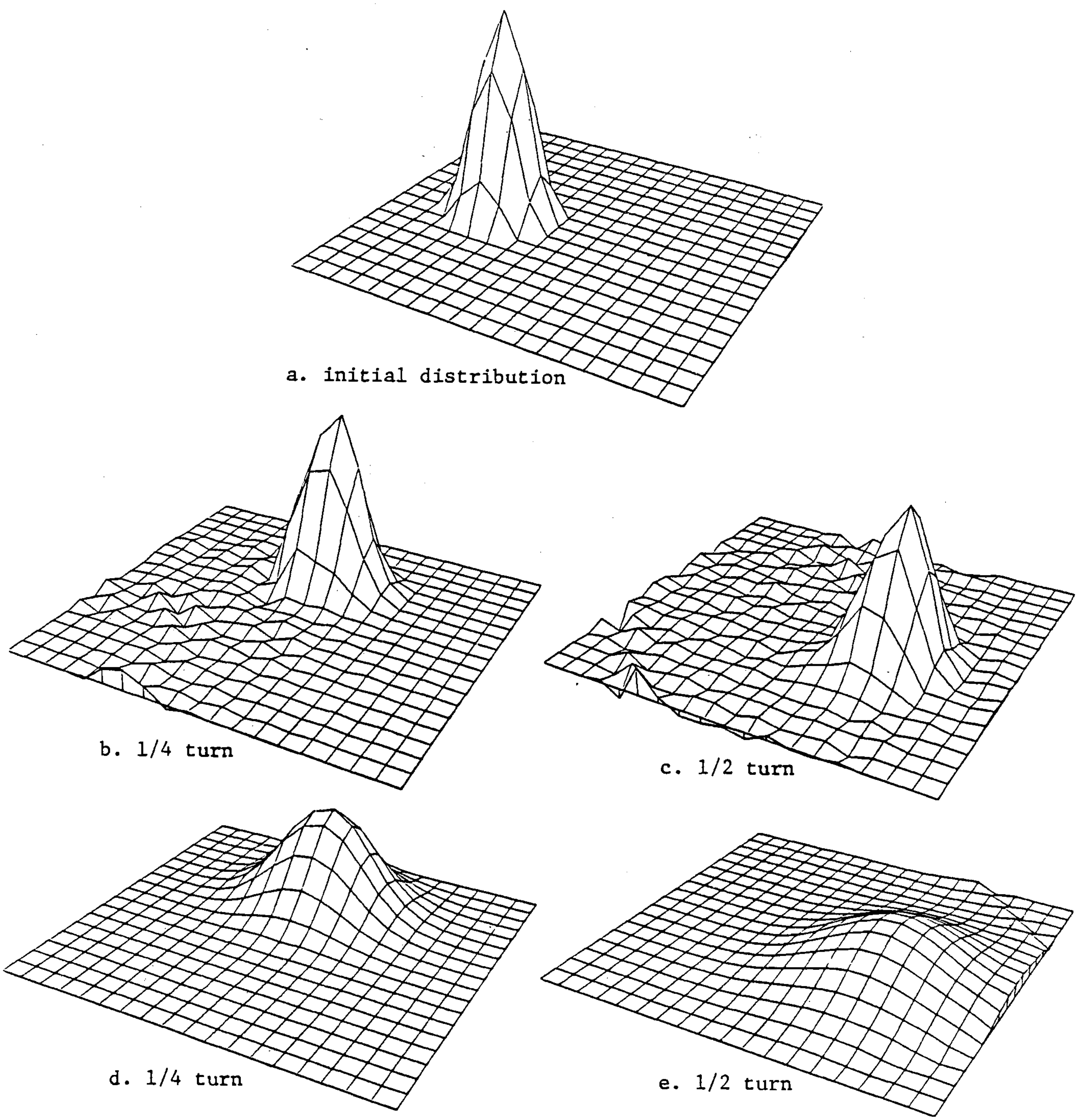


Fig. 3. Concentration Distribution (a) Transported by Pure Advection (b and c) and by Advection and Diffusion (d and e) in Flow Rotating Clockwise as a Solid Body.

sense as the Courant number is not very useful in rotating flows. An arithmetic or even weighted mean velocity is not necessarily a good representative of the advective characteristics of flow. Furthermore, a diffusion tensor calculated according to equation (3) varies widely over the flow field. For the purposes of this investigation, the diffusion coefficient ε_L was used in calculating an overall Peclet number. Similar to the Courant number, the Peclet number is used only as a general indicator and not an absolute measure of numerical diffusion. For the simulations run with solid body rotation, wiggles appeared in the solution for overall Peclet numbers above 7.

With the above information on the performance and limitations of this model the case of mass transport in a shear driven cavity flow was next considered. Shear driven flow within a cavity is highly non-uniform as evidenced by the streamlines shown in Figure 1. Complexity of the flow field prevents simplification of the problem to the point of obtaining a meaningful analytical solution to equation (2). Therefore, the two transport processes, advection and diffusion, can only be examined at a fundamental level. Investigations showed that solution to the system of equations (7) is highly sensitive to the formulation of the diffusion term. Numerical experiments were conducted using the flow generated by equation (8) with a characteristic Reynolds number of 10,000. Convenient time step and grid spacing were selected to achieve an overall Courant number on the order of 0.1. Initial concentration distribution was a cosine hill located at the lower center portion of the flow field.

Simulation results for pure advection and combined advection and diffusion are shown in Figure 4. Concentration pattern for pure advection (no diffusion) at approximately 1/4 turn of the center of mass is given in Figure

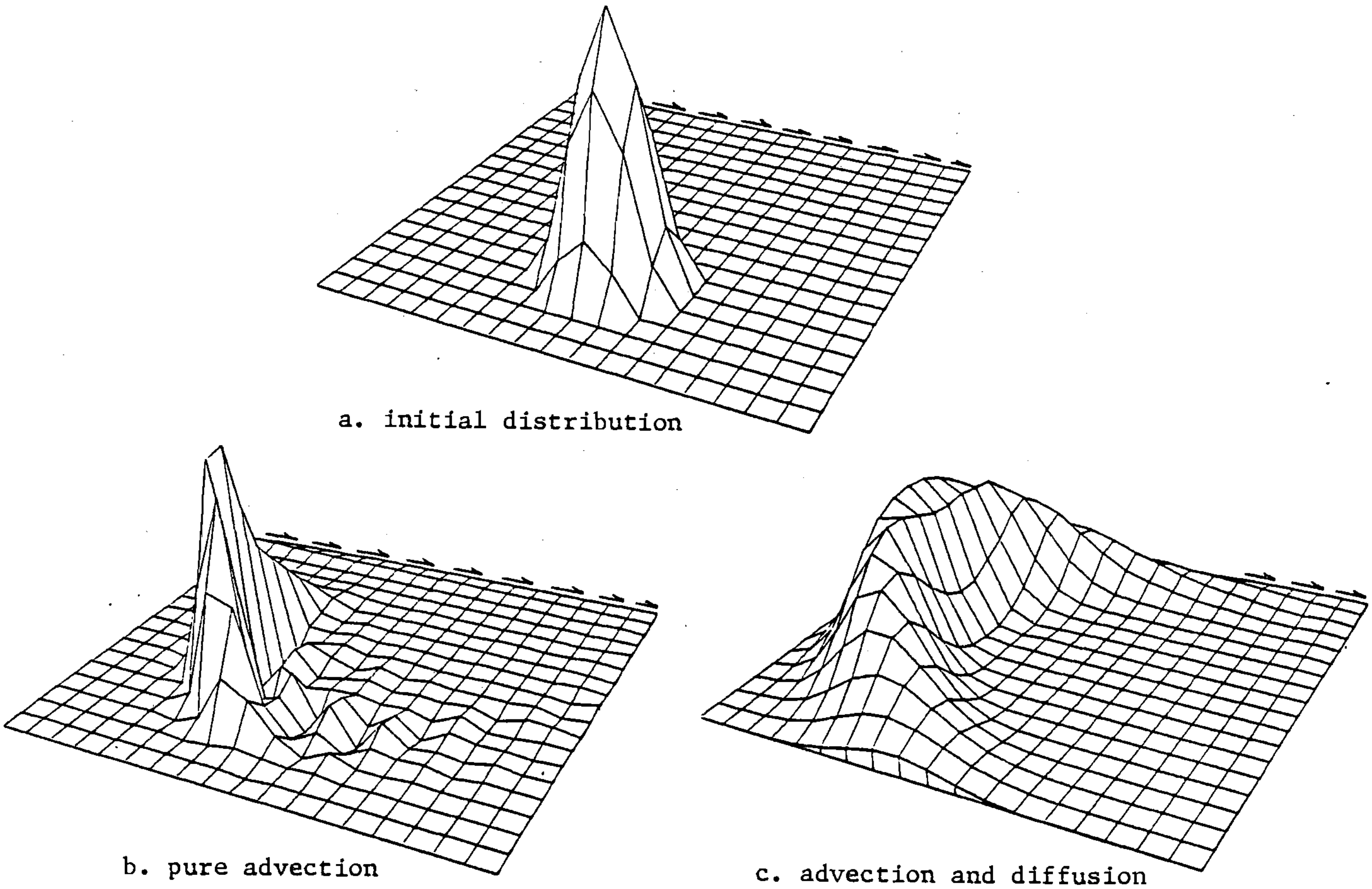


Fig. 4. Concentration Distribution (a) Transported by Pure Advection (b) and by Advection and Diffusion (c) in Shear-Driven Confined Flow for k = 1.0.

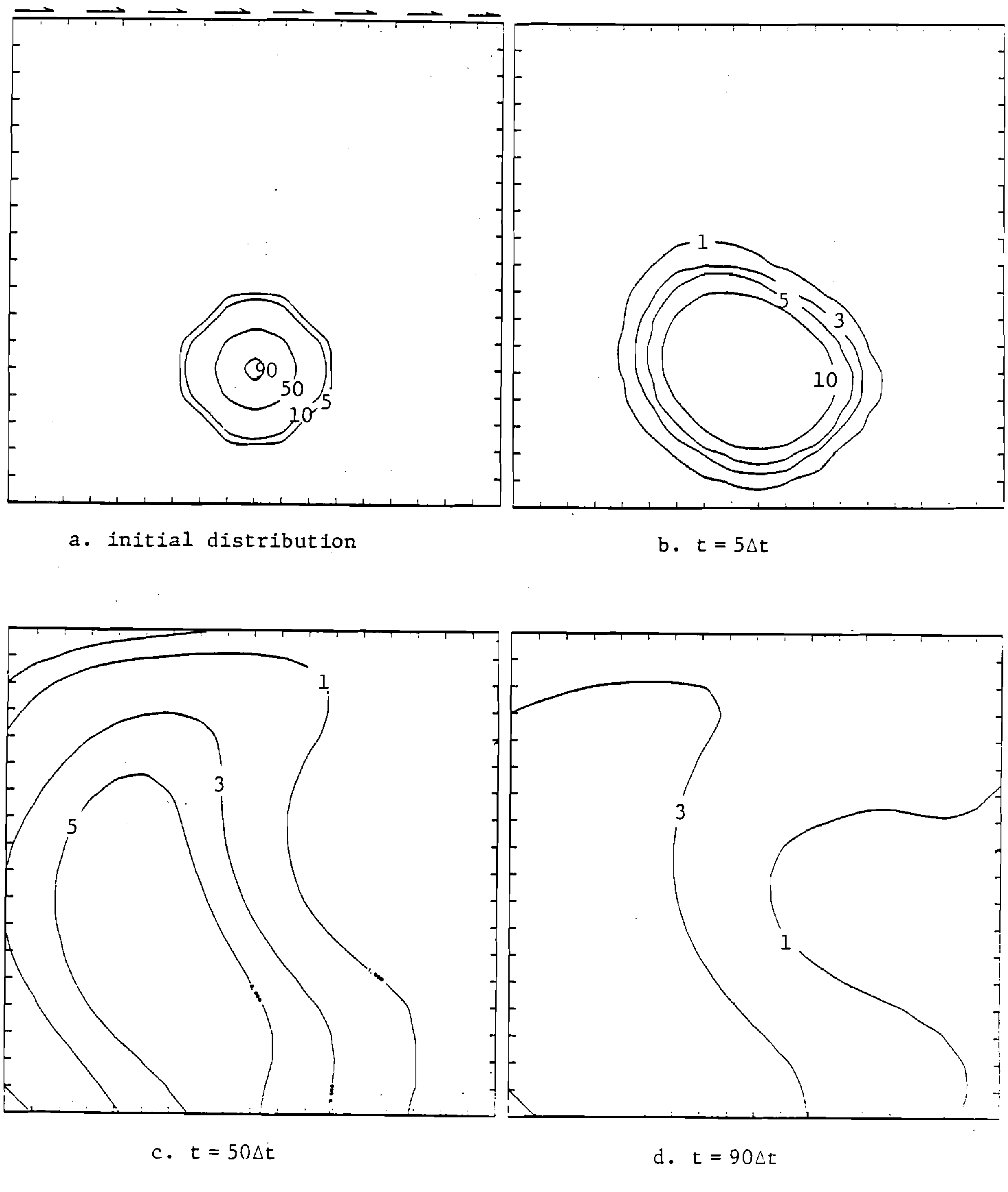
3b. Differential advection caused by the velocity distribution stretches the mass along the streamlines. Oscillating concentration levels, similar to these observed in the solid body rotation tests, trail the distribution. These oscillations (wiggles) become more pronounced in time due to the nature of the numerical scheme and the fact that the tracer approaches the high shear region of flow at the open boundary. Considerable oscillations and loss of mass occurred near the 1/2 turn point of the center of mass rendering the solution meaningless. Mass was not allowed to cross to open boundary and leave the cavity.

Figure 4c shows transport of mass by the combined effects of advection and scalar diffusion. In order to reveal the details, vertical scale in Figure 4c is higher than that in Figure 4b. Initially, spreading is diffusion dominated due to relatively small velocity magnitudes and gradual change in the flow direction. The overall Peclet number for the lower left quadrant of the cavity (Fig. 1) was 0.5. As the spreading mass enters the upper left quadrant, it is transported nearly in the y direction with a Peclet number of about 1. Flow changes direction rather quickly near the open boundary and conforms to the shear induced by the free stream. Transport process becomes highly advection dominated and Peclet number increases rapidly to about 10. In the upper and lower right quadrants, flow decelerates gradually and the associated transport process changes from advection dominated back to diffusion dominated and the scenario repeats itself. Change in transport mechanism from highly diffusive to highly advective over a relatively short distance puts severe strain on any numerical model. Successive solutions given by the present model are reasonable until the mass is in the vicinity of the intense shear zone at the open boundary. At this point, solution is dominated by the discretization errors, associated with central differencing, giving unacceptable mass loss or gain, as well as unreasonable concentration levels along the boundaries.

On the numerical side, the model performance can be improved considerably by using higher order elements and more complex approximation schemes. On the physical side, it is expected that the problems would be less severe if mass were allowed to move through the open boundary and into the free stream via a free shear (exchange) zone. This zone, which is the region of interaction between the driven rotating flow and the driving main flow in practical problems, has been simplified to essentially a line across the open boundary in this investigation. Even though the diffusion tensor is computed for each element, actual diffusion is represented by a constant, ε_L, for the entire flow field. Detailed information on the distribution of eddy diffusion is not available, but it is reasonable for its mean value within the shear zone to vary considerably from that in the rest of the flow. It is expected that the model would give better results for spatially varying diffusion coefficient. Discussion of the flow and transport characteristics of the free shear zone, presently being modeled, is beyond the scope of the investigation reported herein.

Components of the diffusion tensor, computed from equation (3), play a significant role in the geometry of the spreading mass. At $k = 1$, the tensor reduces to a scalar and the cross-variant terms are identically zero. The mass diffuses equally in all directions according to the magnitude of ε_L, as well as being advected along the streamlines. Figures 4c and 4d, discussed earlier, give an example of mass transport with $k = 1$. As k decreases, the transverse diffusion becomes progressively limited and the cross-variant components of the diffusion tensor become more significant. Figure 5 shows an example of evolving concentration pattern for $k = 0.5$. For $k \ll 1$, transverse diffusion is very limited and the mass is distributed in a long, narrow fashion. Simulation results for k = 0.1 are given in Figure 6. Figures 5 and 6, viewed with Figure 1 in mind, show that the concentration pattern is aligned with the general direction of flow while being advected differentially due to non-uniform velocity distribution. The transverse spreading is governed by ε_L and k. It was found that the velocity distribution can actually move the center of mass from one streamline to another, particularly near the open boundary.

Simulations were also conducted with the off diagonal (cross-variant) terms of the diffusion tensor, given in equation (2), artificially set to zero. The results are not significantly different from the runs with D_{xy} and D_{yx} computed according to equation (3). The two solutions deviate from each other around the 45° axes with respect to the center of rotation and quickly merge as the streamline curvatures decrease. Comparisons are at a preliminary level and further investigation of the cross-variant components of the diffusion tensor is necessary.



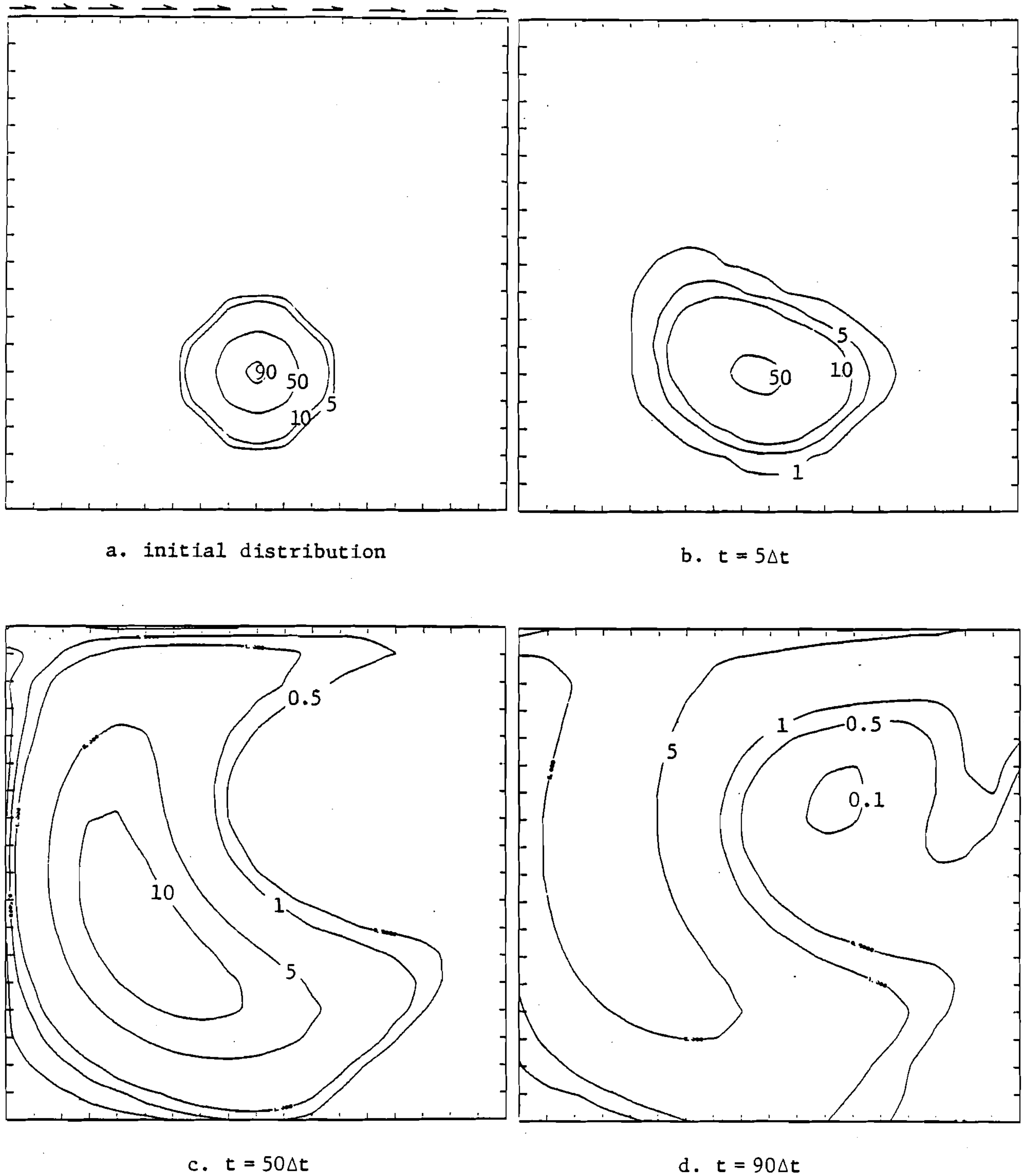
a. initial distribution

b. t = 5Δt

c. t = 50Δt

d. t = 90Δt

Fig. 5. Concentration Patterns for k = 0.5 in Shear-Driven Confined Flow. Contour Levels are based on percent of Initial Concentration.



a. initial distribution

b. t = 5Δt

c. t = 50Δt

d. t = 90Δt

Fig. 6. Concentration Patterns for k = 0.1 in Shear-Driven Confined Flow. Concentration levels are based on percent of Initial Distribution.

6. SUMMARY AND CONCLUSION

Diffusion and advection of mass in steady, two-dimensional rotating flow was investigated by a series of numerical experiments. The unsteady transport equation containing a second-order diffusion tensor was solved using the Galerkin finite element approximation method. Simulating diffusion and advection in uniform flow as well as flow rotating as a solid body provided insight to the the model performance and the descritization errors associated with the numerical scheme used.

Performance of the model in simulating advection and diffusion in shear driven flow is promising. Stable solutions have been obtained for an overall Peclet number as high as 7. Rapid change in the transport mechanism, from diffusion dominated to advection dominated, caused difficulty in the vicinity of the intense shear zone. Increasing the physical diffusion coefficient improves the solution somewhat due to the reduction of the overall Peclet number. It was found that the evolving concentration pattern is sensitive to the order of the diffusion tensor. Scaling the components of the diffusion tensor by the components of the local velocity vector proved to be a reasonable scheme for incorporating changes in the flow direction in the transport process. This, coupled with the ratio of the transverse to longitudinal diffusion coefficients, allows control of the distribution geometry.

Results obtained so far are based on diffusion coefficients selected at mid-range between molecular diffusion and turbulent diffusion coefficients, reported for channel flow. Simulations provide the expected concentration patterns but remain subject to verification by actual data. Unfortunately, published information on mass transport in confined shear driven or recirculating flows is virtually non-existent for model verification at this time.

Refining the computation mesh from 168 triangular elements to 648 showed considerable improvement in the solution. It is expected that finer mesh in the region of intense shear and use of higher order elements will further improve the results. The present numerical scheme does not have any artificial remedies, such as upwinding to supress wiggles or relaxing the troublesome boundary conditions, built into it. These measures may become necessary when actual observed concentration patterns are simulated.

Incorporating an exchange zone, between the rotating flow and the main flow, is being investigated now. In addition to eliminating some of the numerical problems presently experienced at the highly advective open boundary, presence of an exchange zone will enable the model to more realistically simulate mass transport in recirculating flows.

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