THESIS
SYMBOLICAL REPRESENTATION
OF
INVARIANTS AND COVARIANTS
BY
CHARLES W. LEIGH
FOR THE DEGREE OF
B.S.
IN THE COLLEGE OF SCIENCE
UNIVERSITY OF ILLINOIS

Affirmed May 29, 1897.
E.J. Townsend
Associate Professor of Mathematics
1897.
BIBLIOGRAPHY.

American Journal of Mathematics, Vol. XIV.

Bruno's Binäre Formen.

Cayley's Collected Papers, Vol. I.


Gordan's Invariantentheorie.

Salmon's Modern Higher Algebra.
TABLE OF CONTENTS.

Chapter I.

History of invariants and covariants, .................................................. I
Cayley's symbolic method, ......................................................................... 3
General form of the symbol for binary forms, ........................................... II
Application to functions of the second order, .......................................... I4
Simultaneous invariants of the second order, .......................................... I6
Invariants of the second order, ................................................................. I7
Covariants of the second order, ............................................................... I8
Order and degree, .................................................................................... 20
Invariants of the third order, ................................................................. 22
Covariants of the third order, ................................................................. 24
Ternary forms, ....................................................................................... 25

Chapter II.

Symbolic notation of Aronhold, ............................................................... 27
Ternary quadratic quantics, ....................................................................... 30
Simultaneous invariants, ......................................................................... 36
Contravariants, ........................................................................................ 42
Invariants of the ternary cubic, ............................................................... 44
Mixed concomitants, ................................................................................ 50
Covariants, .............................................................................................. 53
Chapter III.

Clebsch's Symbol, .............................................. 54

Binary forms

   Invariants of binary forms, .................................. 54
   Covariants of binary forms, .................................. 61

Ternary forms

   Invariants of ternary forms, .................................. 63
   Covariants of ternary forms, .................................. 69

Certain identities, ............................................. 69

Chapter IV.

Application of the symbols

Polars, ............................................................. 70

Differential method, ............................................ 71

Binomial method, ............................................... 73

Transvection, .................................................... 74

Transvectant of a product of forms, .................................. 75

Transvectant of the polar, ........................................ 77

Chapter V.

Comparison of the symbols, ...................................... 79
Chapter I.

HISTORY OF INVARIANTS AND COVARIANTS.

The first development of the important mathematical principle "Invariance", seems to have been made by La Grange in 1773. In 1801, Gauss took up the general theory of linear transformation, and more particularly established the invariance of some discriminants. Prof. Boole, in a noted paper some time after, showed that all discriminants possessed this property, and he gave a method for deducing other functions of the same kind.

This paper led Prof. Cayley to make an investigation of the functions of the coefficients that possess this invariant property. He found that this property was not peculiar to discriminants, but to certain other functions which he called "Hyperdeterminants." Among the first invariants distinct from discriminants that were thus brought to light were the quadrinvariants of binary forms, and in particular the invariant $S$ of a quartic. These discoveries establish Cayley as the founder of modern geometry. In
1846, he published his symbolical method for finding invariants, the discussion of which is one of the objects of this paper.

The discovery of a principle of so great importance, and a complete novelty, soon attracted the attention of many workers. Mr. Boole discovered the other invariant $T$ of the quartic, and the expression of the discriminant in terms of $S$ and $T$. It is a fact worth noticing that the functions $S$ and $T$ had been used by Eisenstein, but without a knowledge of their invariant properties. Mr. Boole also advanced the principle that in a binary quantic, the operative symbols $\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$ may be substituted for $X$ and $Y$. The principle was extended to quartics in general by Sylvester, to whom is to be ascribed the general statement of the theory of contravariants. To Mr. Boole also belongs the principle that invariants of emanants are covariants of the quartic, which was again generalized by Sylvester.

So that we see while many of the principles had been used in special cases, yet it was Cayley and Sylvester who generalized the principles, and laid the foundation for the theory of invariants and covariants. Indeed the name invariant and much of the nomenclature is due to Sylvester. He also developed the theory of combinants. Salmon, Hesse, and Hermite made substantial additions to the theory. Aronhold was the first to devise a method, for the calculation of invariants symbolically, which has become the favorite method among German workers. In its origin, it is nearly the same as the one devised by Cayley, but the subsequent development, due largely to Clebsch and Gordon, run on lines entire
ly different.

One of the problems that greatly interested Cayley was the determination of the complete system of irreducible invariants of a binary form. In his second memoirs was published an accurate determination of the number of invariants for forms of order 2, 3, 4, 5, and 6, and covariants of order 2, 3, 4. But in regard to the number for forms of higher order, he came to the erroneous conclusion that the respective numbers were infinite. The error was not corrected until 1868, when Prof. Gordon showed that the complete system for a binary quantic of any order contains only a limited number. Prof. Cayley at once returned to the question, and having found his source of error, made further valuable developments in the light of Gordon's result.

Cayley's Symbolic Method.

I

The first appearance of Cayley's symbolical method for the formation of invariants and covariants that I have been able to discover, was in his collected mathematical papers, Vol. I. The question was proposed to find the derivatives of any number of functions which preserve their form after linear transformation. Here the term derivative has the same significance as we are accustomed to assign it at present. These functions Cayley called "Hyperdeterminants," which were afterwards known as invariants and covariants. In the discussion of determining the irreducible invariants, and their relations to the reducible ones, he
introduces his symbol, which we will now develop.

2.

Let us take P series of m variables each

\[ X_1, Y_1, \ldots, X_{l}, Y_{\ell}, \ldots, \ldots, X_{r}, Y_{r}, \ldots, \]

where \( P \geq m \).

Also \( P' \) series of \( m' \) variables each

\[ X'_1, Y'_1, \ldots, X'_2, Y'_2, \ldots, \ldots, X'_{r}, Y'_{r}, \ldots, \]

and \( P' \geq m' \).

Let the variables cogredient to \( X, Y, \ldots, X', Y', \ldots \) be \( \bar{X}, \bar{Y}, \ldots, \bar{X'}, \bar{Y'}, \ldots \). They will then be connected by the linear relation

\[
\begin{align*}
X &= \lambda_1 X + \mu_1 Y + \ldots \quad X' = \nu_1 X' + \rho_1 Y' + \ldots \\
Y &= \lambda_2 X + \mu_2 Y + \ldots \quad Y' = \nu_2 X' + \rho_2 Y' + \ldots
\end{align*}
\]

(1).

where \( X, Y, \ldots \) refer to the series \( P \), and \( X', Y', \ldots \) to the series \( P' \).

The coefficients remain the same in all the systems. For the sake of brevity, we will represent the partial differential coefficients of a function with respect to the variables by the symbol

\[
(2) \quad \frac{\partial}{\partial X} \quad \frac{\partial}{\partial Y} \ldots \ldots = \frac{\partial}{\partial X}, \quad \eta = -\frac{\partial}{\partial Y} \ldots \\
\frac{\partial}{\partial X} \quad \frac{\partial}{\partial Y} \ldots \ldots
\]
Now let $F$ be any function of the above series of variables that we wish to differentiate. Since $X,Y, ..., X',Y'$ ... enter $F$ through $X,Y, ..., X,Y'$ ... we will have for the derivative symbols of $F$ with respect to $X,Y,...$

$$\bar{\bar{g}} = \frac{\partial F}{\partial X} = \frac{\partial F}{\partial X} \cdot \frac{\partial X}{\partial X} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial X} + \ldots$$

$$\bar{\bar{h}} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial X} \cdot \frac{\partial X}{\partial Y} - \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial Y} + \ldots$$

But from equations (1)

$$\frac{\partial X}{\partial x} = \lambda_1, \frac{\partial Y}{\partial x} = \lambda_2 \ldots \frac{\partial X}{\partial y} = \mu_1, \frac{\partial Y}{\partial y} = \mu_2 \ldots$$

and (2)

$$\frac{\partial F}{\partial X} = \bar{\bar{g}}, \quad \frac{\partial F}{\partial Y} = \bar{\bar{h}} \ldots \frac{\partial F}{\partial X} = \bar{\bar{g}}' \ldots$$

Making these substitutions in the equations for $\bar{\bar{g}}, \bar{\bar{h}}$, there results

(3). $\bar{\bar{g}} = \lambda_1 \bar{\bar{g}} + \lambda_2 \bar{\bar{h}} + \ldots$

$$\bar{\bar{h}} = \mu_1 \bar{\bar{g}} + \mu_2 \bar{\bar{h}} + \ldots$$

Again since $X,Y', ..., enter F through X,Y'$ ....

$$\bar{\bar{g}}' = \frac{\partial F}{\partial X} = \frac{\partial F}{\partial X} \cdot \frac{\partial X'}{\partial X} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y'}{\partial X} + \ldots$$

$$\bar{\bar{h}} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial X} \cdot \frac{\partial X'}{\partial Y} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y'}{\partial Y} + \ldots$$

But $\frac{\partial F}{\partial X} = \bar{\bar{g}}', \frac{\partial F}{\partial Y} = \bar{\bar{h}}', \frac{\partial X'}{\partial X} = V_1, \frac{\partial Y'}{\partial X} = V_2 \ldots$
Therefore

\( (3'). \quad \bar{\xi}' = \xi_1 \xi' + \xi_2 \eta' + \ldots \)

\( \bar{\eta}' = \eta_1 \xi' + \eta_2 \eta' + \ldots \)

Equations (3) and (3') easily show the contragredient relation of \( \xi, \eta \ldots \) to \( X, Y \ldots \)

We will represent symbolically the matrices of the partial differential coefficients by \( P \) and \( Q \), or

\[
\begin{pmatrix}
\xi_1 & \xi_2 & \ldots & \xi_p \\
\eta_1 & \eta_2 & \ldots & \eta_p \\
\vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}

(4). \quad P = 
\begin{pmatrix}
\xi_1' & \xi_2' & \ldots & \xi_p' \\
\eta_1' & \eta_2' & \ldots & \eta_p' \\
\vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}

Q =

P contains the partial differential coefficients of \( F \) with respect to the series \( X, Y, \ldots \), while \( Q \) contains those with respect to the series \( P \). The symbols represent the sum of the series of determinants in the first of \( m \) rows and columns, and in the second of \( m' \).

This expression will be true since any matrix can be expanded into a sum of determinants, and \( P \) can never be less than \( m \).

As a special case \( p = m \) we will have but one determinant. \( Q \) has also a similar interpretation for the special case, when \( p' = m' \).

Now let us put

\[
\begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \ldots \\
\mu_1 & \mu_2 & \mu_3 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}

(5). \quad R = 
\begin{pmatrix}
V_1 & V_2 & V_3 & \ldots \\
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}

R =
The $R$ and $R'$ will not be matrices as $P$ and $Q$, as one might suspect at the first thought. In our linear relations from (I), we form our equations of transformation by the introduction of but $m$ new variables, since there are but $m$ in any group of the $p$'s. Then for the $m$ old variables there will correspond $m$ linear relations each containing $m$ terms. Then when we form the determinant of the parameters we have one of $m$ rows and columns, or $m'$ rows and columns according to series of variables we use.

We will now multiply the matrix $P$ by the determinant $R$. By the rules for the multiplication of determinants, we can multiply the rows of one into the successive columns and there results an expression which we will designate $P'$, or

$$
P' = \begin{vmatrix}
\left( \lambda_1 \xi_1 + \lambda_2 \eta_1 + \lambda_3 \xi_1 + \cdots \right) \left( u_{11} \xi_1 + u_{21} \eta_1 + \cdots \right) \\
\left( \lambda_1 \xi_2 + \lambda_2 \eta_2 + \lambda_3 \xi_2 + \cdots \right) \left( u_{12} \xi_2 + u_{22} \eta_2 + \cdots \right) \\
\vdots \\
\left( \lambda_1 \xi_p + \lambda_2 \eta_p + \lambda_3 \xi_p + \cdots \right) \left( u_{1p} \xi_p + u_{2p} \eta_p + \cdots \right)
\end{vmatrix} = RP
$$

Multiplying the rows of $R$ into the columns of $Q$, and calling the result $Q'$, we have

$$
Q' = \begin{vmatrix}
\left( v_1 \xi'_1 + v_2 \eta'_1 + \cdots \right) \left( \rho_1 \xi'_1 + \rho_2 \eta'_1 + \cdots \right) \\
\left( v_1 \xi'_2 + v_2 \eta'_2 + \cdots \right) \left( \rho_1 \xi'_2 + \rho_2 \eta'_2 + \cdots \right) \\
\vdots \\
\left( v_1 \xi'_p + v_2 \eta'_p + \cdots \right) \left( \rho_1 \xi'_p + \rho_2 \eta'_p + \cdots \right)
\end{vmatrix} = RQ
$$
Then if

\[ E = F_\eta \left( P^\sigma, Q^\sigma \right) \]

is a rational homogeneous function of the order \( \sigma \) of the quantities in the series \( P \), and \( \sigma' \) of those in \( Q \), by the above relation

\( (5') \). \[ E' = R^\nu R'^\nu E \]

This relation can be easily seen if we raise the expressions for \( P' \) and \( Q' \) to the \( V \) and \( V' \) th powers respectively and then form the function corresponding to \( E \). The terms \( R^\nu R'^\nu \) occur in form \((5')\), since in forming the function \( E \), \( R^\nu R'^\nu \) is found in every term and can be divided out leaving the desired relation. Now since \((5')\) contains nothing but symbols for the partial differential coefficients, we can define it as a symbol of operation, or in Cayley language "A symbol of hyperdeterminant derivation". Now if \( U \) be any homogeneous function of the variables \( X, Y, \ldots \), which is transformed by the linear substitution into \( U' \), applying the operator \((5')\) to it we have

\[ E' U' = R^\nu R'^\nu E \cdot U. \]

3.

For the sake of clearness let \( A, B, C, \ldots \) represent the different quantities of \( P \), and \( A', B', C', \ldots \) those of \( Q \), or

\[ P = (A + B + C + \ldots) \]
\[ Q = (A' + B' + C' + \ldots) \]

then

\[ E = F_\eta \left( (A + B + C + \ldots)^\nu, (A' + B' + C' + \ldots)^{\nu'} \right) \]
Since $E$ is homogenous in the variables and also the quantities $A, B, \ldots A', E', \ldots$, in the expansion we must have all terms of the type

$$A^\alpha B^\beta C^\nu \ldots A'^{\alpha'} B'^{\beta'} C'^{\nu'} \ldots$$

where

$$\alpha + \beta + \nu + \cdots = \nu$$

$$\alpha' + \beta' + \nu' + \cdots = \nu'$$

then

$$E = \Sigma (A^\alpha B^\beta C^\nu \ldots A'^{\alpha'} B'^{\beta'} C'^{\nu'} \ldots )$$

suppose $U$ of the form

$$U = \Theta 0$$

where $\Theta$ and $0$ are functions of the form

$$F(X, Y, \ldots, X', Y', \ldots)$$

Now it may be supposed that several sets of the variables $X, Y, \ldots X', Y', \ldots$, become identical after the differentiation, so that the functions themselves become absolutely identical. We can then express the general symbol as

$$(6). \quad E \cdot U = \Sigma A^\alpha B^\beta C^\nu \ldots A'^{\alpha'} B'^{\beta'} C'^{\nu'} \ldots \Theta 0.$$  

If after differentiation all variables of any one set become equal to all the others of the $p$ sets, and likewise for the $p$ sets, then $E \cdot U$ refers to a function of but two sets of variables

$$F(X, Y, \ldots X', Y', \ldots)$$

All the preceding has been a discussion of the general theory. Some parts of this may seem hard to understand at first, as no problems have been introduced to illustrate the various steps. Cayley's method of development is to always give the general theory first, and specialize afterwards. So that from this point on problems will be solved to show the application of the
symbol.

For the first special development, we will suppose the \( p' \) series of variables to be neglected, and the \( m \) variables of each set of the \( p \) series to become equal to two. We will then have binary forms under consideration. The functions \( \mathbf{v} \) become \( V_1, V_2, \ldots, V_p \), which are functions of \( X_1Y_1, X_2Y_2, X_3Y_3, \ldots \) respectively. \( M \) then becomes equal to two, and \( P \) is a sum of determinants of the second order, or

\[
P = \left| \begin{array}{cc}
\xi_1 & \xi_2 \\
\eta_1 & \eta_2
\end{array} \right| - \left| \begin{array}{cc}
\xi_3 & \xi_4 \\
\eta_3 & \eta_4
\end{array} \right| + \cdots - \left| \begin{array}{cc}
\xi_p & \xi_{p+1} \\
\eta_p & \eta_{p+1}
\end{array} \right| = \cdots = A + B + C + \cdots
\]

\[
\frac{12}{13} - \frac{13}{14} + \frac{14}{15} + \cdots \pm 23 \pm 24 \cdots
\]

But since the \( p' \) series were removed we have

\[Q = 0\]

Substituting values for \( A, B, C \), in equation (6)

\[(6'). E \mathbf{U} = \sum 12^{\alpha} 13^{\beta} 14^{\gamma} \cdots 23^{\delta} 24^{\epsilon} \cdots 34^{\zeta} V_1 V_2 V_3 V_4 \cdots
\]

The \( \alpha' \beta' \gamma' \cdots \zeta' \) do not refer to the \( A', B' \cdots \) since they are zero but are introduced as new symbols. We can write then for shortness

\[E = \sum \left| \begin{array}{cc}
\xi_i & \xi_k \\
\eta_i & \eta_k
\end{array} \right| = \sum \frac{1}{k}
\]

which is the most general symbolical notation of Cayley. In this symbol it is well to notice that the figures in the symbol refer to the subscript of the \( X \)'s and \( Y \)'s, found in a certain function. For example the figures (I) refer or apply only to \( V_1 \) and so on.
4.

We will now derive the general term for the binary form. Let us write for shortness

\[ \alpha + \beta + \gamma + \cdots + \alpha', \beta', \gamma' + \cdots \]

(7). \[ \alpha + \gamma' + \gamma'' + \cdots + \alpha', \beta', \gamma' + \cdots \]

\[ \beta' + \gamma'' + \cdots + \alpha', \beta', \gamma' + \cdots \]

\[ \cdots \cdots \cdots \]

where \( F \) refers to the number of differentiations with respect to \( X, Y \), which is designated in the symbol

\[ \overline{12}^{\alpha} \overline{13}^{\beta} \overline{14}^{\gamma} \]

by the I's; \( F \) refers likewise to \( X, Y \), and is represented in the symbol by the 2's; \( F \) refers to differentiations with respect to \( X, Y \), being designated by 3, and so on to any number of symbols, or to any number of variables. We will also let

\[ r+s+t+\cdots+r'+s'+t'- \]

\[ N = (-1)^{r+s+t+\cdots+r'+s'+t'} \]

where \( r, s, t, \ldots r', s', t' \ldots \) extend from 0 to \( \alpha, \beta, \gamma, \ldots \beta', \gamma' \ldots \gamma'' \ldots \) respectively, and the expressions \( (\alpha) (s) \ldots \) arise from the expansion of

\[ (\overline{1K})^\alpha = (\xi \eta - \xi \eta')^\alpha \]

by the binomial theorem. Then in general we may say that those expressions are the binomial coefficients. The factor \((-1)^{r+s+t+\cdots+r'+s'+t'}\)

arises from the application of the symbol \((\overline{1K})^\alpha\), for we can readily see that every other term will be negative, and to obtain the sign of the general term, we must take \((-1)^{r+s+t+\cdots+r'+s'+t'}\) as many times as we
have operated on the function.

We will now introduce a derivative symbol, which enables us to express the operator in a general form and simplifies its application very much. We will put

\[
\frac{\partial f}{\partial x^r} \frac{\partial f}{\partial y^r} V = \delta^{f^r} \cdot V = V^r
\]

where \( f \) represents the total number of differentiations performed upon \( V \), by any set of variables \( X, Y \), and the letter \( r \) refers to the number of differentiations with respect to \( Y \) alone. Every term of function \( V \) must be operated upon \( f \) times in all since the functional symbol of operation is homogenous of the \( f \) degree.

Let us now specialize our function \( V \) and put

\[
V_1 = a_0 x^n + a_1 x^{n-1} Y_1 + a_2 x^{n-2} Y_2^2 + \cdots + a_n Y^n
\]

\[
V_2 = a_0 x^n + a_1 x^{n-1} Y_2 + a_2 x^{n-2} Y_2^2 + \cdots + a_n Y^n
\]

\[
V_3 = a_0 x^n + a_1 x^{n-1} Y_3 + a_2 x^{n-2} Y_3^2 + \cdots + a_n Y^n
\]

\[
\vdots
\]

From this it is evident that the symbol \( I \) will apply only to \( V_1 \) which contains \( x, Y_1 \). 2. will apply only to \( V_2 \) and so on. By (6) our general symbol is

\[
E.U = \sum 123^\alpha 13^\beta 14^\gamma \cdots 23^\beta' 24^\gamma' \cdots 34^V\ V_1 \ V_2 \ V_3 \ \ldots
\]

Now by equations (7) and (7') we can write the general term

\[
E.U = \sum N \ f_1 r+s+t, \ f_2 \alpha-r+s+t, \ f_3 \beta-s+\beta'-t' \ y
\]

We have the superscript of \( V_1 \) as \( f_1 \), since \( f_1 \) by equation (7) is the number of times we operate with respect to \( X_1 \) and \( Y_1 \). It can be easily seen why \( Y_1 \) is applied \( r+s+t+\ldots \) where these
quantities vary from 0 to \( \alpha, \beta, \gamma, \ldots \) respectively. If we expand

\[
\mathcal{T}^\alpha = (S_1 n_2 - S_2 n_1)^\alpha = S_1 n_2^\alpha - \alpha S_2 n_1^{\alpha-1} n_2 + \ldots
\]

Here it is evident we would have 0 in the first term for the index of \( Y \); for the second \( I \); the third, \( 2 \) and so on to the last term where we have \( \alpha \). This reasoning can thus be extended to \( \mathcal{T}^6 \) etc. Looking again at the symbol \( \mathcal{T}^\alpha \) for the terms with subscripts 2, we see that for \( X_2, Y_2 \) instead of \( Y_2 \) for instance being applied 0, 1, 2, \ldots, \( \alpha \) times, it is \( \alpha, \alpha-1, \alpha-2 \ldots 2,1,0 \). But since \( r \) passes through all values from 0 to \( \alpha \) we can then express the application of \( Y_2 \) in \( \mathcal{T}^\alpha \) by \( \alpha - r \) which gives the series \( \alpha, \alpha-1, \alpha-2 \ldots 2,1,0 \). We will then have for the symbolical expression \( V_2 \)

\[
\frac{f_2}{V_2} = \frac{\alpha - r + s + t + \ldots}{V_2}
\]

where \( S, t, \ldots \) are accounted for as the symbols in \( V \). With exactly the same reasoning we get

\[
\frac{f_2}{V_2} = \frac{\alpha - s + t + \ldots}{V_2}
\]

When several of the functions become equal, \( (V_1, V_2, V_3 \ldots) \) and consequently some of the \( f \)'s, the operator \( E, U \) refers to a certain number of clearly defined forms. This symbol is homogenous of the degrees \( \theta, \theta' \ldots \) with respect to the differential coefficients of the orders \( f, f' \ldots \) of \( V_1 \), and of the degrees \( \theta_2, \theta'_2 \ldots \) of the differential coefficients of the orders \( f_2, f'_2 \ldots \) of \( V_2 \) and so on. The degree with respect to all will then be
\[ \theta_1 + \theta'_1 + \ldots + \theta_n + \theta'_n + \ldots = P \]

In general single functions only will be considered, and it will be assumed that \( E.U \) contains differential coefficients of the \( f' \)th order and of the \( P' \)th degree only.

5.

The symbolical method will next be applied to the simplest case; that of functions of the second order, and we will then have

\[ E.U \ W = \mathcal{I}^{2^\alpha} U \ W \]

where

\[
U = a_0 x^n + a_1 x^{n-1} y_1 + a_2 x^{n-2} y_1^2 + \ldots \\
W = a'_0 x^n + a'_1 x^{n-1} y_1 + a'_2 x^{n-2} y_1^2 + \ldots
\]

so that \( \xi_1 \eta_2 \) applies to \( U \), and \( \xi_2 \eta_1 \) to \( W \).

We have \( \mathcal{I}^{2^\alpha} \) expanded as follows

\[
\mathcal{I}^{2^\alpha} = (\xi_1 \eta_2 = \xi_2 \eta_1) = \xi^{2^\alpha} \eta_1 = (2^\alpha) \xi^{2^\alpha-1} \eta_2 \eta_1 + (2^\alpha) \xi^{2^\alpha-2} \eta_1^2 
\]

According to (7)

\[
\frac{\partial^{\alpha}}{\partial x^{\alpha}} V = V^{1^0} \\
\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}} V = V^{1^1} \\
\frac{\partial^{\alpha}}{\partial x^{\alpha}} W = W^{1^\alpha} \\
\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}} W = W^{1^\alpha-1}
\]

(9) \[ \mathcal{I}^{2^\alpha} V.W = V^{1^0} W^{1^\alpha} - (\alpha) V^{1^1} W^{1^\alpha-1} + (\alpha) V^{1^2} W^{1^\alpha-2} \ldots \]

When \( \alpha \) is odd we know there will be an even number of terms.

Now if \( V \) and \( W \) were identical functions, it is evident that we would have symmetrically lying terms in \( \mathcal{I}^{2^\alpha} \) equal. But in that case the signs would be opposite and all the terms would destroy each other. For example
\[ \overline{\omega^3} = V^6 W^2 - \left( \frac{5}{2} \right) V^4 W^1 + \left( \frac{7}{2} \right) V^2 W^1 - \left( \frac{3}{2} \right) V W^1 \]

\[ = V^6 W^3 - 3 V^4 W^2 + 3 V^2 W^1 - V W^1 \]

\[ = 0 \text{ when } V = W. \]

\((\alpha')\) Since \( \overline{\omega^3} V, V = V^6 V^3 - 3 V^4 V^2 + 3 V^2 V^1 - V W^1 = 0 \)

If \( \alpha \) is even there will be an odd number of terms, and those symmetrically lying will have the same sign and combine two and two, with the exception of the middle term. Writing the factor 2 on the left hand side, we can write

\[ \overline{\omega^4} V, V = V^6 V^4 + 4 V^4 V^3 + 6 V^2 V^2 - 4 V^3 V^1 + V^2 V^0 \]

\((10)\). \( \frac{1}{2} \overline{\omega^4} V, V = V^6 V^4 + 4 V^4 V^3 + 2 V^2 V^2 \)

Or in general

\((10')\). \( \frac{1}{2} \overline{\omega^n} V V = V^6 V^n - n V^5 V^{n-1} + \left( \frac{n^2}{2} \right) V^4 V^{n-2} \ldots + \frac{1}{2} V^0 V^0 \)

Here \( n = \alpha \) is even.

\[ \overline{\omega^1} = \frac{\partial f}{\partial X_1} - \frac{\partial f}{\partial X_2} \]

\[ \frac{\partial f}{\partial Y_1} - \frac{\partial f}{\partial Y_2} \]

6.

Since the functions \( U, V, W \), on which we have been operating, are homogenous it is evident that their form will remain unchanged by linear transformation; further they must then possess invariant and covariant properties. When \( \alpha = n \), it is also very clear that \( \overline{\omega^\alpha} \) will give us an invariant, since the variables will all be removed by differentiation. If \( \alpha \) is less than \( n \), it will furnish us a covariant. Since
if we apply it to \( f \) any homogenous, we will get the Jacobian of that form. Likewise

\[
\overline{E}^2 = (\zeta_1 \eta_1, \zeta_2 \eta_2)^2 = \left( \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2} - 2 \frac{\partial^2 f}{\partial x \partial y} \right) + \left( \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x^2} \right)
\]

applied to \( f \) gives the Hessian of \( f \).

Simultaneous Invariants of the Second Order.

7.

If we apply \( \overline{E}^n \) to the product of two forms which are of the same degree \( n \), and \( \alpha = n \), the resulting function will be a simultaneous invariant. For example, let

\[
U = a x^2 + 2 b x y + c y^2
\]

\[
V = a' x'^2 + 2 b' x' y' + c' y'^2
\]

To form the invariant \( \overline{E}^2 U V \). Now

\[
\overline{E}^2 U V = U'^2 V'^2 - 2 U' V' + U V'
\]

But

\[
U' = \frac{\partial U}{\partial x} = 2 a \\
V' = \frac{\partial V}{\partial x} = 2 a' \\
U'' = \frac{\partial^2 U}{\partial x^2} = 2 b \\
V'' = \frac{\partial^2 V}{\partial x^2} = 2 b' \\
U'^2 = \frac{\partial U}{\partial y} = 2 c \\
V'^2 = \frac{\partial V}{\partial y} = 2 c'
\]

Making these substitutions in the above we have

\[
\frac{1}{4} (\overline{E}^2 U V) = a c' - 2 b b' + c a'.
\]

which we know to be a simultaneous invariant etc.

We can thus extend the reasoning to two cubics, bi-quadratics.
To obtain the invariants of a single function of the second degree, we will proceed exactly as before, and after the differentiations are performed, remove the subscripts from the coefficients since the two functions are identical.

Example: To compute the invariant of the quadratic.

We have already found the simultaneous invariant of two functions to be

$$\alpha'^2 - 2bb' + \alpha'C$$

Removing the accents from the $\alpha', b', C'$, we have

$$\bar{I}^2 \mathcal{V} = 2(\alpha - b^2)$$

which is the discriminant of the quadratic.

The cubic has no invariant of the second order, as we saw by (9') that

$$\bar{I}^3 \mathcal{V} = 0$$

For the quartic we have $\alpha = 4$, or

$$\bar{I}^4 \mathcal{V}_1, \mathcal{V}_2 = \mathcal{V}_1^0 \mathcal{V}_2^4 - 4 \mathcal{V}_1^1 \mathcal{V}_2^3 + 6 \mathcal{V}_1^2 \mathcal{V}_2^2 - 4 \mathcal{V}_1^3 \mathcal{V}_2^1 + \mathcal{V}_1^4 \mathcal{V}_2^0$$

$$= 2(\mathcal{V}_1^0 \mathcal{V}_2^4 - 4 \mathcal{V}_1^1 \mathcal{V}_2^3 + 3 \mathcal{V}_1^2 \mathcal{V}_2^2)$$

by (10') since $\mathcal{V}_1 = \mathcal{V}_2$, where

$$\mathcal{V}_1 = a_0x_1^4 + 4a_1x_1^3x_1 + 6a_2x_1^2x_1^2 + 4a_3x_1x_1^4 + a_4x_1^4$$

$$\mathcal{V}_2 = a'_0x_2^4 + 4a'_1x_2^3x_1 + 6a'_2x_2^2x_1^2 + 4a'_3x_2x_1^4 + a'_4x_1^4$$

Computing the $\mathcal{V}'s$ as in the quadratic we find

$$\mathcal{V}_1^0 = 24a_0; \mathcal{V}_1^1 = 24a_1; \mathcal{V}_1^2 = 24a_2; \mathcal{V}_1^3 = 24a_3; \mathcal{V}_1^4 = 24a_4$$

$$\mathcal{V}_2^0 = 24a'_0; \mathcal{V}_2^1 = 24a'_1; \mathcal{V}_2^2 = 24a'_2; \mathcal{V}_2^3 = 24a'_3; \mathcal{V}_2^4 = 24a'_4$$

Substituting in the above, and neglecting the factor 24,

$$\frac{1}{2}(\bar{I}^4 \mathcal{V}) = a_0a_4 - 4a_1a_3 + 3a_2^2$$
which comes by dropping the accents from a' ....

**Covariants.**

The general type of covariants of the second is

$$\Phi^{\alpha}$$

where $\alpha$ is less than $n$. This symbol must always be true, for when $\alpha$ is odd it will vanish, and if it were not less than $n$, the variables would all be removed by differentiation and we would have an invariant. This can also be shown unsymbolically, for the index is represented by

$$\eta = \frac{1}{2}(\rho - V)$$

where $\rho$ is the order of the covariant, and $V$ the degree in the variables. For the cubic $\eta = 3$, $\rho = 2$, and $V = \alpha$. For $V = 1$ we get $\eta = 2\frac{1}{2}$ which is absurd; for $V = 2$ we get $\eta = 2$, and this will be found to be the only covariant of the cubic. To show $V = \alpha$, if we expand $\Phi^{\alpha}$ where $\alpha = 2$, we will have each term in the operator of the second degree. When this is applied to the cubic there will remain a linear expression. But the covariant being of the second order, there will be the product of two of such forms in each term. Our final form will then be of the second degree in the variables or $V = \alpha$. For $V = 3$ our covariant vanishes. For $V = 4 = \alpha$, is impossible, since by operating four times, our final result would be zero, and so on for $V$ equal to any higher numbers. Hence, the only covariant of the cubic of the second order is $V = 2$.

We will now calculate the simultaneous covariants of two
cubics.

\[ V = a x_1^3 + 3 b x_1^2 y_1 + 3 c x_1 y_1^2 + d y_1^3 \]

\[ W = a' x_2^3 + 3 b' x_2^2 y_2 + 3 c' x_2 y_2^2 + d' y_2^3 \]

As before

\[ \mathcal{I}^2 = V \cdot W - 2 V^1 W^1 + V^2 W^0 \]

\[ V^0 = \frac{\partial^2 V}{\partial x_1^2} = 6(a x_1 + b y_1) \]
\[ V^1 = \frac{\partial^2 V}{\partial x_1 \partial y_1} = 6(b x_1 + c y_1) \]
\[ V^2 = \frac{\partial^2 V}{\partial y_1^2} = 6(c x_1 + d y_1) \]

\[ W^1 = \frac{\partial^2 W}{\partial x_2 \partial y_2} = 6(b' x_2 + c' y_2) \]
\[ W^2 = \frac{\partial^2 W}{\partial y_2^2} = 6(c' x_2 + d' y_2) \]

\[ \mathcal{I}^2 = \left( a c' - b b' \right) x_1 x_2 + \frac{1}{2} \left( a'd - b'b' \right) x_1 y_1 + \frac{1}{2} \left( a'd - b'b' \right) x_1 y_2 + (b'd - c'c') y_1 y_2 \]

as is obtained by substituting in \( \mathcal{I}^2 \) and neglecting the common factor 6. From this we can easily get the covariant of a single cubic by removing the subscripts from the variables and coefficients, or in other words, put \( V = W \). We then have

\[ \mathcal{I}^2 = \left( a c - b^2 \right) x^2 + \left( a d - b c \right) x y + \left( b d - c^2 \right) y. \]

Applying \( \mathcal{I}^2 \) to the binary quantic we will obtain the only covariant of the second order of the quantic. Since \( \mathcal{I}^3 = 0 \), and \( \mathcal{I}^4 \) would give an invariant, as as four differentiations would remove all the variables.

This same reasoning can be easily applied to the binary quantic - sextic, and we will obtain their system of covariants of the second order.
In applying the operator to any number of functions, it is easily seen that there will be one coefficient in every term of the derivative, for each form on which we operate. This will also be true, if the forms are identical, since we represent the quantic by as many forms of the same degree as there are figures in the operative symbol. These forms differ only in having the subscript for the variables contained in them the same as one of the figures that are found in the symbol. This can be seen by referring to Article 7. We there formed the invariant of the second order of the quadratic, where our operative symbol was \( \Sigma \). We operated upon two forms \( U \) and \( V \) which differed only in \( U \) having the variables \( X, Y \), and \( V \) having \( X, Y \), the subscripts being removed after differentiation. Then it is evident that the order of the invariant in the coefficients will be equal to the number of different figures that appear in the operative symbol. Our general type for invariants of the second order will then be \( \Sigma \).

Let \( U, V, W \), be distinct quantics, which are of the \( n, n', n'' \) degrees respectively. Let the operative symbol be

\[
\begin{array}{cccc}
12 & 13 & 14 & \ldots \\
\alpha^{(1)} & \alpha^{(ii)} & \alpha^{(iii)} & \ldots \\
\beta^{(1)} & \beta^{(ii)} & \beta^{(iii)} & \ldots \\
\gamma^{(1)} & \gamma^{(ii)} & \gamma^{(iii)} & \ldots \\
\end{array}
\]

where

\[
\alpha^{(1)} + \alpha^{(ii)} + \alpha^{(iii)} + \ldots = \alpha
\]

\[
\beta^{(1)} + \beta^{(ii)} + \beta^{(iii)} + \ldots = \beta
\]

\[
\alpha + \beta + \gamma + \ldots = \gamma
\]

Here the subscripts \( 1, 2, 3, 4, \ldots \) refer to \( U, V, W, \ldots \). Now
from the above we see that (I) occurs ⦢ times, which means that 
U is to be differentiated ⦢ times. The degree in the variables
will then obviously be (n - ). Similarly with V the degree
will be (n' - )., W will be of the degree (n'' - ) .......
These functions are homogenous of the degrees (n- ),(n' - ),(n'
(n'' - )...... In every term of the derivative, one term from
each of the above mentioned functions will be found. Then the
degree in X and Y will be the sum of the degrees of the separate
factors or

\[(n - \alpha) + (n' - \beta) + (n'' - \gamma) + \ldots \ldots \]

An illustration of this is given in Article 9 , where we
formed the simultaneous covariant of two cubics. There \(\alpha = \beta = \gamma\)
and \(n = n' = 3\). After the differentiations were performed, the
degree was obviously \(3 - 2 = 1\) & \(3 - 2 = 1\) respectively. Then
when we formed the derivative the degree in X and Y became

\[(n - \alpha) + (n' - \beta) = 1 + 1 = 2.\]

If the forms are identical, and there be P figures in the symbol
we must operate on P forms.

\[U_1, U_2, U_3, \ldots U_p.\]

Therefore the degree in X and Y becomes

\[m^p - (\alpha + \beta + \gamma + \ldots \ldots)\]
since the n's all become equal. If there are r factors of
the type \(1^2\) in the above series, we have

\[\alpha + \beta + \gamma + \ldots \ldots = 2r.\]
since in each factor the degree in the coefficient is two. To
get an invariant it is clear that

\[\alpha = \beta = \gamma = \ldots = n.\]
since $n$ differentiations are necessary to remove the variables.

II.

Invariants of the Third Order.

From the discussion of the preceding paragraph, it is evident that the general type of in-covariants of the third order must be

$$\mathbf{I}_2 \mathbf{3} \mathbf{5}$$

We will have three different figures in the symbol, because we must have three coefficients introduced into each term of the derivative, and each figure can introduce but one. Here we have

$$\mathbf{3} = \mathbf{3}_2 \eta_3 - \mathbf{3}_3 \eta_2, \quad \mathbf{3} = \mathbf{3}_3 \eta_1 - \mathbf{3}_1 \eta_3.$$  

The symbol appears in the form above, because it must be symmetrical, and if, for instance, we wrote $\mathbf{13}$ it would not be true. If the degree of the function on which we operate is $n$ we must then have

$$\alpha + \gamma = \alpha + \beta = \beta + \gamma = n.$$  

in order to obtain an invariant. When we operate on a single function, all the figures in the symbol must appear to the same degree $\alpha$ so that the general type of our symbol is

$$(\mathbf{12} \mathbf{3} \mathbf{3})^\alpha.$$  

When $\alpha$ is odd the function vanishes identically, as has already been shown. So that all invariants of the third order are included in the above form when $\alpha$ is even. Thus $(\mathbf{12} \mathbf{3} \mathbf{3})^\alpha$ is the invariant symbol of the binary quantic, since each subscript
appears with a symbolical exponent four, and our form being of the fourth degree, the variables will all be removed. Again \((\overline{12} \overline{23} \overline{34})^4\) applied to a form of the eighth degree will give an invariant since again each operative symbol appears to the eighth degree. Likewise \((\overline{12} \overline{23} \overline{34})^6\) applied to a form of the twelfth degree will give an invariant. By a comparison of these last results, we see that only forms of the degree \(4m (m = 1, 2, 3, \ldots)\) have invariants of the third order.

Extending this reasoning one step farther, we have for the general type of invariants of the fourth order

\[ (\overline{12} \overline{34})^\alpha (\overline{12} \overline{34})^\beta (\overline{12} \overline{34})^\gamma \]

The order of the invariant is immediately seen, since four figures appear in the symbol. And since each figure appears in each group once, we must have the relation

\[ \alpha + \beta + \gamma = n. \]

in order to obtain an invariant, where \(n\) is the degree of the functions on which we operate. Here

\[ \overline{34} = \xi_3 \eta_4 - \xi_4 \eta_3. \]
Covariants of the Third Order.

There are an unlimited number of covariants of the third order the form of the simplest being

$$\frac{\partial^3 f}{\partial x^3} \frac{\partial^2 f}{\partial y^3} \frac{\partial f}{\partial y} - \frac{\partial^3 f}{\partial x^2 \partial y}$$

$$+ \frac{\partial^3 f}{\partial x^2 \partial y^2} \frac{\partial^2 f}{\partial y^2} \frac{\partial f}{\partial y} + 2 \frac{\partial^3 f}{\partial x^2 \partial y \partial x} \frac{\partial^2 f}{\partial y \partial x} \frac{\partial f}{\partial x}$$

$$= (\xi_1 \eta_2 - \xi_2 \eta_1)^2 (\xi_1 \eta_3 - \xi_3 \eta_1).$$

Applying this to the cubic we obtain

$$\frac{\partial^2 12^2 13}{\partial x^2} \frac{\partial 13}{\partial x} \frac{\partial 13}{\partial y} = (a_0 a_3 - 3a_1 a_2 + 2a_3^3) x_1^3 + 3(a_0 a_1 a_2 - 2a_0 a_3^2 + a_1^3 a_2) x_1 y_1$$

$$- 3(a_0 a_2 a_3 - 2a_0^2 a_2 + a_2^3) x_1 y_1^2 - (a_0^3 - 3a_1 a_2 a_3 + 2a_3^2) y_1^3.$$

Likewise $\frac{\partial^2 12^2 13}{\partial x^2}$ applied to the binary quartic will give a covariant of the sixth degree in the variables. This can be easily seen, for the symbol (I) occurs in all three times. We must then differentiate with respect to $x_1$ and $y_1$ three times which will, of course, leave us a linear expression. The symbol (2) occurs twice, and when we differentiate twice with respect to $x_2 y_2$ we will have a binary quadratic form left. Then since the symbol (3) occurs but once, after its application we will have remaining a binary cubic. Since each of these expressions is homogenous, and also the derivative in their product, we will have as a result of their multiplication, a homogenous function of the sixth degree.
These are the only covariants of the third order of the binary cubic. This can be proven by considering the above symbol. Any other arrangement, as $\overline{23}^2 \overline{12}$, or $\overline{13}^2 \overline{23}$ .... will give the same covariant as the above since the subscripts are removed after differentiations, and our result will be the same. This, of course, would not be true were we forming a simultaneous covariant. We could not have $\overline{12}^2 \overline{13}^2$ as our symbol because in this the subscript (1) occurs to the fourth power, and the result of differentiating the cubic four times is zero, and our covariant will vanish. The same is true for $\overline{13}^2 \overline{13}^3$ or any similar arrangement.

The binary quartic can have other covariants, as $\overline{12}^2 \overline{13}^2$ would give us one of the fourth degree in $X$ and $Y$, and the symbol plainly shows this, since the subscripts (1) (2) (3) occur 4, 2, 2, times respectively, and their product after differentiation will be of the fourth degree.

We can thus extend the meaning to binary quintics and sextics.

Ternary Forms.

Let $X_1 Y_1 Z_1; X_2 Y_2 Z_2; X_3 Y_3 Z_3$ be any cogredient sets of variables. If for the variables we write the differential coefficients, which we designate by $\xi, \eta, \zeta$ with corresponding subscripts, we can form the function

$$\xi_1 (\eta_2 \zeta - \eta_3 \zeta) - \xi_2 (\eta_3 \zeta - \eta_1 \zeta) + \xi_3 (\eta_1 \zeta - \eta_2 \zeta),$$

which is the determinant $(\xi \eta \zeta)$ extended in terms of the elements of the first row. Now according to our symbolical notation,
this can be written

\[ \text{123} \]

This is evidently an invariant symbol of operation just as \[ \text{12} \] was, and when \[ \text{123}^\alpha \] is applied to a function of the degree \( \alpha \) we will get an invariant. If \( \alpha \) is less than \( n \) we will obtain a covariant, as the variables will not all be removed by differentiation. All the laws that have been developed, and formed to hold regarding the order and degree of in-covariants formed by the application of \[ \text{12}^\alpha \], will apply to \[ \text{123} \] with the addition of a new variable in each case. This is the only difference between the two symbols.

(To the student anxious to follow this out further, I would refer him to Salmon's Higher Algebra.)
Chapter II.

SYMBOLIC NOTATION OF ARONHOLD.

I.

In the twenty-eighth volume of Crelle's Mathematical Journal is a paper contributed by Hesse, in which many interesting problems were proposed; problems which are important to algebra, and for the theory of functions of the third degree of three variables. Aronhold began to consider some of the problems very diligently, and as a result of his investigation, we have a symbolical method of representing important functions of the coefficients and variables of a form of the third degree.

Aronhold represents a function of three variables in the form

\[ f(x_1, x_2, x_3) = a_{111} x_1^3 + a_{112} x_1^2 x_2 + a_{113} x_1 x_2^2 + a_{122} x_1^2 x_2 + a_{123} x_1 x_2 x_3 + a_{133} x_2^3 + a_{223} x_2^2 x_3 + a_{233} x_2 x_3^2 + a_{333} x_3^3 \]

where the subscripts of the a's in any term correspond to the subscripts of the X's that are found in that term. That is, if we have \( a_{113} \) in a term we must also have for the variables \( x_1, x_2, x_3 \) where the subscripts of the a's in any term correspond to the subscripts of the X's that are found in that term. That is, if we have \( a_{113} \) in a term we must also have for the variables \( x_1, x_2, x_3 \) we might call the general subscripts of the a's as \( a_{\alpha\beta\gamma} \). Now when \( \alpha, \beta, \gamma \) are all different, and each can take all values from one to three, our term will then be
\[ a_{113}X_1X_2X_3. \]

But we must take all possible combinations of the three things three at a time, which is six. But any two such terms are equal; that is

\[ a_{113}X_1X_2 = a_{113}X_1X_3. \]

Therefore the term of this type in the function has a factor six. When \( K = \lambda \) but different from \( \mu \), we will obtain terms of the type

\[ a_{K\lambda\mu}X_KX_KX_\mu. \]

and when \( K = \lambda = \mu \), there arise terms

\[ a_{KK\lambda}X_KX_KX_K. \]

The terms of the type \( a_{K\lambda\mu}X_KX_\mu \) will contain a factor three since we can form three combinations of \( K, K, \mu \), which however are all equal.

Then for the sake of brevity, and with the above interpretation we can write

\[ f(X_1,X_2,X_3) = \sum a_{K\lambda\mu}X_KX_KX_\mu. \]

where in the summation \( K \neq \mu \) takes all possible combinations from one to three.

Let us now form the second derivatives of \( f(X_1,X_2,X_3) \) with respect to \( X_1^2X_2X_3, X_1X_2X_3 \) and so on. We will represent the derivatives by the symbols \( f''_{x_1x_2}, f''_{x_1x_3}, f''_{x_2x_3}, \ldots \), where the accent applied to subscript variables are those with respect to which the function is derived. Differentiating \( f \) we have

\[ f'_{x_1} = 3a_{111}X_1^2 + 6a_{113}X_1X_3 + 3a_{123}X_2X_3 + 3a_{133}X_1X_3 + 6a_{113}X_1X_3 + 6a_{123}X_2X_3, \]

\[ f'_{x_2} = 3(a_{111}X_1 + a_{113}X_3 + a_{123}X_3). \]
Similarly

$$f_{x_i}'' = \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3)$$

$$f_{y_i}' = \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3)$$

$$f_{x_i}'' = \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3)$$

$$f_{y_i}' = \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3)$$

We will now form the functional determinant

$$\begin{vmatrix}
  f_{x_i}'' & f_{y_i}' & f_{y_i}' \\
  f_{x_i}'' & f_{y_i}' & f_{y_i}' \\
  f_{x_i}'' & f_{y_i}' & f_{y_i}'
\end{vmatrix}
$$

(3)

$$\begin{vmatrix}
  \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) & \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) \\
  \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) & \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) \\
  \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) & \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3)
\end{vmatrix}
$$

(4) =

$$\begin{vmatrix}
  \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) & \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) \\
  \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) & \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) \\
  \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) & \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3)
\end{vmatrix}
$$

In order to show the expansion of this determinant, I will consider the term on the principal diagonal, the details of which will apply to the other terms. We have

$$\begin{vmatrix}
  \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) & \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) \\
  \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) & \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) \\
  \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3) & \theta(a_{111}x_1 + a_{112}x_2 + a_{113}x_3)
\end{vmatrix}
$$

(4') \theta^3 (a_{111}x_1 + a_{112}x_2 + a_{113}x_3)(a_{111}x_1 + a_{112}x_2 + a_{113}x_3)(a_{111}x_1 + a_{112}x_2 + a_{113}x_3)

Let us put

$$b_{1_{123}1_23}x_1x_2x_3 = \theta a_{111}a_{112}a_{113}x_1x_2x_3$$

(a)
or in general

\[ b_{\kappa \lambda \mu} = G a_{\alpha \beta} a_{\gamma} \, x \cdot x \cdot x \cdot \mu \]

where as before the subscripts of the \( b \)'s correspond to the subscripts of the \( x \)'s. If now we divide (4') by \( G^2 \) we will remove the numerical factor \( G^3 \), and we may write

\[ G^3 (a_{111} \, x_1 + a_{112} \, x_2 + a_{113} \, x_3) (a_{122} \, x_1 + a_{222} \, x_2 + a_{223} \, x_3) (a_{133} \, x_1 + a_{333} \, x_2 + a_{333} \, x_3) = I/G^2 (G^2 (b_{113} \, x_1 \, x_2 \, x_3 + b_{123} \, x_1 \, x_2 \, x_3 + b_{223} \, x_2 \, x_3 + \ldots)) \]

This interpretation will apply to every term in (4) and if we put \( A f \) for the value of the expansion of (3') divided by \( G^2 \), there results

\[ A f = I/G^2 \sum f_{k, k', k''} f_{k, k', k''} f_{k, k', k''} = \sum b_{k \lambda \mu} \, x_k \cdot x_k \cdot x_\mu \]

where the \( b \)'s are homogeneous functions of the third degree in the \( a \)'s, as is shown by equation (a).

**Ternary Quadratic Quantics.**

Let us now consider two homogeneous functions of the second degree in \( x_1, x_2, x_3 \)

\[ \sum a_{k \lambda} x_k \cdot x_\lambda \quad \sum b_{k \lambda} x_k \cdot x_\lambda \]

In this case the \( a \)'s and \( b \)'s will have but two subscripts, since the subscripts correspond to the variables in the term, and the functions being of the second degree will have but two variables in each term.
The determinants of the coefficients of these two forms are:

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
\quad \begin{vmatrix}
  b_{11} & b_{12} & b_{13} \\
  b_{21} & b_{22} & b_{23} \\
  b_{31} & b_{32} & b_{33}
\end{vmatrix}
\]

They can be written in another very convenient form by use of the following property. In the form of the term containing \(x_1 x_2\), \(x_2 x_1\), with just the same meaning. Since \(a_{12}\) and \(a_{21}\) are the corresponding coefficients, we will then have

\[(7). \quad a_{12} = a_{21} .\]

Taking the minor of the term \(a_{11}\) in the above determinant and writing it \((aa)^{\prime}\) we have

\[
(aa)^{\prime} = a_{22} a_{33} - a_{32} a_{23}
\]

By relation \((7)\)

\[(8) \quad (aa)^{\prime\prime} = a_{12} a_{33} - a_{23} a_{13} , \quad \text{Likewise}
\]
\[= -(aa)^{12} = a_{12} a_{33} - a_{23} a_{13} = -(a_{23} a_{31} - a_{21} a_{33})
\]

or \((aa)^{12} = a_{13} a_{32} - a_{12} a_{33}\)

\[(9) \quad (bb)^{12} = b_{22} b_{33} - b_{32} b_{23} = b_{22} b_{33} - b_{32} b_{23}
\]
\[(bb)^{12} = b_{13} b_{23} - b_{12} b_{33}\]

Then equation \((6)\) can be written
Let us now differentiate equation (8) with respect to \( a \), and replace the increments by \( (b) \); that is, we will apply to (8) the operator given in Gordan's Invarianten Theorie, as the Aronhold operator, viz

\[
\frac{d}{da} M = \frac{\partial M}{\partial a_0} b_0 + \frac{\partial M}{\partial a_1} b_1 + \frac{\partial M}{\partial a_2} b_2 + \ldots + \frac{\partial M}{\partial a_n} b_n
\]

where \( M \) is here the operand and \( n = 3 \). Then

\[
d(aa) = a_{23} b_{33} + a_{33} b_{23} - a_{22} b_{33} - a_{32} b_{23}
\]

\( (c) \)

\[
d(aa) = a_{23} b_{33} + a_{33} b_{23} - 2a_{23} b_{23} = (ab)^{11}
\]

\( (d) \)

\[
d(aa) = a_{23} b_{33} + a_{33} b_{23} - a_{32} b_{13} - a_{12} b_{33} = (ab)^{12}
\]

\( (e) \)

\[
d(aa) = a_{13} b_{33} + a_{33} b_{13} - a_{32} b_{13} - a_{12} b_{33} = (ab)^{21}
\]

Comparing equations (d) and (e) we have

\[
(ab)^{12} = (ab)^{21}
\]

or in general

\[
(ab)^{k\ell} = (ab)^{\ell k}
\]

Substituting for the constituents of \((IO_a)\) and \((IO_b)\) their values after applying the operator, we have

\[
\begin{array}{c|ccc}
\text{(IO}_a) & (aa)^{11} & (aa)^{12} & (aa)^{13} \\
\text{( IO}_b \text{) } & (bb)^{11} & (bb)^{12} & (bb)^{13} \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{(IO}_a) & (aa)^{21} & (aa)^{22} & (aa)^{23} \\
\text{( IO}_b \text{) } & (bb)^{21} & (bb)^{22} & (bb)^{23} \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{(IO}_a) & (aa)^{31} & (aa)^{32} & (aa)^{33} \\
\text{( IO}_b \text{) } & (bb)^{31} & (bb)^{32} & (bb)^{33} \\
\end{array}
\]
This expression could have been obtained by applying the Aronhold operator to equation (9). This determinant will be shown to be a simultaneous invariant farther on in this paper.

Instead of the assumed homogeneous functions of the second degree, we will choose two, which are the first partial differential quotients of the functions \( f \) and \( \Delta f \) with respect to the variables. Differentiating \( f \) with respect to \( x_1, x_2, x_3 \), respectively, and dividing through by 3 we have

\[
\frac{1}{3} \frac{df}{dx_1} = a_{11} x_1^3 + a_{21} x_2^3 + a_{31} x_3^3 + 2a_{12} x_1 x_2 + 2a_{13} x_1 x_3 + 2a_{23} x_2 x_3
\]

\[
\frac{1}{3} \frac{df}{dx_2} = a_{12} x_1^3 + a_{22} x_2^3 + a_{32} x_3^3 + 2a_{23} x_2 x_3 + 2a_{33} x_3 x_3 + 2a_{13} x_1 x_3
\]

\[
\frac{1}{3} \frac{df}{dx_3} = a_{13} x_1^3 + a_{23} x_2^3 + a_{33} x_3^3 + 2a_{32} x_3 x_2 + 2a_{33} x_3 x_3 + 2a_{12} x_1 x_2
\]

From these differential equations, we see that the first figure in the subscript of each \( a \) is constant for any one equation and is the same as the subscript of the \( x \) in the derivative, while the remaining subscripts are those for the variables found in that term. With this relation we can write for brevity in general

\[
(I3_a) \quad \frac{1}{3} f'^r x_r = a_{r1} x_1^3 + a_{r2} x_2^3 + a_{r3} x_3^3 + 2a_{r2} x_2 x_2 + 2a_{r3} x_3 x_3 + \sum_{i+j+r} a_{ijr} x_i x_j x_r
\]

where \( r \) has the values from 1 to 3. This relation will also hold for \( \Delta f \), since it differs from \( f \) only in the \( a \)'s and \( b \)'s being interchanged. Therefore we can write at once

\[
(I3_b) \quad \frac{1}{3} \Delta f'^r x_r = b_{r1} x_1^3 + b_{r2} x_2^3 + b_{r3} x_3^3 + 2b_{r2} x_2 x_2 + 2b_{r3} x_3 x_3 + \sum_{i+j+r} b_{ijr} x_i x_j x_r
\]
Now for the expression \((ab)^{h^\wedge}\) we can write
\[(a, a_q)^{h^\wedge}\]
if in place of the functions
\[\sum a_k a_l x_k x_l = \sum b_k b_l x_k x_l\]
we select the differential quotients
\[I/3f' x_r\] and \[I/3f' x_q\]
This is true because to obtain the differential equation we have simply replaced the coefficient \(a\) by \(a_r\) and \(a_q\) respectively. Likewise \((a, b)^{h^\wedge}\) will become \((a_r b_q)^{h^\wedge}\)
since the \(b\) in \(\Delta f\) has been replaced by \(b_q\) to obtain \(I/3 \Delta f' x_r\).
The values of \((a_r a_q)\) and \((a_r b_q)\) can be easily formed if we compare equations (c) and (d). We have
\[(a_r a_q)^{h^\wedge} = a_{r,2} a_{q,3} + a_{r,3} a_{q,2} - 2a_{r,2} a_{q,2}\]
\[(a_r b_q)^{h^\wedge} = a_{r,2} b_{q,3} + a_{r,3} b_{q,2} - 2a_{r,2} b_{q,2}\]
The function \((a_r a_q)\) is of the second degree in the \(a\)'s, while \((a_r b_q)\) is of the fourth degree in the \(a\)'s. This is true since we saw by equation (a) that the \(a\)'s themselves are of the third order in the \(a\)'s and therefore the whole determinant of the fourth order.

From equations (II) we have the relations
\[(a_r a_q)^{h^\wedge} = (a_q a_r)^{h^\wedge} \quad (a_r b_q) = (b_q a_r)^{h^\wedge}\]
but we cannot interchange the subscripts and say that
\[(a_r b_q)^{h^\wedge} = (a_q b_r)^{h^\wedge}\]
since the \(q\) refers to the function \(I/3 \Delta f' x_r\), in which the coefficients are \(b\)'s, that are of the third degree in the \(a\)'s. If then we put \(q\) as the subscript of an \(a\) we have changed the
degree of our determinant and therefore its value.

As an example of this symbol, let

\[ S = (a_1 a_1)^{11} (a_1 a_1)^{21} + (a_1 a_1)^{31} (a_1 a_1)^{11} + (a_1 a_1)^{12} (a_1 a_1)^{12} + \cdots \]

\[ = \sum (a_1 a_1)^{k\lambda} (a_k a_\lambda)^{k\lambda} \]

The function \((a_r a_r)^{k\lambda}\) with equal indices can be derived from \((a_r a_q)\) if we will put

\[ q = r \]

But in this case we will have just twice as many terms as before, since we can interchange the \(a\)'s in \((a_r a_r)^{k\lambda}\) and obtain the same term. There will then be two terms for each term \((a_r a_q)^{k\lambda}\) so that our result will contain the factor two, as two terms of each form will be alike.

Then by the relations we have already proven of the interchangeability of the subscripts in the symbols, we may write

\[ S_1 = \sum (a_1 a_1)^{k\lambda} (a_k a_\lambda)^{k\lambda} \]
\[ S_4 = 2 \sum (a_1 a_1)^{k\lambda} (a_k a_\lambda)^{k\lambda} \]
\[ S_2 = \sum (a_1 a_2) (a_k a_\lambda)^{k\lambda} \]
\[ S_5 = 2 \sum (a_1 a_2)^{k\lambda} (a_k a_\lambda)^{k\lambda} \]
\[ S_3 = \sum (a_3 a_3) (a_k a_\lambda)^{k\lambda} \]
\[ S_6 = 2 \sum (a_3 a_3)^{k\lambda} (a_k a_\lambda)^{k\lambda} \]

We have here the relation

\[ S_1 = S_2 = S_3 = S_4 = S_5 = S_6 \]

The proof of this will not be given here, but for the student who is desirous of investigating it farther, I will refer you to Crelle's Mathematical Journal, Vol. 55.

The function \(S\) contains all the coefficients \(a_{k\lambda}\) in respect to which it is homogeneous, and of the second degree. It has, moreover, the property, that its form remains unchanged, if it is
formed from the coefficients of a function \( F \), which in turn is formed from \( f \) by linear substitution.

**Simultaneous Invariants**

Let

\[
\begin{align*}
\xi, \eta, \zeta, \text{ and } u, v, w \end{align*}
\]

be the two systems of original variables, and

\[
\begin{align*}
\xi', \eta', \zeta', \text{ and } u', v', w' \end{align*}
\]

the corresponding system of new variables, which are connected by the relation

\[
\begin{align*}
\xi &= \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 \\
\eta &= \beta_1 \eta_1 + \beta_2 \eta_2 + \beta_3 \eta_3 \\
\zeta &= \gamma_1 \zeta_1 + \gamma_2 \zeta_2 + \gamma_3 \zeta_3 \\
u' &= \alpha'_1 u_1 + \alpha'_2 u_2 + \alpha'_3 u_3 \\
v' &= \beta'_1 v_1 + \beta'_2 v_2 + \beta'_3 v_3 \\
w' &= \gamma'_1 w_1 + \gamma'_2 w_2 + \gamma'_3 w_3
\end{align*}
\]

Let

\[
\begin{align*}
f_1 (\xi, \eta, \zeta) &= \sum a_{\lambda \kappa} \xi_\lambda \eta_\kappa \\
f_2 (\xi, \eta, \zeta) &= \sum b_{\lambda \kappa} \xi_\lambda \eta_\kappa \\
f_3 (\xi, \eta, \zeta) &= \sum c_{\lambda \kappa} \xi_\lambda \eta_\kappa
\end{align*}
\]

be three homogeneous functions of the second order of the variables \( \xi, \eta, \zeta \). By equations (II) we have

\[
\begin{align*}
a_{\lambda \kappa} &= a_{\lambda \kappa} ; \\
b_{\lambda \kappa} &= b_{\lambda \kappa} ; \\
c_{\lambda \kappa} &= c_{\lambda \kappa}
\end{align*}
\]

Then as is well known, we can form for each function, but a single invariant, namely the discriminant, and it can be represented by \( \Delta \) where
Theorem.

If we form the square of the determinant

\[
\begin{vmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3 \\
\end{vmatrix}
\]

(14).

\[
= \sum \pm u_1 v_2 w_3
\]

in which the elements are arbitrary values, and replace the powers and products of the second order by the corresponding coefficients

\[
a_{kk}, b_{kk}, c_{kk}
\]

of the three homogeneous functions \( f_1, f_2, f_3 \), then we obtain a determinant \((a b c)\) which is a simultaneous invariant of the three homogeneous functions.

Proof.

Let the transformed functions of \( f f f f\) be represented by

\[
f'_1(X_1 X_2 X_3) = \sum a'_{kk} X_k X_k
\]

\[
f'_2(X_1 X_2 X_3) = \sum b'_{kk} X_k X_k
\]

\[
f'_3(X_1 X_2 X_3) = \sum c'_{kk} X_k X_k
\]

In order to obtain the coefficients

\[
a'_{kk}, b'_{kk}, c'_{kk}
\]

we must first transform the three linear functions.
by the original substitution. Then we must square the new func-
tions, and introduce the symbolical relation

\( (\alpha). \quad u_\lambda u_\lambda = a_{\lambda\lambda} \quad v_\lambda v_\lambda = b_{\lambda\lambda} \quad w_\lambda w_\lambda = c_{\lambda\lambda} \)

If now we assume the transformed function to be written in
the form

\[
\begin{align*}
U_1 x_1 + U_2 x_2 + U_3 x_3 \\
V_1 x_1 + V_2 x_2 + V_3 x_3 \\
W_1 x_1 + W_2 x_2 + W_3 x_3
\end{align*}
\]

since the function remains unchanged in form by linear transfor-
modation, from equations \( (\alpha) \) we must have another symbolic relation

\( (\beta). \quad U_\lambda U_\lambda = a'_{\lambda\lambda} \quad V_\lambda V_\lambda = b'_{\lambda\lambda} \quad W_\lambda W_\lambda = c'_{\lambda\lambda} \)

From the invariant properties which we know the homogeneous func-
tions \( f_1, f_2, f_3 \), possesses, we have the equations

\[
\begin{align*}
\sum U_1 V_2 W_3 &= r \sum u_1 v_2 w_3 \\
\sum (U_1 V_2 W_3)^2 &= r^2 \sum (u_1 v_2 w_3)^2
\end{align*}
\]

\( r \) is of the first power because \( u_1, v_1, w_1, \ldots \) are linear ex-
pressions, and in transforming each constituent of the modulus \( r \),
will enter only to the first power.

By the definition of the equations \( (\alpha) \) & \( (\beta) \) we can replace
the quantities on the left hand side of the above equation by \( a'b'c' \), and on the right by \( a,b,c \), and there results

\( (a'b'c') = r^2 (a,b,c) \)
We will now proceed to the construction of \((a \ b \ c)\), and to effect this we must expand the determinant \(\sum (u_1 v_2 w_3)\) in terms of the elements of the first row. We then have

\[
\left(\sum (u_1 v_2 w_3)\right)^2 = (u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_2 w_1)^2
\]

But \((v_2 w_3 - v_3 w_2)^2 = v_2^2 w_3^2 + v_3^2 w_2^2 - 2 v_2 v_3 w_2 w_3\)

By \(\alpha\), \(u_1^2 = a_{11}, v_2^2 = b_{22}, w_3^2 = c_{33}\), \(u_1 u_3 = a_{1,2}\)...........

\[
(v_2 w_3 - v_3 w_2)^2 = b_{22} c_{33} - b_{33} c_{22} - 2 b_{23} c_{23}
\]

Also \((v_3 w_1 - v_1 w_3) = v_2 v_3 w_1 w_3 + v_1 v_3 w_1 w_2 \]

The remaining powers and products are similarly formed. Now since the above powers and products arise through the interchange of indices, then we can represent alone the result of the substitution from \(\alpha\) for the powers and products of \(V_k V_l W_m W_n\).

The square of the expression \(\sum (u_1 v_2 w_3)\) consists of terms of the form \(u_1 u_1 (v_2 w_3 - v_3 w_2)^2\) as can be seen by expanding the determinant in terms of the elements of the first row. By equations \(\alpha\)

\[
\begin{align*}
& b_{2,2} c_{3,3} + b_{3,3} c_{2,2} - 2 b_{2,3} c_{2,3} - (b c)^{11} \\
& \vdots \quad (b c)^{12} = (V_l W_3 - V_3 W_l)^2 = b_{2,2} c_{3,3} + b_{3,3} c_{2,2} - 2 b_{2,3} c_{2,3} \\
& \vdots \quad (b c)^{22} = (V_3 W_1 - V_1 W_3)^2 = b_{3,3} c_{1,1} + b_{1,1} c_{3,3} - 2 b_{1,3} c_{1,3} \\
& \vdots \quad (b c)^{33} = (V_1 W_2 - V_2 W_1)^2 = b_{3,3} c_{1,1} + b_{1,1} c_{3,3} + 2 b_{1,3} c_{1,3}.
\end{align*}
\]
\[(b \ c)^{13} = (v_2 w_3 - v_3 w_2)(v_3 w_1 - v_1 w_3)\]
\[
= \ b_{12} c_{13} + b_{13} c_{12} - b_{11} c_{23}
\]
\[(b \ c)^{13} = (v_a w_3 - v_3 w_a)(v_1 w_2 - v_2 w_1)\]
\[
= \ b_{23} c_{13} + b_{13} c_{23} - b_{22} c_{13} - b_{13} c_{22}
\]
and so on. We can then write the above equation in the form
\[
\sum u_k u_\Lambda (b \ c)^{\kappa \Lambda}
\]
But by equations (\#),
\[
u_k u_\Lambda = a_{\kappa \Lambda}
\]
(I5) \[\sum u_k u_\Lambda (b \ c)^{\kappa \Lambda} = \sum a_{\kappa \Lambda} (b \ c)^{\kappa \Lambda} = (a \ b \ c)\]
which is perfectly general and completely represents our form. The invariant \((a \ b \ c)\) is, however, symmetrical with respect to the three systems, \(a_{\kappa \Lambda}, b_{\kappa \Lambda}, c_{\kappa \Lambda}\), and the determinant from which it arises remains unchanged, except as to the sign when we interchange the elements \(u_k v_\Lambda w_\Lambda\). Then it is evident that in the square of the determinant \((u_1 v_2 w_3)\), the interchanging of the elements will not affect it. This being true we may write
\[
\sum a_{\kappa \Lambda} (b \ c)^{\kappa \Lambda} = \sum b_{\kappa \Lambda} (a c)^{\kappa \Lambda} = \sum c_{\kappa \Lambda} (a b)^{\kappa \Lambda}
\]
(I6) \[\sum a_{\kappa \Lambda} (c b)^{\kappa \Lambda} = \sum b_{\kappa \Lambda} (c a)^{\kappa \Lambda} = \sum c_{\kappa \Lambda} (b a)^{\kappa \Lambda}\]
since by equation (I4) the determinant \((bc)^{\kappa \Lambda}\) remains unchanged by the interchange of the elements \(b_{\kappa \Lambda}\) and \(c_{\kappa \Lambda}\).

As a special case, let us put
\[(I6') a_{\kappa \Lambda} = b_{\kappa \Lambda} = c_{\kappa \Lambda}\]
and we can write out ten different invariants, namely
\[
\begin{align*}
(a, a, a) &= \sum a_{k,a}^2 (aa)^{k,a} = \sum b_{k,a} (bb)^{k,a} = \sum c_{k,a} (cc)^{k,a} \\
(a, b, b) &= \sum a_{k,a}^2 (bb)^{k,a} = \sum a_{k,a} (cc)^{k,a} = \sum b_{k,a} (ab)^{k,a} \\
&= \sum c_{k,a} (ac)^{k,a} = \sum b_{k,a} (aa)^{k,a} = \sum c_{k,a} (aa)^{k,a} \\
(a, b, c) &= \sum a_{k,a} (bc)^{k,a}
\end{align*}
\]

But the first expression of (17) is not different from the determinant

\[
\Delta = \sum \pm a_{1,2} a_{2,3} a_{3,3}
\]

If we substitute in (14') the relation (16'), there results

\[
(aa)^{1,1} = a_{2,2} a_{3,3} + a_{3,3} a_{2,2} - 2a_{2,3} a_{3,3} = 2(a_{2,2} a_{3,3} - a_{k,5})
\]

\[
(e')
\]

\[
(aa)^{2,2} = a_{1,1} a_{3,3} + a_{3,3} a_{1,1} - a_{1,1} a_{2,3} - a_{2,3} a_{1,1} = 2(a_{1,2} a_{3,3} a_{2,3})
\]

The expressions (e') are the same as the minor determinants obtained by expanding \(\Delta\) in terms of the elements of the first row, or

\[
\Delta = a_{1,1} (a_{2,2} a_{3,3} - a_{2,3}^2) = a_{1,2} (a_{2,3} a_{3,1} - a_{2,1} a_{3,3}) + a_{1,3} (a_{2,1} a_{3,3} - a_{2,3} a_{3,1})
\]

Interchanging the elements in the middle term to remove the minus sign we have

\[
\Delta = a_{1,1} (a_{2,2} a_{3,3} - a_{2,3}^2) + a_{1,2} (a_{2,3} a_{3,1} - a_{2,1} a_{3,3}) + a_{1,3} (a_{2,1} a_{3,3} - a_{2,3} a_{3,1})
\]

But by (e')

\[
(a_{2,2} a_{3,3} - a_{2,3}^2) = \frac{1}{2} (aa)^{1,1}
\]

\[
(a_{2,2} a_{3,3} - a_{2,3} a_{3,1}) = \frac{1}{2} (aa)^{1,2}
\]

\[
(a_{2,2} a_{3,3} - a_{2,3} a_{3,1}) = \frac{1}{2} (aa)^{1,3}
\]

\[
\therefore \Delta = \frac{1}{2} [a_{1,1} (aa)^{1,1} + a_{1,2} (aa)^{1,2} + a_{1,3} (aa)^{1,3}]
\]

Expanding in terms of the elements of the second row

\[
\Delta = a_{1,1} (a_{2,3} a_{3,3} - a_{2,3} a_{3,1}) - a_{2,3} (a_{1,1} a_{3,3} - a_{1,3} a_{3,1}) + a_{3,3} (a_{2,1} a_{3,3} - a_{2,3} a_{3,1})
\]

\[
- a_{1,2} (a_{2,3} a_{3,1} - a_{2,1} a_{3,3}) + a_{2,3} (a_{1,1} a_{3,3} - a_{1,3} a_{3,1}) + a_{3,3} (a_{1,2} a_{3,3} - a_{1,3} a_{3,1})
\]

But

\[
(a_{2,3} a_{3,1} - a_{2,1} a_{3,3}) = \frac{1}{2} (aa)^{1,2}
\]

\[
(a_{1,3} a_{3,3} - a_{1,1} a_{3,3}) = \frac{1}{2} (aa)^{1,2}
\]

\[
(a_{1,1} a_{3,3} - a_{1,3} a_{3,1}) = \frac{1}{2} (aa)^{1,2}
\]
\[ \Delta = \frac{1}{2} \left[ a_{1,3} (aa)^{1,3} + a_{2,3} (aa)^{2,3} + a_{3,3} (aa)^{3,3} \right] \]

Expanding in terms of the elements of the third row

\[ \Delta = a_{1,3} (a_{1,1} a_{3,1} - a_{3,1} a_{1,1}) - a_{2,3} (a_{1,1} a_{3,2} - a_{3,2} a_{1,1}) + a_{3,3} (a_{1,1} a_{3,3} - a_{3,3} a_{1,1}) \]

\[ = a_{1,3} (a_{1,1} a_{3,1} + a_{3,2} a_{1,2}) + a_{2,3} (a_{1,1} a_{3,1} - a_{3,2} a_{1,2}) + a_{3,3} (a_{1,1} a_{3,1} - a_{3,2} a_{1,2}) \]

But \[ a_{1,1} a_{3,1} - a_{3,2} a_{1,2} = \frac{1}{2} (aa)^{1,3} \]

Adding these three values for \( \Delta \) together we have, after clearing of fractions

\[ 6 \Delta = a_{1,1} (aa)^{1,1} + a_{1,2} (aa)^{1,2} + a_{1,3} (aa)^{1,3} + a_{2,1} (aa)^{2,1} + a_{2,2} (aa)^{2,2} + a_{2,3} (aa)^{2,3} + a_{3,1} (aa)^{3,1} + a_{3,2} (aa)^{3,2} + a_{3,3} (aa)^{3,3} \]

Examining this expression we see that in every term the exponents or superscripts of each determinant is the same as the subscript of the \( a \) that is multiplied into the determinant. We can then express \( 6 \Delta \) as a summation, or

\[ (18) \quad 6 \Delta = \sum a_{x,y} (aa)^{x,y} = (aaa) \]

since it is nothing but the expansion of the determinant \( \Delta \) or the discriminant of the ternary quadratic. It is then the invariant of a single form.

Contravariant. To the forms
also belongs a contravariant form \( T \) (see Salmon Modern Higher Algebra) which when written in full becomes

\[
T = \left\{(a_{2,3} a_{3,3} - a_{3,3}^2) u_1^2 + (a_{3,3} a_{1,1} - a_{1,1}^2) u_2^2 + (a_{1,1} a_{2,2} - a_{2,2}^2) u_3^2 \right. \\
+ 2(a_{1,2} a_{1,3} - a_{1,3} a_{2,3}) u_1 u_2 + 2(a_{1,2} a_{2,3} - a_{2,2} a_{1,3}) u_2 u_3 + \\
\left. 2(a_{1,3} a_{2,3} - a_{1,2} a_{3,3}) u_3 u_2 \right\}
\]

which is of the second order in the quantities \( u_1, u_2, u_3 \) and the coefficients are the minor determinants of \( a_{1,1} a_{2,2} a_{3,3} \) in

\[
\Delta = \begin{vmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{vmatrix}
\]

But as we have seen by equations (e') that

\[
(a_{3,3} a_{3,3} - a_{3,3}^2) = I/2 (aa)^2 \\
(a_{1,1} a_{1,1} - a_{1,1}^2) = I/2 (aa)^3
\]

\[
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdOTSIMULTANEOUS CONTRAVARIANTS
These equations are true, since by definition the contravariant form possesses the invariant property, that

\[ \sum_{\nu} u_{\nu}(a'b')^{\lambda\nu} = r^{\lambda} \sum_{\nu} u_{\nu}(a'b)^{\lambda\nu} \]

which is also a necessary and sufficient condition.

**Invariants of a Ternary Cubic.**

Let

\[ f(x_1, x_2, x_3) = \sum a_{\lambda\mu\nu} x_\lambda x_\mu x_\nu \]

be a homogeneous function of the third degree, where as in function of the second degree we have the relations

(19) \[ a_{\lambda\mu\nu} = a_{\kappa\mu\lambda} = a_{\mu\kappa\lambda} = a_{\mu\lambda\kappa} = a_{\lambda\mu\kappa} = a_{\lambda\kappa\mu} \]

That is, \[ a_{123} = a_{321} = a_{132} = a_{312} = a_{231} = a_{213} \]

When we make all possible combinations of \( K \otimes \mu \), and they are all different, we will have six terms exactly alike, and therefore in the summation, such a term will have the factor 6. Likewise when \( K = \lambda = I \), we will have

\[ a_{\mu, \kappa} = a_{1, 2, 1} = a_{1, 1, 1} \]

Then in the summation, all terms of this form have the factor 3.

**Theorem.** Let us distinguish by
a matrix from which we derive the four minors

\[
\begin{vmatrix}
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3 \\
  w_1 & w_2 & w_3 \\
  p_1 & p_2 & p_3
\end{vmatrix}
\]

Then we will obtain an invariant of \( f \) if we form the product of the determinants

\[
A = \sum \pm u_1 w_1 p_1
\]

\[
B = -\sum \pm u_1 w_2 p_3
\]

\[
C = \sum \pm u_1 v_2 p_3
\]

\[
D = -\sum \pm u_1 v_2 w_3
\]

each of which is of the third degree, and replace the powers and products of

\[
\begin{align*}
  u_k & u_\lambda \ u_\mu \\
  v_k & v_\lambda \ v_\mu \\
  w_k & w_\lambda \ w_\mu \\
  p_k & p_\lambda \ p_\mu
\end{align*}
\]

by the corresponding coefficients of \( f \). We will distinguish the function thus formed by \( GS \) because it will be found to contain a factor \( 6 \). Now

\[
u_1 x_1 + u_2 x_2 + u_3 x_3
\]

is such a function that when transformed, becomes

\[
U_1 x_1 + U_2 x_2 + U_3 x_3 = (u_1 x_1 + u_2 x_2 + u_3 x_3)
\]

We have similar relations for the \( v \)'s, \( w \)'s, and \( p \)'s. And

\[
\sum \pm (U_1 v_3 w_3) = \sum \pm (u_1 v_2 w_3)
\]

Then when we form the product \( A B C D \) we have

\[
GS = \sum \sum \sum \sum \pm (u_1 u_2 u_3 v_3 w_3 v_2 w_2 w_3 p_3 p_3)
\]

and by equation (22) we can write

\[
\sum \sum \sum \sum \pm (U_1 U_2 U_3 V_3 W_3 V_2 W_2 W_3 P_3 P_3 P_3 P_3) = \sum \sum \sum \sum \pm (u_1 u_2 u_3)
\]

\[
v_2 v_3 v_1 w_2 w_3 w_3 p_3 p_3 p_3 p_3
\]
Now when we replace the \( u \)'s \( v \)'s \ldots \) by the \( a \)'s, we will have a homogeneous function of the fourth degree.

We will represent the transformed function of \( f \) by
\[
 f (X_1, X_2, X_3) = \sum a'_{\kappa \lambda \mu} X_\kappa X_\lambda X_\mu
\]
We will then prove that if we form the corresponding function \( S \), we will have the relation
\[
 (23) \quad S' = r^4 S.
\]
Let
\[
 u^1 X_1 + u^2 X_2 + u^3 X_3 \quad \quad V^1 X_1 + V^2 X_2 + V^3 X_3
\]
\[
 W^1 X_1 + W^2 X_2 + W^3 X_3 \quad \quad P^1 X_1 + P^2 X_2 + P^3 X_3
\]
be the four linear functions, and
\[
(24) \quad U^1 X_1 + U^2 X_2 + U^3 X_3 \quad \quad V^1 X_1 + V^2 X_2 + V^3 X_3
\]
\[
 W^1 X_1 + W^2 X_2 + W^3 X_3 \quad \quad P^1 X_1 + P^2 X_2 + P^3 X_3
\]
be the new forms. We will distinguish their determinants by
\[
 A \quad B \quad C \quad D
\]
Then from the relation that exists for a linear substitution we know that
\[
 A' = rA \quad B' = rB \quad C' = rC \quad D' = rD
\]
\[
 (25) \quad A'B'C'D' = r^4 A'B'C'D.
\]
If now we put
\[
 a_{\kappa \lambda \mu} = u^\kappa u^\lambda u^\mu
\]
and call the transformed coefficients \( a' \), we can write
\[
 U^\kappa U^\lambda U^\mu = a'_{\kappa \lambda \mu}
\]
But this relation is valid for the other systems or
\[
 V_\kappa V_\lambda V_\mu = W_\kappa W_\lambda W_\mu = P_\kappa P_\lambda P_\mu = a'_{\kappa \lambda \mu}
\]
since we are dealing with only one form and
\[
 a_{\kappa \lambda \mu} = b_{\kappa \lambda \mu} = c_{\kappa \lambda \mu}
\]
and $f'$ becomes $(U_1X_1 + U_2X_2 + U_3X_3)^3$ from equation (23).

Therefore from (25)

$$C S' = C \mathfrak{r} \mathfrak{s}$$

which proves that $S$ is an invariant.

We will now compute the invariant $S$ in terms of the coefficients $a_{\kappa \lambda \mu}$. We have seen that the invariant of the binary quadratic is written in the form

$$(abc) = \sum a_{\kappa \lambda} (bc)^{\kappa \lambda}$$

where

$$a_{\kappa \lambda} = u_{\kappa} u_{\lambda}; \quad b_{\kappa \lambda} = v_{\kappa} v_{\lambda}; \quad c_{\kappa \lambda} = w_{\kappa} w_{\lambda}$$

was substituted in the equation

$$\sum (u_i v_i w_i)^2 = \sum w_{\kappa} w_{\lambda} (u_i u_i) (v_i v_i)$$

The expression $(u_i u_i v_i v_i)^{\kappa \lambda}$ is formed from

$$(ab)^{\kappa \lambda}$$

according to (I4') in the preceding article. We can represent the expression if we differentiate

$$2\sum \sum \pm u_1 v_2 w_3 w_2 w_3$$

in a similar manner if we differentiate (I) with respect to $w_i$, $w_i$, $w_3$, respectively, and replace the increments by $p_i p_i p_3$. We then have

$$\sum \sum \pm u_1 v_2 w_3 \sum \sum \pm u_1 v_2 p_3$$

But

$$D = -\sum \sum u_1 v_2 w_3; \quad C = \sum \sum u_1 v_2 p_3$$

Therefore by equation (II)

$$\sum D C = -\sum (w_{\kappa} p_{\lambda} + w_{\lambda} p_{\kappa})(uu,vv)^{\kappa \lambda}$$
Equation (I) could have been written
\[ (\Sigma + u_1 w_2 p_3)^\xi = \Sigma u_\xi u_\bar{\xi} (\Phi \Phi \Phi) \]

Differentiating this equation with respect to \( u_1, u_2, u_3 \), and replacing the increments by \( v_1, v_2, v_3 \), respectively, i.e., applying the Aronhold operator \( \mathcal{J} \), we have
\[ (IV) \quad 2\Sigma + (u_1 w_2 p_3) \Sigma + (v_1 w_2 p_3) = \Sigma (u_\xi v_\xi + u_\bar{\xi} v_\bar{\xi}) (\Phi \Phi \Phi)^\xi \]
where \( \xi \) and \( \bar{\xi} \) have the same interpretation as \( K \) and \( \bar{\xi} \). Now
\[ A = \Sigma + v_1 w_2 p_3 \quad \quad B = -\Sigma + u_1 w_2 p_3 \]

by equation (IV). Multiplying (III) and (V) together we have
\[ (VI) \quad 4A B C D = \Sigma (w_K p_K + w_\bar{K} p_\bar{K}) (\Phi \Phi \Phi)^\xi (u_\xi v_\xi + u_\bar{\xi} v_\bar{\xi})(\Phi \Phi \Phi) \]

This equation is in such a form that the substitution for \( a_{\xi \bar{\xi}} \) can be introduced immediately.

We have already seen that the partial derivatives of \( f \) with respect to the three variables respectively can be written in the general form
\[ I/3 f_{x_r} = a_{r,1} x_1^2 + a_{r,2} x_2^2 + a_{r,3} x_3^2 + 2a_{r,12} x_1 x_2 + 2a_{r,13} x_1 x_3 + 2a_{r,23} x_2 x_3 \]
where \( r \) has all values from one to three, the subscripts in each case being the same, as that of the \( x \) with respect to which we differentiate.

We will call these three homogeneous functions \( f_1, f_2, f_3 \), and the laws which we have already found valid for equations of the second degree can then be applied in this case. Here we put
\[ a = a_1; \quad b = a_2; \quad c = a_3 \]
The expression
\[ (bc)^{\xi \bar{\xi}} \]
will then become
And for the expressions
\[ \Sigma a_{k_k^\lambda} (aa)^{\lambda \kappa}; \Sigma a_{k_k^\lambda} (bb)^{\lambda \kappa}; \Sigma a_{k_k^\lambda} (bc)^{\lambda \kappa} \]
respectively we must write

(VII) \[ \Sigma a_{k_k^\lambda} (a_2 a_4)^{\lambda \kappa}; \Sigma a_{k_k^\lambda} (a_2 a_4)^{\lambda \kappa}; \Sigma a_{k_k^\lambda} (a_2 a_4)^{\lambda \kappa} \]
Equations (II) are not invariants of functions of the third order however, but only of functions of the second order; but they have a significance to be developed hereafter.

Now
\[
(u_v v_\sigma + u_\sigma v_e)(uu, vv)^{\lambda 1} - (u_v v_\sigma + u_\sigma v_e)(u_2 v_z + u_z v_2 - 2u_z v_z
\]
\[
= u_2 u_2 v_v v_v + u_2 u_2 v_v v_v - 2u_2 u_2 v_v v_v - 2u_2 u_2 v_v v_v + u_2 u_2 v_v v_v - 2u_2 u_2 v_v v_v
\]

But
\[
u_2 u_2 u_2 = a_{e_2 e_2} v_v v_v = a_{e_2 e_2} \ldots \ldots
\]

\[
= a_{e_2 e_2} a_{e_2 e_2} + a_{e_2 e_2} a_{e_2 e_2} - 2a_{e_2 e_2} a_{e_2 e_2} + a_{e_2 e_2} a_{e_2 e_2} + a_{e_2 e_2} a_{e_2 e_2} - 2a_{e_2 e_2} a_{e_2 e_2} = (a_e a_\sigma)^{\lambda 1} + (a_\sigma a_e)^{\lambda 1} = 2(a_e a_\sigma)^{\lambda 1}
\]
since
\[
a_{e_2 e_2} a_{e_2 e_2} + a_{e_2 e_2} a_{e_2 e_2} - 2a_{e_2 e_2} a_{e_2 e_2} = (a_e a_\sigma)^{\lambda 1}
\]
or in general
\[
(a) \quad (u_v v_\sigma + u_\sigma v_e)(uu, vv)^{\lambda \kappa} = 2(a_e a_\sigma)^{\lambda \kappa}
\]
Therefore from (VI) we have
\[ 4ABCD = 4\Sigma \Sigma (a_e a_\sigma)^{\lambda \kappa} (a_\kappa a_\lambda)^{\rho \sigma} \]
Dividing out the factor 4 and putting
\[ ABCD = 6S \]
we have by expanding the summation for \( \sigma, \rho \), but retaining it for
6S is then an invariant of the fourth order. There are many interesting things to be developed in connection with $S$ and to the student who is anxious for further research, I will refer him to Crelle's Mathematical Journal, Vol. 55.

Mixed Concomitants. **Theorem.** If we square the determinant

$$\sum \pm u_1 v_2 w_3 = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

and multiply it by the two linear factors

$$v = v_1 x_1 + v_2 x_2 + v_3 x_3 \quad \text{and} \quad w = w_1 x_1 + w_2 x_2 + w_3 x_3$$

we obtain

$$vw \left( \sum \pm u_1 v_2 w_3 \right)^2$$

Introducing the substitution

$$v_k v_\lambda v_\mu = a_{k\lambda\mu}; \quad w_k w_\lambda w_\mu = a_{k\lambda\mu}$$

we obtain a function $\varphi$ of the variables $x_1 x_2 x_3, u_1 u_2 u_3$ which is of the second degree in the variables with respect to the two systems, and is a mixed concomitant of $f$. (See Salmon's Modern Higher Algebra)

We will distinguish $\varphi$ as the first mixed concomitant. We have already seen that by a linear substitution, we have

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = U_1 X_1 + U_2 X_2 + U_3 X_3$$

$$v_1 x_1 + v_2 x_2 + v_3 x_3 = V_1 X_1 + V_2 X_2 + V_3 X_3$$

For brevity we will write
\[ V = V_1 X_1 + V_2 X_2 + V_3 X_3 \]
\[ W = W_1 X_1 + W_2 X_2 + W_3 X_3 \]

We know that when such a linear transformation is possible, that two similar functions of the coefficients, the one of the original and the other of the new form, will differ only by a power of the modulus. Therefore we may write

\[ r^2 \left( \sum \pm u_1 V_2 W_3 \right)^2 = \left( \sum \pm U_1 V_2 W_3 \right)^2 \]

Then we have the equation

\[ (26) \quad (\sum \pm U_1 V_2 W_3)^2 (V_1 X_1 + V_2 X_2 + V_3 X_3)(W_1 X_1 + W_2 X_2 + W_3 X_3) = r^2 (\sum \pm u_1 V_2 W_3)^2 (V_1 x_1 + V_2 x_2 + V_3 x_3)(W_1 x_1 + W_2 x_2 + W_3 x_3) \]

For an explicit representation of \( \theta \) we may shorten the above equation by writing it in a different form. If we form the product

\[ (V_1 x_1 + V_2 x_2 + V_3 x_3)(W_1 x_1 + W_2 x_2 + W_3 x_3) = V_1 W_1 x_1^2 + (V_1 W_2 + V_2 W_1) x_1 x_3 + \ldots \]

and notice the formation of each term, we will see that subscripts of the \( x \)'s correspond to the subscripts of the \( v \)'s and \( w \)'s that are found in the coefficients of each term. So that we could write any term in the general form

\[ (v_\rho w_\sigma + v_\sigma w_\rho) x_\rho x_\sigma \]

Then to represent the above product we must form a summation where \( \rho \) and \( \sigma \) take all values from 1 to 3. Now put \( \rho = 1 \) and \( \sigma = 3 \). Then our term becomes

\[ (V_1 W_3 + V_3 W_1) x_1 x_3 \]

Again, put \( \rho = 3 \) and \( \sigma = 1 \), and our term takes the same form

\[ (V_3 W_1 + V_1 W_3) x_1 x_3 \]
This will hold for any other combination, so that in the summation there would be two terms of each type just alike and we would have a common factor 2 which must be removed to obtain the true value of the product, or

\[(v_x x_1 + v_z x_z + v_3 x_3)(w_x x_1 + w_z x_z + w_3 x_3) = I/2 \sum (v_c w_d + v_d w_c) x_c x_d +
\[(v_1 w_2 + v_2 w_1)x_1 x_2 + (v_1 w_3 + v_3 w_1)x_1 x_3 + \ldots.

Also by equation (I) we have

\[(\sum \pm u_1 v_c w_3) = \sum u_k u_\lambda (v_v, w_\mu)^{\kappa\lambda}

Making these substitutions, equation (26) becomes

\[(27) \quad (\sum \pm u_1 v_c w_3)^2(v_x x_1 + v_z x_z + v_3 x_3)(w_x x_1 + w_z x_z + w_3 x_3) =
\[\frac{I}{2} \sum (v_c w_d + v_d w_c)(v_v, w_\mu)^{\kappa\lambda} u_k u_\lambda x_c x_\tau

By equation (a)

\[(v_c w_d + v_d w_c)(v_v, w_\mu)^{\kappa\lambda} = 2 (a_\alpha a_\beta)^{\kappa\lambda}

\[(28) \quad \theta(u_1 u_2 u_3 x_1 x_2 x_3) = \sum \sum (a_\alpha a_\beta)^{\kappa\lambda} u_k u_\lambda x_c x_\tau

By equation (28) we see that \( \theta \) is a homogeneous function of the second degree in the variables, and also the coefficients of \( f \). If we linearly transform (28) we have

\[U_k = u_k; \quad X_c = x_c; \quad (a_\alpha a_\beta) = r^2(a_\alpha a_\beta)

Therefore

\[\sum \sum (a_\alpha a_\beta)^{\kappa\lambda} U_k U_\lambda X_c x_\tau = r^2 \sum \sum (a_\alpha a_\beta)^{\kappa\lambda} u_k u_\lambda x_c x_\tau

\[(29) \quad \theta = r \theta

which proves the covariant properties of \( \theta \).
Covariants. Theorem. If in the product

\[(\sum u_1 v_2 w_3)^\kappa (u_1 x_1 + u_2 x_2 + u_3 x_3)(v_1 x_1 + v_2 x_2 + v_3 x_3)(w_1 x_1 + w_2 x_2 + w_3 x_3)\]

we replace the powers and products of the third order

\[u_\lambda u_\rho u_\mu ; v_\lambda v_\rho v_\mu ; w_\lambda w_\rho w_\mu ;\]

by the corresponding coefficients

\[a_{\lambda\rho\mu}\]

and distinguish this function of \(x_1 x_2 x_3\) by

\[\Delta f (x_1 x_2 x_3)\]

then is \(\Delta f\) a \textbf{covariant} of \(f\).

Making use of the relation

\[(\sum u_1 v_2 w_3)^\kappa = r^\kappa (\sum u_1 v_2 w_3)^\kappa\]

as a consequence of it

\[(30) \quad \Delta f (x_1 x_2 x_3)^\kappa = r^\kappa \Delta f (x_1 x_2 x_3)\]

We will call \(\Delta f\) the first covariant. By comparing the two theorems for \(\Theta\) and \(\Delta f\), it is seen that

\[\Delta f = \Theta (u_1 x_1 + u_2 x_2 + u_3 x_3)\]

under the hypothesis that

\[u_\lambda u_\rho u_\mu = a_{\lambda\rho\mu}\]

Multiplying equation (28) by \((u_1 x_1 + u_2 x_2 + u_3 x_3)\) we have

\[\Delta f = \sum \sum (a_{c \sigma} a_{\cdot \cdot \leftarrow \cdot \sigma})^{\kappa\lambda} u_\lambda u_\sigma x_\sigma (u_1 x_1 + u_2 x_2 + u_3 x_3)\]

Substituting for \(u\) its value in terms of \(a\) there results

\[\Delta f = \sum \sum (a_{c \sigma} a_{\cdot \cdot \leftarrow \cdot \sigma})^{\kappa\lambda} x_c x_\sigma (a_{\leftarrow \leftarrow \cdot \leftarrow \cdot \lambda} a + a_{\leftarrow \leftarrow \cdot \leftarrow \cdot \rho} x_\rho + a_{\leftarrow \leftarrow \cdot \leftarrow \cdot \sigma} x_\sigma) = \sum \sum (a_{c \sigma} a_{\cdot \cdot \leftarrow \cdot \sigma})^{\kappa\lambda} a x x x\]

where in the summation \(c, \sigma\) and \(\lambda, \rho\) have all values from one to three. We therefore have \(\Delta f\) expressed in terms of the variables \(x_1 x_2 x_3\), and the coefficients of \(f\). By equation (30) \(\Delta f\) is not
changed by linear transformation except as to a power of the modulus, which proves that it is a covariant.

To the student anxious to investigate further, many interesting things are found in Crelle's Journal, Vol. 55.

Chapter III.

Clebsch's Symbol.

In chapter II, we saw that Aronhold established a symbolical method for representing invariants and covariants of ternary forms. He stopped at that point, not knowing that his method was perfectly general. Clebsch soon took the matter in hand and found that the symbol was applicable to all forms. He thus established the right to define functional invariants by symbolical methods. He also developed the symbol into a much more simple and abbreviated form which we will now demonstrate.

**Invariants.** Let us take the function

\[ f_n(x) = \tilde{a}_n x_1^n + \binom{n}{1} \tilde{a}_2 x_1^{n-1} x_2 + \binom{n}{2} \tilde{a}_3 x_1^{n-2} x_2^2 + \ldots + \tilde{a}_n x_n^n \]

This expression in the number of terms, the degree of \( x_1 \) and \( x_2 \) in those terms, and in the numerical coefficients resembles the expansion of the binary form.
\[(2) \quad (a_1 x_1 + a_2 x_2)^n = a_1^n x_1^n + \binom{n}{1} a_1^{n-1} a_2 x_1^{n-1} x_2 + \binom{n}{2} a_1^{n-2} a_2^2 x_1^{n-2} x_2^2 + \binom{n}{3} a_1^{n-3} a_2^3 x_1^{n-3} x_2^3 + \cdots + a_2^n x_2^n\]

If now we make a comparison of equation (1) and (2) and establish the relations

\[\overline{a}_n = a_1^n; \quad \overline{a}_1 = a_1^{n-1} a_2; \quad \overline{a}_2 = a_1^{n-2} a_2^2; \ldots\]

where the \(\overline{a}\)'s are the non-symbolical coefficients and \(a_1 a_2\) are the symbolical coefficients, we can write symbolically

\[(2') \quad f_n x = (a_1 x_1 + a_2 x_2)^n = \overline{a}_n^\prime\]

where for the sake of brevity we put

\[a_\gamma = (a_1 x_1 + a_2 x_2)\]

Now \(a_1\) and \(a_2\) are entirely independent of each other and have no meaning at all unless taken in products where the sum of the exponents is \(n\), when they can be expressed in terms of the \(\overline{a}\)'s of the original function \(f_n\).

A simple example of this notation will now be given to show its applicability. Let \(x\) be replaced by \(x'\), through the linear transformation

\[x_1 = \xi_1 x' + \eta_1 x'_1\]
\[x_2 = \xi_2 x' + \eta_2 x'_2\]

where the modulus of transformation \(r\) is defined by the relation

\[r = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix}\]

Let \(f_n(x)\) when transformed be written in the form \(f'_n(x)\) where
If we had expanded (3) in full we would have obtained

\[
f_n'(x) = \frac{a}{a_0} (s_1 x_1' + \eta_1 x_2')^n + \frac{a_1}{a_1} (s_1 x_1' + \eta_1 x_2')^{n-1} (s_2 x_2' + \eta_2 x_2') + \ldots
\]

If we had expanded (3) in full we would have obtained

\[
f_n'(x) = \left( \frac{a}{a_0} s_1'^n + \left( \frac{a_1}{a_1} s_1^{n-1} \eta_1 + \left( \frac{a_2}{a_2} s_2^n \right) \eta_2 + \ldots \right) x_1' + \left( \frac{a_1}{a_1} s_1^{n-1} \eta_1 + \left( \frac{a_2}{a_2} s_2^n \right) \eta_2 + \ldots \right) x_2' \right) + \ldots
\]

Equations (4) show us that the coefficients \( a_i' \)'s are linear functions of the \( a_i \)'s, and homogeneous of the degree \( n \) in \( s_i \eta_i \).

Now by equations (3') we may write

\[
f_n'(x) = \left( a_1 x_1' + a_2 x_2' \right)^n = a^n
\]

if we establish the conditions that

\[
a_i' = a_i'' ; \quad \bar{a}_i' = a_i'^{n-1} a_2' ; \quad \ldots \quad \bar{a}_k' = a_i'^{n-k} a_k'
\]

From our equations of transformation we have the relation by substituting for \( x_1 \) and \( x_2 \)

\[
(a_1 x_1 + a_2 x_2) = a_1 (s_1 x_1' + \eta_1 x_2') + a_2 (s_2 x_1' + \eta_2 x_2')
\]

Then by our notation

\[
(a_1 x_1 + a_2 x_2) = a_5 x_1' + a_7 x_2'
\]

We now have two symbolic expressions for \( f_n'(x') \) namely

\[
(a_1 x_1' + a_2 x_2')^n \quad \text{and} \quad (a_5 x_1' + a_7 x_2')^n
\]

which must then be identically equal. Equating coefficients of like powers of \( x \) we have
Making these substitutions in (4') there results

\[ \overline{a}_i = a_j^\omega ; \overline{a}_1 = a_{5i}^\omega a_j \quad \cdots \quad \overline{a}_k = a_{5i}^\omega a_{\eta}^\omega \]

The simple form and the clearness of the expression of this problem, plainly demonstrates the utility of the notation.

When we wish to express symbolically a power of the \( \overline{a}_i \)'s higher than the first our symbol as explained thus far fails, and we must establish other relations. Take for example the case

\[ \overline{a}_i \overline{a}_n = a_1^\omega a_2 \quad a_1^\omega a_n = a_1^\omega a_2^\omega \]

This is also equal to

\[ a_1^\omega a_2 \cdot a_1^\omega a_2^\omega \]

Now \( \overline{a}_i = a_1^\omega a_2 \) and \( \overline{a}_n = a_1^\omega a_2^\omega \)

or \( \overline{a}_i \overline{a}_n = a_1^\omega a_2 a_1^\omega a_2^\omega = a_1^\omega a_2^\omega \)

Then according to the last relation we could write

\[ \overline{a}_i \overline{a}_n = \overline{a}_i \overline{a}_n \]

But this is not true as is easily seen. To obviate this difficulty we will introduce more symbols subject to the same conditions as the \( a \)'s. That is

\[ \overline{a}_k = a_{1i}^\omega a_{2i}^\omega = b_{1i}^\omega b_{2i}^\omega = c_{1i}^\omega c_{2i}^\omega = \ldots \]

(5) or \( f(x) = a_k^\omega = b_k^\omega = c_k^\omega \)

where the \( a \)'s and \( b \)'s are regarded as equivalent symbols. So that when we represent a product of the \( \overline{a} \)'s we must introduce as many different symbols \( a,b,c,\ldots \) as there are units in the exponent of the \( \overline{a} \)'s. For example

\[ \overline{a}_1 \overline{a}_4 = a_1^\omega a_2 \cdot b_1^\omega b_2 \cdot c_1^\omega c_2 \]
Now from our relation (5) we see that these symbols are inter-
changeable, and we may write

$$a_1^* a_4 = a_1^* a_2 \cdot c_1^* c_2 \cdot b_1^* b_2^*$$

With this condition our case will be perfectly general. By way of
illustration of this principle we will form the discriminant $\Delta$ of
the binary quadratic. We have

$$f_2(x) = a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2 = a_1^* = b_1^* c_1^* = \ldots$$

But

$$\Delta = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} = a_1^* a_2^* = a_1 b_1 (a_1 b_2 - a_2 b_1)$$

The b's are introduced because if we expand $\Delta = a_0 a_2 - a_1^2$, and
by our hypothesis, we must put as many different letters a, b, c, as
we have different a's in each term.

Now $\Delta$ might equally have been expressed in the form

$$\Delta = \begin{vmatrix} b_1^2 & b_1 & b_2^* \\ a_1 & a_2 & a_2^* \end{vmatrix} = b_1 a_2 (b_2 a_2 - a_1 b_2) = -b_1 a_2 (ab)$$

Adding these two values for $\Delta$ together we have

$$2 \Delta = (a_1 b_2 - a_2 b_1) (ab) = (ab)^2$$

$$\therefore \Delta = 1/2 (ab)^2$$

Let us now consider the transformed functions, and call $\Delta$
when operated upon $\Delta'$, there results

$$\Delta' = \begin{vmatrix} a_0' & a_1' \\ a_1' & a_2' \end{vmatrix} = a_1 b_1 (a_1 b_2 - a_2 b_1)$$
\[ \Delta = \begin{vmatrix} b_1^2 & b_1 b_2 \\ a_1 a_2 & a_2^2 \end{vmatrix} = a_1 a_2 (b_1, a_2) = -a_2 b_1 (a_1 b_2) \]

\[ \therefore \quad \Delta' = I/2 (a' b')(a' b') = I/2 (a' b')^2 \]

Now by equation (a)
\[ \begin{vmatrix} a_1' & a_2' \\ b_1' & b_2' \end{vmatrix} = a_1 \xi_1 + a_2 \xi_2, a_1 \eta_1 + a_2 \eta_2 \]
\[ \begin{vmatrix} b_1 \xi_1 + b_2 \xi_2, b_1 \eta_1 + b_2 \eta_2 \end{vmatrix} = \xi_1 \xi_2, \eta_1 \eta_2 \]

But
\[ r = \begin{vmatrix} \xi_1 \xi_2 \\ \eta_1 \eta_2 \end{vmatrix} \]

\[ (a' b')^2 = r^2 (ab)^2 \]

\[ \therefore \quad \Delta' = r^2 \Delta \]

This equation shows us immediately the invariant properties of \( \Delta \) by this symbolical method of representation.

We might have written (5) in the form
\[ (a' b')^m = r^m (ab)^m \]

since \( a, b, c \), are equivalent symbols, we can write the above relation in either of the forms
\[ (a' c')^y = r^y (ab)^y \]
\[ (b' c')^x = r^x (ab)^x \]

Multiplying these three expressions together we have
\[ (a' b')^\mu (a' c')^\nu (b' c')^\psi = r^\mu (ab)^\mu (ac)^\nu (bc)^\psi \]

which is the general expression for an invariant of a form of the \( n \)'th degree. But we know that an invariant is a homogeneous function of the \( n \)'th order in the coefficients. Therefore we must have
\[ \mu + v + \lambda = n \]

Now \( \mu + v + \lambda \) are the number of determinant factors that are found in the general expression, and we can then say in general, that the order of an invariant is the exponent of the modulus of transformation; or it is also equal to the number of determinant factors.

Instead of considering a single function

\[ f_n = a^n \chi = b^n \chi = c^n \chi = \ldots \]

we could consider several forms by the condition

\[ f = a^n \chi ; \quad \psi = b^m \chi ; \quad \gamma = c^r \chi ; \ldots \]

Then

\[ (ab)^\mu (bc)^\nu (bc)^\lambda \]

represents a simultaneous invariant of \( f, \psi, \gamma \). In this case the condition of homogeneity must hold as if each form being considered separately. That is the coefficient of \( f \), must enter to the degree \( n \); those of \( \psi \) to the degree \( m \); and those of \( \gamma \) to the degree \( r \), or

\[ \mu + \lambda = n \]

\[ \mu + v = m \]

\[ v + \lambda = r \]

From this definition it is evident that the resultant of two or more forms, is nothing more nor less than a simultaneous invariant of the forms under consideration. For example, let us calculate the simultaneous invariant of
\[ f = \alpha_0 x_1^2 + 2\alpha_1 x_1 x_2 + \alpha_2 x_2^2 \]
\[ Q = \beta_0 x_1^2 + 2\beta_1 x_1 x_2 + \beta_2 x_2^2 \]
\[ \psi = \gamma_0 x_1^2 + 2\gamma_1 x_1 x_2 + \gamma_2 x_2^2 \]

Forming the resultant, regarding these forms as linear expressions in the quantities \( x_1, x_1 x_2, x_2 \) we have

\[
\begin{vmatrix}
\alpha_0 & \alpha_1 & \alpha_2 \\
\beta_0 & \beta_1 & \beta_2 \\
\gamma_0 & \gamma_1 & \gamma_2 \\
\end{vmatrix}
\]

If now we introduce the symbolical coefficients, \( R \) becomes

\[
R = \frac{a^2}{b^2} \frac{a_1 a_2}{b_1 b_2} \frac{a_3}{b_3} = (ab)(bc)(ac)
\]

Substituting in equation (b) we find

\[
\mu + \lambda = 1 + 1 = 2 ; \quad \mu + v = 2 ; \quad v + \lambda = 2 .
\]

Our condition is satisfied and \( R \) must be an invariant.

**Covariants.** We have already proven that by linear transformation \( a_x \) goes over into \( a'_x \). If now we multiply both sides of equation (5') by these relations we have

\[(a' b') a'_x = r(ab) a_x \]

Raising both sides to the \( \mu \)'th power there results

\[(a' b')^\mu a'^\mu = r^\mu(ab)^\mu a_x^\mu \]

Likewise

\[(b' c')^\nu b'_x = r^\nu(bc)^\nu b_x^\nu \]

\[(a' c')^\nu c'_x = r^\nu(ac)^\nu c_x^\nu \]

Multiplying these relations together,

\[(a' b')^\mu (b' c')^\nu (a' c')^\nu a'^\mu b'_x c'_x = r^\mu \nu \nu(ab)^\mu (bc)^\nu (ac)^\nu a_x^\mu b_x^\nu c_x^\nu \]
We might have used the \( a_x b_x \ldots \) to any other power and our relation would still hold; then we have in general

\[
(a' b')^{\nu}(c' d')^{\lambda}(a' c')^{\tau} b^{\lambda} c_{\lambda} = r^{u v \lambda \nu}(ab)^{\mu}(bc)^{\nu}(ac)^{\tau} a^{r} b^{\mu} c_{\lambda}
\]

which is the general expression for a covariant, providing relations \( \Lambda \) are fulfilled, which will now be given.

Now a covariant is homogeneous in the coefficients of the function under consideration, and since in \( a^x \), the coefficients enter to the degree \( \sigma \), we will have as conditions to be satisfied

\[
\begin{align*}
\mu + \lambda + \sigma &= n \\
\mu + \nu + \rho &= n \\
v + \lambda + \kappa &= n
\end{align*}
\]

since each coefficient must enter to the degree \( n \). If the symbols \( a_x, b_x, c_x, \ldots \) are entirely distinct we will have a simultaneous covariant. When they are equal, then our covariant is one of a single function.

**Example.** Let

\[
\begin{align*}
\phi &= a^3 = a_x x^3 + 3a_1 x^2 y + 3a_2 x y^2 + a_3 y^3 \\
f &= b^3 = b_x x^3 + 3b_1 x^2 y + 3b_2 x y^2 + b_3 y^3 \\
\psi &= c^3 = c_x x^3 + 3c_1 x^2 y + 3c_2 x y^2 + c_3 y^3
\end{align*}
\]

Then

\[(ab)^{\lambda}(ac)^{\tau} b_x c_{\lambda} \ldots\]

is a simultaneous covariant of the three forms, since the coefficients \( a, b, c \) are each found in the expression to the degree three. In these expressions for covariants and invariants, if the degree of any determinant factor is odd, the expression vanishes identically.
Now \((ab)^3\) is the invariant of the cubic, and from the above expression, we can say that the cubic has no invariant of the second order. \((ab)^3\) will be of the second order, for each term in it contains two coefficients. That is

\[(ab)^3 = (a_1 b_2 - a_2 b_1)^3 = a_1^3 b_2^3 - 3a_1^2 b_2 a_1 b_1 + 3a_1 b_2 a_1^2 b_1 - a_1 b_1^3\]

But

\[a_1 = \overline{a}_1; \quad b_2 = \overline{b}_2; \quad a_1 b_1 = \overline{a}_1 \overline{b}_2; \quad b_1 b_1 = \overline{b}_2\]

\[(ab)^3 = \overline{a}_1 \overline{b}_2 - 3a_1 \overline{b}_2 + 3a_1 \overline{b}_1 - a_1 \overline{b}_2 = 0\]

which is evidently of the second order.

**Ternary Forms.**

**Invariants.** The symbolical notation can now be extended very easily to ternary forms, after its application to binary forms has been thoroughly understood. We will represent a ternary form by

\[
f_n(x) = \overline{a}_{n,0} x_1^n + n\overline{a}_{n-1,0} x_1^{n-1} x_2 + n(n-1)\overline{a}_{n-2,1} x_1^{n-2} x_2 x_3 + n(n-1)(n-2)\overline{a}_{n-3,2} x_1^{n-3} x_2^2 x_3 + \ldots + n! \frac{\overline{a}_{n-j,k} x_1^j x_2^k x_3^n}{j!k!}\]

Since the form is homogeneous, and of the \(n\)'th degree with the variables \(x_1 x_2 x_3\), we must have the condition imposed upon each term that

\[i + j + k = n\]

Let us now expand the trinomial
(8) \[ a^n_i = (a_1 x_1 + a_2 x_2 + a_3 x_3)^n = a_1^n x_1^n + na_1^{n-1} a_2 x_1^{n-1} x_2 + \ldots \]
\[ n(n - 1) a_1^{n-2} a_2 a_3 x_1^{n-2} x_2 x_3 + \ldots + \frac{n!}{1! j! k!} a_1^j a_2^k x_1^j x_2^k + \ldots \]

where as before \( i + j + k = n \) and \( a_x = (a_1 x_1 + a_2 x_2 + a_3 x_3) \).

Comparing (7) and (8), we see that the only difference between the two is in the literal coefficients. But we also see that in (8) the literal coefficients are homogeneous in each term to the degree \( n \). Therefore we can equate these coefficients, or

\[ \bar{a}_{n,0} = a_1^{n-1} a_2; \quad \bar{a}_{n,1,0} = a_1^{n-1} a_3; \quad \bar{a}_{n-2,1,1} = a_1^{n-2} a_2 a_3; \]

As in the case of binary forms, the \( a_1 a_2 a_3 \)'s have no meaning, except as they enter in a product to the degree \( n \), when they are completely characterized.

Let us now transform the \( x \)'s by the substitution

\[
\begin{align*}
x_1 &= \xi_1 x_1' + \eta_1 x_2' + \xi_1' x_3' \\
x_2 &= \xi_2 x_1' + \eta_2 x_2' + \xi_2' x_3' \\
x_3 &= \xi_3 x_1' + \eta_3 x_2' + \xi_3' x_3'
\end{align*}
\]

We will represent the determinant of the coefficients \( \xi, \eta \) and \( \xi' \) by \( \sigma \)

\[ \sigma = (\xi, \eta, \xi') \]

Making the above substitution in \( f_n(x) \) we have

\[ f_n(x) = \bar{a}_{n,0} (\xi_1 x_1' + \eta_1 x_2' + \xi_1' x_3')^n + n \bar{a}_{n,1,0} (\xi_1 x_1' + \eta_1 x_2' + \xi_1' x_3')^{n-1} (\xi_2 x_2' + \eta_2 x_2' + \xi_2' x_3') + n(n - 1) \bar{a}_{n-2,1,1} (\xi_1 x_1' + \eta_1 x_2' + \xi_1' x_3')^{n-2} (\xi_2 x_2' + \eta_2 x_2' + \xi_2' x_3') (\xi_3 x_3' + \eta_3 x_3' + \xi_3' x_3') + \ldots + \frac{n!}{1! j! k!} a_{i,j,k} (\xi_1 x_1' + \eta_1 x_2' + \xi_1' x_3')^j (\xi_2 x_2' + \eta_2 x_2' + \xi_2' x_3')^k + \ldots =
\]

\[ \sum \frac{n!}{1! j! k!} \bar{a}_{i,j,k} x_1^i x_2^j x_3^k. \]

We will expand the first term of the above equation, and the con-
clusion drawn in regard to that, will be found to hold for all the remaining terms. Thus

\[ \bar{a}_{n,0} (\xi_1 x'_1 + \eta_1 x'_2 + \zeta_1 x'_3)^n = \bar{a}_{n,0} \xi_1 x'_1^n + \ldots. \]

If we expand all the terms and collect the coefficients of \( x'_1^n \), we will have

\[ (\bar{a}_{n,0} \xi_1 + \bar{a}_{n-1,0} \xi_1 \xi_2 + \ldots) x'_1^n = \bar{a}'_{n,0} \]

and so on.

Again if we represent \( f'(x) \) by the trinomial expression, we have

\[ f'(x) = (a'_1 x'_1 + a'_2 x'_2 + a'_3 x'_3)^n = a'_1 x'_1^n + na'_1 a'_2 x'_1^{n-1} x'_2 + \ldots \]

Again comparing coefficients we may write

\[ a'_1^n = (\bar{a}_{n,0} \xi_1 + \bar{a}_{n-1,0} \xi_1 \xi_2 + \ldots) = a_1^n \]

since the expression is homogeneous and of the degree \( n \) in the \( \xi \)'s.

Likewise it is found that

\[ a'_n = a_n; \quad a'_3 = a_3 \]

But

\[ \bar{a}'_{n,0} = a'_1; \quad \bar{a}'_{n-1,0} = a'_1 a'_2 \ldots \]

Therefore

\[ \bar{a}'_{n,0} = a'_3; \quad \bar{a}'_{n-1,0} = a'_3 a'_n; \quad \bar{a}'_{n-1,0} = a'_3 a'_n a'_3; \ldots \]

As in the case of binary forms, when the function of the \( a \)'s that is to be expressed symbolically is of a higher order than the first, we must introduce additional symbols. That is

\[ \bar{a}'_{n,0} = a'_1 b'_1 \ldots \]
\[ \bar{a}'_{n,0} = a'_1 b'_1 \ldots \]

We will form the discriminant of the quadratic
\[ f_{L}(x) = a_{2,1}x^2 + a_{2,2}x^3 + a_{3,3} + 2a_{1,1}x_3 x_3 + 2a_{1,2}x_3 x_1 + 2a_{1,3}x_1 x_1 = \]
\[ a_{x} = b_{x} = c_{x} \ldots \ldots \]

Calling the discriminant \( \Delta \) we have
\[
\begin{vmatrix}
    a_{2,0,0} & a_{1,1,0} & a_{1,0,1} \\
    a_{1,1,0} & a_{1,0,1} & a_{0,1,1} \\
    a_{1,0,1} & a_{0,1,1} & a_{0,0,1}
\end{vmatrix}
\]

We will replace these values by the symbols, and we obtain
\[
\begin{vmatrix}
    a_{1} & a_{1}a_{1} & a_{1}a_{3} \\
    b_{1}b_{2} & b_{2}b_{3} & b_{3}b_{1} \\
    c_{1}c_{3} & c_{2}c_{3} & c_{3}c_{1}
\end{vmatrix}
= \begin{vmatrix}
    a_{1} & a_{2} & a_{3} \\
    b_{1} & b_{2} & b_{3} \\
    c_{1} & c_{2} & c_{3}
\end{vmatrix}
\]

We make use of \( b \) and \( c \), since by our hypothesis we must introduce a new symbol for each power of the \( a \)'s.

Now by the principles of determinants we can interchange the rows and the sign only will be changed. Since the symbols \( a_{1}a_{1}, b_{1}b_{2}, c_{1}c_{3} \), are found in the first, second, and third rows respectively, interchanging the rows is equivalent to interchanging the symbols \( a, b, c \). Then interchanging these symbols in \( \Delta \) in all possible ways, and dividing out the common factor from each row, our result will be the same as if we simply took the expression \( a_{1}b_{2}c_{3}(abc) \) and interchanged the subscripts \( 1, 2, 3 \) in all possible ways, two at a time, which we know to be six. That is, it can easily be seen that if we were to interchange \( a \) and \( b \) in \( \Delta \) we would have
\[
\Delta = - \begin{vmatrix}
 b_1^2 & b_1 b_2 & b_1 b_3 \\
 a_1 a_2 & a_2 a_3 & a_3 a_1 \\
 c_1 c_3 & c_2 c_3 & c_3^2 \\
\end{vmatrix} = - b_2 a_2 c_3 (abc)
\]

which shows that our result is exactly the same as if we had interchanged (1) and (2) in \( a_1 b_2 c_3 \), and changed the sign. Extending this method to all possible cases and summing up all these values for \( \Delta \) which will be six in number, we have

\[
6 \Delta = (a_1 b_2 c_3 - a_1 c_2 b_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3)abc = (abc)^2
\]

since the expression in parenthesis is nothing but the expansion of

\[
\begin{vmatrix}
 a_1 & a_2 & a_3 \\
 b_1 & b_2 & b_3 \\
 c_1 & c_2 & c_3 \\
\end{vmatrix}
\]

\[
\therefore \quad \Delta = - 1/6 (abc)^2
\]

In general let \( \rho, \sigma, \tau \) ... be symbols for several forms of the type

\[
f = a_\lambda^n = b_\mu^n = c_\nu^n = ... \quad ; \quad \lambda = a_\chi^n = b_\mu^n = c_\nu^n = ...
\]

and \( \rho, \sigma, \tau \) ... the symbols for the corresponding forms \( f' \), \( \lambda' \) ... which have been subjected to linear transformation. But we have proven already that when such an operation is performed
Then

\[
\begin{vmatrix}
\sigma_1' & \sigma_2' & \sigma_3' \\
\tau_1' & \tau_2' & \tau_3' \\
T_n & T_m & T_s
\end{vmatrix} =
\begin{vmatrix}
\sigma_1 & \sigma_2 & \sigma_3 \\
\tau_1 & \tau_2 & \tau_3 \\
S_n & S_m & S_s
\end{vmatrix} =
\begin{vmatrix}
\xi_1 & \xi_2 & \xi_3 \\
\eta_1 & \eta_2 & \eta_3 \\
\eta_1' & \eta_2' & \eta_3'
\end{vmatrix}.
\]

Likewise

\[ (a b c') = r (a b c), \text{ or } (a b c')^2 = r^2 (a b c)^2. \]

That is

\[ \Delta' = r^2 \Delta. \]

which shows the invariant properties of \( \Delta \).

For the cubic

\[ f_3(x) = a_1^3 = b_1 = c_1 = d_1, \]

the discriminant can be written

\[ S = (abc)(abd)(acd)(bcd), \]

where \( a, b, c, \) and \( d \), each appear to the third power. \( S \) must in-
volve four symbols, because if we had only \( a, b, c \), the only way we could include the symbols to the third power, would be in the form

\[(abc)^3\]

which is identically zero. Then since there are four symbols, our invariant is of the fourth order, since there will be four coefficients in each term of the expansion of the above.

**Covariant.** Likewise

\[\Delta = (abc)^3 a_x b_x c_x\]
is the only covariant of \( f_3(x) = a_x^3 = b_x^3 = c_x^3 \). It is a covariant since each coefficient enters to the third degree. It is the only covariant, aside from the function itself, of the cubic since the only formation besides this in which the variables and coefficients enter to the third degree is

\[\Delta = (abc) a_x^3 b_x^3 c_x^3 = 0\]

**Certain Identities.** I will here develop three identical relations, that are extensively used in the application of this symbol. We will take

\[
\begin{vmatrix}
a_1 & a_2 & 0 \\
b_1 & b_2 & 0 \\
c_1 & c_2 & 0 \\
\end{vmatrix} = 0
\]

Now if we multiply the first column by \( x_1 \), and the second by \( x_2 \), and then add their sum to the third column we have

\[
\begin{vmatrix}
a_1 & a_2 & a_x \\
b_1 & b_2 & b_x \\
c_1 & c_2 & c_x \\
\end{vmatrix} = (ab) c_x + (bc) a_x - (ac) b_x = 0
\]
(I) \( (ab) c_x + (bc) a_x + (ca) b_x = 0 \)

If now we put \( x_1 = d_z \), \( x_2 = -d_1 \), we have

\[
\begin{vmatrix}
    a_1 & a_2 & a_1 d_z - a_2 d_1 \\
    b_1 & b_2 & b_1 d_z - b_2 d_1 \\
    c_1 & c_2 & c_1 d_z - c_2 d_1 \\
\end{vmatrix}
= \begin{vmatrix}
    a_1 & a_2 & (ad) \\
    b_1 & b_2 & (bd) \\
    c_1 & c_2 & (cd) \\
\end{vmatrix}
\]

\(- (ab)(cd) + (bc)(ad) - (ac)(bd) = 0\)

(II) \( (ab)(cd) + (bc)(ad) + (ca)(bd) = 0 \)

Again if \( c_1 = y_z \), and \( c_2 = -y_1 \), equation (I) becomes

\[
(ab)(x_1 y_z - x_z y_1) + (b_1 y_1 + b_z y_z) a_x + (a_1 y_1 + a_z y_z) b_x = 0
\]

(III) \( b_x a_x - a_y b_y = (ab)(xy) \).

These same relations can be easily extended to ternary forms, by adding another symbol, but which for the lack of time will not be developed here.

Chapter IV.

Applications of the Symbols.

/.. Polars. Let us consider a homogeneous function of the \( n \)th degree, which we will designate by

\[
f = a_x = a_x a_y \ldots . a_x
\]

to \( n \) factors. If now we remove \( k \) of the factors \( a_x \), and replace them by \( a_y \), we will obtain

\[
a_{1x} a_{2x} \ldots a_{nx} a_{1y} a_{2y} \ldots a_{ny} = a_{x}^{n-k} a_{y}^{k}
\]
This is known as the k'th polar of \( f \), with respect to \( y \), and we express the operation symbolically by

\[
(f_1)_y = a^x_y a^y_x
\]

To show that this is a polar, let us consider \( a^x_x \), and form the first polar with respect to \( y \). Then by our notation we must have

\[
f_y = a^x_y a^y_x = (a_1 x_1 + a_z x_z) y^x + (a_1 y_1 + a_z y_z)
\]

(2)

But

\[
\begin{align*}
a^1_1 &= a^1_z; & a^z_1 &= a^z_z; & a^1_z &= a^1_1; & a^z_z &= a^z_1;
\end{align*}
\]

where

\[
f_y = (a^1_1 x_1 + a^z_z x_z) y^x + 2(a^z_x x_1 + a^1_x x_z) y_1 y_z + (a^1_x x_1 + a^z_x x_z) y_y
\]

Making these substitutions in (2) we have finally

\[
f_y = (a^1_1 x_1 + a^z_z x_z) y^x + 2(a^z_x x_1 + a^1_x x_z) y_1 y_z + (a^1_x x_1 + a^z_x x_z) y_y
\]

which we know to be polar when represented non-symbolically.

(2). Differential Method. Let us form the first differential quotients of \( f = a^x_x \) with respect to \( x_1 \) and \( x_z \).

\[
(3) \quad \frac{d f}{dx_1} = \frac{n a^1_1 x_1^{n-1} + (n-I)(n-1) a^1_z x_z^{n-1} x_1}{1.2} + \frac{n(n-I)(n-2) a^z_z x_z^{n-1} x_1}{1.2}
\]

\[
= \frac{n(a^1_1 x_1^{n-1} + (n-I) a^1_z x_z^{n-1} x_1 + (n-I)(n-2) a^z_z x_z^{n-1} x_1)}{1.2}
\]

\[
= n a^1_1 (a^1_1 x_1 + a^z_z x_z)^{n-1} = n a^1_1 x_1^{n-1} = nf_1.
\]

(4) \quad \frac{d f}{dx_z} = \frac{n a^z_z x_z^{n-1} + 2 n(n-I) a^z_z x_z^{n-1} x_z + 3 n(n-I)(n-2) a^1_1 x_1^{n-1} x_z}{1.2.3}

\[
= \frac{n a^z_z (a^z_z x_z^{n-1} + (n-I) a^1_1 x_1^{n-1} x_z + (n-I)(n-2) x_1^{n-1} x_z)}{2}
\]

\[
= n a^z_z (a_1 x_1 + a_z x_z)^{n-1} = na^z_z x_z^{n-1} = nf_z.
\]

where \( I/n f_1 \), and \( I/n f_z \) represent the derivative of \( f \) with
respect to $x_1$ and $x_2$ respectively. Multiplying (3) and (4) by $y_1$ and $y_2$ respectively and adding, we get

$$(a_1 y_1 + a_2 y_2) a_x^{n-1} = y_1 f_1 + y_2 f_2.$$  

The first polar is then defined by the equation

$$f_y = y_1 f_1 + y_2 f_2 = I/n \left\{ y_1 \frac{\partial f}{\partial x_1} + y_2 \frac{\partial f}{\partial x_2}\right\}$$

Again, by exactly the same operation, we obtain

$$\frac{\partial^2 x}{\partial x_1^2} = n(n-1)a_{11} a_x^{n-2} = n(n-1) f_{11}$$

$$\frac{\partial^2 x}{\partial x_1 \partial x_2} = n(n-1) a_{12} a_x^{n-2} = n(n-1) f_{12}$$

$$\frac{\partial^2 x}{\partial x_2^2} = n(n-1) a_{22} a_x^{n-2} = n(n-1) f_{22}$$

The same can be very easily seen to be true, since we saw by (3) and (4) that when we differentiated $f$ with respect to $x_1$ or $x_2$ we had a common factor $a_1$ or $a_2$. Therefore in the second equation of (5) when we differentiate with respect to $x_1$ and $x_2$ both, we will have the coefficient $a_{12}$. It is also easily seen that there will be a common factor $(n-1)$, since (3) and (4) are homogeneous of the degree $(n-1)$, so that when we operate as shown in (5), each term will contain $(n-1)$ as a factor. If now we multiply the equations (5) by $y_1^2, y_1 y_2, y_2^2$ respectively and add, we will have

$$y_1^2 f_{11} + 2 y_1 y_2 f_{12} + y_2^2 f_{22} = (a_1^2 y_1^2 + 2 a_1 a_2 y_1 y_2 + a_2^2 y_2^2) a_x^{n-2} = a_{11} a_x^{n-2}$$

Since $a_{11} = (a_1 y_1 + a_2 y_2)^2 = a_1^2 y_1^2 + 2 a_1 a_2 y_1 y_2 + a_2^2 y_2^2$

$$f_y = I/n(n-1) \left\{ y_1^2 \frac{\partial^2 f}{\partial x_1^2} + 2 y_1 y_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + y_2^2 \frac{\partial^2 f}{\partial x_2^2}\right\}$$

By this differential method, we can see that if we form the $v$'th polar of a form of the $n$'th degree, it will be zero identically,
when \( v > n \). For differentiating \( n \) times, will remove the variables, and one more operation will give us zero.

Example. To form the first polar of the quadratic

\[
f = a_x^2 = \bar{a}_x x_1^2 + 2\bar{a}_1 x_1 x_2 + \bar{a}_x x_2^2
\]

\[
f_y = a_x a_y = (a_1 x_1 + a_x x_2)(a_1 y_1 + a_x y_2)
\]

\[
= (a_1^2 x_1 + a_1 a_x x_2) y_1 + (a_x a_1 x_1 + a_x^2 x_2) y_2
\]

\[
= (\bar{a}_x x_1 + \bar{a}_1 x_2) y_1 + (\bar{a}_x x_1 + \bar{a}_x x_2) y_2
\]

By the differential method

\[
f_y = \frac{1}{2} \left( y_1 \frac{\partial}{\partial x_1} f + y_2 \frac{\partial}{\partial x_2} f \right) = \frac{1}{2} \left( y_1 (2\bar{a}_x x_1 + 2a_x x_2) 
\right.

\[
+ y_2 (2\bar{a}_1 x_1 + 2\bar{a}_x x_2) \right)
\]

which proves the identity of the two operations.

(3). Binomial Method. In \( a_x \) let us put

\[
x_1 = x_1 + \sqrt[\wedge]{y_1}
\]

\[
x_x = x_x + \sqrt[\wedge]{y_2}
\]

or \( a_x = a_1 x_1 + a_x x_2 = a_1 x_1 + a_x \sqrt[\wedge]{y_1 + a_x x_2 + a_x \sqrt[\wedge]{y_2}} \)

\[
= a_1 x_1 + a_x x_2 + \sqrt[\wedge]{(a_1 y_1 + a_x y_2)} = a_x + \sqrt[\wedge]{a_y}
\]

Then

\[
(a_x + \sqrt[\wedge]{a_y})^n = a_x^n + \binom{n}{1} \sqrt[\wedge]{a_x} a_y + \binom{n}{2} \sqrt[\wedge]{a_x} a_y^2 + \ldots + a_y^n = f(x + \sqrt[\wedge]{y})
\]

The coefficient of \( \sqrt[\wedge]{a}^k \) is

\[
\binom{n}{k} a_x^{n-k} a_y
\]

If we divide this by \( \binom{n}{k} \) we will have the \( k \)'th polar of \( f \) with respect to \( y \). So that we can say in general that if we wish to form the \( m \)'th polar of \( f \), we must make the linear substitutions as above; develop by the binomial theorem in powers of \( \sqrt[\wedge]{a} \), and the coefficient of \( \sqrt[\wedge]{a}^m \) divided by \( \binom{m}{m} \), is the \( m \)'th polar.
Example. Let
\[ f = a_i (x_1 + y)^3 + 3a_1 (x_1 + y)^2 (x_2 + y) + \ldots \]
\[ + 3a_2 (x_1 + y) (x_2 + y)^2 + a_3 (x_2 + y)^3 \]
\[ = a_1 x_1^3 + 3a_1 x_1 y_1 y + 3a_1 x_1 y_2 y^2 + \ldots + 3a_1 x_1 y_3 x_2 y + 6a_2 x_1 x_2 y \]
\[ + 3a_2 x_1 y_3 y^2 + 3a_3 x_2 x_3 x_4 y + 3a_2 x_1 x_2 x_3 y^2 \]
\[ + 3a_3 x_1 y_3 y^2 + 3a_2 x_1 x_2 x_3 y^2 + 3a_2 x_1 x_2 x_3 y^3 + \ldots \]
\[ + 3a_3 x_2 x_3 x_4 y + 3a_3 x_2 x_3 x_4 y^2 + \ldots \]

Here to obtain the second polar with respect to \( y \), we have \( n = 3 \) and \( k = 2 \). Then collecting the terms which contain \( y^2 \), there results
\[ (3) \left( \frac{\partial}{\partial y} \right)^2 f y^2 = 3 \left( \frac{\partial}{\partial y} \right)^2 \left( (a_1 x_1 + a_1 x_2) y^2 + 2y_1 y_2 (a_1 x_1 + a_2 x_2) + y_3 (a_2 x_1 + a_3 x_2) \right) \]
\[ \begin{align*}
&= a_1 a_2^2.
\end{align*} \]
which has already been shown to be identical.

Transvection.

I. Before discussing the transvectant of a function, we will introduce a new operation, that must be here employed. If we have
\[ P = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n \]
and replace \( k \) of the factors \( a_1, a_2, \ldots a_m \) and \( b_1, b_2, \ldots b_n \) respectively by \((ab)^k\), we will have
\[ (C) \quad \mu = (ab)^k a_x^{m-k} b_y^{n-k}. \]
This operation is called the "folding process", and the result obtained in \( (C) \) is known as the transvectant of the two forms \( a_x \)
the transvectant is expressed symbolically
\[ U = (f, Q)^k \]
Now \( k \) can assume all values from 0 to \( n \), where in the two homogeneous functions \( f \) and \( Q \) of the degrees \( n \) and \( m \) respectively, \( n < m \). When \( k = 0 \) we have
\[
(f, Q)^0 = a^m \ b^n \\
(f, Q)^1 = (ab) \ a^{m-1} b^{n-1} \\
\ldots \ldots \ldots \\
(f, Q)^n = (ab)^n b^{m-n}
\]
We see immediately from this equation, that the highest transvectant that we can from, is that of the lowest exponent of the two forms \( a^m \) and \( b^n \). Otherwise they would vanish identically. It is also well to notice that any transvectant in which the exponent of \( (ab) \) is odd is identically zero, because by interchanging \( a \) and \( b \), we change the sign of the determinant.

Transvectant of a product of forms. We will now investigate the case of forming the transvectant of two forms, in which each of these forms consists of a product of simpler forms. The transvectant will then be equal to the sum of several simple transvectants.

Let \( F \) be the product of the forms
\[ f_1 = a^m \ ; \ f_2 = b^n \ ; \ldots \ ; f_k = k^p \]
and \( \Phi \) the product of
\[ \varphi_1 = \alpha^\mu \ ; \ \varphi_2 = \beta^\nu \ ; \ldots \ ; \varphi_k = \xi^\pi \]
For the symbolical factors of the forms \( f_\lambda \), we will introduce the
linear factors \( r \); let the number of them be \( S \), so that \( S \) is equal to the sum of the degrees of \( f_1, f_2, \ldots, f_k \), or

\[ S = m + n + \ldots + p \]

Again let \( \mathfrak{c}_i \) be the linear factors of the forms \( \mathfrak{q}_i \), the number of which is \( \mathfrak{f} \) or

\[ \mathfrak{f} = u + v + \ldots + \pi \]

To obtain the \( k \)'th transvectant of \( F \) over \( \mathfrak{f} \) we will consider the product

\[ \Pi = r_{1, \mathfrak{f}} r_{2, \mathfrak{f}} \ldots r_{s, \mathfrak{f}} \mathfrak{c}_{1, \mathfrak{f}} \mathfrak{c}_{2, \mathfrak{f}} \ldots \mathfrak{c}_{\mathfrak{f}, \mathfrak{f}} \]

We will now make \( k \) foldings of these factors, using in all possible ways, any \( k \) factors of \( r_{1, \mathfrak{f}} \ldots r_{s, \mathfrak{f}} \), with any \( k \) of \( \mathfrak{c}_{1, \mathfrak{f}} \ldots \mathfrak{c}_{\mathfrak{f}, \mathfrak{f}} \), and summing up such terms. But in any term we can permute the \( S \) and \( \mathfrak{c} \) factors \( k \) at a time, all permutations of which will be the same. Now the permutations of \( S \) things \( K \) at a time is \( \frac{S!}{(S-K)!} \), and of \( \mathfrak{c} \) things \( K \) at a time \( \frac{\mathfrak{f}!}{(\mathfrak{f}-K)!} \).

Hence for every term in the summation, by these permutations, in all possible ways we will have \( \frac{S!}{(S-K)!} \cdot \frac{\mathfrak{f}!}{(\mathfrak{f}-K)!} \) terms exactly alike, and this factor must be removed, that is, we must divide by the above factor. Our general term then becomes

\[ (F, \mathfrak{f})^k = \frac{(S-K)! \cdot (\mathfrak{f}-K)!}{S! \mathfrak{f}!} \]

\[ \Sigma (r_{i_1} \mathfrak{c}_{\lambda_1}) (r_{i_2} \mathfrak{c}_{\lambda_2}) \ldots (r_{i_k} \mathfrak{c}_{\lambda_k}) r_{i_{k+1}} x \ldots r_{i_\mathfrak{f}} x \]

\[ \mathfrak{c}_{\lambda_{k+1}} \mathfrak{c}_{\lambda_{\mathfrak{f}}} x \]

If now we replace these linear factors by the symbolical coefficients as follows
and add equal terms together we will have the transvectant of $F$ over $\phi$ expressed in terms of the coefficients of the original form.

Transvectant of the Polar. We introduce here a method for the formation of the transvectant, which may seem very superfluous in comparison with the folding process; but which when an extended investigation of this subject is made, will be found to be very useful. Let us consider the product

$$P - f \cdot \phi = a_x^n \cdot b_x^n$$

Forming the $k$th polar of each of the quantics of this product according to the method developed in paragraph I of this chapter, we have for the product of these $k$th polars;

$$f_y^k \cdot \phi_y^k = a_x^{m-k} a_y^k \cdot b_x^{n-k} b_y^n$$

If now we fold $K$ times each number of this equation according to $y$, we have as a result the following relation

$$(f_y^k \cdot \phi_y^k)^K = (ab)^k a_x^{m-k} b_x^{n-k}$$

where the first number is the $K$th transvectant of the $K$th polar of $f$ over the $K$th polar of $\phi$, while the second member is directly the $K$th transvectant of $a_x^n$ over $b_x^n$. This gives us the following important theorem:

The $K$th transvectant of the $K$th polars of two forms is equal to the $K$th transvectants of those forms.
Example. Let us now form the second transvectant of $f$ over $\mathcal{Q}$ where

$$f = a_\chi = \bar{a}_0 x_1^2 + 2\bar{a}_1 x_1 x_x + \bar{a}_x x_x^2$$
$$\mathcal{Q} = b_\lambda = \bar{b}_0 x_x^2 + 2\bar{b}_x x_1 x_x + \bar{b}_x x_x^2$$

Then $(f, \mathcal{Q})^2 = (f^2, \mathcal{Q}) = (a_\chi, b_\lambda)^2 = (ab)^2$

$$= a_1^2 b_x^2 - 2a_1 b_x a_x b_x + a_x^2 b_1^2 = a_x b_x - 2a_1 b_x + \bar{a}_x \bar{b}_x$$

which we know to be a simultaneous invariant.

(b) To form the third transvectant of

$$f = a^3_\chi ; \quad \mathcal{Q} = b^3_\lambda$$

$$(f, \mathcal{Q})^3 = (a^3_\chi, b^3_\lambda)^3 = (ab)^3 = a_x b_x - 3a_1 b_x + 3a^2 b_x - a_x b_x$$

Here again we have a simultaneous invariant. Then in general we can say that if we form the $n$'th transvectant of two forms of the $n$'th degree, we will get a simultaneous invariant.

We will now form the second transvectant of $f$ over $\mathcal{Q}$. We have already proven that

$$f_{y*} = a_\chi^2 b_\lambda = (\bar{a}_0 x_1 + \bar{a}_x x_x)y_x^2 + 2(\bar{a}_1 x_1 + \bar{a}_x x_x)y_x y_x + (\bar{a}_x x_1 + \bar{a}_x x_x)y_x^2$$

If now we put $b_x$ for $y_x$ and $-b_1$ for $y_x$ and multiply by $b_\chi$, which is equivalent to the folding process, the left side of our equation becomes the form desired, or

$$(ab)^2 a_\chi b_\chi = (\bar{a}_0 x_1 + \bar{a}_x x_x) b_x^2 b_\chi - 2(\bar{a}_1 x_1 + \bar{a}_x x_x) b_1 b_x b_\chi + (\bar{a}_x x_1 + \bar{a}_x x_x) b_x^2 b_\chi$$

Now putting $b_\chi = b_x x_x + b_x x_x$

$$(ab)^2 a_\chi b_\chi = (\bar{a}_0 x_1 + \bar{a}_x x_x) (b_x^2 b_x x_x + b_x^2 x_x) - 2(\bar{a}_1 x_1 + \bar{a}_x x_x) (b_x^2 b_x x_1 + b_1 b_x b_x x_x)$$

But $b_x b_x^2 = \bar{b}_1 ; \quad b_x^2 = \bar{b}_3 ; \quad b_1 b_x = \bar{b}_x ; \quad b_1^3 = \bar{b}_x$

$$(ab)^2 a_\chi b_\chi = (\bar{a}_0 b_x - 2\bar{a}_1 b_x + \bar{a}_x \bar{b}_x) x_x^2 - (\bar{a}_1 \bar{b}_x + \bar{a}_x \bar{b}_1 - \bar{a}_x \bar{b}_x + \bar{a}_x \bar{b}_x) x_x x_1 x_x + (\bar{a}_1 \bar{b}_x - 2\bar{a}_1 \bar{b}_1 + \bar{a}_1 \bar{b}_x) x_x^2$$

This last expression is a simultaneous covariant of two cubics.
Then we can say, in general, that the K'th transvectant of \( f \) over \( \ell \) is a simultaneous covariant, when \( K \leq n \) the degree of \( f \) and \( \ell \).

There is a vast amount of literature in "Gordan's Invariantentheorie," showing the application of this symbol, but I cannot consider the matter farther in this paper.

Chapter V.

Comparison of the Symbols.

In studying the history of these three symbols that I have just discussed, Cayley's symbol seems to have been the first contributed, followed a few years after by that of Aronhold. These symbols were no doubt welcomed by the mathematicians of that time who were investigating in this field, much as the crude telescope constructed by Galileo was welcomed by the earlier astronomers. But with all inventions, improvements have been made, along with the advancement of science. So with these first symbols, they have been replaced by the much greater symbol of Clebsch. While Cayley's symbol is a master production from a mathematical standpoint, yet it lacked one important feature in its application, namely brevity. In applying it to forms for invariants and covariants of the second order, it presents no difficulty; but when we attempt to apply it to forms, for invariants and covariants of a
higher order, the work is very laborious. In fact the work is just as difficult and tedious as the non-symbolic methods of calculation. For this reason it seems to have never been adopted generally by mathematicians. The symbol possesses some good points, however, in this, that it tells us immediately the order of the invariant, and the order and degree of the covariant. For instance, applied to a function of the \( n' \)th degree, will give us a covariant of the second order, and \( (n - \alpha) \)'th degree, in the variable since \( \alpha \) variables are removed by \( \alpha \) differentiations. Cayley's symbol indicates an operation to be performed. Of the first two symbols, Aronhold's seems to have more nearly answered the requirements than Cayley's.

Aronhold contributed an operator

\[
\mathcal{J} = a_0 \frac{\partial f}{\partial \alpha_0} + a_1 \frac{\partial f}{\partial \alpha_1} + \cdots + a_n \frac{\partial f}{\partial \alpha_n}
\]

for the formation of invariants, which has proven to be very valuable in this field of work. In "Gordan's Invariantentheorie" pages 62 to 71, this operator is fully discussed, and given extended applications. But Aronhold failed to pluck the golden apple, when he presented his symbol, for the simple reason that he came to the erroneous conclusion that it was applicable only to ternary quantics of the third degree. Clebsch's symbol is in reality, nothing but an extension of Aronhold's, so that the conclusion we may come to on one, will hold for the other. Clebsch took up the symbol, and found it applicable to any qanctic. It is useful in this, that it expresses an operation already performed. By establishing certain conditions to be imposed on the coefficients, we can break up a form of the \( n' \)th degree into \( n \) linear factors.
which lightens the work of calculation exceedingly. By being able to express a function, we are thus enabled to express the in- and covariants of a function in terms of the roots, since a function can only be so expressed, when we do know its roots. By the use of the symbol, we can express a form, so that certain relations sought can be verified at once, without actually calculating the values of the functions desired. This can be seen at once to be very helpful, since many times that is all that is desired.

On account of the elegance of application, this symbol has come into favor with mathematicians more especially with the Germans. They have gone into the field of modern geometry, and modern higher algebra, and have made rapid progress along those lines. Many valuable discoveries have been made, many of which in all probability would have remained obscure to us to-day had it not been for the application made of this symbol. So that in conclusion we may say, that although Cayley's symbol, and Aronhold's in its original form did not come into much favor, yet they were the material in the match that set the mathematical fire burning in another direction, and it remained for later mathematicians to furnish the fuel. And much credit must be given to Cayley and Aronhold for the high point that investigation in modern higher algebra has already reached.