STATICALLY INDETERMINATE STRESSES IN
RIGIDLY CONNECTED STRUCTURES
OF REINFORCED CONCRETE

BY

MIKISHI ABE
B. E. Tohoku Imperial University, Japan, 1905

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

DOCTOR OF PHILOSOPHY

IN ENGINEERING

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

1914
UNIVERSITY OF ILLINOIS
THE GRADUATE SCHOOL

May 15, 1914.

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

MIKISHI ABE

ENTITLED STATICALLY INDETERMINATE STRESSES IN RIGIDLY CONNECTED STRUCTURES OF REINFORCED CONCRETE

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN ENGINEERING.

AN[14]

In Charge of Major Work

AN[14]

Head of Department

Recommendation concurred in:

AN[14]

Committee on Final Examination

[Signatures of committee members]
CONTENTS

PART I.

I. Introduction.

1. Scope of Investigation and Acknowledgment

2. Recent Tendency in Design and Field of Application of Rigidly Connected Frames of Reinforced Concrete

3. Advantages of the Reinforced Concrete Frame

4. Reliability of Rigidly Connected Reinforced Concrete Frames

II. Fundamental Conceptions and Theorem.

5. Notation

6. Statically Determinate and Indeterminate Systems, and Number of Statically Indeterminates

7. Principle of Least Work

8. Position of Point of Inflection of a Member Fixed at One End when no Load is Applied on the Span

9. Effect of Direct Force on Final Formula for Statically Indeterminates

III. Analysis of Rigidly Connected Frames.

10. Analysis of Simple Frames under Vertical Load

Case 1. Special Frame with Hinged Ends under Concentrated Load

Case 2. Square Frame and Trestle Bent with Hinged Ends under Concentrated Load

Case 3. Special Frame with Hinged Ends under Symmetrical Concentrated Loads
CONTENTS (Continued)

Case 4. Square Frame and Trestle Bent with Hinged Ends under Symmetrical Loads. 25
Case 5. Special Frame with Hinged Ends under Uniform Loads ........................ 26
Case 6. Square Frame and Trestle Bent with Hinged Ends under Uniform Load or Loads. .................... 27
Case 7. Special Frame with Fixed Ends under Symmetrical Concentrated Loads ... 28
Case 8. Square Frame and Trestle Bent with Fixed Ends under Symmetrical Concentrated Loads ............... 30
Case 9. Trestle Bent with Fixed Ends under Concentrated Load ............................ 31
Case 10. Square Frame with Tie at Column Ends under Centrally Concentrated Load, 33
Case 11. Unsymmetrical Frame under Uniform Load ........................................ 33
Case 12. Unsymmetrical Frame with Fixed Ends under Uniform Load ......................... 36

II. Single Story Construction with Three Panels.
Case 13. Frame with Three Panels under Uniform Load (Hinged Ends of Beams and Columns). .................. 39
Case 14. Frame with Three Panels under Uniform Load (Hinged Ends of Beams, Fixed Column Ends). ........... 40
Case 15. Frame with Three Panels under Uniform Load (Hinged Ends of Columns but Fixed Beam Ends). ............ 41
Case 16. Frame with Three Panels under Uniform Load (Fixed Ends of Beams and Columns). ..................... 42
Case 17. Frame with Three Panels under Concentrated Load. ................................ 43
CONTENTS (Continued)

12. Trestle Bent with Tie .............................................. 46
   Case 16. Trestle Bent with Tie under Concentrated Load .............. 46

13. Building Construction with Several Stories and Number of Panels .......... 52
   Case 19. Frame with Hinged Ends. Special cases a, b and c .......... 53
   Case 20. Frame with Fixed Ends. Special cases a, b and c .......... 57

14. Bridge or Viaduct with Three Spans ................................ 62
   Case 21. Frame with Three Spans, Hinged Column Ends. Special cases a and b, 63
   Case 22. Frame with Three Spans, Fixed Column Ends. Special cases a, b, c and d .......... 67

15. Square Frame under Horizontal Load ................................ 71
   Case 23. Square Frame under Concentrated Horizontal Force .......... 71
   Case 24. Square Frame under Uniform Horizontal Load .................. 73
   Case 25. Square Frame having Two Spans, subjected to Concentrated Horizontal Load .............. 75
   Case 26. Water Tank and Reservoir. .................................. 77
     Special case a. Rectangular Tank
     Special case b. Square Tank.

16. Deflection of Frame .............................................. 78

IV. Discussion of Nature of Resulting Formula.

17. Relation between Horizontal Reactions in Symmetrical Frames under Uniform and Concentrated Loads .............. 82

18. Effect of Variation in Moment of Inertia and Relative Height of Frame on Bending Moment in Horizontal Member .............. 88

19. Effect of Variation in Moment of Inertia on Bending Moment in Vertical Member .............................................. 95
## PART II.

### V. Materials, Test Frames, and Method of Testing

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>20. Scope of Tests</td>
<td>104</td>
</tr>
<tr>
<td>Materials</td>
<td>104</td>
</tr>
<tr>
<td>Test Frames</td>
<td>108</td>
</tr>
<tr>
<td>Making Test Frames</td>
<td>109</td>
</tr>
<tr>
<td>Storage</td>
<td>109</td>
</tr>
<tr>
<td>Minor Test Pieces</td>
<td>110</td>
</tr>
<tr>
<td>Method of Testing</td>
<td>110</td>
</tr>
<tr>
<td>Extensometers and Method of Measuring Deformations</td>
<td>110</td>
</tr>
<tr>
<td>Method of Loading</td>
<td>110</td>
</tr>
</tbody>
</table>

### VI. Experimental Data

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>22. Explanation of Table, Diagrams, Drawings and Photographs</td>
<td>117</td>
</tr>
<tr>
<td>23. Phenomena of Frame Tests</td>
<td>117</td>
</tr>
</tbody>
</table>

### VII. Comparison of Theory with Experiment

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>24. General Statement</td>
<td>143</td>
</tr>
<tr>
<td>25. Values of Moduli of Elasticity of Concrete</td>
<td>145</td>
</tr>
<tr>
<td>26. Calculation of Theoretical Stresses</td>
<td>146</td>
</tr>
<tr>
<td>27. Maximum and Minimum Difference between Theoretical and Experimental Values of Stresses</td>
<td>159</td>
</tr>
<tr>
<td>28. Distribution of Stresses in Frames</td>
<td>160</td>
</tr>
<tr>
<td>29. Summary of Comparison</td>
<td>181</td>
</tr>
</tbody>
</table>

### VIII. General Discussion

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>30. Action of Reinforced Concrete Frames under Loads</td>
<td>183</td>
</tr>
<tr>
<td>31. Effect of End Condition of Column on Results</td>
<td>183</td>
</tr>
<tr>
<td>32. Effect of Non-uniformity of Quality of Concrete on Distribution of Stresses</td>
<td>184</td>
</tr>
</tbody>
</table>
V.

33. Distribution of Stress over the Cross Section. 185
34. Position of Point of Inflection in Columns. 188
35. Continuity of the Composing Members of a Frame. 200
36. Stresses at Sharp Corner. 200
37. Critical Point of Failure. 203
38. Deflection. 204

IX. Conclusions and General Comments. 208
# TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>Reactions, Bending Moments and Position of Point of Inflection in Frames with Three Panels</td>
<td>45</td>
</tr>
<tr>
<td>II.</td>
<td>Coefficients of Bending Moment for Frame with Hinged Ends (Case 14)</td>
<td>89</td>
</tr>
<tr>
<td>III.</td>
<td>Coefficients of Bending Moment for Frame with Fixed Ends (Case 14)</td>
<td>91</td>
</tr>
<tr>
<td>IV.</td>
<td>Coefficients of Bending Moment for Frame with Fixed Ends (Case 16)</td>
<td>93</td>
</tr>
<tr>
<td>V.</td>
<td>Coefficient of Bending Moment at Column Ends of Frame (Case 20)</td>
<td>96</td>
</tr>
<tr>
<td>VI.</td>
<td>Coefficients of Bending Moment for Frame of Hinged Ends (Case 21) Special Case b</td>
<td>98</td>
</tr>
<tr>
<td>VII.</td>
<td>Coefficients of Bending Moment for Frame of Fixed Ends (Case 22) Special Case b</td>
<td>100</td>
</tr>
<tr>
<td>VIII.</td>
<td>Tensile Strength of Cement</td>
<td>106</td>
</tr>
<tr>
<td>IX.</td>
<td>Tensile Test of Steel</td>
<td>107</td>
</tr>
<tr>
<td>X.</td>
<td>Data of Frames</td>
<td>108</td>
</tr>
<tr>
<td>XI.</td>
<td>Compression Tests of 6-in. Cubes</td>
<td>112</td>
</tr>
<tr>
<td>XII.</td>
<td>General Results of Tests of Frames</td>
<td>118</td>
</tr>
<tr>
<td>XIII.</td>
<td>Values used in Theoretical Calculations</td>
<td>148–9</td>
</tr>
<tr>
<td>XIV.</td>
<td>Comparison of Theory with Experiment, Frame No. 1</td>
<td>151</td>
</tr>
<tr>
<td>XV.</td>
<td>Comparison of Theory with Experiment, Frame No. 2</td>
<td>152</td>
</tr>
<tr>
<td>XVI.</td>
<td>Comparison of Theory with Experiment, Frame No. 3</td>
<td>153</td>
</tr>
<tr>
<td>XVII.</td>
<td>Comparison of Theory with Experiment, Frame No. 4</td>
<td>154</td>
</tr>
<tr>
<td>XVIII.</td>
<td>Comparison of Theory with Experiment, Frame No. 5</td>
<td>155</td>
</tr>
</tbody>
</table>
### TABLES (Continued)

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>XIX.</td>
<td>Comparison of Theory with Experiment, Frame No. 6</td>
<td>156</td>
</tr>
<tr>
<td>XX.</td>
<td>Comparison of Theory with Experiment, Frame No. 7</td>
<td>157</td>
</tr>
<tr>
<td>XXI.</td>
<td>Comparison of Theory with Experiment, Frame No. 8</td>
<td>158</td>
</tr>
<tr>
<td>XXII.</td>
<td>Maximum and Minimum Difference between Theoretical and Experimental Values of Stresses</td>
<td>159</td>
</tr>
<tr>
<td>XXIII.</td>
<td>Effect of Non-uniformity of Concrete on Distribution of Stress</td>
<td>184</td>
</tr>
<tr>
<td>XXIV.</td>
<td>Flexural Deformation in Column, Frame No. 3</td>
<td>190</td>
</tr>
<tr>
<td>XXV.</td>
<td>Flexural Deformation in Column, Frame No. 5</td>
<td>191</td>
</tr>
<tr>
<td>XXVI.</td>
<td>Flexural Deformation in Column, Frame No. 7</td>
<td>192</td>
</tr>
<tr>
<td>XXVII.</td>
<td>Flexural Deformation in Column, Frame No. 8</td>
<td>193</td>
</tr>
<tr>
<td>XXVIII.</td>
<td>Comparison of Experimental Stress at Sharp Corner with Theoretical Value</td>
<td>202</td>
</tr>
<tr>
<td>XXIX.</td>
<td>Vertical Deflection at the Center of Span</td>
<td>205</td>
</tr>
</tbody>
</table>
VIII.

DIAGRAMS

1. Value of Constants in Case 18 .................................. 50
2. do. .................................. 51
3. Effect of Variation in Moment of Inertia and Relative Height of Frame on Bending Moment in Horizontal Member (Case 13). .......... 90
4. do. (Case 14) .................................. 92
5. do. (Case 16) .................................. 94
6. do. (Case 20) .................................. 97
7. do. (Case 21) .................................. 99
8. do. (Case 22) .................................. 102
9. do. (Case 23) .................................. 103
10-13. Concrete Cylinder Tests .................................. 113-6
14. Diagram showing Points Selected for the Purpose of Comparison of Theoretical Experiments. .............................. 147
15-17. Distribution of Stresses in Frame No. 1, 161-3
18-19. do. in Frame No. 2, 164-5
20-22. do. in Frame No. 3, 166-8
23-24. do. in Frame No. 4, 169-70
25-27. do. in Frame No. 5, 171-3
28-29. do. in Frame No. 6, 174-5
30-31. do. in Frame No. 7, 176-7
32-35. do. in Frame No. 8, 178-81
36. Distribution of Stress over Cross Section, 186
37. do. 187
38-39. Position of Point of Inflection in Frame No. 3 and 7 ............... 194-5
DIAGRAMS (Continued)

40–41. Position of Point of Inflection in Frame No. 5, 196–7
42–43. do. in Frame No. 8, 198–9
44–45. Load–Deflection Diagram . . . . . . . . . . 206–7
46–77. Stresses in Individual Gage Line . . . . . . .213–244
ANALYSIS
STATICALLY INDETERMINATE STRESSES IN RIGIDLY CONNECTED STRUCTURES OF REINFORCED CONCRETE

I. INTRODUCTION.

1. Scope of Investigation and Acknowledgment.—The exact determination of stresses as they actually occur in a rigidly connected frame under load, in spite of its importance in designing a new structure, has but rarely been attempted. The reason for this lies in the fact that it requires considerable time and labor to work out the formulas for statically indeterminate quantities which depend upon the number and fixity of the supports.

In order to obtain, by mathematical calculation, the statically indeterminate stresses due to rigid connection of members, various principles have been advanced.

The writer has deduced the formulas for several types of statically indeterminate structures using the principle of Least Work. In order to put these to practical test, that is the reliability of these formulas for reinforced concrete structures, test specimens of large size were made as a part of the work of the Engineering Experiment Station of the University of Illinois, and the deformations produced in their members by the series of loadings were measured. These results were subjected to investigation and discussion. The work was under the immediate charge of Professor A. II. Talbot, to whom, as well as to other members of the staff, acknowledgments are due for many valuable suggestions and aids.
2. Recent Tendency in Design of Application of Rigidly Connected Frames of Reinforced Concrete.—Reinforced concrete frame constructions were not extensively used in the world before 1905. Since that time they form practically a standard practice in Europe. Many examples can be found in the German texts and magazines. It is also a recent tendency in America to use the reinforced concrete frames for buildings and bridges. A number of very long and high trestles have been constructed during the past few years, one of the longest being the Richmond and Chesapeake Bay Viaduct (2,800 feet long, ranging in height from 18 feet to 70 feet at its highest point).

It is also a tendency in England to use the frame constructions of reinforced concrete during the last few years. A number of reinforced concrete jetties and viaducts were constructed there (Concrete and Constructional Engineering, London).

The field of the application of rigid frames is almost unlimited, for all of the reinforced concrete structures are composed of elements of rigid frames, but it may be subdivided into several classes for the purpose of discussing the applicability. That is to say:

1. Building construction in general,
2. Bridge structure,
3. Trestle and viaduct,
4. Culvert and sewer construction,
5. Subway construction,
6. Retaining walls,
7. Reservoir and water tanks.
1. Every building construction of reinforced concrete may be considered as a rigidly connected frame, for walls, columns, beams and slabs are all rigidly connected with each other, therefore it may be found best to design such structures as a rigid frame under certain assumption. In the continental European countries it is most common to use frames in building constructions, such as roofs, cantilevers, balconies, towers and a building as a whole.

2. Bridge constructions are entirely in the field of the rigid frame. Arches, beam and bent constructions, cantilever bridges and the most of bridge structures can be designed as a frame on rigid analytical basis. In highway bridges, for example, a spandrel braced arch, as shown in the accompanying diagram, is frequently used. In such a case, columns are rigidly connected to the arch ribs and to the superstructure, and therefore they must be designed as a frame.

3. Trestles and viaducts, as shown in the following sketches, must be designed as a frame to secure the safety and to obtain the best proportioning of materials.

4-5. In most cases in practice the monolithic pipe, circular or elliptic form, and box culverts of reinforced concrete are advantageously used. In subway construction, however, a large
4. Box type construction, as shown in the accompanying sketch, is frequently used. In these cases a structure is subject to the earth pressure on its exterior surface and to the live loads on the interior faces. The bending moment, which is statically indeterminate, exists at each rigid joint and these moments vary with the relative stiffness of composing members. Therefore such structures may not be rationally designed without a sufficient knowledge of rigid frames.

6. The retaining walls, dams and bridge abutments of a buttress type are frequently used in the present practice. These will analytically be designed by the aid of rigid frame under proper assumptions.

7. In water tanks and reservoirs of a rectangular or a polygonal form the unknown negative bending moment due to a rigid connection of a wall to wall or a base to wall, as shown in the following sketch, will exist at each corner. These moments are modified by the relative thickness of walls and the dimension of a structure. The knowledge of a rigid frame will suggest the proper method of a solution.

From the above brief statement we can easily see that the monolithic construction falls within the class of rigid frames,
and therefore the only method which develops a proper judgment in the design of such structures will be found in the field of study of rigid frames.

3. Advantages of Reinforced Concrete Frame.—In actual design, insufficient attention is often paid to bending in columns caused by the rigidity of connections. In building construction, bridge work and other structural designs, the bending moment for a beam is frequently taken as an assumed fraction of \( pL \) (where \( p \) is the load and \( L \) is the span), while bending moments at the ends and in the columns are disregarded entirely. The effect of this is either to make the whole structure inadequate or to make one part stronger at the expense of the other. The usual arbitrary practice, provided sufficient attention is paid to negative moment, gives safe results but is not economical in a strict sense.

The reinforced concrete frame has a number of strong points as a structure, for material can be saved and a much better result obtained from the theoretical and structural point of view. In ordinary concrete building construction the entire structure is rigid to a certain extent, but for simplicity this important element of rigidity is usually not fully taken advantage of. With the concrete frame construction, however, advantage may well be taken of the rigidity of the connecting members. Exact analysis is possible, though there are hardly any publications in English which treat of it systematically, and formulas are not in shape for practical use. The rigid frame is capable of exact design, and therefore the economical distribution of materials can be realized.
4. **Reliability of Rigidly Connected Reinforced Concrete Frames.**—Reinforced concrete frames will be perfectly reliable when the following are fulfilled:

a. Perfect continuity or rigidity of joint;

b. Close agreement between theory and experiment;

c. Limited amount of stresses of secondary nature.

So far as the writer is aware no experimental study has been made on this subject and we have no positive or negative proof to these ends.

A partial reason why many engineers hesitate to use concrete frames extensively lies in the fact that they hardly believe in the continuity of the members and doubt the effect of the rigidity. In ordinary design they have no doubt of the resistance to the positive bending moment in the center or the negative moment near the fixed ends of a beam, but they object to allowing reinforced concrete members to take the successive positive and negative bending moments along the axis of a frame.

A question is naturally raised over the fact that reinforced concrete is not homogeneous material, and it is doubted whether or not the formulas deduced from its elastic work of deformation will hold good for such composite members with a fair agreement. Furthermore, secondary stresses may exist in the section of a member. In the actual condition of things, as is well known, the fundamental assumptions which underlie the static considerations can seldom be more than partially fulfilled even under carefully prepared specifications and well executed designs.

All these things must be considered before coming to a con-
elusion on the reliability of rigidly connected reinforced concrete frames. Careful experiments and investigation may decide these questions clearly.
II. FUNDAMENTAL CONCEPTIONS AND THEOREMS.

5. Notation.—The following notation is used throughout the work:

\( A = \) area of cross-section of a member. Numerical suffixes are used for individual members when a frame is composed of members of different size.

\( a = \) distance from the left corner or axis of a frame to the point of application of a concentrated load on a top beam.

\( b = \) distance from the right corner or axis of a frame to the point of application of a concentrated load on the top beam.

\( E = \) modulus of elasticity of the material (considered as constant).

\( H = \) horizontal reaction acting at the end of a column.

\( I = \) moment of inertia in general.

\( I_0 = \) moment of inertia of a member. Other numerical suffixes are used to distinguish one from another.

\( h = \) height of a frame.

\( h_0 = \) total height of a frame.

\( h_1, h_2 = \) subdivision of \( h_0 \).

\( l = \) length of span of a frame having vertical columns.

\( l_0 = \) total length of span.

\( l_1, l_2, \) and \( l_3 = \) subdivision of \( l_0 \).

\( m = \frac{I_l}{I_0} = \) ratio of moment of inertia of horizontal member to that of vertical member.

\( n = \frac{h}{l} = \) ratio of height of frame to length of span.

\( M = \) bending moment in general. Special suffixes are used to denote the bending moment at a specified point.

\( N = \) normal force at a section.

\( P = \) a concentrated load.
p = intensity of a uniformly distributed load. Suffixes are used to denote the different intensities.

V, V₀, V₁, etc. = the vertical reaction at a specified section.

6. **Statically Determine and Indeterminate Systems, and Number of Statically Indeterminates.**—A force is said to be and statically determinate when its direction, magnitude and its point of application are known from the conditions of static equilibrium. The conditions of static equilibrium for any number of forces in a plane are generally well known to be three, that is to say,

(a) \( V = 0 \), or the algebraic sum of all vertical forces acting on a body is equal to zero,

(b) \( H = 0 \), or the algebraic sum of all horizontal forces acting on a body is equal to zero,

(c) \( M = 0 \), or the algebraic sum of the moments of all forces is equal to zero.

The loads to which structures may be subjected are always given. The other external forces are the reactions due to the loads. The reactions are exerted by the supports of the structure, and in order that they may be determined from the statical conditions the total number of unknowns must not exceed three. The ordinary trusses without redundant members are always statically determinate if a frictionless pin is used at each joint. If we consider a case in which two members meet at a joint as shown in the accompanying figure, two unknown forces exist at the joint; that is to say, the vertical and horizontal forces acting at the joint, and therefore the total number of
unknown forces due to the external force $P$ is six. But each mem-
ber will give three statical conditions as stated before, and
therefore this is a statically determinate system.

To conceive of the behavior of statically determinate and
indeterminate systems, it will be found that the conception of
the connection of members by means of joint bars is a convenience.

Now, in Fig. 2, a, the two members are not connected, and therefore
they are merely in loose touch. They are entirely free to move
horizontally and vertically, and also free in rotation; accordingly
it may be called the arrangement having the three freedoms in
motion. An example of this arrangement is a touching joint between

the free ends of cantilever beams as shown in Fig 2,a2. In Fig.
2,b two members are connected by a single bar, and they are pre-
vented from moving vertically, but are free in horizontal motion
and in rotation about a point A. This may be called the arrange-
ment having two freedoms in motion. An example of this arrangement
is the frictionless roller end of a cantilever as shown in Fig.
2,b2. In Fig. 2,c two members are connected by two connecting
bars, so that only one freedom in motion is allowed to take place,
that is to say, the rotation of a member about A, the intersecting
point of two bars. The crown hinge of an arch is an example. In Fig. 2, two members are connected by three joint-bars, and may be called a rigid connection, which allows no freedom in motion. The restrained end of a cantilever beam is an example of this arrangement.

To make a rigid joint it is necessary to have three joint-bars at each member, therefore $3S$ conditions of the equilibrium must be set up to determine $3S$ unknowns when the structure is composed of $S$ members. If the structure is rigidly connected to the ground, more condition than $3S$ are required. Now let $a$ be the number of the joint-bars needed to connect one member to another, and $b$ be the number of the joint bars needed to connect the member to the ground, then we have from the existing $3S$ conditions the following relation to make the structure statically determinate,

$$a + b = 3S.$$  

When the members in the structure are all rigidly connected to each other $a + b$ always exceeds $3S$, and therefore the case falls to a statically indeterminate system, and the relation is modified as follows:

$$a + b - 3S = m,$$

where $m$ represents the number of the statically indeterminate forces. Such system is called $m$-fold statically indeterminate case, and $m$ additional equations of the condition are necessary to determine these unknowns.

Let us consider a few examples.
Case a. \( a+b-3S=5+4-9=0 \), therefore statically determinate,
Case b. \( a+b-3S=6+4-9=1 \), 1-fold statically indeterminate,
Case c. \( a+b-3S=6+12-9=9 \), 9-fold statically indeterminate,
Case d. \( a+b-3S=24+16-21=21 \), 21-fold statically indeterminate,
Case e. \( a+b-3S=36+3-30=9 \), 9-fold statically indeterminate.

It must be borne in mind that there are certain exceptional cases. The accompanying figure shows a case in which two members have three joints. The unknown forces are two member-stresses and four reactions (vertical and horizontal) exerted at both ends. The two members give us six conditional equations, so that this will be a statically determinate system. If we apply a force \( P \) vertically at the center joint, infinitely large lateral stresses are exerted in the member. This state will exist only when the members are perfectly rigid. If this is not the case, a deflection of the middle joint will take place, and thereby the stresses in the members become finite. But they are still indeterminate by the statical conditions, that is to say, the system becomes statically indeterminate. The deviation of
the middle joint has therefore a large influence upon the stresses.

7. Principle of Least Work.—There is a certain law of nature called the principle of least work, that is to say, the resisting forces will store up no more energy than the minimum which is necessary to maintain equilibrium with the external forces, or in other words, the external forces are so adjusted, themselves, as to develop internal forces in the structure which will make the total internal work of resistance in the latter a minimum. When forces act upon an elastic system, in which the deformations are proportional to the stresses, the above principle of least work may be applied to determine the statically indeterminate forces.

The principle of least work has been known for a hundred years, but the first complete announcement of this theorem was given by Castigliano (1879). Cain expresses the principle in the following words:*

"The elastic forces experienced between the molecules after deformation correspond to a minimum of the work of deformation of the system, expressed as a function of certain stresses, taken with respect to these stresses successively, regarded as independent during the differentiation". Professor Hiroi* expresses it in the other words "the partial derivatives of the work of resistance with respect to statically indeterminate forces which are so chosen that the forces themselves perform no work, are equal to zero."

*See Statically Indeterminate Stresses by I. Hiroi.
The total internal work may be subdivided into several classes, those due to bending moment, normal stress, and shearing stress.

The total work due to bending moment $M$ of a member will be

$$
\omega_M = \int \frac{M'dx}{2EI} \tag{a}
$$

For the total internal work due to a normal stress $N$ we have

$$
\omega_N = \int \frac{N'dx}{2EA} \tag{b}
$$

If shearing stress $S$ is uniformly distributed over the cross-section, we have for the internal work due to shearing stress

$$
\omega_S = \int \frac{S'dx}{2GA} \tag{c}
$$

where $G$ expresses the shearing modulus of elasticity of a material. Since, however, the intensity of shear at various points of the cross-section differs with the form of the latter, the expression for the internal work due to shear is modified, and

$$
\omega_{S'} = \int \frac{kS'dx}{2GA} \tag{c}
$$

where $K$ is a known factor for a specified case.

Therefore we have for the total work of resistance,

$$
W = (a) + (b) + (c) = \frac{1}{2} \int \frac{M'dx}{EI} + \frac{1}{2} \int \frac{N'dx}{EA} + \frac{1}{2} \int \frac{kS'dx}{GA}
$$

Let us suppose that there are $n$ statically indeterminate forces $X_1, X_2, \ldots, X_n$ in an elastic system. According to the theorem of Castigliano we have:

$$
\int \frac{M \partial M}{EI \partial X_i} dx + \int \frac{N \partial N}{EA \partial X_i} dx + \int \frac{kS \partial S}{GA \partial X_i} dx = 0
$$

$$
\int \frac{M \partial M}{EI \partial X_2} dx + \int \frac{N \partial N}{EA \partial X_2} dx + \int \frac{kS \partial S}{GA \partial X_2} dx = 0
$$

$$
\int \frac{M \partial M}{EI \partial X_n} dx + \int \frac{N \partial N}{EA \partial X_n} dx + \int \frac{kS \partial S}{GA \partial X_n} dx = 0
$$
These furnish us as many equations of conditions as there are unknown quantities. The solution of these equations will give us the exact formulas for statically indeterminate quantities.

8. Position of Point of Inflection of a Member fixed at One End when no Load is Applied on the Span.—Let us suppose that the member is perfectly fixed at A and supported or partially fixed at B, therefore no deflection at A and B, and, that no load is applied on the member between A and B. Now V, H, and $M_0$ are the vertical and horizontal reactions and the bending moment at the fixed end A due to a force $P$ applied at any point of the member beyond the point B. The bending moment at the point of inflection is obviously zero and

$$M_o + H.a \cdot \sin \theta - V.a \cdot \cos \theta = 0$$

or

$$M_o = V.a \cdot \cos \theta - H.a \cdot \sin \theta$$

The bending moment at any point is

$$M = M_0 - Vx \cos \theta + Hx \sin \theta = EI \frac{dy}{dx^2}$$

or

$$EI \frac{dy}{dx^2} = V(a-x) \cos \theta - H(a-x) \sin \theta$$

Assuming a constant moment of inertia and a constant modulus of elasticity, and integrating this equation, we get

$$EI \frac{dy}{dx} = V(ax - \frac{x^2}{2}) \cos \theta - H(ax - \frac{x^2}{2}) \sin \theta + C$$

but for $x = 0$, $\frac{dy}{dx} = 0$, and therefore $C = 0$. Hence

$$EI \frac{dy}{dx} = V(ax - \frac{x^2}{2}) \cos \theta - H(ax - \frac{x^2}{2}) \sin \theta.$$
Integrating, again
\[ EIy = V\left(\frac{ax^2}{2} - \frac{x^3}{6}\right) \cos \theta - H\left(\frac{ax^2}{2} - \frac{x^3}{6}\right) \sin \theta + C_2, \]
but for \( x = 0 \) and \( x = l \), \( y = 0 \) and therefore,
\[ C_2 = V\left(\frac{l^3}{6} - \frac{al^2}{2}\right) \cos \theta - H\left(\frac{l^3}{6} - \frac{al^2}{2}\right) \sin \theta, \]
and hence
\[ EIy = V\cos \theta \left[\frac{l^3 - x^3}{6} - \frac{a}{2} (l^2 - x^2)\right] - H\sin \theta \left[\frac{l^3 - x^3}{6} - \frac{a}{2} (l^2 - x^2)\right] \]
or
\[ y = \frac{V\cos \theta - H\sin \theta}{2EI} \left[\frac{l^3 - x^3}{3} - a(l^2 - x^2)\right] \]
(1)

This is the equation of the elastic line for all points from A to B.

Now for \( x = 0, \ y = 0 \), and therefore,
\[ \frac{V\cos \theta - H\sin \theta}{2EI} \left(\frac{l}{3} - a\right) = 0, \]
Assuming that \( 2EI \) is not equal to the infinity, and \( l \) is not zero, the above equation is satisfied by the following relation:
\[ V\cos \theta - H\sin \theta = 0, \]
or \( \frac{l}{3} - a = 0 \).

If \( V\cos \theta - H\sin \theta \neq 0 \)
Then we have
\[ a = \frac{1}{3}, \]
(2)
That is to say the point of inflection is always located at \( \frac{1}{3}l \) from the fixed end of the member.
If \( \theta \) becomes zero or the direction of \( H \) coincides with the axis of the member, \( H \sin \theta \) will disappear and the general equation for the elastic curve is
\[ y = \frac{V}{2EI} \left[\frac{l^3 - x^3}{3} - a(l^2 - x^2)\right], \]
(1')
for \( x = 0, \ y = 0 \), and therefore
\[ \frac{VL^2}{2EI} \left(\frac{l}{3} - a\right) = 0, \]
and accordingly \( a = \frac{l}{3} \), for all cases except \( \frac{VL^2}{2EI} = 0 \).
From this important relation, we can express the end moment $M_0$ in terms of $H$ and $V$, and the mathematical operations in the determination of the statically-indeterminate stresses are greatly simplified. We used this relation in the present work.

9. Effect of Direct Force on Final Formula for Statically Indeterminates.—Let us now consider the effect of the direct force which is normal to the cross section, on the final formula for a statically indeterminate stress.

In the general equation

$$\int \frac{M}{EI} \frac{\partial M}{\partial x} \, dx + \int \frac{N}{EA} \frac{\partial N}{\partial x} \, dx + \int \frac{KS}{\sigma A} \, dx = 0$$

we can see that the second term will disappear when $\frac{\partial N}{\partial x}$ is equal to zero, or in other words, when the normal force does not contain any of the statically indeterminate quantities. If the frame has hinges at the ends of columns, the vertical reactions are always statically determinate, and the only statically indeterminate is the horizontal reaction. But if the columns are vertical, then the normal force in the column contains no statically indeterminate. The only member in which a statically indeterminate is contained in the direct force is the top beam. The normal force in this case is the horizontal reaction, but the horizontal reaction due to vertical load or loads on the frame is small in comparison with the vertical reaction, and consequently the effect of the normal force on the final formula for the statically indeterminate is clearly negligible.

When a frame is fixed at its column ends, the vertical and horizontal reactions and the bending moment at the fixed column ends are statically indeterminate. The normal force contains
generally these reactions as a factor. But if the frame is symmetrical in form and in the manner of loading, the vertical reactions become statically determinate, and therefore the horizontal reaction is only term which enters in the expression of the normal force. The normal force in the column, however, contains no factor of statically indeterminates when the fixed columns are vertical. The normal force, which contains the statically indeterminate, is that of the horizontal member (top beam in a simple frame), but this has very small effect on the final results.

From the above statements we can see that the form of frame which will be largely affected by the normal force is that having a sloped column under a vertical load.

The following frame is used to illustrate the method of analysis and to bring out the effect of the direct force on the final expression for the value of the statically indeterminate forces.

In this case \( H \) is the only statically indeterminate force.

Taking the moment of all forces about \( F \) we have,

\[ V_{1}l_{0} - P(b + l_{3}) = 0, \text{ or } V_{1} = \frac{l_{3} + b}{l_{0}}P = K\cdot P, \]

and \[ V_{2} = P - KP, \text{ or } V_{2} = (1 - K)P.\]
Therefore we have in general (neglecting the internal work due to shearing stress)

\[ \int \frac{M}{E I} \frac{\partial M}{\partial H} \, dx + \int \frac{N}{E A} \frac{\partial N}{\partial H} \, dx = 0 \]

All necessary elements in forming the above equation are arranged in the following tabular form.

<table>
<thead>
<tr>
<th>Mem-Number of</th>
<th>Moment Limits of Inertia Integration</th>
<th>Member Specified Member</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB ( I_1 )</td>
<td>From zero to ( S_1 ) ( V_1 x - H y = V_1 \tan \theta_1 y - H y, V_1 \cos \theta_1 + H \sin \theta_1 - y \sin \theta_1 )</td>
<td></td>
</tr>
<tr>
<td>BC ( I_2 )</td>
<td>From zero to ( a ) ( V_1(l_1 + x) - H h ), ( H ) ( -h + 1 )</td>
<td></td>
</tr>
<tr>
<td>CD ( I_2 )</td>
<td>From ( a ) to ( l_2 ) ( V_1(l_1 + x) - H h - P(x - a) ), ( H ) ( -h + 1 )</td>
<td></td>
</tr>
<tr>
<td>DF ( I_3 )</td>
<td>From zero to ( S_2 ) ( V_2 x - H y = V_2 \tan \theta_2 y - H y, V_2 \cos \theta_2 + H \sin \theta_2 - y \sin \theta_2 )</td>
<td></td>
</tr>
</tbody>
</table>

Inserting these values in the general equation we have the following expression in which we have assumed that \( E \) and \( I \) are constant:

\[ \frac{1}{E I_1} \int_0^h (V_1 \tan \theta_1 y - H y)(-y) \sec \theta_1 \, dy + \frac{1}{E I_2} \int_0^a (V_1 l + V_1 x - H h)(-h) \, dx \]

\[ + \frac{1}{E I_2} \int_0^h (V_2 l + V_2 x - H h - P(x - a))(x - a) \, dx + \frac{1}{E I_3} \int_0^h (V_2 \tan \theta_2 y - H y)(y) \sec \theta_2 \, dy \]

\[ + \frac{1}{E A_1} \int_0^h (V_1 \cos \theta_1 + H \sin \theta_1) \sin \theta_1 \, \sec \theta_1 \, dy + \int_0^h \frac{1}{E A_2} \left[ \int_0^a H d x + \int_a^l H d x \right] \]

\[ + \frac{1}{E A_3} \int_0^h \left[ V_2 \cos \theta_2 + H \sin \theta_2 \right] \sin \theta_2 \, \sec \theta_2 \, dy = 0. \]

Integrating and simplifying this equation we get the following general expression for the statically indeterminate force \( H \) in which the effect of the direct force is fully counted.
In this formula, the term which contains $A_1$, $A_2$ and $A_3$ is introduced by taking into consideration the direct forces represented by the second term of the general equation

$$H = \frac{K}{3I_2} \left\{ \frac{S_1}{3I_1} + \frac{S_2}{3I_2} + \frac{S_3}{3I_3} - \frac{S_2}{3I_2} \right\} \left( \frac{b^2}{2I_2} - \frac{S_2}{3I_2} \right) - \left( \frac{S_1}{3I_1} + \frac{S_3}{3I_3} \right) \right\} \left( \frac{b^2}{2I_2} - \frac{S_2}{3I_2} \right)$$

Where $K = \frac{I_2 + b}{l_0}$.

In this formula, the term which contains $A_1$, $A_2$ and $A_3$ is introduced by taking into consideration the direct forces represented by the second term of the general equation

$$\int \frac{M_1}{EI} \partial M_1 \partial x + \int \frac{N_1}{EA} \partial N_1 \partial x = 0.$$ 

If we neglect the effect of the direct force, then we have

$$H = \frac{K}{3I_2} \left\{ \frac{S_1}{3I_1} + \frac{S_2}{3I_2} + \frac{S_3}{3I_3} - \frac{S_2}{3I_2} \right\} \left( \frac{b^2}{2I_2} - \frac{S_2}{3I_2} \right)$$

The frame under consideration is of the form in which the direct forces have larger effect on the final formula for $H$ than any other kind of simple frames.

In most cases, however, $\theta$ will not exceed $30^\circ$, because the increase in $\theta$ will very rapidly increase the horizontal reaction at the end of the column.

Now let us take an example in which $l_1 = l_2 = l_3 = l = 120''$. $\theta_1 = \theta_2 = 45^\circ$. $A_1 = A_2 = A_3 = 10'' \times 12''$, $I_1 = I_2 = I_3 = 1,000$ in. When considered the effect of direct force, we have

$$H = 0.56506 \ P.$$

If we neglect the effect of the direct force, we get

$$H = 0.56466 \ P.$$ 

Accordingly the difference is $0.0004 \ P$ or $\frac{1}{2500} \ P$, which is inconsiderable. This is an example illustrating the effect of the direct force upon the final value of the statically indeterminate.
From the foregoing deduction we have seen that the final formula is very much complicated by taking the direct force into consideration. But we can easily see from the above example that the effect of the internal work due to all direct stresses on the final value for statically indeterminate stresses is very small and is inconsiderable when compared with that of the bending moment. Attention is called to the fact that even although the effect of the internal work of the direct stresses may be neglected in determining the reactions the direct stresses themselves cannot be neglected when calculating the total stress in any member. The direct stresses may algebraically be added after the statically-indeterminate stresses are found.

The deformation due to shear is generally so insignificant when compared with that due to the bending, that it may be entirely neglected without sensible error in the calculation of the internal work.

Therefore these two terms are disregarded in the following deductions, for a sensible error in the calculation is not produced by this negligence, while the final formulas are very much simplified by doing so.
III. ANALYSIS OF RIGIDLY CONNECTED FRAMES.

The method of analysis will now be applied to a variety of several forms of frames, with both concentrated and distributed loading and with hinged and fixed ends. The types of frames are subdivided into the following classes as a convenience of treatment:

10. Analysis of simple frames under vertical load.

11. Analysis of single story constructions with three panels under vertical load.

12. Analysis of a trestle bent with a tie under concentrated vertical load.

13. Analysis of a building construction with several stories and number of panels under vertical load.

14. Analysis of a bridge or viaduct with three spans under vertical load.

15. Analysis of square frames under horizontal load.

10. Analysis of Simple Frames under Vertical Load.

Case 1. Special Frame with Hinged Ends under Concentrated Load.

In this case the statically indeterminate force is H. Neglecting the effect of normal force, the equation of condition is

$$\int \frac{M}{EI} \frac{2M}{bH} \, dx = 0.$$ 

All necessary quantities in forming the equation are arranged in the following tabular form.
The conditional equation is as follows:

\[
\frac{E}{I} \int H \, dy - \frac{2 \sec \theta}{E I_z} \int \left[ \frac{P x}{2} - H (h_1 + x \tan \theta) \right] (h_1 + x \tan \theta) \, dx
\]

\[
- \frac{h_0}{E I} \left\{ \int_0^\frac{l}{2} \left[ \frac{P x}{2} - H h_0 \right] \, dx + \int_\frac{l}{2}^l \left[ \frac{P (l - x)}{2} - H h_0 \right] \, dx \right\} = 0
\]

The solution gives us:

\[
H = \frac{\frac{h_0 l_1}{3 I_1} (l + 4 l_2) + \frac{S l_2 \left( \frac{h_1}{2} + \frac{h_2}{3} \right)}{I_2} \, P}{\frac{2 h_0^2}{3 I_1} + \frac{h_1^2 l_1}{I_1} + \frac{2 S}{I_2} \left[ h_1^2 + h_1 h_2 + \frac{h_2^2}{3} \right]}
\]  

(4)

Case 2. Square Frame and Trestle Bent with Hinged Ends under Concentrated Load.
These are special cases of the foregoing frame and slight consideration will give us the following formulas for $H$.

For Fig. 5

$$H = \frac{\frac{5}{3}I_2 + \frac{L}{8} (L, H_2)}{h \left( \frac{5}{3}I_0 + \frac{L}{8} \right)} P \quad (5)$$

This is also obtained from equation (3) by making $a = b$ and neglecting the effect of normal force.

For Fig. 6

$$H = \frac{\frac{L}{8}}{h} \left[ \frac{2h}{3I_2} + \frac{L}{8} \right] \frac{P}{8} \quad (6)$$

Case 3. Special Frame with Hinged Ends under Two Symmetrical Concentrated Loads.

$H$ is only statically indeterminate in this case and all necessary elements in forming the conditional equations are arranged in the following table:

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>M</th>
<th>$\frac{\partial H}{\partial y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB, A'B'</td>
<td>$I_0$</td>
<td>$-Hy$</td>
<td>$-\frac{h_2}{12} + h_1$</td>
</tr>
<tr>
<td>BC, B'C'</td>
<td>$I_2$</td>
<td>$-H(\frac{h_0}{12}x + h_1) + \frac{F_2}{2}$</td>
<td>$-\frac{h_2}{12} + h_1$</td>
</tr>
<tr>
<td>CC'</td>
<td>$I_1$</td>
<td>$-Hh_0 + \frac{P_2}{2}$</td>
<td>$-h_0$</td>
</tr>
</tbody>
</table>

Fig. 7.
Therefore the conditional equation is as follows:

\[
\frac{P}{EI_0} \int_0^h y^2 \, dy + \frac{P}{EI} \int_0^l \left[ -H \left( h^2 + \frac{b h_x}{l_x} x + \left( \frac{b^2 x^2}{l_x^2} \right) \right) + (\frac{h x}{l_e} + \frac{b h_x^2}{l_x^2}) P \right] \, dx \\
+ \frac{P l_0}{E I} \int_0^l \left[ \frac{P l_0}{E I} - H H_0 \right] \, dx = 0.
\]

The solution gives us the following result:

\[
H = \frac{\left\{ \frac{b h_x}{l_x} + \frac{\frac{P l_0}{E I} \left[ \frac{h_x}{l_x} + \frac{b h_x}{l_x} \right] I_2}{h^2 + h_x h_e + \frac{b h_x^2}{3 l_x^2}} \right\} l_2}{P} \tag{7}
\]

Case 4. Square Frame and Trestle Bent with Hinged Ends

under Symmetrical Loads.

![Diagram of Case 4](image)

Inserting \( h_1 = 0 \) and \( h_2 = h_3 = h \) in the formula 7, we get for the case of Fig. 8:

\[
H = \frac{\left( \frac{l_2}{l_2^2} + \frac{\frac{P l_0}{E I}}{3 l_2^2} \right) l_2}{h \left[ \frac{l_2}{l_x} + \frac{\frac{P l_0}{E I}}{3 l_2^2} \right]} P = \frac{l_2 P}{h \frac{P}{2}} \tag{8}
\]

It is interesting to note that the value of \( H \) is directly proportional to the ratio \( \frac{l_2}{h} \), and for \( l_2 = h \) or \( \theta = 45^\circ \) \( H \) is equal to \( \frac{P}{2} \). It should be noted that in this bent there exists no bending stress throughout the whole bent.
For the rectangular frame (Fig. 9),

\[
H = \frac{(\frac{I_1 + I_2}{I_1}) l_2}{2 h \left[ \frac{2 h}{3 I_1} + \frac{1}{2 I_2} \right]} \cdot P 
\]

If the loads are applied at third points or \( l_1 = l_2 = \frac{1}{3} \),

\[
H = \frac{\frac{1}{2 I_1}}{8 \left( \frac{h}{2} \right) \left[ \frac{2 h}{3 I_1} + \frac{1}{2 I_2} \right]} \cdot P 
\]

Case 5. Special Frame with Hinged Ends under Uniform Loads.

In this case \( H \) is statically indeterminate force as before, and the following table gives us the necessary quantities in the formation of the conditional equation.

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>M</th>
<th>( \frac{\partial M}{\partial H} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB, A'B'</td>
<td>I_0</td>
<td>-H_Y</td>
<td>-Y</td>
</tr>
<tr>
<td>BD, B'D'</td>
<td>I_2</td>
<td>-H(h_1 + x \tan \theta) + \frac{p_1 l_1 x}{2} + p_2 l_2 x - \frac{p_2 x^2}{2}</td>
<td>(h_1 + x \tan \theta)</td>
</tr>
<tr>
<td>DD'</td>
<td>I_1</td>
<td>-H_0 + \frac{p_1 l_1}{2}(l_0 + x) + p_2 l_2(l_2 + x) - \frac{p_2 l_2 x}{2} - \frac{p_1 x^2}{2}</td>
<td>-h</td>
</tr>
</tbody>
</table>
Therefore we have:
\[
\frac{p}{E I_z} \int_{0}^{h_0} H_y^2 dy - \frac{h_0}{E I_z} \int_{0}^{l} \left[ -H h_0 + \frac{P x (x + l_2)}{2} + \frac{P x^2}{2} + P_2 l_2 (l_2 + x) + \frac{P_2 l_2 x}{2} - \frac{P_2 l_2^2}{2} \right] dx = 0
\]

The solution is as follows:
\[
H = \frac{p l_i \left[ \frac{h_0 l_i}{I_i} \left( \frac{l_2}{12} + \frac{l_0}{2} \right) + \frac{S l_i}{I_i} \left( \frac{h_0}{2} + \frac{h_0^2}{3} \right) \right] + P_2 l_2 \left[ \frac{h_0 l_2}{2 l_2} \right] + \frac{S l_2}{I_2} \left( \frac{2 h_0^2}{3} - \frac{h_0^3}{4} \right)}{\frac{2 h_3^3}{3 I_0} + \frac{h_0^2 l_i}{I_i} + \frac{S l_i}{I_2} \left( \frac{h_0^2}{3} + h_0 l_i \right)}
\]

(11)

Case 6. Square Frame and Trestle Bent with Hinged Ends
under Uniform Load or Loads.

The formulas for the special cases as shown in Figs. 11, 12, and 13 are obtained from the general expression for $H$ of the foregoing case.

For Fig. 11,
\[
H = \frac{p l_i}{h \left( \frac{2 h}{3 I_i} + \frac{l_i}{12} \right)} \quad \text{or} \quad H = \frac{p l_i}{h \left( \frac{2}{3 I_i} + \frac{1}{12} \right)}
\]

(12)
For Fig. 12,

\[ H = \frac{pl_1 \left( \frac{5L_2}{3I_0} + \frac{l_2(l_2 + 6l_2)}{12I_0} \right) + \frac{p_2L_2}{2I_0} \left[ \frac{5S\lambda_2}{3I_0} + \frac{l_2^2}{2I_0} \right]}{h \left[ \frac{2S}{3I_0} + \frac{l_2}{I_0} \right]} \]  

(13)

For Fig. 13,

\[ H = \frac{l_2(l_2 + 6l_2) + \frac{5S\lambda_2}{I_0}}{12p_2} \]  

(14)

This agrees with formula (12) when the bent has vertical columns.

Case 7. Special Frame with Fixed Ends under Symmetrical Concentrated Loads.

Statically indeterminate quantities are \( M_0 \) and \( H \) in this case. The following table gives us the necessary quantities in the formation of conditional equations:

<table>
<thead>
<tr>
<th>Member</th>
<th>( I )</th>
<th>( M )</th>
<th>( \frac{\partial M}{\partial H} )</th>
<th>( \frac{\partial M}{\partial M_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB, A'B'</td>
<td>( I_0 )</td>
<td>( M_0 - HY )</td>
<td>(-Y)</td>
<td>(+1)</td>
</tr>
<tr>
<td>BD, B'D'</td>
<td>( I_2 )</td>
<td>( M_0 + \frac{Px}{2} - H(h_1 + xtan\theta) )</td>
<td>(-h_1 + xtan\theta)</td>
<td>(+1)</td>
</tr>
<tr>
<td>DD'</td>
<td>( I_1 )</td>
<td>( M_0 - H_0 + \frac{P_1l_2}{2} )</td>
<td>(-h_0)</td>
<td>(+1)</td>
</tr>
</tbody>
</table>
The conditional equations are as follows:

\[
\begin{align*}
2 \int_0^{h_i} (M_y - H y^2) \, dy + \frac{2 E I}{I_2} \int_0^{l_2} \left[ M_0 + \frac{P x}{2} - H(h_i + x \tan \theta) \right] \, dx \\
+ \frac{h_0}{E I} \int_0^{l_2} \left[ M_0 - H h_0 + \frac{P l_2}{2} \right] \, dx = 0, \quad \text{and} \\
2 \int_0^{h_i} (M_y - H y) \, dy + \frac{2 E I}{I_2} \int_0^{l_2} \left[ M_0 + \frac{P x}{2} - H x \tan \theta - H h_0 \right] \, dx \\
+ \frac{h_0}{E I} \int_0^{l_2} \left( M_0 - H h_0 + \frac{P l_2}{2} \right) \, dx = 0.
\end{align*}
\]

The solution of these equations gives us the following values for \( H \) and \( M_0 \):

\[
H = \frac{K_1 A_2 - K_2 A_1}{A_1 B_2 - A_2 B_1}, \quad \text{(15)}
\]

\[
M_0 = \frac{K_1 B_2 - K_2 B_1}{A_1 B_2 - A_2 B_1}, \quad \text{(16)}
\]

In which

\[
A_1 = \frac{h_i^2}{I_0} + \frac{h_0 l_1}{I_1} + \frac{S l_2}{2 I_2} (h_o + h_i),
\]

\[
A_2 = \frac{h_i^2}{I_0} + \frac{l_1}{I_1} + \frac{S l_2}{2 I_2},
\]

\[
B_1 = \frac{h_i^3}{3 I_0} + \frac{h_0^2 l_1}{I_1} + \frac{S l_2}{2 I_2} (h_i^2 + h_0 h_z + \frac{h_z^2}{3}),
\]

\[
B_2 = \frac{h_i^2}{I_0} + \frac{h_0^2 l_1}{I_1} + \frac{S l_2}{2 I_2} (h_0 + h_i),
\]

\[
K_1 = \left( \frac{h_0 l_1^2}{2 I_1} + \frac{S l_2}{6 I_2} (2 h_0 + h_i) \right) P,
\]

\[
K_2 = \left( \frac{l_1 l_2}{2 I_1} + \frac{S l_2}{2 I_2} \right) P.
\]
Case 8. Square Frame and Trestle Bent with Fixed Ends
under Symmetrical Concentrated Loads.

Putting $h_1 = 0$ in the previous general formula we will readily obtain the following results for the case of square frame under symmetrical concentrated vertical loads (Fig. 15):

$$H = \frac{3 L_2 (L_1 + L_2)}{h^2 \left[ \frac{1}{I_o} + 2 \frac{L}{h} \right]} \frac{P}{2},$$

$$M_o = \frac{L_2 (L_1 + L_2)}{h \left[ \frac{1}{I_o} + 2 \frac{L}{h} \right]} \frac{P}{2}.$$  \hspace{1cm} (17)

If $L_1 = L_2 = \frac{1}{3}$ or loads are applied at third points,

$$H = \frac{\left( \frac{L}{h} \right)^2}{3 \left[ \frac{1}{I_o} + 2 \frac{L}{h} \right]} \cdot P = \frac{1}{4 \left[ 2 + \frac{h}{2 I_o} \right]} \frac{P}{3},$$

$$M_o = \frac{L}{h} \left[ \frac{1}{I_o} + 2 \frac{L}{h} \right] \cdot Pl = \frac{2 + \frac{h}{2 I_o}}{2} \cdot \frac{P l}{9}.$$  \hspace{1cm} (18)

For the case of trestle bent (Fig. 16), we get the following values by putting $h_1 = 0$ in the general formulas:

$$H = \frac{l_2 P}{h^2}, \quad M_o = 0.$$  These must be the case.
Case 9. Trestle Bent with Fixed Ends under Concentrated Load.

Statically indeterminate quantities are II and M\textsubscript{1} in this case.

The following table shows the necessary quantities in the formation of the conditional equations.

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>M</th>
<th>( \frac{\partial M}{\partial H} )</th>
<th>( \frac{\partial M}{\partial M_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB, A'B'</td>
<td>I\textsubscript{0}</td>
<td>(-Hy + M_1 + \frac{1}{2} \frac{y}{h} )</td>
<td>(-y)</td>
<td>+1</td>
</tr>
<tr>
<td>BC</td>
<td>I\textsubscript{1}</td>
<td>(-Hh + M_1 + V(l_2 + x))</td>
<td>(-h)</td>
<td>+1</td>
</tr>
<tr>
<td>B'C</td>
<td>I\textsubscript{1}</td>
<td>(-Hh + M_1 + V(l_2 - x))</td>
<td>(-h)</td>
<td>+1</td>
</tr>
</tbody>
</table>

The conditional equations are as follows:

\[
\frac{E_{I_0}}{h} \int_0^h \left[ H'y^2 M_1, y - V \frac{\partial}{\partial x} \right] dy + \frac{E_{I_1}}{h} \left\{ \left[ H'h - M_1 - V(l_2 + x) \right] dx + \left[ H'h - M_1 - V(l_2 - x) \right] dx \right\} = 0,
\]

\[
\frac{E_{I_0}}{h} \int_0^h (-H'y + M_1 + V \frac{\partial}{\partial x}) dy + \frac{E_{I_1}}{h} \left\{ \left[ H'h + M_1 + V(l_2 + x) \right] dx + \left[ H'h + M_1 + V(l_2 - x) \right] dx \right\} = 0.
\]

Solving these for H and M\textsubscript{1} and simplifying we obtained the following results:

\[
H = \frac{4 \left[ \frac{3}{2}, \frac{l_2}{C} \right] l_2 + (3 l_2 - 8 l_2) \cdot P}{h \left[ 2 + \frac{3}{2}, \frac{l_1}{C} \right]} \cdot \frac{P}{8},
\]

\[
M_1 = \frac{1}{2 + \frac{3}{2}, \frac{l_1}{C}} \cdot \frac{P l_0}{8}, \quad \text{and} \quad h_0 = \frac{h}{3}.
\]
The following case is easily reducible from Case 9.

We can get the following values for statically indeterminate quantities of this case by putting $I_2 = 0$ and making $S = h$:

\[ H = \frac{3}{4} \left[ \frac{1}{2} + \frac{1}{2} \frac{I_1}{I_0} \right] \frac{P \cdot l}{8}, \]

\[ M_1 = \frac{1}{2 + \frac{h}{2} \frac{I_1}{I_0}} \frac{P \cdot l}{8}. \]

These results can easily be obtained from independent analysis.
Case 10. Square Frame with Tie at Column Ends under Centrally Concentrated Load.

The statically indeterminate quantities are \( H \) and \( M_A \) in this case.

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>M</th>
<th>( \frac{\partial M}{\partial H} )</th>
<th>( \frac{\partial M}{\partial M_A} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA'</td>
<td>I_1</td>
<td>( M_A )</td>
<td>0</td>
<td>+1</td>
</tr>
<tr>
<td>AB, A'B'</td>
<td>I_0</td>
<td>( M_A - H_y )</td>
<td>(-y)</td>
<td>+1</td>
</tr>
<tr>
<td>BC</td>
<td>I_2</td>
<td>( M_A - H_h + V_x )</td>
<td>(-h)</td>
<td>+1</td>
</tr>
<tr>
<td>CB'</td>
<td>I_2</td>
<td>( M_A - H_h - \frac{P_x}{2} + \frac{P_l}{2} )</td>
<td>(-h)</td>
<td>+1</td>
</tr>
</tbody>
</table>

The conditional equations are as follows:

\[
\frac{P}{EI_0} \int_0^h (H_y - M_A y) \, dy + \frac{P}{EI_z} \int_0^l \left( \frac{1}{2} H_h - M_A - V_x \right) \, dx + \frac{1}{2} \int_0^l (H_h - M_A + \frac{P_x}{2} - \frac{P_l}{2}) \, dx = 0,
\]

\[
\frac{P}{EI_0} \int_0^h (M_A - H_y) \, dy + \frac{P}{EI_z} \int_0^l M_A \, dx + \frac{P}{EI_z} \int_0^l \left( M_A + H_h + \frac{P_x}{2} \right) \, dx + \frac{1}{2} \int_0^l (M_A - H_h - \frac{P_x}{2} + \frac{P_l}{2}) \, dx = 0.
\]

The solution of these equations gives us the following expressions for \( H \) and \( M_A \):

\[
H = \frac{M_A \left( \frac{b}{I_0} + \frac{c}{I_z} \right) + \frac{P_l}{8 I_z}}{(\frac{b}{I_0} + \frac{c}{I_z}) h},
\]

\[
H = \frac{M_A \left( \frac{b}{I_0} + \frac{c}{I_z} + \frac{l}{I_z} \right) + \frac{P_l}{8 I_z}}{(\frac{b}{I_0} + \frac{l}{I_z}) h}.
\]
Combining these equations we get:

\[ M_A = \frac{b}{I_o I_z} \left( \frac{h}{I_o} \right)^3 + 2 \left( \frac{b}{I_o I_z} \right) + 3 \left( \frac{I_z}{I_z} \right)^2 \cdot \frac{P l}{8} \quad (20) \]

When \( I_1 \) becomes infinity this equation reduces to the form

\[ M_A = \frac{l}{I_o + 2 \left( \frac{l}{I_z} \right)} \frac{P l}{8} \]

Case 11. Unsymmetrical Frame under Uniform Load.

We hitherto considered the symmetrical frames under vertical loads. In this Case and in the next Case we will give the analysis of the unsymmetrical frames.

The statically indeterminate forces are \( H \) and \( V_1 \). All necessary elements in forming the conditional equations are arranged in the following tabular form:

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>M</th>
<th>( \frac{\partial M}{\partial H} )</th>
<th>( \frac{\partial M}{\partial V_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>I_0</td>
<td>-Hy</td>
<td>-y</td>
<td>0</td>
</tr>
<tr>
<td>BC</td>
<td>I_1</td>
<td>( V_1 x - h ) - ( \frac{P x^2}{2} )</td>
<td>-h</td>
<td>+x</td>
</tr>
</tbody>
</table>
Therefore we have for the conditional equations

\[ \frac{1}{EI_0} \int_0^h Hy^2 dy + \frac{h}{EI_0} \int_0^l [Hh - V, x + \frac{P}{2} x^2] dx = 0, \]

\[ \frac{h}{EI_0} \int_0^l [Vx^3 - Hh x - \frac{P}{2} x^2] dx = 0 \]

The solution of these equations is as follows:

\[ H = \frac{3}{4} \left[ \frac{3 + 4 \frac{h L^4}{I_0}}{2} \right] \frac{P l}{12}. \]

\[ V = \frac{3 (1 + \frac{h L^4}{I_0})}{3 + 4 \frac{h L^4}{I_0}} \frac{P l}{2}. \] (21)

The maximum positive bending moment in the top beam occurs at the distance \( x \) from \( C \) and its value is as follows:

\[ x = \frac{3 (1 + \frac{h L^4}{I_0})}{3 + 4 \frac{h L^4}{I_0}} \frac{L}{2}. \] (22)

The value of the maximum positive bending moment is

\[ M_{\text{Max}} = \frac{9 (1 + \frac{h L^4}{I_0})^2 - 2 (3 + 4 \frac{h L^4}{I_0})}{8} \frac{P l^2}{(3 + 4 \frac{h L^4}{I_0})^2}. \] (23)
Case 12. Unsymmetrical Frame with Fixed Ends under Uniform Load.

The statically indeterminate quantities in this case are $H$, $V_1$, and $M_1$. $M_1$ can be expressed in terms of $H$, that is to say, $M_1 = \frac{Hh}{3}$. The number of unknowns are reduced to two from this relation. All necessary elements are given in the following table:

<table>
<thead>
<tr>
<th>Member</th>
<th>$I$</th>
<th>$M$</th>
<th>$\frac{\partial H}{\partial H}$</th>
<th>$\frac{\partial H}{\partial V_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>$I_0$</td>
<td>$M_1 - Hy = H\left(\frac{h}{3} - y\right)$</td>
<td>$\frac{h}{3} - y$</td>
<td>0</td>
</tr>
<tr>
<td>BC</td>
<td>$I_1$</td>
<td>$M_1 - Hh - \frac{pL^2}{2} + V_1 x = H\left(\frac{2h}{3}\right) - \frac{pL^2}{2} + V_1 x$</td>
<td>$-\frac{2h}{3}$</td>
<td>$+x$</td>
</tr>
</tbody>
</table>

Therefore we have the following equations:

$$\frac{1}{EI_0}\int_0^H H\left(\frac{h}{3} - y\right)^2 dy + \frac{2h}{3EI_0} \int_0^L \left[\frac{2h}{3} H + \frac{PX^2}{2} - V_1 x\right] dx = 0,$$

$$\frac{1}{EI_0} \int_0^L \left(V_1 x^2 - \frac{2h}{3} H x - \frac{pL^3}{2}\right) dx = 0.$$
After the mathematical operation we get the following results:

\[
H = \frac{P}{l} \left(1 + \frac{h I_1}{I_0}\right) \frac{P l}{8},
\]

\[
V_i = 4 + 3 \frac{h I_1}{I_0} \frac{P l}{8},
\]

\[
M_i = \frac{1}{1 + \frac{h I_1}{I_0}} \frac{P l}{24},
\]

\[
M_0 = \frac{2 + \frac{3 h I_1}{I_0}}{1 + \frac{h I_1}{I_0}} \frac{P l}{24}.
\]

The formulas for the maximum positive bending moment \(M_{\text{max}}\) in the top beam and its distance \(x\) from \(C\) are as follows:

\[
M_{\text{max}} = 3 \left(1 + \frac{3 h I_1}{I_0}\right)^2 - 2 \left(1 + \frac{h I_1}{I_0}\right)^2 \frac{P l}{24};
\]

\[
x = \frac{1 + \frac{3 h I_1}{I_0}}{1 + \frac{h I_1}{I_0}} \frac{l}{2}.
\]

11. **Single Story Construction with Three Panels**.—In the case of design of a flat slab or a beam-girder construction of single story, many engineers do not take the effect of the bending in columns into consideration. Many authors have tried to analyze the stress distribution in the slab on this assumption. The assumption may be nearly true for cases where the column has a large cross-section and very short length in comparison with the length of span.

But, in practical cases, the moment of inertia of cross
section of a slab is larger than that of a column, and comparatively long columns are frequently used in actual construction. The bending of columns will obviously increase the bending moment at the center of span of a slab for an eccentric loading, and stresses in the slab are greatly modified by the ratios of the moment of inertia of a slab to that of the column, and of height to span length. Therefore it may be concluded that the analysis based on the negligence of column flexure is not applicable for all cases.

In actual cases, there may be twenty or more spans in succession, with different lengths of spans and cross-sections of members, and consequently an exact analysis is hardly possible without any assumption.

If panel AA' in the above figure be loaded, the bending moments in slabs BC, CD, B'C', and C'D' are small and negligible in actual cases. Therefore we may assume that the end condition between of slabs or beams AB and A'B' will, perhaps, be the hinged and the fixed state at B and B' according to the ratio of moments of inertia of the column and slab at that joint.

The following analytical formulas for five different cases are applicable for actual design with proper assumption as the case may be.
Case 13. Frame with Three Panels under Uniform Load over Middle Panel.

Hinged Ends of Beams and Columns.

Statically indeterminate quantities are \( M \) and \( V_1 \) in this case, and the following table will give us the necessary quantities in the formation of the conditional equations.

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>( M )</th>
<th>( \frac{\partial M}{\partial H} )</th>
<th>( \frac{\partial M}{\partial V_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB, A'B'</td>
<td>I₁</td>
<td>( \frac{P lx}{2} - V_1 x )</td>
<td>0</td>
<td>(-x)</td>
</tr>
<tr>
<td>BB'</td>
<td>I₁</td>
<td>(-V_1 l - Hh + \frac{P l^2}{2} + \frac{p l x}{2} - \frac{p x^2}{2} )</td>
<td>(-h)</td>
<td>(-1)</td>
</tr>
<tr>
<td>DB, D'B'</td>
<td>I₀</td>
<td>(-H y)</td>
<td>(-y)</td>
<td>0</td>
</tr>
</tbody>
</table>

The conditional equations are as follows:

\[
\frac{2}{EI} \int_0^l \left( \frac{P l x}{2} - V l x^2 \right) dx + \frac{l}{EI} \int_0^l \left( -V l - H l h + \frac{P l^2}{2} + \frac{p l x}{2} - \frac{p x^2}{2} \right) dx = 0,
\]

\[
\frac{h}{EI} \int_0^l \left( V l - H l + \frac{P l^2}{2} + \frac{p l x}{2} - \frac{p x^2}{2} \right) dx + \frac{2}{EI} \int_0^h (-H y^2) dy = 0.
\]
The final results are as follows:

\[ H = \frac{3}{4l}(3 + 5 \frac{n}{l} \frac{I}{I_0}), \quad \text{or} \quad \frac{1}{4n(3 + 5mn)}Pl \]

\[ V_1 = \frac{6 + 11\frac{n}{l} \frac{I}{I_0}}{4(3 + 5\frac{n}{l} \frac{I}{I_0})} Pl = \frac{6 + 11mn}{4(3 + 5mn)} Pl \]

where \( m = \frac{l_1}{I_0}, \quad n = \frac{h}{l} \)

Having thus found the values of \( H \) and \( V_1 \), the moment and therefore the stress at any section in the beam or columns is determined by ordinary principles of mechanics.


**Hinged Ends of Beams but Fixed at Column Ends.**

![Diagram of the frame with three panels under uniform load.]

The statically indeterminate quantities are three in this case, \( H, V_1, \) and \( M_1 \). \( M_1 \), however, can be expressed in terms of \( H \) owing to the fact that the position of the point of inflection in the column remains unchanged, as proved in equation 1, and \( l_0 = \frac{h}{3} \). Therefore \( M_1 = \frac{Hh}{3} \). This fact greatly reduces the mathematical
operation. The analysis of this frame brings us the following formulas for the statically indeterminate quantities.

\[ H = \frac{1}{2} \left( 4 + \frac{5 \frac{h}{l} I_0}{I} \right) Pl = \frac{1}{2} n(4 + 5mn) Pl \]

\[ V_1 = \frac{8 + 11 \frac{h}{l} I_0}{4(4 + \frac{5 \frac{h}{l} I_0}{I})} Pl = \frac{8 + 11mn}{4(4 + 5mn)} Pl \]

\[ M_0 = \frac{V o l}{3} = \frac{\frac{h}{l} I_0}{1 + 2 \frac{h}{l} I_0} \frac{Pl^2}{18} = \frac{mn}{1 + 2mn} \frac{Pl^2}{18} \]

Case 15. Frame with Three Panels under Uniform Load.

Hinged Ends of Columns but Fixed at Beam Ends.

The statically indeterminates are \( H, V_1 \) and \( M_0 \) in this case. The results of the analysis are as follows without entering the detail of the mathematical calculation:

\[ H = \frac{12 \frac{h}{l} \left( 1 + 2 \frac{\frac{h}{l} I_0}{I} \right)}{1 + 2mn} \]

\[ V_1 = \frac{3 + 7 \frac{h}{l} I_0}{6 \left( 1 + 2 \frac{\frac{h}{l} I_0}{I} \right)} \]

\[ M_0 = \frac{V o l}{3} = \frac{\frac{h}{l} I_0}{1 + 2 \frac{h}{l} I_0} \frac{Pl^2}{18} = \frac{mn}{1 + 2mn} \frac{Pl^2}{18} \]
Case 16. Frame with Three Panels under Uniform Load.

Fixed Ends of Beams and Columns.

This frame has nine statically indeterminate quantities, but they are greatly reduced by the adoption of the symmetrical loading. The statically indeterminate quantities are $H$, $V_1$, $M_0$, and $M_1$ in this case. The results of the analysis give us the following formulas:

\[
H = \frac{1}{4 \frac{h}{l} (2 + 3 \frac{h}{l} \frac{I}{I_0})} P_l = \frac{1}{4n(2 + 3mn)} P_l, \\
V_1 = \frac{4 + 7 \frac{h}{l} \frac{I}{I_0}}{4(2 + 3 \frac{h}{l} \frac{I}{I_0})} P_l = \frac{4 + 7mn}{4(2 + 3mn)} P_l, \\
M_0 = \frac{\frac{h}{l} \frac{I}{I_0}}{2 + 3 \frac{h}{l} \frac{I}{I_0}} \frac{P_l^2}{12} = \frac{mn}{2 + 3mn} \frac{P_l^2}{12}, \\
M_1 = \frac{1}{2 + 3 \frac{h}{l} \frac{I}{I_0}} \frac{P_l^2}{12} = \frac{1}{2 + 3mn} \frac{P_l^2}{12}.
\]
Case 17. Frame with Three Panels under Single Concentrated Load.

This frame has 7 statically indeterminates in general, but they are reduced to two by the adoption of symmetrical load and the relation \( M_0 = \frac{V_0 l}{3} \), as shown in equation 1. The following table gives us the necessary elements for the calculation.

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>( M )</th>
<th>( \frac{\partial M}{\partial H} )</th>
<th>( \frac{\partial M}{\partial V_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BD, B'D'</td>
<td>( I_1 )</td>
<td>( V_1 \left( \frac{l}{3} - x \right) - \frac{P}{2} \left( \frac{l}{3} - x \right) )</td>
<td>0</td>
<td>( \frac{1}{3} - x )</td>
</tr>
<tr>
<td>AB, A'B'</td>
<td>( I_0 )</td>
<td>-Hy</td>
<td>-V</td>
<td>0</td>
</tr>
<tr>
<td>BC</td>
<td>( I_2 )</td>
<td>-V_1 \left( \frac{22}{3} - x \right) - \frac{P}{2} \left( \frac{22}{3} + x \right)</td>
<td>-h</td>
<td>( \frac{22}{3} )</td>
</tr>
<tr>
<td>CB</td>
<td>( I_2 )</td>
<td>-V_1 \left( \frac{22}{3} - x \right) - \frac{P}{2} \left( \frac{22}{3} - x \right)</td>
<td>-h</td>
<td>( \frac{22}{3} )</td>
</tr>
</tbody>
</table>

The conditional equations are as follows:

\[
\frac{P}{EI} \int_0^h H y^2 dy + \frac{h}{EI} \left[ \int_0^{\frac{l}{2}} \left( \frac{22}{3} V + H, h - \frac{P}{2} \left( \frac{22}{3} + x \right) \right) dx \right] + \left[ \int_0^{\frac{l}{2}} \left( \frac{22}{3} V + H, h - \frac{P}{2} \left( \frac{22}{3} + x \right) \right) dx \right] = 0
\]

\[
\frac{P}{EI} \int_0^l \left[ \left( \frac{22}{3} - x \right)^2 - \frac{P}{2} \left( \frac{22}{3} - x \right)^2 \right] dx + \frac{h}{3EI} \left[ \left[ \int_0^{\frac{l}{2}} \left[ \frac{22}{3} V + H, h - \frac{P}{2} \left( \frac{22}{3} + x \right) \right] dx \right] + \left[ \int_0^{\frac{l}{2}} \left( \frac{22}{3} V + H, h - \frac{P}{2} \left( \frac{22}{3} + x \right) \right) dx \right] \right] = 0
\]
The solution gives us:

\[ 16 \left( \frac{L}{I_2} \right) V_I + 8 h \left[ \frac{2h}{I_0} + 3 \left( \frac{L}{I_2} \right) \right] H = 11 \frac{P l^2}{I_2} \]

\[ 8 \left[ \frac{L}{I_2} + \frac{2l}{I_2} \right] V_I + 24 \left( \frac{h}{I_2} \right) H = \left[ 4 \left( \frac{L}{I_2} \right) + 11 \left( \frac{L}{I_2} \right) \right] P \]

Solving these equations for statically indeterminate quantities we get,

\[ V_I = \frac{12 + 8 \left( \frac{L}{I_2} \right) + 22 \left( \frac{h}{I_0} \right) + 33 \left( \frac{L}{I_2} \right)^2 - 33 \left( \frac{L}{I_2} \right)^2}{3 + 2 \left( \frac{h}{I_0} \right) + 4 \left( \frac{h}{I_0} \right)^2 + 6 \left( \frac{L}{I_2} \right) - 6 \left( \frac{L}{I_2} \right)^2} \frac{P}{8} \]

\[ H = \frac{P}{8} \frac{3 \left( \frac{L}{I_2} \right)}{3 + 2 \left( \frac{h}{I_0} \right) + 4 \left( \frac{h}{I_0} \right)^2 + 6 \left( \frac{L}{I_2} \right) - 6 \left( \frac{L}{I_2} \right)^2} \]

\[ V_o = \frac{3 \left[ 2 \left( \frac{h}{I_0} \right) + 3 \left( \frac{L}{I_2} \right)^2 - \left( \frac{L}{I_2} \right)^2 \right]}{3 + 2 \left( \frac{h}{I_0} \right) + 4 \left( \frac{h}{I_0} \right)^2 + 6 \left( \frac{L}{I_2} \right) - 6 \left( \frac{L}{I_2} \right)^2} \frac{P}{8} \]

\[ M_o = \frac{2 \left( \frac{h}{I_0} \right) + 3 \left[ \left( \frac{L}{I_2} \right)^2 - \left( \frac{L}{I_2} \right)^2 \right]}{3 + 2 \left( \frac{h}{I_0} \right) + 4 \left( \frac{h}{I_0} \right)^2 + 6 \left( \frac{L}{I_2} \right) - 6 \left( \frac{L}{I_2} \right)^2} \frac{P L}{8} \]

Special Case. If we put \( I_1 = I_2 \), we will get

\[ V_I = \frac{2 + 5 \left( \frac{h}{I_0} \right)}{1 + 2 \left( \frac{h}{I_0} \right)} \frac{P}{4} \quad V_o = \frac{\left( \frac{h}{I_0} \right)}{1 + 2 \left( \frac{h}{I_0} \right)} \frac{P}{4} \]

\[ H = \frac{P}{8} \frac{1}{3 + 2 \left( \frac{h}{I_0} \right)} \quad M_o = \frac{\left( \frac{h}{I_0} \right)}{3 \left( 1 + 2 \left( \frac{h}{I_0} \right) \right)} \frac{P L}{4} \]
TABLE 1.

Reactions, Bending Moments and Position of Point of Inflection in Frames with Three Panels.

For the convenience of comparison, the results of the analysis for the foregoing four cases are arranged in tabular form as follows:

<table>
<thead>
<tr>
<th>Case 13</th>
<th>Case 14</th>
<th>Case 15</th>
<th>Case 16</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram" /></td>
<td><img src="image2.png" alt="Diagram" /></td>
<td><img src="image3.png" alt="Diagram" /></td>
<td><img src="image4.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>H</th>
<th>$\frac{1}{4n(3+5mn)}Pl$</th>
<th>$\frac{1}{2n(4+5mn)}Pl$</th>
<th>$\frac{1}{12n(1+2mn)}Pl$</th>
<th>$\frac{1}{4n(2+3mn)}Pl$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0$</td>
<td>$\frac{mn}{4(3+5mn)}Pl$</td>
<td>$\frac{mn}{4(4+5mn)}Pl$</td>
<td>$\frac{mn}{6(1+2mn)}Pl$</td>
<td>$\frac{mn}{4(2+3mn)}Pl$</td>
</tr>
<tr>
<td>$V_1$</td>
<td>$\frac{6+11mn}{4(3+5mn)}Pl$</td>
<td>$\frac{8+11mn}{4(4+5mn)}Pl$</td>
<td>$\frac{3+7mn}{6(1+2mn)}Pl$</td>
<td>$\frac{4+7mn}{4(2+3mn)}Pl$</td>
</tr>
<tr>
<td>$M_0$</td>
<td>0</td>
<td>0</td>
<td>$\frac{mn}{1+2mn}Pl^2/18$</td>
<td>$\frac{mn}{2+3mn}Pl^2/12$</td>
</tr>
<tr>
<td>$M_1$</td>
<td>0</td>
<td>$\frac{1}{4+5mn}Pl^2/6$</td>
<td>0</td>
<td>$\frac{1}{2+3mn}Pl^2/12$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$\frac{-mn}{3+5mn}Pl^2/4$</td>
<td>$\frac{-mn}{4+5mn}Pl^2/4$</td>
<td>$\frac{-mn}{1+2mn}Pl^2/9$</td>
<td>$\frac{-mn}{2+3mn}Pl^2/6$</td>
</tr>
<tr>
<td>$M_3$</td>
<td>$\frac{-1}{3+5mn}Pl^2/4$</td>
<td>$\frac{-1}{4+5mn}Pl^2/3$</td>
<td>$\frac{-1}{1+2mn}Pl^2/12$</td>
<td>$\frac{-1}{2+3mn}Pl^2/6$</td>
</tr>
<tr>
<td>$M_4$</td>
<td>$\frac{-(1+mn)}{3+5mn}Pl^2/4$</td>
<td>$\frac{-4+3mn}{4+5mn}Pl^2/12$</td>
<td>$\frac{-3+4mn}{3(1+2mn)}Pl^2/12$</td>
<td>$\frac{-(1+mn)}{2+3mn}Pl^2/6$</td>
</tr>
<tr>
<td>$M_5$</td>
<td>$\frac{1+3mn}{3+5mn}Pl^2/8$</td>
<td>$\frac{4+9mn}{4+5mn}Pl^2/24$</td>
<td>$\frac{3+10mn}{3(1+2mn)}Pl^2/24$</td>
<td>$\frac{2+5mn}{2+3mn}Pl^2/24$</td>
</tr>
</tbody>
</table>

**Distance from Column ends to the point of inflection in mid-beam**

\[ \frac{h}{2} \left( \pm V - \frac{2(1+mn)}{3+5mn} \right) \]

**Distance from Column ends to the point of inflection in mid-beam**

\[ \frac{1}{2} \left[ \pm \sqrt{1 - \frac{24+3mn}{3+5mn}} \right] \]

**Distance from Column ends to the point of inflection in mid-beam**

\[ \frac{1}{2} \left[ \pm \sqrt{1 - \frac{24+3mn}{3+5mn}} \right] \]

\[ \frac{1}{2} \left[ \pm \sqrt{1 - \frac{4(1+mn)}{3(1+2mn)}} \right] \]

\[ m = \frac{h}{2}, \quad n = \frac{h}{4} \]
12. Trestle Bent with a Tie.—The following is an example of a rigid frame which may be used in trestle construction.

Case 18. Trestle Bent with Tie under Concentrated Vertical Load.

The general conditional equations are as follows:

\[ \int \frac{M}{EI} \frac{\partial M}{\partial H_0} \, ds + \int \frac{N}{EA} \frac{\partial N}{\partial H_0} \, ds = 0 \]

\[ \int \frac{M}{EI} \frac{\partial M}{\partial H_1} \, ds + \int \frac{N}{EA} \frac{\partial N}{\partial H_1} \, ds = 0 \]

\[ \int \frac{M}{EI} \frac{\partial M}{\partial M_4} \, ds + \int \frac{N}{EA} \frac{\partial N}{\partial M_4} \, ds = 0 \]

The second terms are neglected as before.

The necessary quantities in the formation of the above equations are as follows:— (In this case, the special notation \( p = \tan \theta, q = \sec \theta \) are used).
The conditional equations are as follows:

\[
\begin{align*}
\frac{2g}{E_1} \int_{0}^{h_0} \left[ (\nabla y - H_0 y) y \, dy + \int (\nabla y - H_0 y - H_1 (y-h_1) + M_4) y \, dy \right] = 0,
\end{align*}
\]

Performing the integration of the above three equations we obtain the following expressions for \(H_0\):

\[
\begin{align*}
H_0 \left[ \frac{2g h_0}{E_1} + \frac{h_0}{I_0} \right] + H_1 K \left[ \frac{2g h_0}{E_1 I_0} (3-k) + \frac{h_0}{I_1} \right] + M_4 \left[ 2k (3-k) + \frac{h_0}{I_0} \right]
\end{align*}
\]
Combining these three equations and solving for $H_1$ we get:

$$H_1 = \frac{P L^2}{8 I_z} \left[ \frac{(4-12 K+12 K^2-3 K^3)}{I_o I_z} + \frac{2 q h z I_z}{I_o I_z} + 2 (1-3 K+3 K^2) \frac{q h z}{I_o I_z} + 3 K \frac{l z}{I_o I_z} \right]$$

$$= \frac{(2-6 K+3 K^2)}{8 I_z} \left[ \frac{q h z I_z}{I_o I_z} \right] + M_4 \left[ \frac{(4-12 K+12 K^2-3 K^3)}{I_o I_z} \right] + 2 q h z I_z + 2 (1-3 K+3 K^2) \frac{q h z I_z}{I_o I_z} + 3 K \frac{l z}{I_o I_z}$$

$$= \left( \frac{(2-6 K+3 K^2)}{8 I_z} \right) \left[ \frac{q h z}{I_o I_z} \right] + \frac{P L^2}{8 I_z} M_4 \left[ \frac{(4-12 K+12 K^2-3 K^3)}{I_o I_z} \right]$$

$$= \left( \frac{(2-6 K+3 K^2)}{8 I_z} \right) \left[ \frac{q h z}{I_o I_z} \right] + \frac{P L^2}{8 I_z} M_4 \left[ \frac{(4-12 K+12 K^2-3 K^3)}{I_o I_z} \right]$$

and

$$H_1 = \frac{(3-2 K) \frac{q h z}{I_o I_z}}{8 I_z} + \frac{P L^2}{8 I_z} M_4 \left[ \frac{(4-12 K+12 K^2-3 K^3)}{I_o I_z} \right]$$

Combining these two equations and solving for $H_4$ we obtain the following expression:

$$M_4 = \frac{2 (2-5 K+4 K^2-3 K^3) \left[ \frac{q h z}{I_o I_z} \right] (I_o I_z) + (4-10 K+5 K^2) \left[ \frac{q h z}{I_o I_z} \right] (I_o I_z)}{(\frac{q h z}{I_o I_z}) + 2 (4-7 K+2 K^2) \left[ \frac{q h z}{I_o I_z} \right] + 2 (1-K) \left[ \frac{q h z}{I_o I_z} \right] + 3 (2-3 K) \left[ \frac{q h z}{I_o I_z} \right]}$$

Knowing the value of $M_4$ it is very easy to determine the stresses at any section of the frame. Now we will proceed to show that the above general equation reduces to equation (20) when $h_o = h_0 = h$, and $l_o = l_2 = l_1$. Substituting $k = 1$ in the above equation, we have

$$M_4 = \frac{I_z}{L^2} \left[ \frac{h_0}{L^2} \right] + 2 \left[ \frac{h_0}{L^2} \right] + 2 \left[ \frac{h_0}{L^2} \right] + 3 \left[ \frac{h_0}{L^2} \right]$$

This is exactly the same as equation (20).
The following form for the expression of $M_4$ is convenient to use in practical computation:

Putting

$$\frac{I_z}{I_h} = A,$$
$$\frac{I_z}{I_h} = B,$$

we have

$$M_4 = \frac{2(2-5k+4k^2-k^3)+(4-10k+5k^2)A^2}{4-10k+8k^2-2k^3+(3+4k^2-k^3)B+2(1-k-k^2)A+3(3k+k^2)AB} \cdot \frac{Pz}{8}$$

$$+ A\left\{ 4-10k+8k^2-3k^3+2(4-7k+2k^2)B+2(1-k-k^2)A+3(3k+k^2)AB \right\}$$

$$= \frac{\lambda A + \mu A^2}{\alpha + \beta B + \gamma A + \delta AB + A(\lambda' + \beta'B + \gamma'A + \delta'AB)} \cdot \frac{Pz}{8} \quad (34)$$

Where $\lambda$, $\beta$, $\gamma$, $\delta$, $\lambda'$, $\beta'$, $\gamma'$, $\delta'$, $\lambda''$, $\mu$ are the function of $k$ and are plotted in the diagrams 1 and 2.

After finding the values of $M_4$ we can successively determine the values for $H_1$ and $H_0$ using the following formulas:

$$H_1 = \frac{(3-2k)A \frac{Pz}{8} - M_4 \{k+(3-k)B+2kA+3AB\}}{(1-k)\{1+A\}h_z} \quad (35)$$

$$H_0 = \frac{[(2-k)\frac{k}{k} + A(2z\frac{z}{z} - I_z)] P + M_4 \left[ \frac{2k}{h_z} + \frac{(A+B)}{h_0} \right] - K(1+A)H_1}{(2-k)+A} \quad (36)$$

Finding these values it is an easy matter to determine the stress at any section of the frame.
Diagram No. 1
Values of Constants for Case 18

\[ \lambda' = 4\cdot10k + 8k^2 - 3k^3 \]
\[ \beta' = 2(4 - 2k + 2k^2) \]
\[ \gamma' = 2(1 - k - k^2) \]
\[ \delta' = 3(2 - 3k) \]
\[ \lambda = 2(2 - 5k + 4k^2 - k^3) \]
\[ \mu = 4 - 10k + 5k^2 \]
Diagram No. 2
Values of Constants
for Case 18.

\[ a = 4 - 10k + 8k^2 - 2k^3 \]
\[ b = 8 - 12k + 1k^2 - k^3 \]
\[ y = 2(1 - k - k^2 + k^3) \]
\[ \delta = 3(2 - 3k + k^2) \]
13. **Building Construction with Several Stories and Number of Panels.**—In the actual building construction of reinforced concrete, it is most common to use a continuous slab for floors supported by a number of columns. The worst loading to produce bending in columns is, of course, an eccentric arrangement, as shown in the accompanying figure. The cross sections and therefore the moments of inertia of columns are ordinarily smaller than those of slabs. Accordingly, the bending moment of the floor slab (CBB'C') is greatly modified by the flexure of columns a, a', b, and b'.

In present practice, attention is hardly paid to this point, and columns are assumed as rigid enough to resist the bending. This may approximately be true for the columns in the lower stories where the columns have large diameters. But this is erroneous for the columns in upper stories, where the cross section of columns is usually small, and serious bending stress will exist in the column due to eccentric load on the floor. The exact analysis is hardly possible owing to the fact that there are so many things to be entered in the formula. From a practical standpoint, it is easily understood that the bending moment in floor slabs, FDD'F' and EAA'E', due to the load on the floor BB' is very small and is inconsiderable if the floors are of moderate thickness. This simply means that the columns a, a', b, and b' are practically
fixed at A, A', D, and D' respectively. If floor slabs are not thick enough to keep column ends in fixed condition, then the end condition of column will, perhaps, be partway between hinged and fixed state. From these assumptions, almost exact analysis is possible as shown in the following. The resulting formulas can be used in the design of building constructions.

**Case 19. Frame with Hinged Ends.**

![Diagram](image)

This is the case having nine statically indeterminate quantities, but the condition of symmetrical loading greatly reduces the number of these quantities. In this analysis it is assumed that the vertical reactions at A and D (also at A' and D') are the same. This may be changed in actual cases, but no effect is produced on bending moments by this assumption. The following table was made as before:
Forming the conditional equations and integrating them we obtain the following results.

\[ H_0 \frac{h^2}{2} + H_2 h_2 \frac{t^2}{2} + V \frac{t^2}{2} = \frac{p l^2}{I_2} , \]

\[ H_0 \frac{h^2}{2} + H_2 h_2 \frac{t^2}{2} + V \frac{t^2}{2} = \frac{p l^2}{I_2} , \]

\[ H_0 \frac{h^2}{2} + H_2 h_2 \frac{t^2}{2} + V \frac{t^2}{2} = \frac{p l^2}{I_2} . \]

The solution of these equations gives us the following formulas,

\[ H_0 = \frac{1}{l^2} \left[ 3 + 3 \frac{h_0 t^3}{h_0 t^3} + 3 \frac{h_0 t^3}{h_0 t^3} + 2 \frac{h^2 t^3}{h^2 t^3} \right] \frac{p l^2}{4} , \]

\[ H_2 = \frac{1}{l^2} \left[ 3 + 3 \frac{h_0 t^3}{h_0 t^3} + 3 \frac{h_0 t^3}{h_0 t^3} + 2 \frac{h^2 t^3}{h^2 t^3} \right] \frac{p l^2}{4} , \]

\[ V_o = \frac{(6 + \frac{t^2}{l^2}) + 6 \frac{t^2}{h, l^2} + 6 \frac{t^2}{h, l^2} + 4 \frac{t^2}{l, l^2} + 4 \frac{t^2}{l, l^2} + 4 \frac{t^2}{l, l^2}}{3 + 3 \frac{t^2}{h, l^2} + 3 \frac{t^2}{h, l^2} + 2 \frac{t^2}{l, l^2} + 2 \frac{t^2}{l, l^2}} \frac{p l^2}{8} , \]

\[ V_l = \frac{\frac{t^2}{l^2}}{3 + 3 \frac{t^2}{h, l^2} + 3 \frac{t^2}{h, l^2} + 2 \frac{t^2}{l, l^2}} \frac{p l^2}{4} . \]
Special Case a. If we assume that $l = l_1 = l_2$, and $l_1 = l_2$, then we get,

$$H_0 = \frac{1}{\bar{h}_2} \left[ 3 + 5 \frac{\bar{h}_1}{l_0} + 3 \frac{\bar{h}_2 l_1}{\bar{h}_2 l_0} \right] \frac{p l}{4},$$

$$H_2 = \frac{1}{\bar{h}_2} \left[ 3 + 5 \frac{\bar{h}_1}{l_0} + 3 \frac{\bar{h}_2 l_1}{\bar{h}_2 l_0} \right] \frac{p l}{4},$$

$$V_0 = \frac{11 + 6 \frac{l_1}{\bar{h}_1 l_1} + 6 \frac{l_1}{\bar{h}_2 l_1}}{5 + 3 \frac{l_1}{\bar{h}_1 l_1} + 3 \frac{l_1}{\bar{h}_2 l_1}} \frac{p l}{4},$$

$$V_1 = \frac{1}{5 + 3 \frac{l_1}{\bar{h}_1 l_1} + 3 \frac{l_1}{\bar{h}_2 l_1}} \frac{p l}{8},$$

$$M_{h_1} = -H_0 h_1, \quad M_{l} = -V_1 l, \quad M_{h_2} = -H_2 h_2.$$

Special Case b. If we, furthermore, assume that $I_3 = 0$, then the form of frame will be

This is transformed to the case (13), and the resulting formulas are as follows:

$$H_0 = \frac{1}{\bar{h}_2} \left[ 3 + 5 \frac{\bar{h}_1}{l_0} \right] \frac{p l}{4},$$

$$V_0 = \frac{11 + 6 \bar{h}_2}{\bar{h}_2 + 6 \frac{l_1}{\bar{h}_1 l_0}} \frac{p l}{4},$$

$$V_1 = \frac{\bar{h}_1}{\bar{h}_2} \left[ 3 + 5 \frac{\bar{h}_1}{l_0} \right] \frac{p l}{4},$$

$$H_2 = 0.$$

These are exactly the same as the results obtained in Case (13).
Special Case c. If we take up the exceptionally special case in which \( l_1 = l_2 = h_1 = h_2 \), and \( I_0 = I_1 = I_2 = I_3 \), we have:

\[
H_0 = \frac{1}{44} P l, \quad H_2 = \frac{1}{44} P l, \\
V_0 = \frac{23}{88} P l, \quad V_1 = \frac{1}{44} P l.
\]

Therefore the bending moments at the joint B (1, 2 and 3) are \( \frac{1}{44} P l^2 \) respectively.
Case 20. Frame with Fixed Ends.

This is the case having fifteen statically indeterminate quantities, but the condition of symmetrical loading greatly reduces the number of these quantities. Assuming that B and B' do not change their original positions vertically and horizontally after deformation due to load, we have the following relations:

\[ V = \frac{3M_1}{l_1}, \quad H_0 = \frac{3M_0}{h_1}, \quad H_2 = \frac{3M_2}{h_2}. \]

The necessary quantities in the formation of the conditional equations are as follows:

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>M</th>
<th>( \frac{\partial M}{\partial M_0} )</th>
<th>( \frac{\partial M}{\partial l_1} )</th>
<th>( \frac{\partial M}{\partial l_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB, A'B'</td>
<td>( I_0 )</td>
<td>( M_0(1-\frac{3x}{h_1}) )</td>
<td>1-( \frac{3x}{h_1} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BC, B'C'</td>
<td>( I_1 )</td>
<td>( M_1(1-\frac{3x}{h_1}) )</td>
<td>0</td>
<td>1-( \frac{3x}{h_1} )</td>
<td>0</td>
</tr>
<tr>
<td>BD, B'D'</td>
<td>( I_2 )</td>
<td>( M_2(1-\frac{3x}{h_2}) )</td>
<td>0</td>
<td>0</td>
<td>1-( \frac{3x}{h_2} )</td>
</tr>
<tr>
<td>BB'</td>
<td>( I_3 )</td>
<td>(-2M_0-2M_1-2M_2+\frac{pl_0x}{2}+\frac{pl_0x}{2} )</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
</tr>
</tbody>
</table>
Forming the equations for the work due to deformation and performing the mathematical operations we get the following equations,

\[ M_0 \left( \frac{P_1}{L_0} + \frac{2L_z}{L_2} \right) + M_1 \frac{2L_z}{L_2} + M_2 \frac{2L_z}{L_2} = \frac{P_i L^3}{12 I_2}, \]
\[ M_0 \frac{2L_z}{L_2} + M_1 \left( \frac{L_i}{L_1} + \frac{2L_z}{L_2} \right) + M_2 \frac{2L_z}{L_2} = \frac{P_i L^3}{12 I_2}, \]
\[ M_0 \frac{2L_z}{L_2} + M_1 \frac{2L_z}{L_2} + M_2 \left[ \frac{L_i}{L_1} + \frac{2L_z}{L_2} \right] = \frac{P_i L^3}{12 I_2}. \]

Solving these equations simultaneously we will get the following formulas for statically indeterminate quantities.

\[ M_0 = \frac{1}{2 + 2 \frac{L_i}{L_1} + 2 \frac{L_i L_3}{L_2 L_0} + \frac{L_i L_2}{L_2 L_0}}, \]
\[ M_1 = \frac{1}{2 + 2 \frac{L_i}{L_1} + 2 \frac{L_i L_3}{L_2 L_1} + \frac{L_i L_2}{L_2 L_1}}, \]
\[ M_2 = \frac{1}{2 + 2 \frac{L_i}{L_1} + 2 \frac{L_i L_3}{L_2 L_3} + \frac{L_i L_2}{L_2 L_3}}, \]
\[ H_0 = \frac{1}{\frac{L_i}{L_2} \left[ 2 + 2 \frac{L_1}{L_1} + 2 \frac{L_1 L_3}{L_2 L_0} + \frac{L_i L_2}{L_2 L_0} \right]}, \]
\[ H_2 = \frac{1}{\frac{L_i}{L_2} \left[ 2 + 2 \frac{L_1}{L_1} + 2 \frac{L_1 L_3}{L_2 L_3} + \frac{L_i L_2}{L_2 L_3} \right]}, \]
\[ V_0 = \frac{(4 + \frac{L_i}{L_1}) + 4 \frac{L_i}{L_1} + 4 \frac{L_i L_3}{L_2 L_1} + 2 \left( \frac{L_i}{L_2 L_1} \right)}{\left[ 2 + 2 \frac{L_i}{L_1} + 2 \frac{L_i L_3}{L_2 L_1} + \frac{L_i L_2}{L_2 L_1} \right]}. \]
Special Case a. If we assume that $l_1 = l_2 = l$, and $I_1 = I_2$ we have the following formulas:

\[
M_0 = \frac{1}{2 + 3 \frac{I_0}{l} + 2 \frac{I_1}{I_0} \frac{P}{l^2}} \times 12,
\]

\[
M_1 = \frac{1}{3 + 2 \frac{I_1}{I_0} + 2 \frac{I_2}{I_1}} \times \frac{P l^2}{12},
\]

\[
M_2 = \frac{1}{2 + 3 \frac{I_2}{I_1} + 2 \frac{I_3}{I_2} \frac{P}{l^2}} \times 12,
\]

\[
H_0 = \frac{1}{\frac{I_0}{l} \left[ 2 + 3 \frac{I_0}{l} + 2 \frac{I_2}{I_0} \right]} \times \frac{P}{4},
\]

\[
H_2 = \frac{1}{\frac{I_2}{l} \left[ 2 + 3 \frac{I_2}{l} + 2 \frac{I_3}{I_2} \right]} \times \frac{P}{4},
\]

\[
V_0 = \frac{7 + 4 \frac{I_0}{I_2} + 4 \frac{I_3}{I_2} \frac{P}{l^2}}{3 + 2 \frac{I_0}{I_2} + 2 \frac{I_3}{I_2}} \times \frac{P}{8},
\]

\[
V_1 = \frac{1}{3 + 2 \frac{I_0}{I_2} + 2 \frac{I_3}{I_2} \frac{P}{l^2}} \times \frac{P}{4},
\]

\[
M_{\hat{n}_1} = -2M_0, \quad M_1 = -2M_0,
\]

\[
M_{\hat{n}_2} = -2M_2.
\]
Special Case b. If we, furthermore, assume that \( I_3 = 0 \), then this is transformed to the Case (16), and resulting formulas are as follows:

\[
M_0 = \frac{1}{2 + 3 \frac{h^2 I_1}{l^2 I_0}} \frac{P l^2}{12},
\]

\[
M_1 = \frac{\frac{h^2 I_1}{l^2 I_0}}{2 + 3 \frac{h^2 I_1}{l^2 I_0}} \frac{P l^2}{12},
\]

\[
M_2 = 0,
\]

\[
H_0 = \frac{1}{4 \left(2 + 3 \frac{h^2 I_1}{l^2 I_0}\right)} \frac{P l}{4},
\]

\[
H_2 = 0,
\]

\[
V_0 = \frac{4 + 7 \frac{h^2 I_1}{l^2 I_0}}{2 + 3 \frac{h^2 I_1}{l^2 I_0}} \frac{P l}{4},
\]

\[
V_1 = \frac{\frac{h^2 I_1}{l^2 I_0}}{2 + 3 \frac{h^2 I_1}{l^2 I_0}} \frac{P l}{4}.
\]

These are exactly the same as the results obtained in Case (16).
Special Case c. If we consider the exceptional special case in which \(l_1 = l_2 = h_1 = h_2\), and \(I_0 = I_1 = I_2 = I_3\), then we have:

\[
M_0 = \frac{1}{84} pl^2, \quad H_0 = \frac{1}{28} pl^2,
\]

\[
M_1 = \frac{1}{84} pl^2, \quad H_1 = \frac{1}{28} pl^2,
\]

\[
M_2 = \frac{1}{84} pl^2, \quad V_0 = \frac{15}{36} pl^2, \quad V_1 = \frac{1}{28} pl^2.
\]

Therefore the bending moment at the joint B (1, 2, and 3) is \(\frac{1}{42} pl^2\) respectively. We can see that the bending moment in the column ends, say point 1 or 2, is not very much affected by the end condition, fixed or hinged, but is nearly the same in both cases.
14. **Bridge or Viaduct with Three Spans.**—In bridge or trestle construction across a long valley or wide stream a number of frames may be grouped together; three spans in a group are frequently used, owing to the expansion provision. One of the examples is the reinforced concrete viaduct (200 meters long) near Stein (Beton u.Eisen, Feb. 26, 1913).

Rigidly connected frames with three spans, equal or unequal, may advantageously be used for bridges and viaducts of moderate spans.

No analytical formulas have, to the writer's knowledge, been published at this time. The following analysis furnishes us with necessary formulas for practical application.
Case 21. Frame with Three Spans, Hinged Column Ends.

Uniform Vertical Loads of Different Intensities.

The statically indeterminate quantities are $V_1$, $H_1$, and $H_0$ in this case. The following table is made as before.

<table>
<thead>
<tr>
<th>Member</th>
<th>$I$</th>
<th>$M$</th>
<th>$\frac{\partial M}{\partial H_0}$</th>
<th>$\frac{\partial M}{\partial H_1}$</th>
<th>$\frac{\partial M}{\partial V_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DF, D'F'</td>
<td>$I_0$</td>
<td>$-V_1 x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>FB, F'B'</td>
<td>$I_1$</td>
<td>$-H_0 h - \frac{2p_1 l_1 + p_2 l_2}{2} x - \frac{p_1 x^2}{2}$</td>
<td>$-h$</td>
<td>0</td>
<td>$-x$</td>
</tr>
<tr>
<td>AAB</td>
<td>$I_2$</td>
<td>$-H_1 V$</td>
<td>0</td>
<td>$-h$</td>
<td>0</td>
</tr>
<tr>
<td>BB'</td>
<td>$I_3$</td>
<td>$-H_0 h - H_1 h - V_1 l_1 + p_1 l_1^2 + p_2 (l_1 l_2 + l_2 x - x^2)$</td>
<td>$-h$</td>
<td>$-h$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

The conditional equations are as follows:

1. \[
\frac{2}{EI_0} \int_0^l H_0 y^2 \, dy - \frac{2h}{EI_0} \int_0^l [-H_0 h + \frac{2p_1 l_1 + p_2 l_2}{2} x - \frac{p_1 x^2}{2} - V_1 x] \, dx = 0,
\]

2. \[
- \frac{h}{EI_0} \int_0^l [-H_0 h - H_1 h - V_1 l_1 + \frac{p_1 l_1^2 + p_2 l_2^2 l_2 x - x^2}{2}] \, dx = 0,
\]

3. \[
\frac{2}{EI_2} \int_0^{l_2} H_0 y_2^2 \, dy - \frac{h}{EI_2} \int_0^{l_2} [-H_0 h - H_1 h - V_1 l_1 + \frac{p_1 l_1^2 + p_2 (l_1 l_2 + l_2 x - x^2)}{2}] \, dx = 0.
\]
Solving these equations and arranging in form we have the following simultaneous equations:

\[ H_0 A_i + H_i B_i + \bar{V}_i C_i = K_i, \]
\[ H_0 A_2 + H_2 B_2 + \bar{V}_2 C_2 = K_2, \]
\[ H_0 A_3 + H_3 B_3 + \bar{V}_3 C_3 = K_3, \]

where

\[ A_i = \bar{H} \left( \frac{2 \ell_x}{3 \ell_i} + \frac{2 \ell_z}{\ell_i} + \frac{\ell_z}{\ell_i} \right), \]
\[ A_2 = \bar{H} \left( \frac{\ell_z}{\ell_i} \right), \]
\[ A_3 = \bar{H} \left[ \frac{\ell_z}{\ell_i} + \frac{\ell_z}{\ell_i} \right], \]
\[ B_2 = \bar{H} \left[ \frac{\ell_z}{\ell_i} + \frac{2 \ell_z}{\ell_i} \right], \]
\[ C_1 = \bar{H} \left[ \frac{\ell_z}{\ell_i} + \frac{\ell_z}{\ell_i} \right], \]
\[ C_2 = \bar{H} \left[ \frac{\ell_z}{\ell_i} \right], \]
\[ C_3 = \bar{H} \left[ \frac{2 \ell_z}{\ell_i} + \frac{\ell_z}{\ell_i} \right], \]

\[ K_1 = p_i \ell_i \left( \frac{2 \ell_z^2}{\ell_i} + \frac{\ell_z}{\ell_i} \right) + p_i \ell_i \left[ \frac{\ell_z^2}{\ell_i} + \frac{\ell_z}{\ell_i} \left( \frac{6 \ell_z + \ell_z}{6} \right) \right], \]
\[ K_2 = p_i \ell_i \left( \frac{\ell_z}{\ell_i} \right) + p_i \ell_i \left[ \frac{\ell_z}{\ell_i} \left( \frac{6 \ell_z + \ell_z}{6} \right) \right], \]
\[ K_3 = p_i \ell_i \left( \frac{5 \ell_z^2}{\ell_i} + \frac{\ell_z}{\ell_i} \right) + p_i \ell_i \left[ \frac{\ell_z^2}{\ell_i} + \frac{\ell_z}{\ell_i} \left( \frac{6 \ell_z + \ell_z}{6} \right) \right]. \]

Solving and simplifying the above equations we get

\[ H_0 = \frac{p_i \ell_i \left[ 3 + 6 \frac{h_{T_i}}{h_{T_0}} + 2 \frac{h_{T_i}}{h_{T_0}} \right] - 2 p_i \ell_i \frac{h_{T_i}}{h_{T_0}}}{4 \frac{h_{T_i}}{h_{T_0}} \Delta}, \]
\[ H_1 = \frac{p_i \ell_i \left[ \frac{\ell_z}{\ell_i} \left( 3 + 4 \frac{h_{T_0}}{h_{T_0}} \right) \right] - 3 p_i \ell_i \left( 1 + \frac{2 h_{T_0}}{h_{T_0}} \right)}{4 \frac{h_{T_0}}{h_{T_0}} \Delta}, \]
\[ \bar{V}_i = \frac{p_i \ell_i \left[ 9 + 15 \frac{h_{T_z}}{h_{T_0}} + 9 \frac{h_{T_z}}{h_{T_0}} + 6 \frac{h_{T_0}}{h_{T_0}} + 2 \frac{h_{T_0}}{h_{T_0}} \left( \frac{6 \frac{h_{T_z}}{h_{T_0}} + 5 \frac{h_{T_z}}{h_{T_0}}}{6} \right) \right]}{2 \Delta} \]
\[ + \frac{p_i \ell_i \left[ 9 + 12 \frac{h_{T_z}}{h_{T_0}} + 12 \frac{h_{T_z}}{h_{T_0}} + 6 \frac{h_{T_0}}{h_{T_0}} + 2 \frac{h_{T_0}}{h_{T_0}} \left( 6 \frac{h_{T_0}}{h_{T_0}} + 4 \frac{h_{T_0}}{h_{T_0}} \right) \right]}{2 \Delta} \]

where \( \Delta = 9 + 12 \frac{h_{T_z}}{h_{T_0}} + 12 \frac{h_{T_z}}{h_{T_0}} + 6 \frac{h_{T_0}}{h_{T_0}} + 2 \frac{h_{T_0}}{h_{T_0}} \left( 6 \frac{h_{T_0}}{h_{T_0}} + 4 \frac{h_{T_0}}{h_{T_0}} \right) \).
Special Case a.

Substituting $l_1 = l_2 = l$, $I_1 = I_2$, $I_0 = I_3$ in the general formulas we obtain the following expressions for special case

$$H_0 = \frac{3\left(1 + 2 \frac{h_1}{I_0}\right)}{4 \frac{h_2}{I_1} \Delta_1} p l,$$

$$H_i = \frac{-\left(\frac{h_i}{I_0}\right)}{2 \cdot \frac{h_2}{I_1} \cdot \Delta_1} p l,$$

$$V_i = \frac{18 + 63 \frac{h_i}{I_0} + 44\left(\frac{h_i}{I_0}\right)^2}{2 \Delta_1} p l,$$

where $\Delta_i = 9 + 30 \frac{h_i}{I_0} + 20\left(\frac{h_i}{I_0}\right)^2$.

Special Case b. Uniform Load on the Middle Span.

For this case we have following formulas:

$$H_0 = (-)\frac{\frac{h_1}{I_0}}{2 \frac{h_2}{I_1} \cdot \Delta_1} p l,$$

$$H_i = \frac{3 + 4 \frac{h_i}{I_0}}{4 \frac{h_2}{I_1} \Delta_1} p l,$$

$$V_i = \frac{9 + 33 \frac{h_i}{I_0} + 22\left(\frac{h_i}{I_0}\right)^2}{2 \Delta_1} p l,$$

$$V_0 = (-)\frac{3 \frac{h_1}{I_0} + 2\left(\frac{h_1}{I_0}\right)^2}{2 \Delta_1} p l,$$

where $\Delta_i = 9 + 30 \frac{h_i}{I_0} + 20\left(\frac{h_i}{I_0}\right)^2$.  

Substituting the following values in the general expressions for \( H \) and \( V_1 \), \( P_1 = 0 \), \( P_2 = P \), \( I_0 = \infty \), \( I_1 = I_2 \), and \( l_1 = l_2 = 1 \), we have:

\[
H_i = \frac{1}{\frac{\delta^2}{2} \left( 1 + 2 \frac{\delta^2}{2} \right)} \frac{P_1}{12},
\]
\[
V_i = \frac{3 + 7 \frac{\delta^2}{2} \delta}{1 + 2 \frac{\delta^2}{2} \delta} \frac{P_1}{6}.
\]

These are exactly the same as the results obtained in Case (15).

Special Case d. If we assume that \( I_0 = I_1 = I_2 = I_3 \),\( l_1 = l_2 = h_1 \), and \( P_1 = P_2 = P \), we will obtain the following values:

\[
H_0 = \frac{9}{236} P_l,
\]
\[
H_i = \frac{-2}{236} P_l,
\]
\[
V_i = \frac{250}{236} P_l,
\]
\[
V_0 = \frac{14}{236} P_l.
\]
Case 22. Frame with Three Spans, Fixed Column Ends.

Uniform Vertical Loads of Different Intensities.

The statically indeterminate quantities are $H_0$, $H_1$, $M_0$, $M_1$, and $V_1$. Using the relation between the horizontal thrusts and the end moments,

$$M_0 = \frac{H_0 h}{3}, \quad M_1 = \frac{H_1 h}{3}$$

we have:

<table>
<thead>
<tr>
<th>Member</th>
<th>$I$</th>
<th>$M$</th>
<th>$\frac{\partial M}{\partial V_1}$</th>
<th>$\frac{\partial M}{\partial M_0}$</th>
<th>$\frac{\partial M}{\partial M_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DF, D'F'</td>
<td>$I_0$</td>
<td>$+M_0 \left(1 - \frac{3V}{h}\right)$</td>
<td>0</td>
<td>$(1 - \frac{3V}{h})$</td>
<td>0</td>
</tr>
<tr>
<td>FB, F'B'</td>
<td>$I_1$</td>
<td>$-2M_0 + \left(\frac{p_1 l_1 + p_2 l_2}{2}\right)x - V_1 x \frac{p_1 x^2}{2}$</td>
<td>$-x$</td>
<td>$-2$</td>
<td>0</td>
</tr>
<tr>
<td>AB, A'B'</td>
<td>$I_3$</td>
<td>$+M_1 \left(1 - \frac{3V}{h}\right)$</td>
<td>0</td>
<td>0</td>
<td>$(1 - \frac{3V}{h})$</td>
</tr>
<tr>
<td>BB'</td>
<td>$I_2$</td>
<td>$-2M_0 - 2M_1 - V_1 l_1 + \left(\frac{p_1 l_1 + p_2 l_2}{2}\right)l_1 + \frac{p_2 l_2 x}{2}$</td>
<td>$- \frac{p_2 x^2}{2}$</td>
<td>$-1 l$</td>
<td>$-2$</td>
</tr>
</tbody>
</table>
Constructing the equations, solving for statically indeterminate quantities, and simplifying we will finally get the following formulas:

\[
M_0 = \frac{P_1 [2 + 3 \frac{h_1}{I_1} + \frac{h_2}{I_2}] - P_2 \frac{I_2}{I_1} \frac{h_2}{I_2}}{12 \Delta},
\]

\[
M_i = \frac{2P_2 \frac{I_2}{I_1} \frac{h_2}{I_2} [1 + \frac{h_1}{I_1}] - P_1 [2 + 15 \frac{h_1}{I_0} + 6 \frac{h_1}{I_1} (1 - 3 \frac{h_1}{I_2})]}{12 \Delta},
\]

\[
V_i = \frac{P_1 \left\{ 8 + 10 \frac{h_1}{I_1} + 6 \frac{h_1}{I_3} \left[ 1 + \frac{h_1}{I_1} \right] + \frac{h_1}{I_2} \left[ 4 + 5 \frac{h_1}{I_1} \right] \right\}}{4 \Delta},
\]

\[+ \frac{P_2 \left\{ 8 \left( 1 + \frac{h_1}{I_1} \right) + 4 \frac{h_1}{I_1} \left[ 1 + \frac{h_1}{I_1} \right] + \frac{h_1}{I_2} \left[ (2 + \frac{h_1}{I_1}) (6 + \frac{h_1}{I_1}) - 4 \right] \right\}}{4 \Delta},\]

(44)

\[
H_0 = \frac{3M_0}{h} = \frac{P_1 \left[ 2 + 3 \frac{h_1}{I_1} + \frac{h_2}{I_2} \right] - P_2 \frac{I_2}{I_1} \frac{h_2}{I_2}}{4 \frac{h_1}{I_1} \Delta},
\]

\[
H_i = \frac{3M_i}{h} = \frac{2P_2 \frac{I_2}{I_1} \frac{h_2}{I_2} [1 + \frac{h_1}{I_1}] - P_1 [2 + 15 \frac{h_1}{I_0} + 6 \frac{h_1}{I_1} (1 - 3 \frac{h_1}{I_2})]}{4 \frac{h_1}{I_1} \Delta},
\]

Where

\[
\Delta = 4 + 2 \frac{h_1}{I_1} + 4 \frac{h_1}{I_1} + 4 \frac{h_1}{I_1} + \frac{h_1}{I_1} \left[ 2 + 3 \frac{h_1}{I_1} \right].
\]
Special Case a. Uniform load of equal intensity, equal length of spans, $I_1 = I_2$, and $I_0 = I_3$.

\[ M_0 = \frac{2 + 3 \frac{h_x}{t_x}}{12 \Delta} P l^2, \]
\[ M_1 = (-) \frac{\frac{h_x}{t_x}}{12 \Delta} P l^2, \]
\[ V_1 = \frac{16 + 42 \frac{h_x}{t_x} + 22 (\frac{h_x}{t_x})^2}{4 \Delta} P l \]
\[ H_0 = \frac{2 + 3 \frac{h_x}{t_x}}{4 \frac{h_x}{t_x}} P l, \]
\[ H_1 = \frac{- \frac{h_x}{t_x}}{4 \frac{h_x}{t_x}} P l, \]

Where $\Delta = 4 + 10 \frac{h_x}{t_x} + 5 (\frac{h_x}{t_x})^2$.

Special Case b. Uniform load in the middle span, equal length of spans, $I_1 = I_2$, and $I_0 = I_3$.

\[ M_0 = (-) \frac{\frac{h_x}{t_x}}{12 \Delta} P l^2, \]
\[ M_1 = \frac{1 + \frac{h_x}{t_x}}{6 \Delta} P l^2, \]
\[ V_0 = (-) \frac{2 \frac{h_x}{t_x} + (\frac{h_x}{t_x})^2}{4 \Delta} P l, \]
\[ H_0 = \frac{- \frac{h_x}{t_x}}{4 \frac{h_x}{t_x}} P l, \]
\[ H_1 = \frac{1 + \frac{h_x}{t_x}}{2 \frac{h_x}{t_x}} P l, \]

Where $\Delta = 4 + 10 \frac{h_x}{t_x} + 5 (\frac{h_x}{t_x})^2$. 
Special Case c.

Putting \( I_0 = \infty \), \( I_1 = I_2 \), \( l_1 = l_2 = 1 \), and \( p_1 = 0 \) in the general equations we obtain the following formulas:

\[
M_0' = -2M_0 = \frac{hI_0}{2 + 3\frac{F}{I_0}} \frac{p_2^2}{12},
\]
\[
M_1 = \frac{1}{2 + 3\frac{h}{I_0}} \frac{p_2^2}{12},
\]
\[
V_1 = \frac{4 + 7\frac{h}{I_0}}{4(2 + 3\frac{h}{I_0})} p_2^2,
\]
\[
H_1 = \frac{1}{4(2 + 3\frac{h}{I_0})} p_2^2.
\]

These are exactly the same as the results which were established in Case 16.

Special Case d.

If we make \( I_0 = I_1 = I_2 = I_3 \), \( h = 1 \), and \( p_1 = p_2 = p \), then we have:

\[
M_0 = \frac{5}{228} p_1^2, \quad H_0 = \frac{5}{76} p_1^2
\]
\[
M_1 = -\frac{1}{228} p_1^2, \quad H_1 = -\frac{1}{76} p_1^2
\]
\[
V_1 = \frac{20}{19} p_1^2
\]
15. **Square Frames under Horizontal Load.**—We have hitherto discussed the case in which the load was applied vertically. It is frequently necessary to determine the statically indeterminate stresses due to a horizontal force, such as a wind pressure and the braking force of a locomotive. The method of determination of these statically unknowns is exactly the same as before. We will take up a few cases as an illustration.

**Case 23. Square Frame under Concentrated Horizontal Force.**

The deflection at B differs from that at C by the shortening of BC due to a direct stress, but it is very small in comparison with the deflection due to the flexure of the columns. H is the only statically indeterminate force, and \( V = \frac{Fkh}{l} \).

The following table contains the necessary quantities.

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>M</th>
<th>( \frac{\partial M}{\partial H} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>( I_o )</td>
<td>(-Hy) ( \sqrt{\cdot} )</td>
<td>(-Y)</td>
</tr>
<tr>
<td>BC</td>
<td>( I_1 )</td>
<td>(-Hh + \frac{Fkh}{2}x) ( \sqrt{\cdot} )</td>
<td>(-h)</td>
</tr>
<tr>
<td>FD</td>
<td>( I_o )</td>
<td>( Hy - P y ) ( \sqrt{\cdot} )</td>
<td>(+Y)</td>
</tr>
<tr>
<td>DC</td>
<td>( I_0 )</td>
<td>( Hy - Pkh ) ( \sqrt{\cdot} )</td>
<td>(+Y)</td>
</tr>
</tbody>
</table>
Therefore the equation of condition is:

\[ \frac{1}{EI_0} \left[ \int_0^h H y^2 \, dy + \int_0^{kh} \left( H y^2 - P^2 \right) \, dy + \int_0^h \left( H y^2 - P k y \right) \, dy \right] + \frac{h}{EI_0} \int_0^l \left( -T_x + H H_x \right) \, dx = 0. \]

The solution gives us the following results:

\[ H = \frac{k \left[ (3 - k^2) \frac{h}{I_0} + 3 \frac{c}{I_1} \right]}{2 \left[ \frac{2h}{I_0} + 3 \frac{c}{I_1} \right]} P = \frac{k \left[ 3 + (3 - k^2) \frac{h I_1}{z I_0} \right]}{2 \left[ 3 + 2 \frac{h I_1}{z I_0} \right]} P. \tag{47} \]

For \( k = 1, I_0 = I_1, \) and \( h = l, \) \( H = \frac{P}{2}. \)

Position of the point of inflection in the top beam is as follows:

\[ l_x = \frac{\left[ 3 + (3 - k^2) \frac{h I_1}{z I_0} \right]}{2 \left[ 3 + 2 \frac{h I_1}{z I_0} \right]} l. \tag{48} \]

For the simplest case in which \( I_0 = I_1, \) \( h = l, \) and \( k = 1, \) we have

\[ l_x = \frac{1}{2} l. \]
Case 24. Square Frame under Uniform Horizontal Load.

H is only statically indeterminate force in this case. Taking the moment of all forces about D, we have

\[ V = \frac{ph^3}{2l} \]

The following table contains the necessary quantities in this case where the effect of the normal forces are neglected.

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>M</th>
<th>( \frac{\partial M}{\partial H} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>I₀</td>
<td>((ph-H)y - \frac{py^2}{2})</td>
<td>-Y</td>
</tr>
<tr>
<td>CD</td>
<td>I₀</td>
<td>(Hy)</td>
<td>-Y</td>
</tr>
<tr>
<td>BC</td>
<td>I₁</td>
<td>((ph-H)h - \frac{ph^2}{2} - \frac{ph^2}{2l}x)</td>
<td>-h</td>
</tr>
</tbody>
</table>

The equation of condition is as follows:

\[
\frac{1}{EI_o} \left[ \int_0^h (H-ph)y^2 + \frac{py^2}{2} dy + H_y^2 dy + \int_0^h Hy^2 dy \right] \\
+ \frac{h^2}{EI_o} \int_0^l [(H-ph) + \frac{ph^2}{2} + \frac{ph^2}{2l}x] dx = 0.
\]
Solving for the statically indeterminate quantity $H$ and simplifying we have the following formula:

$$H = \frac{5h + 6t}{2h + 3t} \frac{Ph}{8},$$

(49)

The point of inflection in the column $AB$ is found by solving the following equation:

$$(Ph - H)h - \frac{Ph^2}{2} = 0,$$

where $h_1$ denotes the distance from $A$ to the point of inflection. The solution gives us the following results:

$$h_1 = 0,$$

and

$$h_2 = \frac{18 + 11 \frac{h_1}{h_0}}{3 + 2 \frac{h_1}{h_0}} \frac{P}{H},$$

(50)

If we take a simple case in which $I_0 = I_1$, $h = 1$, we have

$$h_1 = 0 \text{ or } h_2 = \frac{29}{40} h.$$

The position of the point of inflection in the top beam is found by solving the following equation:

$$\frac{Ph^2}{2} - Hh - \frac{Ph^2}{2} l = 0,$$

where $l_1$ denotes the distance from the point $B$ to the point of inflection, and the result of the solution is

$$l_1 = \frac{3}{4} \left[ \frac{2 + \frac{h_1}{h_0}}{3 + 2 \frac{h_1}{h_0}} \right] l.$$

(51)

For the simplest case where $I_0 = I_1$, and $h = 1$, we have

$$l_1 = \frac{9}{20} l.$$
Case 25. Horizontal Concentrated Load on Square Frame

Having Two Spans.

This frame contains six statically indeterminate quantities, that is to say $H_0$, $H_1$, $V_0$, $V_1$, $M_0$, and $M_1$, and $H_2$, $V_2$, and $M_2$ are expressed as follows:

$$H_2 = P - (H_0 + H_1),$$
$$V_2 = \frac{P h + V_1 l_1 - (M_0 + M_2)}{l_0},$$
$$M_2 = V_0 l_0 + V_1 l_2 + M_0 + M_1 - H h.$$

The following table shows the necessary quantities to form the conditional equations when the effect of normal forces is neglected.

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>M</th>
<th>$\frac{\partial M}{\partial H_0}$</th>
<th>$\frac{\partial M}{\partial H_1}$</th>
<th>$\frac{\partial M}{\partial V_0}$</th>
<th>$\frac{\partial M}{\partial V_1}$</th>
<th>$\frac{\partial M}{\partial M_0}$</th>
<th>$\frac{\partial M}{\partial M_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>$I_0$</td>
<td>$M_0 - H_0 V$</td>
<td>$-y$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BD</td>
<td>$I_1$</td>
<td>$M_0 - H_0 h + V_0 x$</td>
<td>$-h$</td>
<td>0</td>
<td>$+x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>CD</td>
<td>$I_3$</td>
<td>$M_1 - H_1 V$</td>
<td>0</td>
<td>$-y$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>DC</td>
<td>$I_2$</td>
<td>$M_0 + M_1 + V_0 (1 + x) + V_1 x - H_0 h - H_1 h$</td>
<td>$-h$</td>
<td>$(1 + x)$</td>
<td>$+x$</td>
<td>$+1$</td>
<td>$+1$</td>
<td>0</td>
</tr>
<tr>
<td>FG</td>
<td>$I_4$</td>
<td>$M_2 - H_2 y = M_0 + M_1 + V_0 l_0 + V_1 l_2 + H_1 y$</td>
<td>$+1$</td>
<td>$+H_0 V - P(h + y)$</td>
<td>$+y$</td>
<td>$+y$</td>
<td>$+l_0$</td>
<td>$+l_2$</td>
</tr>
</tbody>
</table>
The equations to be solved are as follows:

\( \frac{1}{EI_0} \int_0^h (-M_0y + H_0y^2) \, dy - \frac{h}{EI_0} \int_0^h (M_o - H_o \dot{y} + \bar{T}_0 \dot{x}) \, dx \)

\( - \frac{h}{EI_2} \int_0^h \left[ M_o + M_1 + \bar{T}_0 (l + x) + \bar{T}_1 \dot{x} - H_o \dot{y} - H_1 \dot{y} \right] \, dx \)

\( + \frac{1}{EI_y} \int_0^h \left[ M_o + M_1 + \bar{T}_0 l \dot{y} + \bar{T}_1 \dot{y} + H_o y + H_1 y - P(h + y) \right] dy = 0 \)

\( \frac{h}{EI_0} \int_0^h \left[ M_o + M_1 + \bar{T}_0 (l + x) + \bar{T}_1 \dot{x} - H_o \dot{y} - H_1 \dot{y} \right] \, dx \)

\( + \frac{1}{EI_2} \int_0^h \left[ M_o + M_1 + \bar{T}_0 l \dot{y} + \bar{T}_1 \dot{y} + H_o y + H_1 y - P(h + y) \right] dy = 0 \)

\( \frac{1}{EI_0} \int_0^h \left[ M_o - H_o \dot{y} + \bar{T}_0 \dot{x} \right] \, dx + \frac{1}{EI_2} \int_0^h \left[ M_o + M_1 + \bar{T}_0 (l + x) + \bar{T}_1 \dot{x} - H_o \dot{y} - H_1 \dot{y} \right] \, dx \)

\( + \frac{1}{EI_2} \int_0^h \left[ M_o + M_1 + \bar{T}_0 l \dot{y} + \bar{T}_1 \dot{y} + H_o y + H_1 y - P(h + y) \right] dy = 0 \)

\( \frac{1}{EI_0} \int_0^h \left[ M_o + M_1 + \bar{T}_0 (l + x) + \bar{T}_1 \dot{x} - H_o \dot{y} - H_1 \dot{y} \right] \, dx + \frac{1}{EI_2} \int_0^h \left[ M_o - H_0 \dot{y} \right] \, dx \)

\( + \frac{1}{EI_2} \int_0^h \left[ M_o + M_1 + \bar{T}_0 l \dot{y} + \bar{T}_1 \dot{y} + H_o y + H_1 y - P(h + y) \right] dy = 0 \)

The solution of these six equations will give us six simultaneous equations containing six statically indeterminate quantities. The final results will be very complicated owing to the fact that there are so many constants in the simultaneous equations.
In this case the statically indeterminate quantity is the bending moment \( M_0 \) at the corner of the wall.

Therefore we have as before:

<table>
<thead>
<tr>
<th>Member</th>
<th>I</th>
<th>( M )</th>
<th>( \frac{\partial M}{\partial M_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>( I_1 )</td>
<td>( M_0 + \frac{p l_1}{2} x - \frac{p x^2}{2} )</td>
<td>+1</td>
</tr>
<tr>
<td>BC</td>
<td>( I_2 )</td>
<td>( M_0 + \frac{p l_2}{2} x - \frac{p x^2}{2} )</td>
<td>+1</td>
</tr>
</tbody>
</table>

The conditional equations for a tank having \( n \) sides of \( l_1 \) and \( l_2 \), respectively, are as follows:

\[
\frac{n}{E I_1} \int_0^{l_1} \left[ M_0 + \frac{p l_1}{2} x - \frac{p x^2}{2} \right] dx + \frac{n}{E I_2} \int_0^{l_2} \left[ M_0 + \frac{p l_2}{2} x - \frac{p x^2}{2} \right] dx = 0.
\]

The solution gives the following formula,

\[
M_0 = (-) \frac{l_1^3 + l_2^3 \frac{x_1}{I_1}}{l_1 + l_2 \frac{x_1}{I_2}} \frac{P}{I_2}.
\]

(52)
Special Case a. Rectangular Tank (Fig. 42).

Putting \( l_1 = 1, l_2 = kl \) in the foregoing equation (52) we directly obtain the following formula for the square tank:

\[
M_o = \left( -\frac{1}{12} \right) \frac{1 + k \frac{l_1^3}{l_2^3}}{1 + k} \frac{p l^2}{12} \tag{53}
\]

Special Case b. Square Tank.

In this case \( k = 1 \), and we have

\[
M_o = \left( -\frac{1}{12} \right) p l^2.
\]

The maximum positive moment at the center of the wall of the tank is

\[
M_{\text{max}} = \frac{1}{24} p l^2.
\]

16. Deflection of Frame.—There are several methods of deriving analytically the formula for deflection of a member subject to flexure:

(a) by deriving the equation of its elastic curve,

(b) by means of the work due to an auxiliary load of unity,

(c) by analyzing its bending moment diagram, or by means of area-moments.

The first method is based on the well-known relation between the radius of curvature of elastic line and bending moment. This
method is commonly used by many authors on Strength of Materials. This is most convenient when the deflection at all points in a member is desired, but it is very laborious in all but the simplest cases like simple beams.

The second method is more convenient when the general expression for the deflection at any definite point of a straight member is desired. The easiest and most flexible method of finding the deflection at any definite point in a member is the last one, area-moments, which was first enunciated by Professor Charles E. Greene in 1873.*

To deduce the deflection formulas for frames, the method of area-moment will be used. Omitting elementary conceptions we know that the angle between tangents drawn at any two points A and B (Fig. 43) of the elastic member is the summation of \( \phi \) between those two limits, or if the member has the constant moment of inertia,

\[
\phi = \int_A^B \frac{d\phi}{dx} = \frac{1}{EI} \int_A^B M dx.
\]

*See Michigan Technic, p. 21.
But \(\int_{a}^{b} M\,dx\) is the area of the moment diagram between two limits, and if this area is known and then divided by EI, the result is the total change of angle, \(\phi\), in radians.

To find the displacement of a point B on the elastic line from a tangent at A imagine that the portion AB, when in an unstressed condition, had the position AB' and that the point B' has been bent around to B through a distance \(\Delta\). The path from B' to B is, strictly speaking, not a straight vertical line, but in the case of engineering structures the curvature is very small and accordingly the path of B deviates but little from the right line. The elementary displacement caused by a bending of element dx is

\[d\Delta = x\,d\phi\]

and the total displacement of a member of constant cross section is

\[\Delta = \int_{a}^{b} x\,d\phi = \frac{1}{EI} \int_{a}^{b} M\cdot dx \cdot x = \frac{1}{EI} \bar{x} \int_{a}^{b} M\,dx\]

where \(\bar{x}\) is the distance from the centroid of the moment diagram to the ordinate B'. The product of the moment—area by its arm is called the area—moment and when divided by EI is the displacement \(\Delta\) of the point B from a tangent at A.

If the cross section of a member is not constant, the following formula must be used:

\[\Delta = \frac{\bar{x}}{E} \int_{a}^{b} \frac{M}{I}\,dx\]

Consequently a diagram of \(\frac{M}{I}\) is to be constructed, and \(\phi\) and \(\Delta\) can easily be found as before.
A few examples of finding the deflection formulas will be given for the purpose of checking the observed deflections.

Deflection at the Center of Span of Simple Frame under Symmetrical Concentrated Loads.

Deflection at the center of span is as follows:

\[
\Delta_{\text{max.}} = \frac{1}{EI} \int_0^l M dx = \frac{1}{EI} \left\{ (M_1 + M_2) \left[ \frac{l^2}{2} - \frac{a^3}{2} + \frac{a}{2} \left( \frac{l^2}{2} - \frac{2a}{3} \right) - M_2 \frac{l^2}{4} \right] \right\}
\]

\[
= \frac{1}{EI} \left\{ \frac{P l}{6} \left[ \frac{l^2}{4} - \frac{a^3}{2} + \frac{a^2}{4} + \frac{a^2}{2} \right] - M_2 \frac{l^2}{8} \right\}
\]

\[
= \frac{1}{EI} \left\{ \frac{P l}{144} \left( 3l^2 - 6al + 4a^2 \right) - M_2 \frac{l^2}{8} \right\}
\]  

(54)

For \( a = \frac{1}{3} \),

\[
\Delta_{\text{max.}} = \frac{1}{EI} \int_0^l M dx = \frac{1}{EI} \left\{ \frac{P l}{6} \left[ \frac{l^2}{8} - \frac{l^2}{12} + \frac{a^2}{8} \right] - \frac{M_2 l^2}{8} \right\} = \frac{1}{EI} \left\{ \frac{13}{1296} P l^2 - \frac{l^2}{8} M_2 \right\}
\]  

(55)

For centrally concentrated load,

\[
\Delta_{\text{max.}} = \frac{1}{EI} \int_0^l M dx = \frac{1}{EI} \left\{ \frac{P l}{4} \left[ \frac{l^3}{4} - \frac{l^2}{2} + \frac{a^2}{2} \right] - M_2 \frac{l^2}{4} \right\}
\]

\[
= \frac{1}{EI} \left[ \frac{P l^3}{48} - \frac{a^2}{8} M_2 \right]
\]  

(56)
For uniformly distributed load on the top beam,

\[ \Delta_{\text{max}} = \frac{1}{E I} \int \frac{x}{M} \, dx \]

\[ = \frac{1}{E I} \left[ \frac{P^2}{8} \left( \frac{2}{3} \frac{3^3}{16} - M_2 \frac{7}{2^4} \right) \right] \]

\[ = \frac{1}{E I} \left[ \frac{P^2}{128} - M_2 \frac{7^2}{8} \right] \quad (57) \]

This method of finding deflection can be extended to any frame in which statically indeterminate forces are otherwise determined.

IV. DISCUSSION OF NATURE OF RESULTING FORMULA.

17. Relation between Horizontal Reactions in Symmetrical Frame under Uniform and Concentrated Load.—It is interesting and important to note from the results of the foregoing analysis that there is a fixed relation between the horizontal reactions in the symmetrical frame under distributed and concentrated loads. To see this fixed relation we have selected a few cases as an illustration.

Case

Horizontal Thrust at Column Ends, \( H \)

\( \left( m = \frac{3}{11}, \frac{n}{10} \right) \)

\( \frac{3}{n(3+2mn)} \frac{P^2}{12} \)

A.

\( \frac{3}{n(3+2mn)} \frac{P}{8} \)

A'.
Case Horizontal Thrust at the Column Ends:

\[ \frac{3}{n(2+mn) \frac{P^2}{12}} \]

\[ \frac{3}{n(2+mn) \frac{P}{8}} \]

\[ \frac{1}{n(1+2mn) \frac{P}{8}} \]

\[ \frac{3}{n(3+4mn) \frac{P^2}{12}} \]

\[ \frac{3}{n(3+4mn) \frac{P}{8}} \]
We have already stated that there is also a fixed relation between the horizontal thrust and the bending moment at the fixed column or beam ends, and the bending moment can be expressed in terms of the horizontal thrust. The bending moment at any section of a frame is a function of the horizontal thrust. Therefore we may say that the statically indeterminate stresses in the symmetrical frame have a fixed relation under distributed and concentrated loads.

From the foregoing illustrations we can express the horizontal thrusts at the column ends under a uniform and a single centrally concentrated load as the following forms:

Under uniform load \[ H = \kappa \frac{P}{12}, \]
Under centrally concentrated load \[ H = \kappa \frac{P}{8}, \]

where \( \kappa \) is exactly the same in both cases, but varies as the form of frame. Therefore we can directly write down the formula for the horizontal thrust in the frame under concentrated load if we know the formula for that in the frame under uniform load. Consequently the analysis of the statically indeterminate forces is to find out the form of a function.

Now we will proceed to interpret why the horizontal thrust has \( \frac{P}{12} \) or \( \frac{P}{8} \) as a factor. Taking the moment of all forces about the top corner of the frame we have for the bending moment at that corner:

For Case A \[ M = \frac{3}{3+2mn} \frac{Pl^2}{12} = K_1 \frac{Pl^2}{12}, \]
For Case A' \[ M = \frac{3}{3+2mn} \frac{P}{B} = K_1 \frac{P}{B}, \]
For Case B \[ M = \frac{2}{2+mn} \frac{Pl^2}{12} = K_2 \frac{Pl^2}{12}, \]
For Case B' \[ M = \frac{2}{2+mn} \frac{P}{B} = K_2 \frac{P}{B}. \]
It is a well known fact that when a beam is perfectly fixed at its ends, the negative bending moments due to a distributed load and a centrally concentrated load are $\frac{P_1}{12}$ and $\frac{P_1}{8}$ respectively.

We know, therefore, that the bending moment is obtained from the value of the end bending moment of a fixed beam by multiplying by $K$, a coefficient which depends upon the form of the frame but is independent of the method of loading.

Now returning to the nature of the formula for the horizontal thrust at the column end of a frame, we will further notice that the above fixed relation between the values of the horizontal thrusts for a frame under a distributed load and a concentrated load still holds for the case in which a frame is subjected to a non-symmetrical load. The following simple frame will be sufficient to show the statement.

Case. Horizontal Thrust at the Column End, H.

$$\frac{3}{n(3+2mn)} \frac{P_1}{12} = K_1 \frac{P_1}{12},$$

$$\frac{3}{n(3+2mn)} \frac{ab P}{12} = K_2 \frac{ab P}{12},$$

$$\frac{3}{n(2+mn)} \frac{P_1}{12} = K_2 \frac{P_1}{12},$$

$$\frac{3}{n(2+mn)} \frac{ab P}{12} = K_2 \frac{ab P}{12}.$$
We, therefore, conclude that the coefficient $K$ in general remains constant for the same frame and is independent of the method of loading. This statement can easily be extended for the case of multiple concentrated loads, for in that case the horizontal thrust is a sum of the horizontal thrust due to an individual concentrated load.

Note:—It is seen that this statement still holds for a non-symmetrical frame from Cases D and D'.

It will be of interest to find the locus of $y_0$, the point of intersection of the reactions with the load.

As is clear, $H$ (Fig. 44) is a function of $l$, $h$, $I$, and $P$ in a given case. In a complicated form of frame, there are, of course, many statically indeterminate quantities, but $H$ is an important one. The remaining statically indeterminates have always the same denominator as $H$. Therefore it is very interesting to know the nature of $H$.

Since the moments at $A$ and $B$ (Fig. 44) are zero, we know that the equilibrium polygon for the load $P$ must pass through these points. Taking the moments of $H$ and $V_1$ about the point $C$ we have:

$$V_1a - Hy_0 = 0,$$

or

$$y_0 = \frac{V_1a}{H}.$$
H and \( V_1 \) are known in this case when \( P \) and \( a \) are given, and

\[
H = \frac{(\frac{2}{h})^2}{2(\frac{2}{3} + \frac{1}{h})} \frac{ab}{2^2 P},
\]

\[
V_1 = \frac{6}{2} P.
\]

Therefore we have

\[
y_0 = \frac{ab}{2} \frac{P}{H} = \frac{2(\frac{2}{3} + \frac{1}{h})}{(-\frac{2}{h})^2},
\]

or

\[
y_0 = 2h \left( 1 + \frac{2h}{3} \right).
\]

(58)

This equation is entirely free from \( a \), \( b \), and \( P \). Therefore \( y_0 \) is a constant quantity for a given frame and is not changed by the change of the point of application of a load \( P \). Accordingly the locus of the point \( O \) is a straight line parallel to \( AB \).

If we take an example, in which \( \frac{M_1}{I_0} = 1 \) and \( \frac{h}{1} = 1 \),

then

\[
y_0 = \frac{10}{3} h.
\]

For \( \frac{h}{r} = 2 \), \( \frac{h}{r} = 1 \), \( y_0 = \frac{16}{3} h \).

Equation (58) enables us to determine the position of loads for the maximum reaction and stress in any member.

The same method can be extended to any case, remembering that when a column is fixed at its end the point of application of the reaction deviates from the neutral line of the column by

\[
\frac{M_1}{V_1}
\]

where \( M_1 \) is the end moment and \( V_1 \) is the vertical reaction at that point.
18. **Effect of Variation in Moment of Inertia and Relative Height of Frame on Bending Moment in Horizontal Member.**—To see what effect is produced on bending moment by the relative variation of the moment of inertia of cross-section of members and relative height of frame itself, the following tables II to IV and diagrams 3 to 6 were made for several kinds of frames.

From the practical standpoint it is only necessary to consider the bending moment at the center of the span where the load is applied eccentrically with respect to the columns to produce maximum moments and thrusts. Curves of the same nature as those shown in the diagrams can be formed for any kind of frames having the same kind of loading.

From the general nature of the curves shown we may draw the following conclusions:

1. The bending moment is rapidly increased from \( \frac{I_1}{I_0} = 0 \) to \( \frac{I_1}{I_0} = 1.5 \) by the increase of ratio \( \frac{I_1}{I_0} \), but from that point the increase is comparatively small.

2. The increase in height of frame has the same effect on bending moment as the ratio \( \frac{I_1}{I_0} \).

3. The variation in coefficient of bending moment is much wider in the frame hinged at ends of columns and beams than in the case of fixed ends.

4. By the fixity of ends the coefficient of positive bending moment is slightly decreased.

5. In most common cases, where the ratio \( \frac{h}{l} \) is not very far from 1 and \( \frac{I_1}{I_0} \) varies from 1.5 to 3.0, the bending moment at the center of span (case of equal spans) varies from \( \frac{1}{16} \) to \( \frac{1}{14} \) of \( pl^2 \), and may be briefly assumed as \( \frac{1}{16} pl^2 \).
### TABLE II.

Coefficients of Bending Moment for Frame with Hinged Ends.

Case 13.
See Fig. 21.

![Bending moment at the end of middle span.](image)

\[
M_0 = - \frac{1 + \frac{h_2}{3l_2}}{3 + 5 \frac{h_2}{l_2}} \frac{pl^2}{4} = -\alpha p l^2
\]

\[
M_c = \frac{1 + \frac{h_2}{3l_2}}{3 + 5 \frac{h_2}{l_2}} \frac{pl^2}{8} = \beta p l^2
\]

= Bending moment at the center of middle span.

<table>
<thead>
<tr>
<th>(\frac{1}{12}\lambda)</th>
<th>(\frac{h}{l})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda)</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>.0833</td>
</tr>
<tr>
<td>1.0</td>
<td>&quot;</td>
</tr>
<tr>
<td>1.5</td>
<td>&quot;</td>
</tr>
<tr>
<td>2.0</td>
<td>&quot;</td>
</tr>
<tr>
<td>2.5</td>
<td>&quot;</td>
</tr>
<tr>
<td>3.0</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\frac{1}{24}\beta)</th>
<th>(\frac{h}{l})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta)</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>.0417</td>
</tr>
<tr>
<td>1.0</td>
<td>.0417</td>
</tr>
<tr>
<td>1.5</td>
<td>&quot;</td>
</tr>
<tr>
<td>2.0</td>
<td>&quot;</td>
</tr>
<tr>
<td>2.5</td>
<td>&quot;</td>
</tr>
<tr>
<td>3.0</td>
<td>&quot;</td>
</tr>
</tbody>
</table>
Diagram No 3
Variation of Coef. B.

For $\beta = 1.50$

For $\beta = 1.25$

For $\beta = 1.00$

For $\beta = 0.75$

For $\beta = 0.50$

For $\beta = 0.20$

(Case of Fixed Column Ends)

(Case of Fixed Column & Beam Ends)
TABLE III.

Coefficients of Bending Moment for Frames with Fixed and Hinged Ends.

Case 14.

See Fig. 22.

\[ M_c = \text{Bending moment at the fixed end of column} = \frac{1}{4+5\frac{h}{L}} \frac{pL}{6} = \alpha pL^2, \]
\[ M_e = \text{" " " " center of middle span} = \frac{4+9\frac{h}{L}}{4+5\frac{h}{L}} \frac{pL^2}{12} = \beta pL^2, \]
\[ M_c = \text{" " " " end} = \frac{4+9\frac{h}{L}}{4+5\frac{h}{L}} \frac{pL^2}{24} = \delta pL^2. \]

<table>
<thead>
<tr>
<th>( \frac{h}{L} )</th>
<th>0.20</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/24=</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.0417</td>
<td>0.0371</td>
<td>0.0318</td>
<td>0.0284</td>
<td>0.0257</td>
<td>0.0234</td>
</tr>
<tr>
<td>1.0</td>
<td>&quot;</td>
<td>0.0334</td>
<td>0.0257</td>
<td>0.0215</td>
<td>0.0185</td>
<td>0.0163</td>
</tr>
<tr>
<td>1.5</td>
<td>&quot;</td>
<td>0.0303</td>
<td>0.0215</td>
<td>0.0173</td>
<td>0.0145</td>
<td>0.0125</td>
</tr>
<tr>
<td>2.0</td>
<td>&quot;</td>
<td>0.0276</td>
<td>0.0185</td>
<td>0.0145</td>
<td>0.0119</td>
<td>0.0101</td>
</tr>
<tr>
<td>2.5</td>
<td>&quot;</td>
<td>0.0257</td>
<td>0.0163</td>
<td>0.0125</td>
<td>0.0101</td>
<td>0.0085</td>
</tr>
<tr>
<td>4.0</td>
<td>&quot;</td>
<td>0.0238</td>
<td>0.0145</td>
<td>0.0109</td>
<td>0.0088</td>
<td>0.0073</td>
</tr>
<tr>
<td>1/12=</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.0833</td>
<td>0.0796</td>
<td>0.0754</td>
<td>0.0728</td>
<td>0.0706</td>
<td>0.0687</td>
</tr>
<tr>
<td>1.0</td>
<td>&quot;</td>
<td>0.0767</td>
<td>0.0706</td>
<td>0.0672</td>
<td>0.0647</td>
<td>0.0630</td>
</tr>
<tr>
<td>1.5</td>
<td>&quot;</td>
<td>0.0743</td>
<td>0.0672</td>
<td>0.0637</td>
<td>0.0616</td>
<td>0.0600</td>
</tr>
<tr>
<td>2.0</td>
<td>&quot;</td>
<td>0.0723</td>
<td>0.0648</td>
<td>0.0616</td>
<td>0.0595</td>
<td>0.0581</td>
</tr>
<tr>
<td>2.5</td>
<td>&quot;</td>
<td>0.0706</td>
<td>0.0630</td>
<td>0.0600</td>
<td>0.0581</td>
<td>0.0568</td>
</tr>
<tr>
<td>3.0</td>
<td>&quot;</td>
<td>0.0690</td>
<td>0.0616</td>
<td>0.0588</td>
<td>0.0571</td>
<td>0.0559</td>
</tr>
</tbody>
</table>
Diagram No. 4.
Variation of Coefficient \( \gamma \).
<table>
<thead>
<tr>
<th>$l_1$</th>
<th>$l_0$</th>
<th>$h/1$</th>
<th>0</th>
<th>0.20</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1/24=</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.0417</td>
<td>0.0454</td>
<td>0.0498</td>
<td>0.0524</td>
<td>0.0545</td>
<td>0.0563</td>
<td>0.0578</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>&quot;</td>
<td>0.0484</td>
<td>0.0545</td>
<td>0.0578</td>
<td>0.0600</td>
<td>0.0618</td>
<td>0.0634</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>&quot;</td>
<td>0.0508</td>
<td>0.0578</td>
<td>0.0610</td>
<td>0.0634</td>
<td>0.0650</td>
<td>0.0662</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>&quot;</td>
<td>0.0528</td>
<td>0.0600</td>
<td>0.0634</td>
<td>0.0654</td>
<td>0.0668</td>
<td>0.0681</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>&quot;</td>
<td>0.0545</td>
<td>0.0620</td>
<td>0.0650</td>
<td>0.0669</td>
<td>0.0683</td>
<td>0.0692</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>&quot;</td>
<td>0.0560</td>
<td>0.0635</td>
<td>0.0663</td>
<td>0.0680</td>
<td>0.0690</td>
<td>0.0698</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE IV.**

Coefficients of Bending Moment for Frame with Fixed Ends.

Case 16.

See Fig. 24.

$M_c =$ Bending moment at the fixed end of column

\[ M_c = \frac{1}{2 + 3 \frac{h}{l_0}} \frac{p l^2}{24} = \alpha p^2, \]

$M_b =$ at the end of middle span beam

\[ M_b = \frac{1}{2 + 3 \frac{h}{l_0}} \frac{p l^2}{6} = -\beta p^2, \]

$M_c =$ center of middle span beam

\[ M_c = \frac{2 + 5 \frac{h}{l_0}}{2 + 3 \frac{h}{l_0}} \frac{p l^2}{24} = \gamma p^2. \]
Diagram No. 5

Variation of Coef. $\gamma$

Coef. of $\phi$, $\gamma$

Limit for Case of Hinged Ends.
19. Effect of Variation in Moment of Inertia on Bending Moment in Vertical Member.—To see the variation in bending moments in column ends due to the variation of moment of inertia of cross section of columns, several values of $I_{h1}$ and $I_{h2}$ are computed for various values of moment of inertia for the Case 20. In the following computation, the heights and spans of frame are assumed thus:

$$h_1 = h_2, \text{ and } l_1 = l_2 = 2h_1,$$

and consequently $\frac{h_1}{l} = 0.5, \frac{h_1}{l} = 1, \text{ and } \frac{h_2}{l} = 0.5$.

The simplified formulas are given in Table V.
TABLE V.

Coefficient of Bending Moment at Column Ends.

Case 20.

See Fig. 29.

\( M_{h1} = \) bending moment at the top of lower column,

\[
M_{h1} = \frac{1}{2 + 1.5 \frac{I_1}{I_0} + 2 \frac{I_0}{I_3}} \cdot \frac{P^2}{6} = \lambda P^2.
\]

\( M_{h2} = \) bending moment at the foot of upper column,

\[
M_{h2} = \frac{1}{2 + 1.5 \frac{I_1}{I_0} + 2 \frac{I_0}{I_3}} \cdot \frac{P^2}{6} = \beta P^2.
\]

<table>
<thead>
<tr>
<th>( \frac{I_1}{I_0} \frac{I_0}{I_3} )</th>
<th>( \frac{I_3}{I_0} \frac{I_0}{I_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0330</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0287</td>
</tr>
<tr>
<td>2.5</td>
<td>0.0255</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0228</td>
</tr>
</tbody>
</table>

These results are plotted in the diagram

It is easily seen from the diagram that comparatively higher bending stress will exist at the top of lower column than at the foot of upper column in actual structures. It is also seen that the bending moment due to the variation of ratios of moment of inertia varies widely in case of lower column while it is very slight in upper column.

Therefore sufficient attention should be paid on this matter in the case of the design of a structure.
TABLE VI.
Coefficients of Bending Moment for Frame with Hinged Ends.

Case 21.
Special Case b.

(See Fig. 32)

\[ M_b = \text{Bending moment at the end of middle span beam} = \frac{9 + 24\left(\frac{h}{L}\right) + 12\left(\frac{h}{L}\right)^2}{9 + 30\left(\frac{h}{L}\right) + 20\left(\frac{h}{L}\right)^2} \beta L^2 \]

\[ M_c = \text{Bending moment at the center of middle span beam} = \frac{9 + 42\left(\frac{h}{L}\right) + 36\left(\frac{h}{L}\right)^2}{9 + 30\left(\frac{h}{L}\right) + 20\left(\frac{h}{L}\right)^2} \beta L^2 \]

<table>
<thead>
<tr>
<th>( \frac{L_1}{L_0} )</th>
<th>( \frac{h}{L} )</th>
<th>0</th>
<th>0.20</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/12 = 0.5</td>
<td>0.0833</td>
<td>.0739</td>
<td>.0711</td>
<td>.0689</td>
<td>.0672</td>
<td>.0658</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td>.0689</td>
<td>.0658</td>
<td>.0634</td>
<td>.0620</td>
<td>.0605</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
<td>.0656</td>
<td>.0625</td>
<td>.0605</td>
<td>.0591</td>
<td>.0580</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td></td>
<td>.0634</td>
<td>.0605</td>
<td>.0587</td>
<td>.0573</td>
<td>.0565</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td></td>
<td>.0607</td>
<td>.0579</td>
<td>.0565</td>
<td>.0550</td>
<td>.0547</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| 1/24 = 0.5          | 0.0417              | .0511 | .0539 | .0561 | .0578 | .0592 |
| 1.0                 |                      | .0560 | .0592 | .0616 | .0630 | .0645 |
| 1.5                 |                      | .0594 | .0625 | .0645 | .0659 | .0670 |
| 2.0                 |                      | .0616 | .0645 | .0663 | .0677 | .0685 |
| 3.0                 |                      | .0643 | .0671 | .0685 | .0700 | .0703 |
Diagram No. 7.
Variation of Coef. \( \beta \).
### Table VII.

**Coefficients of Bending Moment for Frame with Fixed Ends.**

**Case 22.**

**Special Case b.**

*(See Fig. 35)*

\[ M_b = \text{Bending moment at the end of middle span beam} = \frac{4+8\left(\frac{h}{h_0}\right)+3\left(\frac{h}{h_0}\right)^2}{4+10\left(\frac{h}{h_0}\right)+5\left(\frac{h}{h_0}\right)^2} \cdot \beta^2,\]

\[ M_c = \text{Bending moment at the center of middle span beam} = \frac{4+14\left(\frac{h}{h_0}\right)+9\left(\frac{h}{h_0}\right)^2}{4+10\left(\frac{h}{h_0}\right)+5\left(\frac{h}{h_0}\right)^2} \cdot \beta^2,\]

\[ M_l = \text{Bending moment at the fixed end of column} = \frac{1+\frac{h}{h_0}}{4+10\left(\frac{h}{h_0}\right)+5\left(\frac{h}{h_0}\right)^2} \cdot \beta^2.\]

<table>
<thead>
<tr>
<th>( l_1 \frac{z_{nc}}{l_0} )</th>
<th>( \frac{h}{l} )</th>
<th>0</th>
<th>0.20</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 1/12=</td>
<td></td>
<td></td>
<td>.0758</td>
<td>.0732</td>
<td>.0711</td>
<td>.0694</td>
<td>.0680</td>
<td></td>
</tr>
<tr>
<td>1.0  &quot;</td>
<td></td>
<td></td>
<td>.0711</td>
<td>.0680</td>
<td>.0658</td>
<td>.0640</td>
<td>.0626</td>
<td></td>
</tr>
<tr>
<td>1.5  &quot;</td>
<td></td>
<td></td>
<td>.0680</td>
<td>.0648</td>
<td>.0626</td>
<td>.0610</td>
<td>.0597</td>
<td></td>
</tr>
<tr>
<td>2.0  &quot;</td>
<td></td>
<td></td>
<td>.0658</td>
<td>.0626</td>
<td>.0605</td>
<td>.0592</td>
<td>.0580</td>
<td></td>
</tr>
<tr>
<td>3.0  &quot;</td>
<td></td>
<td></td>
<td>.0626</td>
<td>.0598</td>
<td>.0580</td>
<td>.0567</td>
<td>.0559</td>
<td></td>
</tr>
</tbody>
</table>

<p>| 0.5 1/24= |        |    | .0492 | .0518 | .0539 | .0556 | .0570 |
| 1.0  &quot;   |        |    | .0539 | .0570 | .0592 | .0610 | .0624 |
| 1.5  &quot;   |        |    | .0570 | .0602 | .0624 | .0640 | .0653 |
| 2.0  &quot;   |        |    | .0592 | .0624 | .0645 | .0658 | .0670 |
| 3.0  &quot;   |        |    | .0624 | .0652 | .0670 | .0683 | .0691 |</p>
<table>
<thead>
<tr>
<th>$\frac{I_1}{I_0}$</th>
<th>$\frac{J_{ec}'}{J_{ec}}$</th>
<th>$\frac{h}{l}$</th>
<th>0</th>
<th>0.20</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.0417</td>
<td></td>
<td>0.0305</td>
<td>0.0271</td>
<td>0.0244</td>
<td>0.0222</td>
<td>0.0204</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>&quot;</td>
<td></td>
<td>0.0244</td>
<td>0.0204</td>
<td>0.0176</td>
<td>0.0154</td>
<td>0.0138</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>&quot;</td>
<td></td>
<td>0.0204</td>
<td>0.0165</td>
<td>0.0138</td>
<td>0.0119</td>
<td>0.0105</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>&quot;</td>
<td></td>
<td>0.0176</td>
<td>0.0138</td>
<td>0.0114</td>
<td>0.0097</td>
<td>0.0084</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>&quot;</td>
<td></td>
<td>0.0138</td>
<td>0.0105</td>
<td>0.0084</td>
<td>0.0071</td>
<td>0.0061</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Diagram No. 8.
Variation of Coef. B.
EXPERIMENTAL INVESTIGATION
PART II.

V. MATERIALS, TEST FRAMES, AND METHOD OF TESTING.

20. Scope of Tests.—The investigation described in this part of the thesis was taken up with a view of determining experimentally:

(1) The elastic action of the reinforced concrete frames under external loads;

(2) The amount and the distribution of stresses in the steel and concrete and over the cross section of members;

(3) The continuity of the composing members of a frame;

(4) The critical point of failure;

(5) The reliability of a reinforced concrete frame;

(6) The applicability of the theoretical formulas in the design of frames;

(7) Something of the information relating to the design of reinforced concrete frames.

To do this the eight specimens of five different types of the reinforced concrete frames were designed according to the theoretical analysis and were made in the latter part of the year 1913 in the laboratory of the University of Illinois. The tests were carried out in January, February, and March 1914. The experimental results were then worked out. In planning the tests the relation of the various phenomena was kept in mind, and the results are discussed and compared with the theoretical values.


Materials.—The materials used in making the test frames were
similar to those used in the ordinary reinforced concrete structures in this country. The sand, stone and cement were obtained in the open market.

**Stone.**—The stone was a good quality of crushed limestone ordered to pass over a 1/2-inch screen and through a 1-inch screen. It contained about 50 % of voids.

**Sand.**—The sand was of good quality, hard, sharp, well graded, and generally clean.

**Cement.**—Universal portland cement was used for all specimens. It was good in quality and well calcined. Table VIII gives the strength of standard briquettes of neat cement and of 1-3 mortar for ages of 7 and 28 days. The results of tests with Vicat needle showed the following time of setting:

Initial set — 3 hours and 15 minutes.
Final set — 6 hours and 0 minutes.
### TABLE VIII.

**TENSILE STRENGTH OF CEMENT.**

<table>
<thead>
<tr>
<th>Ref. No.</th>
<th>Heat Cement</th>
<th>1:3 Mortar with Ottawa Sand</th>
<th>1:3 Mortar with Building Sand</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Age 7 days</td>
<td>Age 28 days</td>
<td>Age 7 days</td>
</tr>
<tr>
<td>Percent</td>
<td>24.5</td>
<td>25.3</td>
<td>25.3</td>
</tr>
<tr>
<td>of Water</td>
<td>5</td>
<td>9.6</td>
<td>9.6</td>
</tr>
<tr>
<td>1</td>
<td>600</td>
<td>230</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>570</td>
<td>200</td>
<td>200</td>
</tr>
<tr>
<td>3</td>
<td>600</td>
<td>190</td>
<td>210</td>
</tr>
<tr>
<td>4</td>
<td>590</td>
<td>210</td>
<td>200</td>
</tr>
<tr>
<td>5</td>
<td>—</td>
<td>230</td>
<td>—</td>
</tr>
</tbody>
</table>

**Average:** 590 661 660 212 245 279 353
### TABLE X.

**DATA OF FRAMES.**

<table>
<thead>
<tr>
<th>Frame No.</th>
<th>Span Length a. to a.</th>
<th>Cross Section</th>
<th>Longitudinal Reinforcement</th>
<th>Stirrups</th>
<th>Nominal Height</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>At Top At Bottom</td>
<td>Column in. x in.</td>
<td>Beam in. x in.</td>
<td>Column (Top)</td>
<td>Beam (Center)</td>
</tr>
<tr>
<td>1</td>
<td>6 - 0 6 - 0</td>
<td>8 x 10</td>
<td>8 x 11 1/4</td>
<td>3 - 1/2 in.</td>
<td>0.82</td>
</tr>
<tr>
<td>2</td>
<td>6 - 0 6 - 0</td>
<td>8 x 10</td>
<td>8 x 9 1/4</td>
<td>3 - 1-in.</td>
<td>0.82</td>
</tr>
<tr>
<td>3</td>
<td>6 - 0 6 - 0</td>
<td>8 x 10</td>
<td>8 x 11 1/4</td>
<td>4 - 1/2-in.</td>
<td>1.09</td>
</tr>
<tr>
<td>4</td>
<td>6 - 0 6 - 0</td>
<td>8 x 10</td>
<td>8 x 9 1/4</td>
<td>4 - 1/2-in.</td>
<td>0.82</td>
</tr>
<tr>
<td>5</td>
<td>3 - 6 6 - 0</td>
<td>8 1/2 x 8 1/2</td>
<td>8 1/2 x 17 3/8</td>
<td>4 - 1/2-in.</td>
<td>1.28</td>
</tr>
<tr>
<td>6</td>
<td>6 - 0 6 - 0</td>
<td>8 x 10</td>
<td>8 x 11 1/4</td>
<td>3 - 1/2-in.</td>
<td>0.82</td>
</tr>
<tr>
<td>7</td>
<td>6 - 0 6 - 0</td>
<td>8 x 10</td>
<td>8 x 11 1/4</td>
<td>4 - 1/2-in.</td>
<td>1.09</td>
</tr>
<tr>
<td>8</td>
<td>4 - 8 8 - 8</td>
<td>8 x 8</td>
<td>8 x 11 1/4</td>
<td>4 - 1/2-in.</td>
<td>1.40</td>
</tr>
</tbody>
</table>

* Double loop.
* Three equal spans.
* Only used in the middle span.

All reinforcement of plain round bars.
All concrete of 1-2-4 mix.
Concrete.—Men skilled in making concrete were employed in the work. Care was taken in measuring, mixing and tamping to secure as uniform a concrete as possible. All of the concrete was made of the proportions 1 of cement, 2 of sand, and 4 of stone by volume. The mixing was done with a concrete mixing machine.

Steel.—The reinforcing bars were plain round rods of open hearth mild steel. Test pieces were taken from the test frames after the test. Table 9 gives the results of the tests.

Test Frames.—Five different types of frame were selected for the tests. The cross section of various members varied from 8 x 8 in. to 8 1/2 x 17 3/8 in. The length of span of the frames was 6 ft. on centers except Frame No. 8, which had three spans of 4-ft., and 8-in. The height of the frames varied from about 5 ft. to about 10 ft. The general arrangement of the bars and details are shown in Fig. 43.

<table>
<thead>
<tr>
<th>Nominal Size inches</th>
<th>Yield Point lb. per sq.in.</th>
<th>Ultimate Strength lb. per sq.in.</th>
<th>Percent Elongation in 8 in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>36 200</td>
<td>55 100</td>
<td>26.3</td>
</tr>
<tr>
<td>&quot;</td>
<td>36 200</td>
<td>54 200</td>
<td>26.9</td>
</tr>
<tr>
<td>&quot;</td>
<td>36 900</td>
<td>54 700</td>
<td>26.7</td>
</tr>
<tr>
<td>&quot;</td>
<td>36 700</td>
<td>54 600</td>
<td>26.9</td>
</tr>
<tr>
<td>&quot;</td>
<td>37 700</td>
<td>54 200</td>
<td>25.0</td>
</tr>
<tr>
<td>&quot;</td>
<td>37 400</td>
<td>55 900</td>
<td>30.0</td>
</tr>
</tbody>
</table>

Average
Care was taken in designing the test specimens to secure the continuity of connected members and to develop the high bending stresses in the columns and beams at nearly the same time. The ends of the steel reinforcing bars were bent in hooks in the case of the fixed columns at the ends. Continuous bars from one end to another were used for all frames. Several bars were welded and these welds were located at the point where the bending moment was very small.

In the frames with stirrups, U-shaped or double U-shaped stirrups were used. They passed under the longitudinal bars and extended to the top of the beam. The size and spacing of the stirrups are given in Table X.

The radius of the bends of the main rods was about 5 in.

Making of the Frames.—It was hoped to make the frames in a vertical position, similar to that in practice. There was a little difficulty in doing this and more expense was needed to build the forms. Instead, all the frames were built directly on the concrete floor of the laboratory in a horizontal position with a strip of building paper beneath the forms. The forms were generally removed after seven days, and the frames were lifted from the horizontal position after thirty days and were kept in a vertical position in the laboratory until the date of test.

Storage.—The frames were left in the concrete mixing room until the date of test. They were dampened every morning for two weeks after making to prevent too rapid drying, and were dampened occasionally after that time. The temperature ranged from 55°F to 70°F.
Minor Test Pieces.—Tests were made on 6-in. cubes from the concrete used in the frames. The results are given in Table XI. In addition to this, tests were made on concrete cylinders 6 in. in diameter and 16 in. long, to give a means of judging of the modulus of elasticity of the concrete used in the frames. The results of cylinder tests are shown in Diagrams from 10 to 13.

Method of Testing.—The methods of loading and the bedding of the plates for the different frames are shown in Fig. 45 to 49. The specimens were tested in the 600,000-lb. Riehle testing machine in the Laboratory of Applied Mechanics of the University of Illinois. Deflections were read on some of the frames. The deformation of the steel and concrete were measured at the various parts of the frames.

Extensometers and Method of Measuring Deformation and Deflections.—Extensometers of the Berry type, modified at the University of Illinois, were used in measuring the deformations. With these instruments, the deformations as small as from .0000167 in. to .00005 in. can be measured directly with considerable accuracy. The method of using these instruments is described in Bulletin No. 64 of Engineering Experiment Station of University of Illinois, and in a paper, "The Use of the Strain Gage in the Testing of Materials," Proceedings of American Society for Testing Materials for 1913. Variation in temperature is sufficient to cause a change in the length of the instrument. Hence observations on an unstressed standard bar of invar steel were taken for the purpose of making temperature corrections. Small steel plugs, about 1 inch long, were set in plaster of paris in the concrete,
where the concrete deformations were to be measured. Small gage holes, .055 in. in diameter, were drilled in the reinforcing bars and in the steel plugs. Two sets of initial readings were taken before the application of load. A complete set of the observations of the deformations was taken at each increment of load. Temperature corrections based on the linear variation of the time were made to get the final results, and the stresses due to the load were deduced from them as described in the papers above referred to.

**Method of Loading.**—To develop high stresses in the beam and in the columns nearly at the same time one-third point loadings were used for many of the frames. In frame No. 5 the centrally concentrated load was used to develop as high a flexural stress in the columns as possible.

In frame No. 8 in order to see the effect of the eccentric load on the adjacent spans and at the same time to produce high bending stresses in the middle beam and in the column, a uniform load on the middle span was selected.
<table>
<thead>
<tr>
<th>Weeks</th>
<th>Days</th>
<th>3 in. Cube</th>
<th>8 x 16 in. Cylinder</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>64</td>
<td>1780</td>
<td>1150</td>
</tr>
<tr>
<td>1</td>
<td>64</td>
<td>1750</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>64</td>
<td>1680</td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>1740</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>1850</td>
</tr>
</tbody>
</table>
### TABLE XI.

**COMPRESSION TESTS OF 6-IN. CUBES.**

**1:2:4 Concrete.**

<table>
<thead>
<tr>
<th>Frame No.</th>
<th>Age at Test days</th>
<th>Maximum Load lb. per sq.in.</th>
<th>Frame No.</th>
<th>Age at Test days</th>
<th>Maximum Load lb. per sq.in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>64</td>
<td>1780</td>
<td>5</td>
<td>61</td>
<td>3070</td>
</tr>
<tr>
<td>1</td>
<td>64</td>
<td>1750</td>
<td>5</td>
<td>61</td>
<td>3100</td>
</tr>
<tr>
<td>1</td>
<td>64</td>
<td>1680</td>
<td>5</td>
<td>61</td>
<td>2580</td>
</tr>
<tr>
<td>Av.</td>
<td>64</td>
<td>1740</td>
<td>Av.</td>
<td>61</td>
<td>2920</td>
</tr>
<tr>
<td>2</td>
<td>62</td>
<td>2210</td>
<td>6</td>
<td>62</td>
<td>2605</td>
</tr>
<tr>
<td>2</td>
<td>62</td>
<td>2250</td>
<td>6</td>
<td>62</td>
<td>2445</td>
</tr>
<tr>
<td>2</td>
<td>62</td>
<td>2540</td>
<td>6</td>
<td>62</td>
<td>2510</td>
</tr>
<tr>
<td>Av.</td>
<td>62</td>
<td>2330</td>
<td>Av.</td>
<td>62</td>
<td>2520</td>
</tr>
<tr>
<td>3</td>
<td>73</td>
<td>2860</td>
<td>7</td>
<td>60</td>
<td>2140</td>
</tr>
<tr>
<td>3</td>
<td>73</td>
<td>2820</td>
<td>7</td>
<td>60</td>
<td>2390</td>
</tr>
<tr>
<td>3</td>
<td>73</td>
<td>2840</td>
<td>7</td>
<td>60</td>
<td>2220</td>
</tr>
<tr>
<td>Av.</td>
<td>73</td>
<td>2840</td>
<td>Av.</td>
<td>60</td>
<td>2250</td>
</tr>
<tr>
<td>4</td>
<td>66</td>
<td>2600</td>
<td>8</td>
<td>63</td>
<td>3268</td>
</tr>
<tr>
<td>4</td>
<td>66</td>
<td>2580</td>
<td>8</td>
<td>63</td>
<td>3900</td>
</tr>
<tr>
<td>4</td>
<td>66</td>
<td>2570</td>
<td>8</td>
<td>63</td>
<td>3653</td>
</tr>
<tr>
<td>Av.</td>
<td>66</td>
<td>2580</td>
<td>Av.</td>
<td>63</td>
<td>3614</td>
</tr>
</tbody>
</table>
Concrete Cylinder Test

Frame No. 1.
Age 64 days

Max. Unit Load = 1150 lb.

Frame No. 2.
Age 62 days

Max. Load = 1850 lb.
Diagram No. 11

Unit Load in lb. per sq.in.

Frame No. 4
Max Load = 1910 lb.
Age 66 days

Frame No. 3
Max Load = 2050 lb.
Age 73 days

Unit Deformation, in. per in.
Diagram No. 12.

Frame No. 5.
Age 61 days.
Unit Max. Load = 2670 lb.

Frame No. 6.
Age 62 days.
Unit Max. Load = 2310 lb.
Frame No. 7.
Age 60 days.
Max. Load = 1870 lb.

Frame No. 8.
Age 63 days.
Max. Load = 3060 lb.

Unit Load, lbs per sq. in.

Unit Deformation, in. per in.
22. Explanation of Tables, Diagrams, Drawings, and Photographs.—Table XII contains data of the tests of the test frames. All other tables are self-explanatory. The loads given in the tables are the loads applied by the testing machine, and do not include the weight of the frame itself. The load at first crack is the load noted when the first fine crack was observed during the test. The ultimate load is the highest load applied to the specimen just before the load carried began to decrease slowly. The maximum tensile and compressive stresses in Table 12 are the highest stresses observed at the points specified. The vertical shearing stress was calculated with the ordinary formula \( v = \frac{V}{bd'} \), where \( v \) represents the vertical shearing stress in the concrete, \( V \) represents the total vertical shear at the end of the beam, \( b \) is the breadth of the beam, and \( d' \) is the distance from the center of the steel to the center of compression in the concrete. The bond stress in the beam was computed with the formula \( u = \frac{V}{\text{mod} \cdot} \), where \( u \) is the bond stress per unit of area on the surface of the reinforcing steel, \( m \) is the number of reinforcing bars, and \( \phi \) is the circumference or periphery of one reinforcing bar. The values of \( d' \) were selected with reference to the amount of reinforcing steel and the modulus of elasticity of the concrete.

The diagrams, drawings, and photographs are self-explanatory.

25. Phenomena of Frame Tests.—As may be expected, in reinforced concrete members, the tensile stresses in the steel were
<table>
<thead>
<tr>
<th>Frame No.</th>
<th>Age (days)</th>
<th>Method of Loading</th>
<th>First Crack</th>
<th>Maximum Load</th>
<th>Ultimate Deformation</th>
<th>Maximum Load at Failure</th>
<th>Ultimate</th>
<th>Maximum Stress, lb. per sq.in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>63</td>
<td>Concentrated</td>
<td>12 000</td>
<td>36 000</td>
<td>40 500</td>
<td>36 000</td>
<td>21 700</td>
<td>35 200 2320* 3150*</td>
</tr>
<tr>
<td>2</td>
<td>62</td>
<td>Load at Center</td>
<td>8 000</td>
<td>14 000</td>
<td>32 900</td>
<td>28 900</td>
<td>12 300</td>
<td>3340* 1570</td>
</tr>
<tr>
<td>3</td>
<td>63</td>
<td>Concentrated</td>
<td>21 000</td>
<td>46 000</td>
<td>61 000</td>
<td>39 500</td>
<td>20 300</td>
<td>15 700 23 800 1500 3050</td>
</tr>
<tr>
<td>4</td>
<td>63</td>
<td>do.</td>
<td>10 000</td>
<td>30 000</td>
<td>50 000</td>
<td>25 300</td>
<td>21 800</td>
<td>27 600 2590 2920</td>
</tr>
<tr>
<td>5</td>
<td>61</td>
<td>Load at Center</td>
<td>40 000</td>
<td>100 000</td>
<td>146 000</td>
<td>36 400</td>
<td>5 700</td>
<td>11 800 12200*</td>
</tr>
<tr>
<td>6</td>
<td>58</td>
<td>Concentrated</td>
<td>18 000</td>
<td>36 000</td>
<td>46 000</td>
<td>29 800</td>
<td>7 400</td>
<td>10 300 27 000 2750 3340*</td>
</tr>
<tr>
<td>7</td>
<td>62</td>
<td>do.</td>
<td>21 000</td>
<td>46 000</td>
<td>61 000</td>
<td>44 400</td>
<td>25 500</td>
<td>18 100 30 200 3700* 4130*</td>
</tr>
<tr>
<td>8</td>
<td>61</td>
<td>Uniform Load over</td>
<td>45 000</td>
<td>60 000</td>
<td>124 000</td>
<td>30 900</td>
<td>13 200</td>
<td>14 700 2540* 14700*</td>
</tr>
</tbody>
</table>

* Not reliable, elastic limit exceeded.
+ Not maximum load.
$ Steel stress in compression side of column.
# Stress at the part one inch distant from extreme fiber.
very small at low loads. Undoubtedly this effect was largely due to the ability of the concrete to carry tensile stress. As soon as the concrete on the tension side of the member was sufficiently stretched, a vertical tension crack formed on the part underneath the load and then a crack formed at the side near the juncture of the column and the beam, in most cases. After the formation of these cracks, the tension in these parts is taken mainly by the reinforcing bars. As the loads were increased the cracks developed and new cracks appeared on the tension side between the points of application of the load on the beam, and horizontal cracks formed at regular intervals in the columns.

The tensile stress due to the negative bending moment at the juncture of the beam and the column was small, and tension cracks did not form in many frames until the high loads were applied. The bent up bars in the beam came into action as soon as tension cracks formed at the vicinity. Tensile stress, as high as 22,000 pounds per square inch, was developed. The tensile stresses in the steel at the fixed ends of the columns were rather low. The tensile strength of the concrete in this part apparently reduced the tensile stress in the steel.

High compressive stresses were developed in the concrete at the top of the columns, and the maximum compression was observed along the sharp corner at the juncture of the beam and the columns, as might be expected. This is due to the curved beam action at that rigid joint. In all frames the maximum load was higher than the load expected.

The location of the cracks is shown in Photographs No. 2, 3, 4, 5, 7, 8 and 9.
Photograph No. 1.

View of Frame No. 1 under Testing Machine.
Photograph No. 2.

View of Frame No. 1.
Photograph No. 3.

View of Frame No. 2.
Photograph No. 4.

View of Frame No. 3.
Photograph No. 6.

View of Frame No. 5 under Testing Machine.
Photograph No. 7.

View of Frame No. 5.

Max Load 146000 lb.

No5.
Photograph No. 8.

View of Frame No. 6.
Photograph No. 9.

View of Frame No. 7.
Photograph No. 10.

View of Frame No. 8 under Testing Machine.
Photograph No. 11.

View of Frame No. 8 under Testing Machine.
Detail of Frame No. 1 and 6.

Location of Gauge-Lines.

Scale: 3" to 32"
Detail of Frame No. 2 and 4.

Scale, 3" to 32".

Location of Gauge Lines in Frame No. 2.
- Concrete Readings
- Steel Readings
  1) Gauge Lines on Back Side.

Location of Gauge Lines in Frame No. 4.
- Concrete Readings
- Steel Readings
  1) Gauge Lines on Back Side.

Location of Gauge Lines.
Scale. 3'10" 32"
Detail of Frame No. 8 and Location of Gauge Lines.

Scale 1" = 16'

Remarks:
Number with [ ] indicates Gauge Lines for Concrete Deformation Readings.
Number with ( ) indicates Gauge Lines for Strut-Deformation Readings.
Number without Parenthesis indicates Gauge Lines for Strut-Deformation Readings on Back Side of Frame.

Reinforcement in Beam.
The general phenomena of the test of the individual frame are given in the following brief notes.

Frame No. 1. (Fig. 45).—Square frame with supported column ends. Nominal span length 6 ft. Total height 5 ft. 2 in. Loaded at the one-third points. At 12,000 lb. the first fine crack appeared at the part directly under the load and extended from the bottom vertically 2 in. to the level of the reinforcement. At the same load the first noticeable cracks appeared, one in the outside edge of the column on a level with the bottom surface of the beam and one at 2 ft. 5 in. from the bottom of each column end. These cracks extended into the steel. At 18,000 lb. the old cracks had extended to a point 5 in. from the edges, and three new cracks appeared in the beam directly under the left side load point. At the same load new cracks in the column were formed at the place where the vertical steel was bent diagonally toward the inside of the columns and extended to the steel. At 24,000 lb. the cracks in the beam extended vertically up to the center of the beam depth and new cracks appeared at the place where the longitudinal steel was bent up. At the same time the cracks in the column extended inward and new cracks were formed in the left hand column. These cracks extended inward as the load increased. No crack appeared in the top side of the beam until the load was increased to 36,000 lb. At that load cracks appeared 8 in. from the top corner of the frame and extended vertically. The frame carried 40,500 lb. and the load was held for a few minutes and then dropped very slowly. The cracks were well distributed in the tension zone of the frame and no crack due to diagonal tension was formed. The frame failed by tension in the steel of the top beam.
Frame No. 2, (Fig. 46).—Special frame with supported column ends. Nominal span length 6 ft. Total height 6 ft. Loaded at the center of the span. At 8,000 lb. two cracks appeared 2 in. on each side of the center of the top beam and extended upward 2 in. and 3 in. respectively. At 12,000 lb. these cracks had extended vertically 6 in. from the bottom surface of the beam. A new crack appeared at the right hand inside the first corner 10 1/2 in. from the center of the beam and extended diagonally toward the load point. At the same load four new cracks appeared at both shoulders 6 1/2 in. inward from the top corners of the columns and extended toward the centers of the rigid joints. At 14,000 lb. the cracks had extended. Unfortunately the column on one side slipped about 1/4 in. outward due to the lack of enough friction to resist the horizontal thrust at the support. However, satisfactory information was obtained because very high tensile stress (32,900 lb. per sq.in.) had been developed at the center of the beam before the slipping occurred.

Frame No. 3, (Fig. 47).—Square frame with fixed column ends. Nominal span length 6 ft. Total height of the frame from the fixed column ends 4 ft. 11 in. Loaded at the one-third point. No noticeable cracks appeared until 21,000 lb. were applied. At this load three cracks formed in the top beam and extended about 4 in. upward from the bottom surface of the beam. Several cracks appeared in both columns at this load. At 30,000 lb. the cracks extended further and new cracks formed in the beam and columns, and one crack appeared at the end of the beam near the extension of the inner face of the column. The cracks in the
beam were located between the points of loading and no cracks were seen outside of these points. Two cracks due to the negative bending moment at the end of the beam near the extension of the inner face of the column were developed. The cracks in the upper part of the columns were located within 14 in. downward from the extended line of the bottom face of the beam. No crack was observed at the fixed ends of the columns. The frame carried 60,000 lb. and the load was held for a few minutes, then dropped very slowly, and there appeared to be no danger of sudden failure. No diagonal tension crack appeared in the beam, and the frame failed by the tension in the longitudinal steel of the beam.

Frame No. 4, (Fig. 46).—Special frame supported at the column ends. Nominal span length 6 ft. Total height of the frame 6 ft. 3 in. from the supported end. Loaded at the one-third points. At 10,000 lb. the first noticeable crack appeared at the left hand inside top corner, and extended 2 1/2 in. upward. At 15,000 lb. two more cracks appeared in the horizontal beam between the loaded points. At the same load two cracks formed around the rigid joint between the right hand column and the beam. The cracks extended further as the load increased, and several new cracks appeared in the beam and columns. Accidentally the frame was built slightly out of form, as shown in the accompanying sketch, and more stress was thrown to the right hand column than to the other. The distribution of the cracks shows this clearly. The frame, however, carried comparatively high load (50,000 lb.). The frame failed by ten-
sion in the steel in the horizontal beam and at the rigid joint between the columns and the sloped beam.

Frame No. 5, (Fig. 48).—Trestle bent with a tie. Span length center to center at the supported column ends 6 ft. Total height from the base to the top of the frame 10 ft. 1 1/2 in. Loaded at the center of the top beam. The cross section of top beam 8 1/2 x 16 in. and column section 8 1/2 x 8 1/2 in. At 40,000 lb. the first two noticeable cracks appeared under the load point of the beam. One extended 5 in. and the other 3 in. from the bottom face of the beam. At 60,000 lb. three additional cracks appeared in the beam and the middle one extended as deep as 2 in. from the bottom face of the beam. These cracks had extended diagonally almost to the top of the beam. At 100,000 lb. the first cracks in the column appeared in the right hand column at its connection with the beam. At 140,000 lb. other cracks appeared in the same part of both columns. At this load a crack suddenly occurred at the right hand rigid joint between the tie and the column with a breaking sound. The frame carried 146,000 lb. and the load gradually dropped. The maximum load was controlled by the failure of the top beam which failed by tension in the steel reinforcement.

Frame No. 6, (Fig. 45).—Same as No. 1. Third-point loading. At 18000 lb. four noticeable cracks appeared, two of them under the load points, one near the center of the beam, and one at the upper part of the right hand column. At 24,000 lb. the cracks extended further and the additional cracks appeared in the beam and columns at regular intervals. At 30,000 and 36,000 lb. new
cracks appeared in the beam where the longitudinal bars were bent up and these cracks ran diagonally almost to the load points. No crack appeared on the top side of the beam ends. The frame carried 46,000 lb. and after a few seconds the load dropped slowly. The frame failed by tension in the longitudinal steel in the beam.

**Frame No. 7, (Fig. 47).**—Same as No. 3. Loaded at one-third points. The first noticeable cracks appeared at 21,000 lb., three in the beam and three in the columns. At 30,000 lb. the cracks had extended further and two cracks due to the negative bending moment appeared at the ends 8 in. from the outside face of the columns. At the same load three cracks formed in the bottom half of the beam. As the load increased the crack, located on the outside of the left hand load point, extended diagonally almost to the load point, and the cracks at both ends of the beam extended vertically downward nearly to the bottom side of the beam. The ultimate load carried by the frame was 61,000 lb. At this load sudden failure took place at both inside corners of the lower ends of the columns and the cracks extended horizontally and vertically almost through the concrete base and almost through the columns. This fact shows that the considerable positive bending was developed there. The frame failed by tension in the longitudinal steel of the top beam and also at the concrete base.

**Frame No. 8, (Fig. 49).**—Frame with three spans. Span lengths 4 ft. 8 in. center to center. Total height of the frame 6 ft. 7 1/4 in. Uniform load on the middle span. No cracks were seen until the load had reached 45,000 lb. At this load three cracks
appeared in the middle span and one on the top side just outside of each intermediate column. The former is due to the positive bending moment and the latter is due to the negative moment as might be expected. These cracks were located symmetrically and they extended vertically about 6 in. At 60,000 lb. they extended deeper. The frame was subjected to this load over 20 hours, but the fall in the applied load was only 300 lb. This shows that the frame was still strong enough to resist the external load. At 75,000 lb. several new cracks appeared at both ends of the top beam and also in the upper part of the intermediate columns. At this load the steel reinforcement at the bottom of the middle beam was stressed in tension beyond the elastic limit of the steel. The frame, however, carried the load in good condition and the highest load was 134,000 lb. At this load the crack at the center of the middle span was opened considerably and the steel at this part was scaled, and showed the failure by tension in the steel. At the same time the concrete at the top of the intermediate columns was crushed. Also the concrete base was cracked at the bottom end of the right hand intermediate column. It is noted that the stresses in the outside columns were very low, even at the maximum load.

Note.—The base of the frame was 15 ft. 4 in. in length, while the length of the base of the testing machine is 10 ft. 6 in. Consequently the ends of the frame projected beyond the base of the testing machine. To observe the end condition of the frame under test an Ames dial was attached at each end of the frame as indicated in Fig. 50. The movements of both ends were observed as
the load increased. The maximum movements (upward) were observed at 60,000 lb. and the amounts were as follows:

\[
\frac{1}{263} \text{ in. at E-end,} \quad \frac{1}{300} \text{ in. at W-end.}
\]

therefore we can judge that steel stress in the beam of the side span may be slightly modified by the movement, but the structure as a whole is not affected.
VII. COMPARISON OF THEORY WITH EXPERIMENT.

24. General Statement.—For the purpose of comparison the theoretical stresses at several points in the frame were computed by the ordinary formulas for resisting moment (straight-line relation) and summarized in Table XIII. There are several points to be considered before starting the calculation.

(a) It must be noted that the quality of the concrete used in making the frames was not uniform over the cross section of the member owing to the fact that the frames were made in a flat position on the floor of the laboratory, instead of a vertical position, as is usual in practice. The concrete on the back side of the frame (or the bottom side when the frame was made) was richer, while that on the front side (or the top side when the frame was made) was poorer, as already stated, therefore the back side was stronger than the front side and there was a distinct tendency to throw more stress in the steel of the back side than in that of the front side. The steel stresses were modified by this fact. Accordingly, it will be more reasonable to take the average value of the observed stresses in the part in question for the purpose of comparison.

(b) The steel stresses are greatly modified by the presence of tension in the concrete for the low loads. Therefore we must consider the two cases in comparing the experimental results with theory,—Case I, in which the concrete is allowed to take the tension; and Case II, in which the concrete is considered to be broken in tension. These two cases are used in the following comparison.
(c) To develop high stresses in the members the cross sections of the test frames were larger in proportion to the span than would commonly be used in practice. In most of the test pieces, the column width occupied nearly one-seventh \((1/7)\) of the nominal span \((c.\ to\ c.\ distance\ of\ the\ columns)\). In addition to this the corner at the juncture of the beam and the column was provided with a fillet. The bending moment at the center of the beam was lessened by these facts. The nominal span and height of the frame \((c.\ to\ c.\ distances)\) were used in computing the horizontal reactions, and then the bending moment at any desired point was computed in terms of the nominal span length \(l\). In finding the numerical values of the bending moment in the beam and also of those in the columns having fixed ends, the nominal span length \(l\) was replaced by the clear length of the span.

(d) The frame No. 8 consisted of six members, and the reactions and the bending moments are obviously modified by the relative stiffness of the composing members (the concrete base is assumed to be so rigid that the column ends are kept in the fixed condition. This assumption was practically true in the present tests). When the concrete at any point is broken in tension, say in the column, the moment of inertia of the cross section is never the same as before, and therefore the entire condition of flexure will be modified. This must be true for all cases, but the other frames except No. 8 were so designed as to develop tension cracks in the beam and also in the columns nearly at the same time. Frame No. 8, however, has different conditions from the others, and the bending moments in the columns were
small, even in the intermediate columns, while the bending moment in the middle beam was so large that tension cracks developed at a low load. Consequently there are two cases to be considered in comparing experimental results with theory,—Case A, in which tension in the concrete holds in all members; and Case B, in which the beams and the intermediate columns are cracked on the tension side. The bending moment in the outside column is very small, and there is no chance for tension cracks. The moments in the intermediate columns and in the beams of the side spans are also small except at the extreme end, and only one crack appeared in this member. Therefore it is not the correct assumption to neglect tension in the entire cross section on the outside of the tension rods in these two members in the calculation of the moment of inertia of the cross section when Case B has been reached. The most probable value for the moment of inertia of these members will be an average between using the full cross section and a section which neglects the part outside of the tension rods. This assumption was made in the numerical computation of the theoretical values of the frame No. 8.

All these considerations were made in the calculation of the theoretical values.

25. Values of Moduli of Elasticity $E_c$ of Concrete.—The values for $E_c$ of the various kinds of the concrete, with which the test specimens were made, were obtained by the tests of the control cylinders of the standard size (8 inches in diameter and 16 inches in length). Those are as follows:—
<table>
<thead>
<tr>
<th>No. of Frame</th>
<th>Value of $E_c$, lb. per sq. in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2,100,000</td>
</tr>
<tr>
<td>2</td>
<td>3,600,000</td>
</tr>
<tr>
<td>3</td>
<td>3,300,000</td>
</tr>
<tr>
<td>4</td>
<td>3,600,000</td>
</tr>
<tr>
<td>5</td>
<td>4,500,000</td>
</tr>
<tr>
<td>6</td>
<td>3,900,000</td>
</tr>
<tr>
<td>7</td>
<td>3,700,000</td>
</tr>
<tr>
<td>8</td>
<td>4,000,000</td>
</tr>
</tbody>
</table>

These values were used in the following computations except for the cases of Frames No. 5 and 8. The concrete of No. 5 and 8 was not so rich, as shown by the cylinder tests, and therefore the following values were used by judgment.

$$\frac{E_s}{E_c} = 10 \text{ for No. 5 and } \frac{E_s}{E_c} = 9 \text{ for No. 8.}$$

26. Calculation of Theoretical Stresses.—All considerations stated in paragraph 24 were used in the calculation of the theoretical stresses at the selected points in the individual frames.

In addition to the stresses due to flexure, the stresses due to the direct force must be taken into consideration. The direct stress due to the horizontal thrust is comparatively small, but that due to vertical reaction is large. The stresses due to all direct forces were computed by the usual formulas for a reinforced concrete column, and the computed results are shown in Table XIII.

From these values the resultant stresses were computed and the results are given in Tables XIV to XXI, together with the experimental results.
Diagram No. 14.

Diagram Showing Points Selected for the Purpose of Comparison of Theory with Experiment.
![TABLE XIII.

VALUES USED IN THEORETICAL CALCULATIONS.](image)

<table>
<thead>
<tr>
<th>No. of Frame</th>
<th>Point</th>
<th>H (in)</th>
<th>P</th>
<th>Direct Stress (in)</th>
<th>Case I.</th>
<th>Case II.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Steel</td>
<td>Concrete</td>
<td>k</td>
</tr>
<tr>
<td>1</td>
<td>A</td>
<td>-3.43P</td>
<td>.0068</td>
<td>.070P</td>
<td>.0049P</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.062P</td>
<td>-4.04P</td>
<td>.0109</td>
<td>.070P</td>
<td>.0049P</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>+7.43P</td>
<td>.0098</td>
<td>.012P</td>
<td>.0009P</td>
<td>0.41</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>-4.84P</td>
<td>.0082</td>
<td>.049P</td>
<td>.0059P</td>
<td>0.58</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.121P</td>
<td>+5.28P</td>
<td>.0123</td>
<td>.015P</td>
<td>.0018P</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>+9.78P</td>
<td>.0123</td>
<td>.015P</td>
<td>.0018P</td>
<td>0.36</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
<td>+1.37P</td>
<td>.0027</td>
<td>.053P</td>
<td>.0059P</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.145P</td>
<td>-3.02P</td>
<td>.0109</td>
<td>.052P</td>
<td>.0058P</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>+5.40P</td>
<td>.0098</td>
<td>.015P</td>
<td>.0017P</td>
<td>0.60</td>
</tr>
<tr>
<td>4</td>
<td>A</td>
<td>-4.30P</td>
<td>.0082</td>
<td>.049P</td>
<td>.0059P</td>
<td>0.58</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.100P</td>
<td>+4.90P</td>
<td>.0123</td>
<td>.013P</td>
<td>.0015P</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>+4.90P</td>
<td>.0123</td>
<td>.013P</td>
<td>.0015P</td>
<td>—</td>
</tr>
</tbody>
</table>
### TABLE XIII (Continued).

VALUES USED IN THEORETICAL CALCULATIONS.

<table>
<thead>
<tr>
<th>No. of Frame</th>
<th>Point</th>
<th>(H) in inch unit</th>
<th>(P)</th>
<th>Direct Stress in</th>
<th>Case I</th>
<th>Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Steel Concrete</td>
<td>(k)</td>
<td>(f_s)</td>
</tr>
<tr>
<td>5</td>
<td>(A_{1})</td>
<td>.023P (P^2=0.0064)</td>
<td>.060P</td>
<td>—</td>
<td>0.50</td>
<td>0.047P</td>
</tr>
<tr>
<td></td>
<td>(A_{2})</td>
<td>0.10P do.</td>
<td>.060P</td>
<td>—</td>
<td>0.50</td>
<td>0.055P</td>
</tr>
<tr>
<td></td>
<td>(B)</td>
<td>0.62P do.</td>
<td>.060P</td>
<td>—</td>
<td>0.50</td>
<td>0.024P</td>
</tr>
<tr>
<td></td>
<td>(C)</td>
<td>8.40P .013</td>
<td>.003P</td>
<td>—</td>
<td>0.57</td>
<td>0.150P</td>
</tr>
<tr>
<td>6</td>
<td>(A)</td>
<td>-3.42P .0068</td>
<td>.040P</td>
<td>.0052P</td>
<td>0.58</td>
<td>0.10P</td>
</tr>
<tr>
<td></td>
<td>(B)</td>
<td>0.077P -4.04P .0109</td>
<td>.040P</td>
<td>.0052P</td>
<td>0.59</td>
<td>0.12P</td>
</tr>
<tr>
<td></td>
<td>(C)</td>
<td>+7.46P .0098</td>
<td>.007P</td>
<td>.0009P</td>
<td>0.59</td>
<td>0.22P</td>
</tr>
<tr>
<td>7</td>
<td>(A)</td>
<td>2.03P (P^2=0.0054) (P^3=0.0027)</td>
<td>.048P</td>
<td>.0059P</td>
<td>0.52</td>
<td>0.178P</td>
</tr>
<tr>
<td></td>
<td>(B)</td>
<td>0.145P -3.02P .0109</td>
<td>.047P</td>
<td>.0058P</td>
<td>0.54</td>
<td>0.13P</td>
</tr>
<tr>
<td></td>
<td>(C)</td>
<td>+5.40P .0098</td>
<td>.014P</td>
<td>.0017P</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>8</td>
<td>(A_{1})</td>
<td>.0098</td>
<td>Neglected</td>
<td>—</td>
<td>0.33</td>
<td>0.46P</td>
</tr>
<tr>
<td></td>
<td>(A_{2})</td>
<td>(P^2=0.0073) (P^3=0.0049)</td>
<td>Neglected</td>
<td>—</td>
<td>0.58</td>
<td>0.047P</td>
</tr>
<tr>
<td></td>
<td>(B)</td>
<td>(P^2=0.0062) (P^3=0.0049)</td>
<td>.070P</td>
<td>—</td>
<td>0.50</td>
<td>0.166P</td>
</tr>
<tr>
<td></td>
<td>(C)</td>
<td>(P^2=0.0062) (P^3=0.0049)</td>
<td>Neglected</td>
<td>—</td>
<td>0.50</td>
<td>0.06P</td>
</tr>
</tbody>
</table>

* Compressive stress in steel, \(P\)'s.  + Shown on next page.  Case All.  # Case BII.
**REACTIONS AND BENDING MOMENTS IN FRAME NO. 8.**

**COEFFICIENTS OF BENDING MOMENT AND REACTION.**

<table>
<thead>
<tr>
<th>Coefficient of</th>
<th>Case A Fraction</th>
<th>Case A Decimal</th>
<th>Case B Fraction</th>
<th>Case B Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0 \div \text{pl.}$</td>
<td>$\frac{1}{115}$</td>
<td>$0.0087$</td>
<td>$\frac{1}{103}$</td>
<td>$0.0097$</td>
</tr>
<tr>
<td>$H_1 \div \text{do.}$</td>
<td>$\frac{1}{43.0}$</td>
<td>$0.0033$</td>
<td>$\frac{1}{43.5}$</td>
<td>$0.0230$</td>
</tr>
<tr>
<td>$V_o \div \text{do.}$</td>
<td>$\frac{1}{21.3}$</td>
<td>$0.0470$</td>
<td>$\frac{1}{25.6}$</td>
<td>$0.0390$</td>
</tr>
<tr>
<td>$V_1 \div \text{do.}$</td>
<td>$\frac{1}{1.83}$</td>
<td>$0.5470$</td>
<td>$\frac{1}{1.85}$</td>
<td>$0.5390$</td>
</tr>
<tr>
<td>$M_0 \div \text{pl.}$</td>
<td>$\frac{1}{314}$</td>
<td>$0.0032$</td>
<td>$\frac{1}{283}$</td>
<td>$0.0035$</td>
</tr>
<tr>
<td>$M'_o \div \text{do.}$</td>
<td>$\frac{1}{157}$</td>
<td>$0.0064$</td>
<td>$\frac{1}{142}$</td>
<td>$0.0070$</td>
</tr>
<tr>
<td>$M_1 \div \text{do.}$</td>
<td>$\frac{1}{117}$</td>
<td>$0.0085$</td>
<td>$\frac{1}{119}$</td>
<td>$0.0084$</td>
</tr>
<tr>
<td>$M'_1 \div \text{do.}$</td>
<td>$\frac{1}{58.5}$</td>
<td>$0.0170$</td>
<td>$\frac{1}{60}$</td>
<td>$0.0167$</td>
</tr>
<tr>
<td>$M_3 \div \text{do.}$</td>
<td>$\frac{1}{24.6}$</td>
<td>$0.0406$</td>
<td>$\frac{1}{30.4}$</td>
<td>$0.0393$</td>
</tr>
<tr>
<td>$M_8 \div \text{do.}$</td>
<td>$\frac{1}{17.3}$</td>
<td>$0.0577$</td>
<td>$\frac{1}{20.2}$</td>
<td>$0.0495$</td>
</tr>
<tr>
<td>$M_{Al} \div \text{do.}$</td>
<td>$\frac{1}{14.9}$</td>
<td>$0.0673$</td>
<td>$\frac{1}{13.3}$</td>
<td>$0.0751$</td>
</tr>
<tr>
<td>$M_{A2} \div \text{do.}$</td>
<td>$\frac{1}{38.1}$</td>
<td>$0.0262$</td>
<td>$\frac{1}{26.6}$</td>
<td>$0.0376$</td>
</tr>
<tr>
<td>$M_B \div \text{do.}$</td>
<td>$\frac{1}{87.2}$</td>
<td>$0.0115$</td>
<td>$\frac{1}{88.3}$</td>
<td>$0.0113$</td>
</tr>
<tr>
<td>$H_C \div \text{do.}$</td>
<td>$\frac{1}{31.7}$</td>
<td>$0.0316$</td>
<td>$\frac{1}{42.6}$</td>
<td>$0.0234$</td>
</tr>
</tbody>
</table>
**TABLE XIV.**

**COMPARISON OF THEORY WITH EXPERIMENT.**

<table>
<thead>
<tr>
<th>Point</th>
<th>Case</th>
<th>Applied Load on Specimen (lb.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>12 000</td>
</tr>
<tr>
<td></td>
<td>$f_s$</td>
<td>$f_c$</td>
</tr>
<tr>
<td></td>
<td>$+_{s}$</td>
<td>$+_{c}$</td>
</tr>
<tr>
<td>A</td>
<td>Theoretical II</td>
<td>9 800</td>
</tr>
<tr>
<td></td>
<td>Experimental Av.</td>
<td>9 700</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td>+100</td>
</tr>
<tr>
<td>B</td>
<td>Theoretical II</td>
<td>7 200</td>
</tr>
<tr>
<td></td>
<td>Experimental Av.</td>
<td>5 400</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td>+1 800</td>
</tr>
<tr>
<td>C</td>
<td>Theoretical II</td>
<td>11300</td>
</tr>
<tr>
<td></td>
<td>Experimental Av.</td>
<td>10 500</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td>+800</td>
</tr>
</tbody>
</table>
### TABLE XV.

**COMPARISON OF EXPERIMENT WITH THEORY.**

**Frame No. 2.**

<table>
<thead>
<tr>
<th>Point</th>
<th>Case</th>
<th>Applied Load on Specimen (lb.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>8,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>①s</td>
</tr>
<tr>
<td>Theoretical</td>
<td>I</td>
<td>1200</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>7700</td>
</tr>
<tr>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental</td>
<td>Av.4400</td>
<td>400</td>
</tr>
<tr>
<td></td>
<td>Av.2700</td>
<td>550</td>
</tr>
<tr>
<td>Diff.</td>
<td>①1700</td>
<td>-150</td>
</tr>
<tr>
<td>Theoretical</td>
<td>II</td>
<td>7400</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>6100</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>①1300</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>Av.11500</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>①2500</td>
</tr>
</tbody>
</table>
### Table XVI.

**Comparison of Experiment with Theory.**

**Frame No. 3.**

<table>
<thead>
<tr>
<th>Point</th>
<th>Case</th>
<th>21,000</th>
<th>30,000</th>
<th>38,000</th>
<th>46,000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$T_s$</td>
<td>$T_c$</td>
<td>$T_s$</td>
<td>$T_c$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$T_s$</td>
<td>$T_c$</td>
<td>$T_s$</td>
<td>$T_c$</td>
</tr>
<tr>
<td>Theoretical I</td>
<td>500</td>
<td>333</td>
<td>600</td>
<td>480</td>
<td>900</td>
</tr>
<tr>
<td>A</td>
<td>Experimental Av.</td>
<td>1500</td>
<td>190</td>
<td>1400</td>
<td>250</td>
</tr>
<tr>
<td>Diff.</td>
<td>-1000</td>
<td>+140</td>
<td>-800</td>
<td>-230</td>
<td>-1100</td>
</tr>
<tr>
<td>Theoretical I</td>
<td>2300</td>
<td>610</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>9200</td>
<td>760</td>
<td>13200</td>
<td>1080</td>
<td>16700</td>
</tr>
<tr>
<td>Av.</td>
<td>5800</td>
<td>690</td>
<td>13200</td>
<td>1080</td>
<td>16700</td>
</tr>
<tr>
<td>B</td>
<td>Experimental Av.</td>
<td>5300</td>
<td>1100</td>
<td>11900</td>
<td>1520</td>
</tr>
<tr>
<td>Diff.</td>
<td>+500</td>
<td>-410</td>
<td>+1300</td>
<td>-440</td>
<td>0</td>
</tr>
<tr>
<td>Theoretical I</td>
<td>3600</td>
<td>670</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>16000</td>
<td>940</td>
<td>22800</td>
<td>1350</td>
<td>28900</td>
</tr>
<tr>
<td>Av.</td>
<td>9800</td>
<td>800</td>
<td>22800</td>
<td>1350</td>
<td>28900</td>
</tr>
<tr>
<td>C</td>
<td>Experimental Av.</td>
<td>10200</td>
<td>670</td>
<td>18600</td>
<td>820</td>
</tr>
<tr>
<td>Diff.</td>
<td>-400</td>
<td>+130</td>
<td>+4200</td>
<td>+530</td>
<td>+3200</td>
</tr>
</tbody>
</table>
### TABLE XVII.

**COMPARISON OF EXPERIMENT WITH THEORY.**

**Frame No. 4.**

<table>
<thead>
<tr>
<th>Point</th>
<th>Case</th>
<th>Applied Load on Frame (lb.)</th>
<th>10 000</th>
<th>15 000</th>
<th>21 000</th>
<th>30 000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$T_s$</td>
<td>$T_c$</td>
<td>$T_s$</td>
<td>$T_c$</td>
<td>$T_s$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>Theoretical</td>
<td>I</td>
<td>1 500</td>
<td>410</td>
<td>2 300</td>
<td>620</td>
<td>3 200</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>6 500</td>
<td>540</td>
<td>12 700</td>
<td>810</td>
<td>17 800</td>
</tr>
<tr>
<td></td>
<td>Av.</td>
<td>5 000</td>
<td>480</td>
<td>7 500</td>
<td>720</td>
<td>10 500</td>
</tr>
<tr>
<td>A Experimental</td>
<td></td>
<td>3 400</td>
<td>590</td>
<td>7 600</td>
<td>1 000</td>
<td>11 700</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>+1600</td>
<td>-110</td>
<td>-100</td>
<td>-280</td>
<td>-1 200</td>
</tr>
<tr>
<td>Theoretical</td>
<td>I</td>
<td>2 000</td>
<td>340</td>
<td>3 000</td>
<td>510</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>8 700</td>
<td>630</td>
<td>13 100</td>
<td>940</td>
<td>18 300</td>
</tr>
<tr>
<td></td>
<td>Av.</td>
<td>5 300</td>
<td>490</td>
<td>6 000</td>
<td>730</td>
<td>18 300</td>
</tr>
<tr>
<td>B Experimental</td>
<td></td>
<td>3 500</td>
<td>680</td>
<td>7 200</td>
<td>1 050</td>
<td>16 300</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>+1 800</td>
<td>-190</td>
<td>+800</td>
<td>-320</td>
<td>+2 000</td>
</tr>
<tr>
<td>Theoretical</td>
<td>II</td>
<td>8 700</td>
<td>630</td>
<td>13 100</td>
<td>940</td>
<td>18 300</td>
</tr>
<tr>
<td>C Experimental</td>
<td></td>
<td>7 200</td>
<td>730</td>
<td>11 400</td>
<td>1 020</td>
<td>19 300</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>+1 500</td>
<td>-100</td>
<td>+1 700</td>
<td>-80</td>
<td>-1 000</td>
</tr>
</tbody>
</table>

* Experimental value is taken from the average of observed stresses of the gage lines 84, 88, 100 and 104.

- Average of gage lines 11, 15, 59 and 63.
## TABLE XVIII.

**COMPARISON OF EXPERIMENT WITH THEORY.**

**Frame No. 5.**

<table>
<thead>
<tr>
<th>Point</th>
<th>Case</th>
<th>20,000</th>
<th>40,000</th>
<th>60,000</th>
<th>80,000</th>
<th>100,000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$f_c'$</td>
<td>$f_s'$</td>
<td>$f_c'$</td>
<td>$f_s'$</td>
<td>$f_c'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+T$</td>
<td>$-T$</td>
<td>$+T$</td>
<td>$-T$</td>
<td>$+T$</td>
</tr>
<tr>
<td>A1</td>
<td>Theoretical I</td>
<td>-900</td>
<td>-1500</td>
<td>-1900</td>
<td>-5000</td>
<td>-2600</td>
</tr>
<tr>
<td></td>
<td>Experimental</td>
<td>-200</td>
<td>-1200</td>
<td>-2000</td>
<td>-3700</td>
<td>-1300</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>+700</td>
<td>+500</td>
<td>-100</td>
<td>-700</td>
<td>+1500</td>
</tr>
<tr>
<td>A2</td>
<td>Theoretical I</td>
<td>-1100</td>
<td>-1300</td>
<td>-2200</td>
<td>-2700</td>
<td>-3300</td>
</tr>
<tr>
<td></td>
<td>Experimental</td>
<td>-800</td>
<td>-1800</td>
<td>-2500</td>
<td>-3700</td>
<td>-3300</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>+300</td>
<td>-500</td>
<td>-300</td>
<td>-1000</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>Theoretical I</td>
<td>-500</td>
<td>-1900</td>
<td>-900</td>
<td>-3900</td>
<td>-1400</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>-11400</td>
<td>-4000</td>
<td>+15200</td>
<td>-5300</td>
<td>-19000</td>
</tr>
<tr>
<td></td>
<td>Av.</td>
<td>-500</td>
<td>-1900</td>
<td>-900</td>
<td>-3900</td>
<td>+5000</td>
</tr>
<tr>
<td></td>
<td>Experimental</td>
<td>-300</td>
<td>-1200</td>
<td>+1300</td>
<td>-2000</td>
<td>+2200</td>
</tr>
<tr>
<td></td>
<td>(Col. No. 1)</td>
<td>+200</td>
<td>+700</td>
<td>-2200</td>
<td>+1700</td>
<td>+2800</td>
</tr>
<tr>
<td>C</td>
<td>Theoretical II</td>
<td>3000</td>
<td>12000</td>
<td>16100</td>
<td>24000</td>
<td>30000</td>
</tr>
<tr>
<td></td>
<td>Experimental</td>
<td>3100</td>
<td>10100</td>
<td>16300</td>
<td>24800</td>
<td>33500</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>-100</td>
<td>+1900</td>
<td>-200</td>
<td>-800</td>
<td>-2500</td>
</tr>
</tbody>
</table>
TABLE XIX.

COMPARISON OF EXPERIMENT WITH THEORY.

Frame No. 6.

<table>
<thead>
<tr>
<th>Point</th>
<th>Cage Line</th>
<th>Applied Load on Specimen (lb.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>10 000</td>
</tr>
<tr>
<td></td>
<td>$T_s$</td>
<td>$T_c$</td>
</tr>
<tr>
<td></td>
<td>$T_s$</td>
<td>$T_c$</td>
</tr>
</tbody>
</table>

| A      | Theoretical Case I | 1000 | 300 | 1800 | 540 | 2400 | 720 | 3000 | 900 |
|        | Experimental Case I | 1000 | 500 | 3700 | 900 | 9000 | 1630 | 14800 | 2460 |
| Diff.  | 0 | -200 | -1900 | -360 | +1900 | -730 | -1100 | -560 | +5600 | -530 |

| B      | Theoretical Case I | 1200 | 330 | 2200 | 600 | 14400 | 1100 | 18000 | 1360 |
|        | Experimental Case I | 1200 | 330 | 6500 | 720 | 14400 | 1100 | 18000 | 1360 |
| Diff.  | 0 | -200 | -1900 | -360 | +1900 | -730 | -1100 | -560 | +5600 | -530 |

| C      | Theoretical Case I | 2200 | 420 | 4000 | 760 | 25400 | 1560 | 31800 | 1950 |
|        | Experimental Case I | 2200 | 420 | 19600 | 1170 | 25400 | 1560 | 31800 | 1950 |
| Diff.  | 0 | -200 | -1900 | -360 | +1900 | -730 | -1100 | -560 | +5600 | -530 |
### TABLE XI

**COMPARISON OF EXPERIMENT WITH THEORY.**

**Frame No. 7.**

<table>
<thead>
<tr>
<th>Point</th>
<th>Gage Line</th>
<th>Applied Load on Frame (lb.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>21 000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30 000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>38 000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>46 000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( T )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( T )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( T )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( T )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( T )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( T )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( T )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theoretical Case I</th>
<th>900</th>
<th>3700</th>
<th>1300</th>
<th>5300</th>
<th>1600</th>
<th>6800</th>
<th>1900</th>
<th>8200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical Case II</td>
<td>2700</td>
<td>610</td>
<td>3900</td>
<td>870</td>
<td>4900</td>
<td>1100</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Theoretical Av.</td>
<td>5900</td>
<td>710</td>
<td>8400</td>
<td>1060</td>
<td>10600</td>
<td>19800</td>
<td>1750</td>
<td>1750</td>
</tr>
<tr>
<td>Theoretical Case I</td>
<td>1600</td>
<td>1000</td>
<td>1440</td>
<td>28900</td>
<td>2210</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theoretical Av.</td>
<td>13100</td>
<td>1530</td>
<td>20300</td>
<td>2400</td>
<td>27500</td>
<td>29000</td>
<td>37000</td>
<td>Ev. E</td>
</tr>
<tr>
<td>Diff.</td>
<td>2900</td>
<td>-530</td>
<td>2500</td>
<td>-960</td>
<td>-1400</td>
<td>-1070</td>
<td>-2000</td>
<td>---</td>
</tr>
</tbody>
</table>


- \( f_c \) or the compressive stress in the steel.
- \( * \) over elastic limit.
### TABLE XXI.

**COMPARISON OF EXPERIMENT WITH THEORY.**

Frame No. 8.

<table>
<thead>
<tr>
<th>Point</th>
<th>Gage Line</th>
<th>Applied Load in Pounds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>30,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_t$</td>
</tr>
<tr>
<td>A 1</td>
<td>Theoretical</td>
<td>8900</td>
</tr>
<tr>
<td></td>
<td>Experimental</td>
<td>7000</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>+1900</td>
</tr>
<tr>
<td>A 2</td>
<td>Theoretical</td>
<td>1400</td>
</tr>
<tr>
<td></td>
<td>Experimental</td>
<td>2300</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>-900</td>
</tr>
<tr>
<td>B</td>
<td>Theoretical</td>
<td>5000</td>
</tr>
<tr>
<td></td>
<td>Experimental</td>
<td>1300</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>+3700</td>
</tr>
<tr>
<td>C</td>
<td>Theoretical</td>
<td>1800</td>
</tr>
<tr>
<td></td>
<td>B II</td>
<td>13800</td>
</tr>
<tr>
<td></td>
<td>Av.</td>
<td>1800</td>
</tr>
<tr>
<td></td>
<td>Diff.</td>
<td>-300</td>
</tr>
</tbody>
</table>

* Tension in concrete is taken into consideration.

† Compressive stress in the steel plate.

‡ Average stress of Case A II and Case B II.
The following table was made to show the maximum difference between the theoretical and experimental values of stresses.

**TABLE XXII.**

MAXIMUM AND MINIMUM DIFFERENCE BETWEEN THEORETICAL AND EXPERIMENTAL VALUES OF STRESSES.

<table>
<thead>
<tr>
<th>Number of Test Frame.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>At Center of Loaded Beam</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max. Stress</td>
<td>39500</td>
<td>37500</td>
<td>36400</td>
<td>36000</td>
<td>29800</td>
<td>44400</td>
<td>37500</td>
<td></td>
</tr>
<tr>
<td>observed</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diff. Max.</td>
<td>-2200</td>
<td>+2500</td>
<td>+4200</td>
<td>-3800</td>
<td>+2500</td>
<td>+5600</td>
<td>+2900</td>
<td>+3200</td>
</tr>
<tr>
<td>Min.</td>
<td>-400</td>
<td>-100</td>
<td>-600</td>
<td>-100</td>
<td>-100</td>
<td>-600</td>
<td>+1400</td>
<td>+1000</td>
</tr>
<tr>
<td><strong>STRESS IN STEEL</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>At upper part of column</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max. Stress</td>
<td>35200</td>
<td>12300</td>
<td>25400</td>
<td>31000</td>
<td>35600</td>
<td>23500</td>
<td>22300</td>
<td>19200</td>
</tr>
<tr>
<td>observed</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diff. Max.</td>
<td>-2400</td>
<td>+1700</td>
<td>+2100</td>
<td>+4100</td>
<td>+2800</td>
<td>+5600</td>
<td>+3300</td>
<td>+5400</td>
</tr>
<tr>
<td>Min.</td>
<td>+100</td>
<td>-400</td>
<td>0</td>
<td>-100</td>
<td>200</td>
<td>0</td>
<td>+500</td>
<td>-100</td>
</tr>
<tr>
<td>At lower part of column</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max. Stress</td>
<td>-----</td>
<td>-----</td>
<td>6000</td>
<td>-----</td>
<td>19800</td>
<td>-----</td>
<td>9400</td>
<td>-----</td>
</tr>
<tr>
<td>observed</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-1700</td>
<td>-----</td>
<td>-1000</td>
<td>-----</td>
<td>+1800</td>
</tr>
<tr>
<td>Diff. Max.</td>
<td>-----</td>
<td>-----</td>
<td>-1700</td>
<td>-----</td>
<td>-1000</td>
<td>-----</td>
<td>+1800</td>
<td>-----</td>
</tr>
<tr>
<td>Min.</td>
<td>-----</td>
<td>-----</td>
<td>0</td>
<td>-----</td>
<td>+300</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td><strong>At center of Loaded Beam</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max. Stress</td>
<td>3320</td>
<td>-----</td>
<td>1500</td>
<td>2590</td>
<td>-----</td>
<td>2750</td>
<td>3750</td>
<td>2750</td>
</tr>
<tr>
<td>observed</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diff. Max.</td>
<td>-1100</td>
<td>-----</td>
<td>+570</td>
<td>-660</td>
<td>-----</td>
<td>-730</td>
<td>-1070</td>
<td>-990</td>
</tr>
<tr>
<td>Min.</td>
<td>-90</td>
<td>-----</td>
<td>+130</td>
<td>-80</td>
<td>-----</td>
<td>-240</td>
<td>-530</td>
<td>-290</td>
</tr>
<tr>
<td><strong>COMPRESSIVE STRESS IN CONCRETE</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>At upper part of column</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max. Stress</td>
<td>3130</td>
<td>1570</td>
<td>3050</td>
<td>2920</td>
<td>----</td>
<td>3840</td>
<td>2880</td>
<td>-----</td>
</tr>
<tr>
<td>observed</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diff. Max.</td>
<td>-1380</td>
<td>-150</td>
<td>-470</td>
<td>-730</td>
<td>-----</td>
<td>-730</td>
<td>-1500</td>
<td>-----</td>
</tr>
<tr>
<td>Min.</td>
<td>-30</td>
<td>+70</td>
<td>-190</td>
<td>-110</td>
<td>-----</td>
<td>-200</td>
<td>-640</td>
<td>-----</td>
</tr>
</tbody>
</table>

+ Sign is used when the theoretical value exceeds the experimental value.
- Sign is used when the experimental value exceeds the theoretical value.
28. Distribution of Stresses in Frames.—To aid the eye, the observed values of the stresses were plotted in Diagrams 15-35. These diagrams show clearly the distribution of the stresses along the individual member of the frames and the critical points of failure.
Stress Distribution Diagram 15. F. No. 1.

$P = 18,000 \text{ lb.}$

$1'' = 2000 \text{ lb. per sq. in. for steel}; 1'' = 2000 \text{ lb. per sq. in. for concrete.}

--- Theoretical Value.
Stress Distribution Diagram

\[ P = 24,000 \text{ lb} \]

1" = 20,000 lb per sq.in. for Steel
1" = 2,000 lb per sq.in. for Concrete

Theoretical Value.
Stress Distribution Diagram XVII  F No 1

\[ P = 30,000 \text{ lb.} \]

1" = 20,000 lb per sq.in for Steel
1" = 2,000 lb per sq.in for Concrete

Theoretical Value.
Stress Distribution Diagram XVIII. F. No 2.

\[ P = 8000 \text{ lb.} \]

1" = 20,000 lb. per sq. in. for Steel; 1" = 2,000 lb. per sq. in. for Conc.

--- Theoretical Value.
Stress Distribution Diagram XIX. F. No 2.

\[ P = 12,000 \text{ lb.} \]

Theoretical Value:
- \( 1'' = 20,000 \text{ lb per sq in for Steel} \)
- \( 1'' = 2,000 \text{ lb per sq in for Concrete} \)
Stress Distribution Diagram XX20. NO 3.

\[ P = 30,000 \text{ lb.} \]

\[ \frac{P}{2} \]

\[ \frac{P}{2} \]

\[ \tau = 820 \text{ lb.} \]

\[ \tau = 22100 \text{ lb.} \]

\[ 1'' = 20,000 \text{ lb. per sq.in. for Steel} \]

\[ 1'' = 2,000 \text{ lb. per sq.in. for Concrete} \]

--- Theoretical Value.
Stress Distribution Diagram XX121, F. NO 3.

\[ P = 38,000 \text{ lb.} \]

1" = 20,000 \text{ lb./per sq.in. for Steel.}

1" = 2,000 \text{ lb./per sq.in. for Concrete.}

Theoretical Value.
Stress Distribution Diagram 22. F. no. 3.

\[ P = 46,000 \text{ lb.} \]

\[ \frac{P}{2} \]

\[ f = 1500 \text{ lb.} \]

1" = 20,000 lb. per sq. in. for Steel,
1" = 2000 lb. per sq. in. for Concrete.

--- Theoretical Value.
Stress Distribution Diagram 23.
F. No. 4.
P = 21000 lb.

1" = 20,000 lb. per sq. in. for Steel; 1" = 2000 lb. per sq. in. for Concrete.

Theoretical Value
Stress Distribution Diagram 24.

F No. 4  
P = 30,000 lb.

1" = 20,000 lb. per sq. in. for Steel; 1" = 2000 lb. per sq. in. for Concrete.

Theoretical Value.
Stress Distribution Diagram XXV 25.
F. NO 5.

P = 60,000 lb.

1" = 20,000 lb./per sq.in.

--- Theoretical Value ---
Stress Distribution Diagram XXVI26.
F. No 5.
P = 80,000 lb.

--- Theoretical Value.

1" = 20,000 lb.persq.in.
Stress Distribution Diagram XXVII27.

F. NO 5.

P = 100,000 lb. per sq. in.

1" = 20,000 lb. per sq. in.

--- Theoretical Value.
Stress Distribution Diagram 29.

F. No 6 ,  P = 30,000 lb.

1" = 20,000 lb/sq.in. for Steel; 1" = 2000 lb/sq.in. for concrete.

Theoretical Value.
Stress Distribution Diagram. XXVII. F. No. 6.

\[ P = 36,000 \text{ lb.} \]

1" = 20,000 lb. per sq. in. for Steel.
1" = 2,000 lb. per sq. in. for Concrete.

Theoretical Value.
Stress Distribution Diagram \( \frac{30}{XXX} \), F. No. 7.

\( f_c = 2,900 \text{ lb.} \)

\( P = 38,000 \text{ lb. per sq. in.} \)

1" = 20,000 lb. per sq. in. for Steel

1" = 2,000 lb. per sq. in. for Concrete

--- Theoretical Value ---

Front Bar

Back Bar
Stress Distribution Diagram 31. F. No. 7.

P = 46,000 lb.

1" = 29,000 lb. per sq. in. for Steel; 1" = 2,000 lb. per sq. in. for Concrete.

Theoretical Value
Stress Distribution Diagram 32.
F No 8.
Load = 30,000 lb.

1" = 20,000 lb per sq. in. for Steel; 1" = 2,000 lb per sq. in. for Conc.

--- Theoretical Value.
Stress Distribution Diagram 33.

F. No. 8.

Load = 45,000 lb.

f = 2260 lb.

1" = 2000 lb. per sq. in. for Steel; 1" = 2000 lb. per sq. in. for Conc.

--- Theoretical Value ---
Stress Distribution Diagram 34.

F No 8.

Load = 60,000 lb.

\[ f_b = 2540 \text{ lb.} \]

1" = 20000 lb per sq. in. for steel; 1" = 2000 lb per sq. in. for concrete.

Theoretical Value.
Stress Distribution Diagram 35.

F. No. 8.

Load = 75,000 lb.

\[ f_s = 2750 \text{lb.} \]

1\% = 29,000 lb. per sq. in. for steel; 1\% = 2900 lb. per sq. in. for concrete.

--- Theoretical Value ---
29. **Summary of Comparison.**—From the above comparison the following conclusions may be drawn:

1. **Steel stress at the center of the loaded top beam.**

   The experimental and theoretical values are in fair agreement for all kinds of the tested frames except a few cases where the load was comparatively low or extremely high. The maximum difference between experimental and theoretical values is about nineteen percent of the observed highest stress.

2. **The steel stresses in the columns.**

   The above statement holds for this case, but the percent of the maximum difference is higher than the last case owing to the fact that the direct stress is not equally distributed over the cross section of the column.

3. **Compressive stresses in the concrete.**

   The observed compressive stresses in the concrete at the low loads which developed the unit stress up to about 800 lb. per sq.in. agree with the theoretical values, but in most cases the observed concrete stresses were higher than the theoretical stresses. The discrepancy ran frequently up to fifty percent. The wide difference is partly due to the rigid connection of the members accompanying a certain secondary stress and is partly due to the fact that the moduli of elasticity of the concrete were determined by the control cylinder tests.

   **Note.**—The moduli of elasticity of the concrete determined by the control cylinder tests were rather high and they varied from 2,100,000 to 4,500,000 lb. per sq.in., and in
most cases their values were in the vicinity of 3,500,000 lb. per sq.in. These values were used in the computation of the concrete stresses. But it seems that the concrete used in the frames was not so stiff as indicated by the cylinder tests. If the actual modulus of elasticity of the concrete is assumed to be about 2,500,000 lb. per sq.in., the observed concrete stresses will be reduced about thirty percent from the values given in the tables.
VIII. GENERAL DISCUSSION.

30. Action of Reinforced Concrete Frames under Loads.—In the analysis of a frame the fundamental consideration was the form of the elastic curve of the axis of a frame under load. The form of an elastic curve evidently depends upon the method of loading and the form of the frame under consideration. One reason why many engineers hesitate to use formulas based on elastic deformation is that they question the elastic action of a frame under load. The reinforced concrete frame is not of course a purely elastic system, but the tests herein described showed that the action of a frame under load follows fairly the theoretical analysis. This condition is practically the same until the yield point of one of the composing materials (steel and concrete) has been reached.

31. Effect of the End Condition of Column on Results.—In advance it had been hoped that a frictionless pin might be devised for the frames having free ends and a massive concrete base for frames No. 3, 7 and 8. Neither of the two base arrangements used for test frames 1, 2, 4 and 6 may be called ideal devices, and there may exist certain secondary stresses due to friction or bending. These secondary stresses were actually very small and may be entirely neglected without appreciable error.

The concrete bases used for the frames No. 3, 7 and 8 to secure the fixity of the column ends may also not be an ideal arrangement and a slight bending in the base due to a load may be expected to have an influence on the bending in the other
members. The deformation readings at the middle point of the base were taken at each increase in the load. The results of the observation showed that there was practically no bending stress for all loads except the ultimate load.

32. Effect of Non-uniformity of Quality of Concrete on Distribution of Stresses.—The frames were made in the flat position on the laboratory floor, as has been described. Consequently on the side which was at the bottom during making, denser concrete was obtained than on the top side. When the load was applied to the specimen, a lateral bending took place in the columns and more tension was thrown on the rich concrete than on the poor concrete. All these things modified the stress distribution in the reinforcing steel. The effect of the non-uniformity of the quality of the concrete on the distribution of stresses will be clear from the following table.

TABLE XXIII.

EFFECT OF NON-UNIFORMITY OF CONCRETE ON DISTRIBUTION OF STRESS.

<table>
<thead>
<tr>
<th>Frame No.</th>
<th>At Center of Span</th>
<th>At Upper Part of Column</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Observed Stress</td>
<td>Difference</td>
</tr>
<tr>
<td></td>
<td>in Steel</td>
<td>in Steel</td>
</tr>
<tr>
<td></td>
<td>On Bottom Side</td>
<td>On Top Side</td>
</tr>
<tr>
<td>1</td>
<td>22 200*</td>
<td>27 100*</td>
</tr>
<tr>
<td>2</td>
<td>32 900</td>
<td>27 400</td>
</tr>
<tr>
<td>3</td>
<td>39 500</td>
<td>29 000</td>
</tr>
<tr>
<td>4</td>
<td>34 400</td>
<td>24 000</td>
</tr>
<tr>
<td>5</td>
<td>36 400</td>
<td>30 600</td>
</tr>
<tr>
<td>6</td>
<td>29 800</td>
<td>26 400</td>
</tr>
<tr>
<td>7</td>
<td>23 500*</td>
<td>21 500*</td>
</tr>
<tr>
<td>8</td>
<td>30 900*</td>
<td>18 400*</td>
</tr>
</tbody>
</table>

* Not at the maximum load.
33. Distribution of Stress over the Cross Section.—It is most common to assume that the stresses are uniformly distributed over the cross section when a member is subjected to a pure compression or pure tension and that stresses are symmetrically distributed over the cross section when a material is subjected to flexure. Even in the case of steel members this is not actually realized and in the case of a built-up column the excess of a maximum fiber stress over an average stress occasionally runs up to fifty percent. Therefore it will be worth while to make a discussion on this point. As already stated, the steel stresses in the columns and beams were modified by the non-uniformity of the quality of the concrete used. Consequently we can easily see that the stress distribution over the cross section is not uniform. To see this more clearly and definitely, special measurements were made in the columns of frame No. 6 and 7. The location of the gage lines was selected at a place where the bending is not sufficiently large to produce tension cracks in the concrete. The gage lines were located on the four faces of the column and the observations were made at each increase of the load. Figures 50 & 51 show the results of observations. The parallelogram drawn with heavy lines shows a cross section of the column, and the tensile and compressive deformations are shown by the red and black lines, respectively, to aid the eye. The shaded area is a compression zone under the load 14000 lb. in Fig. 51. From these figures it is quite clear that the highest tensile stress was developed at the outer corner of the rich concrete side and the highest compressive stress was developed at the opposite corner.
Fig. 50.

Distribution of Stress over the Cross-Section of Column, Frame No. 6.

17-27 Gauge Lines.
Deformation, 1" = .0001" (or 400 lb. per sq. in.)
Fig. 51.

Distribution of Stress over the Cross-Section of Column, Frame No. 7.
of the poor concrete side. This distribution of the stresses was not much altered by the increase of the load. Such a distribution of stresses is natural when the material is composed of a harder part and a softer part.

34. **Position of Point of Inflection in Columns.**—The position of the point of inflection in the member of a structure is an important element for use in designing the frame and in discussing the theory. The constancy of the position of the point of inflection for different loads shows that there is a constant ratio between the positive and negative bending moments. We have already stated that the point of inflection in a column fixed at one end is located at one-third of its length from the fixed end. This is the statement according to the theory. It is worth while, however, to make sure of this important fact by determining this point by the experiment. To do this, it is convenient to resolve the observed deformation of a fiber into two parts, (a) the deformation due to the flexure alone; (b) the deformation due to the direct force.

In doing this we will assume a straight line relation of the deformation due to flexure.

The following additional notation will be used:

- \( D \) = deformation of an extreme fiber due to flexure alone,
- \( D_c \) = observed deformation of inmost concrete or steel fiber,
- \( D_p \) = observed deformation of the outmost steel fiber,
- \( D_p \) = deduced deformation due to direct stress.

Using these notations these are easily obtained the following expressions from the accompanying figures:
When the stress is wholly compressive:

\[ D = \frac{D_a - D_b}{2}, \quad \text{and} \quad D_P = \frac{D_a + D_b}{2} \]

When there exist tensile and compressive stresses in the section:

\[ D = \frac{D_a + D_b}{2}, \quad \text{and} \quad D_P = \frac{D_a - D_b}{2} \]

The following series of Tables (XXIV—XXVII) are the results of the numerical computation from the observed data of the frames No. 3, 5, 7 and 8. In these tables the deformations are expressed in millionths of an inch. These results are plotted in Diagrams (38-43) to obtain the actual position of the point of inflection. From the foregoing Diagrams (38-43) we can see that the point of inflection remained practically unchanged by the increase of the load for all kinds of frames here described. In the cases of the frames No. 3 and 7, the deviation of the actual point of inflection from the theoretical point was limited within an inch. Therefore we may draw the conclusion that the actual position of the point of inflection in the column of a frame fairly agrees with its theoretical position, consequently the elastic action of reinforced concrete frames follows closely the theory.
TABLE XXIV.
FLEXURAL DEFORMATION IN COLUMN.
Frame No. 3.

<table>
<thead>
<tr>
<th>Gage Line</th>
<th>Applied Load in Pounds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Inside</td>
</tr>
<tr>
<td></td>
<td>Dc  Ds  D  Dc  Ds  D  Dc  Ds  D</td>
</tr>
<tr>
<td></td>
<td>Outside</td>
</tr>
<tr>
<td>Frame No. 3.</td>
<td>21,000</td>
</tr>
</tbody>
</table>

|                  | 20, 23 | 28 | +48 | -77 | 63 | +73 | -145 | 109 | +101 | -165 | 133 |
| COLUMN NUMBER 1. | 21, 24 | 29 | -35 | -78 | 22*| -28 | 85  | 29*| -25  | 77  | 26*|
|                  | 25 | 30 | -180| +35 | 107| -169| 83  | 111| -190 | +112 | 151|
| COLUMN NUMBER 2. | 26 | -- | -235| +128*| 182| -277| +240*| 259| -340 | +350*| 345|
|                  | 27 | 2, 17 | -700| +226 | 463| -914| +427 | 670| -1063| +593 | 828|
|                  | 32, 35 | 40 | +15 | -100| 58 | +25 | -88 | 57 | +43  | -155 | 99 |
|                  | 33, 36 | 41 | -69 | -65 | 2*| -57 | -83 | 13 | -42  | -63 | 11 |
| COLUMN NUMBER 2. | 37 | 42 | -143| +12 | 78 | -119| +98 | 108| -140 | +78  | 109|
|                  | 38 | -- | -282| +65*| 173| -364| +220*| 292| -397 | +295*| 346|
|                  | 39 | 8, 11 | -346| +122| 234| -539| +358 | 449| -692 | +518 | 605|

* Here D = \( \frac{Dc - Ds}{2} \), and Dp = \( \frac{Dc + Ds}{2} \)

* Not observed but estimated by proportion.
<table>
<thead>
<tr>
<th>Gage Line</th>
<th>Inside</th>
<th>Outside</th>
<th>Inside</th>
<th>Outside</th>
<th>Inside</th>
<th>Outside</th>
<th>Inside</th>
<th>Outside</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A 1</td>
<td>-75</td>
<td>-117</td>
<td>16</td>
<td>-83</td>
<td>-127</td>
<td>22</td>
<td>-115</td>
</tr>
<tr>
<td>B</td>
<td>B 1</td>
<td>-127</td>
<td>-107</td>
<td>10</td>
<td>-140</td>
<td>-162</td>
<td>11</td>
<td>-160</td>
</tr>
<tr>
<td>C</td>
<td>C 1</td>
<td>-130</td>
<td>-68</td>
<td>31</td>
<td>-227</td>
<td>-120</td>
<td>54</td>
<td>-265</td>
</tr>
<tr>
<td>D</td>
<td>D 1</td>
<td>-145</td>
<td>-120</td>
<td>13</td>
<td>-227</td>
<td>-157</td>
<td>35</td>
<td>-295</td>
</tr>
<tr>
<td>E</td>
<td>E 1</td>
<td>-180</td>
<td>-173</td>
<td>58</td>
<td>-180</td>
<td>-173</td>
<td>58</td>
<td>-180</td>
</tr>
<tr>
<td>F</td>
<td>F 1</td>
<td>-162</td>
<td>-50</td>
<td>56</td>
<td>-250</td>
<td>-60</td>
<td>95</td>
<td>-306</td>
</tr>
<tr>
<td>G</td>
<td>G 1</td>
<td>-153</td>
<td>-145</td>
<td>4</td>
<td>-200</td>
<td>-145</td>
<td>27</td>
<td>-235</td>
</tr>
<tr>
<td>H</td>
<td>H 1</td>
<td>-75</td>
<td>-145</td>
<td>35</td>
<td>-125</td>
<td>-215</td>
<td>45</td>
<td>-173</td>
</tr>
<tr>
<td>I</td>
<td>I 1</td>
<td>-58</td>
<td>-173</td>
<td>58</td>
<td>-68</td>
<td>-252</td>
<td>92</td>
<td>-40</td>
</tr>
<tr>
<td>J</td>
<td>J 1</td>
<td>+25</td>
<td>-204</td>
<td>115</td>
<td>+33</td>
<td>-296</td>
<td>166</td>
<td>+80</td>
</tr>
<tr>
<td>K</td>
<td>K 1</td>
<td>+70</td>
<td>-170</td>
<td>120</td>
<td>+150</td>
<td>-282</td>
<td>216</td>
<td>+310</td>
</tr>
<tr>
<td>L</td>
<td>L 1</td>
<td>--------</td>
<td>---------</td>
<td>--------</td>
<td>--------</td>
<td>---------</td>
<td>--------</td>
<td>---------</td>
</tr>
<tr>
<td>a</td>
<td>a 1</td>
<td>-190</td>
<td>-40</td>
<td>75</td>
<td>-60</td>
<td>-105</td>
<td>23</td>
<td>-120</td>
</tr>
<tr>
<td>b</td>
<td>b 1</td>
<td>-230</td>
<td>-23</td>
<td>54</td>
<td>-273</td>
<td>-35</td>
<td>120</td>
<td>-275</td>
</tr>
<tr>
<td>c</td>
<td>c 1</td>
<td>-233</td>
<td>-23</td>
<td>105</td>
<td>-260</td>
<td>-45</td>
<td>108</td>
<td>-343</td>
</tr>
<tr>
<td>d</td>
<td>d 1</td>
<td>-220</td>
<td>+63</td>
<td>142</td>
<td>-322</td>
<td>0</td>
<td>161</td>
<td>-374</td>
</tr>
<tr>
<td>e</td>
<td>e 1</td>
<td>-198</td>
<td>------</td>
<td>---------</td>
<td>-295</td>
<td>------</td>
<td>---------</td>
<td>-345</td>
</tr>
<tr>
<td>f</td>
<td>f 1</td>
<td>-133</td>
<td>-23</td>
<td>55</td>
<td>-275</td>
<td>0</td>
<td>138</td>
<td>-325</td>
</tr>
<tr>
<td>g</td>
<td>g 1</td>
<td>-140</td>
<td>-76</td>
<td>31</td>
<td>-212</td>
<td>-163</td>
<td>25</td>
<td>-240</td>
</tr>
<tr>
<td>h</td>
<td>h 1</td>
<td>-138</td>
<td>-135</td>
<td>2</td>
<td>-190</td>
<td>-188</td>
<td>1</td>
<td>-180</td>
</tr>
<tr>
<td>i</td>
<td>i 1</td>
<td>-140</td>
<td>-183</td>
<td>22</td>
<td>-180</td>
<td>-237</td>
<td>58</td>
<td>-128</td>
</tr>
<tr>
<td>j</td>
<td>j 1</td>
<td>-70</td>
<td>-198</td>
<td>64</td>
<td>-70</td>
<td>-250</td>
<td>90</td>
<td>-23</td>
</tr>
<tr>
<td>k</td>
<td>k 1</td>
<td>-23</td>
<td>-227</td>
<td>102</td>
<td>+30</td>
<td>-290</td>
<td>160</td>
<td>+65</td>
</tr>
<tr>
<td>l</td>
<td>l 1</td>
<td>--------</td>
<td>---------</td>
<td>--------</td>
<td>--------</td>
<td>---------</td>
<td>--------</td>
<td>---------</td>
</tr>
</tbody>
</table>

**Table XXV:**
FLEXURAL DEFORMATION IN COLUMN
Frame No. 5.

<table>
<thead>
<tr>
<th>Applied Load in Pounds</th>
<th>60000</th>
<th>100000</th>
<th>146000</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-115</td>
<td>-150</td>
<td>18</td>
</tr>
<tr>
<td>B</td>
<td>-160</td>
<td>-216</td>
<td>29</td>
</tr>
<tr>
<td>C</td>
<td>-265</td>
<td>-120</td>
<td>83</td>
</tr>
<tr>
<td>D</td>
<td>-295</td>
<td>-203</td>
<td>46</td>
</tr>
<tr>
<td>E</td>
<td>-327</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>F</td>
<td>-306</td>
<td>-157</td>
<td>74</td>
</tr>
<tr>
<td>G</td>
<td>-235</td>
<td>-202</td>
<td>16</td>
</tr>
<tr>
<td>H</td>
<td>-173</td>
<td>-310</td>
<td>69</td>
</tr>
<tr>
<td>I</td>
<td>-40</td>
<td>-365</td>
<td>163</td>
</tr>
<tr>
<td>J</td>
<td>+80</td>
<td>-370</td>
<td>225</td>
</tr>
<tr>
<td>K</td>
<td>+310</td>
<td>-365</td>
<td>338</td>
</tr>
<tr>
<td>L</td>
<td>+135</td>
<td>------</td>
<td>------</td>
</tr>
</tbody>
</table>

**Legend:**
- A, B, C, D: Inside and Outside Gages
- Applied Load in Pounds
- 60000, 100000, 146000: Load Levels
### TABLE XXVI.

**FLEXURAL DEFORMATION IN COLUMN.**

Frame No. 7.

<table>
<thead>
<tr>
<th>Gage Line</th>
<th>Inside</th>
<th>Outside</th>
<th>( \Delta C )</th>
<th>( \Delta S )</th>
<th>( \Delta )</th>
<th>( \Delta C )</th>
<th>( \Delta S )</th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>21 200</td>
<td>21 000</td>
<td>30 000</td>
<td>24</td>
<td>5</td>
<td>-365</td>
<td>+147</td>
<td>256</td>
<td>-535</td>
</tr>
<tr>
<td>23</td>
<td>4</td>
<td></td>
<td>-202</td>
<td>+117</td>
<td>160</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>3</td>
<td></td>
<td>-125</td>
<td>+18</td>
<td>71</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>2</td>
<td></td>
<td>-55</td>
<td>-27</td>
<td>14*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td></td>
<td>+23</td>
<td>-112</td>
<td>68</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>89</td>
<td>46</td>
<td></td>
<td>+60</td>
<td>88</td>
<td>74</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>15</td>
<td></td>
<td>-355</td>
<td>+215</td>
<td>285</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>14</td>
<td></td>
<td>-205</td>
<td>+95</td>
<td>150</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>13</td>
<td></td>
<td>-127</td>
<td>+85</td>
<td>106</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>12</td>
<td></td>
<td>-58</td>
<td>-10</td>
<td>24*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>11</td>
<td></td>
<td>+5*</td>
<td>-53</td>
<td>29</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>76</td>
<td>73</td>
<td></td>
<td>+57</td>
<td>-163</td>
<td>110</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Note: \( \Delta \) represents applied load in pounds.*
### TABLE XXVII.

**FLEXURAL DEFORMATION IN COLUMN.**

Frame No. 8.

<table>
<thead>
<tr>
<th>Cage Point</th>
<th>Inside Point</th>
<th>Outside Point</th>
<th>( D_c )</th>
<th>( D_s )</th>
<th>( D )</th>
<th>( D_p )</th>
<th>( D_c )</th>
<th>( D_s )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A'</td>
<td></td>
<td>-420</td>
<td>+250</td>
<td>335</td>
<td>85</td>
<td>-700</td>
<td>+845</td>
<td>261</td>
</tr>
<tr>
<td>B</td>
<td>B'</td>
<td></td>
<td>-280</td>
<td>+ 60</td>
<td>170</td>
<td>110</td>
<td>-377</td>
<td>+145</td>
<td>261</td>
</tr>
<tr>
<td>C</td>
<td>C'</td>
<td></td>
<td>-167</td>
<td>- 28</td>
<td>70</td>
<td>98</td>
<td>-275</td>
<td>+ 55</td>
<td>165</td>
</tr>
<tr>
<td>Col. D</td>
<td>D'</td>
<td></td>
<td>-140</td>
<td>-30</td>
<td>55</td>
<td>85</td>
<td>-217</td>
<td>+ 43</td>
<td>130</td>
</tr>
<tr>
<td>No. 2 E</td>
<td>E'</td>
<td></td>
<td>- 75</td>
<td>-10</td>
<td>34</td>
<td>44</td>
<td>-125</td>
<td>-120</td>
<td>53</td>
</tr>
<tr>
<td>F</td>
<td>F'</td>
<td></td>
<td>-58</td>
<td>- 80</td>
<td>11</td>
<td>69</td>
<td>-150</td>
<td>-118</td>
<td>134</td>
</tr>
<tr>
<td>G</td>
<td>G'</td>
<td></td>
<td>- 90</td>
<td>-110</td>
<td>10</td>
<td>100</td>
<td>+150</td>
<td>-163</td>
<td>157</td>
</tr>
<tr>
<td>H</td>
<td>H'</td>
<td></td>
<td>+ 65</td>
<td>- 80</td>
<td>73</td>
<td>8</td>
<td>+ 50</td>
<td>-205</td>
<td>128</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Applied Load in Pounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>45,000</td>
</tr>
<tr>
<td>60,000</td>
</tr>
<tr>
<td>75,000</td>
</tr>
</tbody>
</table>

- Over elastic limit and not reliable.
Diagram No. 38.

Position of Point of Inflection in Frames Nos. 3 and 7.

Unit Deformation, in. per in.

I, = Deformation in Col. No. 1 under 2,000 lb.
II, = " Col. No. 2 "
I, = " Col. No. 1 " 3,000 lb.
II, = " Col. No. 2 "

Horizontal scale: 1" = 8".

I, = Deformation in Col. No. 1 under 2,000 lb.
II, = " Col. No. 2 "
I, = " Col. No. 1 " 3,000 lb.
II, = " Col. No. 2 "

Horizontal scale: 1" = 8".
Diagram No. 39.
Position of Point of Inflection in Frame No. 3.
Diagram No. 40.

Deformation in Column No. 1

Frame No. 5

Unit Deformation, or Stress in Steel, lb. per sq. in.

Hor. Scale, 1 in. to 16

Supported End of Column

Top of Column

Center of Tie

Actual Point of Inflection

P = 148,000 lb.

P = 100,000 lb.

P = 80,000 lb.

P = 60,000 lb.

P = 46,000 lb.

P = 30,000 lb.

P = 20,000 lb.

P = 10,000 lb.

P = 5,000 lb.

P = 1,000 lb.

P = 0 lb.
Diagram No. 41.

Deformation in Column No. 2

Frame No. 5

Actual Point of Inflexion

Center of Tie

Hor. Scale, 1" to 16"
Diagram No. 42.

Position of Point of Inflection,
Col. No. 2, Frame No. 8.

Flexural Deformation.

Theoretical Point of Inflection:
Actual Point of Inflection:

Fixed End

Observed Steel Deflection in Inside of Column No. 2

Horizontal Scale 1' to 8"
Diagram No. 43.
Position of Point of Inflection.
Col. No. 3, From No. 8.

Unit Deformation, in. per in.

Flaxural Deformation.

P=75,000 lb.
P=60,000 lb.
P=45,000 lb.
P=30,000 lb.
P=15,000 lb.

Observed Steel Deformation in Insid of Column No 3

Fixed End

Horizontal Scale 1" to 8"

S. OF S. FORM 5
35. **Continuity of the Composing Members of a Frame.**—The continuity of the members is one of the most important subjects in reinforced concrete frames. The results of the present tests showed that there was no sign of the discontinuity of members under the load. The moment, and therefore the stresses, were well transmitted by the rigid connection. When the frame is free to turn at the lower column ends there is a tendency to form the first crack near the juncture at A (Fig. 53), but cracks did not appear at B until a very high load was applied. When the frame is fixed at its lower column ends, the crack appeared at B under comparatively low loads. The place of formation of first cracks, of course, depends on the relative stiffness of the connecting members, but it may be said that when the vertical member is fixed at its end due attention should be paid to the negative bending moment at the ends of the horizontal member to secure perfect continuity.

36. **Stresses at Sharp Corner.**—In a frame construction a square corner, as shown at A in Fig. 54, should be avoided for all connections, for it is a well known fact that theoretically the material at the corner can offer no resistance to bending. Therefore it is most common to design such corners as shown in Fig. 55. In this case, however, the portion abed can not be treated as a
straight member, for the length of fibers along be is consider­ably shorter than that of the outside fibers. The rigid analysis of this case may be very complicated, but a close approximation is to treat this as a curved member in the calculation. There are many analytical methods of curved members. The following is one of the analytical results which is given in Slocum and Hancock's Strength of Materials (p. 200):

\[ \varepsilon_{\text{max}} = \frac{M(h+2d)}{(h+2\rho)Ad} + \frac{P}{A}, \]

in which

- \( A \) = area of cross section of a member;
- \( d \) = distance between gravity axis and neutral axis;
- \( h \) = depth of member;
- \( \rho \) = radius of curvature;
- \( M \) = bending moment in any given section;
- \( P \) = normal force acting at any given section.

High compressive stresses were developed at the sharp corners of the tested frames. To compare these results with the theoretical value, calculations were made by assuming the center of curvature at the center of a circle inscribed at the inside sharp corner. \( h \) is taken from B to D neglecting the concrete area outside of the steel, and \( \rho \) is taken from the assumed center of curvature to C which is halfway between B and D. The results of computation are given in Table XXVIII, together with the experimental results. The results are in close agreement, considering the errors involved in the assumption.
TABLE XXVIII.
COMPARISON OF EXPERIMENTAL STRESS AT SHARP CORNER WITH THEORETICAL VALUE.

<table>
<thead>
<tr>
<th>Load</th>
<th>Stress in lb. per sq. in.</th>
<th>Experimental</th>
<th>Theoretical</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Frame (No. 3 (Gage Line 81)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 000</td>
<td>300</td>
<td>280</td>
<td>-20</td>
<td></td>
</tr>
<tr>
<td>14 000</td>
<td>720</td>
<td>570</td>
<td>-150</td>
<td></td>
</tr>
<tr>
<td>21 000</td>
<td>760</td>
<td>850</td>
<td>+90</td>
<td></td>
</tr>
<tr>
<td>30 000</td>
<td>1460</td>
<td>1210</td>
<td>-250</td>
<td></td>
</tr>
<tr>
<td>38 000</td>
<td>2120</td>
<td>1540</td>
<td>-580</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Frame (No. 6 (Gage Line 71)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 000</td>
<td>310</td>
<td>330</td>
<td>+20</td>
<td></td>
</tr>
<tr>
<td>18 000</td>
<td>450</td>
<td>600</td>
<td>+150</td>
<td></td>
</tr>
<tr>
<td>24 000</td>
<td>900</td>
<td>800</td>
<td>-100</td>
<td></td>
</tr>
<tr>
<td>-30 000</td>
<td>1100</td>
<td>1000</td>
<td>-100</td>
<td></td>
</tr>
<tr>
<td>-36 000</td>
<td>1320</td>
<td>1200</td>
<td>-120</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Frame (No. 1 (Gage Line 69)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>120 000</td>
<td>80</td>
<td>330</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18 000</td>
<td>670</td>
<td>600</td>
<td>-70</td>
<td></td>
</tr>
<tr>
<td>24 000</td>
<td>810</td>
<td>800</td>
<td>-10</td>
<td></td>
</tr>
<tr>
<td>-30 000</td>
<td>1000</td>
<td>1000</td>
<td>-100</td>
<td></td>
</tr>
<tr>
<td>-36 000</td>
<td>1410</td>
<td>1200</td>
<td>-210</td>
<td></td>
</tr>
</tbody>
</table>
37. The Critical Point of Failure.—As has already been described, the horizontal section of the vertical member at its juncture with the horizontal member is a critical section in the type formed in frames No. 1 and 6. When the bottom ends of the vertical members are fixed the rigidity of these members is greatly increased, and the vertical section of the horizontal member above the inner column face is the critical section for failure, instead of the horizontal section of the vertical member just referred to. In a similar type of frame (as No. 2 or 4), the horizontal deflection at the level of the top of the columns is remarkably large, and accordingly the inside top corner or the section between the horizontal member and the inclined member is obviously a critical point of failure. A frame having an inclined column is generally very strong. This is due to the fact that the eccentricity of the resultant force is reduced by the inclination of the column, and therefore the moment in the column is generally small. The moment in the column is also reduced by an increase in cross section of a tie, but in this case due attention should be paid to the section at the juncture of the tie and the column, for a critical point of failure is at this joint. In frame No. 8, the bending moment in the intermediate column is rather small, even at its top. Therefore the frame is liable to be cracked by the negative moment in the horizontal member just outside the intermediate column.

The sections referred to, together with that at the center of the loaded horizontal member, are the critical points for bending moments in frame construction.
38. Deflections.—The observed deflections are shown in Table XXIX. These are also plotted in Diagrams 44 and 45. These values were checked by the theoretical formulas and the actual deflections agreed fairly with the theoretical values at low loads. The higher the load the wider the discrepancy. This is due to the fact that the moment of inertia of the cross section of the reinforced concrete members is continually reduced with an increase in load. The general nature of the load–deflection curves indicates no abrupt change in their directions. In frames No. 3 and 7, more or less sudden changes in the direction of the curve are seen at 14,000 lb. load. This is due to the crack formation on the top side of the beam produced by the negative bending moment at that load. In frames No. 1, 2 and 4 the horizontal deflections at the upper part of the columns were measured. The load–deflection curve is almost linear. The sudden change of the load–deflection curve of No. 4 is due to the crack formation in the frame.
## TABLE XXIX.

VERTICAL DEFLECTION AT THE CENTER OF SPAN.

<table>
<thead>
<tr>
<th>Load in lb.</th>
<th>Deflection in inches</th>
<th>Load in lb.</th>
<th>Deflection in inches</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Frame No. 1</strong></td>
<td></td>
<td><strong>Frame No. 6</strong></td>
<td></td>
</tr>
<tr>
<td>6 000</td>
<td>.0130</td>
<td>10 000</td>
<td>.036</td>
</tr>
<tr>
<td>12 000</td>
<td>.0270</td>
<td>18 000</td>
<td>.052</td>
</tr>
<tr>
<td>18 000</td>
<td>.0454</td>
<td>24 000</td>
<td>.070</td>
</tr>
<tr>
<td>24 000</td>
<td>.0694</td>
<td>30 000</td>
<td>.100</td>
</tr>
<tr>
<td>30 000</td>
<td>.0902</td>
<td>36 000</td>
<td>.139</td>
</tr>
<tr>
<td>36 000</td>
<td>.1273</td>
<td>40 000</td>
<td>.162</td>
</tr>
<tr>
<td>40 500</td>
<td>.2422</td>
<td><strong>Frame No. 2</strong></td>
<td></td>
</tr>
<tr>
<td>4 000</td>
<td>.020</td>
<td>14 000</td>
<td>.0155</td>
</tr>
<tr>
<td>8 000</td>
<td>.056</td>
<td>21 000</td>
<td>.0345</td>
</tr>
<tr>
<td>12 000</td>
<td>.082</td>
<td>30 000</td>
<td>.0627</td>
</tr>
<tr>
<td>18 000</td>
<td>.101</td>
<td>38 000</td>
<td>.084</td>
</tr>
<tr>
<td>24 000</td>
<td>.125</td>
<td>46 000</td>
<td>.1226</td>
</tr>
<tr>
<td>30 000</td>
<td>.149</td>
<td><strong>Ultimate</strong></td>
<td>.9310</td>
</tr>
<tr>
<td><strong>Frame No. 3</strong></td>
<td></td>
<td><strong>Frame No. 8</strong></td>
<td></td>
</tr>
<tr>
<td>7 000</td>
<td>.0070</td>
<td>15 000</td>
<td>.0045</td>
</tr>
<tr>
<td>14 000</td>
<td>.0125</td>
<td>30 000</td>
<td>.0192</td>
</tr>
<tr>
<td>21 000</td>
<td>.0305</td>
<td>45 000</td>
<td>.0358</td>
</tr>
<tr>
<td>30 000</td>
<td>.0585</td>
<td>60 000</td>
<td>.0532</td>
</tr>
<tr>
<td>38 000</td>
<td>.0845</td>
<td>75 000</td>
<td>.0822</td>
</tr>
<tr>
<td>46 000</td>
<td>.1210</td>
<td><strong>Frame No. 4</strong></td>
<td></td>
</tr>
<tr>
<td>5 000</td>
<td>.008</td>
<td>10 000</td>
<td>.018</td>
</tr>
<tr>
<td>10 000</td>
<td>.018</td>
<td>15 000</td>
<td>.059</td>
</tr>
<tr>
<td>15 000</td>
<td>.059</td>
<td>21 000</td>
<td>.118</td>
</tr>
<tr>
<td>21 000</td>
<td>.118</td>
<td>30 000</td>
<td>.168</td>
</tr>
</tbody>
</table>
Diagram No. 45.
Load-Deflection Diagrams.

Horizontal Deflection at The Top of Columns.

Frame No. 2.

Frame No. 4.

Frame No. 8.
Vertical Deflection at The Center of Middle Span.

Load in Pounds.

Deflection in Inches.
IX. CONCLUSIONS AND GENERAL COMMENTS.

From the tests and the discussions it would seem evident that among the facts brought out are the following:

1. Considering the errors involved in the measurement of the deformations performed, and in the determination of the modulus of elasticity of the concrete, as well as those due to assumptions with reference to the distribution of stresses across the section and over the gage length, the foregoing indicates a fair agreement between analyses and tests and justifies the conclusion that the formulas, given in this thesis, for the statically indeterminate stresses as they apply to concrete and reinforced concrete structures will give values for the stresses in the members well within the limit of accuracy required in design.

2. The elastic action of a frame and the manner of stress distribution along the frame axis under external loads fairly agrees with the theoretical consideration.

3. The point of inflection in the member of a frame under loads agrees with that of the theory with high degree of accuracy.

4. The important stresses of a secondary nature were limited to compressive stress in the concrete. The variation between calculated and observed stresses was frequently as much as fifty percent of the observed stress. This is due partly to the rigid connection of members and partly to the fact that the moduli of elasticity of the concrete were determined by the control cylinder test instead of the tests with actual frames.
5. The high compressive stress in the concrete in the tests may be due partly to an uneven distribution of the direct stress over the cross section of the member. It seems that when a load is applied to the beam of a frame at one side of a supporting column more stress (due to a direct force alone) is exerted along the nearest fibers than the farthest fibers of the column, as may be expected.

6. If a frame is carefully designed and well reinforced, there need be no anxiety as to the rigidity of a joint, and a perfect continuity of members has been proven by the tests.

7. No sudden failure took place in the frames tested. The increase in the deflection was very uniform, indicating for reinforced concrete frames as great reliability as for steel structures.

8. The critical load, at which the first appreciable fine crack will appear, is increased by the fixity of the column ends of a frame. This is obviously due to the increase in horizontal thrusts at column ends.

9. At sharp inside corners, high compressive stresses were developed in the concrete due to so-called curved beam action and in several cases the frame failed by the crushing of the concrete at these corners under the high loads.

10. A slight deviation of the axis of vertical members from a vertical line, that is to say, a slight "out-of-form" of the vertical columns, produces a remarkable variation in the stress distribution in the frame.

11. With frames having a thickness smaller than the width of the composing members, there was a tendency toward the bending
of the frame laterally, and the additional stresses in the reinforcing steel and concrete accompanying the local bending were large.

12. If a frame is made of concrete which is poorer on one side than on the other, there is likely to be a local bending of a member owing to the local weakness. This fact will modify the stress distribution over the section of the member. The variation from the average stress was frequently fifty percent in the test.

13. In a rigid joint, careful attention should be paid to the design of the connecting members at that joint to give an equal rigidity. If the connecting members, Fig. 56, have the same cross sections at A and B, more stress is carried at B than at A when the vertical member is free to turn at its end, and more stress is likely to be developed at A than at B when the vertical member is fixed at its end.

14. Owing to the existence of a horizontal thrust (which varies from $\frac{1P}{10}$ to $\frac{1P}{18}$ in most common cases of simple frames) at the ends of a vertical member, it is advisable to use a column which is sloped slightly toward the direction of the reaction at the end. Such arrangement will greatly reduce a bending stress in a vertical member. If this arrangement is not practicable, a slight increase in the top width of a vertical member and slight decrease in its bottom width, as shown in Fig. 57, will materially add to the rigidity of a frame.
without an increase in the amount of material used.

15. For a frame having a sloped column, it is possible to select the form of a frame in such a way that the column takes no bending stress throughout its length as is remarked in the analysis (See Case 4).

16. Due attention should be paid to the rigid joint of a tie member to insure the stiff connection with a main member. There was a marked tendency to cause a sudden breaking of such a joint with the increase of bending moment in a main member (a reference may conveniently be made to the frame No. 5 and 7.)

17. The use of a footing rigidly connected to the lower end of a vertical member is advisable, for it will materially reduce the bending moment at B and C (Shown in Fig. 58). A frame having such a footing is solvable analytically, and it approximately falls to the case which is halfway between hinged and fixed end of the vertical member provided the foundation is unyielding. A little consideration is needed to arrange proper reinforcement at A.

18. The stirrups in the frames did not come into action until diagonal cracks had formed. In this respect the tests showed results consistent with results of tests of many simple beams.
APPENDIX

LOAD–STRESS DIAGRAMS.
Frame No. 1

Steel Stress in Column No. 1

34.1

32

30.5

Steel Stress in Column No. 2

17

15

13

Unit Stress, $1'' = 20,000$ lb. per sq. in.
Frame No. 1
Concrete Stress

Unit Stress, 1" = 2000 lb per sq.in.
Frame No. 1

Stress in Steel of Top Beam

Load in Thousands of Pounds

Unit Stress, 1" = 20,000 per sq.in.
Frame No. 2
Steel Stress

Load in Thousands of Pounds

Unit Stress, $1'' = 10,000$ lb. per sq. in.
Frame No. 2
Steel Stress

Unit Stress 1" = 10,000 lb. per sq.in.
Frame № 2.
Stress in Stirrups.

Load in Thousands of Pounds.

Unit Stress, 1" = 10,000 lb./sq. in.
Frame No. 2
Concrete Stress

Top Side

Bottom Side

Load in Thousands of Pounds

Unit Stress 1" = 1,000 lb. per sq. in
Frame No. 2
Steel Stress

Column No. 1

85 86
87 83
84 85
85 89

Load in Thousands of Pounds

Unit Stress, 1" = 10,000 lb./persq.in.

Column No. 2

101 103
102 106
105 101
100 104
Frame No. 2
Concrete Stress

Column No. 2

Column No. 1

Load in Thousands of Pounds

Unit Stress, 1" = 1,000 lb. per sq.in.
Frame No. 3. Steel Stress in Column No. 1

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 3. Concrete Stress in Column No. 1.

Unit Stress, 1" = 1,000 lb. per sq.in.
Frame No. 3. Steel Stress in Top Beam.

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 3. Steel Stress in Back Side of Beam.

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 3.

Steel and Concrete Stresses in Beam.

Concrete Stress.

Steel Stress.

Unit Stress, 1" = 1,000 lb. per sq.in. for Concrete
1" = 10,000 lb. per sq.in. for Steel
Frame No. 4.

Steel Stress in Beam.

Load in Thousands of Pounds

30
20
10
0

30
20
10
0

Unit Stress. 1" = 10,000 lb. per sq.in.
Frame No. 4
Steel Stress

Load in Thousands of Pounds

Stress in Stirrups

Unit Stress, 1" = 10,000 lb. per Sq. in.
Frame No. 4 Stress in Stirrups

Stress in Inclined Beam (Bottom Side)

Load in Thousands of Pounds

Stress in Concrete

Unit Stress, 1" = 10,000 lb per sq. in for Steel.
Frame No 4 Concrete Stress.

Unit Stress, 1" = 2,000 lb per sq in.
Frame No. 5.

Stress in Column No. 1.

Unit Stress, 1" = 10,000 lb. per sq.in.

Load in Thousands of Pounds
Frame No. 5.
Stress in Column No. 1.

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 5.

Stress in Column No. 2.

Load in Thousands of Pounds:

0  40  80  120  160

Unit Stress. 1" = 10,000 lb. per sq.in.
Frame No. 5.

Stress in Column No. 2.

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 5.

Stress in Beam.

Bottom.

Top.

Unit Stress, 1" = 10,000 lb. per sq. in.
Frame No. 5.

Stress in Tie and Stirrups.

Stress in Tie.

Stress in Stirrups.

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 5.

Stress in Concrete.

Load in Thousands of Pounds

160
120
80
40
0

160
120
80
40
0

116 117
120 121
123 124

125 126 127 128 129

Unit Stress, 1" = 2,000 lb. per sq.in.
Frame No. 6. Stress in Column No. 1.

Steel Stress in Column No. 2.

Steel Stress in Top Beam.

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 6. Stress in Concrete (Columns).

Concrete Stress Around Column.

Concrete Stress in Beam.

Unit Stress, 1" = 2,000 lb. per sq.in.
Frame No. 7.

Steel Stresses.

Column No. 1.

Column No. 2.

Top Beam.

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 7.
Concrete Stress.

Column No. 1.

Column No. 1 and 2.

Top Beam

Unit Stress, 1" = 1,000 lb. per sq.in.
Frame No. 8.
Column No. 1.

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 8.
Steel Stress.
Column No. 2.

Load in Thousands of Pounds

Steel Stress in Top Beam at Bottom.
Span A.

Unit Stress, 1" = 10,000 lb. per sq.in.
Frame No. 8.
Steel Stress.
Column No. 3.

Steel Stress in Top Beam at Bottom.
Span C.

Load in Thousands of Pounds

75 159 50 60 61 62 64 65
60
30
0

67 68 69 70 72 73
75 150
60
30
0

151 147 75 77 78 80 81 149
75 150
60
30
0

Unit Stress, 1'' = 10,000 lb. per sq.in.
Frame No. 8.

Stress in Top Beam
(Middle Span)
(Tensile Stress in Front Steel)

Unit Stress, 1\" = 10,000 lb. per sq. in. in steel.
1\" = 2,000 lb. per sq. in. in concrete.
Vita.

The candidate was born in 1883, in Iwate Prefecture, Japan. He was prepared in the High School of his own Prefecture. He attended the Tohoku Imperial University, Japan, holding a scholarship during attendance, and graduated from the Civil Engineering Department, and received his degree of Bachelor of Engineering in 1905 with honors.

During 1905 and 1906 he was an assistant engineer in the designing office of the Maintenance of Way Department, Hokuriku Division, Imperial Government Railways of Japan. In 1907, he was an assistant engineer to the division superintendent of Maintenance of Way and Construction, Tokyo Division, Imperial Government Railways of Japan, and had the charge of the design of the yard and terminal improvements relating the electrification of the Yamanote line, the Suburban railways of Tokyo City. During 1908, he was connected with the double tracking and yard improvements of Tokaido Trunk line, and designed the Sekibe tunnel of 3300 feet length and several extension works of the bridge constructions and earth works for the double tracking. During 1909 and 1910, he was connected with the general works of the maintenance of way and construction, and he also designed and erected the superstructures of the Fuji Railway bridge having nine spans of two hundred foot span. The accompanying blueprint is a view of the bridge.

In 1911, he was appointed by the Imperial Government Railways of Japan and the Department of Agriculture and Commerce to study reinforced concrete in the United States of America as a Government
Student, and came to this country in January, 1912. In February, 1912, he was admitted to the Graduate School of the University of Illinois and entered upon his graduate work in the same institution.