A SMALL PRESENTATION FOR MORAVA E-THEORY POWER OPERATIONS

BY

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DISSERTATION

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Abstract

Let E denote a Morava E-theory at a prime p and height h. We characterize the power operations on π_0 of a $K(h)$-local E_∞-E-algebra in terms of a small amount of algebraic data. This involves only the E-cohomology of two groups, namely the symmetric groups on p and p^2 letters. Along the way, we also define and explore a notion of coquadratic comonad.

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Chapter 1

Introduction

We begin with a story about the Witt vectors. Let R be a $\mathbb{Z}_p$-algebra, though for some this, $\mathbb{Z}$ will work perfectly well. Classically, the p-typical Witt vectors $\mathbb{W}(R)$ are defined as follows. As a *set*, they are

$$\mathbb{W}(R) = \prod_{i=0}^{\infty} R.$$

The $\mathbb{Z}_p$-algebra structure is more complicated, but is usually given by declaring that a certain map is a ring homomorphism. Define a map $\mathbb{W}(R) \to \prod_{i=0}^{\infty} R$ whose component in the n-th entry is $\Sigma_{i=0}^{n} p^i r_i^{p^{n-i}}$. There is a unique $\mathbb{Z}_p$-algebra structure on $\mathbb{W}(R)$ so that this map is a $\mathbb{Z}_p$-algebra homomorphism. Now, a general element of $\mathbb{W}(R)$ will be quite complicated, as it's specified by infinitely much data. In addition, $\mathbb{W}$ admits the structure of a comonad on $\mathbb{Z}_p$-algebras, and similarly, to specify $\mathbb{W}$-coalgebra structure on R would require quite a lot of information. However, it turns out that only a small piece of this data determines the rest. To that end, we make a definition.

Definition 1.1. A θ-ring is a $\mathbb{Z}_p$ algebra R, equipped with an operation $\theta : R \to R$ which satisfies the following formulas.

- $\theta(1) = 0$
- $\theta(a + b) = \theta(a) + \theta(b) - \frac{1}{p} \sum_{i=1}^{p-1} \binom{p}{i} a^i b^{p-i}$
- $\theta(ab) = a^p \theta(b) + b^p \theta(a) + p\theta(a)\theta(p)$.

We could also phrase this in more categorical language. Let V be the endofunctor on $\mathbb{Z}_p$-algebras which, as a set is $R \times R$. We endow this with a $\mathbb{Z}_p$-algebra structure much as we did with the Witt vectors, but more explicitly, via the following formulas.

$$(r, r') + (s, s') = (r + s, r' + s' - \frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i} r^i s^{p-i})$$

$$(r, r')(s, s') = (rs, r^p s' + s^p r' + pr's')$$

$$a(r, r') = (ar, ar' + r^p \frac{(a - a^p)}{p}).$$

A coalgebra for the endofunctor V is a $\mathbb{Z}_p$-algebra equipped with a structure map $R \to V(R)$ so that the composite $R \to V(R) \to R$ is the identity.

It's fairly clear from the definitions that a V-coalgebra structure on R is the same thing as a θ-ring structure.

In the early 80's, Joyal ([Joy85]) studied the relationship between these θ-rings and the Witt vectors (though he used the term δ-rings). Indeed, he proved the following theorem.

Theorem 1.1. *The forgetful functor from θ-rings to $\mathbb{Z}_p$-algebras not only has a left adjoint, but also a right adjoint. Further, this right adjoint is the p-typical Witt vector functor from above.*

One can rephrase this theorem as saying that $\mathbb{W}$ is the cofree comonad on the endofunctor V. In this thesis, we give something of a generalization of this theorem. Of course, that requires saying how we intend to generalize it, and for that we need to introduce some algebraic topology.

Let $K_p^\wedge$ denote the E_∞ ring spectrum of p-adic K-theory. As an E_∞ ring, it comes with a notion of *power operations* on the homotopy of its E_∞ algebras. It's often more convenient, though, to consider its $K(1)$-local algebras. The action of these power operations has been studied by several people, most notably, McClure ([BMMS86]), Bousfield ([Bou96]) and Hopkins ([DFHH14], Chapter 16). They prove, in particular, the following characterization of the action of the power operations on π_0.

Theorem 1.2. *The zeroth homotopy group of a $K(1)$-local E_∞ $K_p^\wedge$-algebra admits a natural θ-ring structure.*

Further, we can construct the functor V in a "topological" fashion. Let R be a $\mathbb{Z}_p = \pi_0(K_p^\wedge)$-algebra. Define $V(R)$ as the pullback of the diagram

$$\begin{array}{ccc} V(R) & \longrightarrow & (K_p^\wedge)^0(B\Sigma_p) \otimes_{\mathbb{Z}_p} R \\ \downarrow & & \downarrow \scriptstyle{\mathrm{aug} \otimes \mathrm{id}} \\ R & \xrightarrow{(-)^p} & R. \end{array}$$

The definition of V from before (at least, as a commutative monoid under multiplication) can be recovered from this by noticing that $(K_p^\wedge)^0(B\Sigma_p) \cong \mathbb{Z}_p[\![T]\!]/(T^2 - pT)$, which is a free $\mathbb{Z}_p$ algebra with basis $\{1, T\}$ This is a result of McClure ([BMMS86] Proposition IX.5.3).

We wish to generalize the above discussion, but first we need to say how. The cohomology theory $K_p^\wedge$ is the first in a sequence of cohomology theories called Morava E-theories, denoted here with the letter E. Attached to these are two numbers: a prime p and a positive integer h, called "height." More will be said about these in Section 2.5. These all come with a notion of power operations, and so one could contemplate the structure these provide on homotopy groups of E-algebras.

Rezk provided an algebraic context to think about such problems.

Theorem 1.3 (Rezk, [Rez09])**.** *Let E be a Morava E-theory at a prime p and height h. There is a monad $\mathbb{T}$ on the category of graded E_*-modules so that the homotopy groups of $K(h)$-local E_∞-E-algebras naturally take values in $\mathbb{T}$-algebras (which are, in particular, commutative E_*-algebras).*

In the case of $h = 1$, E is $K_p^\wedge$ and a $\mathbb{T}$-algebra is what Bousfield ([Bou96]) calls a $\mathbb{Z}/2$-graded θ-ring.

As before, there is a surprising right adjoint.

Theorem 1.4 ([Rez09], Proposition 4.23)**.** *The forgetful functor* $\mathbb{T}$-Alg $\to$ E_*-Alg *has a right adjoint W (as well as the usual left adjoint F).*

We can of course restrict the input and output of these three functors $\mathbb{T}, \mathbb{F} = UF$, and $\mathbb{W} = UW$. If M is an E_0-module, we can consider it as a E_*-module concentrated in degree 0, apply $\mathbb{T}$, and only take the degree zero part of the value. We can also do similar things on the algebra category and get functors $\mathbb{F}$ and $\mathbb{W}$ on the category of E_0-algebras. For the rest of this paper, we will restrict our attention to the category of E_0-algebras and (abusively) use undecorated notation for the restricted versions of these functors. In the height one case, the functor $\mathbb{W}$ now recovers the p-typical Witt vectors.

As in the case with the Witt vectors, we can now wonder how much data, really is a $\mathbb{F}$-algebra structure on a given E_0-algebra, and is there a "small" functor, like V, whose coalgebras model $\mathbb{T}$-algebras? In fact, replacing $K_p^\wedge$ with Morava E-theory of arbitrary height in the pullback square at the end of that section is a perfectly valid thing to do, and will continue to define an endofunctor on the category of E_0-algebras. Unfortunately, its coalgebras are not the same thing as $\mathbb{T}$-algebras. Intuitively, this is because there are generally *relations* between operations, like the Adem relations in the Steenrod algebra. We need to incorporate this data as well. In Section 4.1, we will define a notion of a "coquadratic pair" of functors (V, V_2), where $V_2(R)$ can, roughly, be thought of as an admissible basis for the elements of $VV(R)$ under some family of quadratic relations. There is a well-defined notion of coalgebra for a coquadratic pair, and also a notion of "cofree comonad" on a coquadratic pair. The primary goal of this thesis will then be to prove the following theorem, which is something of a Morava E-theory analog of Joyal's theorem above.

Theorem 1.5. *There is a coquadratic pair (V, V_2) on the category of E_0-algebras so that $U\mathbb{W}$ is the cofree comonad on (V, V_2). As a consequence, the data of a (V, V_2)-coalgebra on an E_0-algebra R, is precisely the same data as the data of a $\mathbb{F}$-algebra or a $\mathbb{W}$-coalgebra on R.*

This thesis is organized as follows. Section 2.1 establishes a few conventions, and Section 2.2 is a brief bit of algebra. Section 2.3 introduces some useful maps, and Section 2.4 provides some background on power operations. Section 2.5 briefly introduces Morava E-theory and its associated power operations. Section 2.7 establishes some background and a few new results on the Morava E-cohomology of symmetric groups. Sections 3.1 and 3.2 introduce the functors V and V_2, as well as the auxilliary functors $\overline{V}$ and $\overline{V_2}$, and studies some of their properties and interactions. Section 4.1 and 4.2 are a categorical interlude which define the notion of coquadratic pairs, as well as their coalgebras and associated cofree comonads. Section 4.3 explicitly builds the cofree comonad on the coquadratic pair (V, V_2) and Section 5.1 identifies $(\overline{V}, \overline{V_2})$-coalgebras with something more tractable. Finally, Section 5.2 provides a proof of Theorem 1.5

Chapter 2

Preliminaries

2.1 Notation and Conventions

Let Σ_n be the symmetric group on some ordered set S with n letters. We'll sometimes talk about $\Sigma_i \times \Sigma_j$ as a subgroup of Σ_n for $i + j = n$. We will always consider this to be via the embedding where Σ_i permutes the first i elements of S and where Σ_j permutes the last j elements.

If G is a finite group and $H \leq \Sigma_n$ is a subgroup of a symmetric group (often Σ_n itself), we'll use the symbols $G \wr H$ to denote the semidirect product $G^n \rtimes H$, with H acting on G^n by permuting the factors. In other words, $G \wr H$ fits into a short exact sequence

$$1 \longrightarrow G^n \longrightarrow G \wr H \longrightarrow H \longrightarrow 1.$$

Via the embeddings $\Sigma_i \times \Sigma_j \leq \Sigma_n$ above, we can consider two distiguished classes of subgroups of $\Sigma_m \wr \Sigma_n$, namely the subgroups $(\Sigma_i \times \Sigma_j) \wr \Sigma_n$ (with $i + j = m$) and the subgroups $\Sigma_m \wr (\Sigma_k \times \Sigma_l)$ (with $k + l = n$).

For a commutative ring k, we'll often need to talk about more than one k-algebra structure on a ring R. Sometimes, if we have two such algebra structures in mind simultaneously, one will be a left algebra structure, and one will be a right algebra structure. If $f : k \to R$ and $g : k \to R$ are structure maps for a left k-algebra and a right k-algebra respectively, we'll denote this situation with superscripts on the appropriate side. For example, ${}^f R^g$. If we only have one algebra structure in mind but the side matters, we'll use similar notation on the appropriate side. Tensor products over k will always be taken as k-k-bimodule tensor products.

If k has characteristic p, with Frobenius endomorphism ϕ, then we can base-change a right k-algebra R^f along ϕ to get another k-algebra $\phi^*(R)$.

If R^f has characteristic p, but k doesn't, we can still base-change along the Frobenius on k/p and get a "twisted" k-algebra structure on R. We'll denote this similarly to the case when k does have characteristic p, or we might also use the notation $R^{f(p)}$.

There are also a few slightly non-standard operations that we will want to apply to k-algebras.

If A and B are k-algebras, and $f : A \to B$ is a map (which is possibly just a map of sets) we can form an "exterior" tensor power $f^{\boxtimes n} : A \to B^{\otimes n}$ where

$$f^{\boxtimes n}(a) = f(a) \otimes \ldots \otimes f(a).$$

If A, B,and C are k-algebras, and $f : A \to B \otimes C$ is a set map, we can further form a "half-exterior power" $f^{\ltimes n} : A \to B^{\otimes n} \otimes C$ as the composite

$$A \xrightarrow{f^{\boxtimes n}} (B \otimes C)^{\otimes n} \xrightarrow{\cong} B^{\otimes n} \otimes C^{\otimes n} \xrightarrow{\mathrm{id} \otimes \mathrm{mult}} B^{\otimes n} \otimes C.$$

Clearly, even if f is additive, $f^{\boxtimes n}$ and $f^{\ltimes n}$ might fail to be. However, both constructions preserve multiplicative maps.

Unless otherwise stated, all tensor products will be over the same base ring E_0, defined in Section 2.5. Because of this, we will generally neglect to write the base.

2.2 Bialgebras

Let k be a commutative ring. On occasion, it will be useful to talk about the sort of thing that could represent an endofunctor on the category of k-algebras. To that end, we make the following definition, following [BW05b].

Definition 2.1. A k-bialgebra is a commutative k-algebra A, together with three extra pieces of structure:

- a cocommutative coassociative coproduct $\Delta^+ : A \to A \otimes_k A$ ("coaddition"), with a counit $\varepsilon^+ : A \to k$ and antipode $\sigma A \to A$, endowing A with the structure of a cocommutative Hopf algebra over k.

- a second cocommutative coassociative coproduct $\Delta^\times : A \to A \otimes_k A$ ("comultiplication"), with a counit $\varepsilon^\times$, and which codistributes over Δ^+.

- a ring map $\beta : k \to k\text{-Alg}(A, k)$ ("co-k-linear structure"). The ring structure on the target is given by the previous two items.

More succinctly, A represents a commutative k-algebra scheme over $\mathrm{Spec}(k)$, or a k-algebra valued functor on k-algebras.

By definition, a k-bialgebra A defines a functor $k\text{-Alg}(A, -) : k\text{-Alg} \to k\text{-Alg}$. Borger and Wieland ([BW05b]), and earlier, Tall and Wraith [TW70]

Proposition 2.1. *This functor has a left adjoint $A \odot -$. If A represents a functor $F : k\text{-Alg} \to k\text{-Alg}$ and B represents a functor $G : k\text{-Alg} \to k\text{-Alg}$, then $A \odot B$ represents $F \circ G$.*

This provides the category of k-bialgebras with a (nonsymmetric) monoidal product. The unit is $k[e]$, with $\Delta^+(e) = 1 \otimes e + e \otimes 1, \Delta^\times(e) = e \otimes e, \beta(c)(e) = c, \varepsilon^+(e) = 0, \varepsilon^\times(e) = 1$ and $\sigma(e) = -e$. This represents the identity functor.

In fact, given a k-algebra B, both of the cited papers give an explicit construction of $A \odot B$, as follows.

For $a \in A$, let $\Delta^+(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ and $\Delta^\times(a) = \sum_i a_i^{[1]} \otimes a_i^{[2]}$. Then (as a k-algebra) $A \odot B$ is the k-algebra generated by symbols of the form $a \odot b$ (for all $a \in A$, $b \in B$,) subject to the relations

1. $(a + a') \odot b = a \odot b + a' \odot b$
2. $(aa') \odot b = (a \odot b)(a' \odot b)$
3. $c \odot b = c$
4. $a \odot (b + b') = \sum_i (a_i^{(1)} \odot b)(a_i^{(2)} \odot b')$
5. $a \odot (bb') = \sum_i (a_i^{[1]} \odot b)(a_i^{[2]} \odot b')$
6. $a \odot c = \beta(c)(a)$,

for all $a, a' \in A$, $b, b' \in B$ and $c \in k$.

We end the section with an observation we will need later.

Lemma 2.1. *If F and G are two endofunctors of k-Alg, represented by polynomial rings $A = k[x_1, \ldots, x_n]$ and $B = k[y_1, \ldots y_m]$, then as a k-algebra, $A \odot B$ (representing $F \circ G$) is a polynomial algebra in mn variables, and these variables can be taken to be the $x_i \odot y_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.*

We learned the proof of the second part from Charles Rezk.

Proof. First, we show the claim about polynomial algebras. The fact that F and G are represented by polynomial algebras is the same thing as saying that, as sets $F(R)$ and $G(R)$ are naturally isomorphic to R^n and R^m, respectively. We then have that $F(G(R)) = R^{mn}$ as sets, and the claim is immediate from there.

For the statement about variables, it suffices to show that the elements $x_i \odot y_j$ generate $A \odot B$. Let C be the subalgebra generated by these elements, and consider the set $S = \{g \in B | f \odot g \in C, \text{ for all } f \in A\}$. We wish to show that S is all of B. Let f denote an arbitrary element of A. The constants are in S, since $f \odot c = \beta(c)(f) \in k$. Further, S is closed under addition, since relation (4) in the construction of $A \otimes B$ and the definition of S present $f \odot (g + h)$ (for $g, h \in S$) as a sum of products of elements of C. Similarly, S is closed under multiplication. Finally, the y_j are in S, by relations (1), (2) and (3) above, and we are done. □

2.3 Transfers

We'll talk a lot about a certain flavor of generalized group cohomology later. To do this, we'll need to use some "wrong-way" or transfer maps.

Definition 2.2. If $p : X \to Y$ is a finite-sheeted covering space map, there is a stable map $p^! : \Sigma^\infty_+ Y \to \Sigma^\infty_+ Y$ called the "stable transfer map"

These maps have several properties which we will recall when/if they become necessary. For more information, see [Ada78], Chapter 4, and [BMMS86], Chapter 2.

We do note that if H is a subgroup of a finite group G, then $BH \to BG$ is a covering space (with degree equal to the index of H in G.) For some multiplicative cohomology theory E, we will rename the induced map on cohomology $E^0(BG) \to E^0(BH)$ as "restriction", and often abbreviate this to res. Where there's little possibility of confusion, we'll call the associated map $E^0(BH) \to E^0(BG)$ "transfer," and denote it tr. We do note one useful property now, which is proven in Chapter 4 of [Ada78].

Proposition 2.2. *The composite* $\mathrm{tr} \circ \mathrm{res}$ *is given by multiplication by* $\mathrm{tr}(1)$.

In the case of the symmetric groups Σ_{p^k}, there are two special ideals in $E^0(B\Sigma_{p^k})$. The first of these, denoted I_n, is generated by all of the images of all of the transfer maps $E^0(B(\Sigma_i \times \Sigma_{n-i})) \to E^0(B\Sigma_n)$ as i ranges from 1 to $(n-1)$. This is called the "transfer ideal." The other is the intersection of all of the kernels of the restriction maps to all of those same subgroups, and will be denoted by J_n. We will sometimes also denote the ring $E^0(B\Sigma_n)/I_n$ by $\mathcal{O}_n$.

Similarly, in the cohomology $E^0(B\Sigma_m \wr \Sigma_n)$ of a wreath product, we can consider an analogous transfer ideal $I_{m,n}$, generated by the images of all transfers from the subgroups of the form $(\Sigma_i \times \Sigma_j) \wr \Sigma_n$ or $\Sigma_m \wr (\Sigma_k \times \Sigma_l)$, with $i + j = m$ and $k + l = n$.

2.4 Power Operations

An E_∞-ring structure on a spectrum E is quite a lot of data. In particular, for each n there is a "multiplication" map

$$\mu_n : \mathbb{P}_n(E) = (\Sigma^\infty_+ E\Sigma_n) \wedge_{\Sigma_n} E^{\wedge n} \to E,$$

Further, given an E_∞ ring E, there is a well-behaved notion E-modules, and similarly well-behaved notion of smash product $\wedge_E$ on the category of E-modules (see, e.g. [EKMM97]). This allows us to define the notion of an E_∞-E-algebra. This is via the free E_∞-E-algebra monad, similar to the functors $\mathbb{P}_n$ above, just with all of the smash products taken over E. To wit,

$$\mathbb{P}^E(M) = \bigvee_{n \geq 0} \mathbb{P}^E_n(M) = \bigvee_{n \geq 0} (E \wedge \Sigma^\infty_+ E\Sigma_n) \wedge_{E,\Sigma_n} M^{\wedge_E n}.$$

The monad structure maps here come from the operad structure on the E-modules $E \wedge \Sigma^\infty_+ E\Sigma_n$, inherited from the category of spaces. Now, an E_∞-E-algebra A is just an algebra from this monad on the category of E-modules. In particular, A comes with structure maps

$$\mu_n^A : \mathbb{P}_n^E(A) \to A.$$

The author would like to apologize for the proliferation of uses for the letter E here. Historical circumstances seem to force it to have several different meanings in close context. We hope this does not cause too much confusion.

In any case one might start to wonder what manifestation all this extra structure has on homotopy groups. The answer is power operations. We can construct these as follows. For simplicity and focus, we concentrate on the operations on π_0. These will be the main concern this thesis anyway. Consider A an E_∞-E-algebra. Let α be an element of $E_0(B\Sigma_n)$, represented by an E-module map $E \to E \wedge \Sigma^\infty_+ B\Sigma_n \cong \mathbb{P}_n^E(E)$ and let $x \in \pi_0(A)$ be a homotopy class represented by an E-module map $E \to A$. We can now consider the composite

$$E \xrightarrow{\alpha} \mathbb{P}_n^E(E) \xrightarrow{\mathbb{P}_n^E(x)} \mathbb{P}_n^E(A) \xrightarrow{\mu_n^E} E.$$

Via this construction E-homology classes of symmetric groups parametrize certain sorts of operations on the homotopy of E_∞-E-algebras.

There's a cohomological flavor of power operations that we'll also make use of. Let X be a *space*. We first write $\mathbb{P}_n(X)$ for $\mathbb{P}_n(\Sigma^\infty_+ X) \cong \Sigma^\infty_+(\mathbb{P}_n X)$. Now given a *cohomology class* $x \in E^0(X)$, we can construct another cohomology class $\mathcal{P}_n(x) \in E^0(\mathbb{P}(X))$ as the composite

$$\mathbb{P}_n(\Sigma^\infty_+ X) \xrightarrow{\mathbb{P}_n(x)} \mathbb{P}_n(E) \xrightarrow{\mu_n} E.$$

This defines a map $\mathcal{P}_n : E^0(X) \to E^0(\mathbb{P}_n X)$, which we call the n-th total (exterior) power operation. In general it is *not* a ring map. However, it is multiplicative, and there is a formula for what it does to a sum of classes. These and some other properties are summarized in the following proposition, which is part of [BMMS86], Proposition VIII.1.1

Proposition 2.3. *The following properties of the maps $\mathcal{P}_n$ hold.*

- $\mathcal{P}_n(xy) = \mathcal{P}_n(x)\mathcal{P}_n(y)$
- $\mathcal{P}_n(1) = 1$
- $$\mathcal{P}_n(x+y) = \mathcal{P}_n(x)\mathcal{P}_n(y) + \sum_{0<i<n} \operatorname{tr}^{\Sigma_n}_{\Sigma_i \times \Sigma_{n-i}} \mathcal{P}_i(x)\mathcal{P}_{n-i}(y).$$
- *If i is the "inclusion" $X^{\wedge n} \to \mathbb{P}_n(X)$ and $\Delta : X \to X^{\wedge n}$ is the diagonal, then $i^*\mathcal{P}_n(x) = x^{\boxtimes n}$ and $\Delta^* i^* \mathcal{P}_n(x) = x^n$.*

We should also note that in the case $X = *$, this construction specializes to a map $\mathcal{P}_n : E_0 \to E^0(B\Sigma_n)$, and in the case $X = B\Sigma_m$, we get a map $\mathcal{P}_n : E^0(B\Sigma_m) \to E^0(B\Sigma_m \wr \Sigma_n)$. In particular, if $I_n \leq E^0(B\Sigma_n)$ and $I_{n,m} \leq E^0(B\Sigma_n \wr \Sigma_m)$ are the transfer ideals from before, the induced maps $\overline{\mathcal{P}}_n : E_0 \to E^0(B\Sigma_n)/I_n$ and $\overline{\mathcal{P}}_n : E_0(B\Sigma_m) \to E^0(B\Sigma_m \wr \Sigma_n)/I_{m,n}$ are ring homomorphisms. However, they are *not* E_0-algebra homomorpisms.

This does mean though, that the rings $E^0(B\Sigma_n)/I_n$ and $E_0(B\Sigma_m) \to E^0(B\Sigma_m \wr \Sigma_n)/I_{m,n}$ each acquire a *second* E_0-algebra structure, via these ring maps. In both cases, we'll consider this to be a *left* algebra structure (even though E_0 is commutative), and use the letter t for the structure map. We'll use the letter s for the standard algebra structure maps, and consider these to be right algebra structures.

For spaces, there's an internal version of these cohomology operations as well. The diagonal inclusion $\Sigma^\infty_+ X \to (\Sigma^\infty_+ X)^{\wedge n}$ induces a map $\Delta : \Sigma^\infty_+(B\Sigma_n \times X) \to \mathbb{P}_n(X)$. The resulting composite

$$E^0(X) \xrightarrow{\mathcal{P}_n} E^0(\mathbb{P}_n(X)) \xrightarrow{\Delta^*} E^0(B\Sigma_n \times X)$$

gives a map P_n, the total internal n-th power operations. In favorable cases (e.g. when the cohomology of $B\Sigma_n$ is a finite free E_0-module), the target of P_n is isomorphic to $E^0(B\Sigma_n) \otimes E^0(X)$. This will be always be the case in what follows. This means that we can obtain operations $E^0(X) \to E^0(X)$ by further composing the total internal power operation with linear functionals $E^0(B\Sigma_n) \to E_0$.

These internal operations satisfy properties similar to those enjoyed by the external ones. See, e.g. [BMMS86] Proposition VIII.1.4 for more details.

2.5 Morava E-theory and its Power Operations

For the remainder of this thesis, we'll consider operations on one particular family of ring spectra. We won't need need anything in particular about the construction of these, so we'll only briefly recall it here.

Let k be a perfect field of characteristic p, and let Γ be a formal group over k of height h. There is a notion of "deformation" of such things, and the moduli problem of such deformations is representable, by a ring which is (non-canonically) isomorphic to $\mathbb{W}(k)[\![u_1, \ldots, u_{h-1}]\!]$ ([LT66]). Here $\mathbb{W}(k)$ is the ring of (p-typical) Witt vectors on k. It turns out that one can functorially construct a ring spectrum $E_{(k,\Gamma)}$ out of such data, whose homotopy groups are $\mathbb{W}(k)[\![u_1, \ldots, u_{h-1}]\!][u^{\pm 1}]$. (Here, u is in degree -2, and everything else is in degree 0.)

For more details about the construction, see [Rez98].

From now on, we fix the ring spectrum E to be Morava E-theory associated to some formal group Γ of height h over a perfect field of characteristic p, and supress both the field and the formal group. Implicitly, p and h will also be fixed.

It is a celebrated theorem of Goerss, Hopkins and Miller ([GH04]) that Morava E-theory admits an essentially unique E_∞ ring structure. Thus, its algebras come equipped with a notion of power operations as in Section 2.4. One is now led to wonder what can be said about the algebra of all such operations. The answer turns out to be "quite a lot," at least, if one makes a restriction about what sort of algebras are admissible. In the remainder of this thesis, we will focus exclusively on the case of $K(h)$-local E-algebras.

Rezk ([Rez09]) constructed a monad $\mathbb{T}$ on the category of graded E_*-modules which attempts to algebraically model the power operation structure on the homotopy of $K(h)$-local E_∞-E-algebras. We recall a few of its properties now. (See also [Rez12].)

Proposition 2.4. *Let M be an E-module and A_* a graded E_*-module. The following properties of the functor $\mathbb{T}$ hold.*

1. *There is a comparison map*

$$\alpha : \mathbb{T}(\pi_* L_{K(h)} M) \to \pi_* L_{K(h)} \mathbb{P}^E M,$$

which is natural in M

2. $\mathbb{T}$ *decomposes as a direct sum of functors* $\bigoplus_{m\geq 0} \mathbb{T}_m$.

3. *Each of the functors* $\mathbb{T}_m$ *preserve finite free* E_**-modules.*

4. *The comparison map* α *respects this direct sum decomposition. That is, for each* m*, there is a map*

$$\alpha_m : \mathbb{T}_m(\pi_* L_{K(h)} M) \to \pi_* L_{K(h)} \mathbb{P}^E_m M.$$

Further, if M *is a finite free* E*-module, this map is an* isomorphism.

5. *As a consequence of the previous items,* $\mathbb{T}_m(E_*) = E_*(B\Sigma_m)$*, and* $\mathbb{T}(E_*) = \bigoplus_{m\geq 0} E_*(B\Sigma_m)$

6. *If* A_* *is concentrated in even degrees, so is* $\mathbb{T}A_*$.

7. *If* R *is an* E_∞*-*E*-algebra (that is, and algebra for the monad* $\mathbb{P}^E$*), then, via the comparison map* α*, the* E_**-module* $\pi_* L_{K(h)} R$ *inherits a natural* $\mathbb{T}$*-algebra structure. That is,* $\pi_* L_{K(h)}$ *defines a functor* $\mathbb{P}^E\text{-Alg} \to \mathbb{T}\text{-Alg}$.

8. *A* $\mathbb{T}$*-algebra structure on* A_* *is, in part, the structure of a commutative* E_**-algebra on* A_*

Remark 2.1. As already stated, for the remainder of this thesis, we will concentrate on the operations which act on π_0. To do this with the functor $\mathbb{T}$, we can regard an E_0-module A_0 as a graded E_*-module concentrated in degree 0, apply $\mathbb{T}$ (or $\mathbb{T}_m$) and forget all but the degree 0 part. When we talk about $\mathbb{T}$ (or $\mathbb{T}_m$) as an endofunctor on E_0-modules, this is what we mean. The previous proposition holds with all of the $*$'s replaced with 0's.

One of the insights of [Rez09] is that the forgetful functor $U : \mathbb{T}\text{-Alg} \to E_*\text{-Alg}$ is more structured than it might appear at a glance.

Proposition 2.5. *The forgetful functor* $U : \mathbb{T}\text{-Alg} \to E_*\text{-Alg}$ *has a left adjoint* F and a right adjoint W*. The analogous statement on* π_0 *also holds. Further, the adjunction* $U \dashv W$ *is comonadic.*

Remark 2.2. We'll denote the associated monad on E_0-algebras by $\mathbb{F}$ and the associated comonad on E_0-algebras by $\mathbb{W}$. One might argue that, in fact, $\mathbb{W}$ is the main object of study here.

Rezk also considers an "additive" version. He constructs a graded E_0-subalgebra $\Gamma = \oplus_k \Gamma[k]$ of $\mathbb{T}(E_0)$. The inclusion $\Gamma[k] \to \mathbb{T}(E_0)$ in fact lands in $\mathbb{T}_{p^k}(E_0)$. A Γ-algebra is then simply an E_0-algebra equipped with an right algebra structure over Γ. However, it further turns out that Γ is a Koszul E_0-algebra ([Rez12]). This, in particular, means that a Γ-algebra structure on an E_0-algebra A is equivalent to an E_0-algebra map $P : A \otimes {}^s\Gamma[1]^t \to A$ so that there exists a map $P_2 : A \otimes {}^s\Gamma[2]^t \to A$ making the following diagram commute.

$$\begin{array}{ccc} A \otimes {}^s\Gamma[1]^t \otimes {}^s\Gamma[1]^t & \xrightarrow{\mathrm{id}\otimes \mathrm{mult}} & A \otimes {}^s\Gamma[2]^t \\ \downarrow{\scriptstyle P\otimes \mathrm{id}} & & \downarrow{\scriptstyle P_2} \\ A \otimes {}^s\Gamma[1]^t & \xrightarrow{P} & A. \end{array}$$

The map P_2 is unique if it exists.

There is a forgetful functor from $\mathbb{T}$-algebras to Γ-algebras, and Rezk considers the question of when a Γ-algebra structure on A lifts to a $\mathbb{T}$-algebra structure. Theorem A of [Rez09] gives a condition for when this holds, encapsulated by the following statement.

Theorem 2.1. *There is an element $\sigma \in \Gamma[1]$ so that, if A is a p-torsion-free Γ-algebra, the Γ-algebra structure on A lifts to a $\mathbb{T}$-algebra structure precisely when $x\sigma \equiv x^p \bmod pA$ for all $x \in A$. Such a lift is necessarily unique.*

The condition in this theorem (and some of its variants) is called the *Frobenius congruence.*

2.6 Hopkins-Kuhn-Ravenel Character Theory

We'll occasionally make use of the generalized character theory of Hopkins, Kuhn and Ravenel ([HKR00]). This section will provide a brief review of the points we will need. Their proofs work in slightly more generality than we will need, but we specialize to the case of Morava E-theory of height h at the prime p here.

Let G be a finite group, and consider the set of group homomorphisms $\mathbb{Z}_p^h \to G$. The group G acts on this set by conjugation.

Definition 2.3. A "height h conjugacy class" on G to be an orbit of $\mathrm{Hom}(\mathbb{Z}_p^h, G)$ under this conjugation action.

More concretely, by choosing the standard $\mathbb{Z}_p$-basis $e_1, \ldots, e_h$ for $\mathbb{Z}_p^h$, we can consider a height h conjugacy class $[\alpha]$ as an equivalence class of h-tuples $(\alpha(e_1), \ldots, \alpha(e_h))$ of commuting p-power order elements of G. Again, G acts on such h-tuples by conjugation in each coordinate.

Remark 2.3. If we used $\mathbb{Z}$ instead of $\mathbb{Z}_p$ and considered the case $h = 1$, we would recover the usual definition of conjugacy class in a group.

This gives a notion of "class function"

Definition 2.4. Let R be a commutative ring. An R-valued "height h class function" on G is simply a function ψ from the set of height h conjugacy classes in G to R. The set of all such things will be denoted by $Cl_h(G, R)$

We can of course think of a height h class function as a function $G^h \to R$ which is constant on height h classes.

Similarly to the cohomology of a group, the R-valued class functions on G come with a notion of restriction and transfer. The second of these is also sometimes known as induction.

For H a subgroup of G, define transfer and restriction maps, tr_H^G and res_H^G on height h class functions by the following formulas.

$$\mathrm{tr}_H^G(\psi)(\alpha) = \sum_{gH \in (G/H)^{\mathrm{im}\alpha}} \psi(g^{-1}\alpha g)$$

$$\mathrm{res}_H^G(\phi)(\beta) = \phi(i \circ \beta).$$

Here, ϕ is a height h class function on G ψ is a height h class function on H, $\alpha : \mathbb{Z}_p^h \to G$ represents a height h conjugacy class in G, $\beta : \mathbb{Z}_p^h \to G$ represents a height h conjugacy class in H and $i : H \to G$ is the inclusion. From these definitions, it is easy to check the usual formula for composing transfer and restriction.

Proposition 2.6.

$$\phi\, \mathrm{tr}_H^G(\psi) = \mathrm{tr}_H^G(\mathrm{res}_H^G(\phi)\psi).$$

In particular, $\mathrm{tr}_H^G(\mathrm{res}_H^G(\phi)) = \phi\,\mathrm{tr}_H^G(1)$*, where* 1 *is the constant class function with value* 1.

Hopkins, Kuhn and Ravenel construct a faithfully flat $p^{-1}E_0$-algebra C_0 (there called $L(E^*)$), and they show that after a base-change to C_0, we can identify $E^0(BG)$ with the ring of C_0-valued class functions on G.

Theorem 2.2. *There is an isomorphism*

$$\chi : C_0 \otimes E^0(BG) \to Cl_h(G, C_0),$$

which is natural in the group G

This is Theorem C in their paper. An immediate consequence is that this isomorphism is compatible with the restriction maps on cohomology and class functions. They also prove (Theorem D) that it's compatible with transfers on both sides. Indeed,

Theorem 2.3. *If* $H \leq G$*, the following diagram commutes.*

$$\begin{array}{ccc} C_0 \otimes E^0(BH) & \xrightarrow{C_0 \otimes \mathrm{tr}_H^G} & C_0 \otimes E^0(G) \\ \downarrow \chi & & \downarrow \chi \\ Cl_h(H, C_0) & \xrightarrow{\mathrm{tr}_H^G} & Cl_h(G, C_0) \end{array}$$

Here, the top arrow is the stable transfer map on cohomology assoctiated to the covering space $BH \to BG$*, and the bottom arrow is the class function transfer constructed above.*

.

We should also note a slightly different interpretation for the symmetric group

Proposition 2.7. *As an* C_0*-module,* $C_0 \otimes E^0(B\Sigma_m)$ *is the free* C_0*-module on the isomorphism classes of* $\mathbb{Z}_p^h$ *sets of order* m.

This is precisely because we can think of a group homomorphism $\mathbb{Z}_p^h \to \Sigma_m$ as an action of Z_p^h on the set $\{1, \dots, m\}$.

2.7 Some preliminaries on E-cohomology of symmetric groups

As in Section 2.4, the study of power operations for an E_∞ ring spectrum often relies on detailed knowledge of the cohomology groups of classifying spaces of symmetric groups. In Proposition 2.4, we also saw that this is the case for $\mathbb{T}$-algebras. Thus, to study these, we need recall much that is already in the literature on the Morava E-theory of symmetric groups.

First, Hopkins, Kuhn and Ravenel ([HKR00]) give a condition (Theorem E there) under which the Morava K-theory cohomology of the classifying space of a group is concentrated in even degrees, which in particular applies to symmetric groups. Strickland ([Str98]) later provided a very detailed analysis of the Morava E-theory cohomology of symmetric groups of p-th power order. He builds upon the work of Hopkins, Kuhn and Ravenel to show, among many things that the Morava E-cohomology of symmetric groups is free and concentrated in even degrees.

It turns out that for Morava E-theory, the ideal I_{p^k} and J_{p^k} have descriptions in terms of a single subgroup.

Lemma 2.2 ([Str98], Lemma 8.11)**.** *The ideals I_{p^k} and J_{p^k} are equal to the image of the transfer map $E^0(B\Sigma^p_{p^{k-1}}) \to E^0(B\Sigma_{p^k})$, and the kernel of the restriction map $E^0(B\Sigma_{p^k}) \to E^0(B\Sigma^p_{p^{k-1}})$, respectively.*

Strickland then shows the following.

Theorem 2.4 ([Str98], Theorem 8.6)**.** *The quotient maps*

$$E^0(B\Sigma_{p^k}) \to E^0(B\Sigma_{p^k})/I_{p^k}$$

and

$$E^0(B\Sigma_{p^k} \to E^0(B\Sigma_{p^k})/J_{p^k}$$

are split surjective. As a consequence, those quotients are free E_0-modules. Further, J_{p^k} is a free $E^0(B\Sigma_{p^k})/I_{p^k}$-module on one generator.

Finally, he also provides an algebro-geometric interpretation of the ring $E^0(B\Sigma_{p^k})/I_{p^k}$.

Theorem 2.5 ([Str98], Theorem 8.6)**.** *Let (Γ, K) be the formal group over K, for which $(E_\Gamma)_0$*

classifies deformations. Then the E_0-algebra $E^0(B\Sigma_{p^k})/I_{p^k}$ is naturally isomorphic to the $(E_\Gamma)_0$-algebra $\mathcal{O}_{p^k}$ which classifies isogenies of degree p^k between deformations of Γ.

Later, we will need some knowledge of the structure of the *whole* E-cohomology of certain symmetric groups. As a first step, we note the following proposition, which is Proposition 10.5 of [Rez09].

Proposition 2.8. *The augmentation map $E^0(B\Sigma_p) \to E_0$ passes to a map $\mathcal{O}_p \to E_0/p$. This classifies the Frobenius isogeny, and further, the following square is a pullback of E_0-modules.*

$$\begin{array}{ccc} E^0(B\Sigma_p) & \longrightarrow & \mathcal{O}_p \\ \downarrow & & \downarrow \\ E_0 & \longrightarrow & E_0/p. \end{array}$$

There is also something of an analog for symmetric groups of arbitrary p-th power order.

Proposition 2.9. *Let I_{p^k} be the transfer ideal in $E^0(B\Sigma_{p^k})$ and J_{p^k} be the kernel of the restriction map $E^0(B\Sigma_{p^k}) \to E^0(B\Sigma_{p^{k-1}}^p)$. The square of quotients*

$$\begin{array}{ccc} E^0(B\Sigma_{p^k}) & \longrightarrow & E^0(B\Sigma_{p^k})/I_{p^k} \\ \downarrow & & \downarrow \\ E^0(B\Sigma_{p^k})/J_{p^k} & \longrightarrow & E^0(B\Sigma_{p^k})/(I_{p^k} + J_{p^k}) \end{array}$$

is a pullback (and since all the maps are surjective ring maps, also a pushout) .Further, the quotient $E^0(B\Sigma_{p^k})/I_{p^k} + J_{p^k}$ is p^r-torsion for some r

Proof. For both claims, it suffices to show that the combined projection map

$$\pi : E^0(B\Sigma_{p^k}) \to E^0(B\Sigma_{p^k})/I_{p^k} \oplus E^0(B\Sigma_{p^k})/J_{p^k}$$

becomes an isomorphism after inverting p. This means that π is injective before inverting p, so that $I_{p^k} \cap J_{p^k}$ vanishes. A standard generalization of the Chinese Remainder Theorem then gives the pullback claim. Further, the cokernel of π also vanishes after inverting p. As it's a finitely generated E_0-module, some power of p kills it.

To prove the rational isomorphism, we use the character theory of [HKR00]. As noted in

Section 2.6, there is a faithfully flat $p^{-1}E_0$-algebra C_0 so that $C_0 \otimes E^0(B\Sigma_{p^k})$ is isomorphic to the ring of C_0-valued height h class functions on Σ_{p^k}. With Proposition 2.7 we can identify this as a ring with $\prod C_0$, where the product ranges over isomorphism classes of Z_p^h-sets of order p^k. By the formula for transfers (Theorem 2.3), or as shown by Strickland and Turner ([ST97], Theorem 4.2), the quotient by the transfer ideal corresponds to the projection onto the product indexed by the *transitive* Z_p^h-sets. Further, the quotient by the kernel of the restriction map corresponds to the projection onto the remaining factors of the product decomposition. We immediately get the required isomorphism after base-change to C_0. But since C_0 is faithfully flat, the map π is an isormorphism after inverting p as desired.

□

Proposition 2.10. *The square from Proposition 2.9 remains a pullback after tensoring up to any p-torsion-free E_0-algebra*

Proof. It's enough to check that the square is a pullback in E_0-modules. Let R be a p-torsion-free E_0-algebra. Proposition 2.9 gives a short exact sequence of E_0-modules

$$0 \longrightarrow E^0(B\Sigma_{p^2}) \longrightarrow E^0(B\Sigma_{p^k})/I_{p^k} \oplus E^0(B\Sigma_{p^k})/J_{p^k} \longrightarrow E^0(B\Sigma_{p^k})/(I_{p^k} + J_{p^k}) \longrightarrow 0.$$

Base changing to R gives an exact sequence

$$E^0(B\Sigma_{p^2}) \otimes R \to (E^0(B\Sigma_{p^k})/I_{p^k} \oplus E^0(B\Sigma_{p^k})/J_{p^k}) \otimes R \to E^0(B\Sigma_{p^k})/(I_{p^k} + J_{p^k}) \otimes R \longrightarrow 0.$$

But the first two terms in this sequence are free R-modules of the same rank, as they're free E_0-modules of the same rank before tensoring up to R. Since R is p-torsion-free, the third term is a torsion R-module. These facts together give that the first map in the sequence is injective, proving the claim. □

We now go about identifying one of the bottom terms in square from Proposition 2.9 in the case $k = 2$.

Proposition 2.11. *The restriction map $E^0(B\Sigma_{p^2}) \to (E^0(B\Sigma_p)^{\otimes p})^{\Sigma_p}$ is split surjective as a map of E_0-modules. As a consequence, $(E^0(B\Sigma_p)^{\otimes p})^{\Sigma_p} \cong E^0(B\Sigma_{p^2})/J_{p^2}$*

Proof. In the course of the proof of theorem 8.6 of [Str97], Strickland shows that the intersection of the decomposables in $\mathbb{T}(E_0)$ with $E_0(B\Sigma_m)$ is a retract, and so a summand. Dualizing the statement that we want, we wish to identify this with the p-th symmetric power of $E_0(B\Sigma_p)$. By the dual of lemma 8.11 there, we know that every decomposable element will be in the image of the multiplication map $\mathrm{Sym}^p(E_0(B\Sigma_p)) \to E_0(B\Sigma_{p^2})$. Thus, we can now carefully select a basis for $E_0(B\Sigma_{p^2})$. Let $x^p, y_1, \ldots, y_n$ be a basis for $E_0(B\Sigma_p)$, by Proposition 2.8 (where x is a generator for $E_0(B\Sigma_1)$, and the y_i are orthogonal to x^p). Then every element of the decomposables in $E_0(B\Sigma_{p^2})$ is of the form

$$y_{i_1} \ldots y_{i_k} x^{p^{p-k}} \tag{2.1}$$

for some k. From Strickland's results we can count the E_0-ranks of the modules involved here. Using Hopkins-Kuhn-Ravenel character theory he identifies the rank of $E^0(B\Sigma_{p^k})$ as the number of $\mathbb{Z}_p^h$-sets of order p^k. In the case $k = 1, 2$, this results in

$$\mathrm{rnk}(E_0(B\Sigma_p)) = \frac{p^h - 1}{p - 1} + 1$$

$$\mathrm{rnk}(E_0(B\Sigma_{p^2})) = \frac{(p^{h+1} - 1)(p^h - 1)}{(p^2 - 1)(p - 1)} + \binom{\frac{p^h-1}{p-1} + p}{p}$$

and that the rank of the indecomposables in $E_0(B\Sigma_{p^2})$ is the first term in the above formula. We now have that the rank of the decomposables is the same as the rank of $\mathrm{Sym}^p(E_0(B\Sigma_p))$ (which is the second term). Thus, all of the elements of the form (2.1) are distinct, and come in a unique way from one of the basis elements of $\mathrm{Sym}^p(E_0(B\Sigma_p))$. We now have that the multiplication map $\mathrm{Sym}^p(E_0(B\Sigma_p)) \to E_0(B\Sigma_{p^2})$ is split monic (as a map of E_0-modules). Dualizing, we get that the restriction map is split surjective. □

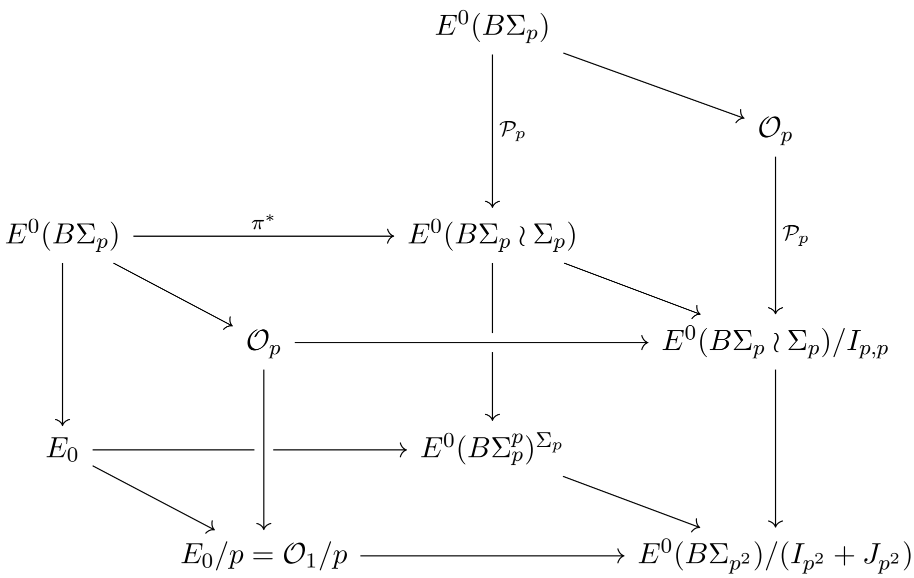
Remark 2.4. Perhaps it's not clear where we used anything specific about the case p^2 here. In the case of p^k for $k > 2$ the rank of the decomposables will be smaller than the rank of the p-th symmetric power of $E_0(B\Sigma_p)$. This is already seen when $p = 2$ and $h = 2$, where the decomposables have rank 133, and the symmetric power has rank 153. Intuitively, what's happening here is that the elements in $E_0(B\Sigma_p)$ which "come from" lower degrees only form a rank one submodule. If it were larger, products of such elements might come from more than one element of the symmetric power.

Remark 2.5. The E_0-algebra $E^0(B\Sigma_{p^k})/(I_{p^k} + J_{p^k})$ inherits a second E_0-algebra structure from the E_0-algebra t on $\mathcal{O}_{p^2}$. This will also be denoted by a superscript t on the left.

We say one more thing about the above pullback square that we'll need later.

Proposition 2.12. *The map $\mathcal{O}_{p^2} \to E^0(B\Sigma_{p^2})/(I_{p^2} + J_{p^2})$ factors through a map $\mathcal{O}_{p^2} \to \mathcal{O}_p^s \otimes {}^{\mathrm{Frob}}\mathcal{O}_1/p = \phi^*(\mathcal{O}_p/p)$.*

Proof. The wreath product $\Sigma_p \wr \Sigma_p$ has a quotient map to the second wreath factor. Call this π and consider the following diagram.



The left face is the pullback square for $E^0(B\Sigma_p)$, the bottom right square commutes as its top edge is the restriction from the quotient $E^0(B\Sigma_{p^2}) \to \mathcal{O}_{p^2}$, and the top left square commutes by definition of $I_{p,p}$. That the top left square commutes is a standard fact about the interaction of transfers with wreath products. The two maps out of $\mathcal{O}_p$ combine together to give a map ${}^t\mathcal{O}_p^s \otimes {}^t\mathcal{O}_p^s \to E^0(B\Sigma_p \wr \Sigma_p)/I_{p,p}$ which factors the map $\mathcal{O}_{p^2} \to E^0(B\Sigma_p \wr \Sigma_p)/I_{p,p}$. This of course gives a map ${}^t\mathcal{O}_p^s \otimes {}^t\mathcal{O}_p^s \to E^0(B\Sigma_{p^2})/(I_{p^2} + J_{p^2})$. Note that the factor of $\mathcal{O}_p$ coming from the left side of the diagram is the *second* factor in the tensor product. Then the fact that the front face of the above diagram commutes provides the desired factorization

$$ {}^t\mathcal{O}_p^s \otimes {}^t\mathcal{O}_p^s \to \mathcal{O}_p^s \otimes \mathcal{O}_1/p \to E^0(B\Sigma_{p^2})/(I_{p^2} + J_{p^2}). $$

□

Chapter 3

Two Functors

3.1 The functors V and V_2

We now proceed to construct the promised functors V and V_2 from the main theorem. We also give a few auxilliary functors, which are perhaps more tractable. These will be often be used to prove things about our main targets.

The first definition is immediately inspired by the pullback square of the introduction.

Definition 3.1. Define a functor $V : E_0\text{-Alg} \to \text{Sets}$ as follows. For an E_0-algebra R, $V(R)$ is the pullback

$$\begin{array}{ccc} V(R) & \longrightarrow & E^0(B\Sigma_p)\otimes_{E_0} R \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{aug}\otimes\mathrm{id}} \\ R & \xrightarrow{(-)^p} & R \end{array}$$

For a map $f : R \to S$, $V(f)$ is defined via the universal property of $V(S)$, that is "componentwise."

Of course, the maps in the definition of V are not E_0-algebra maps, or even ring maps. They are, however, multiplicative. As a result, $V(R)$ is only defined as a commutative monoid (under multiplication) for now. Later we will in fact show that it actually does take values in commutative E_0-algebras. For now, we prove two useful facts about V.

Proposition 3.1. *The functor V is representable by a polynomial algebra over E_0.*

Proof. The augmentation map $\mathrm{aug} : E^0(B\Sigma_p) \to E_0$ splits as a map of E_0-modules, by Theorem 2.4. In fact, we may choose a basis $x, x_1, \ldots, x_d$ of $E^0(B\Sigma_p)$, so that $\mathrm{aug}(x) = 1$ and $\mathrm{aug}(x_i) = 0$ for all i. From the definition, of V, we can write a general element of $V(R)$ as $(r, \sum_{i=0}^d r_i \otimes x_i)$ with $r_0 = r^p$ (and all of the r's elements of R). In other words, we can write $V(R)$ as $\{(r, r^0 \otimes x_0 + \sum_{i=1}^d r_i \otimes x_i)\}$. As a set, this is patently isomorphic to R^{1+d}. □

Proposition 3.2. *The functor V preserves monomorphisms.*

Proof. This follows immediately from the previous proposition. □

To prove basically anything else about V, we're going to need another functor. To that end, we make the following definintion.

Definition 3.2. Define another functor $\overline{V} : E_0\text{-Alg} \to \text{Ring}$ as the pullback

$$\begin{array}{ccc} \overline{V}(R) & \longrightarrow & {}^{t}(E^0(B\Sigma_p)/I_p^s \otimes_{E_0} R \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \text{aug} \otimes \text{id}} \\ R & \xrightarrow{(-)^p \bmod p} & {}^{s(p)}R/p \end{array}$$

Note that this functor *is* actually ring-valued, since the bottom and right maps are ring homomorphisms. In fact, we can give it an E_0-algebra structure as well.

Proposition 3.3. *There is a natural E_0-algebra structure on $\overline{V}(R)$ for all E_0-algebras R.*

Proof. We've already noted that it has a ring structure. For the E_0-algebra structure, we use the *left* algebra structures on everything. In the case of $E^0(B\Sigma_p)/I_p$, this is the algebra structure coming from the additivized p-th power operation, as in Section 2.4. In the case of R/p, this is the usual algebra structure twisted by the p-th power. The bottom map in the definition of $\overline{V}(R)$ is now clearly an algebra map, and the left map is by Proposition 2.3. □

$\overline{V}$ is very closely related to V; they are isomorphic on E_0-algebras that have no p-torsion.

Proposition 3.4. *There is a natural transformation $V \to \overline{V}$. In addition, if R is a p-torsion-free E_0-algebra, the induced map $V(R) \to \overline{V}(R)$ is an isomorphism (of commutative monoids).*

Proof. From Proposition 2.8 we have a pullback square of E_0-modules taking the form

$$\begin{array}{ccc} E^0(B\Sigma_p) & \xrightarrow{q} & {}^{t}(E^0(B\Sigma_p)/I_p)^s \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \sigma} \\ E_0 & \xrightarrow{q} & {}^{s(p)}E_0/p. \end{array} \tag{3.1}$$

Tensoring with R and attaching this to the pullback square defining $V(R)$ gives a composite commutative square

$$\begin{array}{ccccc} V(R) & \longrightarrow & E^0(B\Sigma_p)\otimes R & \xrightarrow{q\otimes \mathrm{id}} & {}^t(E^0(B\Sigma_p)/I_p)^s\otimes R \\ \downarrow & & \downarrow{\scriptstyle i^*\otimes\mathrm{id}} & & \downarrow{\scriptstyle \sigma\otimes\mathrm{id}} \\ R & \xrightarrow{(-)^p} & E_0\otimes R & \xrightarrow{q\otimes\mathrm{id}} & {}^{s(p)}(E_0/p)^s\otimes R. \end{array} \tag{3.2}$$

But the pullback of the right and bottom legs of this composite square is defined to be $\overline{V}(R)$; this supplies the natural transformation required.

For the isomorphism, let R be a p-torsion-free E_0-algebra. Note first that all of the corners of the square (3.1) except the bottom right are free E_0-modules – the top left and top right by Strickland ([Str98]). The bottom right corner, on the other hand, has only p-torsion. Thus, the square (3.1) remains a pullback after tensoring up to R. This gives that the composite square (3.2) is also a pullback, proving the isomorphism.

□

This isomorphism now allows us to provide the promised E_0-algebra structure on $V(R)$ for arbitrary R.

Proposition 3.5. *For any E_0-algebra R, $V(R)$ has a natural E_0-algebra structure.*

Proof. Combining Propositions 3.3 and 3.4, we immediately get the required E_0-algebra structure when R is p-torsion-free. Now note that if R is p-torsion-free, then so is $\overline{V}(R)$. Indeed, the pullback definition of $\overline{V}(R)$ presents it as a submodule of $R\times(E^0(B\Sigma_p)/I_p)^s\otimes R$; this is p-torsion-free since R and $E^0(B\Sigma_p)/I_p$ are. This means that V restricts to an endofunctor (which we will still call V) on the subcategory $E_0\text{-Alg}^{\mathrm{ptf}}$ of p-torsion-free E_0-algebras. Proposition 3.1 gives that V is representable, not just on the whole category $E_0\text{-Alg}$, but also on $E_0\text{-Alg}^{\mathrm{ptf}}$, by an object P. Since V takes values in E_0-algebras on $E_0\text{-Alg}^{\mathrm{ptf}}$, P gains all of the maps and identities of an E_0-bialgebra, with the slight caveat, that all of those are only taking place in the subcategory of p-torsion-free algebras. But all of these structure maps exist just as well in the whole category $E_0\text{-Alg}$, endowing the functor V (on the whole category now) with the structure of an E_0-algebra-scheme. That is, $V(R)$ has a natural E_0-algebra structure, for arbitrary R. □

The trick used in this proof – show a claim in the p-torsion-free case, and then recover the full claim via some form of representability – will show up several times in this thesis. The author tends to think of it as a more elaborate version of the old trick of writing down formulas rationally, and then noticing that they are in fact defined over the integers. Here, we may be unable to write down the formulas explicitly, but the same idea applies.

We should also note the following, which was needed and shown in the course of the proof of the previous proposition.

Proposition 3.6. *If R is a p-torsion-free E_0-algebra, then so is $V(R)$.*

We noted in the introduction that coalgebras for V (whatever that might mean) fail to fully capture the notion of $\mathbb{T}$-algebras. We also claimed there that we can remedy this deficiency by introducing another functor. We do that now.

Definition 3.3. Define another functor $V_2 : E_0\text{-Alg} \to \text{Sets}$ as follows, for an E_0-algebra R, $V_2(R)$ is the limit of the diagram

$$\begin{array}{ccccc} & & & & E^0(B\Sigma_{p^2}) \otimes_{E_0} R \\ & & & & \big\downarrow{\scriptstyle \mathrm{res}\otimes id} \\ & & E^0(B\Sigma_p) \otimes_{E_0} R & \xrightarrow{\mathrm{id}^{\ltimes p}} & (E^0(B\Sigma_p)^{\otimes p})^{\Sigma_p} \otimes_{E_0} R \\ & & \big\downarrow{\scriptstyle aug\otimes\mathrm{id}} & & \\ R & \xrightarrow{(-)^p} & R & & \end{array}$$

Again, a map $f : R \to S$, $V_2(f)$ is defined in "the obvious way," via the universal property of limits.

Similarly to the case with the functor V, this is only *a priori* defined as a commutative monoid. We now proceed to relitigate the series of propositions and definitions that followed that definition, with subscript 2's everywhere.

Proposition 3.7. *The functor V_2 is representable by a polynomial algebra.*

Proof. By Proposition 2.11, we can choose a basis $y_1, \ldots y_k, z_1, \ldots z_l$ of $E^0(B\Sigma_{p^2})$ (as an E_0-module) so that the z_j are in the kernel of the restriction and so that the images of the y_i under res form a basis for $(E^0(B\Sigma_p)^{\otimes p})^{\Sigma_p}$. A generic element of $V_2(R)$ can then be written in the form $(r, \sum x_i \otimes a_i, \sum y_j \otimes b_j + \sum z_k \otimes c_k)$. (where the x_i are as in the proof of Proposition 3.1, and r, as well as the a_i, b_j and c_k are in R,) subject to the condition that $\phi(x \otimes r^p + \sum x_i \otimes a_i) = \sum \mathrm{res}(y_j) \otimes b_j)$. This condition determines the b_j's. Thus, $V_2(R)$is, as a set, isomorphic to $1 + d + l$ copies of R. □

Just like we had with V, we want a somewhat more managable functor to compare V_2 with.

Definition 3.4. Define a functor $\overline{V_2} : E_0\text{-Alg} \to \text{Rings}$ as follows. If R is an E_0-algebra, let $\overline{V_2}(R)$ be the limit of the diagram

$$\begin{array}{ccccc} & & & & {}^t\mathcal{O}^s_{p^2}\otimes R \\ & & & & \big\downarrow{\scriptstyle (\mathrm{res}\circ c)\otimes\mathrm{id}} \\ & & {}^t\mathcal{O}^s_p\otimes R & \xrightarrow{\overline{\mathcal{P}_p}\otimes(-)^p} & {}^tA_{p^2}/(I_{p^2}+J_{p^2})^s\otimes R \\ & & \big\downarrow{\scriptstyle q\otimes\mathrm{id}} & & \\ R & \xrightarrow{(-)^p} & {}^{s(p)}R/p^s & & . \end{array}$$

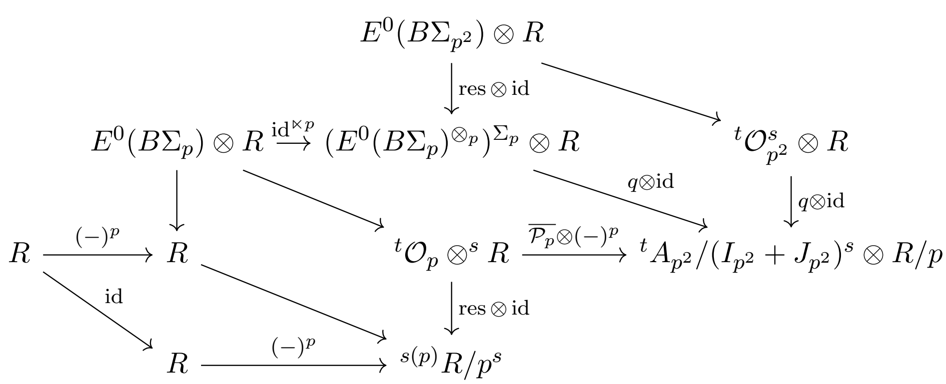
Here, we have saved on space by defining A_{p^2} as $E^0(B\Sigma_{p^2})$. Additionally, $\overline{\mathcal{P}_p}$ is the composite given by $\mathcal{O}_p \to E^0(B\Sigma_p \wr \Sigma_p)/I_{p,p} \to {}^tA_{p^2}/(I_{p^2}+J_{p^2})^s$ from the proof of Proposition 2.12, and q is the defining quotient map.

Proposition 3.8. *$\overline{V_2}$ is valued in E_0-algebras.*

Proof. The only thing in question is the algebra structure, as all of the maps involved in the definition of $\overline{V_2}$ are ring homomorphisms. If we twist the usual module structure on R/p and use the second algebra structure t on $A_{p^2}/(I_{p^2}+J_{p^2})$ (as in Remark 2.5), we get E_0-algebra maps. □

Proposition 3.9. *There is a natural homomorpism $V_2 \to \overline{V_2}$. If R is p-torsion-free, the map $V_2(R) \to \overline{V_2}(R)$ is an isomorphism (of commutative monoids).*

Proof. Consider the following diagram.



The unlabeled arrows are reasonably obvious quotients. The limit of the upper zigzag is $V_2(R)$, and the limit of the lower zigzag is $\overline{V_2}(R)$. This immediately gives the map. For the isomorphism, let R be a p-torsion-free E_0-algebra. We already know (by Proposition 3.4) that the map $V(R) \to \overline{V}(R)$ is an isomorphism, so it suffices to show that the square

$$\begin{array}{ccccc} V_2(R) & \longrightarrow & E^0(B\Sigma_{p^2}) \otimes R & \longrightarrow & {}^t\mathcal{O}_{p^2} \otimes^s R \\ \downarrow & & \downarrow & & \downarrow \\ V(R) \longrightarrow E^0(B\Sigma_p) & \xrightarrow{\mathrm{id}^{\times p}} & (E^0(B\Sigma_p)^{\otimes p})^{\Sigma_p} \otimes R & \xrightarrow{q \otimes \mathrm{id}} & {}^tA_{p^2}/(I_{p^2} + J_{p^2})^s \otimes R \end{array}$$

is a pullback. By definition the left hand square is a pullback, Proposition 2.10 gives us that the right hand square is a pullback, and we are done. □

Proposition 3.10. *The functors V_2 and $\overline{V_2}$ preserve p-torsion-free objects.*

Proof. If R is p-torsion-free, the definition of $\overline{V_2}(R)$ presents it as a submodule of a p-torsion-free module. This gives the claim for $\overline{V_2}$. The claim for V_2 now follows by Proposition 3.10. □

Proposition 3.11. *For any E_0-algebra R, $V_2(R)$ has a natural E_0-algebra structure.*

Proof. This follows from Propositions 3.8, 3.9 and 3.10 exactly as in the proof of Proposition 3.5. □

3.2 The map $V_2 \to VV$.

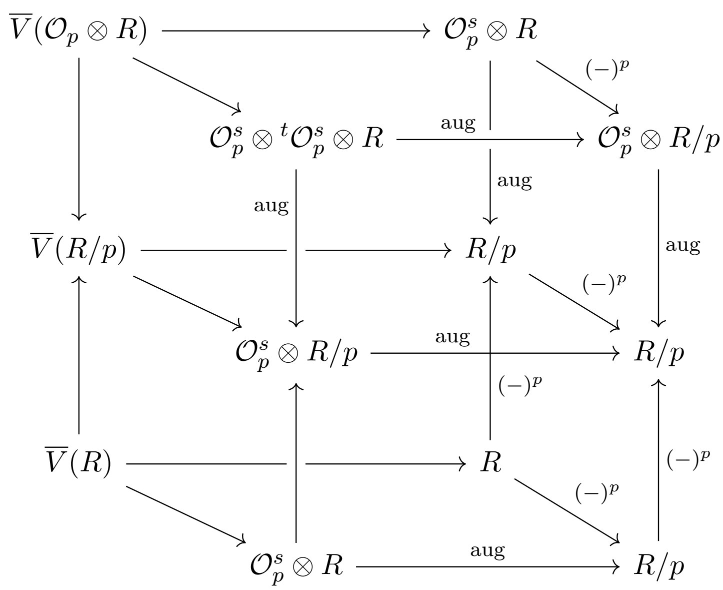
To consider the functor V_2 as some sort of "admissibles" in VV, we need a map that allows us to do so. We construct this map now.

Proposition 3.12. *There is a natural transformation $V_2 \to VV$.*

Proof. By the same arguments as in Proposition 3.5, it suffices to construct a map $\overline{V_2} \to \overline{VV}$. Now note that $\overline{V}$ preserves pullbacks (and so products); $\overline{V}$ is defined as a limit of sets, and limits of E_0-algebras are computed as sets. Thus, if R is an E_0-algebra, we can compute $\overline{VV}(R)$ as a pullback

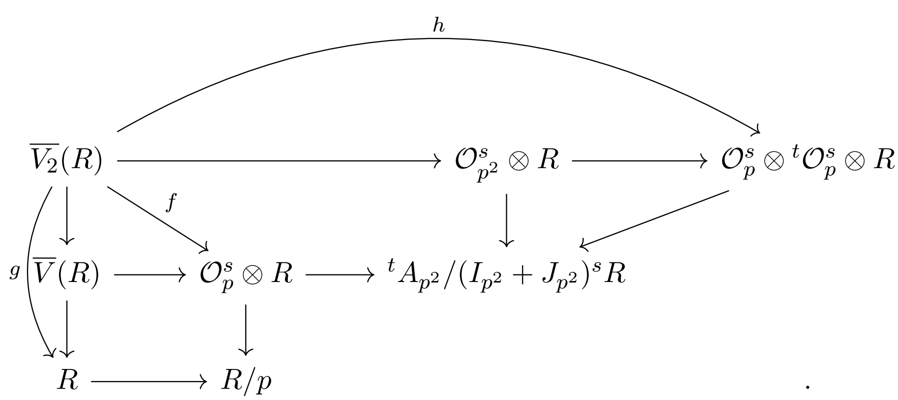
$$\begin{array}{ccc} \overline{VV}(R) & \longrightarrow & \overline{V}(\mathcal{O}_p^s \otimes R) \\ \downarrow & \lrcorner & \downarrow \\ \overline{V}(R) & \longrightarrow & \overline{V}(R/p), \end{array}$$

given by applying $\overline{V}$ to the definition of itself. Thus, to give a map $\overline{V_2}(R) \to \overline{V}(R)$, we just need to give compatible maps to $\overline{V}(R)$ and $\overline{V}(\mathcal{O}_p^s \otimes R)$. The first map is clear; by definition $\overline{V_2}(R)$ comes with a map ψ to $\overline{V}(A)$. Expanding out the pullback above, we get a diagram



We need to give a pair of maps $\overline{V_2}(R)$ to the upper right and bottom right corners making the whole diagram commute. This amounts to maps three maps from $\overline{V_2}(R)$ to $O_p^s \otimes {}^t\mathcal{O}_p^s \otimes R$, $O_p^s \otimes R$ and R satisfying certain conditions.

Consider the defining diagram of $\overline{V_2}(R)$ (with a few extra things added)



The arrows labeled f, g, and h are defined to be the evident composites with the respective sources and targets. These three maps give us the required maps to various corners of the defining diagram above for $\overline{VV}(R)$. We now have three conditions that we'd like to be satisfied. First, $\text{aug} \circ f =$

$(-)^p \circ g$ by definition. Second, $(-)^p \circ f = \operatorname{aug} \circ h$ by the factorization in Proposition 2.12, and $(-)^p \circ f = \operatorname{aug} \circ h$ by the factorization in that proposition and consideration of the relative Frobenius on ${}^t(\mathcal{O}_p/p)^s$.

□

Note that the functors V and $\overline{V}$ come equipped (by definition) with augmentations $V \to \mathrm{id}$ and $\overline{V} \to \mathrm{id}$. This means in turn that the composite functors VV and $\overline{VV}$ come with *two* maps to V and $\overline{V}$, respectively, by applying this augmentation in either factor. Let E and $\overline{E}$ be the equalizers.

Proposition 3.13. *The natural transformation $V_2 \to VV$ of the previous proposition factors through the equalizer E, and is split monic.*

Proof. It suffices to show this for the map $\overline{V_2} \to \overline{VV}$ (and the equalizer $\overline{E}$), by the now-usual representablity argument. Fix an E_0-algebra R. By defintion of $\overline{V_2}(R)$, a generic element of $\overline{V_2}(R)$ is of the form $(r, x, y) \in R \times (\mathcal{O}_p^s \otimes R) \times (\mathcal{O}_{p^2}^s \otimes R)$, subject to two equalities. We also saw in the proof of the previous proposition that a generic element of $\overline{VV}(R)$ is of the form $(s, w, w', z) \in R \times (\mathcal{O}_p^s \otimes R) \times (\mathcal{O}_p^s \otimes R) \times (\mathcal{O}_p^s \otimes {}^t\mathcal{O}_p^s \otimes R)$ (subject to conditions).

We saw before that there is a map $c : \mathcal{O}_{p^2} \to \mathcal{O}_p^s \otimes {}^t\mathcal{O}_p$, and the map in the previous proposition is constructed so that

$$(r, x, y) \mapsto (r, x, x, (c \otimes \mathrm{id})(y)).$$

The claim about the equalizer is now clear.

For the other claim, we note that Rezk ([Rez12]) showed that c is a split monomorphism of E_0-modules. This immediately gives the injectivity claim. Finally, let $\gamma : \mathcal{O}_p^s \otimes {}^t\mathcal{O}_p \to \mathcal{O}_{p^2}$ be any splitting of c. Then the map

$$(s, w, w', z) \mapsto (s, w, (\gamma \otimes \mathrm{id})(z))$$

provides the required splitting of $\overline{V_2}(R) \to \overline{VV}(R)$. □

Chapter 4

Coquadratic Pairs and Comonads

4.1 Coquadratic Pairs and Their Coalgebras

In the introduction we claimed that we could think of a certain comonad as being the "cofree comonad" on certain data that we called a "coquadratic pair". This section will set up the framework of coquadratic pairs, and discuss a few examples that are the primary subject of this thesis. Of course, there is also a monadic version. We don't need that version here, so we leave that for a later date.

We get on with the main definition of this section.

Definition 4.1. Let $\mathcal{C}$ be a category. A *coquadratic pair* on $\mathcal{C}$ consists of two functors $F, F_2 : \mathcal{C} \to \mathcal{C}$, together with three natural transformations:

- $\epsilon : F \to Id_{\mathcal{C}}$, ($F$ is equipped with an augmentation)
- $\epsilon_2 : F_2 \to Id_{\mathcal{C}}$, ($F_2$ is also equipped with an augmentation)
- and a structure map $\alpha : F_2 \to F \circ F$, which commutes with the augmentations. ($F \circ F$ inherits a natural augmentation from that on F.)

We'll think of $F(R)$ as the "generators" of some larger functor of R and $F_2(R)$ as "admissible" two-fold composites of these generators.

Sometimes we'll denote a coquadratic pair by a blackboard bold letter, and its constituent functors by the same letter in ordinary typeface. For example: $\mathbb{F} = (F, F_2, \epsilon, \epsilon_2, \alpha)$. We will also often drop the structure maps from the notation.

We've already seen a few examples.

Example 4.1. (V, V_2), and $(\overline{V}, \overline{V_2})$ from the previous sections give examples of coquadratic pairs on the category of E_0-algebras.

We can also associate a coquadratic pair to a comonad.

Example 4.2. If W is a comonad on $\mathcal{C}$ with augmentation ϵ and structure map $\Delta : W \to WW$, we can define its "underlying" coquadratic pair as $UW = (W, W, \epsilon, \epsilon, \Delta)$.

Of course, we can assemble coquadratic pairs into a category.

Definition 4.2. Let $\mathbb{F} = (F, F_2, \epsilon, \epsilon_2, \alpha)$ and $\mathbb{G} = (G, G_2, \eta, \eta_2, \beta)$ be coquadratic pairs in a category $\mathcal{C}$. A morphism $\Phi : \mathbb{F} \to \mathbb{G}$ is a pair of natural transformations $\phi : F \to G, \phi_2 : F_2 \to G_2$ which are morphisms of augmented endofunctors so that the following diagram commutes:

$$\begin{array}{ccc} F_2 & \xrightarrow{\phi_2} & G_2 \\ \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \beta} \\ FF & \xrightarrow{\phi\phi} & GG. \end{array}$$

Remark 4.1. Example 4.2 defines a forgetful functor U from the category of comonads $\mathrm{Cmd}(\mathcal{C})$ on $\mathcal{C}$ to the category of coquadratic pairs $CQ(\mathcal{C})$ on $\mathcal{C}$. That is, a morphism $f : W \to C$ of comonads gives a morphism (f, f) of coquadratic pairs. The definition of morphism of comonads ensures that the necessary diagram commutes.

We are really interested in the notion of a *coalgebra* for a coquadratic pair.

Definition 4.3. Let $\mathbb{F} = (F, F_2, \epsilon, \epsilon_2, \alpha)$ be a coquadraic pair on $\mathcal{C}$, and let c be an object in $\mathcal{C}$. The structure of an $\mathbb{F}$-coalgebra on c is a pair of morphisms $\phi : c \to Fc, \phi_2 : c \to F_2c$, so that the following diagrams commute:

$$\begin{array}{ccc} c & \xrightarrow{\phi} & Fc \\ & {\scriptstyle id}\searrow & \big\downarrow{\scriptstyle \epsilon_c} \\ & & c \end{array} \qquad \begin{array}{ccc} c & \xrightarrow{\phi_2} & F_2c \\ & {\scriptstyle id}\searrow & \big\downarrow{\scriptstyle (\epsilon_2)_c} \\ & & c \end{array}$$

$$\begin{array}{ccc} c & \xrightarrow{\phi} & Fc \\ \big\downarrow{\scriptstyle \phi_2} & & \big\downarrow{\scriptstyle F(\phi)} \\ F_2c & \xrightarrow{\alpha_c} & FFc. \end{array}$$

Remark 4.2. In the case that the structure map $F_2 \to FF$ is monic, the morphism ϕ_2 in the definition of $\mathbb{F}$-coalgebra is unique if it exists. Thus, in this case, being an $\mathbb{F}$-coalgebra becomes a *condition* on F-coalgebras.

Remark 4.3. If W is a comonad, we can regard a W-coalgebra (C, f) as a UW-coalgebra, with structure maps (f, f). However, the structure of a UW-coalgebra on C is *not* necessarily the same thing as a W-coalgebra structure, since the diagonal on W need not be monic.

Example 4.3. A $(\overline{V}, \overline{V_2})$-coalgebra is precisely a Γ-algebra which satisfies the Frobenius congruence. We'll prove this in secton 5.1.

Of course there is a notion of " quadratic pair" and of an algebra for such data. A terse definition would be a coquadratic pair in the opposite category. We have no need of such a notion here, and so will not say anything further about it.

4.2 Cofree and coquadratic comonads

We saw in the previous section that there is a forgetful functor U from the category of comonads on $\mathcal{C}$ to the category of coquadratic pairs on $\mathcal{C}$. In this section, we'll be concerned with when a comonad is determined by a coquadratic pair.

Let $\mathbb{F}$ be a quadratic pair, and consider the category $\mathrm{Cmd}(\mathcal{C})_{/\mathbb{F}}$ whose objects are comonads W equipped with a morphism of coquadratic pairs $f : UW \to \mathbb{F}$, and whose objects are maps of comonads $\phi : W \to W'$ so that the diagram

$$\begin{array}{ccc} UW & \xrightarrow{U\phi} & UW' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & \mathbb{F} & \end{array}$$

commutes in the category of coquadratic pairs on $\mathcal{C}$.

Definition 4.4. If $\mathbb{F}$ is a coquadratic pair, a *cofree comonad* on $\mathbb{F}$, is a terminal object of $\mathrm{Cmd}(\mathcal{C})_{/\mathbb{F}}$.

Remark 4.4. We are not claiming that such a thing exists for a general coquadratic pair. In fact, part of the claim of Theorem 5.1 is the existence. Later, we will prove this by constructing it directly.

Also, the terminology is abusive, since the objects of $\mathrm{Cmd}(\mathcal{C})_{/\mathbb{F}}$ are more than just a comonad. However, we will generally be even more abusive, and refer to this terminal object as *the* cofree comonad on $\mathbb{F}$. (As is usual, all such things are isomorphic.)

Definition 4.5. A comonad W on $\mathcal{C}$ is *coquadratic* if there exists a coquadratic pair $\mathbb{F}$ and a morphism f of coquadratic pairs $UW \to \mathbb{F}$, realizing W as the cofree comonad on $\mathbb{F}$.

Remark 4.5. It may not be the case that W is the cofree comonad on UW.

Remark 4.6. It may be the case that a comonad W is coquadratic in more than one way. That is W could be the cofree comonad on more than one coquadratic pair.

We'll also need a comparison between coalgebras for a coquadratic pair and coalgebras between its cofree comonad.

Proposition 4.1. *If $\mathcal{C}$ is a category with all coproducts, and if W is the cofree comonad on a coquadratic pair (F, F_2) in $\mathcal{C}$, then the induced functor W-Coalg $\to (F, F_2)$-Coalg is an isomorphism, which is the identity on underlying objects.*

Proof. The hypothesis on the category ensures that, for any object x of $\mathcal{C}$, the left Kan extension of $1 \to \mathcal{C}$ along itself exists. This is a comonad by standard properties of left Kan extensions. We will call it the "endomorphism comonad" of x, and denote it by $\mathcal{E}^x$. The definition of left Kan extension also ensures that for any endofunctor G of $\mathcal{C}$, a map $\phi : x \to Gx$ is the same as a natural transformation $\Phi : \mathcal{E}^x \to G$. If G is a comonad and ϕ gives x the structure of a G-coalgebra, then Φ is a comonad morphism. If (F, F_2) is a coquadratic pair, and x is an (F, F_2)-coalgebra,, with structure maps ϕ and ϕ_2, then we get a morphism of coquadratic pairs $(\Phi, \Phi_2) : U\mathcal{E}^x \to (F, F_2)$ in the same way. Now, if W is the cofree coalgebra on (F, F_2), then its universal property gives that the map $(\Phi, \Phi_2) : U\mathcal{E}^x \to (F, F_2)$ factors uniquely through a comonad morphism $\mathcal{E}^x \to W$, giving x a W-coalgebra structure. We now have a functors W-Coalg $\to (F, F_2)$-Coalg (by neglect of structure) and (F, F_2)-Coalg $\to W$-Coalg (by this construction), both of which are the identity on underlying objects, and which will give the identity when composed in either order. □

4.3 The cofree comonad on (V, V_2)

The goal of this section is to prove the existence, by explicit construction, of the cofree comonad on $\mathbb{V} = (V, V_2)$. We roughly follow the presentations of [BW05a] and [Rez96], with several differences. First, those sources construct free monads, whereas we want to construct cofree comonads; this is mostly a matter of replacing colimits with limits. Second, as V is already augmented, we must build this into our construction, instead of freely adjoining an augmentation. Third, and most importantly, we must also incorporate our additional functor V_2.

On the other hand, since we are only interested in doing this for a specific case, we can take advantage of a few things that are special to our situation.

Consider the iterated composites $V^{\circ n}$ as $n \geq 1$ varies. The idea will be to take the limit of these, incorporating various identities. Since V is an augmented endofunctor, we get n natural maps $V^{\circ n} \to V^{\circ n-1}$ by inserting the augmentation into various positions. We can define E_n to be the equalizer of these n maps, and note that this gives us a single natural map $E_n \to E_{n-1}$. Now we know that the inclusion $V_2 \to VV$ factors through E_2. Then we can define another functor $V_{(n)}$ (for $n \geq 2$) inductively as $V_{(2)} = V_2$ and V_n is the pullback of the angle $V_{(n-1)} \to E_{n-1} \leftarrow E_n$. Alternatively, we can define V_n as the limit of the diagram formed by the n maps $V^{\circ n} \to V^{\circ n-1}$ and the map $V_{n-1} \to V^{\circ n-1}$. By definition, these come equipped with maps $V_{(n)} \to V_{(n-1)}$. Finally, we define $C_{\mathbb{V}}$ as the limit of the $V_{(n)}$. We note that the maps $V_{(n)} \to V^{\circ n}$ are monic, as pullbacks of monic maps.

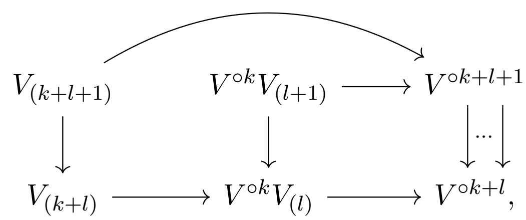
Before we prove the required things about this construction, note that the functors V and V_2 commute with limits, as they are representable. In the course of the next few propositions, we will use this fact repeatedly, and often without further comment.

To show that this is indeed the cofree comonad on (V, V_2) we'll need to use a few lemmas.

Lemma 4.1. *The inclusion $V_{(k+l)} \to V^{\circ k+l}$ factors through both of the inclusions $V^{\circ k} V_{(l)} \to V^{\circ k+l}$ and $V_{(k)} V^{\circ l} \to V^{\circ k+l}$.*

Proof. The result is tautological if either k or l are 1. To factor through the first inclusion, we fix

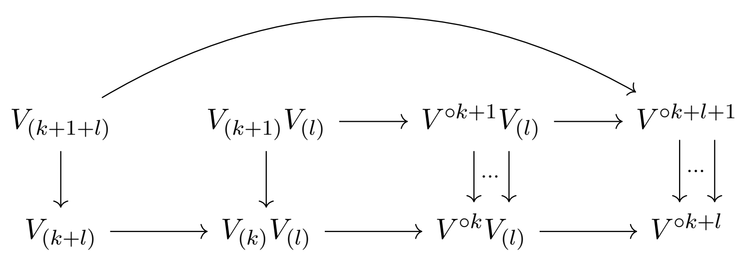
k and induct on l, so assume the result for the pair (k, l). We have a commutative diagram



and we want to fill in the top row, making the diagram continue to commute. Note that the bottom composite is the inclusion, by inductive hypothesis. The right hand "square" is a limit diagram (since V preserves limits), so we get our desired map. The other inclusion follows very similarly. □

Lemma 4.2. *The inclusion* $V_{(k+l)} \to V^{\circ k+l}$ *factors through the inclusion* $V_{(k)}V_{(l)} \to V^{\circ k+l}$

Proof. Again, we induct on k and l (separately). For notation's sake, set $V_{(1)} = V$. The claim is tautologically true if $k = l = 1$.. Now assume the lemma for a pair of integers (k, l). We wish to establish it for the pairs $(k+1, l)$ and $(k, l+1)$. We cover the case of increasing k first. Consider the following diagram.



We again wish to complete the diagram with a map $V_{(k+1+l)} \to V_{(k+1)}V_{(l)}$, and as in the previous lemma, the composite along the bottom is the inclusion by inductive hypothesis. By the previous lemma, the curved arrow on the top factors through a map $V_{(k+1+l)} \to V^{\circ k+1}V_{(l)}$ (with the diagram remaining commutative). Then since the middle "square" is a limit diagram (since V preserves limits), we get the desired factorization. Induction on l is very similar, except that we also need to use that $V_{(k)}$ preserves limits, as it's a limit of limit-preserving functors. □

Proposition 4.2. *The functor* $C_{\mathbb{V}}$ *carries a natural comonad structure. Further, it's underlying coquadratic pair has a natural map (of coquadratic pairs) to* (V, V_2).

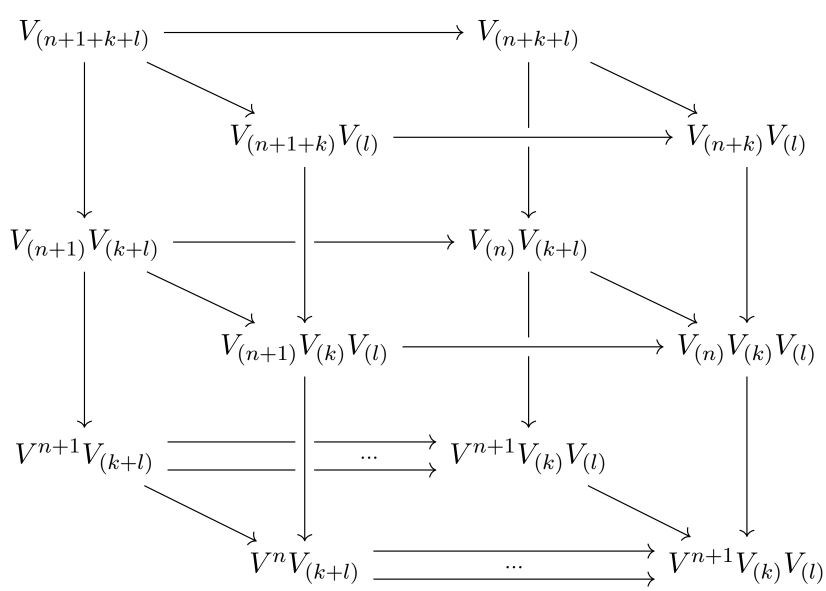
Proof. The second part of the proposition is evident by construction. For the first part, define the augmentation $C_{\mathbb{V}} \to \mathrm{id}$ as the composite $C_{\mathbb{V}} \to V \to \mathrm{id}$. (There is only one evident map $C_{\mathbb{V}} \to V$ by construction.) Define the comultiplication as follows. We want a natural transformation $\Delta : \lim_n V_{(n)} \to \lim_{k,l} V_{(k)} V_{(l)}$. (We used the fact that V commutes with limits here.) So it suffices to construct compatible maps $\lim_n V_{(n)} \to V_{(k)} V_{(l)}$. We get this by considering the composite $\lim_n V_{(n)} \to V_{(k+l)} \to V^{\circ k+l}$ and applying Lemma 4.2. For future usage, write $\Delta_{k,l}$ for these maps $C_{\mathbb{V}} \to V_{(k)} V_{(l)}$. The counit diagram is now reasonably clear, and we need to check the coassociavity axiom.

That is, we need to check, for all n, k, l, that the square

$$\begin{array}{ccc} V_{(n+k+l)} & \xrightarrow{\Delta_{n,k+l}} & V_{(n)} V_{(k+l)} \\ \downarrow{\scriptstyle \Delta_{n+k,l}} & & \downarrow{\scriptstyle V_{(n)}(\Delta_{k,l})} \\ V_{(n+k)} V_{(l)} & \xrightarrow{\Delta_{n,k} V_{(l)}} & V_{(n)} V_{(k)} V_{(l)} \end{array}$$

commutes. To do that, we induct (again) on n, k and l. The case where all three are 1 follows from the definition of $V_{(3)}$.

Let's consider the case of increasing n first. Assuming that the above square commutes for (n, k, l), we want to show it for $(n+1, k, l)$. As with previous inductive arguments, consider the following diagram.



The front right hand square is a limit diagram, and the whole outer diagram commutes, as well as all of the faces except possibly the top left. The bottom left square commutes by assumption. Thus, both of the ways around the desired commutative square (top left) are the same, by definition of $V_{(n+1)}$ as a limit. The other two cases follow similarly. □

Proposition 4.3. *With the the structure from the previous proposition, $C_{\mathbb{V}}$ is the cofree comonad on $\mathbb{V}$.*

Proof. We check the universal property. Let W be a comonad equipped with a morpism of co-quadratic pairs $(f, f_2) : UW \to (V, V_2)$. The required maps $W \to V^{\circ n}$ are given by $f^n \circ \Delta^{\circ n-1}$, and the required map $W \to V_2$ is provided by f_2. These maps are compatible with the structure map $\alpha : V_2 \to VV$ since (f, f_2) is a morphism of coquadratic pairs. Since $C_{\mathbb{V}}$ was defined as a limit of these things, we get our desired natural transformation $\phi : W \to C_{\mathbb{V}}$. For later use, let's name the map $W \to V_{(k)}$ as $\phi_{(k)}$. It's clear that ϕ is the only such natural transformation which is compatible with (f, f_2), since $C_{\mathbb{V}}$ is defined as a limit. It's also clear that this is a morphism of augmented functors, so we now only need to check that it is, in fact a morphism of comonads. That is, we need to check that the square

$$\begin{array}{ccc} W & \xrightarrow{\phi} & C_{\mathbb{V}} \\ \downarrow{\scriptstyle \Delta^W} & & \downarrow{\scriptstyle \Delta^{C_{\mathbb{V}}}} \\ WW & \xrightarrow{\phi\phi} & C_{\mathbb{V}}C_{\mathbb{V}} \end{array}$$

commutes. Unpacking this, we see that the right-and-then-down composite $\Delta^{C_{\mathbb{V}}} \circ \phi$ is given on components by taking the map $\phi_{(k+l)} : W \to V_{(k+l)} \to V^{\circ k+l}$ and factoring it through $V_{(k)} \circ V_{(l)}$. On the other hand, the down-and-then-right composite is given as the comultiplication on W followed by $\phi_{(k)}\phi_{(l)}$. This also is a factorization of a certain map from $W \to V^{\circ k+l}$. Thus we need to check that the two maps $f^{k+l} \circ (\Delta^W)^{\circ k+l-1}$ and $((f^k(\Delta^W)^{\circ k-1})(f^l(\Delta^W)^{\circ l-1}))\Delta^W$ are equal. This follows from the interchange law for horizontal and vertical compostions of natural transformations. □

Chapter 5

Proof of the Main Theorem

5.1 A characterization of $(\overline{V}, \overline{V_2})$-coalgebras

In section 4.1, we promised a characterization of $(\overline{V}, \overline{V_2})$-coalgebras. This section exists to fulfill that promise. Before we need to slightly recast the notion of Γ-algebra.

Definition 5.1. A Γ-algebra structure on R is data of an E_0-algebra homomorphism $P : R \to \mathcal{O}_p^s \otimes_{E_0} R$, subject to the existence (and consequently, uniqueness) of a dotted arrow P_2 making the following diagram commute.

$$\begin{array}{ccc} R & \xrightarrow{\quad P \quad} & \mathcal{O}_p^s \otimes R \\ {\scriptstyle P_2}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes P} \\ O^s_{p^2} \otimes R & \xrightarrow{c \otimes \mathrm{id}} & {}^t\mathcal{O}_p^s \otimes {}^t\mathcal{O}_p^s \otimes R. \end{array}$$

By E_0-linear duality, this amounts to the same thing as before.

A Γ-algebra R satisfies the *Frobenius congruence* (following [Rez09]) if the composite $(\mathrm{aug} \otimes \mathrm{id} \circ P : R \to R/p$ is the p-th power map. Again, this is straightforwardly equivalent to the definition from before.

Now we can state the main result of this section.

Proposition 5.1. *A $(\overline{V}, \overline{V_2})$-coalgebra is precisely a Γ-algebra which satisfies the Frobenius congruence. In other words, a $(\overline{V}, \overline{V_2})$-coalgebra structure on R determines and is determined by the structure on R of a Frobenius Γ-algebra.*

Proof. Given a Γ-algebra structure on R, satisfying the Frobenius congruence, we wish to create a $(\overline{V}, \overline{V_2})$-coalgebra structure on R. This is a pair of maps $\psi : R \to \overline{V}(R), \psi_2 : R \to \overline{V_2}(R)$, satisfying certain conditions. We'll provide the $\overline{V}$-coalgebra structure first. By the limit definition of $\overline{V}$, we only need to provide a map $R \to \mathcal{O}_p^s \otimes R$, so that

$$\begin{array}{ccc} R & \longrightarrow & \mathcal{O}_p^s \otimes R \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{aug}\otimes\mathrm{id}} \\ R & \xrightarrow{(-)^p} & R/p \end{array}$$

commutes. This is provided by P. The commutativity of the diagram is exactly the Frobenius congruence. Now for the $\overline{V_2}$-coalgebra structure. Again, by the limit definition, we need a map $R \to \mathcal{O}_p^s \otimes R$ (already provided by P), as well as a map $R \to \mathcal{O}_{p^2}^s \otimes R$, subject to some more conditions (which we will check in a moment). This second map is provided by P_2. We need to check that this actually produces a map $R \to \overline{V_2}(R)$ That is, we need that the diagram

$$\begin{array}{cccc} R & & \xrightarrow{\qquad\qquad P_2 \qquad\qquad} & \mathcal{O}_{p^2} \otimes R \\ \Big\| & \overset{P}{\searrow} & & \big\downarrow{\scriptstyle q\otimes\mathrm{id}} \\ \Big\| & & \mathcal{O}_p \otimes R \xrightarrow{\overline{\mathcal{P}}_p \otimes (-)^p} & {}^tA_{p^2}/(I_{p^2}+J_{p^2})^s \otimes R \\ \Big\| & & \big\downarrow{\scriptstyle q\otimes\mathrm{id}} & \\ R & \xrightarrow{(-)^p} & R/p & \end{array}$$

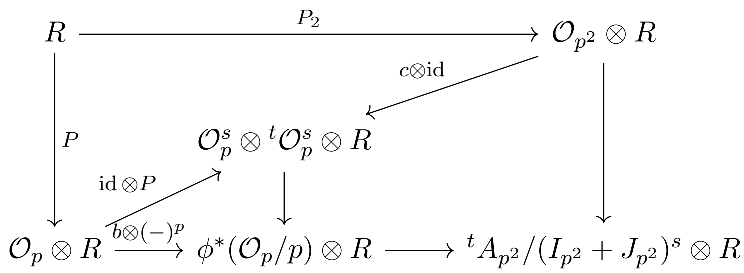
commutes. The lower left portion is guaranteed to commute, since this is the Frobenius congruence. The upper portion is slightly more complicated.

First, recall that $\phi : \mathcal{O}_1/p \to \mathcal{O}_1/p$ denotes the p-th power map, and $\phi^*\mathcal{O}_p/p = (\mathcal{O}_p/p)^s \otimes^{\phi} \mathcal{O}_1/p$ is the base changed $\mathcal{O}_p/p$. The relative Frobenius map $\mathrm{Frob} : \phi^*\mathcal{O}_p/p \to {}^{t(p)}\mathcal{O}_p/p$ factors the p-the power map $\mathcal{O}_p/p \to \mathcal{O}_p/p$ as a composite

$$\mathcal{O}_p/p \xrightarrow{a\mapsto a\otimes 1} \phi^*\mathcal{O}_p/p \xrightarrow{\mathrm{Frob}} {}^{t(p)}\mathcal{O}_p/p.$$

We'll call the first map b in what follows.

We can insert the commutativity diagram for a Γ-algebra into that top portion as shown below.

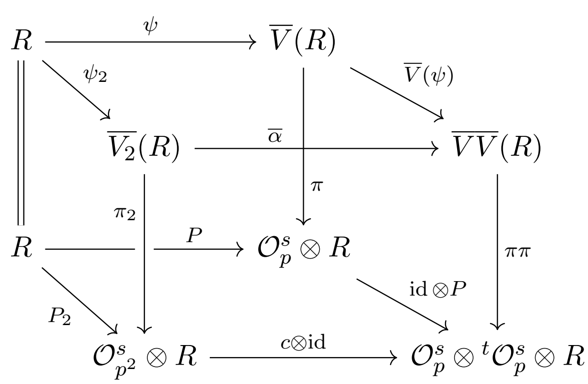


Now, the top left sector commutes, since R is a Γ-algebra, the right sector commutes by Proposition 2.12, and the bottom left sector commutes precisely because the Γ-algebra structure on R satisfies the Frobenius congruence.

To go the other way, from a $(\overline{V}, \overline{V_2})$ coalgebra to a Γ-algebra, we note that the definitions of $\overline{V}$ and $\overline{V_2}$ endow these functors with natural transformations $\pi : \overline{V} \to \mathcal{O}_p^s \otimes -$ and $\pi_2 : \overline{V_2} \to \mathcal{O}_{p^2}^s \otimes -$. Composing these with the structure maps $\psi : R \to \overline{V}(R)$ and $\psi_2 : R \to \overline{V_2}(R)$, we get the desired Γ-algebra structure maps P and P_2.

We now need to check that the commutativity diagram for Γ-algebras does in fact commute, or, going the other way, that the analogous diagram for $(\overline{V}, \overline{V_2})$-coalgebras commutes.

Consider the following diagram.



(5.1)

Given that either the top or bottom face commutes, we need to check that the other one does. Let's assume first that the top face commutes. Then the left and right faces commute by the definition of π and the construction of P and P_2 from ψ and ψ_2 The back face also commutes by the construction of P from ψ Then we only need to check that the front face commutes, but this is due to the construction of the map $\overline{\alpha} : \overline{V_2} \to \overline{V}\overline{V}$ given in Proposition 3.12.

Now assume that the bottom face commutes. We want to verify that the top does. But this is due to the construction of $\overline{\alpha}$, again as in the proof of Proposition 3.12.

These two processes give functors between the categories of Frobenius Γ-algebras and $(\overline{V}, \overline{V_2})$-coalgbras, which are the identity on underlying objects, and evidently the identity when composed in either order. This proves the proposition.

□

5.2 Comparison of (V, V_2)-coalgebras and $\mathbb{T}$-Algebras

In this section, we'll set up a comparison between (V, V_2)-coalgebras and $\mathbb{T}$-algebras. More precisely, we'll show the following theorem.

Theorem 5.1. *Let U be the forgetful functor from $\mathbb{T}$-algebras to E_0-algebras, and W its right adjoint. Then the comonad $\mathbb{W} = UW$ on E_0-algebras is the cofree comonad on the augmented endofunctor with relations (V, V_2).*

We first show a version of this theorem on p-torsion-free objects.

To set notation, let $C_{\mathbb{V}}$ denote the cofree comonad on (V, V_2). In addition, let $(C_{\mathbb{V}}\text{-Coalg})^{\text{ptf}}$ and $(\mathbb{W}\text{-Coalg})^{\text{ptf}}$ be the full subcategories of $C_{\mathbb{V}}$-coalgebras (resp. $\mathbb{W}$-coalgebras) consisting of the objects which are p-torsion-free (as underlying E_0-algebras.)

Proposition 5.2. *There is an isomorphism of categories between $(C_{\mathbb{V}}\text{-Coalg})^{\text{ptf}}$ and $(\mathbb{W}\text{-Coalg})^{\text{ptf}}$, which is the identity on underlying E_0-algebras.*

Proof. Consider the following diagram of functors between categories of algebras or coalgebras.

$$
\begin{array}{ccc}
C_{\mathbb{V}}\text{-Coalg}^{\text{ptf}} & \hookrightarrow & C_{\mathbb{V}}\text{-Coalg} \\
\| & & \|{\scriptstyle (1)} \\
(V, V_2)\text{-Coalg}^{\text{ptf}} & \hookrightarrow & (V, V_2)\text{-Coalg} \\
\|{\scriptstyle (4)} & & \downarrow{\scriptstyle \text{forget}} \\
(\overline{V}, \overline{V_2})\text{-Coalg}^{\text{ptf}} & \hookrightarrow & (\overline{V}, \overline{V_2})\text{-Coalg} \\
\| & & \|{\scriptstyle (2)} \\
\Gamma\text{-Alg}^{\text{Frob,ptf}} & \hookrightarrow & \Gamma\text{-Alg}^{\text{Frob}} \\
\|{\scriptstyle (5)} & & \uparrow{\scriptstyle \text{forget}} \\
\mathbb{T}\text{-Alg}^{\text{ptf}} & \hookrightarrow & \mathbb{T}\text{-Alg} \\
\| & & \|{\scriptstyle (3)} \\
\mathbb{W}\text{-Coalg}^{\text{ptf}} & \hookrightarrow & \mathbb{W}\text{-Coalg}
\end{array}
$$

All of the maps are identities (or inclusions) on underlying E_0-algebras. The numbered isomorphisms may need some justification. The isomorphism (1) is given by Propositin 4.1. Proposition 5.1 gives (2), and playing around with adjunctions gives (3). The functors V and $\overline{V}$ are the same functor on the p-torsion-free subcategory, and the same is true for V_2 and $\overline{V_2}$; this gives (4). Finally, (5) is precisely Theorem A of [Rez09]. Composition along the left gives the required homomorphism. □

To finish the proof of the theorem, we need a sequence of propositions that allow us to reduce to the p-torsion-free case just established.

Proposition 5.3. *The comonad $C_{\mathbb{V}}$ on E_0-algebras preserves p-torsion-free objects.*

Proof. The functors V and V_2 preserve p-torsion-free objects, by Propositions 3.6 and 3.10. Now, $C_{\mathbb{V}}$ is constructed as a limit out of iterates of these, so it also preserves p-torsion-free objects. □

Proposition 5.4. *The comonad $C_{\mathbb{V}}$ is representable, by a p-torsion-free object.*

Proof. We know by Propositions 3.1 and 3.7 that the functors V and V_2 are representable by polynomial rings P and Q over E_0. Lemma 2.1 gives that the iterates $V^{\circ n}$ are also represented by the polynomial rings $P^{\odot n}$ and that the various induced maps $P^{\odot n-1} \to P^{\odot n}$ are just given by inclusions of subsets of variables. Since the map $V_2 \to VV$ is split monic, the induced map $P_2 \to Q$ is split epic. The result follows from the limit definition of $C_{\mathbb{V}}$ and this splitting. □

Proposition 5.5. *The comonad* $\mathbb{W}$ *is representable, by a* p*-torsion-free object.*

Proof. Recall that $\mathbb{W}$ was defined as the composite UW, where U was the forgetful functor from $\mathbb{T}$-algebras to E_0-algebras, and W was its right adjoint. A straightforward manipulation of adjunctions now says that $\mathbb{W}$ is representable by $\mathbb{T}(E_0)$, which is p-torsion-free. □

Proposition 5.6. *The comonad* $\mathbb{W}$ *on* E_0*-algebras preserves* p*-torsion-free objects.*

Proof. We'd like to show that the map (of E_0-modules) "multiplication by p" on $\mathbb{W}(A)$ is monic. As $\mathbb{W}$ is representable by $\mathbb{T}(E_0)$, we can consider the following diagram.

$$\begin{array}{ccc}
\mathrm{Hom}_{E_0\text{-Alg}}(\mathbb{T}(E_0), A) & \xrightarrow{\cdot p} & \mathrm{Hom}_{E_0\text{-Alg}}(\mathbb{T}(E_0), A) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{E_0\text{-Alg}}(\mathbb{T}(E_0), p^{-1}A) & \xrightarrow{\cdot p} & \mathrm{Hom}_{E_0\text{-Alg}}(\mathbb{T}(E_0), p^{-1}A) \\
\uparrow \wr & & \uparrow \wr \\
\mathrm{Hom}_{E_0\text{-Alg}}(p^{-1}\mathbb{T}(E_0), p^{-1}A) & \xrightarrow{\cdot p} & \mathrm{Hom}_{E_0\text{-Alg}}(p^{-1}\mathbb{T}(E_0), p^{-1}A)
\end{array}$$

The vertical arrows on the top are monic, since A is p-torsion free. Thus, the top horizonal arrow is monic if the bottom horizontal arrow is. By representability, the natural transformation $\cdot p : \mathbb{W}(-) \to \mathbb{W}(-)$ is induced by a map $[p] : \mathbb{T}(E_0) \to \mathbb{T}(E_0)$. That the bottom horizontal arrow in the diagram is monic now follows from the next lemma. □

Lemma 5.1. *The map* $[p] : \mathbb{T}(E_0) \to \mathbb{T}(E_0)$ *is an isomorphism after inverting* p*.*

Proof. As and E_0-module, $\mathbb{T}(E_0)$ decomposes as

$$\mathbb{T}(E_0) \simeq \bigoplus_{n \geq 0} E_0^{\wedge}(B\Sigma_n).$$

The map $[p]$ preserves this decomposition, and so we can decompose $[p]$ and dualize to get a family of maps $[p]_n^{\vee} : E^0(B\Sigma_n) \to E^0(B\Sigma_n)$. Since the coproduct Δ_+ on $T(E_0)$ representing addition on

$\mathbb{W}$ is given by stable transfers in the homology of symmetric groups, we can write

$$[p]_n^\vee = \sum_\lambda \operatorname{tr}_{\Sigma_\lambda}^{\Sigma_n} \operatorname{res}_{\Sigma_\lambda}^{\Sigma_n},$$

where the sum is over all ordered p-tuples $\lambda = (\lambda_1, \ldots, \lambda_p)$ of non-negative integers whose sum is n, and $\Sigma_\lambda = \Sigma_{\lambda_1} \times \ldots \times \Sigma_{\lambda_p} \leq \Sigma_n$. We wish to show (for each n) that this is an isomorphism after inverting p. To do this, we'll use the character theory of Hopkins, Kuhn and Ravenel ([HKR00].)

The upshot of all of this is we can define a map $[p]_n^{cl} : Cl_h(G, C_0) \to Cl_h(G, C_0)$ by the same formula as before;

$$[p]_n^{cl}(\phi) = \sum_\lambda \operatorname{tr}_{\Sigma_\lambda}^{\Sigma_n} \operatorname{res}_{\Sigma_\lambda}^{\Sigma_n}(\phi) = \phi\left(\sum_\lambda \operatorname{tr}_{\Sigma_\lambda}^{\Sigma_n}(1)\right).$$

By the discussion above, the diagram

$$\begin{array}{ccc} C_0 \otimes E^0(BG) & \xrightarrow{C_0 \otimes [p]_n^\vee} & C_0 \otimes E^0(G) \\ \big\downarrow \chi & & \big\downarrow \chi \\ Cl_h(H, C_0) & \xrightarrow{[p]_n^{cl}} & Cl_h(G, C_0) \end{array}$$

commutes. Since C_0 is a faithfully flat $p^{-1}E_0$-algebra, $[p]_n^\vee : p^{-1}E^0(B\Sigma_n) \to p^{-1}E^0(B\Sigma_n)$ is an isomorphism if and only if $C_0 \otimes [p]_n^\vee$ is. It thus suffices to check that $[p]_n^{cl}$ is an isomorphism, i.e., that the class function

$$\sum_\lambda \operatorname{tr}_{\Sigma_\lambda}^{\Sigma_n}(1)$$

is a unit in $Cl_h(G, C_0)$

A height h class function ϕ is a unit if and only if, for all height h conjugacy classes α, $\phi(\alpha)$ is a unit in C_0. Now, by the definition of transfers for class functions given above,

$$\operatorname{tr}_{\Sigma_\lambda}^{\Sigma_n}(1)(\alpha) = \sum_{g\Sigma_\lambda \in (\Sigma_n/\Sigma_\lambda)^{\mathrm{im}\alpha}} 1 = \left|(\Sigma_n/\Sigma_\lambda)^{\mathrm{im}\alpha}\right|,$$

which is a non-negative integer. Thus

$$\sum_\lambda \operatorname{tr}_{\Sigma_\lambda}^{\Sigma_n}(1)(\alpha) = \sum_\lambda \left|(\Sigma_n/\Sigma_\lambda)^{\mathrm{im}\alpha}\right|$$

is a non-negative integer. The term given by $\lambda = (n, 0, \ldots, 0)$ contributes a 1 to the sum, and so we see that this is actually a *positive* integer. As C_0 is a $\mathbb{Q}$-algebra (since $p^{-1}E_0$ is), all positive integers are units in C_0, and we are done. □

Proof of Theorem 5.1. By Propositions 5.3 and 5.6, the comonads $C_{\mathbb{V}}$ and $\mathbb{W}$ restrict to comonads on the full subcategory $(E_0\text{-Alg})^{ptf}$. Let $(C_{\mathbb{V}}\text{-Coalg})^{\mathrm{ptf}}$ denote the full subcategory of $C_{\mathbb{V}}$-coalgebras whose underlying objects are p-torsion-free, and similarly for $(\mathbb{W}\text{-Coalg})^{\mathrm{ptf}}$. That is, these are the categories of coalgebras for the comonads $C_{\mathbb{V}}$ and $\mathbb{W}$ considered as comonads on the category $(E_0\text{-Alg})^{\mathrm{ptf}}$. By Proposition 5.2 we have an isomorphism $\Phi' : (C_{\mathbb{V}}\text{-Coalg})^{\mathrm{ptf}} \to (\mathbb{W}\text{-Coalg})^{\mathrm{ptf}}$ of categories which is the identity on underlying objects. Then by Theorem 3.6.3 of [BW05a] (dualized), this induces and is induced by an isomorphism $\varphi' : C_{\mathbb{V}} \to \mathbb{W}$ of comonads on $(E_0\text{-Alg})^{\mathrm{ptf}}$. Since $C_{\mathbb{V}}$ and $\mathbb{W}$ are representable by p-torsion-free objects $P_{\mathbb{V}}$ and $P_{\mathbb{W}}$, respectively (Propositions 5.4 and 5.5), we get an isomorphism $P_{\mathbb{W}} \xrightarrow{\sim} P_{\mathbb{V}}$. This in turn induces an isomorphism $\varphi : C_{\mathbb{V}} \to \mathbb{W}$ of comonads on the *whole* category E_0-Alg, and so by Theorem 3.6.3 of [BW05a] again, we have the desired isomorpism $\Phi : C_{\mathbb{V}}\text{-Coalg} \to \mathbb{W}\text{-Coalg}$. □

Corollary 5.1. *A $\mathbb{T}$-algebra structure on an E_0-algebra A is the same data as a (V, V_2)-coalgebra structure on A. More precisely, there is an isomorphism of categories between* $\mathbb{T}$-Alg *and* (V, V_2)-Coalg*, which is the identity on underlying E_0-algebras.*

Proof. By adjuctions, a $\mathbb{T}$-algebra structure on R is precisely the same thing as a $\mathbb{W}$-coalgebra structure on R. By definition, the structure of a coalgebra for $C_{\mathbb{V}}$ is precisely the same thing as the structure of a (V, V_2)-coalgebra. □

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