COMPUTATIONAL ANALYSIS OF THERMO-ACOUSTIC INSTABILITIES IN COMBUSTION CHAMBERS AND AFTERBURNERS

BY

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THESIS

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A simplified model is introduced to study thermo-acoustic instabilities in asymmetric combustion chambers. Such instabilities can be triggered when correlations between heat-release and pressure oscillations exist, leading to undesirable effects. Gas turbine designs typically consist of a periodic assembly of \( N \) identical units; as evidenced by documented studies, the coupling across sectors may give rise to unstable modes, which are the highlight of this study. In the proposed model, the governing equations are linearized in the acoustic limit, with each burner modeled as a one-dimensional system, featuring acoustic damping and a compact heat source. The coupling between the burners is accounted for by solving the two-dimensional wave equation over an annular region, perpendicular to the burners, representing the chamber’s geometry. The discretization of these equations results in a set of coupled delay-differential equations, that depends on a finite set of parameters. Furthermore, \( N \)-periodic geometries commonly prone to such instabilities include annular combustion chamber and afterburner configurations, hence, apart from the effect of model parameters the effect of geometry on the overall stability of the system is considered in this article. The system’s periodicity is leveraged using a recently developed root-of-unity formalism (Schmid et al, 2015). This results in a linear system, which is then subjected to modal and non-modal analysis to explore the influence of the coupled behavior of the burners on the system’s stability and receptivity.
To my parents and sister.
ACKNOWLEDGMENTS

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<td>Partial differential equation</td>
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\( j \) \( j^{th} \) root of unity or \( j^{th} \) azimuthal boundary condition across a single sector of the \( N \)-periodic geometry.

\( \omega \) Non-dimensional angular forcing frequency.

\( R_j(\omega) \) Resolvent norm corresponding to a forcing frequency \( \omega \) and a root of unity \( j \).

\( u_z \) Axial velocity along the burner element.

\( p_z \) Pressure along the burner.

\( u_\theta \) Azimuthal velocity across the chamber.

\( u_r \) Radial velocity across the chamber.

\( p \) Pressure across the chamber; Also represents fluid pressure in the derivation of the wave equation equations.

\( \tau \) Time delay associated with heat transfer between the flame and fluid flow.

\( K \) Strength of the heat source associated with heat transfer between the flame and fluid flow.

\( L \) Length of the burners.

\( z_f \) Distance between the closed end of the burner and heater.

\( N \) Number of periodic units within a annular combustion chamber or afterburner geometry.

\( n \) Represents the number of degrees of freedom required to capture the flow in one unit.

\( \rho \) Density.

\( T \) Temperature.

\( \vec{U} \) Velocity.
s  Entropy.
c  Speed of sound.
M  Mach number associated with the fluid velocity.
t  Time.
γ  Ratio of the specific heat at constant pressure to that at constant volume.
q  Heat release per unit volume and unit time.
ρ'  Density perturbation.
U'  Velocity perturbation.
p'  Pressure perturbation.
s'  Entropy perturbation.
q'  Heat release perturbation per unit volume and unit time.
ρ  Mean density.
U  Mean velocity.
p  Mean pressure.
ρ*  Non-dimensional density.
T*  Non-dimensional temperature.
U*  Non-dimensional velocity.
Q*  Non-dimensional heat release perturbation per unit volume and unit time.
Lw  Length of the hot wire, in the flame model.
d  Diameter, in the flame model.
Tw  Temperature of the hot wire, in the flame model.
κ  Heat conductivity of the air surrounding the hot wire, in the flame model.
τ  Time lag between heat transfer and flow velocity as a result of thermal inertia, in the flame model.
ξ  Term representing the frequency dependent damping.
c_1 and c_2  Coefficients associated with the frequency dependent damping parameter.
\vec{e}_\theta and \vec{e}_r  Unit vectors in the radial and azimuthal directions.
$\nu$ Artificial viscosity applied to the chamber pressure.

$G_{eb}$ Coupling kernel that communicates the influence of the chamber onto the burner.

$G_{bc}$ Coupling kernel that communicates the influence of the burner onto the chamber.

$\vec{z}_0$ Represents the location at which the compact kernel $G_{eb}$ is positioned along the burner.

$\vec{r}_0$ Represents the location at which the compact kernel $G_{bc}$ is positioned along the chamber.

$\vec{z}_{max}$ Represents the radius of the compact kernel $G_{eb}$ is positioned along the burner.

$\vec{r}_{max}$ Represents the radius of the compact kernel $G_{bc}$ is positioned along the chamber.

A Block circulant matrix, representing the global dynamics.

$A_k$ Matrix representing individual unit dynamics.

$\rho_j$ The $j^{th}$ root of unity.

$\vec{x}$ Discretized global state vector.

$\vec{x}_j$ Discretized state vector which completely describes the flow in the $(j+1)^{th}$ unit.

D Represents the finite dimensional spatial differential operator describing the global dynamics of an $N-$periodic system.

$D_j$ Represents the finite dimensional spatial differential operator describing the dynamics of the eigenmodes corresponding to the $j^{th}$ root of unity.

M Represents the resolvent operator.

$M_j$ Represents the resolvent operator corresponding to the $j^{th}$ root of unity.

$\lambda$ Eigenvalues.

$\sigma$ Singular values.

U Left eigenvectors of the matrix $MM^*$.

S Diagonal matrix containing the resolvent operators singular values.

V Right eigenvectors of the matrix $M^*M$.

$\alpha$ Padé scheme coefficient.

$Ker_b$ Shape of the un-weighted kernel over the burner.

$Ker_c$ Shape of the un-weighted kernel over the chamber.
$\omega_b$ Kernel weights over the burner.

$\omega_c$ Kernel weights over the chamber.

$B_0$ Matrix without delay.

$B_1$ Matrix with delay.

$\vec{v}$ General representation of an eigenvector.

$C_N$ is the $N \times (N + 1)$ Chebyshev differentiation matrix, with all real entries.

$A_N$ Matrix used to obtain the time delayed eigenstructure of the system $(-\lambda I + B_0 + B_1 e^{-\tau \lambda}) \vec{v} = 0$.

$A_{left}^\pm$, $A_{left}^\pm$ Represent amplitude coefficients of the left and right running waves, respectively.

$R_{left}$, $R_{right}$ Represent reflection factors in the context of the analytical investigation of the acoustics within a single burner.

$k$ Non-dimensionalized wave number.

$J_n$ Bessel function of the first kind.

$Y_n$ Bessel function of the second kind.
Energy conversion devices, such as gas turbines, can exhibit large power densities accompanied by combustion instabilities ([1], [2]), formed as a result of a resonant feedback between the acoustic waves and flow dynamics ([3], [4], [5]), which ultimately results in unwanted high pressure and heat release oscillations.

The driving mechanism behind such thermo-acoustic instabilities was recognized in the nineteenth century by Lord Rayleigh [6], who observed that if pressure and heat release fluctuations were in phase, the oscillations were enhanced. This principle in relation to combustion instabilities can be seen as an unsteady flow interacting with a fluctuating thermal source, potentially caused by turbulent combustion, resulting in an unsteady heat release. The unsteady heat release can in turn act as an acoustic source in the system, resulting in self-sustained oscillations [7].

Thermo-acoustic instabilities severely affect the life and performance of energy conversion devices, especially true in the context of high-performance engines ([8], [9]), whose $NO_x$ emissions are governed by stringent environmental regulations. Formation of $NO_x$ has an exponential dependence on the (local) temperature in the combustion zone, thus motivating an uniform temperature in the reaction zone in order to achieve the required low emission levels. This uniform reaction zone temperature shifts the operating conditions towards the lean-burn parameter range. In addition, the absence of by-pass air in convectively cooled combustion systems results in a decrease of acoustic damping and an increase in pulsation amplitudes, which can have devastating effects over the structural integrity of the energy conversion devices. Hence, a combination of lean-burn parameters and convectively cooled combustion chambers (low emission design) has made the energy conversion device more prone to combustion driven oscillations, or thermo-acoustic instabilities. Furthermore,
article [10] shows that thermo-acoustic phenomenon are inherently non-normal. Thus fueling the design and study of models that can predict the stability and receptivity of such self-sustained oscillations ([11], [1], [2]).

The possible approaches for the study of combustion instabilities range from theoretical models, through low-order network methods, to full scale Large Eddy Simulations. The cost and complexity of each modeling approach decreases with an increase in the level of modeling required. Full scale experimental work in this field is difficult ([12], [13], [14]). Therefore, simplified configurations are used to study these instabilities both in the longitudinal and azimuthal direction. Studies performed by [15], [16] and [17] have investigated longitudinal combustion instabilities. More recently azimuthal modes in the simplified annular Rijke tube configuration, using heating grids as the unsteady heat source, have also been studied (Moeck et al. [18] and Gelbert et al.[19]). Dawson et al. [20] and Worth et al. [21] have used an annular configuration with swirled premixed flames to study the effects of the interaction between flames and mean swirl on the stability and nature of azimuthal modes. In [22] an annular combustion chamber is specifically designed to avoid complexities that hinder the study of thermo-acoustic phenomenon. Hence, this chamber configuration is ideal for a systematic investigation of the thermo-acoustic oscillations coupled either by longitudinal or azimuthal modes, and will be used in chapter 5 of this study to confirm the results of the modal analysis. Although experimental investigations can capture the entire physics, the cost and effort required to understand and control these instabilities can become prohibitive ([4], [23], [24]).

Three-dimensional numerical simulations, such as Large Eddy Simulations (LES), are capable of reproducing natural combustion instabilities in complex geometries, by taking into account every mechanism involved in the combustion-acoustic coupling. Recent studies such as [25], [26] have successfully simulated azimuthal instabilities in real gas turbine combustion chambers, using LES. Although computationally intensive LES type simulations can reproduce natural combustion instabilities, they are not sufficient to understand and control the unstable modes that manifest in energy conversion devices ([4], [23], [24]), hence, low-order models and theory on simplified geometries are needed to guide such computationally intensive simulations or experimental investigations.
Theoretical methods have been used to investigate the underlying phenomenon governing the mutual interaction between combustion and acoustics, leading to thermo-acoustic instabilities ([27], [4], [28], [29]). However, these methods require a certain number of hypothesis to be satisfied in order to simplify the problem, hence only highly simplified configurations can be studied.

Finally, low-order network models are used in order to investigate complex geometries (where theoretical studies fail and, experimental and numerically intensive methods become expensive) by decomposing the real geometry into a set of constant density lumped acoustic elements and compact heat sources. Each of these elements is modeled by means of a linear transfer function matrix, which can be solved analytically ([29], [30], [31]). In recent studies such as [32] a semi-analytical approach using a 1D network model has been proposed for the study of azimuthal modes in a one-ring configuration. This study has been extended to a 2D network for the study of two-ring configurations in [33]. Such studies obtain fast responses and provide a phenomenological interpretation. However, the main drawback is the low level of geometric complexity that can be taken into account when using these models. Helmholtz solvers are a suitable alternative to the network models when it is essential to take into account the detailed effects such as a spatially distributed flame response, especially when these features are not acoustically compact. In a recent study [34], a thermo-acoustic model based on the Helmholtz equation is used to exploit the discrete rotational symmetry common to most energy conversion devices, by recognizing that such a model admits special solutions of the so-called Bloch type. These solutions are obtained by considering a single representative segment, with the appropriate Bloch-type boundary conditions, of the geometry and hence significantly reduce the computational effort involved in determining the eigenstructure of such systems. However, a break in the rotational symmetry of the azimuthal modal solutions, caused for example by amplitude dependent flame dynamics, results in the loss of the so called Bloch-wave structure.

Thermo-acoustic instabilities result from the coupling between unsteady heat release and the lowest natural acoustic eigenmodes of the configuration [35]. Since the radial and longitudinal dimensions of most combustion chambers are shorter than their azimuthal dimension, the lowest frequency modes correspond to the azimuthal modes [36]. Therefore, in light of
the inherently non-normal and azimuthal nature of the unstable modes of $N$–periodic energy conversion devices, this article chooses to focus on the manner in which unsteady pressure in the chamber (azimuthal modes) can lead to unsteady flow rates within the burners, in the linear limit, and thus couple the burner elements of an $N$–periodic energy conversion device. In order to model this phenomenon we consider a network model approach accompanied by the roots of unity formalism [37] which exploits the $N$–periodic nature of the problem, similar to [34], in order to efficiently compute the eigenstructure and features arising from their superposition (non-normal behavior). The block circulant structure of the entire $N$–component geometry is recognized by the roots of unity formalism, which then reduces it to a modified single-unit system, thus achieving the computational cost of a isolated-unit periodic analyses while correctly modeling the full interaction with the $N−1$ subcomponents. Furthermore, as described in [37], features arising from the superposition of modal solutions can exhibit non-normal behavior only as a result of sub-unit (with appropriate root of unity type boundary conditions) dynamics, allowing the receptivity of the global $N$–periodic energy conversion device to be analyzed as the superposition of $N$–local (smaller-scale) problems. This approach ensures computational efficiency in both modal and non-modal analysis.

The method employed in this article employs the network model approach accompanied by the roots of unity formalism, offering a framework that can efficiently ([37]) analyze the stability and receptivity of an $N$–periodic energy conversion device in the linear limit. Thus providing a flexibility, lacking in computationally intensive simulations and experimental investigations, required to understand and control the inherently non-normal thermo-acoustic instabilities that develop in such $N$–periodic energy conversion devices.
CHAPTER 2

PHYSICAL MODEL

One-dimensional ducts with a compact heat source (Rijke tube model) are used as a convenient framework for studying the fundamental principles of thermo-acoustic instabilities [38], [4] and [7]. The specific situation where the geometry of $N$–periodic energy conversion devices (such as gas turbines) lead to azimuthal, coupled, thermo-acoustic instabilities requires a model that can account for both the dynamics of a single burner and the instabilities sustained due to azimuthal coupling of the burner elements. These instabilities occur in many gas turbines [2] and are often the strongest instabilities [29] in such configurations.

In order to study these non-normal, azimuthal, thermo-acoustic instabilities in such $N$–periodic energy conversion devices, this article utilizes a model consisting of vertical one-dimensional ducts, with a compact heat source representing the burner elements, coupled together by a two-dimensional annulus/disc representing the combustor/afterburner. The following sections describe these configurations in greater detail followed by a derivation of the linear delayed differential equations that model the dynamics of the flame.

2.1 Model Configuration

In this article, annular combustion chamber and afterburner configurations are used to study the azimuthal thermo-acoustic instabilities in $N$–periodic energy conversion devices. A detailed description of these configurations is provided in the rest of this section.

An example of a gas turbine combustion chamber, in which azimuthal thermo-acoustic instabilities appear, is shown in the figure 2.1a. The study of thermo-acoustic instabilities in such an engineering setup is challenging, owing to the interactions among multiple aspects of the system’s dynamics such as turbulence, separation and combustion. This motivates a
Figure 2.1: Energy conversion device configurations.

study of a setup (also used in experimental studies) specifically designed to avoid any complexities associated with swirling turbulent flames and, facilitate a systematic investigation of thermo-acoustic oscillations, coupled together by azimuthal modes, in such \( N \)-periodic configurations (figure 2.1b). The experimental setup described in [22] have multiple flames formed by matrix injectors having laminar and have well-documented describing functions, thus the full configuration can be considered as a periodic assembly of \( N \) identical sectors.

Leveraging this \( N \)-periodicity, by using the roots of unity formalism [37], as described in section 2.3.2, the globally periodic eigenmodes are numerically computed using a single unit sector of an entire configuration. A schematic of such a unit and its boundary conditions are presented in figures 2.2 and 2.3 for the annular combustor and afterburner configurations. In each schematic, modeled after the setup in [22], glass walls of the chamber are represented by radial wall boundary conditions while the azimuthal boundary conditions are accounted for by the roots of unity formalism (section 3.0.6). Along the vertical direction of each schematic, the top boundary is modeled as being open to the atmosphere whilst the bottom boundary is modeled as another wall. The experimental setup in article [22] features a
plenum, which couples the burners from the bottom, however, this has not been accounted for in the current article. Finally, an $n - \tau$ model is used to represent the flame within each burner element.

![Figure 2.2: Annular Combustion Chamber Configuration](image)

![Figure 2.3: Afterburner Configuration](image)

The annular chamber and afterburner configurations, in figure 2.2 and 2.3, are further simplified by representing the three-dimensional chamber as a two-dimensional annulus/disc and the burner elements as one-dimensional vertical ducts, each with a compact heat source in figure 2.4 and 2.5. This heat source is modeled as a simple electric-mesh heater ([39], [40]) that transfers heat to the surrounding flow field according to the Heckl [39] modified Kings law [41]. The burner and chamber elements are coupled together using a compact coupling
kernel, along the burner in the region entirely above the heating element, and across the geometric center of each sector of the 2D chamber (the extent of each 2D kernel represents the size of the matrix injectors as shown in figure 2.1b).

Figure 2.4: 3D schematic to Coupled 1D burner to 2D annular chamber schematic

Figure 2.5: 3D schematic to Coupled 1D burner to 2D afterburner schematic

2.2 Governing Equations

The governing equations derived in the following sections aim to capture the coupled thermo-acoustic behavior of $N$—periodically placed burner elements in both the afterburner and annular combustion chamber configurations described in figures 2.4 and 2.5.
Convective wave equation governing the acoustic wave propagation

Consider the equations describing the conservation of mass, momentum and energy along with the state equation of an inviscid fluid, in the absence of any source terms:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0, \]  
\[ \rho \left[ \frac{\partial \vec{U}}{\partial t} + \vec{U} \cdot \nabla \vec{U} \right] + \nabla p = 0, \]  
\[ \rho T \left[ \frac{\partial s}{\partial t} + \vec{U} \cdot \nabla s \right] = 0 \]  
\[ p = p(\rho, s) \]

As mentioned, equations 2.1, 2.2 and 2.3 describe a fluid without mass addition, external body forces, viscous dissipation and heat input. Although these assumptions do not directly apply to a full scale annular combustion chamber or afterburner, a model using these simplified equations ([42], [5], [43]) have been shown to capture the overall dynamical behavior of such instabilities. As a result, in this work we also consider the entire system consisting of an array of linearly coupled sub-systems governed by the above set of equations. This approach, known as the network method, allows the combustion and local dissipation processes to be represented along interfaces separating the fresh mixture and high temperature reaction products, hence incorporating the seemingly omitted source terms. As a result of this modeling approach, the acoustic field on either side of the interface need not contain any source terms. The equations 2.5 to 2.9 are now linearized, in the limit of small amplitude perturbations, to arrive at the standard wave equation. The jump relations across the interfacial sub-systems, representing the combustion and dissipative phenomenon, further modify these wave equations in order to arrive at a description of the thermo-acoustic behavior in such \( N \)-periodic energy conversion systems (figures 2.2 and 2.3).

The procedure is further simplified by considering a zero base flow, which is valid in the
context of thermo-acoustic modeling [38], [4] and [7]. Thus pressure \( p = \bar{p} + p' = \rho' \), density \( \rho = \bar{\rho} + \rho' = \rho' \) and velocity \( \bar{U} = \bar{U} + \bar{U}' = \bar{U}' \), expressed as a sum of their mean and fluctuating parts, are substituted into equations 2.1 and 2.2 while retaining only the first order terms to yield:

\[
\frac{\partial \rho'}{\partial t} + \bar{p} \nabla \cdot (\bar{U}') = 0, \tag{2.5}
\]

\[
\frac{\partial \bar{U}'}{\partial t} + \nabla p' = 0 \tag{2.6}
\]

In this study, acoustic behavior is being investigated in the vicinity of a stationary, zero base flow, solution on the system’s phase space. Therefore the mean density and speed of sound are constants with respect to time. The spatial variation of these terms is also neglected, resulting in a description of the acoustics over the components of the system’s network free of heat sources or dissipative phenomenon, and close to a zero base flow condition. The interfaces separating any two such components hold jump relations that describe the spatial gradients of the neglected heat source or dissipative phenomenon. By taking these facts into consideration the time derivate of the acoustic continuity equation 2.5 and the spatial derivative of the momentum equations 2.6 are given as follows:

\[
\frac{\partial^2 \rho'}{\partial t^2} + \bar{p} \nabla \cdot (\frac{\partial \bar{U}'}{\partial t}) = 0, \tag{2.7}
\]

\[
\frac{\partial \nabla \cdot \bar{U}'}{\partial t} + \nabla^2 p' = 0, \tag{2.8}
\]

Also, the equation of state is expressed as:

\[
\frac{Dp}{Dt} = \left( \frac{\partial p}{\partial \rho} \right)_s \frac{D\rho}{Dt} + \left( \frac{\partial p}{\partial s} \right)_\rho \frac{Ds}{Dt} \tag{2.9}
\]

where, under isentropic conditions (no dissipation) \( \left( \frac{\partial p}{\partial \rho} \right)_s = c^2 \), \( c \) being the speed of sound, equation 2.9 becomes:
\[
\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt},
\]  

(2.10)

In linearized form this equation becomes:

\[
\frac{\partial p'}{\partial t} = c^2 \frac{\partial \rho'}{\partial t}.
\]  

(2.11)

Now, subtracting equation 2.8 from 2.7, and utilizing equation 2.5 yields the following form of the wave equation:

\[
\frac{\partial^2 p'}{\partial t^2} - c^2 \nabla^2 p' = 0.
\]  

(2.12)

2.2.1 First order non-dimensional form of the wave equation

Equation 2.12 is a second order, linear partial differential equation which can be simplified to a first order form as follows:

\[
\bar{p} \frac{\partial \vec{U}'}{\partial t} + \nabla p' = 0
\]  

(2.13)

\[
\frac{\partial p'}{\partial t} + c^2 \bar{p} \nabla \cdot \vec{U}' = 0
\]  

(2.14)

Further, non-dimensionalizing the above equation with respect to the mean pressure, density and velocity of the system yields:

\[
\begin{cases}
\frac{\partial \vec{U}^*}{\partial t} + \frac{1}{\gamma M} \nabla p^* = 0, \\
\frac{\partial p^*}{\partial t} + \gamma M \nabla \cdot \vec{U}^* = 0
\end{cases}
\]  

(2.15)

where the terms with a * represent the non-dimensionalized perturbation terms and \(M\) is the Mach number corresponding to the acoustic scaling of the velocity.
2.2.2 Flame Model

The equation of state given by equation 2.11 is no longer valid when heat is added to the system due to the presence of the flame, instead the non-isentropic linearized form of the equation of state given below is used

\[
\frac{\partial p'}{\partial t} = c^2 \frac{\partial \rho'}{\partial t} + (\gamma - 1)q',
\]

(2.16)

where, the term \(q' = \rho T \frac{\partial s'}{\partial t}\)

Subtracting equation 2.8 from 2.7 yields:

\[
\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = 0,
\]

(2.17)

Now, substituting equation 2.16 in the above equation yields:

\[
\frac{\partial^2 p'}{\partial t^2} - (\gamma - 1) \frac{\partial q'}{\partial t} - c^2 \nabla^2 p' = 0.
\]

(2.18)

This equation can then be re-written in the form of equations 2.15 as follows:

\[
\frac{\partial \tilde{U}^*}{\partial t} + \frac{1}{\gamma M} \nabla p^* = 0,
\]

(2.19)

\[
\frac{\partial p^*}{\partial t} + \gamma M \nabla \cdot \tilde{U}^* - \gamma M Q^* = 0,
\]

(2.20)

where the terms with a * represent the non-dimensionalized perturbation terms, \(q'\) is the heat release perturbation per unit volume per unit time and \(Q^* = \frac{(\gamma - 1)Lq'}{\rho c^2 U}\).

The model adopted for the term \(Q^*\) is a modified version of the non-linear model employing King’s law [41] as proposed by Balasubramanian and Sujith [10]. This model was used in [7] and is briefly described here for convenience. The modifications reflect the observations of Heckl [39] who found that non-linear behavior begins at velocity fluctuations of approximately one third the mean velocity, resulting in
\[
\bar{Q} + Q' = L_w (T_w - T) \left[ \kappa + 2 \sqrt{\pi \kappa c_v \rho d} \left( 1 - \frac{1}{3\sqrt{3}} \right) \sqrt{u_z} + \frac{1}{\sqrt{3}} \sqrt{\frac{u_z}{3} + u_z(t - \tau)} \right]
\] (2.21)

where, \(L_w\) and \(d\) represent the length and diameter of the hot-wire. \(T_w\) and \(T\) represent the wire and surrounding air temperatures, the term \(\kappa\) is the heat conductivity of air and \(c_v\) is the specific heat of air per unit mass at constant volume. \(\tau\) represents the time lag between the heat transfer and the flow velocity as a result of thermal inertia. The equation representing the heat release perturbations per unit volume per unit time in non-dimensional form is given as

\[
Q = Q_f (t - \tau) \delta(z - z_f) = \frac{K}{2} \left[ \sqrt{\frac{1}{3} + (u_z)_{zf}(t - \tau)} - \sqrt{\frac{1}{3}} \right] \delta(z - z_f)
\] (2.22)

where \(\delta\) denotes the standard Dirac distribution and \(K\) is the non-dimensional strength of the heater. Usually, \((u_z)_{zf}\) is taken to be the velocity value on the cold side \((z \leq z_f)\) of the heater. In this manner, the system of equations is closed.

### 2.2.3 Frequency Dependent Damping

The wave equation is further modified to incorporate dissipative phenomena into the system. Dissipation may be introduced locally to model the finite hydrodynamic region in the vicinity of the heater (Heckl and Howe 2007), or globally to include end losses and viscous effects associated with the boundary layer, for example. Although the growth of the latter has been neglected, we nonetheless include a frequency-dependent term in the energy equation, finally leading to

\[
\begin{aligned}
\frac{\partial \bar{U}}{\partial t} + \frac{1}{\gamma M} \nabla p &= 0, \\
\frac{\partial p}{\partial t} + \frac{\gamma M}{\rho} \nabla \cdot \bar{U} - \gamma MQ + \xi \ast p &= 0
\end{aligned}
\] (2.23)

The non-dimensional terms in the above equations are no longer indicated with the help of an Astrix, and the \(*\) symbol represents the convolution operator that captures the frequency
dependent damping, that accounts for the acoustic losses in the form of imperfect reflections off of the system boundaries and the boundary layer losses. The $\xi$ term is represented according to Sterling and Zukoski (1991) and Matveev and Culick (2003a) by

$$\xi_j = c_1 j^2 + c_2 \sqrt{j}$$ (2.24)

where $j$ represents the wavenumber and parameters $c_1 = 0.1$ and $c_2 = 0.06$ ([44], [45]) are kept constant.

2.2.4 Coupled Chamber-Burner Governing Equations

The complete set of governing equations representing thermo-acoustic phenomenon within annular combustion chambers or afterburners, as described in section 2.1, is arrived at by coupling (using kernels) a modified form of the wave equation (eq: 2.23) in the burner configuration, with the standard wave equations (eq: 2.15) inside the 2D annulus, allowing the interactions of the burners through the annular system.

These kernels are used to impose a weighted average of the velocity gradients across the burner and chamber elements and thus coupling the two sections. Figure 2.6 details the structure and location of the kernels over their respective elements. This coupling is pictorially illustrated through figure 2.6a which shows that the weighted gradient $\left( \nabla \cdot (u_\theta \vec{e}_\theta + u_r \vec{e}_r) \right)_c$ (computed using kernel $G_{bc}$) is distributed over the burner using the kernel $G_{cb}$ and similarly in 2.6b the weighted gradient $\left( \nabla \cdot \vec{e}_z \right)_b$ (computed using kernel $G_{cb}$) is distributed over the annulus using kernel $G_{bc}$.

The mathematical representation of the kernels and the manner in which they influence equations 2.23 and 2.15 are as shown below:
\begin{align}
\frac{\partial u_z}{\partial t} + \frac{1}{\gamma M} \frac{\partial p_z}{\partial z} &= 0, \\
\frac{\partial p_z}{\partial t} + \gamma M \frac{\partial u_z}{\partial z} + \gamma M G_{cb}(\vec{z}, \vec{z}_0, z_{max}) \left( \nabla \cdot (u_\theta \vec{e}_\theta + u_r \vec{e}_r) \right)_c + \xi * p_z - \gamma MQ &= 0, \\
\frac{\partial u_\theta}{\partial t} + \frac{1}{\gamma M} \frac{\partial p}{\partial r} &= 0, \\
\frac{\partial u_r}{\partial t} + \frac{1}{\gamma M} \frac{\partial p}{\partial \theta} &= 0, \\
\frac{\partial p}{\partial t} + \gamma M \frac{\partial u_z}{\partial r} + \gamma M \frac{\partial u_\theta}{\partial \theta} + \gamma M G_{bc}(\vec{r}, \vec{r}_0, r_{max}) \left( \nabla \cdot u_z \vec{e}_z \right)_b - \nu \left( \nabla^2 p \right)_c &= 0,
\end{align}

(2.25)

where, terms $p_z, u_z, u_\theta, u_r$ and $p$ represent the pressure and velocity along the burner, the azimuthal and radial velocities of the chamber and the chamber pressure, respectively. Terms $G_{cb}$ and $G_{bc}$ represent functions of the kernels that communicate velocity gradient information from the chamber to the burner and, from the burner to the chamber respectively, and the terms $\vec{z}_0$ and $\vec{r}_0$ represent the locations at which the compact kernels are positioned along the burner and in the chamber. Terms $\left( \nabla \cdot (u_\theta \vec{e}_\theta + u_r \vec{e}_r) \right)_c$ and $\left( \nabla \cdot u_z \vec{e}_z \right)_b$ denote the weighted average of the chamber velocity gradient surrounding $\vec{r}_0$ within a radius of $r_{max}$ and the weighted average of the burner velocity gradient surrounding $\vec{z}_0$ within a radius of $z_{max}$. The term $\nu$ is the coefficient of artificial viscosity applied to the chamber pressure, in order to suppress the influence of high frequency oscillations (figure 2.6b). The
full numerical implementation of this coupling is discussed in chapter 3.

2.3 Global Dynamics From Single Periodic Unit (Root of unity formalism)

The aim of this study is to investigate the linear stability of a thermo-acoustic system formed by the coupling of multiple burner elements, through a chamber geometry similar to an annular combustion chamber or an afterburner. The inherent $N$-periodic structure of such systems prompts an analysis of a single periodic unit, by assuming periodic azimuthal boundary conditions. However, this analysis fails to capture modal solutions that are, globally periodic and yet, non-periodic over the single periodic representative unit. Without
Figure 2.8: The azimuthal component of the eigenmodes corresponding to each of the three roots of unity.

utilizing the roots of unity formalism the only alternative is to perform a global analysis by considering the entire $N$–unit geometry. This global analysis can quickly become computationally prohibitive, owing to the $O((N \times n)^3)$ nature of the system (where $n$ represents the degrees of freedom required to capture the flow in one unit). The following subsections detail the manner in which the roots of unity formalism helps reduce the computational cost in analyzing the modal and non-modal behavior of such $N$–periodic elements.

2.3.1 Modal analysis

The novel roots of unity formalism (Schmid et al, 2015) can be utilized to obtain the globally periodic eigenmodes of the system from a single unit of the geometry, by subjecting this representative unit to $N$ different azimuthal boundary conditions, and capturing all the eigenmodes of the $N$–periodic system at $\frac{1}{N^2}$ the computational cost. In figure 2.8 the first column, corresponding to a root of unity $j = 0$, represent global solutions corresponding to a periodic boundary conditions across a single unit (indicated by the yellow area). Whilst, rest of the columns represent modes that are not periodic over a single unit and would hence be impossible to describe using purely periodic boundary conditions.
To illustrate the formalism an \((N = 3)\)--periodic system is considered, as shown in figure 2.7b. Figure 2.7a indicates the three different complex values of the parameter \(\rho_j\), that represent the \(N\) different azimuthal boundary conditions, applied across (azimuthal direction) the single unit sector in order to arrive at the systems globally periodic eigenmodes. Furthermore, the figure 2.8, representing the globally periodic eigenmodes of the \((N = 3)\)-periodic 2D annular chamber, are computed numerically at a computational cost of the order \(O(N \ast n^3)\). However, the same globally periodic eigenmodes could be constructed by considering the entire \(N\)-unit geometry at a computational cost of the order \(O((N \ast n)^3)\), hence, in this example the computational cost is reduced by a factor of \(\frac{1}{N^2} = \frac{1}{9}\) by employing the roots of unity formalism. In figure 2.8, \(j = 0, 1, ..., N - 1\) signifies the different roots of unity. The complex term \(\rho_j\), that represents the azimuthal boundary condition across a single unit, can also be interpreted as the phase difference between adjacent periodic units. This facilitates the construction of the global eigenmodes, seen in figure 2.8, from the eigenmodes over a single periodic unit.

The mathematical representation of this procedure, as described in [37], is illustrated in the rest of this section. Consider a linearized fluid system consisting of \(N\) identical units connected by appropriate boundary conditions. Also, let the discretized state vector that completely describes the flow in the \(k\)th unit be represented by \(\vec{x}_k\). Then, the discrete dynamical system representation of this system is given by:

\[
\frac{\partial}{\partial t} \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \\ . \\ . \\ \vec{x}_N \end{bmatrix} = \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \\ . \\ . \\ \vec{x}_N \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & \cdots & A_N \\ A_N & A_1 & \cdots & A_{N-1} \\ & A_2 & \cdots & A_1 \end{bmatrix} \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \\ . \\ . \\ \vec{x}_N \end{bmatrix}
\]

(2.26)

In the above matrix equation, the global matrix \(A\) is block circulant, on account of the \(N\)--periodic configuration, where \(A_k \in \mathbb{C}^{n \times n}, k = 1, ..., N\), denoting the dynamics of an individual unit.

In order to assess the modal stability of the global matrix \(A\), its eigenvalues and eigenvectors need to be computed. Therefore the global eigenvector of a matrix \(A\) corresponding
to the \((j + 1)^{th}\) azimuthal boundary condition can be formulated as:

\[
\tilde{x} = \begin{bmatrix}
\vec{v}_j \\
\rho_j \ast \vec{v}_j \\
\rho_j^2 \ast \vec{v}_j \\
\vdots \\
\rho_j^N \ast \vec{v}_j
\end{bmatrix}
\] (2.27)

where \(\tilde{x}\) is an eigenvector of \(A\) i.e. \((A \tilde{x} = \lambda \tilde{x})\) and \(\vec{v}_j\) satisfies the eigenvalue problem

\[
D_j \vec{v}_j = \lambda \vec{v}_j
\] (2.28)

where, \(D_j\) represents the finite dimensional spatial differential operator given by the following summation of the individual dynamic matrices \(A_{j+1} \in \mathbb{C}^{n \times n}\) (\(n = \text{number of degrees of freedom per sector}\)) of each unit sector with respect to the single periodic unit sector under consideration:

\[
D_j = A_1 + \rho_j A_2 + \rho_j^2 A_3 + \ldots + \rho_j^{N-1} A_N
\] (2.29)

where, \(\rho_j = e^{(ij \frac{2\pi}{N})}\).

Therefore, the modal analysis of the global matrix \((A)\) is computationally simplified from an \(O((N \ast n)^3)\) problem to an \(O(N \ast n^3)\) problem, by solving for the eigenmodes of the finite dimensional form of the dynamical system (eq: 2.25), over an individual-periodic-unit and for each root of unity \(j = 0, 1, \ldots, N - 1\), given by:
where $\rho_j \ast [u_{z,1}, p_{z,1}, u_{\theta,1}, u_{r,1}, p_1 \ldots u_{z,n}, p_{z,n}, u_{\theta,n}, u_{r,n}, p_n]^T = \vec{x}_j$ is the discretized state vector which completely describes the flow in the $(j + 1)^{th}$ unit.

### 2.3.2 Non-modal analysis

Thermo-acoustic systems are inherently non-normal [46]. Hence a system with asymptotically stable eigenvalues can display unstable behavior when subject to small-amplitude harmonic perturbations. This section describes the most sensitive/receptive eigenmodes of the system by quantifying this non-normal behavior of the long-time response, also called the resolvent norm [47].

Therefore, the resolvent operator, and hence resolvent norm, of the system is computed by considering a general harmonically forced problem at frequency $\omega [47]$:  

$$\frac{\partial \vec{x}}{\partial t} = \vec{D} \vec{x} + \vec{x}_f e^{i\omega t}$$  

(2.31)

where, $\omega \in R$ and $\vec{x}$ represents the variables of the system, $\vec{D}$ represents the finite dimensional differential global operator matrix, that governs the acoustic behavior of the inherently non-normal ([48]) $N$-periodic thermoacoustic systems under investigation and,
\( \vec{x}_f e^{i\omega t} \) represents the harmonic perturbation/forcing. The general solution to this problem consists of a transient and long-time solution, given as

\[
\vec{x}(t) = e^{t\mathbf{D}}\vec{x}_0 - (\mathbf{D} - i\omega\mathbf{I})^{-1} \vec{x}_f e^{i\omega t} \quad (2.32)
\]

The quantity \(( -i\mathbf{D} - \omega\mathbf{I})^{-1} \) is known as the resolvent. This quantity is the maximum possible response that a linear system described by \( \mathbf{D} \) can exhibit, subject to the forcing frequency \( \omega \) and, a measure of its norm is given by the following expression

\[
R(\omega) = \| (-i\mathbf{D} - \omega\mathbf{I})^{-1} \| = \| F \text{diag} \left( \frac{1}{\lambda_1 - \omega}, \ldots, \frac{1}{\lambda_{n*N} - \omega} \right) F^{-1} \| \quad (2.33)
\]

The norm described by the above equation is most conveniently calculated as the maximum singular value of the expression

\[
R(\omega) = \sigma_1 \left\{ F \text{diag} \left( \frac{1}{\lambda_1 - \omega}, \ldots, \frac{1}{\lambda_{n*N} - \omega} \right) F^{-1} \right\} \quad (2.34)
\]

where, \( R(\omega) \) represents the norm of the maximum response for a forcing frequency \( \omega \).

Furthermore, a singular value decomposition of the resolvent operator \( \mathbf{M} = ( -i\mathbf{D} - \omega\mathbf{I})^{-1} \) yields two sets of orthogonal basis vectors \( \mathbf{U} \) and \( \mathbf{V} \) which are the left and right eigenvectors of the matrices \( \mathbf{MM}^* \) and \( \mathbf{M}^*\mathbf{M} \), respectively. These, orthogonal basis span the domain (\( \mathbf{U} \)) and range (\( \mathbf{V} \)) of all perturbation that the system may be subject to, and the magnitude of their response (receptivity) is again captured by the corresponding singular values \( \text{diag}(\mathbf{S}) \), where \( \mathbf{USV}^T = \mathbf{M} \).

Now, that we have a qualitative measure of a system’s non-normal response and the structure of \( q_f \) corresponding to the most responsive disturbances, we can apply this analysis to our \( N \)-periodic system with the help of the roots of unity formalism. We should note that there exists we identify the two types of non-normal behavior in the context of \( N \)-periodic systems, (i) the superimposition of modal solutions within a single sub-unit, governed by the eigenstructure of \( D_0 \), and (ii) the superposition of modal solutions from all sub-units,
taking into account the eigenstructures of $D_j, j = 0, 1, \ldots, N - 1$. The latter case is of particular interest, as it may described trans-unit dynamics which is absent in the former case. It is important to recall that circulant matrices are normal and thus have orthogonal eigenvectors. Consequently, the matrix exponential norm is fully governed by the least stable eigenvalue. For block-circulant matrices, as in our case, transient effects can arise due to the non-normality of each individual sub-unit dynamics, but no other transient effects arise from the superposition across units as the eigenvectors of block-circulant matrices are block-orthogonal. This latter statement can easily be verified by forming the scalar product of two eigenvectors of the form 2.27. We assume that the two eigenvectors stem from two different values of $\rho$, which we take as $\rho_j$ and $\rho_k$ with $j \neq k$. Denoting by $\vec{v}_j$ an eigenvector of $D_j$ and by $\vec{v}_k$ an eigenvector of $D_k$ we arrive at

$$
\vec{x}^H \vec{x} = \begin{bmatrix}
\vec{v}_j \\
\rho_j * \vec{v}_j \\
\rho_j^2 * \vec{v}_j \\
\vdots \\
\rho_j^N * \vec{v}_j
\end{bmatrix}
H
\begin{bmatrix}
\vec{v}_k \\
\rho_k * \vec{v}_k \\
\rho_k^2 * \vec{v}_k \\
\vdots \\
\rho_k^N * \vec{v}_k
\end{bmatrix}
= (1 + \eta + \eta^2 + \ldots + \eta^{N-1}) \vec{v}_j^H \vec{v}_k = \frac{1 - \eta^N}{1 - \eta} \vec{v}_j^H \vec{v}_k
$$

(2.35)

with $\eta = \exp(i(k - j) \frac{2\pi}{N})$. Recalling that $\eta^N = 1$ for $j \neq k$, we conclude that two global eigenvectors corresponding to two different roots-of-unity are mutually orthogonal. This eigenvector structure allows the treatment of the global (potentially large-scale) problem as a superposition of $N$ local (smaller-scale) problems. Therefore a complete description of the systems non-normal behavior can be obtained by subjecting the resolvent operators $M_j$, corresponding to each root of unity, to a singular value decomposition, in order to obtain the system’s most responsive disturbances. The resolvent and resolvent norm corresponding to each root of unity is given below:

- Resolvent corresponding to the root of unity with index $j$
\[ M_j = (-iD_j - \omega I)^{-1} \]  \hspace{1cm} (2.36)

- Resolvent norm corresponding to the root of unity with index \( j \)

\[ R_j(\omega) = \sigma_{j,1} \left\{ F_j \text{ diag}\left( \frac{1}{\lambda_{j,1} - \omega}, ..., \frac{1}{\lambda_{j,n} - \omega} \right) F_j^{-1} \right\} \]  \hspace{1cm} (2.37)
This section describes the spatial discretization scheme used to represent the governing equations in section 2.23, and the manner in which the partial differential equation formulation of delay differential equations (DDE) is used to approximate the eigenvalues and eigenvectors of these equations in the limit of small-amplitude disturbances. Finally, a description of the modal and non-modal analysis is provided.

3.1 Finite Difference Formulation

The spatial discretization scheme used to represent the differential equations describing the thermo-acoustic behavior within the burner and chamber elements of an $N$–periodic energy conversion device are described in the following subsections.

3.1.1 Discretized Equations: Burner

A staggered arrangement of the variables is used to provide a numerical approximation of the continuous burner system (with velocities and pressure nodes on the cell centers and cell faces, respectively). The burner elements in these systems have closed and open boundaries. The closed boundary is represented by a Dirichlet boundary condition for the velocity and a Neumann boundary condition for the pressure. The conditions are reversed for the open boundary. The spatial derivatives along the burner are represented using a fourth order compact Padé approximation in order to limit the dispersion errors detrimental to the propagation of acoustic waves. Furthermore, to achieve a first-order accuracy across the singularity, the heater location is chosen to coincide with a pressure node. The space
discretization is completed by the approximation of the damping, performed by discrete cosine transforms of type-VIII in order to alleviate Gibbs phenomenon-related artifacts. These approximations, presented in this section, eventually lead to a system of coupled linear delay difference equations.

In order to retain the compact character of the acoustic source a ghost-fluid treatment is employed [49]. Thus the nonlinear term is preserved in the spatial discretization scheme by recasting the second equation in (2.25) as

\[
\frac{\partial p}{\partial t} + \gamma M \frac{\partial}{\partial z} \left( u_z - Q_f(t - \tau) H(z - z_f) \right) + G_{cb}(\vec{z}, \vec{z}_0, z_{max}) \left( \nabla \cdot \left( u_\theta \vec{e}_\theta + u_r \vec{e}_r \right) \right)_c + \xi * p = 0 \quad (3.1)
\]

In the above, \( H \) stands for the Heaviside function, which means that the velocity field contains a discontinuity of amplitude \( Q_f(t - \tau) \) at the location of the heater. Therefore, the above equation is in tune with the coupled sub-system approach mentioned in section 2.2. Furthermore, for enhanced accuracy we force the heater location to coincide with a cell face or pressure node, denoted by the index \( i_f \) and coordinate \( z_f = z_{i_f} \). Therefore, the magnitude of the delayed velocity across the heater, in the expression for \( Q_f \), can be approximated as

\[
(u_z)_f(t - \tau) \simeq \frac{(u_z)_{i_f - 1/2}(t - \tau) + (u_z)_{i_f + 1/2}(t - \tau)}{2}. \quad (3.2)
\]

Before stating the semi-discretized from of the governing equations inside the 1D duct model of the burners, the kernels that are used for coupling purposes are briefly described. The normalized weights of the kernel used to couple the chamber onto the burner is defined as:

\[
\omega_b(i) = \frac{Ker_b(|z_i - z_0|, z_{max})}{\sum_i Ker_b(|z_i - z_0|, z_{max})} \quad (3.3)
\]

where, \( Ker_b \) represents the kernel, given by \( \frac{1 + \cos \left( \pi \frac{|z_i - z_0|}{z_{max}} \right)}{2} \). Term \( i \) represents the \( i^{th} \) staggered pressure grid-point, inside the 1D duct, coinciding with a portion of the kernel \( Ker_b \) that is centered at a distance \( z_0 \) from the closed end. The equations that follow represent the semi-discretized governing equations of the burner at the interior nodes and
the two boundaries:

- Semi-discrete form of the equations (2.25) along the interior points of the burner is given below:

\[
\alpha \frac{\partial (u_z)_{i-\frac{1}{2}}}{\partial t} + (1 - 2\alpha) \frac{\partial (u_z)_{i+\frac{1}{2}}}{\partial t} + \alpha \frac{\partial (u_z)_{i+\frac{3}{2}}}{\partial t} + \frac{1}{\gamma M} \frac{(p_z)_{i+1} - (p_z)_i}{\Delta z} = 0, \tag{3.4}
\]

\[
\alpha \frac{\partial (p_z)_{i-1}}{\partial t} + (1 - 2\alpha) \frac{\partial (p_z)_i}{\partial t} + \alpha \frac{\partial (p_z)_{i+1}}{\partial t} + \gamma M \frac{(u_z)_{i+\frac{1}{2}} - (u_z)_{i-\frac{1}{2}}}{\Delta z}
\]
\[
+ d_i = \gamma MQ \delta(z_i - z_f) - \gamma M \left( \frac{z_{\text{max}}}{2\Delta z} \right) (\omega_b(i)) \sum_{j,k} \left( (\nabla \cdot u_{j,k})_{cyl} \omega_c(j, k) \right). \tag{3.5}
\]

where, \(2 z_{\text{max}}\) represents the width of the kernel \((Ker_b)\). The Padé scheme, with \(\alpha = 1/24\), results in fourth order accuracy for all interior points. \((\nabla \cdot u_{j,k})_{cyl}\) represents the divergence of chamber velocity in cylindrical co-ordinates, \(\Delta z\) represents the distance between consecutive pressure nodes along the 1D duct.

- The closed and open boundaries of the 1D duct, in semi-discretized form are given below:

  - Closed Boundary:

\[
(1 - 2\alpha) \frac{\partial (u_z)_{\frac{3}{2}}}{\partial t} + \alpha \frac{\partial (u_z)_{\frac{5}{2}}}{\partial t} + \frac{1}{\gamma M} \frac{(p_z)_2 - (p_z)_1}{\Delta z} = 0, \tag{3.6}
\]

\[
\frac{\partial (p_z)_1}{\partial t} + \gamma M \frac{(u_z)_{\frac{3}{2}}}{\Delta z} + d_1 = \gamma MQ \delta(z_1 - z_f)
\]
\[
- \gamma M \left( \frac{z_{\text{max}}}{2\Delta z} \right) (\omega_b(1)) \sum_{j,k} \left( (\nabla \cdot u_{j,k})_{cyl} \omega_c(j, k) \right) \tag{3.7}
\]

  - Open Boundary:

\[
\alpha \frac{\partial (u_z)_{nb + \frac{1}{2}}}{\partial t} - \frac{1}{\gamma M} \frac{(p_z)_{nb}}{\Delta z} = 0, \tag{3.8}
\]
\[
\alpha \frac{\partial (p_z)_{nb-1}}{\partial t} + (1 - 2\alpha) \frac{\partial (p_z)_{nb}}{\partial t} + \frac{\gamma M}{\Delta z} \left( \frac{u_{z}^n_{nb} - u_{z}^n_{nb-\frac{1}{2}}}{\Delta z} \right) + d_{nb} = \\
\gamma M Q \delta (z_{nb} - z_f) - \gamma M \left( \frac{z_{max}}{2\Delta z} \right) \left( \omega_b (nb) \right) \sum_{j,k} \left( (\nabla \cdot u_{j,k})_{cyl} \omega_c (j, k) \right) \\
\text{(3.9)}
\]

where, \( nb \) represents the number of pressure nodes along the burner geometry.

### 3.1.2 Discretized Equations: Chamber

Similarly, a staggered arrangement of the variables (angular and radial velocities, and pressure) is used to provide a numerical approximation of the continuous chamber system. The velocities and pressure nodes are located on the cell faces and cell centers, respectively. The chamber geometry has closed boundaries along the radial direction, and the azimuthal boundary conditions are as specified by the root of unit formalism. The closed boundary is represented by a Dirichlet boundary condition on the velocity and a Neumann boundary condition on the pressure. The spatial derivatives across the chamber are represented using a second order central difference approximation. The discretization, presented in this section, eventually lead to a system of coupled linear delay difference equations.

The normalized weights of the kernel that are used to distribute the weighted sum of the velocity divergence from the burner onto the chamber is defined as:

\[
\omega_c (j, k) = \frac{|r_{j,k}^-| \cdot Ker_c (|r_{j,k}^- - r_0|, r_{max})}{\sum_{j,k} (|r_{j,k}^-| \cdot Ker_c (|r_{j,k}^- - r_0|, r_{max}) } ,
\]

\text{ (3.10)}

where, \( Ker_c \) represents the kernel, given by \( \frac{1 + \cos \left( \frac{\pi |r_{j,k}^- - r_0|}{r_{max}} \right) }{2} \), and terms \( j \) and \( k \) represent the \( j^{th} \) radial and \( k^{th} \) azimuthal staggered pressure grid-point, across the 2D chamber, coinciding with a portion of the kernel \( Ker_c \) that is centered at a the location \( r_0 \) measured from the center of the 2D chamber. The equations that follow represent the semi-discretized governing equations of the chamber at the interior nodes and boundaries:

- Semi-discrete form of the equations (2.25) along the interior points of the chamber is
given below:

\[
\frac{\partial (u_r)_{j-\frac{1}{2},k}}{\partial t} + \frac{1}{\gamma M} \frac{p_{j,k} - p_{j-1,k}}{\Delta r} = 0, \tag{3.11}
\]

\[
\frac{\partial (u_\theta)_{j,k+\frac{1}{2}}}{\partial t} + \frac{1}{\gamma M} \frac{p_{j,k+1} - p_{j,k-1}}{r_j \Delta \theta} = 0, \tag{3.12}
\]

\[
\frac{\partial p_{j,k}}{\partial t} + \gamma M * \left( \frac{r_{j+\frac{1}{2}} (u_r)_{j+\frac{1}{2},k} - r_{j-\frac{1}{2}} (u_r)_{j-\frac{1}{2},k}}{r_j \Delta r} \right) + \gamma M \frac{(u_\theta)_{j,k+\frac{1}{2}} - (u_\theta)_{j,k-\frac{1}{2}}}{r_j \Delta \theta} =
\]

\[
- \gamma M \left( \frac{\pi^2 - 4}{2\pi |r_{j,k}| \Delta \theta \Delta r} \right) (\omega_c(j,k)) \sum_i \left( (\nabla \cdot u_i)_{\text{burner}} \omega_b(i) \right) - \frac{\nu}{r_{i,j} \Delta r \Delta \theta} \left( \frac{p_{i+1,j} - p_{i,j}}{\Delta r} r_{i,j} \Delta \theta + \frac{p_{i,j} - p_{i-1,j}}{\Delta r} (r_{i,j} - \Delta r) \Delta \theta \right) +
\]

\[
\frac{\nu}{r_{i,j} \Delta r \Delta \theta} \left( \frac{p_{i+1,j} - p_{i,j}}{r_{i,j} \Delta \theta} \Delta r - \frac{p_{i,j} - p_{i,j-1}}{r_{i,j} \Delta \theta} \Delta r \right).
\]

where, \(2 r_{max}\) represents width of the kernel. The term \((\nabla \cdot u_i)_{\text{burner}}\) is the divergence of burner velocity in Cartesian coordinates. \(\Delta r\) and \(\Delta \theta\) are the radial and azimuthal distance between consecutive pressure nodes across the chamber.

- Semi-discrete form of the equations (2.25) at the radial and azimuthal boundaries of the system are:

  - Inner Radius:

    \[
    \frac{\partial (u_r)_{-\frac{1}{2},k}}{\partial t} + \frac{1}{\gamma M} \frac{p_{1,k}}{\Delta r} = 0, \tag{3.14}
    \]

    where, \(n\) represents the number of pressure nodes along the radial direction of the chamber.

  - Outer Radius:

    \[
    \frac{\partial (u_r)_{n+\frac{1}{2},k}}{\partial t} - \frac{1}{\gamma M} \frac{p_{n,k}}{\Delta r} = 0, \tag{3.15}
    \]

  - Left Azimuthal Boundary:
\[
\frac{\partial (u_{\theta})_{j,m+\frac{1}{2}}}{\partial t} + \frac{1}{\gamma M} \rho_j \frac{(p_{j,1}) - p_{j,m-1}}{r_j \Delta \theta} = 0,
\]  
(3.16)

where, \( m \) represents the number of pressure nodes along the azimuthal direction of a single sector of the chamber geometry and \( \rho_j = e^{(lj \frac{2\pi}{N})} \) (\( N \) = number of identical sectors in the chamber).

- Right Azimuthal Boundary:

\[
\frac{\partial (u_{\theta})_{j,1+\frac{1}{2}}}{\partial t} + \frac{1}{\gamma M} \frac{p_{j,2} - p_{j,m}}{\rho_j} = 0,
\]  
(3.17)

The above equations simply list the modifications to 3.11 required to implement the boundary conditions across each sector, the remaining equations are identical to those listed for the interior points.

3.2 Spectral and Bifurcation Analysis

The equations described in the previous section are non-linear delay difference equations. The stability of such systems in the limit of small-amplitude perturbations can be analyzed, by utilizing the partial-differential-equation (PDE) method proposed by [50]. The analysis is performed on the discretized system of equations described in (3.5) with a linearized source term given by

\[
Q_f(t - \tau) \equiv \frac{K}{2} \left[ \sqrt{\frac{1}{3} + (u_z)_{f}(t - \tau)} - \sqrt{\frac{1}{3}} \right] \approx \sqrt{3} \frac{K}{4} u_f(t - \tau).
\]  
(3.18)

The parameters of interest in the ensuing spectral and bifurcation analysis are the non-dimensional heater strength \( K \), its location \( z_f \), the delay term \( \tau \) and terms such as \( z_{\text{max}} \) and \( r_{\text{max}} \) that describe the nature of the coupling between burner and chamber elements. Semi-discrete delay difference equations, described in the previous section, can be recast in the matrix form as

\[
\ddot{x} = B_0 \dot{x} + B_1 \ddot{x}(t - \tau) \equiv B_j \ddot{x},
\]  
(3.19)
where, \( \vec{x} = [u_z, p_z, u_\theta, u_r, p]^T \) represents the governing variables of the entire system, \( \mathbf{B}_0 \) is the matrix containing differentiation and damping operators, \( \mathbf{B}_1 \) includes the linearized source term and \( \mathbf{B}_j \) represents the finite dimensional spatial differential operator corresponding to the \( j^{th} \) azimuthal boundary condition. The eigenvectors and eigenvalues of this matrix satisfy the following relation [50]

\[
(-\lambda \mathbf{I} + \mathbf{B}_0 + \mathbf{B}_1 e^{-\tau \lambda}) \vec{v} = 0 \quad (3.20)
\]

where, \( \vec{v} \) stands for the eigenvectors and \( \lambda \) for the eigenvalues of the above set of coupled delay differential equations. Equation (3.20) is a non-linear eigenvalue problem that can be solved numerically, using the PDE method suggested by [50]. The methods suggested by [50] are incorporated in the software package DDE-BIFTOOL ([51]) which is employed in this article to obtain neutral curves corresponding to the different roots of unity, against the parameters \( K \) and \( \tau \), for a given geometry, coupling parameters and heater location. This method is briefly described below.

The numerical approximation to the eigenvectors of the matrix \( \mathbf{B}_j \) are given by the last block row (size: \( n \times n(N + 1) \)) of the right eigenvectors of \( \mathbf{A}_N \) (size: \( n(N + 1) \times n(N + 1) \)), whilst the eigenvalues of both matrices are the same. Matrix \( \mathbf{A}_N \) is given by

\[
\mathbf{A}_N = \begin{bmatrix}
\mathbf{C}_N & \otimes & \mathbf{I}_n \\
\mathbf{B}_1 & 0 & \ldots & 0 \\
& \mathbf{B}_0
\end{bmatrix}
\quad (3.21)
\]

where, \( \otimes \) represents the Kronecker tensor product, \( \mathbf{I}_n \) is the \( n \times n \) identity matrix and, \( \mathbf{C}_N \) is the \( N \times (N + 1) \) Chebyshev differentiation matrix, with all real entries, given by

\[
\mathbf{C}_N = \frac{N}{\tau} \begin{bmatrix}
-1 & 1 \\
& \ddots & \ddots \\
& & -1 & 1
\end{bmatrix}
\quad (3.22)
\]
In this chapter, the numerical framework described in chapter 3 is validated against the analytical solutions of an undamped 1D duct, with a compact heat source, and simple annular chamber model. The annular chamber test case validates the proposed framework in the context of an $N$-periodic geometry and, the undamped 1D duct model of a stand-alone burner validates the framework’s ability to accurately capture thermo-acoustic phenomenon. Increasingly finer spatial discretizations of the system are computed to ensure that the solutions, obtained using this framework, converge.

4.1 Analytical validation of the 1D duct in the limit of zero damping

The modal shapes and frequencies of the acoustic waves in a 1D duct geometry are used to validate the numerical approximation. The following sub-sections describe the derivation of the analytical solution followed by a description of the the proposed framework as applied to this test case. This comparison not only confirms the 1D discretization implemented as a part of the framework, but more importantly confirms its ability to accurately represent the time delayed heat source term.

Analytical solution

In the proposed model the 1D duct is represented as a damped linear systems with a time delayed heat source. In the absence of singularities, the traveling wave solutions on either side of the heat source are given, using $n - \tau$ model [29], by
Figure 4.1: Superimposition of the analytical and numerical eigenspectra of the 1D duct in the limit of zero damping.

\[
\begin{align*}
\frac{p_\epsilon}{\epsilon} &= A_\epsilon^+ e^{ikz'} - i\omega t + A_\epsilon^- e^{-ikz'} - i\omega t, \\
\frac{u_\epsilon}{\epsilon} &= A_\epsilon^+ e^{ikz'} - i\omega t - A_\epsilon^- e^{-ikz'} - i\omega t
\end{align*}
\]

(4.1)

for \( \epsilon \in \{\text{left}, \text{right}\} \), \( z' = z - z_f \) if \( \epsilon = \text{right} \) else \( z' = z_f \). In non-dimensional form \( \omega = k \). The burner is modeled as an open-closed rijke tube and hence the coefficients \( A_\epsilon^\pm \) are related to the reflection factors at each end of the tube as follows:

\[
\begin{align*}
\text{Closed boundary} : \quad R_{\text{left}} &= \frac{A_{\epsilon \text{left}}^+}{A_{\epsilon \text{left}}^-} = 1, \\
\text{Open boundary} : \quad R_{\text{right}} &= \frac{A_{\epsilon \text{right}}^+}{A_{\epsilon \text{right}}^-} e^{2ik(1-z_f)} = -1
\end{align*}
\]

(4.2)

the remaining relations are obtained by assuming a constant pressure profile and a gradient preserving jump in the velocity profile across the heating element, as shown below:
\[
\begin{aligned}
p_{\text{right}} - p_{\text{left}} &= 0, \\
u_{\text{right}} - u_{\text{left}} &= \frac{\sqrt{3}}{4} K [W u_{\text{left}}(z_f, t_\tau) + (1 - W) u_{\text{right}}(z_f, t_\tau)]
\end{aligned}
\] (4.3)

where \( W \in [0, 1] \). In this scenario the velocity values on both side are available and hence an average value of \( W = \frac{1}{2} \) is used. Now, for a non-trival solution to exist, \( \omega \) must satisfy a dispersion relation of the type

\[
\left( 1 - K \frac{\sqrt{3}}{8} e^{i \omega \tau} \right) \cos(\omega z_f) \cos(\omega (1 - z_f)) - \left( 1 + K \frac{\sqrt{3}}{8} e^{i \omega \tau} \right) \sin(\omega z_f) \sin(\omega (1 - z_f)) = 0.
\] (4.4)

This relation is then numerically inverted to arrive at the complex eigenvalues \(-i \omega\) and eigenvectors of the system. These eigenmodes are then compared with the corresponding mode shapes and frequencies obtained using the numerical framework applied to this test case. The results of this comparison are shown in figure 4.1, which displays the eigenspectrum obtained from the analytical and numerical methods, in the range \( \Re = [-0.3, 0.3] \) and \( \Im = [-40, 50] \), with a relative error of the order \( 10^{-5} \).

4.2 Analytical validation of the annular chamber in the linear limit

The structure and frequency of the modes resulting from the 2D wave equation over the annular domain are used to validate the undamped-2D-linear numerical framework. The following sub-sections describe the derivation of the analytical solution followed by a description of the the proposed framework applied to this test case. Finally, application of the roots of unity formalism to the numerical solution of this test case is discussed.

Since the validation test case aims to compare the acoustic modes over an annular domain, the non-dimensionalized version of the equation 2.12 is expressed in cylindrical coordinates as shown below:

\[
\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial^2 p}{\partial t^2} - \frac{1}{r} \frac{\partial p}{\partial r} - \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} = 0.
\] (4.5)
The above differential equation can be solved by assuming that the solution in general can be written as a product of the three independent variables \( r, \theta \) and \( t \) as shown below:

\[
p(r, \theta, t) = R(r)\Phi(\theta)T(t). \tag{4.6}
\]

Substituting 4.6 into equation 4.5 and dividing by \( p \) produces

\[
\frac{1}{T} \frac{d^2T}{dt^2} - \frac{1}{R} \left( \frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) - \frac{1}{r^2 \Phi} \frac{d^2\Phi}{d\theta^2} = 0. \tag{4.7}
\]

Each of the three parts \( \left[ \frac{1}{T} \frac{d^2T}{dt^2}, \frac{1}{R} \left( \frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right), \frac{1}{r^2 \Phi} \frac{d^2\Phi}{d\theta^2} \right] \) in the above expression is a function of only one variable. Hence the remaining expressions can be treated as constants. Thus the following equations can be solved to arrive at the expressions for \( R, \theta \) and \( T \)

\[
\frac{1}{T} \frac{d^2T}{dt^2} = -k^2, \tag{4.8}
\]

\[
\frac{1}{\Phi} \frac{d^2\Phi}{d\theta^2} = -n^2, \tag{4.9}
\]

\[
\frac{1}{R} \left( \frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) = \left( \frac{n^2}{r^2} - k^2 \right), \tag{4.10}
\]

where, \( k \) and \( n \) are constants. Thus equations 4.8 to 4.10 indicate the variables \( R, \theta \) and \( T \) can vary independent of each another. The first equation is simply the harmonic oscillator equation and its solution is simply given in the following manner:

\[
T(t) = T_0 e^{ikt}. \tag{4.11}
\]

\( T_0 \) depends on the initial conditions and \( k \) is considered to be a real number as the acoustic wave solution oscillates in time. Furthermore, equation 4.9 is similar to 4.8 and therefore has a similar solution given by:

\[
\Phi(\theta) = \Phi_0 e^{in\theta}, \tag{4.12}
\]
where, $\Phi_0$ depends on the initial conditions, and $n$ is an integer, resulting from the global periodic boundary conditions.

4.10, can be modified to the standard form of the Bessel’s equation by considering a new form of the independent variable ($r$) as $s = kr$. This results in the following differential equation

$$s^2 \frac{d^2 R(s)}{ds^2} + s \frac{dR(s)}{ds} + (s^2 - n^2)R(s) = 0.$$  \hspace{1cm} (4.13)

Since 4.13 is the standard Bessel’s equation, the general form of its solution is a linear combination of the integer-order Bessel functions of the first ($J_n(s)$) and second kind ($Y_n(s)$), as shown below

$$R(r) = R_J J_n(kr) + R_Y Y_n(kr),$$  \hspace{1cm} (4.14)

where, the values of $k$ and $n$ are the same as before and the values of $R_J$ and $R_Y$ depend of the boundary conditions of the system.

Putting the three functions together, we have a general expression to the solution of the 2D wave equation in polar coordinates:

$$p(r, \theta, t) = e^{ikt} \left( AJ_n(kr) + BY_n(kr) \right) e^{in\theta},$$  \hspace{1cm} (4.15)

This equation is completed by applying the boundary conditions corresponding to a model 2D combustion chamber. Zero velocity at the radial boundaries (walls) of the annulus and a zero gradient in pressure as

$$\frac{\partial p}{\partial r} \bigg|_{r=r_i} = k e^{ikt} \left( AJ_n'(kr_i) + BY_n'(kr_i) \right) e^{in\theta} = 0,$$  \hspace{1cm} (4.16)

$$\frac{\partial p}{\partial r} \bigg|_{r=r_o} = k e^{ikt} \left( AJ_n'(kr_o) + BY_n'(kr_o) \right) e^{in\theta} = 0.$$  \hspace{1cm} (4.17)

Equations 4.15 to 4.17 yield the eigenvalues and eigenvectors of the system. As an example, figure 4.2a shows the analytical eigenvector corresponding to a non-dimensional
frequency of 3.1645, calculated using these equations.

\[ Y'_n(k r_o) J'_n(k r_i) - Y'_n(k r_i) J'_n(k r_o) = 0 \] \hspace{1cm} (4.18)

The numerical counterpart of the pressure eigenvector, corresponding to the same angular frequency of \( \omega = 3.1645 \), is shown in figure 4.2b. The discretization scheme described in chapter 3 is applied to the entire \( 2\pi \) domain of the annular combustion chamber, resulting in a single matrix representation of the systems finite dimensional spatial differential operator. The comparison of the analytical and numerical eigenvalues in figure 4.2a shows good agreement. This agreement validates the numerical discretization along the polar spatial coordinates representing the chamber.

The pressure eigenvector, for the angular frequency of \( \omega = 3.1645 \), is shown in figure 4.2c. The discretization scheme described in chapter 3 is applied to a single periodic unit of the \( N \)-periodic chamber geometry, resulting in a matrix representation of the systems spatial differential operator over a single sector, with different azimuthal phase boundary conditions represented by the \( N \) different roots of unity. Figure 4.2b shows the analytical and numerical eigenvalues super-imposed over each other to indicate the validity of this approximation. Therefore, the comparison between analytical and numerical eigenmodes confirm that same eigenvalues are extracted using this framework with reduced cost. A similar correspondence is seen between the eigenvectors presented in 4.2 from each method of analysis (they differ simply by a complex factor).
(a) Superimposition of the analytical and numerical eigenspectra, without the roots of unity formalism, of the annular chamber in the linear limit ($\omega = 3.1645$ is highlighted).

(b) Superimposition of the analytical and root of unity enabled numerical eigenspectra of the annular chamber in the linear limit ($\omega = 3.1645$ is highlighted).

(c) Analytical solution.

(d) Numerical solution.

(e) Numerical solution with roots of unity.

Figure 4.2: Eigenstructure of the annular chamber in the linear limit.
4.3 Convergence Study

The eigenspectrum obtained from the matrix representation of the linear spatial differential operator (eq: 2.29), described in chapter 3, is utilized to determine whether the algorithm used to compute the linear stability of the system converges as the number of nodes per unit sector of the $N$-periodic domain are increased. The results of this analysis, as shown in figure 4.3, indicate how the order of accuracy as a result of the numerical approximations, made within the chamber and duct models, manifest in the infinity-norm, 2-norm and 1-norm convergence diagrams. From chapter 3 we know that the radial boundary conditions across the 2D chamber, the end boundary conditions of the 1D duct and the ghost-fluid model...
across the heat source are all first order accurate approximations. The discretization of the acoustic equations within the inner domain of the burner elements and the 2D chamber, are fourth-order and second-order accurate, respectively. The figures in 4.3 each have three lines indicating the rate of first (pink, \( \Delta \)), second (black, square) and fourth (green, \( \nabla \)) order convergence. Therefore, from the figure it is seen that the resulting order of convergence, for each norm, is bounded between the second and fourth order constant convergence lines.
CHAPTER 5
RESULTS AND DISCUSSIONS

The previous chapters propose and validate a linear model representing the coupled behavior of $N$-periodically assembled one-dimensional burners (thermo-acoustic elements) within a simple two-dimensional chamber. In this chapter, the linear model is applied to two geometries representing an annular combustion chamber and an afterburner (both $N$-periodic in nature).

In this study we investigate the performance of the model by changing the parameters governing the instabilities of the system, such as the geometry of the chamber, heater strength, the time delay associated with the heater and the location of the heater (both inside the 1D duct and across the 2D chamber). This investigation is motivated by previous studies, such as [10] and [7], which indicate that these parameters, and in particular the heat source, have significant influence on the overall instability of the thermo-acoustic system.

Each test case aims to provide information regarding the linear modal stability of the annular combustion chamber and afterburner system, utilizing the spectral and bifurcation analysis as described in section 3.2. This is followed by a comparison of the frequency and modal shape of the most unstable eigenmode with the results of the experimental study [22]. Finally, the system’s receptivity to harmonic forcing is computed in order to visualize the non-normal behavior of the system (as described in section 2.3.2).

5.1 Modal Analysis: Annular Combustion Chamber

This section details the linear stability of the acoustic modes corresponding to an annular combustion chamber, modeled as described in chapter 2. Two test models are chosen and the nature of their instability is studied, using the systems eigenspectrum and bifurcation
Figure 5.1: Schematic of the Annular combustion chamber $An_1$

diagrams, as a function of the system properties such as heater strength and associated time delay, for specific heater locations (both along the burner and chamber geometry).

5.1.1 Annular combustion chamber, case $An_1$:

Figure 5.1 illustrates the geometry of a $(N = 16)$—sector annular combustion chamber, with a heat source located inside each burner element (length $L$) at a distance $z_f = \frac{5L}{8}$ from its closed boundary. The burner elements themselves are located at the geometric center of each periodic segment. The inner and outer radii of the chamber is assumed to be three and four times the length $z_f$, respectively. This configuration is denoted as $An_1$.

The linear stability of this configuration for a non-dimensional heater strength $K = 2$ and time delay $\tau = 0.355$, is illustrated in figure 5.2a. Only a subset of roots of unity $j = 0, 1, \ldots, 8$ are considered since, roots of unity $j = a$ and $j = N - a$ (where $a$ is an integer $\in [1, N - 1]$) yield the same eigenspectrum. Therefore, 5.2a represents the configuration’s global linear stability. Furthermore, the most unstable non-dimensional frequencies of this configuration, across the global eigenmodes of the system, are in the vicinity of the value 4.7 (shown in the enlarged subplot on figure 5.2a). This value of non-dimensional frequency corresponds to the most unstable eigenvalue of a stand-alone burner with the same heater location (eigenspectrum shown in figure 5.2b). This suggests that the , azimuthal modes along the chamber are of the same frequency of the most unstable mode of the burner itself.

Next, we investigate the stability of this configuration as a function of the non-dimensional
strength $K$ and time delay $\tau$ associated with the heat source, in the vicinity of the non-dimensional frequency 4.7. This information is represented in the form of a bifurcation diagram as shown in figure 5.3. Each neutral curve in figure 5.3 represents the locus of Hopf bifurcations, that model $\text{An}_1$ will transitions through, across the phase space $[K, \tau]$. Furthermore, each neutral curve corresponds to a unique root of unity. Therefore the combination of all neutral curves represents the global transition in the stability of the system as a function of parameters $K$ and $\tau$.

Finally, the bifurcation diagram is accompanied by a visualization of the azimuthal eigenmodes (representing the heat release rate) corresponding to the roots of unity with the most unstable neutral curve ($j = 0, 1, 2, 3, 4$) across the phase space $[K, \tau]$. Among the azimuthal eigenmodes illustrated in 5.3, the $j = 0$ mode signifies a unit periodic eigenmode. This repeating structure is seen clearly in the embedded figure $j = 0$ within figure 5.3. The roots of unity $j = 1, 2, 3$ and 4 each represent solutions that are not periodic over a single unit and, the specific change in phase that these modes incur across a single unit is given by the expression $\theta = j * \frac{2\pi}{N}$. Furthermore, the phase space $[K,\tau]$ surrounding the neutral curve corresponding to the root of unity $j = 0$, shown in the enlarged portion of the bifurcation
Figure 5.3: Bifurcation diagram and unstable eigenvectors of the Annular combustion chamber $An_1$. 
Figure 5.4: Schematic of the Annular combustion chamber $An_2$

Diagram, is populated with neutral curves of the roots of unity $j = 1, 2, 3$ and $4$. Thus indicating that the azimuthal modal shapes that dominate the systems transition from stable to unstable behavior will resemble modal shapes shown in 5.3.

5.1.2 Annular combustion chamber with a modified heater location along the burner elements, case $An_2$:

Figure 5.4 illustrates the geometry of a ($N = 16$)–sector annular combustion chamber, with a heat source located inside each burner element (length $L$) at a distance $z_f = \frac{L}{3}$ from its closed boundary. The burner elements themselves are located at the geometric center of each periodic segment. The inner and outer radii of the chamber is assumed to be three and four times the length $z_f$, respectively. This configuration is denoted as $An_2$.

The eigenspectrum of this configuration, corresponding to a non-dimensional heater strength $K = 2.58$ and time delay $\tau = 0.2$, is illustrated in figure 5.5a. Furthermore, the most unstable non-dimensional frequencies of this configuration, across the global eigenmodes of the system, are in the vicinity of the value 7.8 (shown in the enlarged subplot on figure 5.5a). This value of non-dimensional frequency corresponds to the most unstable eigenvalue of a stand-alone burner with the same heater location (eigenspectrum shown in figure 5.5b). This result agrees with the previous observations in $An_1$.

Next, we investigate the stability of this configuration as a function of the non-dimensional strength $K$ and time delay $\tau$ associated with the heat source, in the vicinity of the non-
Figure 5.5: Eigenspectra of the configuration $A n_2$ and a stand alone burner with the same $K$, $\tau$ and $z_f$

dimensional frequency 7.8. This information is represented in the form of a bifurcation diagram as shown in figure 5.6.

Finally, the bifurcation diagram is accompanied by a visualization of the azimuthal eigen-modes (representing the heat release rate) corresponding to the roots of unity with the most unstable neutral curve ($j = 7$, 8) over phase space $[K, \tau]$. The neutral curves, corresponding to each root of unity, transition in an order that is different from the one observed in figure 5.3. In this bifurcation diagram the modes corresponding to the roots of unity $j = 7$ and $j = 8$ are the most unstable, where as modes corresponding to $j = 0, 1, 2, 3$ and 4 have a larger region of phase space $[K, \tau]$ corresponding to stable behavior. This is in direct contrast with the results of the configuration $A n_1$. Furthermore, the most unstable modal solutions are non-periodic over a single unit and incur a phase difference $\theta = 7 \times \frac{2\pi}{16}$ and $\theta = 8 \times \frac{2\pi}{16}$, across a unit sector, in case of the modal solutions corresponding to roots of unity $j = 7$ and $j = 8$ respectively.
Figure 5.6: Bifurcation diagram and unstable eigenvectors of the Annular combustion chamber $An_2$. 
5.2 Modal Analysis: Afterburner

This section details the linear stability of the acoustic modes corresponding to an afterburner, modeled mentioned in chapter 2. Three models are used to illustrate the effect of heater location along the burner and chamber elements of the afterburner. Model $A_{f1}$ is similar to the annular combustion chamber model $A_{n1}$, with the exception that the chamber now resembles a disc instead of an annulus. Similarly, model $A_{f2}$ resembles the annular combustion chamber model $A_{n2}$. Finally, model $A_{f3}$ studies the modification of case $A_{f1}$ when the burner is moved closer to the center of the afterburner chamber by a distance equal to the length $z_f$ (distance between the closed end of the burner and heater, inside each burner). The configuration of each model is illustrated with the help of a schematic. The system’s eigenspectrum and bifurcation diagrams are used to represent each models stability characteristics. The bifurcation diagrams contain, in addition to the neutral curves corresponding to each root of unity, the eigenmodes corresponding to the most unstable roots of unity, as shown in the figure 5.8.

5.2.1 Afterburner similar to model $A_{n1}$, case $A_{f1}$:

Figure 5.7 illustrates the geometry of a ($N = 16$)—sector afterburner, with a heat source located inside each burner element (length $L$) at a distance $z_f = \frac{5L}{8}$ from its closed boundary. The burner elements themselves are located at a radius 3.5 times the length $z_f$. The outer
Figure 5.8: Eigenspectra of the configuration $A_f^1$ and a stand alone burner with the same $K$, $\tau$ and $z_f$.

radius of each afterburner is equal to 4 times the length $z_f$. This configuration is denoted as $A_f^1$.

The eigenspectrum of this configuration, with a non-dimensional heater strength $K = 2.05$ and time delay $\tau = 0.365$, is illustrated in figure 5.8a. Furthermore, the most unstable non-dimensional frequencies of this configuration, across the global eigenmodes of the system, are in the vicinity of the value 4.7 (shown in the enlarged subplot on figure 5.8a). This value of non-dimensional frequency corresponds to the most unstable eigenvalue of a stand-alone burner with the same heater location (eigenspectrum shown in figure 5.8b). This result agrees with the previous observation in $A_n^1$.

Next, we investigate the stability of this configuration as a function of the non-dimensional strength $K$ and time delay $\tau$ for this most unstable frequency. This information is represented in the form of a bifurcation diagram as shown in figure 5.9.

Finally, the bifurcation diagram is accompanied by a visualization of the azimuthal eigenmodes (representing the heat release rate) corresponding to the roots of unity with the most unstable neutral curve ($j = 1, 2, 3, 4, 6$) over the phase space [$K$, $\tau$]. The neutral curves, corresponding to each root of unity, transition in an order that is similar to that observed...
Figure 5.9: Bifurcation diagram and unstable eigenvectors of the Afterburner $A_{f_1}$. 
in figure 5.3 (case $An_1$). In this bifurcation diagram the modes corresponding to the roots of unity $j = 0, 1, 2, 3, 4$ and $j = 6$ are the most unstable, and agree with the observations in case of $An_1$. Furthermore, the most unstable modal solutions are non-periodic over a single unit and incur a phase difference $\theta = [1, 2, 3, 4, 6] \times \frac{2\pi}{16}$, across a unit sector, in case of the modal solutions corresponding to the roots of unity $j = 1, 2, 3, 4$ and $j = 6$, respectively.

5.2.2 Afterburner with a modified heater location along the burner elements, case $Af_2$:

Figure 5.10 illustrates the geometry of a $(N = 16)$—sector afterburner, with a heat source located inside each burner element (length $L$) at a distance $z_f = \frac{L}{3}$ from its closed boundary. The burner elements themselves are located at a radius 3.5 times the length $z_f$. The outer radius of each afterburner is equal to 4 times the length $z_f$. This configuration is denoted as $Af_2$ and, is similar to the case $An_2$.

The eigenspectrum of this configuration, for heater strength $K = 3.25$ and time delay $\tau = 0.2$, is illustrated in figure 5.11a. The most unstable non-dimensional frequencies of this configuration, across the global eigenmodes of the system, is approximately 7.8 (shown in the enlarged subplot on figure 5.11a). This result agrees with the previous observations in $An_2$.

Next, we investigate the stability of this configuration as a function of the non-dimensional strength $K$ and time delay $\tau$ for the non-dimensional frequency 7.8. This information is rep-
Figure 5.11: Eigenspectra of the configuration $Af_2$ and a stand alone burner with the same $K$, $\tau$ and $z_f$ represented in the form of a bifurcation diagram as shown in figure 5.12.

Finally, the bifurcation diagram is accompanied by a visualization of the azimuthal eigen-modes (representing the heat release rate) corresponding to the roots of unity with the most unstable neutral curve ($j = 0, 5, 7, 8$) over the domain of $K$ and $\tau$. The neutral curves, corresponding to each root of unity, transition in an order that is similar to that observed in figure 5.6. In this bifurcation diagram the modes corresponding to the roots of unity $j = 0, 5, 7$ and $j = 8$ form the most unstable. The bifurcation diagram indicates an order of transition from stable to unstable behavior that resembles the case $An_2$. Here, the neutral curves corresponding to the roots of unity $j = 0, 5$ have a larger range of $\tau$ values for which they are stable and hence form the most unstable neutral curves at lower and higher values of $\tau$ respectively.

As mentioned in the previous section, the modal solutions represented here are non-periodic over a single unit through a phase difference given by $\theta = j \times \frac{2\pi}{N}$ (for $j = 1, 2, 3, 4, 6$ and $N = 16$), and therefore the pattern of heat release is similar to case $An_1$. 

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Figure 5.12: Bifurcation diagram and unstable eigenvectors of the Afterburner $Af_2$.

Figure 5.13: Schematic of the Afterburner $Af_3$. 
5.2.3 Afterburner with a modified heater location across the chamber, case $Af_3$:

Figure 5.13 illustrates the geometry of a $(N = 16)$—sector afterburner, with a heat source located inside each burner element (length $L$) at a distance $z_f = \frac{5L}{8}$ from its closed boundary. The burner elements themselves are located at a radius 2.5 times the length $z_f$. The outer radius of each afterburner is equal to 4 times the length $z_f$. This configuration is denoted as $Af_3$. This case is selected to investigate the effect of burner location in the chamber of an afterburner geometry.

The eigenspectrum of this configuration, for non-dimensional heater strength $K = 2.05$ and time delay $\tau = 0.355$, is illustrated in figure 5.14a. The most unstable non-dimensional frequencies of this configuration, across the global eigenmodes of the system, is approximately 4.7 (shown in the enlarged subplot on figure 5.14a). Although the location of the burners are altered inside the chamber, the most unstable frequency stays the same as $An_1$ and $Af_1$. Thus confirming the conclusion that the most unstable frequency of the burner governs the overall dynamics.

Next, we investigate the stability of this configuration as a function of the non-dimensional
Figure 5.15: Bifurcation diagram and unstable eigenvectors of the Afterburner $Af_3$. 
strength $K$ and time delay $\tau$ associated with the heat source, in the vicinity of the non-dimensional frequency 4.7. This information is represented in the form of a bifurcation diagram as shown in figure 5.15.

Finally, the bifurcation diagram is accompanied by a visualization of the azimuthal eigenmodes (representing the heat release rate) corresponding to the roots of unity with the most unstable neutral curve $(j = 6, 8)$ over phase space $[K, \tau]$. The neutral curves, corresponding to each root of unity, transition in an order that is different from the one observed in figure 5.3. In this bifurcation diagram the modes corresponding to the roots of unity $j = 6$ and $j = 8$ are the most unstable, where as modes corresponding to $j = 0, 1, 2, 4$ and especially $3$ have a larger region of phase space $[K, \tau]$ corresponding to stable behavior. This is in direct contrast with the results of the configuration $An_1$. Furthermore, the most unstable modal solutions are non-periodic over a single unit and incur a phase difference $\theta = 6 \times \frac{2\pi}{16}$ and $\theta = 8 \times \frac{2\pi}{16}$, across a unit sector, in case of the modal solutions corresponding to roots of unity $j = 6$ and $j = 8$ respectively.

Bifurcation diagrams representing the most unstable azimuthal boundary conditions (roots of unity) from each of the above mentioned models are merged together into a single diagram, in order to gain an understanding of the stability of each model in relation to the other. This information is given in figure 5.16. Cases $An_1$, $Af_1$ and $Af_3$ (group 1) have similar combined neutral curve (combination of the most unstable portions of each neutral curve) behavior. This is also true when comparing the combined neutral curves of cases $An_2$ and $Af_2$ (group 2). Cases in a single group share the same region of combined stable and unstable behavior over the phase space $[K, \tau]$, and have the same heater location along the burner elements. Thus, in addition to determining the most unstable frequency of the system’s global eigenmodes, the heater location along the burner elements also determines the portion of phase space that corresponds to unstable system behavior.
5.3 Comparison between the most unstable eigenmodes and experimentally observed limit cycle behavior.

The aim of this section is to verify if there is a correspondence between the limit cycle behavior of the experimental setup [22] and, the unstable eigenmodes obtained from a modal analysis of the model representing the experimental setup. The eigenmode corresponding to the root of unity $j = 1$, of the model, is found to have the same structure as that of the limit cycle obtained in the experiment, as shown in figure REF. Thus indicating the current frameworks potential to guide numerically intensive computations and experimental investigations of the thermo-acoustic behavior of such systems.
(a) Unstable azimuthal mode as observed in [22], direction of reading is from left to right and from top to bottom.

(b) Unstable azimuthal mode corresponding to $j = 1$ for a configuration similar to the experimental setup in [22], direction of reading is from left to right and from top to bottom.

Figure 5.17: Comparison between the most unstable eigenmode and experimentally observed limit cycle behavior.
5.4 Non-Modal Analysis:

This section details the receptivity of $N$–periodic systems, subject to a harmonic forcing, under linearly stable conditions. Cases $A_{n_1}$ and $A_{f_1}$, are selected for comparison. The receptivity of these configurations is explored over the linearly stable region of the systems bifurcation diagrams to observe its dependence on system parameters, such as $K$ and $\tau$. Finally, a comparison between the global receptivity of the annular combustion chamber and afterburner systems, indicates the influence of chamber geometry over the non-normality of such thermo-acoustic systems.

5.4.1 Receptivity of the annular combustion chamber, model $A_{n_1}$:

$A_{n_1}$ represents an annular combustion chamber. The system’s response to small amplitude harmonic perturbations/forcing, with asymptotically stable eigenmodes, is measured as the maximum response of the corresponding long-term solution [47]. This long-term solution is
described in section 2.3.2 (and in [47]) as the operator $M_j = (-iD_j - \omega I)^{-1}$ acting on the harmonic perturbation $\vec{x}_f e^{i\omega t}$. The above mentioned operator is called the resolvent and its norm, called the resolvent norm, is the systems maximum response at the forcing frequency $\omega$. Furthermore, the systems response to such a forcing, with real values of $\omega$, is equivalent to computing the resolvent operator $M_j$ along the imaginary axis of the system’s eigenspectrum. Hence, a plot of the system’s resolvent norm at each point along the imaginary axis of the systems eigenspectrum yields a representation of the systems receptivity to harmonic forcing (systems non-normality).

In this section, we evaluate the systems resolvent norm (figure 5.19) over a select range of non-dimensional frequencies ($\omega = [0.1, 6.1]$). The two locations where the resolvent plots are extracted are highlighted by $p1$ and $p2$ on the bifurcation plot of figure 5.18. These locations on $[K, \tau]$ correspond to linearly stable parameters and illustrate distinct non-normal behavior of the system. To maintain clarity, only unique resolvent norm corresponding to the first 9 roots of unity ($j = 0, 1, ..., 8$), of the 16-periodic system $An_1$, are displayed. The resolvent norm diagram at point $p1$ shows only the frequencies corresponding to the low frequency eigenvalues of the afterburner (The first 4 are indicated by circles), where as the resolvent norm at point $p2$ shows this spectrum super imposed over the non-normal behavior of the system, where the highly non-normal regions are enclosed within two rectangular boxes. Finally, figures 5.20 to 5.23 represent the forcing and response (along the 1D duct and the 2D chamber), corresponding to the maximum singular values of the resolvent for the most locally unstable root of unity. The velocity field along the 1D duct and the pressure field along the 2D annular chamber is visualized in each of these figures. It is also observed that the most unstable disturbance along the velocity field of the 1D duct represents a Dirac delta impulse accompanied by the most dominant eigenmode at that frequency. Dirac impulse remains only when every mode is equally dominant. A similar behavior is seen in case of the azimuthal pressure modes of this configuration. Here, the azimuthal mode is simply the burners influence (coupling kernel) over the annular chamber, in case every mode is equally dominant. In case a specific eigenmode is dominant, a superposition of this eigenmode and the coupling kernel manifest as the most responsive disturbance.
5.4.2 Receptivity of the afterburner, model $Af_1$:

$Af_1$ is similar to the annular combustion chamber $An_1$, except for the disc shaped chamber geometry. The model’s response, at the same locations as the previous case, are displayed in figure 5.25 over a select range of non-dimensional frequencies ($\omega = [0.1, 6.1]$). The resolvent diagrams indicate that the most prominent non-normal behavior is close to point $p_2$, in the phase space $[K, \tau]$. Finally, figures 5.26 to 5.29 represent the maximum disturbance and response (along the 1D duct and the 2D chamber), corresponding to the maximum singular value of the resolvent operator and, most locally unstable root of unity. The nature of the maximum disturbance agrees with the results as described in the non-modal analysis of $An_1$. 
(a) \( p_1 \): \( K = 1.37028 \); \( \tau = 0.638783 \); Circles: Chamber eigenvalues close to the imaginary axis.

(b) \( p_2 \): \( K = 1.40073 \); \( \tau = 0.147331 \); Circles: Chamber eigenvalues close to the imaginary axis; Box: Non-normal behavior.

Figure 5.19: Resolvent norm corresponding to case \( Am_1 \), for roots of unity \( j = 0 \) through 8, over the non-dimensional frequencies \( \omega = [0.1, 6.1] \).
(a) Structure of the disturbance over the chamber; \( j = 3 \).
(b) Structure of the response over the chamber; \( j = 3 \).

(c) Structure of the disturbance over the burner; \( j = 3 \).
(d) Structure of the response over the burner; \( j = 3 \).

Figure 5.20: Most responsive disturbance and the corresponding response for case \( An_1 \), at point the p1: \( K = 1.37028 \), \( \tau = 0.638783 \) on the bifurcation diagram 5.18 and at the frequency \( \omega = 1.26 \) on the resolvent norm diagram 5.19a.
(a) Structure of the disturbance over the chamber; $j = 1$.
(b) Structure of the response over the chamber; $j = 1$.
(c) Structure of the disturbance over the burner; $j = 1$.
(d) Structure of the response over the burner; $j = 1$.

Figure 5.21: Most responsive disturbance and the corresponding response for case $An_1$, at point the $p1$: $K = 1.37028 \, \tau = 0.638783$ on the bifurcation diagram 5.18 and at the frequency $\omega = 4.7$ on the resolvent norm diagram 5.19a.
(a) Structure of the disturbance over the chamber; $j = 3$.
(b) Structure of the response over the chamber; $j = 3$.

c) Structure of the disturbance over the burner; $j = 3$.
(d) Structure of the response over the burner; $j = 3$.

Figure 5.22: Most responsive disturbance and the corresponding response for case $An_1$, at point the $p2$: $K = 1.40073$ $\tau = 0.147331$ on the bifurcation diagram 5.18 and at the frequency $\omega = 1.26$ on the resolvent norm diagram 5.19a.
(a) Structure of the disturbance over the chamber; \( j = 1 \).

(b) Structure of the response over the chamber; \( j = 1 \).

(c) Structure of the disturbance over the burner; \( j = 1 \).

(d) Structure of the response over the burner; \( j = 1 \).

Figure 5.23: Most responsive disturbance and the corresponding response for case \( An_1 \), at point the p2: \( K = 1.40073 \) \( \tau = 0.147331 \) on the bifurcation diagram 5.18 and at the frequency \( \omega = 4.7 \) on the resolvent norm diagram 5.19a.
Figure 5.24: Bifurcation diagram corresponding to case $A_{f_1}$. 
Figure 5.25: Resolvent norm corresponding to case $Af_1$, for roots of unity $j = 0$ through 8, over the non-dimensional frequencies $\omega = [0.1, 6.1]$. 

(a) $p1$: $K = 1.37028 \tau = 0.638783$; Circles: Chamber eigenvalues close to the imaginary axis.

(b) $p2$: $K = 1.40073 \tau = 0.147331$; Circles: Chamber eigenvalues close to the imaginary axis; Box: Non-normal behavior.
(a) Structure of the disturbance over the chamber; \( j = 2 \).

(b) Structure of the response over the chamber; \( j = 2 \).

(c) Structure of the disturbance over the burner; \( j = 2 \).

(d) Structure of the response over the burner; \( j = 2 \).

Figure 5.26: Most responsive disturbance and the corresponding response for case \( Af_1 \), at point the \( p_1 \): \( K = 1.37028 \tau = 0.638783 \) on the bifurcation diagram 5.24 and at the frequency \( \omega = 1.26 \) on the resolvent norm diagram 5.25a.
Figure 5.27: Most responsive disturbance and the corresponding response for case $Af_1$, at point the $p1$: $K = 1.37028 \; \tau = 0.638783$ on the bifurcation diagram 5.24 and at the frequency $\omega = 4.63$ on the resolvent norm diagram 5.25a.
(a) Structure of the disturbance over the chamber; $j = 2$.

(b) Structure of the response over the chamber; $j = 2$.

(c) Structure of the disturbance over the burner; $j = 2$.

(d) Structure of the response over the burner; $j = 2$.

Figure 5.28: Most responsive disturbance and the corresponding response for case $Af_1$, at point the $p_2$: $K = 1.40073 \, \tau = 0.147331$ on the bifurcation diagram 5.24 and at the frequency $\omega = 1.26$ on the resolvent norm diagram 5.25a.
Figure 5.29: Most responsive disturbance and the corresponding response for case $Af_1$, at point $p_2$: $K = 1.40073 \quad \tau = 0.147331$ on the bifurcation diagram 5.24 and at the frequency $\omega = 4.63$ on the resolvent norm diagram 5.25a.
Using a framework similar to the network model approach, accompanied by the roots of unity formalism [37], an efficient methodology to compute thermo-acoustic instabilities in $N$–periodic energy conversion devices was presented. The efficiency of this framework rests on the realization that the equations governing the thermo-acoustic behavior in $N$–periodic geometries forms a block-circulant system. The roots of unity formalism then reduces the entire $N$–periodic geometry to a modified single-unit system, thus achieving the computational cost of a isolated-unit periodic analyses while correctly modeling the full interaction with the $N - 1$ subcomponents. Furthermore, demonstrated in [37], features arising from the superposition of modal solutions can exhibit non-normal behavior only as a result of sub-unit (with appropriate root of unity type boundary conditions) dynamics. This allowed the receptivity of the global $N$–periodic energy conversion device to be analyzed as the superposition of $N$–local (smaller-scale) problems. Ensuring computational efficiency both in case of the modal and non-modal analyses.

Applying this framework to configurations (inspired by the experimental setup in [22]) the azimuthally coupled behavior of the periodically arranged burner elements and, the variation of this behavior as a function of system parameters such as heater strength ($K$), time delay ($\tau$) and location ($z_f$), has been presented. The modal analysis is summarized in a combined bifurcation diagram. Together these results show that the most unstable frequency and region of instability over the phase space $[K, \tau]$, is strongly dependent on the heat source’s location within the burner elements. Receptivity of an annular combustion chamber and afterburner geometry, measured using the resolvent norm, demonstrated the inherent non-normal behavior of such thermo-acoustic systems. These results indicate the variation in non-normal behavior over the asymptotically stable region of the system’s phase space $[K, \tau]$. 
Furthermore, a singular value decomposition of the system’s resolvent operator yields the most responsive disturbances and their corresponding response mode shapes. Finally, results of modal analysis over a geometry very similar to the experimental setup in [22] are shown to be in agreement with one another.

Therefore, the above mentioned results enforce the conclusion that the current framework is capable of guiding numerically intensive computations and experimental investigations of the complex thermo-acoustic behavior of such $N$–periodic energy conversion devices.
REFERENCES


[37] Peter J. Schmid, Miguel Fosas de Pando, and Nigel Peake. Stability analysis for n-periodic arrays of fluid systems. *In preparation*.


