

The Geometry of Voting: From Lincoln's Presidential Election to College Football Polls

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Voting Methods

- **Plurality Voting:**
Each voter selects one candidate. The candidate receiving the largest number of votes wins.
- **Vote for Two (Three, etc.):**
Generalization of plurality voting. Each voter selects exactly two (three, etc.) candidates. The candidate receiving the largest number of votes wins.
- **Borda Count:**
We assign 0 points to the last-ranked candidate, 1 point to the second-last candidate, and so on. The candidate receiving the largest number of points wins.
- **Pairwise Comparison:**
A candidate who beats all other candidates in pairwise comparison is called Condorcet winner.

Voting Paradoxes

- **Existence of Cycles (Condorcet Paradox):**
In a pairwise comparison of 3 candidates A , B and C , it is possible that a majority prefers A over B , a majority prefers B over C , and a majority prefers C over A .
- **Multiple Outcomes Paradox:**
Different voting methods can produce different election outcomes.

References

- William V. Gehrlein, *Condorcet's Paradox*, Theory and Decision (1983), 15(2), 161-197
- William V. Gehrlein & Peter C. Fishburn, *The Probability of the Paradox of Voting: A Computable Solution*, Journal of Economic Theory (1976), 13, 14-25
- Donald G. Saari, *Chaotic elections: A Mathematician Looks at Voting*, American Mathematical Society (2001), 33-69
- Donald G. Saari & Fabrice Valognes, *Geometry, Voting, and Paradoxes*, Mathematics Magazine (1998), 71(4), 243-259
- A. Tabarrok & L. Spector, *Would the Borda Count Have Avoided the Civil War?*, Journal of Theoretical Politics (1999), 11(2), 261

Real World Example: AP College Football Polls

2015 AP College Football Polls

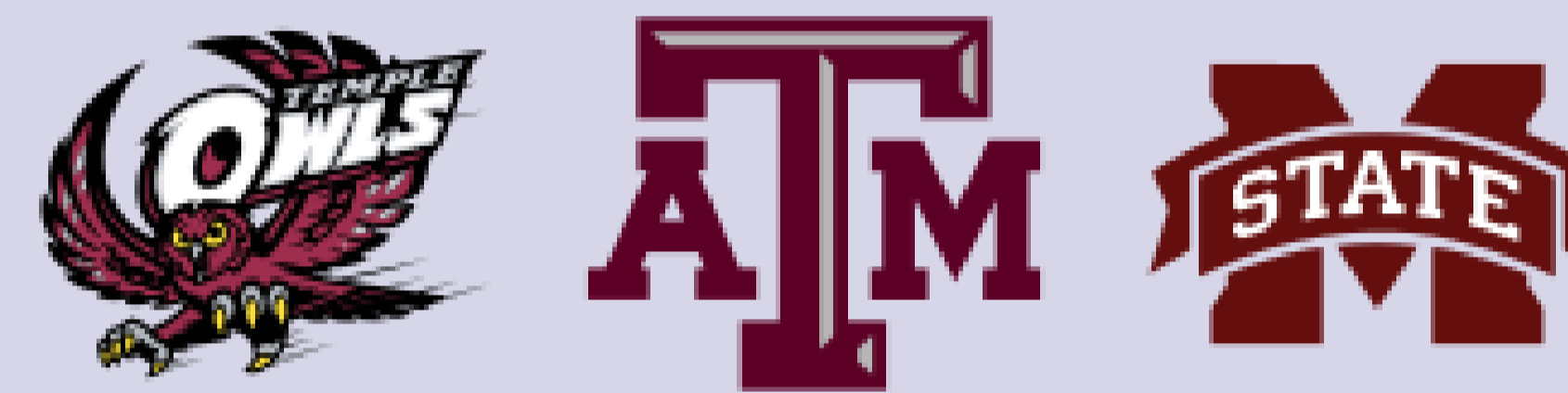


- 128 teams
- 61 voters
- Each voter submits a ranked list of top 25 teams
- Voting method used: Borda Count
- 16 weeks ("elections")
- 24400 observations altogether

Voting Paradoxes in AP Poll Data

- **Voting Methods Analyzed:**
 - **Borda- n** gives n points for 1st place, $n - 1$ for 2nd, ..., 0 points for n th place and beyond
 - **Example:** Borda-3 assigns 3 points for 1st place, 2 points for 2nd place, and 1 point for 3rd place.
 - **Borda-25** is the official voting method used by the AP Poll
- **Multiple Outcomes Paradox:**
 - Multiple Top 5 rankings observed in 5 out of 16 weeks
 - Borda-5 and Borda-6 had the highest number of discrepancies
 - 23 different Top 5 rankings out of all 16 weeks
- **Existence of Cycles:**
 - **Total number of cycles in all 16 polls:** 303
 - **Average number of cycles per poll:** 19
 - **Maximum number of cycles:** 42 (Week 4)
 - **Minimum number of cycles:** 9 (Week 1 and 15)

Example of a Cycle



Week 10 AP Top 25 Poll (Nov. 1, 2015)

- **Official Ranking:**
 - 23. Temple
 - 24. Mississippi State University (MSU)
 - 25. Texas A&M (TAMU)
 - **Ranking:** Temple > TAMU > MSU
- **Pairwise Comparison:**
 - Temple beat TAMU by 6 points
 - TAMU beat MSU by 11 points
 - MSU beat Temple by 1 point
 - **Cycle:** Temple > TAMU > MSU > Temple

Example of Multiple Outcome Paradox

Week 4 AP Top 25 Poll (Sept. 20, 2015)

- Winner: Ohio State University
- The teams ranked second and lower depend on the voting method used:

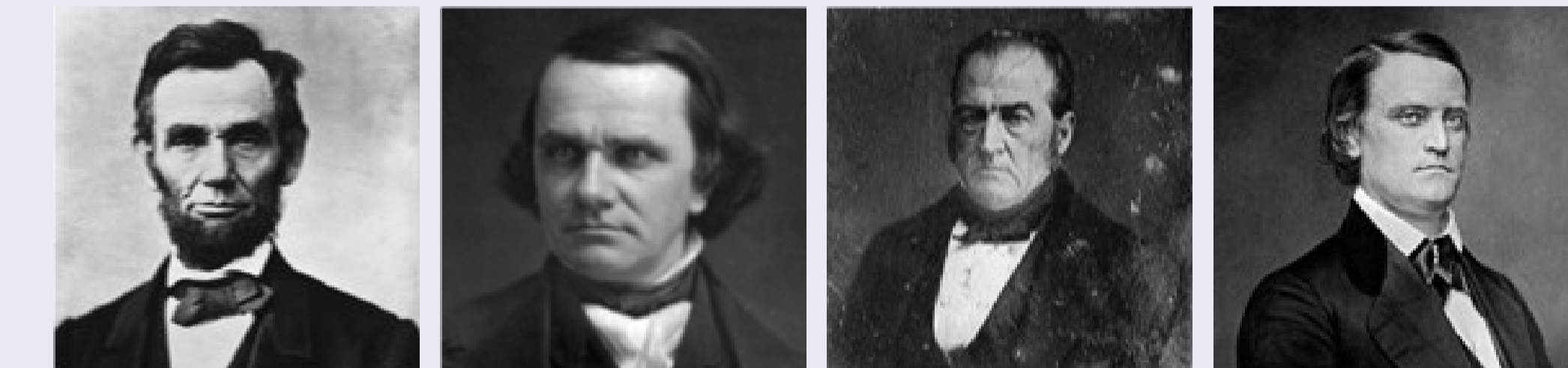
Method	Voting Weights	2nd	3rd	4th	5th
Vote for 1	(1, 0, ..., 0)	UMS	MSU	LSU	Air Force
Borda 3	(3, 2, ..., 0)	MSU	UMS	TCU	Georgia
Borda 5	(5, 4, ..., 0)	MSU	TCU	UMS	Baylor
Official Borda	(25, 24, ..., 1)	MSU	UMS	TCU	Baylor

MSU: Michigan State University
TCU: Texas Christian University
LSU: Louisiana State University
UMS: University of Mississippi



Lincoln's 1860 Election

Candidates in the 1860 US presidential election:



Lincoln Douglas Bell Breckinridge

- **Outcome of the Election:**
Lincoln won with 39.78% of the plurality vote, while the second place candidate, Douglas, received only 29.36%.
- **Mean Historian Profile:**
The mean of 13 prominent historians' estimates of the number of voters for each ranking of the 4 candidates.
- **Outcome under Different Election Methods:**
Using the mean historian profile, Tabarrok and Spector calculated the outcomes of the election under different voting methods.

Method	Voting Weights	Winner	2nd Place	3rd Place
Vote for 1	(1, 0, 0, 0)	Lincoln	Douglas	Breckinridge
Vote for 2	(1, 1, 0, 0)	Douglas	Bell	Lincoln
Vote for 3	(1, 1, 1, 0)	Douglas	Bell	Lincoln
Borda Count	(3, 2, 1, 0)	Douglas	Bell	Lincoln

The chart below shows the proportion of votes for each of the 4 candidates under different voting methods:

