METHOD IN NUMBER

BY

CHARLES JEFFERSON WAITS, A. B.

INDIANA UNIVERSITY

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The following paper is an attempt to present a method in number which shall be both logical and psychological. The prevailing number "fads" all claim a psychological basis and development, but each of these methods selects one of the three mental acts involved in all number ideas as the principal act and subordinates the other two. The "ratio fad" selects the relational element. The "observation method" selects the synthetic act. The "analytic method" selects the analytic act. The truth is that the three mental acts are coordinate.

The following discussion has for its basis Dewey's Psychology of Number, Brooks Philosophy of Arithmetic, Newby's Science of Number, and Sandison's Theory of the School.

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The mere fact that many objects exist or that these objects are presented to the mind through sense-perception does not account for a consciousness of number. Many objects are observed by the eye, many different sounds recognized by the ear, many sensations experienced by the sense of touch, but the idea of number may not be in consciousness. Five objects may be placed before the child in order to interest him and still the idea five be unknown to him. The perception of individual objects aids the mind in conceiving numerical ideas but the bare perception of them does not constitute number.

In order that the idea of number may arise in consciousness the objects perceived must be compared and related in a certain way. In conceiving the numerical idea in a definite group of objects, the individuals are compared as to their separate unity and also as to their collective unity. For example, in the conception of four objects as four, the mind recognizes the four distinct individuals and at the same time the one unity made up of four things. In this mental act, as in all acts of knowing - there is present the element of analysis followed by synthesis. These acts may be
greatly assisted by the choice of objects. In teaching the idea of four, the four objects used should have greater likeness than differences, since the separate unit's may be marked by special differences. If the differences are greater than the likeness, it is more difficult to think the objects as one unity. The principle, "only like units can be thought together," should be recognized in the first stages of number development.

In the process of analysis, two errors are common. One fails to analyze sufficiently. It assumes that "concrete objects before the senses must produce concrete knowledge in the mind." If five objects are before the child's eyes, he must of necessity have the idea five. This is known as the "observation method." The other error is in carrying the analysis too far. It makes analysis an end instead of a means. It follows the formula: first, the definite particular; second, the interconnections; third, the organic whole. By this method, the child is forced into a phase of thinking beyond his capacity. The attempt is made to give him definite ideas when he is capable of only vague ideas. To assume that the child must know "all that can be done with the number five" before he can think six is to disregard his natural mental development. Just as the child's concept horse is corrected by a knowledge of other animals, so is his concept five corrected and enriched by a vague knowledge of six. Again, this method attempts to teach all the process simultaneously and thus disregards one process as
Following the act of analysis in a numerical conception is the synthetic act. By this act of mind the separate individuals found by analysis are put together and held in consciousness as a unity composed of separate parts. If the child has three objects before him, he must hold in mind the three ones and see them as a unity, in order to think three.

The stress of life out of which number grows is the necessity for measurement in order to make an indefinite magnitude a definite quantity. In order to do this, the vague magnitude must be measured by a known magnitude of the same kind. For example, if the length of a room is to be determined, it must be broken up into known units of length. This is done by applying a known unit of length to the vague length and noting the times of repetition necessary to measure it, or observing the "how many" known units in the vague length. The process of measurement gives rise to the idea of number.

Number may be defined, then, as the ratio of one quantity to another of the same kind taken as a standard; or it may be defined as a unit or group of like units thought together. According to the first definition, the emphasis of attention is given to the times of repetition of the measuring unit, while the second emphasizes the...
the aggregate of the whole as made up of like known parts. The conception of measuring units and times of repetition is inseparable from number as expressing the numerical value of a quantity. The conception of ten dollars may be obtained by thinking one dollar as repeated ten times, or by thinking ten dollars as a whole of ten parts taken one time. The numerical process and the numerical value are the same in each act, the difference being only a difference of emphasis. Number as showing "how much" of a measured quantity represents the quantity as an aggregation of known parts, or units, which equal the whole, while number as "how many" represents the times of repetition of a known unit necessary to equal or measure a given whole. In every problem requiring measurement both elements are involved.

The "ratio method" centers the attention upon the "how many" in all measurements and makes it the controlling idea in all number processes. It disregards the fact that the child in his elementary number work is not naturally conscious of the ratio of measured quantity to the measuring unit. He sees ratio together with the measuring unit, the two thought together. This method attempts to make explicit from the beginning the ratio idea, and in attempting to do this forces the child to a phase of thinking beyond his capacity. He sees number as ratio only after much experience in number work.

From the nature and origin of number it is seen there are
three factors involved in the process of measurement, the whole, or unity, the unit of measurement, and the number, showing the relation of the whole to the unit of measurement. A change in the nature of these factors gives rise to the different number processes.

In the first stage of number work, the whole is vague and the unit of measurement is a known individual object or quantity. With the child's knowledge of "ones" as a basis, he proceeds to determine the number of ones in familiar groups or wholes around him. Example: the number of pupils in his row, the number of boys in his class, the number of steps in the length of the room. Continue the work until the pupil can readily count objects from one to ten. By this means he gets a vague idea of numbers from one to ten, and this vague knowledge is the basis for the succeeding step.

The work of the succeeding stage is to correct and enrich the vague ideas already obtained. This is done by a process of analysis and recombination. The pupil has been thinking three as three ones only, but by separation and recombination he is able to see three as two and one. By the process of analysis and recombination, a group of four objects of four units of measurement is seen as three and one, and as two and two. The vague ideas of numbers from one to ten may be made less vague by analyzing the whole into its elements and recombining into a new whole. In the process of addition, the whole is vague, and the parts are definite (being measured by a known individual unit). The result of the operation
must be to make the whole or sum definite. The stress of attention is placed on the whole or sum which is seen as a unity composed of known parts or addends.

In the process of subtraction, the whole, or sum, is definite and one of the parts is known, while the other part is vague. By analyzing the whole, the other part becomes definite. Subtraction differs from addition in that the stress of attention in addition is on the sum, while in subtraction it is on the parts or addends.

In each the individual thing or quantity, thought as one whole, the unit of measurement.

In finding the different combinations in the numbers from one to ten, the whole or sum was separated into only two parts but a need arises for a separation into more than two parts. For example, a boy had a certain amount of money. He spent four cents for a top, three cents for a pencil, and had five cents left, how many cents had he at first? Many practical problems should be given involving addition with three or more addends, changing from the variable addend to the constant addend. Lead the pupil by means of these to see the relation of addition and multiplication. In the problem 2 cents plus 2 cents plus 2 cents plus 2 cents plus 2 cents equals 10 cents, the pupil reaches the result by a process of addition; but after the result is reached, he must rethink ten as composed of five twos. In like manner, new meaning is put into the former conception of the numbers 4, 6, 8, 9, and 10. In multiplication
the emphasis of attention is placed upon the size of the measuring unit and its times of repetition in order to equal the whole. The unit of measurement has changed from the single unit to a group of single units. In multiplication, as in addition, a vague whole is made definite.

Division is the correlate of multiplication and is a process of continuous subtraction using the same subtrahend. Prepare the pupil for division by giving him practical problems in continuous subtraction with a variable subtrahend; as, John had 10 cents. He spent 3 cents for a pencil, 2 cents for candy, 4 cents for a top, and 1 cent for a marble, how many cents had he after each purchase? Use the same process with a constant subtrahend; as, John had 10 cents. He spent 2 cents for a pencil, 2 cents for candy, 2 cents for a top, 2 cents for marbles, and 2 cents for a ball; how many cents had he left? How many purchases did he make? In the last process fix the attention upon the times of spending 2 cents. By this means new meaning is put into the conception ten in that it can be made into five twos. The pupil should be led to see that multiplication and division differ only in the known factors. In division the whole and the measuring unit is definite, while the times of repetition is vague, and the conscious test for division is continuous subtraction with the same subtrahend.

The ideas of 1/2, 1/3, 1/4, and 1/5 are implicit in the division process with numbers to 10, and should be made explicit at this
stage of the work. The pupil should be led to see that the idea $1/2$ is not only one of two equal parts of one orange, or one apple, but one of two equal parts of any unity; as, 8 apples, 4 feet, or 6 gallons. The fractional idea represents the ratio of the unit of measurement to the measured whole. What before was implicit in all the number processes has become explicit.

The conscious recognition of the ratio of the unit of measurement to the measured whole gives rise to the phase of division called partition. It differs from the other elementary processes in that the unit of measurement is unknown. Before a known quantity can be separated into equal parts it must have a unit of measurement. For example, eight apples are divided equally among four pupils; how many apples does each pupil receive? The terms given are eight apples and four pupils. In order to perform the act of division, each pupil must be given one apple, and the process repeated until the apples are divided or the number left is smaller than the number of pupils. By performing the process of division, the pupil soon realizes the fact that the measuring unit is not four boys but four apples, and that the times of repetition of the measuring unit is the number of apples given to each boy. He sees that four apples taken twice is the same as two apples taken four times. If the pupil thinks the division process, this phase of the work will not be confusing to him. In the above problem, he read-
illy sees that it requires four apples to give one to each pupil and that each pupil will receive as many apples as there are four apples in eight apples.

The pupil advances from the elementary processes involved in numbers from one to ten to the same processes involved in numbers from ten to twenty. In advancing from two to ten he found a growing difficulty in thinking the individuals as a whole, and more time was required to count them. A need is felt for "economy of effort," which gives rise to grouping of objects into groups of ten each. The ten objects composing the group are thought of as one unity. Eleven is seen as one group of ten objects and one object; eleven cents is seen as one dime and one cent; sixteen, as one ten group and six objects; twenty, as two ten groups. The different combinations of addition and subtraction can be easily seen by taking objects from the group of ten and putting them with the separate individuals. After twenty is reached, the advance to one hundred is only a repetition of the preceding process. The combinations in multiplication and division should follow the same principles as set forth in the work with numbers below ten.

The numbers from 20 to 100 should be thought of as composed of two groups of units, the ten unit group and the single unit. Give exercise in finding the sum of 2 ten dollars, 6 dollars, and 7 dollars; $35 and $2; $45 and $7; $65 and $12; $78 and $2; $84 and $7. Each problem should be analyzed into its unit groups.
The advance is made from problems involving a single process to those involving two or more processes with units of measurement of the same kind as the quantities measured. Example: (1) James had 3 cts. and his father gave him 6 cts., after spending 10 cts., how many cents had he left? (2) Three groups of pennies containing 8 pennies each were made into four equal groups, how many pennies in each group? After much drill in miscellaneous exercises illustrated above, the unit of measurement of a different kind from the quantities measured may be introduced. Example, A farmer exchanged 4 dozen of eggs for coffee. A common unit of measurement different from either the eggs or the coffee is necessary. This common unit is the value of a limited amount of each. If the eggs are valued at 10 cents a dozen, and the coffee at 20 cents a pound, the eggs and the coffee are related by means of the common unit, cents. The equivalent of one can be expressed in terms of the other.

Many practical problems involving this unit of measurement should be given the pupil, since it frees him from his limitation to particular units of measurement and gives him the key to the business world.

The logical and psychological order of development is from idea to name, and from name to symbol. The symbol three represents three units of measurement whether of individual things or groups of things; as, three inches, three two inches, or three tens. The pupil thinks three tens and five units as one group of three.
tens and one group of five individuals; and a shorter expression would be 35, the mere position of the symbol indicating the character of the unit. Drill should be given in the shorter expression of 4 tens 7 units; 6 tens 3 units; 9 tens 3 units; 1 hundred 3 tens 5 units; 2 hundred dollars, 4 ten dollars 6 dollars. If the idea of grouping is made definite, the formal notation will be easily grasped.

The formal process of addition and subtraction emphasizes the principle already observed, that only like units or groups of units can be thought together. 3 tens 5 units and 4 tens 3 units are 7 tens 8 units. This fact in regard to integers is the underlying principle in addition and subtraction of fractions.

In the formal process of multiplication, the pupil should have clearly in mind what each symbol in the multiplicand and multiplier represents and then think the process of multiplication. For example, if one barrel of flour weighs 106 pounds find the weight of 7 barrels. 106 pounds is 1 hundred 3 tens 6 units. 1 hundred 3 tens 6 units is multiplied by 7. 7 times the six units are 42 units, or 4 tens and 2 units. 7 times 3 tens are 63 tens; 63 tens and 4 tens are 67 tens, or 6 hundreds and 7 tens. 7 times the 1 hundred is 7 hundreds; 7 hundreds and 6 hundreds are 13 hundreds, or 1 thousand 3 hundreds. The entire weight, then, is one thousand 3 hundreds, 7 tens 2 units, or 1372 pounds. Again, if one bar-
rel of flour weigh 106 pounds, find the weight of 24 barrels. The first step in the process is the same as the preceding problem, but two ten times 106 pounds gives the weight in the 10 unit group, instead of the single unit. That result would be 302 ten pounds which, combined with 4 times 106 pounds, or 784 pounds, is the required weight.

In the process of division, the same principles apply. Example, if 4 acres of land cost 352 dollars, what is the cost of one acre? It is required to make 352 dollars into 4 equal parts.

$\frac{1}{4}$ of 3 hundreds is 3 hundreds. $\frac{1}{4}$ of 5 tens is 1 ten, and 1 ten undivided. The 1 ten and 2 units are still undivided, and these thought together are 12 units. 12 units divided into 4 equal parts gives 3 units. The cost of 1 acre, then, is 2 hundreds 1 ten 3 units dollars, or 213 dollars. Again, 43 acres of land are sold for 1161 dollars; find the cost of one acre. The cost of one acre 1 thousand 1 hundred 6 tens 1 unit divided by 43. The 1 thousand 1 hundred 6 tens is 116 tens, which divided by 43 gives 2 tens, with 30 tens and 1 unit not divided. 30 tens and 1 unit thought together is 301 units, which divided by 43 is 7 units. The cost of one acre is found to be 2 tens and 7 units dollars, or 27 dollars. The fact that the formal process is also a thought process must not be disregarded in any phase of number work.

If the work in multiplication and division is clearly seen as a process of measurement, the process in greatest common measur-
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The relation of greatest common divisor to simple division can be readily shown by a few concrete problems. Require the pupil to find what units of measurement greater than one inch will exactly measure 12 inches; 10 inches; both ten and twelve inches. What units of measurement will exactly measure 12 inches and 18 inches? Give the pupil many problems requiring the greatest common measure and lead him to see that the greatest common measure of two numbers is the product of their common prime factors used in continuous multiplication; and that it always exactly measures the sum of the two numbers, their difference, or any number of times either of the numbers.

In the process of finding the greatest common measure, the pupil is given the quantities to be measured and is required to find the greatest common measure. In least common multiple, he is given the units of measurement, and is required to find the least quantity that can be exactly measured by each of them. The idea of least common multiple should be made clear by concrete problems. Example, What quantities between 3 inches and 25 inches will a 3 inch measure exactly measure? A 4 inch measure? What quantities between 3 inches and 25 inches will either a 3 inch or a 4 inch measure exactly measure? The least quantity? Separate the multiple and the factors into their prime factors and compare the results. Continue the examples until the pupil sees the fact
that the least common multiple always contains only the prime fac-
tors of the measuring quantities, no factor being repeated in the 
multiple more times than it occurs in one of the measuring quanti-
ties.

In his number work so far the pupil has been dealing with the integral unit or groups of integral units of measurement, but he should see that the units of measurement may be fractional units or groups of like fractional units. Instead of the one-foot, or three-foot unit of measurement, one-half foot or one third of a foot may be used as well. The expression 3/4 of a foot or 7/5 feet is no more mysterious than 3 feet or 7 pounds, each being 3 or 7 times their respective units of measurement. The fractional notation of three fourths (3/4) or (3/4) is in perfect harmony with the idea.

In the process of simple addition only quantities having like units of measurement could be thought together; in fractional addition the units of measurement must be alike and equal. As preparation for fractions the pupil should be required to think 1 quart and 1 pint in terms of the smallest unit of measurement. Also, 12 feet and 6 inches; 3 feet and 4 inches; 1 yard and 2 feet. Continue this exercise until the process can be easily thought. Re-think the same in terms of the largest unit of measurement. Re-
quire the pupil to think 1 ft. and 1/4 ft. in terms of the 1/4 unit;
Two and one half feet in terms of the \( \frac{1}{2} \) ft; \( \frac{3}{4} \) ft in terms of \( \frac{1}{4} \) ft. By concrete problems, make clear the idea of the fractional unit of measurement and its relation to the integral unit.

In simple addition it was seen that only quantities having the same unit of measurement could be thought together, so in fractional addition the quantities must have the same unit of measurement. \( \frac{1}{2} \) ft. and \( \frac{1}{4} \) ft. must be thought in terms of the \( \frac{1}{4} \) ft. in order to be thought together. Again, in order to think one half \( \frac{1}{2} \) yard and \( \frac{1}{3} \) yard into one sum, both quantities must be measured by a common unit of measurement. This common unit may be easily found by drawing each length, and indicate the subdivisions, feet and inches. The largest common unit of measurement will be discovered and the number of these in the \( \frac{1}{2} \) yard and \( \frac{1}{3} \) yard. The pupil should be led to see that the largest common unit of measurement of two or more fractions is a fractional unit which has for its denominator the least common multiple of the denominators of the given fractions.

In multiplication and division of fractions, the ratio idea involved in all number work becomes explicit. A conscious comparison is made between the fractional unit and the primary unit, i.e. the fractional unit and the unit of the fraction. In multiplication of fractions, the comparison is made between the fractional unit of the multiplier and its primary unit; in division of frac-
tions, between the fractional unit of the divisor and its primary unit.

The first phase of multiplication of fractions, a fraction by a whole number, presents no difficulty. Its close relation to addition of fractions and simple multiplication makes the process self-evident. The pupil will experience no difficulty in seeing that $\frac{3}{4} \times 3 = \frac{9}{4}$.

The second phase, the multiplication of a whole number by a fraction, should be presented by means of concrete problems in which the process involved may be easily thought. For example, if one yard of cloth costs 60 cents, how much will $\frac{1}{2}$ yard cost? $\frac{1}{4}$ yard, $\frac{1}{3}$ yard, $\frac{2}{3}$ yard, $\frac{3}{4}$ yard, three and five sixths yards? In each problem the pupil finds the cost of the fractional unit of measurement from its relation, or ratio, to the primary unit and multiplies this cost by the number of fractional units given.

The method to be followed when both multiplicand and multiplier are fractional in form involves nothing different from the preceding phase. In each phase the basis is the relation of the fractional unit of the multiplier to its primary unit. The pupil should be led to see in the problems given the shorter or formal process of multiplication of fractions.

Since division is the inverse of multiplication, the same method applies to division of fractions as to multiplication. In
dividing a whole number by a fraction, or a fraction by a fraction, the thought process is the advance from the given number of fractional units of the divisor to the fractional unit, and thence to the primary unit. Example, if 3/4 of a yard of silk cost $6/5 how much will one yard cost? Since 3/4 of a yard costs $6/5, 1/4 of a yard will cost 1/3 of $6/5, or $2/5: 4/4 of a yard, or one yard, will cost 4 times $2/5, or $8/5. By comparing the result with the given terms, the formal process of division of fractions will be discovered by the pupil.

In order to emphasize the facts already developed and to give skill in thinking the fractional process, many miscellaneous problems should be given involving two or more phases of the work. For example, if 3/4 of a yard of silk cost $7/8, how much will 5/6 of a yard cost?

Since the Decimal fraction is only a particular kind of Common fraction, a knowledge of the latter must be the basis for the mastery of the former. A Decimal fraction is a fraction which has for its fractional unit one-tenth, or some power of one-tenth; as 1/10, 1/100, 7/1000. The new idea in the decimal fraction is its notation. The basis for this new idea is the knowledge which the pupil already has of the number scale as applied to integers.

The relation between units of successive places in the representative scale is seen to be as one to ten. In passing from right to left, each successive unit is ten times the value of its preceding unit. Then, in passing from left to right, each successive unit is one-tenth of the value of the unit preceding it. According to this law of the representative scale, the fraction one-tenth would
be written by placing 1 in the first place to the right of units place, one-tenth of one-tenth or one-hundredth would be written in the second place to the right of units place, or the first place to the right of tenths. Each place is named from its fractional unit. The notation of decimals is only an extension of the notation with which the pupil is already familiar. The reading of decimals and integers follow the same general law. 3 tens and 4 units are read 34 units; 3 tenths and 4 hundredths are 34 hundredths, in each the smallest unit gives name to the expression.

The operations of addition and subtraction of decimals are not different from the same operations with integers, but multiplication and division will require special consideration. Since decimal fractions are particular common fractions, the basis for multiplication of decimal fractions should be the knowledge which the pupil has of multiplication of common fractions, together with his knowledge of decimal notation. Require the pupil to multiply a decimal fraction in common fraction form by an integer; as 27/10 by 5; 324/100 by 26; 7/1000 by 439, and express the product in each case by the decimal notation. Find the relation in each between the number of decimal places in the product to the number of ciphers in the denominator of the product when expressed in common fraction form. Again, let it be required to multiply 36/10 by 4/10; 15/10 by 48/10; 237/100 by 6/10; 4338/100 by 23/100; 9/1000 by 7/100 and express the result in each problem by the decimal notation. Find the relation between the number of decimal places in the product to the number of ciphers in the denominators of the multiplicand and multiplier. State the general truth.

As in multiplication of decimals the basis is multiplication
of common fractions, so in division of decimals the basis is division of common fractions. The law for locating the decimal point in the quotient should be developed from division of common fractions similar to the development of the law for "pointing off" in multiplication of decimals.

Since the decimal fraction is only a particular common fraction,—a common fraction having a denominator of ten or some power of ten—the change from a common to a decimal fraction, or vice versa, will offer no difficulties. If the denominator of the common fraction is not a power of ten, perform such operation on both numerator and denominator as will give a decimal denominator, and write the result by means of the decimal notation.

Percentage is a process of computation involving a particular decimal fraction, the hundredth. It is no new mysterious process but an old process assuming particular form. If the pupil has mastered decimal notation and the fundamental processes involving decimals, the operations in percentage present nothing new. He finds here only new names for old ideas. He is already familiar with the terms multiplicand, multiplier and product, but he now finds that, the multiplicand, multiplier, and product are called base, rate per cent, and percentage respectively, if the multiplier takes the form of hundredths. He has already learned that the processes of multiplication and division involve three terms, any two of which being given, the third may be easily found. In the applications of percentage he finds this same fact. His percentage problem gives him two factors to find the product, or a product and one factor to find the other. If multiplication and division of decimals are mastered, the processes of percentage will offer no difficulties. The hardest
work in percentage applications is the interpretation of the problem. The pupil fails to see in the new terms their old meaning. In Profit and Loss, Commission, Brokerage, Stocks and Bonds, Insurance, Duties, and Takes the association of the new terms with their corresponding terms in multiplication and division should receive special emphasis.

The element of time gives rise to a special class of percentage applications; as, Interest, True and Bank Discount, etc. The point to be emphasized in this work is the necessity in certain transactions for this new element, and its influence in computation. If A agrees to pay B each year six hundredths of the amount of money loaned him by B for the use of it, the influence of the time element is readily seen. When the pupil sees this influence, his work is only a repetition of the preceding work in percentage.

The subject of Evolution in arithmetic is doubtless more mysterious to the average seventh or eighth grade pupil than any other phase of number. This is due, I think, to the fact that it is usually presented in an isolated and mechanical form. The relation is not shown to the former processes. Evolution is the inverse of Involution, and Involution is a particular continuous multiplication. For example, \(2 \times 3 \times 5 \times 7\) is a problem in continuous multiplication, or "Composition," and \(3 \times 3 \times 3 \times 3\) is a problem in continuous multiplication using the same multiplier, or Involution. The square, or second power of a number is a problem in multiplication in which the multiplicand and multiplier are the same number.

Evolution should be given an arithmetical instead of geometrical basis. In order to do this and to show its relation to Involution, require the pupil to square numbers of one, two, three, four, and five places, and compare the number of places in the product, or
power, to the number of places in the factor. Lead the pupil to find the law and state it. Again, require him to square numbers of two places and keep the partial products separated. For example, $34 \times 34 = 1200 + 1200 + 900$, which is the units squared plus twice the product of the tens and units plus the tens squared. The same process may be extended to numbers of three or more places or to decimals. After the principle is found and formulated the pupil is prepared to view evolution as the inverse process.

Let it be required to find the square root of 7396. The pupil has already learned that the square of a number of two terms gives three or four places in its square. Since the given number has four terms, it must have two places, tens and units, in its square root. He has also learned that the square of a number consisting of tens and units is composed of the square of the tens plus twice the product of the tens by the units plus the square of the units. Since the square of tens is hundreds, the hundreds of the given number contain the square of the tens of the root. The hundreds of the number are 73. The greatest square in 73 hundreds is 64 hundreds, the sq. root of which is 8 tens. 64 hundreds taken from the given number leaves 996. This remainder contains a product of which twice the tens of the root is one factor, and the units of the root the other. Since tens multiplied by units give tens, the 99 tens of the remainder must contain this product. By dividing 99 tens by 16 tens the supposed units of the root are found to be 6. 16 tens multiplied by six units, taken from the 996 leaves 36 units for the remaining part of the given number. But this remainder must contain the square of the units of the root. The square of 6 equals 36, which taken from the remaining part of the number leaves nothing. The square root of 7396 is found to
be 86. The method of procedure is the inverse of that for finding the second power.

The method for finding the cube root is similar to that for square root. Let it be required to find the third power, or cube of 34. This is found by multiplying the square of 34 by 34. It has already been learned that the square of 34 is \(30^2 + 2(30 \times 4) + 4^2\).

The square multiplied by \((30+4)\) gives as partial products \(30^3 + 2 (30^2 \times 4) + (30 \times 4^2) + 2 (30 \times 4^2) + 4^3\).

Collecting like terms, the result is \(30^3 + 3 (30^2 \times 4) + 3 (30 \times 4^2) + 4^3\). Thus the cube of any number consisting of tens and units may be seen to be composed of the cube of the tens plus three times the square of the tens multiplied by the units plus three times the tens multiplied by the square of the units plus the cube of the units.

By observing the cubes of numbers of one, two, three, four, or more places it is seen that the cube of units may consist of one, two, or three places; cube of tens, four, five or six; cube of hundreds seven, eight, or nine terms; etc.

Example: Find the cube root of 593704.

Since there are six terms in the power, there must be two terms in its cube root, tens and units.

The cube of a number consisting of tens and units is composed of the cube of the tens plus three times the square of the tens multiplied by the units plus three times the tens multiplied by the square of the units plus the cube of the units. The cube of tens is thousands, then the thousands of the power must contain the cube of the tens of the root. The greatest cube number of thousands in 593\(\times\) is 512 thousands, the cube root of which is 8 tens. 512 thousands taken from the power leaves 80704. This remainder contains a product
one factor of which is three times the square of the tens and the other the units of the root. 3 times the square of 8 tens is 192 hundreds. Since the product of hundreds by units is hundreds, the hundreds of the remaining part of the power contain the product of 192 hundreds by the units of the root. 807 hundreds divided by 192 hundreds equals 4, the supposed units of the root. 4 times 192 hundreds taken from 80704 leaves 3904. This remainder contains the product of 3 times the tens by the square of the units of the root.

24 tens by 16 = 384 tens. This taken from the remaining part of the power leaves 64. This remainder must contain the cube of the units of the root. The cube of 4 is 64, which taken from the remaining part of the power leaves nothing. Thus the cube root of 592704 is found to be 84.

The principle for finding the square root or cube root of a number may be extended to higher roots; as, the fourth root or fifth root.

In conclusion, the pupil should rethink the number processes and see the subject as an organized whole. The synthetic processes of Addition, Multiplication, Composition, (or Multiples), and Involution all have their root in addition. Each process is a particular phase of the one just preceding it. The analytic processes of Subtraction, Division, Disposition, (or Factoring), and Evolution are related as are the synthetic processes. Each phase of the analytic processes finds its correlate in the synthetic.