

A CONSTRUCTION OF EXTENDED
TOPOLOGICAL FIELD THEORIES

BY

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DISSERTATION

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Abstract

We give an explicit construction of extended topological field theories over a manifold taking values in the deloopings of $U(1)$ from the data of differential forms on the manifold. More specifically, for a manifold M , using a version of the Dold-Kan construction, we create from the Deligne complex a Kan complex, $|\mathfrak{D}_n|^+(M)$, and show that there is a natural map into $\text{Fun}^\otimes(\mathbf{Bord}_n^{\text{or}}(M), \mathbf{B}^n U(1))$. The main theorem asserts that this map becomes a weak equivalence after restricting to the framed bordisms. The construction is on the point-set level, so immediately can be refined to the setting where the Deligne complex is considered discretely, which is known as a model for the differential cohomology.

As a part of the proof we show that the geometric realization of the bordism (∞, n) -category is the n -fold delooping of the infinite loop space of the Madsen-Weiss spectrum $\text{MTO}(n)$. The direct attempts to generalize the Galatius-Madsen-Tillmann-Weiss proof of $n = 1$ case fail due the globularity restriction on the bordism (∞, n) -category. To overcome this, we use the abstract transversality argument of Bökstedt and Madsen, Gromov's theory of microflexible sheaves and Rezk's argument on realization fibrations.

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1 Higher Category Theory

1.1 Introduction

The language of higher category theory that we are going to use is that of quasi-categories. Following [Lu1], we are going to call quasi-categories, ∞ -categories. This particular theory of higher categories will play a special role in this paper. We will use the theory of ∞ -categories as the underlying framework of all of our constructions. It should not be very difficult to formulate the constructions and theorems in this paper using other higher category theories, e.g. model categories. Most of the constructions that we are going to make are going to involve a background manifold. These constructions are natural with respect to this manifold, therefore, they define a pre(co)sheaves over the site of manifolds. In fact, some of them, will even be (co)sheaves and that will be of great importance to us. The theory of ∞ -topoi provides a convenient framework of formulating and working with constructions of this nature.

We will also introduce the notion of n -fold Segal spaces due to Barwick (see [BaSP1]). These are the higher categorical generalizations of n -categories. We will call these categories (∞, n) -categories. In particular, the theory of $(\infty, 1)$ -categories is equivalent to that of ∞ -categories. We will also introduce the notion of (∞, n) -precategories. The latter notion is easier to work with than (∞, n) -categories. The bordisms categories naturally admit an (∞, n) -precategorical structure. Obtaining the bordism (∞, n) -categories amounts to performing the Rezk completion procedure.

We would like to use this introduction to provide a motivation for higher category theory through the lens of moduli problems. Higher category theory is some sort of amalgam of category theory with homotopy theory. Its purpose was a more natural integration of (higher) automorphisms into moduli problems. For instance, in algebraic geometry one would quite often find oneself in a situation where a particular moduli problem is not representable because of existence of automorphisms. Interestingly enough in algebraic topology the same problems tend to have quite well-behaved moduli. The algebraic topologists owe this to the fact there is a notion of homotopy and the definition of representability changes, so as to incorporate these homotopies. The algebraic geometers (of old times) were not as lucky due to the rigidity of the category of schemes, so injection of homotopies were usually quite hard to do. Higher category theory has tools that allow one to incorporate homotopy theory in rigid situations, such as schemes.

I would like to present the story of moduli in a little detail below – laying out the story of algebraic geometry and that of algebraic topology. This will be partly a heuristic discussion, so most of it can be skipped. There will be some definitions though. I will incorporate some remarks about manifolds in both stories as they can be viewed both from the algebro-geometric and algebro-topological points of view. We do this, since the main focus of the paper are manifolds.

We will begin with the story of algebraic geometry. Let us begin by an abstract formulation of a rigid moduli problem. We have a category of interest \mathcal{C} and a presheaf $M : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$. A moduli problem consists of two questions:

- 1) Is the presheaf M representable?
- 2) If it is what is the representing object?

The categories \mathcal{C} of main interest to use will be Sch , Aff , Man – the categories of schemes, affine schemes, manifolds, respectively. As it happens quite often there are lots of examples where M fails to be representable. In fact, there is a simple obstruction to representability that we need to address. Usually the category \mathcal{C} comes with the additional structure of Grothendieck topology. We will recall this definition later in Section 1.2.2. The categories Sch and Aff have a whole zoo of topologies: Zariski, étale, Nisnevich, fppf. The category Man has the regular topology and the surjective submersion topology. It is fairly straightforward to show that representable functors are sheaves over Sch_{Zar} : this amounts to showing that maps between schemes can be constructed locally via gluing. See [Ha]. For the rest of the sites on Sch mentioned above the technique of faithfully flat descent can show that representable functors are sheaves [De]. For Man the situation is easier and could be argued topologically. This definition forces us to focus on sheaves on a particular site. This observation though being necessary will not suffice, of course, there are lots of examples of non-representable sheaves over the categories mentioned above. Nevertheless, we can see that the right place to look for representable functors is within sheaves over sites. Therefore, we enlarge the categories enough so that all the sheaves become representable. This is the notion of the so called topos. There are several equivalent definitions of topoi. The simplest definition is the following.

DEFINITION 1. A category is a topos if it is equivalent to a category of sheaves over some Grothendieck site.

Now let us suppose we are given a sheaf M over Sch_{Zar} and we are trying to find a representing object for it. We say that M actually lives in $\text{Sh}(\text{Sch}_{\text{Zar}})$, and so does Sch_{Zar} , but as a subcategory. Therefore, the question simply reduces to understanding whether M is equivalent to something in Sch_{Zar} . In itself this approach may seem tautological, however, the sheaves sometimes do possess geometry and our task is to recognize that. Even if the answer is negative and the sheaf is not representable, we can still somehow hope to talk about the geometry of M and manage to extract some useful information. For instance, there is the Grothendieck site Aff_{Zar} of affine schemes with Zariski topology. A scheme X can be view as sheaves on Aff_{Zar} via the inclusion into Sch_{Zar} and then composing with $\text{Hom}(-, X)$. One can show that the resulting functor $\text{Sch}_{\text{Zar}} \rightarrow \text{Sh}(\text{Aff}_{\text{Zar}})$ is an embedding, in fact, a version of Yoneda embedding. This example somehow ought to show that some sheaves on the affine Zariski site possess some geometry, since supposedly the schemes do. Thus, it is not inconceivable that the sheaves on schemes would do that. The reason for this is even stronger,

since there is an equivalence $\mathrm{Sh}(\mathrm{Aff}_{\mathrm{Zar}}) \simeq \mathrm{Sh}(\mathrm{Sch}_{\mathrm{Zar}})$ – there may be some rigid geometric constructs which are not schemes.

To give a more substantial example let us remind the reader of the notion of algebraic spaces. An algebraic space is sheaf X on $\mathrm{Sch}_{\mathrm{fppf}}$ satisfy the following properties. First, the diagonal map $X \rightarrow X \times X$ is representable, meaning that that if we consider the pullback to a scheme we obtain another scheme (i.e. a representable functor). Second, there is a surjective étale map from a scheme to X . Again, this means that the pullback to a scheme is surjective étale. There is a full subcategory of algebraic spaces $\mathcal{A}\mathrm{lsp}$ of $\mathrm{Sh}(\mathrm{Sch}_{\mathrm{fppf}})$. The schemes are contained in $\mathcal{A}\mathrm{lsp}$. One interesting feature of algebraic spaces is that it is closed under the free group action quotients. The schemes are not closed under these quotients as demonstrated by the Hironaka’s example. See Appendix B of [Ha] for further details.

The advantage of topoi is the fact that any categorical construction on sets can be performed on the objects of a topos. In particular, this category is complete and cocomplete. For instance, the space \mathbb{R}^∞ is not a manifold in the conventional sense. It’s only fault is that it is infinite-dimensional. However, in the topos of sheaves over manifolds \mathbb{R}^∞ does exist! It is simply the colimit of inclusions of affine spaces, i.e. $\mathbb{R}^\infty \simeq \mathrm{colim} \mathbb{R}^n$. Mind you that this not going to be a pointwise colimit. Namely, the natural map $\mathrm{colim} \mathrm{Hom}(X, \mathbb{R}^n) \rightarrow \mathrm{Hom}(X, \mathbb{R}^\infty)$ is not an isomorphism. There is a direct way of describing \mathbb{R}^∞ : to a manifold X it assigns a countably infinite collection of smooth functions (f_1, f_2, \dots) , such that for each compact subspace $C \subset X$ finitely many of these functions are non-zero on C .

Recall that the issue of pointwise colimits not matching with the colimit in sheaves comes from very general considerations. We have an inclusion i of $\mathrm{Sh}(\mathcal{C})$ into $\mathcal{P}\mathrm{sh}(\mathcal{C})$, but we also have a functor in the reverse direction $^+ : \mathcal{P}\mathrm{sh}(\mathcal{C}) \rightarrow \mathrm{Sh}(\mathcal{C})$, which is commonly referred to as the *sheafification* functor. These functors form an adjoint pair $(^+, i)$. Since i is right adjoint we can only expect it to preserve limits: preservation of colimits may not happen. The functor $^+$, on the other hand, preserves colimits by virtue of being a left adjoint. In fact, it also preserves finite limits. The reason for this is that certain finite limits and directed colimits commute. We will call the functors preserving finite limits left exact. This means that $\mathrm{Sh}(\mathcal{C})$ is a left exact localization of $\mathcal{P}\mathrm{sh}(\mathcal{C})$. We claim that we can recover the Grothendieck topology from this adjunction. Suppose that we have subcategory \mathcal{E} of $\mathcal{P}\mathrm{sh}(\mathcal{C})$, such that the inclusion admits a left adjoint $^+$, which preserves finite limits. Given a collection of maps $\{U_\alpha \rightarrow X\}$ in \mathcal{C} , we will call it a covering family if the diagram

$$X^+ \leftarrow \coprod U_\alpha^+ \rightrightarrows \coprod U_{\alpha\beta}^+$$

is a coequalizer in \mathcal{E} . In the diagram, we implicitly used the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$. One can show that these coverings satisfy the axioms of a Grothendieck pretopology, so they define a Grothendieck topology on \mathcal{C} . The sheaves on this Grothendieck site are the objects of \mathcal{E} . This basically gives another characterization of topoi .

PROPOSITION 1. *A category \mathcal{E} is a topos if and only if it is a left exact localization of a presheaf category.*

In fact, this will be the definition that will be generalized to the higher categorical setting. The notion of Grothendieck topology will be indispensable, since the examples of ∞ -topoi of interest to us will come from Grothendieck sites.

The characterizations of topoi that we have seen up to this point were external, namely they required some reference to a Grothendieck site or a presheaf category. The interesting thing is that they possess internal categorical properties that characterize them. There are the so called *Giraud axioms*, which we begin to describe. Suppose \mathcal{E} is an arbitrary small category. The Giraud axioms for \mathcal{E} are the following:

- 1) \mathcal{E} is presentable,
- 2) the colimits in \mathcal{E} are universal,
- 3) the sums in \mathcal{E} are disjoint,
- 4) all the equivalence relations are effective in \mathcal{E} .

Let me explain the terminology involved in these axioms. We will begin with the easier ones.

We begin with condition 2). Consider an object $X \in \mathcal{E}$, and a morphism $f : Y \rightarrow X$. There is a map of slice categories $f^* : \mathcal{E}_{/X} \rightarrow \mathcal{E}_{/Y}$, that assigns to $Z \rightarrow X$ the pullback $Y \times_X Z \rightarrow Y$. We know that this functor is right adjoint with its left adjoint being f_* – the functor obtained by composing with f . As a consequence, f^* preserves limits. The colimits in \mathcal{E} are universal if f^* preserves colimits for all morphisms f . This can be written as a formula that resembles the distributive law:

$$Y \times_X (\operatorname{colim} Z_\alpha) \simeq \operatorname{colim} (Y \times_X Z_\alpha).$$

Now we consider condition 3). Consider two objects $X, Y \in \mathcal{E}$. Then we can consider their fiber product over their coproduct $X \times_X \coprod_Y Y$, and the sums being disjoint means that this object is equivalent to the initial object \emptyset of \mathcal{E} for any X and Y in \mathcal{E} .

For 4) we first recall that in set-theory an equivalence relation on a set X can be given as a subset $R \subset X \times X$ subject to various conditions. Similarly, if X now is an object of \mathcal{E} , then we can define an equivalence relation as a subobject R of $X \times X$, so that the inclusion of sets

$$\operatorname{Hom}(Y, R) \hookrightarrow \operatorname{Hom}(Y, X \times X) \simeq \operatorname{Hom}(Y, X) \times \operatorname{Hom}(Y, X)$$

is an equivalence relation for all $Y \in \mathcal{E}$. Again taking our cue from sets we can consider the coequalizer X/R of the two maps coming from the inclusions $R \rightrightarrows X$. Then we can consider the natural morphism $R \rightarrow X \times_{X/R} X$. The equivalence relation R is effective if the latter morphism is an equivalence.

Finally, let me get to 1). This condition is more involved. Essentially, one ought to think of presentability

of a category as some sort of a condition that tells us that the category is determined by a much smaller full subcategory, even though the category itself may be huge. A category \mathcal{E} is presentable if the following conditions are satisfied:

- a) \mathcal{E} is locally small,
- b) \mathcal{E} is cocomplete,
- c) there is a set S of objects that generates \mathcal{E} under filtered small colimits,
- d) any object of \mathcal{E} is compact.

If we assume c), then d) is equivalent to requiring that all objects in S are compact. Using the fact that \mathcal{E} is presentable, one may not be able to recover the categorical structure on \mathcal{E} from that of the full subcategory S . However, one can always enlarge S to \overline{S} while still remaining a set, so that the categorical structure of \mathcal{E} can be reconstructed from that of \overline{S} . The idea is that there is a cardinal κ , such that all objects in S are κ -compact. We include all the ordinal colimits of size less than or equal to κ of objects in S into \overline{S} . This subcategory “controls” the categorical structure of \mathcal{E} . Suppose that Y is an object in \mathcal{E} outside of \overline{S} . This means that Y can be written down as a κ -filtered colimit of objects in S , $\text{colim}_\beta Y_\beta$. Now the maps from any $X = \text{colim}_\alpha X_\alpha$, where X_α are in S , can be determined by the following formula $\text{Hom}(X, Y) \simeq \lim_\beta \text{colim}_\alpha \text{Hom}(X_\alpha, Y_\beta)$.

Giraud axioms are satisfied by the category of sets. In fact, they are satisfied by any topos as we mentioned previously, so long as \mathcal{C} is small. The category Man is presentable; in fact, it is essentially small. The reason is that any manifold can be embedded into \mathbb{R}^∞ . On the other hand, Sch and Aff suffer from the problem that they are not small. However, we can limit ourselves to considering the spectra of finitely presentable rings.

THEOREM 1. A category \mathcal{E} is left exact localization of a small presheaf category if and only if it satisfies Giraud’s axioms.

If \mathcal{E} is a left exact localization of a small presheaf category, then it is fairly straightforward to show that Giraud’s axioms hold. The converse can be proved by first noting that since \mathcal{E} is presentable, then \mathcal{E} is a localization of a presheaf category, namely the small subcategory S that generates \mathcal{E} under filtered colimits. This inclusion preserves limits, so there is a localization functor from $\mathcal{Psh}(S)$ to \mathcal{E} . The rest of Giraud’s axioms guarantee that this localization is left exact. Basically the point is that the small limits are generated by products and equalizer diagrams. We won’t need this theorem strictly speaking, but we will discuss its higher categorical version.

We called the formulation of the moduli problem above rigid. The problems of this sort have more chance of being representable, however, a vast array of objects that we would like to have some representatives do not form sheaves. Let me focus on an example of line bundles. Consider the functor $\widetilde{\text{Pic}} : \text{Sch}^{\text{op}} \rightarrow \text{Set}$ that assigns to the scheme X the set of equivalence classes of line bundles over X , with the images of morphisms given by

pullbacks. It would definitely be desirable for us to have a geometric object that represents $\widetilde{\text{Pic}}$. However, we have a problem of applying the analysis above, since $\widetilde{\text{Pic}}$ fails to be a sheaf! We cannot look for it in the topos $\text{Sh}(\text{Sch}_{\text{zar}})$. How exactly does it fail to be a sheaf? Consider the case of \mathbb{P}^1 . We can decompose it as a union of two Zariski open sets U_+ and U_- , each isomorphic to \mathbb{A}^1 . Over \mathbb{A}^1 the only invertible sheaf is $\mathcal{O}_{\mathbb{A}^1}$. However, as we know $\widetilde{\text{Pic}}(\mathbb{P}^1) \simeq \mathbb{Z}$ generated by twisting sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$. This demonstrates how the presheaf $\widetilde{\text{Pic}}$ is not a sheaf. But to get a better grip on what just happened let us analyze the proof of the above isomorphisms. Suppose \mathcal{L} is an invertible sheaf over \mathbb{P}^1 . Then we know that $\mathcal{L}|_{U_+}$ and $\mathcal{L}|_{U_-}$ are isomorphic to \mathcal{O}_{U_+} and \mathcal{O}_{U_-} via isomorphisms φ_+ and φ_- , respectively. Let U_{\pm} denote the intersection $U_+ \cap U_-$. Then we can consider the composite $\varphi_-|_{U_{\pm}} \circ \varphi_+^{-1}|_{U_{\pm}} : \mathcal{O}_{U_{\pm}} \longrightarrow \mathcal{L}|_{U_{\pm}} \longrightarrow \mathcal{O}_{U_{\pm}}$, which is invertible. The latter is equivalent to giving a unit in $\mathcal{O}(U_{\pm})^{\times} \simeq (\mathbb{Z}[x^{\pm 1}])^{\times} \simeq \mathbb{Z}/2 \times \mathbb{Z}$. However, this element depends on the choice of φ_+ and φ_- . We can vary φ_+ up to a unit in $\mathcal{O}(U_+)^{\times} \simeq (\mathbb{Z}[x])^{\times} \simeq \mathbb{Z}/2$, and a similar thing happens for φ_- . This ambiguity says that the unit obtained in $\mathbb{Z}/2 \times \mathbb{Z}$ is not independent of the choice. However, the second component is independent! This procedure gives us a map $\widetilde{\text{Pic}}(\mathbb{P}^1) \longrightarrow \mathbb{Z}$. The analysis also demonstrates that this map is an isomorphism. The same type of analysis holds for manifolds, and functors $\widetilde{\text{Pic}}(-, \mathbb{C})$ and $\widetilde{\text{Pic}}(-, \mathbb{R})$ that assign to the manifold the equivalence classes of complex and real line bundles over the manifold. In fact, what we were showing was that the equivalence of functors

Schemes	Manifolds
$\widetilde{\text{Pic}} \simeq \text{H}^1(-, \mathbb{G}_m)$	$\widetilde{\text{Pic}}(-, \mathbb{C}) \simeq \text{H}^2(-, \mathbb{Z})$
	$\widetilde{\text{Pic}}(-, \mathbb{R}) \simeq \text{H}^1(-, \mathbb{F}_2)$

As you could see in the above analysis, we relied on the fact there were local automorphisms of the invertible sheaves. The point is that when we quotient by the equivalence relations we lose all this local data and hence the ability to build invertible sheaves. Therefore, the consideration of the functor $\widetilde{\text{Pic}}$ is not the right thing to do. It does not possess enough internal data, so that we can build it from ground up.

The keyword that we need to implement here is stacks. These are a generalization of sheaves that possess enough structure, so that we can recover $\widetilde{\text{Pic}}$. Suppose we are given a Grothendieck site \mathcal{C} . Let \mathfrak{Spd} denote the category of groupoids. First we would like to talk about descent. Let $S : \mathcal{C}^{\text{op}} \longrightarrow \mathfrak{Spd}$ be cofunctor. Let us say that we are given a covering of $\mathfrak{U} = \{U_{\alpha} \longrightarrow X\}$ in \mathcal{C} . Then we can form the following partial simplicial diagram.

$$\prod_{\alpha} S(U_{\alpha}) \rightrightarrows \prod_{\alpha\beta} S(U_{\alpha\beta}) \rightrightarrows \prod_{\alpha\beta\gamma} S(U_{\alpha\beta\gamma})$$

This diagram is a functor from the category $\Delta_{\leq 2}$ – the full subcategory of objects $[0]$, $[1]$ and $[2]$. We will write this functor as $S_{\mathfrak{U}}$. There exists a functor $\nabla_{\leq 2}$ in $\text{Cat}^{\Delta_{\leq 2}^{\text{op}}}$, such that $\nabla_{\leq 2}([k])$ is the category of k composable morphisms. Then we define $D_{\mathfrak{U}}(S) = \text{Hom}(\nabla_{\leq 2}, S_{\mathfrak{U}})$ and we call this the descent groupoid of S

with respect to the covering \mathfrak{U} . Note that $S(X)$ has a natural functor to $D_{\mathfrak{U}}(S)$, which comes from restriction maps.

DEFINITION 2. A cofunctor $S : \mathcal{C}^{\text{op}} \rightarrow \mathcal{G}\text{pd}$ is called a stack if the functor $S(X) \rightarrow D_{\mathfrak{U}}(S)$ is an equivalence for all coverings \mathfrak{U} .

The cofunctors from \mathcal{C}^{op} to $\mathcal{G}\text{pd}$ are called prestacks, and the functor category $\mathcal{G}\text{pd}^{\mathcal{C}^{\text{op}}}$ is usually labeled as $\mathcal{P}\text{st}(\mathcal{C})$. Within this category we have the full subcategory formed by stacks, which we will write as $\text{St}(\mathcal{C})$. Now let me argue that we have not lost anything, since $\text{St}(\mathcal{C})$ contains $\text{Sh}(\mathcal{C})$. This is simply coming from the inclusion $\text{Set} \hookrightarrow \mathcal{G}\text{pd}$ that sends a set to the discrete groupoid corresponding to it, and the descent condition for stacks simply reduces to the sheaf condition on presheaves. In other words, we have a commutative diagram:

$$\begin{array}{ccc} \text{Sh}(\mathcal{C}) & \xrightarrow{\quad \dots \quad} & \text{St}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{P}\text{sh}(\mathcal{C}) & \hookrightarrow & \mathcal{P}\text{st}(\mathcal{C}) \end{array}$$

Technically, it is commutative up to a natural transformation, but there is no need to worry about that.

REMARK 1. As a remark, though it will become more important later, we could have replaced $\Delta_{\leq 2}$ with full Δ . Along with this we can form the Čech simplicial object by considering arbitrarily many intersections of open sets. The stacks do not change because of this modification. The fundamental limitation really lies in the fact the target of our functors are groupoids. Let me contrast this with the previous situation where we just considered sets.

Let us now go back to the case where $\mathcal{C} = \text{Sch}_{\text{Zar}}$. We consider the following functor $\text{Pic} : \text{Sch}_{\text{Zar}}^{\text{op}} \rightarrow \mathcal{G}\text{pd}$, where the objects of $\text{Pic}(X)$ are the invertible sheaves over X and the morphisms are the isomorphisms of invertible sheaves. Now there is a set-theoretic consideration: the invertible sheaves do not form a set! However, we can avoid it by requiring, for instance, that the invertible sheaf be a subbundle of sufficiently big direct sum of structure sheaves, where the cardinality of the direct sum may depend on the scheme. Note this not a rigidification, since the inclusion is not part of the structure – this is a purely a set-theoretic trick! Now Pic is a stack! In fact, we will establish a connection with \mathbb{G}_m – the multiplicative group. Given any sheaf of groups on Sch_{Zar} we can construct a stack out of it. This is a general construction valid over any Grothendieck site. Now note that $\widetilde{\text{Pic}}$ is a functor equivalent to $\pi_0(\text{Pic})$, where π_0 assigns to a groupoid its connected components. In particular this means that π_0 of a stack is not a sheaf. However, the statement is true for π_1 , though we do need to choose basepoints for this. This allows us to consider stacks for any group object in $\text{Sh}(\text{Sch}_{\text{Zar}})$, in particular, for group schemes.

EXAMPLE 1. Besides the Picard stack, Pic , there are also the moduli stacks of elliptic curves, \mathcal{M}_{ell} , with

their compactifications. This is the stack that assigns to a scheme X , the groupoid whose objects are proper smooth maps $E \rightarrow X$ with geometrically connected fibers of genus 1, along with a section $X \rightarrow E$, and whose morphisms are isomorphism $E \rightarrow F$ which commute with maps to X . It is very clear that by this definition that \mathcal{M}_{ell} is a stack. There is also the stack, $\mathcal{M}_{\text{ell}}^{\text{inv}}$, which assigns to S an relative elliptic curve $E \rightarrow X$ with a family of invariant 1-forms. Clearly, there is forgetful map $\mathcal{M}_{\text{ell}}^{\text{inv}} \rightarrow \mathcal{M}_{\text{ell}}$. There is also a map $\mathcal{M}_{\text{ell}}^{\text{inv}} \rightarrow \text{Pic}$, which assigns to X the invertible sheaf of invariant 1-forms on X . We will see that $\mathcal{M}_{\text{ell}}^{\text{inv}} \simeq \mathcal{M}_{\text{ell}} \times_{\text{Pic}} *$.

EXAMPLE 2. Similarly, there is the moduli stack of 1-dimensional formal groups, \mathcal{M}_{fg} . The groupoid $\mathcal{M}_{\text{fg}}(X)$ is a map of formal schemes $\widehat{G} \rightarrow X$ for which there exists a Zariski cover $\{\text{Spec}(R_\alpha)\}$ of X , such that the pullback to \widehat{G} is isomorphic to $\text{Spf}(R_\alpha[[x]])$. In addition we endow this map with a relative formal group scheme on this map, i.e. a unit map $X \rightarrow \widehat{G}$ and a multiplication $\widehat{G} \times_X \widehat{G} \rightarrow \widehat{G}$ satisfying the group axioms. From any relative group scheme one can obtain formal group via the completion at the ideal corresponding to the unit. Since elliptic curves are group schemes, then there is a morphism of stacks $\mathcal{M}_{\text{ell}} \rightarrow \mathcal{M}_{\text{fg}}$.

EXAMPLE 3. Now let us consider an example of a moduli stack from differential geometry. Say that G is a Lie group with Lie algebra \mathfrak{g} . Of course, we can form the moduli stack of principal G -bundles, BG as we will see below. However, we can also form the moduli stack of principal G -bundles with connection, BG_{∇} , where $BG_{\nabla}(X)$ is the groupoid of principal G -bundles with connection 1-form on the bundles. In a completely analogous fashion, we can consider the stack of principal G -bundles with flat connections, BG_{∇}^{\flat} . There is a natural morphism $BG_{\nabla}^{\flat} \rightarrow BG_{\nabla} \rightarrow BG$. As we will see later, BG_{∇} comes with a map to $\mathcal{A}^2(-, \mathfrak{g})$, 2-forms valued in \mathfrak{g} . The stack BG_{∇}^{\flat} is the fiber product $BG_{\nabla} \times_{\mathcal{A}^2(-, \mathfrak{g})} *$. Finally, we have the stack of flat principal G -bundles $B(G_{\text{disc}})$, where one is allowed to use transition maps that are constant on the overlaps.

Let us now describe how one may obtain stacks from group actions. Suppose that G is a discrete group with a right action on the set X . Then we can form a groupoid $X//G$, whose objects are the elements of X and whose morphisms are pairs $(x, g) \in X \times G$ going from x to gx . In a more general setting suppose that G is a group object in a Grothendieck site \mathcal{C} , which acts on X . This means that for any Y in \mathcal{C} , the group $\text{Hom}(Y, G)$ acts on the set $\text{Hom}(Y, X)$. We can form the presheaf of groupoids (prestack) $\widetilde{X//G}$, which when evaluated on Y gives the groupoid $\text{Hom}(Y, X)//\text{Hom}(Y, G)$. We then define the quotient stack to be the stackification of this stack, $X//G = (\widetilde{X//G})^+$. In some sense, all the interesting stacks are formed via this construction. One important thing to mention is the if a stack can be written as $X//G$ it may be written so in multiple ways. For instance, if G is acting freely on X , i.e. this property holds when we evaluate it on an object Y , then we can consider the sheaf quotient X/G . If we perform then stack quotient with respect to the trivial group $*$, then we get the stack corresponding to X/G . This is equivalent to the stack $X//G$. Thus, we see that in this scenario $(X/G)//* \simeq X//G$. Also, a piece of notation if X is equivalent to the $*$, then we

write BG for $*//G$.

REMARK 2. For example, the stack Pic is equivalent to BG_m . Similarly, the stacks BGL_n is equivalent to the stack assigning rank n vector bundles to the schemes. The moduli stack of elliptic curves, \mathcal{M}_{ell} has the Weierstrass presentation as quotient stack $\text{Spec}(\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}])//\text{Spec}(\mathbb{Z}[u^{\pm 1}, \lambda, \mu, \nu])$, where Δ is the discriminant of the projective curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. In a similar way, the moduli stack of formal groups, \mathcal{M}_{fg} , can be written as a quotient stack as follows $\text{Spec}(\mathbb{Z}[a_{i,j}]_{i,j=1}^{\infty})//\text{Spec}(\mathbb{Z}[b_0^{\pm 1}, b_i]_{i=1}^{\infty})$. Somehow, from these presentations it is clear that for instance \mathcal{M}_{ell} are geometrically small, meanwhile \mathcal{M}_{fg} is quite big. The point is that quotient presentations can provide useful information about the geometry of the stack.

REMARK 3. Now let us consider the differential setting. As in any setting the stack BG is equivalent to $*//G$. Let us consider the sheaf of \mathfrak{g} -valued 1-forms, $\mathcal{A}^1(-, \mathfrak{g})$. The group G acts on this sheaf as follows. Suppose that X is manifold, then the map $g : X \rightarrow G$ acts on $\omega \in \mathcal{A}^1(X, \mathfrak{g})$, by the following formula $g \cdot \omega = \text{ad}_g \omega + g^* \mu_G$, where μ_G is the Maurer-Cartan form on G . We can thus form the quotient stack $\mathcal{A}^1(-, \mathfrak{g})//G$ is equivalent to BG_{∇} . The flat forms $\mathcal{A}_b^1(-, \mathfrak{g})$ within $\mathcal{A}^1(-, \mathfrak{g})$ are G -equivariant, so we can form the quotient stack $\mathcal{A}_b^1(-, \mathfrak{g})//G$. This stack is equivalent to BG_{∇}^b . One thing to note is that $\mathcal{A}_b^1(-, \mathfrak{g})$ can be represented as a quotient sheaf $\mathfrak{g}/\mathfrak{g}_{\text{disc}}$, where $\mathfrak{g}_{\text{disc}}$ is the sheaf that assigns to a manifold the constant maps into \mathfrak{g} . As we will see later, expressing these stacks as quotients will give us a great deal of computational leverage.

To make this last statement more precise, let us discuss a little on what the geometry of a stack is. To do this we will need to discuss the notion of representability. Suppose that \mathcal{C} is a Grothendieck topology and $\text{St}(\mathcal{C})$ is the category of stacks over \mathcal{C} . The pullbacks from stacks are defined in slightly more refined way. Suppose that we have a diagram of stacks

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \mathcal{Y} & \xrightarrow{g} & \mathcal{Z} \end{array}$$

The stack pullback $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is defined so that when evaluated on object $X \in \mathcal{C}$, the objects of the resulting groupoid is a triple (x, y, φ) , where x is an object in $\mathcal{X}(X)$, y is an object in $\mathcal{Y}(X)$, and φ is a morphism from $f(x)$ to $g(y)$, and a morphism from (x, y, φ) to (x', y', φ') is a pair (χ, v) , where χ is morphism from x to x' , v is a morphism from y to y' , such that $\varphi' f(\chi) = g(v)\varphi$. If for all \mathcal{Y} represented by an object in \mathcal{C} (meaning, in particular, that \mathcal{Y} is a sheaf), the stack $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is also represented by an object in \mathcal{C} , then we say that the morphism f is representable. Suppose that a certain property \mathbf{P} of morphisms is stable under pullbacks. Such properties include openness, closedness, surjectiveness, flatness, being étale, smooth, etc. We say that a representable morphisms of stacks f has property \mathbf{P} if for all map $g : Y \rightarrow Z$, where Y is an object in \mathcal{C} ,

the pullback map $\mathcal{X} \times_Z Y \rightarrow Y$ has property **P**.

For the purposes of this text, we would call a stack \mathcal{X} in $\text{St}(\mathcal{C})$ geometric if the following conditions hold:

- 1) the diagonal map $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable;
- 2) there is an object in $U \in \mathcal{C}$ that admits a representable surjective map $p : U \rightarrow \mathcal{X}$.

This means that the stack is basically obtained as quotient of stack. In the algebro-geometric setting, there is additional requirement that \mathcal{X} be quasi-compact. In this setting, if the covering is p can be chosen to be étale, then we call the stack, a *Deligne-Mumford stack*. If p is smooth, then \mathcal{X} is an *Artin stack*.

REMARK 4. The moduli stack of elliptic curves with an invariant 1-form, $\mathcal{M}_{\text{ell}}^{\text{inv}}$ is a Deligne-Mumford stack. It is not very clear from the Weierstrass presentation, but nevertheless locally finite presentations can be achieved. More specifically, one can localize the equation at primes 2 and 3. When one localizes with respect to these primes, one may complete squares and cubes to reduce the number of variables. Eventually, one gets a finite group acting on schemes. For more details, see [KM].

REMARK 5. If G is a Lie group, then BG is a geometric stack. To show that the diagonal map is representable, we consider a map $X \rightarrow BG \times BG$, which classifies the principal bundles E and F . If we consider the pullback stack $P = X \times_{BG \times BG} BG$, when evaluated at Y , it produces the following category:

- the objects of $P(Y)$ are the quadruples (f, D, ϕ, ψ) , where f is map from Y to X , D is a principal G -bundle over Y , and $\phi : f^*E \rightarrow D$ and $\psi : f^*F \rightarrow D$ are isomorphism of principal G -bundles;
- a morphism from (f, D, ϕ, ψ) to (f', D', ϕ', ψ') is an isomorphism $\iota : D \rightarrow D'$, so that $\iota\phi = \phi'$ and $\iota\psi = \psi'$.

From this description one can easily see that $P(Y)$ is equivalent to a set. In fact, P is represented by the following fiber product $(E \times_X F) \times_{G \times G} G$, where the action of $G \times G$ on G is given by the formula $(g, h) \cdot x = gxh^{-1}$. Thus, the diagonal for BG is representable.

To show that BG is geometric we show that $p : * \rightarrow BG$ is submersive. If $X \rightarrow BG$ classifies the principal G -bundle E , then $X \times_{BG} *$ is equivalent to E , which maps submersively onto X .

REMARK 6. One can apply this framework to get a more conceptual understanding of Chern-Weil theory. What Chern-Weil theory tries to do is to compute the de Rham forms of the BG_{∇} , i.e. find a way of attaching de Rham characteristic classes for principal G -bundles with connections. In this language, this amounts to computing $\mathcal{A}^n(BG_{\nabla})$. Recall that $BG_{\nabla} \simeq \mathcal{A}^1(-, \mathfrak{g})//G$, and since \mathcal{A}^n is 0-truncated, we see that $\mathcal{A}^n(BG_{\nabla}) = \mathcal{A}^n(\mathcal{A}^1(-, \mathfrak{g}))^G$. See, [FH], for further information.

Basically, we see that for the transition from the setting of sheaves to the setting of stacks, we had to replace the category of sets with the category of groupoids, and also modify the notion of descent that we have. Now both sets and groupoid can be interpreted in the context of homotopy theory in the following way. Recall that a space X is called k -truncated if $\pi_n(X, x) \simeq *$ for all $x \in X$ and $n > k$. Let us consider

0-truncated spaces. It is fairly clear that up to weak homotopy, such a space is completely characterized by its connected components which form a set. Thus, the theory 0-truncated spaces is equivalent to that of sets. Similarly, a 1-truncated space is completely characterized by its fundamental groupoid. The fundamental groupoid functor establishes an equivalence between 1-truncated spaces and groupoids. This observation is the point where all the higher categorical framework kicks in. The idea is that topological spaces are the natural generalizations of sets and groupoids, and the generalizations of presheaves and sheaves should be taking values in topological spaces. Intuitively such a procedure will give us a higher analogue of a stack that besides keeping track of automorphisms keep track of higher automorphisms.

It is important to argue why we need to have objects that keep track of higher automorphism. One basic reason is that if we decide to consider parametrized families of stacks, then the notion of a 2-stack would be unavoidable. Let me talk about a specific example, where this came in handy. Let us say that X is a manifold. Then it is well known that hermitian line bundles are in one-to-one correspondence with degree 2 integral cohomology classes of X via the second Chern class. This is essentially because there is a short exact sequence $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1)$, \mathbb{R} is flabby, and $U(1)$ is the structure group of hermitian line bundles. Using the same set of ideas, we can see that the isomorphism classes of $BU(1)$ -bundles over X would correspond to degree 3 integral cohomology classes of X . These are precisely, the gerbes over X bound by $U(1)$. More specifically, there is a 2-truncated object in $\text{Sh}_\infty(\text{Man})$ called $B^2U(1)$, which assigns to each manifold X the 2-truncated space of gerbes bound by $U(1)$. The connected components of $B^2U(1)(X)$ are in one-to-one correspondence with degree 3 integral cohomology classes of X . Thus, the higher setting gives a natural place, where one can geometrically realize the cohomology classes of X . More on gerbes can be found in [BM1] and [BM2].

EXAMPLE 4. This line of thought can be applied to give geometric representation of sheaf cohomology. Suppose that \mathcal{F} is a sheaf of abelian groups on some topological space, X . Then we can assign a presheaf of n -truncated spaces $\widetilde{B}^n\mathcal{F}$ on X , which assigns to an open set U the topological space $B^n(\mathcal{F}(U))$. This presheaf is not going to be sheaf in general. Let us denote by $B^n\mathcal{F}$ the sheafification of $\widetilde{B}^n\mathcal{F}$. The connected components of $B^n\mathcal{F}(X)$ are in bijective correspondence with the sheaf cohomology classes of $H^n(X, \mathcal{F})$.

EXAMPLE 5. Algebraic K-theory furnishes another example, where higher category theory can be of conceptual and practical use. Recall that for any scheme X one can define the K-theory spectrum $K(X)$ space, which can be defined by performing the Quillen Q-construction on the category of finitely generated projective \mathcal{O}_X -modules. If X is a regular, separated, Noetherian scheme, then this presheaf of spectra is a sheaf of spectra. There are various modifications and enhancements of this statement that can be found in [TT].

1.2 ∞ -Categories

1.2.1 Basic Definitions and Concepts

In this subsection we give a definition of the notion of ∞ -categories. We will then interpret the definition as saying that ∞ -categories are fibrant objects in the Joyal model category. After that we introduce several definitions of important maps of simplicial sets, which will be useful later. It will be necessary to provide ourselves with basic extensions of ordinary categorical notions to ∞ -categories. In the end we introduce the homotopy category of simplicial sets, and state the analogue of Whitehead's theorem for ∞ -categories.

We begin with a quick motivation. The relation of ordinary categories with ∞ -categories comes from the so called nerve construction. Suppose that we are given an ordinary category \mathcal{C} . Then we can form a simplicial set called the nerve, $N(\mathcal{C})$, such that $N(\mathcal{C})_n = \text{Fun}([n], \mathcal{C})$, where $[n]$ is the ordered set $\{0 \leq 1 \leq 2 \leq \dots \leq n\}$. Not all simplicial sets arise this way. To describe which ones do we need several definitions.

Recall that there is a cosimplicial simplicial set $\Delta^\bullet : \Delta \rightarrow \text{Set}_\Delta$ that is the adjoint of the functor $\text{Hom}(-, -) : \Delta \times \Delta^{\text{op}} \rightarrow \text{Set}$. In other words, $\Delta^n = \Delta^\bullet([n]) = \text{Hom}_\Delta(-, [n])$. By Yoneda lemma, there is a bijection $K_n \simeq \text{Hom}(\Delta^n, K)$. Note that Δ^n has a distinguished n -simplex, namely the one corresponding to the identity $\mathbb{1}_{\Delta^n}$, which is the only n -simplex that is not degenerate. All the simplices of dimension higher than n are degenerate. Therefore, we can remove the simplex corresponding to $\mathbb{1}_{\Delta^n}$ along with its degeneracies, and obtain a new simplicial set. We will write this new simplicial set as $\partial\Delta^n$. Within $\partial\Delta^n$ there are n non-degenerate $(n-1)$ -simplices, given by the face maps $d_k \in \text{Hom}(\Delta^{n-1}, \Delta^n) \simeq \text{Hom}(\Delta^{n-1}, \partial\Delta^n)$. Just as above we can remove the simplex corresponding to d_k along with all of its degeneracies from $\partial\Delta^n$. We will write this new simplicial set as Λ_k^n . These are the so called n -horns.

The nerves of categories are characterized by the following property: the extension problem

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & N(\mathcal{C}) \\ \downarrow & \nearrow & \uparrow \\ \Delta^n & & \end{array}$$

is uniquely solvable for $0 < k < n$. The inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ are called inner horns if $0 < k < n$. The condition above is usually called *unique inner horn filling condition*. In fact, the nerve construction provides an equivalence between categories and the simplicial sets with this property. The inverse equivalence to the nerve construction can be described as follows. Suppose that we are given a simplicial set K that satisfies the unique horn filling condition. We construct an ordinary category $U(K)$ out of it. We declare the objects of $U(K)$ to be the elements of the set K_0 . If we are given two objects X and Y in $U(K)$, then we define $\text{Hom}(X, Y)$ to be the set $(d_0 \times d_1)^{-1}\{(X, Y)\}$, where d_0 and d_1 are the face maps from K_1 to K_0 . We need to define composition of $f \in \text{Hom}(X, Y)$ and $g \in \text{Hom}(Y, Z)$. Observe that f and g can be interpreted as maps from Δ^1 to K . The fact that the endpoint of f matches the origin of g allows us to construct the right

vertical map

$$\begin{array}{ccc}
 \Delta^2 & \longleftarrow & \Lambda_1^2 \\
 \uparrow d_1 & \dashrightarrow & \downarrow f \cup_Y g \\
 \Delta^1 & \longrightarrow & K
 \end{array}$$

The lift (and hence the diagram) exists and is unique by the unique horn filling condition. The element in K_1 , that the horizontal map on the bottom corresponds to, can be checked to be in $\text{Hom}(X, Z)$. The fact that this composition is unital and associative is a further exercise in the horn filling condition, which the readers can easily do on their own. It is also done in proposition 1.1.2.2. of [Lu1]. The latter proves that the nerve construction gives us an equivalence between categories and simplicial sets satisfying the unique horn filling condition.

As we could see, the uniqueness of the extension should be thought of as saying that giving a pair of appropriate morphisms we can compose them. The higher dimensional unique horn filling conditions provide with the associativity. With ∞ -categories we would like to have a composition, but we do not care for it to give us a single output. What we desire that the ambiguities form a space and that this space is contractible. The associativity of multiplication is again a matter of further contractibilities. In fact, all these conditions are encoded in the formalism of simplicial sets. All we need to do is to drop the uniqueness assumption.

DEFINITION 3. A simplicial set \mathcal{C} is called an ∞ -category if the following extension problem

$$\begin{array}{ccc}
 \Lambda_k^n & \longrightarrow & \mathcal{C} \\
 \downarrow & \dashrightarrow & \uparrow \\
 \Delta^n & &
 \end{array}$$

has a solution for $0 < k < n$.

Notice that the if we include the $k = 0, n$ then we obtain a Kan complex. As we will see later this additional conditions turn an ∞ -category into an ∞ -groupoid, which in any reasonable theory should correspond to spaces.

We know that Kan complexes are fibrant objects in Set_Δ under the Kan model structure, Theorem 3.6.5 of [Ho]. From the statement of the theorem we notice that the model structure is cofibrantly generated by trivial cofibrations $\partial\Delta^n \rightarrow \Delta^n$ and cofibrations $\Lambda_k^n \rightarrow \Delta^n$, where $0 \leq k \leq n$. In fact more is true—this model structure is combinatorial. In a way one can say that the Kan model structure “filters” the Kan complexes, which are the fibrant objects in this model structure. There is a model structure, called the Joyal model structure, on Set_Δ that “filters” ∞ -categories which we introduce below.

Before doing so, however, we need to introduce the notion of categorical equivalence. In the next section, we are going to define an adjunction $\mathfrak{C}[-] : \text{Set}_\Delta \rightleftarrows \text{Cat}_\Delta : \mathbb{N}$, where Cat_Δ is the category of simplicial categories.

One ought to think of $\mathcal{C}at_\Delta$ as another model for higher categories. This could be made precise by giving $\mathcal{C}at_\Delta$ with a model structure. However, for our discussion below we need to talk about the weak equivalences, which we dub as categorical equivalences. If \mathcal{C} is a simplicial category, then we can define its homotopy category $\widehat{\text{Ho}}(\mathcal{C})$. This category has the same objects as \mathcal{C} . The morphisms are obtained from the monoidal functors, $\text{Set}_\Delta \rightarrow \text{Top} \rightarrow \mathcal{H}$, where the first map is the geometric realization and \mathcal{H} is the homotopy category of topological spaces. Thus, \mathcal{C} is an \mathcal{H} -enriched category. A functor between simplicial categories, gives an \mathcal{H} -enriched functor between their homotopy categories. A functor is a *categorical equivalence* if the induced map on the homotopy categories is an equivalence. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between ∞ -categories is called a categorical equivalence if $\mathcal{C}[F]$ is a categorical equivalence. At the end of this subsection we will provide a direct construction of a homotopy category for ∞ -categories. These constructions do not provide the same \mathcal{H} -enriched category, but they are equivalent.

The following proposition is essentially the theorem 2.2.5.1 and 2.4.6.1 of [Lu1].

THEOREM 2. *There is a left proper combinatorial model structure on Set_Δ , called the Joyal model structure, such that*

- 1) *the cofibrations are the monomorphisms of simplicial sets,*
- 2) *the weak equivalences are the categorical equivalences,*
- 3) *the fibrant objects of this category are precisely ∞ -categories.*

A map is called an *inner fibration* if it has right lifting property with respect to the inner horn inclusions, i.e. the maps $\Lambda_k^n \rightarrow \Delta^n$ for $0 < k < n$. An *inner anodyne* map is one that has the left lifting property with respect to inner fibrations. By the small object argument, corollary 2.1.15 of [Ho], the class of inner anodyne maps is the weakly saturated class of morphisms of Set_Δ generated by the inner horn inclusions. Lemma 2.2.5.2 of [Lu1] shows that the inner horn inclusions are trivial cofibrations in this model structure. Thus, we can see that the fibrations in the Joyal model category are inner fibrations. The fibrations in the Joyal model category are usually called *categorical fibrations*. The converse statement that inner fibrations are categorical is false. For instance, one can contemplate about the inclusion of $\mathbf{N}([1])$ into $\mathbf{N}(E)$, where E denotes the ordinary category consisting of a single isomorphism using corollary 2.4.5.6 of [Lu1].

REMARK 7. Even though Set_Δ is a simplicial category, this simplicial structure is *not* compatible with the Joyal model structure. It is easy to see as the mapping simplicial set between ∞ -categories are not necessarily Kan complexes. This reflects a more refined structure of the higher category of ∞ -categories, namely, that it is implicitly an $(\infty, 2)$ -category. There is the formalism of marked simplicial sets, which provides an equivalent model category, which is simplicial and the fibrant objects correspond to ∞ -categories. See section 3.1 of [Lu2].

Interestingly, the notions of ordinary category theory lift to higher categories. We will begin with discussion

of a couple simple ordinary categorical notions generalized to ∞ -categories. Then we will give a several constructions of new ∞ -categories from old ones.

Suppose \mathcal{C} is an ∞ -category. We call the elements of \mathcal{C}_0 the *objects* of \mathcal{C} , and the elements of \mathcal{C}_1 the *morphisms* of \mathcal{C} . The source and target maps are given by the structure maps d_1 and d_0 , respectively. The degeneracy map $\mathcal{C}_0 \rightarrow \mathcal{C}_1$ corresponds to the identity morphisms. The identity morphism for X will be labeled as $\mathbb{1}_X$. However, since we do not have a well-defined composition—this statement is more like a nomenclature. In fact, any object X can be viewed as an element in \mathcal{C}_n by using the degeneracy map $\mathcal{C}_0 \rightarrow \mathcal{C}_n$. We will write $\mathbb{1}_X^n$ this degenerate simplex in \mathcal{C}_n .

Let us address the composition briefly. Let's say we are given two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. This means that we can form a map $\Lambda_1^2 \simeq \Delta^1 \cup_{\Delta^0} \Delta^1 \rightarrow \mathcal{C}$ by gluing f and g together. By the inner horn filling condition we can extend this map to a map $\Delta^2 \rightarrow \mathcal{C}$. We will call the morphism $f \circ d_1 : \Delta^1 \rightarrow \mathcal{C}$ a *composition* of $g \circ f$. Note that the choice comes from the non-uniqueness of the horn filling. Further, we can establish homotopies this way. We will say $f_1, f_2 : X \rightarrow Y$ are *homotopic* if f_2 is a composition of f_1 and $\mathbb{1}_Y$. One can show that this is an equivalence relation. We will write \simeq for this equivalence relation. If instead of requiring f_2 to be a composition of f_1 and $\mathbb{1}_Y$, we required it to be a composition of $\mathbb{1}_X$ and f_1 , we would obtain the same relation. This notion is very handy, since it establishes a connection between all compositions. If $g \circ_1 f$ and $g \circ_2 f$ are compositions of f and g , then they are homotopic. Conversely, anything homotopic to a composition of f and g is itself a composition. Furthermore, any choice of compositions respects the equivalence relation.

A morphism $f : X \rightarrow Y$ is called an equivalence if there exists a morphism $g : Y \rightarrow X$, so that $g \circ f \simeq \mathbb{1}_X$ and $f \circ g \simeq \mathbb{1}_Y$ for some (in fact, any) choices of composition. In some sense, equivalent objects can be treated on equal footing from the ∞ -categorical standpoint, however, what we have talked about so far does not reflect that yet, since we have not taken into account the higher dimensional structure. One ought to think of object, morphisms and homotopies between them as the most important part of an ∞ -category. The higher dimensional simplices give tools for homotopy coherence. Sometimes certain higher dimensional aspects can be detected just on the level of morphisms. Here is an example of something of that form, which follows from Prop. 1.2.5.1 in [Lu1].

PROPOSITION 2. *An ∞ -category is a Kan complex if and only if all of its morphisms are invertible.*

Therefore, we will write the category of Kan complexes by Spd_∞ . This proposition shows that the ∞ -categories that we have defined adhere to the well-established principle, that the theory of ∞ -groupoids coincides with spaces. It also fulfills our desire stated in the introduction of having a context, where ordinary categories and spaces are on equal footing. One can use this observation to make the following claim.

PROPOSITION 3. *The Kan model structure is the Bousfield localization of the Joyal structure with respect*

to a single map $\Lambda_0^2 \longrightarrow \Delta^2$ (or $\Lambda_2^2 \longrightarrow \Delta^2$).

In particular, the inclusion functor $\mathfrak{Gpd}_\infty \longrightarrow \mathfrak{Cat}_\infty$ has a left adjoint, which we will write as $-\hat{}$. This functor also has a right adjoint $-\sim : \mathfrak{Cat}_\infty \longrightarrow \mathfrak{Gpd}_\infty$. One can think of \mathcal{K}^\sim as the maximal groupoid contained in \mathcal{K} . More concretely, $(\mathcal{K}^\sim)_n$ consists of all the simplices of $\Delta^n \longrightarrow \mathcal{K}$, such that all the possible morphisms that we get from face maps $\Delta^1 \longrightarrow \Delta^n \longrightarrow \mathcal{K}$ are equivalences in \mathcal{K} . The resulting simplicial set is a Kan complex.

In ordinary category theory we can take the opposite of a category. Similarly here, given an ∞ -category \mathcal{C} , we can form the opposite ∞ -category \mathcal{C}^{op} . An efficient way of describing it would be by noticing that the category Δ has an involution r . It can be defined as sending $[n]$ to itself, however, it sends $d_k : [n] \longrightarrow [n+1]$ to d_{n+1-k} and $s_k : [n+1] \longrightarrow [n]$ to s_{n-k} . We define \mathcal{C}^{op} to be the composition of \mathcal{C} with r . This respects the ordinary categorical construction of the opposite categories in the sense that $\mathbf{N}(\mathcal{C}^{\text{op}}) \simeq \mathbf{N}(\mathcal{C})^{\text{op}}$.

Another simple construction that one can is the construction of the product category. Suppose that \mathcal{C} and \mathcal{D} are ∞ -categories, then $\mathcal{C} \times \mathcal{D}$ is simply their product as simplicial sets. This simplicial set is an ∞ -category. There is also the adjoint procedure. Suppose \mathcal{K} is a simplicial set and \mathcal{C} is an ∞ -category. Then the simplicial set $\text{Map}_{\text{Set}_\Delta}(\mathcal{K}, \mathcal{C})$ is an ∞ -category. This amounts to showing that the functor $-\times \mathcal{K}$ preserves inner anodyne maps—see, for instance, proposition 1.2.7.3. of [Lu1]. If \mathcal{K} is also an ∞ -category, then we will write $\text{Fun}(\mathcal{K}, \mathcal{C})$ for $\text{Map}_{\text{Set}_\Delta}(\mathcal{K}, \mathcal{C})$, and it will be called *the functor ∞ -category*. The objects of $\text{Fun}(\mathcal{K}, \mathcal{C})$ are the functors from \mathcal{K} to \mathcal{C} , and the morphisms are the natural transformations. In simpler terms a functor from \mathcal{K} to \mathcal{C} is a map of simplicial sets $F : \mathcal{K} \longrightarrow \mathcal{C}$. Taking our cue from the discussion above, we can define F to be a *homotopical equivalence* if there exists $G : \mathcal{C} \longrightarrow \mathcal{K}$, such that $\mathbb{1}_{\mathcal{K}} \simeq G \circ F$ in $\text{Fun}(\mathcal{K}, \mathcal{K})$ and $\mathbb{1}_{\mathcal{C}} \simeq F \circ G$ in $\text{Fun}(\mathcal{C}, \mathcal{C})$. We will show that the notions of homotopical equivalences and categorical equivalences coincide for maps of ∞ -categories.

The category of ∞ -categories \mathfrak{Cat}_∞ can be enriched in simplicial sets. In fact, this category is an $(\infty, 2)$ -category, since it is enriched in ∞ -categories. However, we can throw away the non-invertible 2-morphisms, which should result in getting an ∞ -category. We can accomplish this by declaring the mapping simplicial set $\text{Map}_{\mathfrak{Cat}_\infty}(\mathcal{C}, \mathcal{D})$ to be $\text{Fun}(\mathcal{C}, \mathcal{D})^\sim$. This turns \mathfrak{Cat}_∞ into a fibrant simplicial category. This allows us to have an ∞ -category of ∞ -categories. We will discuss more of this in next subsection.

The most important invariant of an ∞ -category is its homotopy category. One can define the ordinary homotopy category as follows. Suppose that \mathcal{C} is an ∞ -category. We define the objects of $\widetilde{\text{Ho}}(\mathcal{C})$ to be the same as the objects of \mathcal{C} . The morphisms from X to Y are equivalence classes of morphisms from X to Y in \mathcal{C} under the homotopy equivalence relation defined above. Once we quotient by this relation the composition becomes well-defined, unital and associative, and $\widetilde{\text{Ho}}(\mathcal{C})$ becomes an ordinary category. This category is an invariant of an ∞ -category in the sense that if given a categorical equivalence $F : \mathcal{C} \longrightarrow \mathcal{D}$ the induced map $\widetilde{\text{Ho}}(F) : \widetilde{\text{Ho}}(\mathcal{C}) \longrightarrow \widetilde{\text{Ho}}(\mathcal{D})$ is an equivalence of categories.

This invariant is not as strong as we would like it to be. It is not strong enough to detect categorical equivalences between ∞ -categories. The right thing to do is to consider the enriched version of the category above. For a general simplicial set \mathcal{K} , we already have seen the invariant $\widehat{\text{Ho}}(\mathcal{C}[\mathcal{K}])$, which is an \mathcal{H} -enriched category, where \mathcal{H} is the homotopy category of topological spaces. We present an explicit construction for ∞ -categories.

Suppose that \mathcal{C} is an ∞ -category. The objects of $\text{Ho}(\mathcal{C})$ are the objects of \mathcal{C} . Given any two objects X and Y of $\text{Ho}(\mathcal{C})$, we can form a *mapping* simplicial set $\text{Map}(X, Y)$. It is the fiber of (X, Y) under the map $\mathcal{C}^{\Delta^1} \rightarrow \mathcal{C}^{\partial\Delta^1} \simeq \mathcal{C}^{\mathbb{I}^2\Delta^0}$. The mapping simplicial set is, in fact, a Kan complex, which follows from Prop. 1.2.2.3 and Cor. 4.2.1.8. We would like to forget this simplicial set and remember only the homotopy type of $|\text{Map}(X, Y)|$. We define the latter to be $\text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y)$.

We now proceed to describe the composition. The idea follows the intuition that the composition is really an extension along Δ^2 of the inner horn Λ_1^2 . Let us denote by $\text{Map}(X, Y, Z)$ the fiber of the point (X, Y, Z) with respect to the map $\mathcal{C}^{\Delta^2} \rightarrow \mathcal{C}^{\mathbb{I}^3\Delta^0}$, where the map is induced by the inclusion of the three vertices of Δ^2 . Then the zigzag of inclusions $\Lambda_1^2 \hookrightarrow \Delta^2 \leftarrow \Delta_{\{0,2\}}^1$ gives a zigzag of Kan complexes

$$\begin{array}{ccc} & \text{Map}(X, Y, Z) & \\ \simeq \swarrow & & \searrow \\ \text{Map}(X, Y) \times \text{Map}(Y, Z) & & \text{Map}(X, Z) \end{array}$$

The left map is trivial Kan fibration, since it is an inner fibration of Kan complexes. Therefore, once apply simplicial realization and pass to the homotopy category we obtain a well defined map

$$|\text{Map}(X, Y)| \times |\text{Map}(Y, Z)| \rightarrow |\text{Map}(X, Z)|.$$

The unit of the composition is given by $\mathbb{1}_X \in \text{Map}(X, X)$.

To show that this composition is associative we need to construct yet another space. For X, Y, Z and W in \mathcal{C} , we consider the fiber of (X, Y, Z, W) projection $\mathcal{C}^{\Delta^3} \rightarrow \mathcal{C}^{\mathbb{I}^4\Delta^0}$, again induced by the inclusion of the vertices. Let us call this fiber $\text{Map}(X, Y, Z, W)$. We can then write the following diagram

$$\begin{array}{ccccc} \text{Map}(X, Z) \times \text{Map}(Z, W) & \xleftarrow{\simeq} & \text{Map}(X, Z, W) & \xrightarrow{\quad} & \text{Map}(X, W) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Map}(X, Y, Z) \times \text{Map}(Z, W) & \xleftarrow{\simeq} & \text{Map}(X, Y, Z, W) & \xrightarrow{\quad} & \text{Map}(X, Y, W) \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ \text{Map}(X, Y) \times \text{Map}(Y, Z) \times \text{Map}(Z, W) & \xleftarrow{\simeq} & \text{Map}(X, Y) \times \text{Map}(Y, Z, W) & \xrightarrow{\quad} & \text{Map}(X, Y) \times \text{Map}(Y, W) \end{array}$$

which shows that the composition in $\text{Ho}(\mathcal{C})$ is indeed associative.

We would like to promote this construction to a functor between 2-categories $\text{Ho} : \text{Cat}_\infty \rightarrow \text{Cat}_{\mathcal{H}}$, where $\text{Cat}_{\mathcal{H}}$ denotes the 2-category of \mathcal{H} -enriched categories, functors and natural transformations. The 2-morphisms of Cat_∞ are given by the equivalence classes of natural transformations. We have already defined what Ho does on objects. Now suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ is functor. We define $\text{Ho}(F) : \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{D})$, so that it sends the object X to FX , and on morphisms the map $|\text{Map}(X, Y)| \rightarrow |\text{Map}(FX, FY)|$ is the appropriate restriction of the map $F^{\Delta^1} : \mathcal{C}^{\Delta^1} \rightarrow \mathcal{D}^{\Delta^1}$. Finally if we are given a natural transformation ν from F to G , then it gives a point ν_X of $\text{Map}(FX, GX)$, which after passing to homotopy category gives a natural transformation from $\text{Ho}(F)$ to $\text{Ho}(G)$. This natural transformation is $\text{Ho}(\nu)$. This is not yet well-defined – we need to verify that if $\nu \simeq \mu$, then $\text{Ho}(\nu) = \text{Ho}(\mu)$. This follows from the fact that the points ν_X and μ_X are homotopic in $\text{Map}(FX, GX)$, which is a direct consequence of definitions. As a consequence of this discussion we obtain the following proposition.

PROPOSITION 4. *If the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence, then $\text{Ho}(F) : \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{D})$ is an equivalence of \mathcal{H} -enriched categories.*

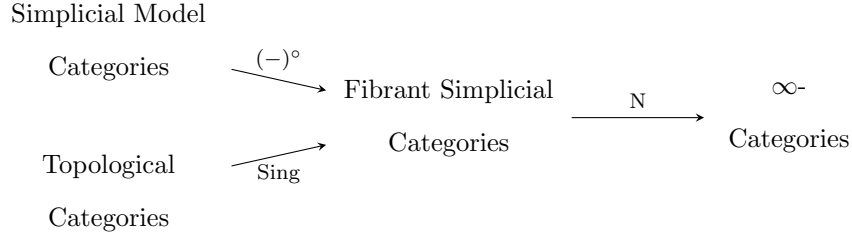
This should remind one of classical homotopy theory. The homotopy category can be thought of as the analogue of the homotopy groups of spaces. One could convert the homotopy category into a category enriched in graded abelian group by applying the functor π_* , but we will not do that. Whitehead’s theorem in classical homotopy theory the homotopy groups *detect* homotopy equivalences. The fact that an analogous statement works for ∞ -categories is what makes this comparison more substantial. In fact, the proof is very much reminiscent of the classical Whitehead’s theorem.

PROPOSITION 5. *A functor F is an equivalence of ∞ -categories if and only if $\text{Ho}(F)$ is an equivalence.*

1.2.2 Constructions of ∞ -Categories

Strangely enough one encounters ∞ -categories in an indirect fashion. Nevertheless, they are convenient to work with in the sense that all the ordinary categorical notions can be transported verbatim to the ∞ -categorical setting. Therefore, there are several tools of converting other notions of higher categories to ∞ -categories. The convenience of ∞ -categories is that we can think of them as ordinary categories with extra structure. One can imagine that the theory of ∞ -categories is a packaging agency, that takes your product and beautifully packages for the use of others, but it rarely occurs that the product is of its own. This fact was pointed out to me by Chris Kapulkin.

Here is a diagram to guide us through conversion procedure



We begin by constructing the functor N , which we will call *simplicial nerve*. In fact, the simplicial nerve functor will be a part of a Quillen adjunction, $\mathfrak{C}[-] : \text{Set}_\Delta \rightleftarrows \text{Cat}_\Delta : N$. The model structure on Set_Δ is the Joyal model structure, the one on Cat_Δ needs more elaboration.

First, let us show that Cat_Δ is complete and cocomplete. To show this we use the adjunction $\widetilde{\text{Ho}} : \text{Set}_\Delta \rightleftarrows \text{Cat} : N$, which exhibits Cat as an ordinary categorical Bousfield localization with respect to the inner horn inclusions. Furthermore, $\widetilde{\text{Ho}} \circ N$ is equivalent to the identity functor. Since Set_Δ is complete, then Cat is also complete, since localization preserves limits. The category Cat is cocomplete, since if we are given a (small) diagram \mathcal{C}_α indexed over some small category, then $\widetilde{\text{Ho}}(\text{colim } N(\mathcal{C}_\alpha))$ is the colimit of the diagram. Then it follows, that diagram category $\text{Cat}^{\Delta^{\text{op}}}$ is also complete and cocomplete. The functor $\widetilde{\text{Ho}}$ send 0-simplices to the objects of the resulting category, and N performs the inverse operation. Therefore, $\text{Ob}(\text{colim } \mathcal{C}_\alpha) \simeq \text{colim } \text{Ob}(\mathcal{C}_\alpha)$ and $\text{Ob}(\text{lim } \mathcal{C}_\alpha) \simeq \text{lim } \text{Ob}(\mathcal{C}_\alpha)$. This implies that the subcategory Cat_Δ in $\text{Cat}^{\Delta^{\text{op}}}$ is closed under limits and colimits, so it is complete and cocomplete.

Note that for any simplicial set K we can construct a category $\langle K \rangle$, which has two objects 0 and 1, the endomorphisms are trivial, $\text{Hom}(0, 1) = K$ and $\text{Hom}(1, 0) = \emptyset$. Let \emptyset denote the initial simplicial category and $*$ the terminal simplicial category. The following theorem is a consequence of A.3.2.4 of [Lu1].

THEOREM 3. *There is a left proper combinatorial model structure on Cat_Δ , called the Bergner model structure, such that*

- 1) *the cofibrations are the weakly saturated class of morphisms generated by $\emptyset \longrightarrow *$ and $\langle \partial \Delta^n \rangle \longrightarrow \langle \Delta^n \rangle$,*
- 2) *the weak equivalences are the categorical equivalences of simplicial categories,*
- 3) *the fibrant objects are the non-empty simplicial categories, where all the mapping simplicial sets are Kan complexes.*

Now we construct a cosimplicial simplicial category. $\mathfrak{C}^\bullet : \Delta \longrightarrow \text{Cat}_\Delta$. The following is a definition of a strict 2-category $\overline{\mathfrak{C}^n} \text{---} \mathfrak{C}^n$ is obtained by applying the functor $N : \text{Cat} \longrightarrow \text{Set}_\Delta$.

- 1) the objects are the elements of the set $\{0, \dots, n\}$,
- 2) the morphisms are the finite subsets of $\{0, \dots, n\}$ —the starting point of a subset is its minimum and the endpoint is its maximum,
- 3) the 2-morphisms are the inclusions of finite subsets—the vertical composition is the inclusion and the horizontal composition is the union.

Notice that this definition is functorial in order preserving maps, whence we obtain a cosimplicial simplicial category \mathfrak{C}^\bullet . Recall, that we also have a simplicial simplicial set category $\Delta^\bullet : \Delta \longrightarrow \text{Set}_\Delta$. We define $\mathfrak{C}[-]$ to be the left Kan extension of \mathfrak{C}^\bullet across Δ^\bullet . This functor has a right adjoint $N : \text{Cat}_\Delta \longrightarrow \text{Set}_\Delta$, which is given by the following explicit formula: $N(\mathcal{C})_k \simeq \text{Hom}_{\text{Cat}_\Delta}(\mathfrak{C}^k, \mathcal{C})$. Simplicial nerve construction can be viewed as a generalization of the ordinary nerve construction in the following way. We can regard ordinary categories as simplicial categories if we consider the mapping simplicial sets to be constant. This is actually fully faithful embedding of ordinary categories into simplicial categories. The constancy of mapping simplicial sets forces the function $\text{Hom}(\mathfrak{C}^n, \mathcal{C}) \longrightarrow \text{Hom}([n], \mathcal{C})$ to be an isomorphism for an ordinary category \mathcal{C} . This shows that the nerve constructions agree.

Note that to show that this adjunction is a Quillen adjunction we can check that it preserves cofibrations. The reason for this is the definition of equivalences in the Joyal model structure. To show that it preserves the cofibrations it suffices to check for $\mathfrak{C}[\partial\Delta^n] \longrightarrow \mathfrak{C}[\Delta^n] \simeq \mathfrak{C}^n$. A simple check can verify that $\mathfrak{C}[\partial\Delta^n]$ is a simplicial subcategory of \mathfrak{C}^n , where all the mapping simplicial sets coincide except $\text{Map}_{\mathfrak{C}[\partial\Delta^n]}(0, n)$ is the boundary of the square $\text{Map}_{\mathfrak{C}^n}(0, n)$. This is a cofibration, because we have the following pushout square

$$\begin{array}{ccc} \langle \partial(\Delta^1)^n \rangle & \longrightarrow & \mathfrak{C}[\partial\Delta^n] \\ \downarrow & & \downarrow \\ \langle (\Delta^1)^n \rangle & \longrightarrow & \mathfrak{C}^n \end{array}$$

This demonstrates that we have a Quillen adjunction, but also more is true (Thm. 2.2.5.1 [Lu1]).

THEOREM 4. *The Quillen adjunction $(\mathfrak{C}[-], N)$ is a Quillen equivalence.*

This theorem is very important for the following reasons. First of all it allows one to construct ∞ -categories out of simplicial categories. If \mathcal{C} is a fibrant simplicial category, then $N(\mathcal{C})$ is an ∞ -category. Secondly, the theorem demonstrates that the theory of ∞ -categories is equivalent to the theory of simplicial categories. One ought to think of simplicial categories as some sort of a rigid version of ∞ -categories. Whenever, one may need a more a rigid construction, one can go to Cat_Δ and then come back to Set_Δ using the adjunction above.

EXAMPLE 6. Recall that the category of simplicial sets itself is simplicially enriched. Unfortunately, it is not fibrant. Indeed, if K and S are not fibrant, then the maps $\text{Hom}_{\text{Set}_\Delta}(K, S)$ is not going to be fibrant in general. However, the trick is to restrict to the full subcategory of fibrant simplicial sets under the Kan model structure, which we will write as $(\text{Set}_\Delta^{\text{Kan}})^\circ$. Now $\text{Hom}_{(\text{Set}_\Delta^{\text{Kan}})^\circ}(K, S)$ is a fibrant simplicial set, so $(\text{Set}_\Delta^{\text{Kan}})^\circ$ is fibrant simplicial category. Therefore, the nerve of it will give us an ∞ -category. We will write \mathcal{S} for the ∞ -category $N((\text{Set}_\Delta^{\text{Kan}})^\circ)$, and we will call it the ∞ -category of spaces. The terminology will be explained below.

EXAMPLE 7. Suppose that \mathcal{C} is an ∞ -category. Then we can form the presheaf ∞ -category as the functor category $\mathcal{Psh}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$. If \mathcal{C} came from an ordinary category, then the resulting category is the category of simplicial presheaves on \mathcal{C} taking values in Kan complexes. This particular construction of an ∞ -category is very important to current work for two reasons:

- we will need them to define (∞, n) -categories,
- they will be used to define ∞ -topoi.

EXAMPLE 8. We can generalize the construction above to arbitrary simplicial categories. Suppose that \mathcal{C} is a simplicial category. Then we can construct another simplicial category \mathcal{C}^\sim , by declaring its objects to be the same as those of \mathcal{C} , and $\text{Map}_{\mathcal{C}^\sim}(X, Y)$ to be $\text{Map}_{\mathcal{C}}(X, Y)^\sim$. The compositions and the units are defined in natural fashion. After applying the simplicial nerve construction we obtain an ∞ -category, $N(\mathcal{C}^\sim)$. We can apply this construction, to the case of fibrant objects in $\text{Set}_\Delta^{\text{Joyal}}$. We denote $N(((\text{Set}_\Delta^{\text{Joyal}})^\circ)^\sim)$ by Cat_∞ , the (big) ∞ -category of ∞ -categories. Note that \mathcal{S} can be realized as an ∞ -subcategory of Cat_∞ .

As we could see from the examples above the constructions follow the pattern of considering a model category and passing to the subcategory of cofibrant-fibrant objects. If we are given a simplicial model category \mathbf{M} , we can consider the subcategory of cofibrant-fibrant objects, \mathbf{M}° . The following proposition holds.

PROPOSITION 6. *The simplicial category \mathbf{M}° is fibrant.*¹

Proof. Let us write the maps for the simplicial structure by $\otimes : \text{Set}_\Delta \times \mathbf{M} \longrightarrow \mathbf{M}$ and $\text{Map}(-, -) : \mathbf{M}^{\text{op}} \times \mathbf{M} \longrightarrow \text{Set}_\Delta$. The map $\Lambda_k^n \longrightarrow \Delta^n$ is a trivial cofibration and if X is cofibrant, then $\emptyset \longrightarrow X$ is also a cofibration. Then, since the maps above are a part of a Quillen bifunctor, then we can conclude that the map $\Lambda_k^n \otimes X \longrightarrow \Delta^n \otimes X$ is a trivial cofibration. Therefore, it has the left lifting property with respect to any fibrant object Y . Using the fact that the maps above are adjoint, we conclude that $\text{Map}(X, Y)$ has the right lifting property with respect to the inclusions $\Lambda_k^n \longrightarrow \Delta^n$, so it is a Kan complex. \square

There is a reason for considering simplicial model categories as a starting point. We will see in the next section that the construction of limits and colimits in the setting of ∞ -categories is hard, they are given only by their universal property. What we will see is that the construction of homotopy limit and colimits in model categories is the thing that allows one to talk of existence of limits and colimits in ∞ -categories.

EXAMPLE 9. Another quite interesting example that ties to the introduction of this chapter is the derived category of an abelian category. Suppose that \mathcal{A} is a presentable abelian category. We consider the category $\text{Ch}^+(\mathcal{A})$ of chain complexes in \mathcal{A} concentrated in non-negative degrees. Since the category is presentable we can endow $\text{Ch}^+(\mathcal{A})$ with the injective model structure, where the cofibrations are the injections and the weak

¹Incidentally, this shows that the Joyal model structure cannot be endowed with a simplicial model structure.

equivalences are given by quasi-isomorphisms. This model category structure is cofibrantly generated. One can show that this category is tensored and cotensored over Set_Δ . Furthermore, this gives a simplicial model category structure on $\text{Ch}^+(\mathcal{A})$. Thus, we can apply the construction above and get $\mathcal{D}(\mathcal{A}) = \text{N}(\text{Ch}^+(\mathcal{A}))$, the derived ∞ -category of \mathcal{A} . This is an example of a stable ∞ -category. See, [Lu2], for further details.

Finally, let us discuss how one obtains an ∞ -category out of topological categories. Suppose that \mathcal{T} is a topological category. Then we can convert it into a simplicial category by applying the functor $\text{Sing} : \text{Top} \rightarrow \text{Set}_\Delta$. The resulting category $\widehat{\mathcal{T}}$ is automatically fibrant, therefore, the simplicial set $\text{N}(\widehat{\mathcal{T}})$ is an ∞ -category.

EXAMPLE 10. The category of topological spaces itself is topologically enriched. In this case again we will be interested in the cofibrant-fibrant subcategory $(\text{Top})^\circ$. This category is topologically enriched and does not contain any pathologies that we do not like. We construct the ∞ -category $\text{N}(\widehat{(\text{Top})^\circ})$. This ∞ -category is equivalent to \mathcal{S} . To see this note that the adjunction $\| : ((\text{Set}_\Delta))^\circ \rightleftarrows \widehat{(\text{Top})^\circ} : \text{Sing}$ is an equivalence of simplicial categories. Therefore, the $\text{N}(\text{Sing})$ gives an equivalence from $\text{N}(\widehat{(\text{Top})^\circ})$ to \mathcal{S} .

EXAMPLE 11. The following example is not going to be used in the future—though it is a good example. We can topologize $\text{Hom}_{\text{Man}}(X, Y)$ as a subspace of $\text{Hom}_{\text{Top}}(X, Y)$ by remembering that we have a forgetful functor $\text{Man} \rightarrow \text{Top}$. This turns Man into a topological category. We will denote the nerve of this category by \mathcal{M} . This is what we will call the ∞ -category of manifolds. One can think of these ∞ -categories as of manifolds but only considered up to homotopy type. One would naturally expect that the underlying geometry would be ruined by the homotopy invariance.

1.2.3 Limits, Colimits and Localization

Limits and colimits are extremely important notions in ordinary category theory. It is not surprising that they are of central importance in higher category theory as well. We begin this section by defining the notion of overcategories and undercategories. Then we will define what it means to be a limit and colimit in ∞ -categories. We will finish the section by showing how the constructions in the previous section convert homotopy limits and colimits to limits and colimits in ∞ -categories.

First we need a construction. There is pullback functor $i^* : \text{Set}_\Delta^{\Delta^1} \rightarrow \text{Set}_\Delta^{\Delta^0} \amalg \Delta^0 \simeq (\text{Set}_\Delta)^2$. This map has a right adjoint, which is called the join operation $-\star- : (\text{Set}_\Delta)^2 \rightarrow \text{Set}_\Delta^{\Delta^1}$. Even though, the joining lands us in the overcategory of Δ^1 , $-\star-$ will denote the object mapping to Δ^1 , or equivalently, whatever we get by applying the forgetful functor $\text{Set}_\Delta^{\Delta^1} \rightarrow \text{Set}_\Delta$. There is a nice formula that we will need $\Delta^n \star \Delta^m = \Delta^{n+m+1}$. In particular, we can form the so called cones constructions using the join. The left cone of simplicial set K is $\Delta^0 \star K$, which we will write as K^\triangleleft , and the right cone is $K \star \Delta^0$, written as K^\triangleright . Both K^\triangleleft and K^\triangleright have distinguished point, which we will call a cone point.

The join can also be described via an explicit formula. We leave it to the reader that this construction satisfies the properties above. We write D_n for the set of Dedekind cuts of the ordinal $[n]$. If given $\lambda \in D_n$, then we denote by λ_- and λ_+ , the smaller and the bigger subsets respectively, both of which we regard as ordinals. The set $(K \star S)_n$ is given by the following union

$$\coprod_{\lambda \in D_n} K_{\lambda_-} \times S_{\lambda_+}.$$

The structure maps are obtained from the realization that the Dedekind cuts pullback, via the inverse image construction. Thus, given an order preserving map $f : [m] \rightarrow [n]$, we get a function $f^* : D_n \rightarrow D_m$, and we have two restrictions $f_{\pm} : f^*(\lambda)_{\pm} \rightarrow \lambda_{\pm}$. Then, we get a natural map $K_{\lambda_-} \times S_{\lambda_+} \rightarrow K_{f^*(\lambda)_-} \times S_{f^*(\lambda)_+}$. For this construction to work uniformly regard K_{\emptyset} and S_{\emptyset} to be singletons.

Now let us discuss the notion of slice ∞ -categories. These become important if we want to pass to relative settings. Suppose that we are given a simplicial set K . We can form the undercategory of K in simplicial sets $\text{Set}_{\Delta}^{K/}$, which is again a simplicial category. There exists a functor $- \star K : \text{Set}_{\Delta} \rightarrow \text{Set}_{\Delta}^{K/}$, where \star denotes the join operation.

LEMMA 1. *The functor $- \star K$ preserves colimits.*

Proof. Suppose that we are given a diagram S^{α} of simplicial sets labeled by any small category. Then we know that $(\text{colim } S^{\alpha})_n \simeq \text{colim } S_n^{\alpha}$. Let D_n° denote the subset of D_n consisting of those cuts, such that the smaller set is non-empty. Then we can write

$$\begin{aligned} (\text{colim } S^{\alpha} \star K)_n &\simeq \coprod_{\lambda \in D_n} (\text{colim } S^{\alpha})_{\lambda_-} \times K_{\lambda_+} \simeq \left(\coprod_{\lambda \in D_n^{\circ}} (\text{colim } S^{\alpha})_{\lambda_-} \times K_{\lambda_+} \right) \coprod K_n \\ &\simeq (\text{colim } \coprod_{\lambda \in D_n^{\circ}} S_{\lambda_-}^{\alpha} \times K_{\lambda_+}) \coprod K_n \simeq \text{colim}_{K_n/} \coprod_{\lambda \in D_n} S_{\lambda_-}^{\alpha} \times K_{\lambda_+}. \end{aligned}$$

Last expression is precisely the set of n -simplices of $\text{colim}_{K_n/}(S^{\alpha} \star K)$. \square

Using the adjoint functor theorem, we can construct a right adjoint – the overcategory construction. More concretely, suppose that $F : K \rightarrow \mathcal{C}$ is in the undercategory of K . Then $(\mathcal{C}/_F)_n$ is the set $\text{Hom}_{\text{Set}_{\Delta}^{K/}}(\Delta^n \star K, \mathcal{C})$. The undercategory $(-/_F)$ functor is the right adjoint to $K \star -$ functor. One has to demonstrate that if \mathcal{C} is an ∞ -category, then so are $\mathcal{C}/_F$ and \mathcal{C}_F according to Cor. 2.1.2.2 in [Lu1].

Let \mathcal{C} be an ∞ -category and let $F : K \rightarrow \mathcal{C}$ be a diagram.

DEFINITION 4. *An object $\emptyset : \Delta^0 \rightarrow \mathcal{C}$ is initial if the undercategory $\mathcal{C}_{\emptyset/}$ corresponding to it is a contractible simplicial set. Dually, if $*$: $\Delta^0 \rightarrow \mathcal{C}$ is final if $\mathcal{C}_{/*}$ is a contractible Kan complex. A colimit of F is an initial object of $\mathcal{C}/_F$ and a limit of F is a final object of \mathcal{C}_F .*

Recall from the previous subsection that the homotopy theory of ∞ -categories and fibrant simplicial categories are equivalent. Therefore, we need to discuss the analogous notion of colimit and limits in the setting of simplicial categories. Turns out that the appropriate notion that we need to consider is that of a homotopy colimit and homotopy limit. This is important, since construction of homotopy limits and colimits is easy. Once, we have an explicit procedure of converting homotopy limits and colimits into limits and colimits of ∞ -categories, we can make claims of completeness and cocompleteness of some ∞ -categories.

We begin by recalling the notion of a homotopy limit and colimit for model categories. First of all suppose \mathbf{M} is combinatorial model category. Let \mathcal{C} be a small simplicial category. Then we can form the diagram category $\mathbf{M}^{\mathcal{C}}$. This category has two model structures according to the following proposition A.2.8.2 from [Lu1].

PROPOSITION 7. *The category $\mathbf{M}^{\mathcal{C}}$ has two combinatorial model structures:*

- 1) *The projective model structure, where the weak equivalence and fibrations are given levelwise.*
- 2) *The injective model structure, where the weak equivalence and cofibrations are given levelwise.*

These model structures are equivalent to one another via the identity adjunction, since the class of weak equivalences is the same in both cases. They usually do not coincide, except possibly in the case where \mathcal{C} is a point. Sometimes if \mathcal{C} has the extra structure of a Reedy category, we can also consider the Reedy model structure on $\mathbf{M}^{\mathcal{C}}$, but we will not consider it here.

There exists an adjoint pair of functors $\text{colim} : \mathbf{M}^{\mathcal{C}} \rightleftarrows \mathbf{M} : \text{const}$, where const is the constant diagram functor. This adjunction is a Quillen adjunction if we endow $\mathbf{M}^{\mathcal{C}}$ with the projective model structure. Then one defines the homotopy colimit functor hocolim to be the left derived functor $\mathbb{L}\text{colim} : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}$. We need to make a choice of taking a cofibrant replacement functor $\mathcal{Q} : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}^{\mathcal{C}}$. The left derived functor $\mathbb{L}\text{colim}$ is simply the composition of $\text{colim} \circ \mathcal{Q}$. Notice that by the virtue of being a derived functor hocolim is homotopy invariant in the sense that if we have two diagrams F and G that are levelwise weakly equivalent, then $\text{hocolim } F \simeq \text{hocolim } G$. This fact is of central importance, and the reason why we are interested in studying this construction. In particular, this shows that all further discussion will be independent of the choice of the cofibrant replacement functor. In general, we will be interested in homotopy colimits only up to weak equivalence.

Notice that from the definition that we have given there exists a natural map $\text{hocolim } F \rightarrow \text{colim } F$, which comes from the natural transformation from \mathcal{Q} to $\mathbb{1}_{\mathbf{M}^{\mathcal{C}}}$. Suppose that we are given a cone functor $\overline{F} : \mathcal{C}^{\triangleright} \rightarrow \mathcal{M}$. We will say that \overline{F} is a *homotopy colimit diagram* if there is a weak equivalence $\overline{F}(\ast) \rightarrow \text{hocolim } F$, and such that after the composing with $\text{hocolim } F \rightarrow \text{colim } F$, the resulting composite on \mathcal{C} is F .

With this construction, in certain situations we can form functor ∞ -categories in two ways. If S is a fibrant simplicial model category, then we can simply take the functor ∞ -category $\text{Fun}(\mathbf{N}(S), \mathbf{N}(\mathbf{M}^{\circ}))$. The

other way of forming an ∞ -category is by consider $N((\mathbf{M}^S)^\circ)$. Note that if $F : S \rightarrow \mathbf{M}$ is cofibrant-fibrant then $F(C)$ is cofibrant-fibrant for all C in S . Thus, $(\mathbf{M}^S)^\circ \rightarrow \mathbf{M}^S$ factors through $(\mathbf{M}^\circ)^S$. This gives an evaluation map $(\mathbf{M}^S)^\circ \times S \rightarrow \mathbf{M}^\circ$. The following proposition is a trivial corollary of Prop. 4.2.4.4 of [Lu1].

PROPOSITION 8. *The adjoint of the nerve of the evaluation map*

$$N((\mathbf{M}^S)^\circ) \rightarrow \text{Fun}(N(S), N(\mathbf{M}^\circ))$$

is an equivalence of ∞ -categories.

If \mathcal{C} is an ∞ -category and $\mathbf{M} = \text{Set}_\Delta^{\text{Kan}}$, then the functor above gives the following equivalence

$$N\left(\left(\text{Set}_\Delta^{\mathfrak{C}(\mathcal{C})}\right)^\circ\right) \rightarrow \text{Fun}(N(\mathfrak{C}(\mathcal{C})), \mathcal{S}) \simeq \mathcal{P}\text{sh}(\mathcal{C}).$$

Using this equivalence we can construct the Yoneda embedding functor, $j : \mathcal{C} \rightarrow \mathcal{P}\text{sh}(\mathcal{C})$. On the level of simplicial categories the Yoneda embedding is easily defined by using the representing functors. Thus, in particular, we obtain a functor $\mathfrak{C}(\mathcal{C}) \rightarrow \text{Set}_\Delta^{\mathfrak{C}(\mathcal{C})}$. However, the representing functors will be fibrant since \mathcal{C} is an ∞ -category, however, the cofibrancy may not be guaranteed. Because of this we apply the cofibrant replace functor on the diagrams to obtain a simplicial functor $\mathfrak{C}(\mathcal{C}) \rightarrow (\text{Set}_\Delta^{\mathfrak{C}(\mathcal{C})})^\circ$. Then the Yoneda embedding is defined as the following composite

$$\mathcal{C} \rightarrow N(\mathfrak{C}(\mathcal{C})) \rightarrow N\left(\left(\text{Set}_\Delta^{\mathfrak{C}(\mathcal{C})}\right)^\circ\right) \simeq \mathcal{P}\text{sh}(\mathcal{C}).$$

The fact that it is an embedding is Prop. 5.1.3.1 in [Lu1].

Now let us discuss the limits and colimits in an ∞ -category. These correspond to *homotopy* limits and colimits in model categories. Defining them is straightforward, however, to show that certain ∞ -categories admit these limits and colimits construction in diagram model categories.

As we mentioned earlier a colimit of a diagram implicitly corresponds to homotopy colimit in model categories. By essentially using the fact that all small homotopy colimits and limits can be constructed in a simplicial model category, then one can show the following (Prop. 4.2.4.8 in [Lu1]).

PROPOSITION 9. *Let \mathbf{M} be a combinatorial simplicial model category. The associated ∞ -category $S = N(\mathbf{M}^\circ)$ admits small limits and colimits.*

COROLLARY 1. *The presheaf ∞ -category $\mathcal{P}\text{sh}(\mathcal{C})$ is complete and cocomplete.*

With this corollary we can freely talk about limits and colimits of diagrams in presheaf ∞ -categories. Furthermore, we will see that the localizations of the presheaf categories will also admit all the small limits

and colimits. Let us review the notion of localization in the ∞ -categorical setting.

DEFINITION 5. *Let \mathcal{C} and \mathcal{D} be ∞ -categories. Then a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is left adjoint to $g : \mathcal{D} \rightarrow \mathcal{C}$ (or g is a right adjoint to f), if there is a unit transformation $u : \mathbb{1}_{\mathcal{C}} \rightarrow g \circ f$, such that the composite*

$$\mathrm{Map}_{\mathrm{Ho}(\mathcal{D})}(f(C), D) \longrightarrow \mathrm{Map}_{\mathrm{Ho}(\mathcal{C})}((g \circ f)(C), g(D)) \longrightarrow \mathrm{Map}_{\mathrm{Ho}(\mathcal{C})}(C, g(D))$$

is an isomorphism in \mathcal{H} for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

In analogy with ordinary category theory, provided one of adjoints the other is uniquely determined up to a contractible space of choices. We need adjunctions in order to talk about (Bousfield) localization. Localization in its most general incarnation is the process of inverting a class of morphisms. One can do this completely formally, however, the resulting ∞ -categories are difficult to work with. However, in some situations one can realize the localization *within* the ∞ -category, namely, as a *full* subcategory of the original category. This makes the process of working with them much easier.

DEFINITION 6. *A functor $L : \mathcal{C} \rightarrow \mathcal{C}$ is a localization if the restriction of the codomain to essential image is a left adjoint to the inclusion.*

A localization functor is completely determined by the set of morphisms it inverts. This follows from the fact that $L : \mathcal{C} \rightarrow \mathcal{C}$ is determined by its essential image (by virtue of being a left adjoint) and Prop. 5.5.4.2 (a) in [Lu1], which states that $L\mathcal{C}$ is the full subcategory of S -local objects, where S is the collection of morphisms that get sent to equivalences under L . An object $X \in \mathcal{C}$ is S -local if $\mathrm{Map}_{\mathrm{Ho}(\mathcal{C})}(f, X)$ is an equivalence of Kan complexes for all $f \in S$. The set of inverted morphisms is not an arbitrary collection of morphisms. It satisfies a *strong saturation condition*, which means the following:

- if $f : C \rightarrow D$ is in S and the following diagram is a pushout, then g is also in S ;

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow & & \downarrow \\ E & \xrightarrow{g} & F, \end{array}$$

- the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by S is closed under taking colimits;
- the 2-out-of-3 property holds for morphisms in S .

It is clear from that $\cap_{\alpha} S_{\alpha}$ is strongly saturated if S_{α} 's are strongly saturated. This means that it makes sense to talk about the smallest strongly saturated collection of morphisms containing a given collection of morphisms. If S_0 is a collection of morphisms, then let $\overline{S_0}$ denote the smallest strongly saturated collection of morphisms containing S_0 . If $S = \overline{S_0}$ for some *set* S_0 , then we call S of *small generation*. Prop. 5.5.4.15 of [Lu2] states that one can construct a localization functor $L : \mathcal{C} \rightarrow \mathcal{C}$ exhibiting S as the inverted set of

morphisms, so long as \mathcal{C} is a presentable ∞ -category and S is strongly saturated and of small generation. For instance, if $\mathcal{E} = \mathcal{Psh}(\mathcal{C})$, where \mathcal{C} is a small ∞ -category, then \mathcal{E} is presentable. We will discuss presentability of ∞ -categories in the next subsection.

1.2.4 Accessibility and Presentability

A lot of times the categories or ∞ -categories that we are dealing with are very big, but their structure can be derived from small data. This means that if we are given a “big” category, \mathcal{E} , there is a small subcategory \mathcal{C} , that we can understand better, and from the categorical structure of \mathcal{C} we can recover that of \mathcal{E} . There are two components to making this more precise. The first is that any object in \mathcal{E} can be represented as some kind of colimit of objects in \mathcal{C} . One should keep in mind the example of sets: every set is generated by the singleton under filtered colimits and disjoint unions. The second component is the commuting of certain colimits. The point is that $\text{Hom}(\text{colim}_{\alpha} X_{\alpha}, \text{colim}_{\beta} Y_{\beta}) \simeq \lim_{\alpha} \text{Hom}(X, \text{colim}_{\beta} Y_{\beta})$, but the second colimit cannot be necessarily be move past Hom , so we need more information to make this move possible. This is done by requiring existence of certain compact objects in the generating subcategory. To facilitate the discussion we need a few preliminary concepts.

Recall that there is the Yoneda embedding $\mathcal{C} \rightarrow \mathcal{Psh}(\mathcal{C})$. The presheaf category $\mathcal{Psh}(\mathcal{C})$ is cocomplete. Let us denote by $\text{Ind}_{\kappa}(\mathcal{C})$ the full subcategory of $\mathcal{Psh}(\mathcal{C})$ generated by \mathcal{C} by κ -filtered colimits.

DEFINITION 7. *An ∞ -category \mathcal{C} is κ -accessible, for a regular cardinal κ , if there is an equivalence $\text{Ind}_{\kappa}(\mathcal{C}^0) \simeq \mathcal{C}$ and \mathcal{C}^0 is a small ∞ -category. The category \mathcal{C} is accessible if it is κ -accessible for some regular cardinal κ .*

EXAMPLE 12. The ∞ -category of spaces is accessible. Let κ be an uncountable cardinal. Indeed, if X is a Kan complex, let $\text{Cof}_{\kappa}(X)$ denote the ordinary category of Kan subcomplexes of S of X , which are equivalent to κ -small Kan complex, and the morphisms are inclusions. The colimit of $\text{Cof}_{\kappa}(X)$ in Set_{Δ} is X is, in fact, a homotopy colimit, Prop 7.3 in [Du]. The category $\text{Cof}_{\kappa}(X)$ has either a terminal object or is κ -filtered. This shows that \mathcal{S} is κ -accessible.

Once we have an example of an accessible ∞ -category we generated many more. One such procedure is the construction of functor categories (Prop. 5.4.4.3, [Lu1]).

PROPOSITION 10. *If \mathcal{C} is an accessible ∞ -category, then so is $\text{Fun}(K, \mathcal{C})$ for any simplicial set K .*

From this proposition, we have the following corollary.

COROLLARY 2. *If \mathcal{C} is a small category, then $\mathcal{Psh}(\mathcal{C})$ is accessible.*

DEFINITION 8. *An ∞ -category \mathcal{C} is presentable if it is accessible and cocomplete.*

Note that we instantly, see that $\mathcal{Psh}(\mathcal{C})$ is a presentable ∞ -category. There is another characterization of

presentable ∞ -categories. An ∞ -category \mathcal{E} is presentable if and only if there is a small ∞ -category \mathcal{C} and an accessible localization $\mathcal{Psh}(\mathcal{D}) \rightarrow \mathcal{E}$. A localization is accessible if \mathcal{E} is accessible.

One interesting thing about presentable categories is that they are also complete, Cor. 5.5.2.4 in [Lu1]. We need the following theorem about presentable ∞ -categories, Prop. 5.5.4.15 in [Lu1].

THEOREM 5. *If \mathcal{C} is a presentable ∞ -category and S is a small collection of morphisms, then there exists a localization which inverts the strongly saturated class of morphisms generated by S .*

1.3 ∞ -Topoi

1.3.1 Definitions and Some Properties

In ordinary category theory topoi form a context, where one can sensibly do “set-theory” in a coherent over some “space”, which satisfy certain local-to-global conditions. In particular, things like sheaf cohomology pop-up from such general contexts. One can think of homotopy theory as a generalization of set-theory, in the sense that sets are simply discrete simplicial sets up to homotopy. One should think of ∞ -topoi as homotopy theoretic extension of ordinary topoi. Having such a thing allows one to perform homotopy theory over a “space”, while maintaining the local-to-global properties. In addition to this ∞ -topoi subsume the classical topos theory. One can think of ∞ -topoi as the ∞ -category of sheaves of Kan complexes on some “space”. This particular definition is hard to generalize to higher categorical setting, since Grothendieck topologies do not produce a general enough class of ∞ -topoi. Nevertheless, ∞ -topoi that we will need come from Grothendieck topologies. The definition from ordinary category theory that generalizes easily is the left exact localization definition.

DEFINITION 9. *Suppose \mathcal{X} is an ∞ -category. We will call \mathcal{X} an ∞ -topos if there exists a small ∞ -category \mathcal{C} and an accessible left exact localization $\mathcal{Psh}(\mathcal{C}) \rightarrow \mathcal{X}$.*

REMARK 8. One subtlety that sneaks in is the accessibility of the localization. In ordinary categories left exact localizations are automatically accessible. However, it is not known whether the same statement is true in the ∞ -categorical setting.

Even though, we will not use Giraud’s characterization of ∞ -topoi, we write them down for the sake of completeness and they also show that ∞ -topoi, much like the ordinary topoi can be described internally. Before stating the theorem, we need to explain the higher categorical generalizations of Giraud axioms. Let us fix an ∞ -category \mathcal{E} . Giraud’s axioms for \mathcal{E} are the following:

- 1) \mathcal{E} is presentable,
- 2) the colimits in \mathcal{E} are universal,
- 3) the sums in \mathcal{E} are disjoint,

4) all the groupoids in \mathcal{E} are effective.

We discussed what presentability means in the previous section. The conditions 2) and 3) are straightforward. Condition 4) will require some explanation. We can view the ordinary simplicial category Δ as an ∞ -category by applying the nerve construction. A simplicial object in \mathcal{E} is a functor $S : N(\Delta)^{\text{op}} \rightarrow \mathcal{E}$. A simplicial object S will be called a groupoid if the following diagram is a pullback

$$\begin{array}{ccc} U_{m+n} & \longrightarrow & U_m \\ \downarrow & & \downarrow \\ U_n & \longrightarrow & U_0 \end{array}$$

for the maps $[n], [m] \rightarrow [n+m]$, whose images intersect at single point, corresponding to points $[0] \rightarrow [n], [m]$. Let Δ_+ denote the category of finite ordered sets with the empty set included. We will write $[-1]$ for the empty set. Let $\Delta_+^{\leq n}$ denote the full subcategory of Δ_+ spanned by the objects $[j]$, such that $j \leq n$. Then we consider the diagram

$$\begin{array}{ccc} N(\Delta)^{\text{op}} & \xrightarrow{S} & \mathcal{E} \\ \downarrow & \nearrow S_! & \uparrow S_!|_{\Delta_+^{\leq 0}} \\ N(\Delta_+)^{\text{op}} & \longleftarrow & N(\Delta_+^{\leq 0})^{\text{op}} \end{array}$$

where $S_!$ is the appropriate left Kan extension of S . We will call a simplicial object S *effective* if the natural transformation $S_! \rightarrow (S_!|_{\Delta_+^{\leq 0}})^1$ is an equivalence of functors. It is true that all the effective simplicial objects are groupoids. The converse may not hold in general, but it does hold for \mathcal{E} an ∞ -topos, Prop. 6.1.3.19 and Prop. 6.1.5.3 in [Lu1].

THEOREM 6. *The ∞ -category \mathcal{X} is an ∞ -topos if and only if \mathcal{X} satisfies Giraud's axioms.*

Recall that a Kan complex K is n -truncated if $\pi_k(K) \simeq *$ for $k > n$. Let us denote by $\tau_{\leq n}\mathcal{S}$ the ∞ -subcategory of n -truncated Kan complexes. Given an ∞ -category \mathcal{C} an object X is n -truncated if the functor $\text{Map}(-, X) : \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}$ factors through $\tau_{\leq n}\mathcal{S}$. Let us denote by $\tau_{\leq n}\mathcal{C}$ the ∞ -subcategory of n -truncated objects. There is a potential abuse of notation for $\tau_{\leq n}\mathcal{S}$, but it is clear that the two notations agree. One thing that comes out immediately is that $\tau_{\leq 0}\mathcal{C}$ is equivalent to a nerve of an ordinary category. In fact, $\tau_{\leq 0}\mathcal{C} \simeq N(\widetilde{\text{Ho}}(\tau_{\leq 0}\mathcal{C}))$. This is because the mapping spaces between objects in $\tau_{\leq 0}\mathcal{C}$ are equivalent to *discrete* Kan complexes. When we are dealing with an ∞ -topos we have the following proposition (Thm. 6.4.1.5 in [Lu1]).

PROPOSITION 11. *An ordinary category \mathcal{E} is a topos if and only if there is an ∞ -topos \mathcal{X} and an equivalence $\mathcal{E} \rightarrow \tau_{\leq 0}\mathcal{X}$.*

REMARK 9. Let us now demonstrate how the notion of sheaf cohomology can be viewed through the ∞ -categorical lens. Recall that if \mathcal{E} is an ordinary topos and $A \in \mathcal{E}$ is an abelian group object, then we can construct the sheaf cohomology $H^n(X, A)$ for any object X in \mathcal{E} using injective resolutions. One other thing that one may do is to take \mathcal{E} and present it as the subcategory of 0-truncated objects of some ∞ -category \mathcal{X} . Now within \mathcal{X} we can perform the bar-construction on the abelian group object A as follows. Let \mathcal{X} be a left exact localization of $\mathcal{Psh}(\mathcal{C})$ (in this case \mathcal{C} may even be an ordinary category). Then A gives an abelian group object in $\mathcal{Psh}(\mathcal{C})$. This implies that each $C \in \mathcal{C}$ evaluates to an abelian simplicial set. There is an n -fold bar construction functor from $B^n : Ab(\mathcal{S}) \rightarrow \mathcal{S}$. Composing $B^n \circ A$ gives a presheaf on \mathcal{C} . Applying sheafification yields $B^n A = (B^n \circ A)^+$. Then the sheaf cohomology will be simply $H^n(X, A) \simeq \pi_0(\text{Map}_{\mathcal{X}}(X, B^n A))$. So as we can see in the setting of ∞ -topoi the sheaf cohomology becomes more similar to the homotopy theoretic approach to cohomology.

1.3.2 Grothendieck Topologies and Constructions of ∞ -Topoi

As you might have noticed we have not mentioned Grothendieck topologies in the definition of ∞ -topoi. The next thing we would like to discuss is how one can a ∞ -topos from a Grothendieck topology on an ∞ -category. However, first we need to define the latter.

We begin by defining a notion of sieve for an ∞ -category. A *sieve* on \mathcal{C} is a full subcategory \mathcal{C}^0 of \mathcal{C} , such that if $C \rightarrow D$ is a morphism in \mathcal{C} and D belongs to \mathcal{C}^0 , then $C \in \mathcal{C}^0$. Alternatively, one can think of a sieve as a functor from \mathcal{C} to $N(\Delta_{\mp}^{\leq 0})$. The correspondence is given by sending the objects of \mathcal{C}^0 to the object $[0]$ and the rest of the objects to $[1]$. This gives a bijective correspondence between sieves on \mathcal{C} and the objects of $\text{Fun}(\mathcal{C}, N(\Delta_{\mp}^{\leq 0}))$. From the second description it is clear that sieves can be pulled back. More concretely, let $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor and \mathcal{D}^0 is a sieve on \mathcal{D} , then it can be easily checked that the preimage $F^{-1}(\mathcal{D}^0)$ is a sieve on \mathcal{C} .

A sieve on an object C in an ∞ -category \mathcal{C} , is simply a sieve on $\mathcal{C}_{/C}$. Let Cov be an assignment of a collection of sieves over a given object. That is for any object $C \in \mathcal{C}$ we assign a collection $\text{Cov}(C)$, whose elements are sieves on C . Such a specification will be called a collection of covering sieves if the following conditions are satisfied:

- 1) $\mathcal{C}_{/C}$ is in $\text{Cov}(C)$ for all $C \in \mathcal{C}$,
- 2) given any morphisms $f : C \rightarrow D$ and $\mathcal{C}_{/D}^0$ is in $\text{Cov}(D)$, then the pullback of $f^*(\mathcal{C}_{/D}^0)$ is in $\text{Cov}(C)$,
- 3) suppose that $\mathcal{C}_{/C}^0$ is in $\text{Cov}(C)$ and $\mathcal{C}_{/C}^1$ is an arbitrary sieve on C ; if for any $f : D \rightarrow C$ in $\mathcal{C}_{/C}^0$ the pullback $f^*\mathcal{C}_{/C}^1$ is in $\text{Cov}(D)$, then $\mathcal{C}_{/C}^1$ is in $\text{Cov}(C)$.

Using these notions we give the following definition.

DEFINITION 10. A *Grothendieck topology* in an ∞ -category \mathcal{C} is a choice of a collection of covering sieves.

One can check that for ordinary categories this defines the conventional notion of a Grothendieck topology. In fact, one can show that giving a Grothendieck topology on \mathcal{C} is equivalent to giving a Grothendieck topology on $\widetilde{\text{Ho}}(\mathcal{C})$, where $\widetilde{\text{Ho}}(\mathcal{C})$ is the ordinary (unenriched) homotopy category of \mathcal{C} . This means that Grothendieck topologies do not incorporate much of the higher categorical information of \mathcal{C} . As we will see later, Grothendieck topologies produce left exact localizations of $\mathcal{P}\text{sh}(\mathcal{C})$, but, due to the previous defect, they do not produce all such localizations. As another remark, we will be mostly interested in the case where \mathcal{C} is a nerve of an ordinary category, which seems to be the most appropriate setting for Grothendieck topologies.

Now let \mathcal{C}^0 be a sieve on \mathcal{C} . As we have observed earlier, such a choice is equivalent to a functor $\mathcal{C} \rightarrow \Delta_+^{\leq 0}$. We can consider such a functor as a presheaf on \mathcal{C} , since we have a functor $\Delta_+^{\leq 0}$ to \mathcal{S} . Notice that the presheaf is (-1) -truncated. Conversely, suppose that \mathcal{F} is a (-1) -truncated presheaf on \mathcal{C} , we can construct a sieve on \mathcal{C} , by realizing that such a presheaf is equivalent to one that factors through $\Delta_+^{\leq 0}$.

Given an ordinary Grothendieck site \mathcal{C} we get an ∞ -topos $\text{Sh}(\mathcal{C})$. The 0-truncated objects of $\tau_{\leq 0}\text{Sh}(\mathcal{C})$ produce an ordinary category, which will coincide with the nerve of an ordinary topos corresponding to the Grothendieck site. The category $\tau_{\leq 1}\text{Sh}(\mathcal{C})$ coincides with 2-category of stacks over \mathcal{C} . As it was promised previously we can extract all the objects of the introduction from this ∞ -categorical construction.

The main example that we will be interested in is the ∞ -topos of sheaves on manifolds. Let us consider the Man the ordinary category of manifolds equipped with the open cover topology. The construction extracts an ∞ -topos $\text{Sh}(\text{Man})$ out of such information of sheaves on this Grothendieck site, which we will write as \mathcal{M} . Recall that there is topologically enriched category Man_∞ , and the same construction on Man_∞ would produce a radically different result.

1.4 (∞, n) -Categories

1.4.1 The Model of Θ_n -Spaces

It seems that there are quite a few approaches to (∞, n) -categories. In all cases, it seems that they are powered by some version of higher category theory. One of the reasons why we introduced ∞ -categories separately is that we are going to use them as a *framework* for the definition of (∞, n) -categories.

The most intuitive approach to (∞, n) -categories implements the notion of Θ_n categories, which we begin describe. Joyal's description in [Jo1] is conceptual in the sense that it allows one to have pictures, so we will begin with that definition. We first define what an interval bundle is. It is a surjective map of finite sets $p : X \rightarrow B$, with the additional information of ordering on fibers of p . In particular, we have natural sections d_0 and d_1 corresponding to the sections of minima and of maxima, respectively. The singular set B^* is the set of points with fiber a single element. We can also define the boundary of X , ∂X , to be the union

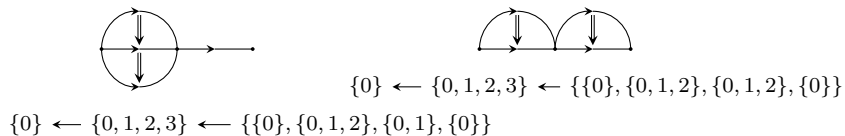
of fiberwise minima and maxima.

DEFINITION 11. A combinatorial n -disk is a sequence of interval bundles:

$$\{0\} = D_0 \longleftarrow D_1 \longleftarrow \dots \longleftarrow D_{n-1} \longleftarrow D_n,$$

such that $(D_k)^* = \partial D_{k-1}$ for $0 \leq k < n$. We adopt the convention that $\partial D_0 = \emptyset$. A morphism is a structure preserving map between such sequences. Let us denote this category \mathcal{D}_n . The category Θ_n is defined to be $(\mathcal{D}_n)^{\text{op}}$.

One can imagine this category using certain categorical diagrams. For instance, we can draw the following pictures for the case $n = 2$:



The way one draws this diagrams is by using the inner Dedekind cuts of the ordered sets. For each inner cut of D_1 put a vertex. For each $x \in D_1$, note that $p^{-1}(x) \subset D_2$ has an order by definition. For each cut of $p^{-1}(x)$ attach an arrow from the points corresponding to cuts ∂_-x and ∂_+x . When done with D_1 , move to points of D_2 , then D_3 , etc. One done with all the points we obtain a diagram representing an element in Θ_n .

What we have done above was a description of a construction of a strict n -category. The category Θ_n can be interpreted as a subcategory of $\text{Cat}_n^{\text{strict}}$, which is indeed the case. The embedding is done inductively. For Θ_0 is the trivial category and $\text{Cat}_0^{\text{strict}}$ is the category of sets: the embedding is that of the terminal object. Now suppose that we have an embedding $\Theta_n \hookrightarrow \text{Cat}_n^{\text{strict}}$. Now consider that $n+1$ -categories \mathcal{C} , with objects $\{0, \dots, k\}$, and if $s \geq 0$ then $\text{Hom}(l, l+s) = \prod_{i=l}^{l+s-1} \theta_i$, where $\theta_i \in \Theta_n$, and if $s < 0$ then $\text{Hom}(l, l+s) = \emptyset$. The full subcategory of these objects is equivalent to Θ_{n+1} . This interpretation of the Θ_n categories is the most convenient one for us.

There are several notions concerning strict n -categories that we need to discuss. Firstly, there is an embedding $\text{Cat}_{n-1}^{\text{strict}} \hookrightarrow \text{Cat}_n^{\text{strict}}$, which is constructed inductively starting from the cartesian functor $\text{Set} \rightarrow \text{Cat}$, which sends a set to the discrete category. It is relatively easy to check that this functor can be restricted to an embedding $\Theta_{n-1} \hookrightarrow \Theta_n$. There is a right adjoint to the inclusion functor $\text{Cat}_{n-1}^{\text{strict}} \rightarrow \text{Cat}_n^{\text{strict}}$, which can be constructed inductively by starting with $\text{Cat} \rightarrow \text{Set}$ functor, where we disregard the morphisms and consider the set of objects. We will write this right adjoint as $\partial : \text{Cat}_n^{\text{strict}} \rightarrow \text{Cat}_{n-1}^{\text{strict}}$. This operation does not descend to a functor from Θ_n to Θ_{n-1} .

There is also a suspension functor $\Sigma : \text{Cat}_{n-1}^{\text{strict}} \rightarrow \text{Cat}_n^{\text{strict}}$, which sends the $(n-1)$ -category \mathcal{C} to the n -category $\Sigma(\mathcal{C})$ of two objects $[0]$ and $[1]$, and \mathcal{C} -worth of morphisms from $[0]$ to $[1]$. This functor restricts to a functor $\Sigma : \Theta_{n-1} \rightarrow \Theta_n$.

We can inductively define the cells \mathfrak{e}_k by setting \mathfrak{e}_0 to be the singleton set and \mathfrak{e}_k to be $\Sigma(\mathfrak{e}_{k-1})$. The cells generate $\mathcal{C}at_n^{\text{strict}}$ under colimits according to proposition 2.4 of [BaSPr]. Let us denote by \mathcal{P}_n the full subcategory of $\mathcal{C}at_n^{\text{strict}}$ spanned by cells.

Notice that \mathcal{P}_n is subcategory of Θ_n for $k \leq n$. Therefore, we see that cells generate Θ_n under colimits. However, more is true Θ_n is generated by the pushout diagrams in \mathcal{P}_n . For each object $X \in \Theta_n$, we have the comma category $(\mathcal{P}_n \downarrow X)$ and a natural diagram $\Lambda_X : (\mathcal{P}_n \downarrow X) \rightarrow \Theta_n$. The discussion above can be summarized by the formula $\text{colim } \Lambda_X \simeq X$.

We now start to describe a general framework for (∞, n) -categories. We postulate first that there is an ∞ -category of (∞, n) -categories, which we will write as $\mathcal{C}at_{(\infty, n)}$. Naturally, the ∞ -category of spaces, \mathcal{S} ought to be an ∞ -subcategory of $\mathcal{C}at_{(\infty, n)}$. Furthermore, we want there to be a right adjoint to this inclusion, $(-)^{\sim} : \mathcal{C}at_{(\infty, n)} \rightarrow \mathcal{S}$. One should think of X^{\sim} as the maximal ∞ -subgroupoid within X . We also would like this category to be Cartesian closed: that is we have products $- \times -$ and mapping objects $\text{Map}(-, -)$. In addition to this, we should have strict n -categories, $\mathbf{N}(\mathcal{C}at_n^{\text{strict}})$ as a part of $\mathcal{C}at_{(\infty, n)}$. This implies that $\mathbf{N}(\Theta_n)$ is also a subcategory of $\mathcal{C}at_{(\infty, n)}$. As a consequence, we obtain a functor $\mathcal{C}at_{(\infty, n)} \rightarrow \mathcal{P}sh(\mathbf{N}(\Theta_n))$ by sending the (∞, n) -category to $\text{Map}(-, X)^{\sim}$. The first requirement is that this functor is an embedding. This reduces us to the setting where we have to describe a subcategory of $\mathcal{P}sh(\mathbf{N}(\Theta_n))$. In fact, we would like there to be a localization functor. As we have seen earlier an object in Θ_n can be described as a pushout of a diagram of disks, namely $\text{colim } \Lambda_X = X$. Of course, in the presheaf category this may fail to be a colimit. Nevertheless, we still retain a map $\text{colim } \Lambda_X \rightarrow X$. These are the segalization morphisms. Namely, we define Segal_n to be the collection of morphisms in $\mathcal{P}sh(\mathbf{N}(\Theta_n))$, $\{\Lambda_X \rightarrow X\}_{X \in \Theta_n}$. The next step is the Rezk completion. Recall that there is a fully faithful embedding $\mathcal{S} \hookrightarrow \mathcal{P}sh(\mathbf{N}(\Theta_n))$. Thus functor has a left adjoint $^+ : \mathcal{P}sh(\mathbf{N}(\Theta_n)) \rightarrow \mathcal{S}$. We define Rezk_n to be $\{* \rightarrow (\mathfrak{e}_k)^+\}_{k=1}^n$. We then define $\text{Segal}_n^{\wedge} = \text{Segal}_n \cup \text{Rezk}_n$. Then the category $\mathcal{C}at_{(\infty, n)}$ is defined as the localization category $(\text{Segal}_n^{\wedge})^{-1}(\mathcal{P}sh(\mathbf{N}(\Theta_n)))$.

DEFINITION 12. *We define the ∞ -category of Θ_n -spaces to be $(\text{Segal}_n^{\wedge})^{-1}(\mathcal{P}sh(\mathbf{N}(\Theta_n)))$. We will write it as $\Theta_n \mathcal{S}$.*

From the description it is understood that the category Θ_n is not extremely essentially. It is just a tool that facilitates all sorts of composition that we can have in the (∞, n) -category. The homotopic nature of the composition is provided by the framework of ∞ -categories. We will touch on several other reformulations of what we mean by (∞, n) -categories. All of them, will bear the flavor of the definition above. All of these ∞ -categories will be equivalent to one another according to the remarkable result of Barwick and Schommer-Pries, [BaSPr].

There is a procedure of constructing a Θ_n -space from a simplicial category. Suppose \mathcal{X} is simplicial category. Then by regarding Θ_n as a subcategory of $\mathcal{C}at_{\Delta}$, for each $X \in \Theta_n$, we can assign the space

$N(\text{Map}(X, \mathcal{X}))^\sim$. The Segal condition can be easily checked, once we notice that the colimit diagram Λ_X consists of cofibrations. The presheaf is also Rezk complete: $\text{Map}(\mathfrak{e}_k^+, \mathcal{X}) \simeq \text{Map}(\mathfrak{e}_k^+, \mathcal{X}^\sim) \simeq \text{Map}(*, \mathcal{X}^\sim)$. Thus, we obtain a functor $\text{Cat}_\Delta \rightarrow \Theta_n \mathcal{S}$. On the subcategory of fibrant simplicial categories this functor takes values in $\Theta_1 \mathcal{S}$. We can use this observation to construct a functor $\text{Cat}_\infty \rightarrow \Theta_1 \mathcal{S}$. This functor is an equivalence of ∞ -categories.

EXAMPLE 13. The following example will be important later in the discussion. This is an explicit version of the Dold-Kan construction, where we take a chain complex and turn into a category. Suppose that we are given a chain complex of topological abelian groups A_* of length $n + 1$:

$$A_0 \leftarrow A_1 \leftarrow \dots \leftarrow A_n.$$

The construction will be done by inducting on n . If $n = 0$, the groupoid that we are looking for will be A_0 itself. Now suppose that the construction has been done for $n - 1$, and say it assigns to the chain complex A_* a Θ_{n-1} -space \widehat{A} . We also need to make the assumption that given a chain complex of length n , there is functor $d : \widehat{A}^{\geq 1}(X) \rightarrow A_0$ for X in Θ_{n-1} , where we regard A_0 as the constant functor on Θ_{n-1} . Here $\widehat{A}_*^{\geq 1}$ the chain complex obtained by discarding A_0 . Given an object $[n](X_1, \dots, X_n)$ in Θ_n , we assign to it the space $A_0 \times \prod_{i=1}^n \widehat{A}^{\geq 1}(X_i)$. Suppose we are given a morphism $\varphi : [n](X_1, \dots, X_n) \rightarrow [m](Y_1, \dots, Y_m)$, i.e. $\varphi_* : [n] \rightarrow [m]$ and maps $\varphi_{(k)} : X_k \rightarrow Y_{\varphi_*(k)}$. The corresponding map $\widehat{A}(\varphi) : A_0 \times \prod_{i=1}^m \widehat{A}^{\geq 1}(Y_i) \rightarrow A_0 \times \prod_{i=1}^n \widehat{A}^{\geq 1}(X_i)$ sends (x_0, x_1, \dots, x_m) to $(x_0 + \sum_{j=1}^{\varphi_*^{-1}(0)} dx_j, \sum_{j \in \varphi_*^{-1}(1)} \widehat{A}^{\geq 1}(\varphi_{(j)}), \dots, \sum_{j \in \varphi_*^{-1}(n)} \widehat{A}^{\geq 1}(\varphi_{(j)}))$. This construction will appear in its n -fold Segal space incarnation as a version of the Dold-Kan construction, and will play an important role in current work.

1.4.2 The Unicity Theorem

In [BaSPr], Barwick and Schommer-Pries prove that given a set of four axioms that ∞ -category satisfies one can show that it is equivalent to Rezk's Θ_n -spaces. We have hinted at the idea earlier, but let us spell it out in detail. We begin with some notation.

Let $\text{Cat}_n^{\text{strict}}$ denote the ordinary category of strict n -categories. We make the convention that $\text{Cat}_0^{\text{strict}}$ is the category of sets. As we have mentioned earlier this category should be a part of the theory of (∞, n) -categories, namely the 0-truncated part. There are several operations within this category. Within this category we have a subcategory Gaunt_n , which is the full subcategory of strict n -categories for which the invertible k -morphisms are identities for $k \leq n$. Let Glob_n be the subcategory of Gaunt_n consisting of the \mathcal{D}_k for $0 \leq k \leq n$. Finally, we consider the category Υ_n , which is the full subcategory of Gaunt_n containing Glob_n and closed under fiber products over the cells \mathfrak{e}_k . Now we can state the axioms of definition 6.8 in [BaSPr].

DEFINITION 13. The ∞ -category \mathcal{C} is a homotopy theory of (∞, n) -categories if there is a fully faithful functor $f : \Upsilon_n \rightarrow \tau_{\leq 0}\mathcal{C}$, so that the following axioms are satisfied.

- 1) The composite $\Upsilon_n \rightarrow \tau_{\leq 0}\mathcal{C} \rightarrow \mathcal{C}$ strongly generates \mathcal{C} . In particular \mathcal{C} is presentable.
- 2) For any morphism $\eta : X \rightarrow f(\mathbf{e}_i)$ of \mathcal{C} , the fiber product functor

$$\eta^* : \mathcal{C}_{/f(\mathbf{e}_i)} \rightarrow \mathcal{C}_{/X},$$

preserves colimits. Since \mathcal{C} is presentable this is equivalent to the existence of internal homs for the categories of correspondences $\mathcal{C}_{/f(\mathbf{e}_i)}$.

- 3) The following equivalences are satisfied:

- a) For each integer $1 \leq i \leq n-1$ the natural morphism

$$f(\mathbf{e}_{i-1}) \coprod_{f(\partial\mathbf{e}_{i-1})} f(\mathbf{e}_{i-1}) \rightarrow f(\partial\mathbf{e}_i)$$

is an equivalence, as is the natural map from the empty colimit to $f(0)$.

- b) For each pair of integers $0 \leq i < j \leq n$, the natural morphism

$$f(\mathbf{e}_j) \coprod_{f(\mathbf{e}_i)} f(\mathbf{e}_j) \rightarrow f(\mathbf{e}_j \coprod_{\mathbf{e}_i} \mathbf{e}_j)$$

is equivalence.

- c) For each $0 \leq i \leq n$, each $0 < j, k \leq n-i$, and every nondegenerate morphism $\mathbf{e}_{i+j} \rightarrow \mathbf{e}_i$ and $\mathbf{e}_{i+j} \rightarrow \mathbf{e}_i$, the natural morphism

$$f(\mathbf{e}_{i+j} \coprod_{\mathbf{e}_i} \mathbf{e}_{i+k}) \coprod_{f(\Sigma^{i+1}(\mathbf{e}_{j-1} \times \mathbf{e}_{k-1}))} f(\mathbf{e}_{i+k} \coprod_{f(\mathbf{e}_i)} \mathbf{e}_{i+j}) \rightarrow f(\mathbf{e}_{i+j} \coprod_{\mathbf{e}_i} \mathbf{e}_{i+k})$$

is an equivalence.

- d) For any $0 \leq k \leq n$, the natural morphism

$$(f(\mathbf{e}_k) \coprod f(\mathbf{e}_k)) \coprod_{f(\Sigma^k \mathbf{e}_1) \coprod f(\Sigma^k \mathbf{e}_1)} f(\Sigma^k[3]) \rightarrow f(\mathbf{e}_k)$$

is an equivalence.

- 4) For any quasicategory \mathcal{D} and any full faithful functor $g : \Upsilon_n \rightarrow \tau_{\leq 0}\mathcal{D}$ satisfying the conditions above, there exists a localization $L : \mathcal{C} \rightarrow \mathcal{D}$ and an equivalence $L \circ f \simeq g$.

In this situation, we say that the functor f exhibits \mathcal{C} as a homotopy theory of (∞, n) -categories.

Let $\text{Thy}_{(\infty, n)}$ denote the maximal Kan complex contained in the full subcategory of the ∞ -category of

∞ -categories spanned by the homotopy theories of (∞, n) -categories.

One has to show that such an ∞ -category is non-empty. From the universality it will follow that $\text{Thy}_{(\infty, n)}$ is connected. However, the uniqueness is not canonical, since there are automorphisms of the homotopy theory of (∞, n) -categories. Nevertheless, the homotopy type of $\text{Thy}_{(\infty, n)}$ is quite simple.

THEOREM 7. *The Kan complex $\text{Thy}_{(\infty, n)}$ is equivalent to $B(\mathbb{Z}/2)^n$.*

The theorem above is the unicity theorem. It implies that any two models for (∞, n) -categories are “almost” canonically equivalent.

To show that the theory of Θ_n -spaces and the n -fold Segal spaces are equivalent one needs to show that they both satisfy the axioms above. Theorems 11.15 and 12.6 in [BaSPr] show precisely that. We move on to describe n -fold Segal spaces.

1.4.3 n -Fold Segal Spaces

In this section we introduce the ∞ -category of an n -fold complete Segal space. This model also satisfies the unicity axioms. We will follow [Lu3] and [BaSPr].

Following [BaSPr] we write

$$\Delta^{\times n} = \underbrace{\Delta \times \cdots \times \Delta}_{n \text{ times}}.$$

Let us consider the ∞ -category of presheaves $\mathcal{Psh}(\mathbb{N}(\Delta^{\times n}))$. The ∞ -category of (small) (∞, n) -categories is a localization of this presheaf category. As an intermediate construction we will also define the ∞ -category of (∞, n) -precategories. We know that $\mathcal{Psh}(\mathbb{N}(\Delta^{\times n}))$ is presentable. Therefore, we need to specify the strongly saturated set of small generation in order to define these localizations. We will specify the generating set, which is a union of three sets Glob_n , Segal_n and Comp_n . The definition of Glob_n is direct, meanwhile the definitions of Segal_n and Comp_n are inductive.

Suppose that we are given $[m] = ([m_1], \dots, [m_n]) \in \Delta^{\times n}$. Then we can form a new object $[\widehat{m}] = ([\widehat{m}_1], \dots, [\widehat{m}_n]) \in \Delta^{\times n}$, where

$$\widehat{m}_j = \begin{cases} 0 & \text{if there exists } i \leq j, \text{ so that } m_i = 0 \\ m_j & \text{otherwise} \end{cases}$$

Let Glob_n be the image of the set of morphisms $\{[m] \longrightarrow [\widehat{m}] \mid [m] \in \Delta^{\times n}\}$ in $\mathcal{Psh}(\mathbb{N}(\Delta^{\times n}))$ under the Yoneda embedding.

Consider the diagram

$$\begin{array}{ccc}
[0] & \xrightarrow{0} & [m-k] \\
k \downarrow & & \downarrow \\
[k] & \cdots \cdots \cdots & [m]
\end{array}$$

is a pushout diagram $\mathcal{N}(\Delta)$. This diagram is not a pushout diagram in $\mathcal{Psh}(\mathcal{N}(\Delta))$. Let $\langle k, m-k \rangle$ be the pushout of the image of the solid arrow diagram in $\mathcal{Psh}(\mathcal{N}(\Delta))$ under the Yoneda embedding. There is a morphism $\langle k, m-k \rangle \rightarrow [m]$. Let us denote by Segal_1 the set of all such morphisms in $\mathcal{Psh}(\mathcal{N}(\Delta))$.

There is an ordinary functor $\times : \Delta \times \Delta^{\times(n-1)} \rightarrow \Delta^{\times n}$ sending $[k] \times [m]$ to $([k], [m])$. There is an essentially unique functor

$$\times : \mathcal{Psh}(\mathcal{N}(\Delta)) \times \mathcal{Psh}(\mathcal{N}(\Delta^{\times(n-1)})) \rightarrow \mathcal{Psh}(\mathcal{N}(\Delta^{\times n}))$$

that preserves colimits in each of the variables, and so that $[k] \times [m] = ([k], [m])$. The definition of Segal_n is given by the following expression,

$$\text{Segal}_n = \{\text{Segal}_1 \times [m] \mid [m] \in \Delta^{\times(n-1)}\} \cup \{[k] \times \text{Segal}_{n-1} \mid [k] \in \Delta\}.$$

Let K denote the nerve the category consisting of two objects and two isomorphisms, inverse to each other, going from each of the points to the other. The simplicial set $K : \Delta^{\text{op}} \rightarrow \text{Set}$ can be converted into a simplicial space by composing it with the functor $i : \text{Set} \rightarrow \text{Set}_\Delta$, where send each set to the discrete space corresponding to this set. Since discrete spaces are fibrant in Set_Δ , then $i \circ K$ lands in $(\text{Set}_\Delta)^\circ$. We then apply the nerve construction to obtain a map $\mathcal{N}(i \circ K) : \mathcal{N}(\Delta^{\text{op}}) \rightarrow \mathcal{N}(\text{Set}_\Delta^\circ) = \mathcal{S}$. Let us refer to this element in $\mathcal{Psh}(\mathcal{N}(\Delta))$ as E . The set Comp_1 consists of a single element $\{E \rightarrow [0]\}$. The inductive part is given by the formula

$$\text{Comp}_n = \{\text{Comp}_1 \times [0]\} \cup \{[k] \times \text{Comp}_{n-1} \mid [k] \in \Delta\}.$$

DEFINITION 14. *The localization of $\mathcal{Psh}(\Delta^{\times n})$ with respect to the union of morphisms Segal_n and Glob_n is called the ∞ -category of (small) (∞, n) -precategories, and is denoted by $\mathcal{PCat}_{(\infty, n)}$. The objects of $\mathcal{PCat}_{(\infty, n)}$ are referred to as (∞, n) -precategories. The localization of $\mathcal{PCat}_{(\infty, n)}$ with respect to the image of Comp_n is called the category of (∞, n) -categories. The objects of $\text{Cat}_{(\infty, n)}$ are called (∞, n) -categories. We denote by $\wedge : \mathcal{PCat}_{(\infty, n)} \rightarrow \text{Cat}_{(\infty, n)}$ the localization functor which is called the Rezk completion.*

EXAMPLE 14. We can endow $(\text{Set}_\Delta)^{\Delta^{\text{op}}}$ with the Rezk model structure, which is the localization of the Reedy model structure on the diagram category, which is equivalent to the projective model structure. The fibrant objects in this model category are the $(\infty, 1)$ -categories. A result due to Joyal and Tierney, [JT],

states there is a Quillen equivalence,

$$F : \mathrm{Set}_\Delta^{\mathrm{Joyal}} \rightleftarrows (\mathrm{Set}_\Delta)_{\mathrm{Rezk}}^{\Delta^{\mathrm{op}}} : G,$$

where $F(X)_{n,m} = X_n$ and $G(Y)_n = Y_{n,0}$. This statement demonstrates that the theory of ∞ -categories and $(\infty, 1)$ -categories are equivalent.

REMARK 10. As with ∞ -categories, one can define geometric realization for (∞, n) -categories. Recall that there is a functor $\Delta^{[\bullet]} : \Delta^{\times n} \rightarrow \mathrm{Set}_\Delta$, sending $([m_1], \dots, [m_n])$ to $\Delta^{m_1} \times \dots \times \Delta^{m_n}$. Given a functor $F : (\Delta^{\times n})^{\mathrm{op}} \rightarrow \mathrm{Set}_\Delta$, we can form the coend $\int^\Delta \Delta^{[\bullet]} \times F$, which we will write as $|F|$. Thus, we obtain a functor $|-| : (\mathrm{Set}_\Delta)^{(\Delta^{\times n})^{\mathrm{op}}} \rightarrow \mathrm{Set}_\Delta$. This functor admits a right adjoint, $G : \mathrm{Set}_\Delta \rightarrow (\mathrm{Set}_\Delta)^{(\Delta^{\times n})^{\mathrm{op}}}$, such that $G(X)([m_1], \dots, [m_n]) = \mathrm{Map}_{\mathrm{Set}_\Delta}(\Delta^{m_1} \times \dots \times \Delta^{m_n}, X)$. In fact, the pair $(|-, G|)$ is a Quillen adjunction if $(\mathrm{Set}_\Delta)^{(\Delta^{\times n})^{\mathrm{op}}}$ is endowed with the projective model structure and Set_Δ with the Kan model structure. This implies existence of the functor

$$|| : \mathcal{S}^{\mathrm{N}(\Delta^{\times n})^{\mathrm{op}}} \simeq \mathrm{N}\left(\left((\mathrm{Set}_\Delta)^{(\Delta^{\times n})^{\mathrm{op}}}\right)^\circ\right) \rightarrow \mathrm{N}\left((\mathrm{Set}_\Delta)^\circ\right) \simeq \mathcal{S}.$$

If we consider the composite of G with the cofibrant replace functor, we obtain a functor between fibrant simplicial categories. On the level of ∞ -categories we get a functor

$$G : \mathcal{S} \rightarrow \mathcal{S}^{\mathrm{N}(\Delta^{\times n})^{\mathrm{op}}},$$

which is right adjoint to the geometric realization functor. In fact, G is fully faithful, and therefore, spaces can be thought of (∞, n) -categories. In fact, they can be thought of as the (∞, n) -categories, where all the morphisms are invertible. Thus, the geometric realization can be thought of as inverting all the morphisms.

1.4.4 Symmetric Monoidal Structures

In this section we formalize the notion of a symmetric monoidal (∞, n) -category. For this we need define some additional notions. First, we consider the category Υ of finite pointed sets. We will use the symbol $*$ for basepoints, and write $\langle n \rangle$ for the set $\{*, 1, \dots, n\}$. Notice that there are n maps from $\iota^k : \langle n \rangle \rightarrow \langle 1 \rangle$, such that $(\iota^k)^{-1}(1) = \{k\}$.

DEFINITION 15. A symmetric monoidal (∞, n) -precategory is a functor $\mathcal{C}^\otimes : \mathrm{N}(\Upsilon) \rightarrow (\mathcal{P})\mathrm{Cat}_{(\infty, n)}$, such that the functors $\iota_*^k : \mathcal{C}^\otimes(\langle n \rangle) \rightarrow \mathcal{C}^\otimes(\langle 1 \rangle)$ determine an equivalence $\mathcal{C}^\otimes(\langle n \rangle) \simeq \mathcal{C}^\otimes(\langle 1 \rangle)^n$. We will write \mathcal{C} for $\mathcal{C}^\otimes(\langle 1 \rangle)$ and call it the underlying (∞, n) -precategory of \mathcal{C}^\otimes . Also, for notational convenience we will write $\mathcal{C}_{\langle n \rangle}^\otimes$ instead of $\mathcal{C}^\otimes(\langle n \rangle)$. The full subcategory ∞ -precategory of $(\mathcal{P})\mathrm{Cat}_{(\infty, n)}^{\mathrm{N}(\Upsilon)}$ spanned by symmetric monoidal (∞, n) -categories will be written as $(\mathcal{P})\mathrm{Cat}_{(\infty, n)}^\otimes$.

REMARK 11. Let $\mathcal{C}^\otimes : \mathbf{N}(\Upsilon) \rightarrow \mathcal{P}\text{Cat}_{(\infty, n)}$ be a symmetric monoidal (∞, n) -precategory. We can compose this functor with the Rezk completion functor to obtain $(\mathcal{C}^\otimes)^\wedge : \mathbf{N}(\Upsilon) \rightarrow \text{Cat}_{(\infty, n)}$. The resulting functor is an (∞, n) -category, since Rezk completion commutes with products, which follows from the fact that there is a Cartesian presentation of the model categories for (∞, n) -precategories and (∞, n) -categories, [Re2].

EXAMPLE 15. Let \mathcal{C} be an ordinary category and let A be an abelian group object in \mathcal{C} . We can define $A^\otimes : \Upsilon \rightarrow \mathcal{C}$ sending $\langle n \rangle$ to A^n and the map $p : \langle n \rangle \rightarrow \langle m \rangle$, $p_* : A^n \rightarrow A^m$ is given by the composition

$$A^n \simeq \prod_{j \in \{1, \dots, m\}} \prod_{i \in p^{-1}(j)} A \longrightarrow \prod_{j \in \{1, \dots, m\}} A \simeq A^m,$$

where the middle map corresponds to the multiplication. Now suppose that $\mathcal{C} = (\text{Set}_\Delta)^{(\Delta^{\times n})^{\text{op}}}$ and let $A : (\Delta^{\times n})^{\text{op}} \rightarrow \text{Ab}_\Delta$ be an abelian group object in it, then we can form a functor $A^\otimes : \Upsilon \rightarrow (\text{Set}_\Delta)^{(\Delta^{\times n})^{\text{op}}}$. If A is fibrant, then the functor A^\otimes lands in $(\text{Set}_\Delta^\circ)^{(\Delta^{\times n})^{\text{op}}}$, so we get a functor

$$\mathbf{N}(A)^\otimes = \mathbf{N}(A^\otimes) : \mathbf{N}(\Upsilon) \rightarrow \mathcal{P}\text{sh}(\mathbf{N}(\Delta^{\times n})).$$

To obtain a symmetric monoidal (∞, n) -category we apply the Rezk completion functor. We label the latter as $\mathbf{N}(A)^\otimes$ by the abuse of notation.

EXAMPLE 16. Let $n = 0$, then $\text{Cat}_{(\infty, 0)} \simeq \mathcal{S}$. Symmetric monoidal $(\infty, 0)$ -categories correspond to infinite loopspaces. Recall that infinite loopspaces correspond to Γ -spaces—functors of form $\Upsilon \rightarrow \text{Top}$. If we apply the singularization functor to get a functor $\Upsilon \rightarrow \text{Set}_\Delta^\circ$. The nerve construction gives us a functor $\mathbf{N}(\Upsilon) \rightarrow \mathcal{S}$. The original functor being a Γ -space implies that $\mathbf{N}(\Upsilon) \rightarrow \mathcal{S}$ is symmetric monoidal. The fact that this procedure gives an equivalence can be viewed as a strictification theorem.

There is another way of obtaining symmetric monoidal (∞, n) -categories using the theory of operads. Let \mathcal{P} be an ordinary category with products. Then an operad, \mathbb{O} , in \mathcal{P} is a collection of objects, $\mathbb{O}(n)$ with an action of the n -th symmetric group Σ_n , for each non-negative integer n along with maps of the form

$$\varphi_{k_1, \dots, k_n}^n : \mathbb{O}(n) \times (\mathbb{O}(k_1) \times \dots \times \mathbb{O}(k_n)) \longrightarrow \mathbb{O}(k_1 + \dots + k_n),$$

satisfying various axioms. See [M] for the list of axioms. The object \mathbb{O} encodes the n -ary operations, and the axioms encode the natural requirements that such operations would have.

EXAMPLE 17. The endomorphism operad is the prototypical example of an operad. Let X be a pointed object in \mathcal{C} . The operad $\text{End}(X)$ is defined by the expression $\mathbb{E}\text{nd}(X)(n) = \text{Hom}_{\mathcal{C}_*}(X^n, X)$, and the structure maps are given by composition.

EXAMPLE 18. In $\mathcal{C} = \mathcal{T}\text{op}$ there is the so-called little n -cubes operad, \mathbb{C}_n , defined so that $\mathbb{C}_n(k)$ is the spaces of framed embeddings of $\coprod^k I^n$ into I^n . The maps come from composing embeddings. It also makes sense to set $n = \infty$, in which case, we will obtain \mathbb{C}_∞ , the colimit of the operads, \mathbb{C}_n , which embed into one another by crossing the embeddings with I .

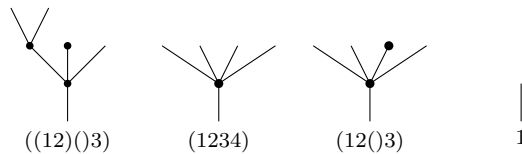
EXAMPLE 19. Let \mathbb{I} denote the embedding operad for \mathbb{R}^∞ , which we will need in the next chapter. It is defined, so that $\mathbb{I}(n)$ is the space of smooth embeddings of $\coprod^n \mathbb{R}^\infty$ into \mathbb{R}^∞ .

An \mathbb{O} -algebra structure $A \in \mathcal{P}$ is a collection of Σ_n -invariant morphisms $\psi_n : \mathbb{O}(n) \times A^n \rightarrow A$, for each non-negative integer n , where Σ_n acts on $\mathbb{O}(n) \times A^n$ diagonally. These morphisms satisfy addition set of axioms, which also can be found in the reference above. However, one can reformulate the above definition, by saying that an \mathbb{O} -algebra structure on A in \mathcal{P} is a morphism of operads $\mathbb{O} \rightarrow \text{End}(A)$. An operad in $\mathcal{T}\text{op}$ will be called a *topological operad*, and an operad in Set_Δ° will be called a *simplicial operad*. A topological or simplicial operad, \mathbb{E} , is called an E_∞ -operad if the $\mathbb{E}(n)$ are contractible and the action of Σ_n on $\mathbb{E}(n)$ is free.

We would like to demonstrate that an algebra over a topological E_∞ -operad gives rise to a commutative algebra objects in ∞ -categories. Suppose that \mathcal{C} is a category tensored and cotensored over topological spaces, which also admits finite products. Let \mathbb{E} be a topological E_∞ -operad and X be an object of \mathcal{C} . Since \mathcal{C} is tensored over $\mathcal{T}\text{op}$, then we can make sense out of the action of \mathbb{E} on X . In other words, it makes sense to write down maps $\mathbb{E}(n) \times X^n \rightarrow X$. We can convert \mathcal{C} into an ∞ -category by applying the nerve construction, $\mathbb{N}(\mathcal{C})$. The resulting category has products. We would like to construct a functor $\mathbb{N}(\mathcal{Y}) \rightarrow \mathbb{N}(\mathcal{C})$, so that the image of $\langle 1 \rangle$ is equivalent to X , and that of $\langle n \rangle$ is equivalent to X^n as in the definition. Doing this directly is difficult, and we will perform a roundabout construction. For that we need some definitions.

We define the category of dendroids following [HHM], Ω . The objects of this category are finite planar trees with distinguished root. Not all the edges have two vertices. The edges that have only a single vertex are called *external*, and the ones that have two vertices are called *internal*. The root is an external edge. The rest of the external edges are not output edges are called *leaves*. The root defines a direction for the tree (towards the root). Then if a vertex belongs to an edge, then we can make sense of what it means to be either an input edge or an output edge. The number of input edges of vertex is called the *valence* of the vertex. A vertex of 0 valence is called a *stump*. To define morphisms we define S_T to be the union of all full subtrees and stumps of T . This set is closed under intersections. We define a map from W to T to be an intersection preserving map from S_W to S_T . colored check this!

Here are a few examples



Notice that below each tree we wrote down a parenthesized word corresponding to the tree. The numbers in the expression label the leaves of the tree (from left to right). For each vertex we put parenthesis grouping the incoming edges and the resulting parenthesized expression is thought of as the outgoing edge.

Recall that Υ^{op} is the category (unpointed) finite sets with morphisms being the partially defined maps between these sets. There is a functor $L : \Omega^{\text{op}} \rightarrow \Upsilon$, which sends a tree T to the set of its leaves, L_T . Given a morphism $\gamma : W \rightarrow T$ we let $L_\gamma : L_T \rightarrow L_W$ be defined on leaves $l \in L_T$, such that there is a $\ell \in L_T$, such that l is in the full subtree defined by $\gamma(\ell)$ and $L_\gamma(l) = \ell$. Let us denote by S the set of all morphisms in Ω^{op} that get sent to isomorphism under L .

PROPOSITION 12. *The functor L exhibits Υ as $S^{-1}(\Omega^{\text{op}})$.*

Proof. The functor L inverts the morphisms S by definition. Now suppose that we are given a functor $F : \Omega^{\text{op}} \rightarrow \mathcal{C}$, such that all the morphisms in S get sent to isomorphisms. We can construct a functor $\tilde{F} : \Upsilon \rightarrow \mathcal{C}$ as follows. The value of \tilde{F} on $\langle n \rangle$ is $F((12 \dots n))$. Let $f : \langle n \rangle \rightarrow \langle m \rangle$ be a morphism in Υ . There is a zigzag $(12 \dots m) \rightarrow T_f = ((f^{-1}(1))(f^{-1}(2)) \dots (f^{-1}(m))f^{-1}(*)) \leftarrow (12 \dots n)$, where the second arrow is in S . If we apply the functor L , the resulting zigzag $\langle n \rangle \leftarrow L(T_f) \rightarrow \langle m \rangle$ commutes with f . Furthermore, T_f is initial among such zigzags. We define $\tilde{F}(f)$ to be the composite of the inverse of $T_f \leftarrow (12 \dots n)$ and $(12 \dots m) \rightarrow T_f$. One can show using the universality of T_f that $\tilde{F} \circ L$ is naturally isomorphic to F . \square

PROPOSITION 13. *Let \mathcal{C} is an ordinary category and let S be a subset of morphisms of \mathcal{C} . Then there is an equivalence $\mathbb{N}(S^{-1}\mathcal{C}) = S^{-1}\mathbb{N}(\mathcal{C})$.*

Proof. Using the formalism of marked simplicial sets in Chap. 3 of [Lu1], we consider $(\mathbb{N}(\mathcal{C}), S)$. There is a sequence of marked anodyne maps such that the composite is $(\mathbb{N}(\mathcal{C}), S) \rightarrow S^{-1}\mathbb{N}(\mathcal{C})^\sharp$, such that at each stage the ∞ -category is a nerve of an ordinary category. \square

COROLLARY 3. *There is an equivalence $\mathbb{N}(L)$ exhibits $\mathbb{N}(\Upsilon)$ as $S^{-1}\mathbb{N}(\Omega^{\text{op}})$.*

Now let us go back to \mathcal{C} a category tensored and cotensored over $\mathcal{J}\text{op}$, and X an \mathbb{E} -algebra. We can construct a functor $X^\otimes : \Omega^{\text{op}} \rightarrow \mathcal{C}$ as follows. Let T be a tree, and V_T be the set of its vertices and L_T be the set of its leaves. We define

$$X(T) = \prod_{L_T} X \times \prod_{v \in V_T} \mathbb{E}(\text{val}(v)),$$

where $\text{val}(v)$ is the valence of the vertex. The operad structure maps allow one to define the what X^\otimes does on morphisms. Let us explain on example. Let us consider the tree map $(1234) \rightarrow ((12)())(34)$ given by parenthesizing (12) and sending 3 to $()$ and 4 to (34) . The map from $\mathbb{E}(3) \times (\mathbb{E}(2) \times X^2) \times (\mathbb{E}(0) \times *) \times (\mathbb{E}(2) \times X^2)$ to $\mathbb{E}(4) \times X^4$ is formed by sending $\mathbb{E}(0) \times *$ to the basepoint of X , $\mathbb{E}(2) \times X^2$ to composite X , and $\mathbb{E}(3) \times (\mathbb{E}(2) \times \mathbb{E}(0) \times \mathbb{E}(0)) \rightarrow \mathbb{E}(3) \times (\mathbb{E}(2) \times \mathbb{E}(1) \times \mathbb{E}(1)) \rightarrow \mathbb{E}(4)$.

Note that X^\otimes sends the maps that induce isomorphisms on leaves to equivalence in \mathcal{C} . The same thing will happen if we apply the nerve construction $N(X^\otimes) : N(\Omega^{\text{op}}) \rightarrow N(\mathcal{C})$. Because of this we get a functor $N(\Upsilon) \rightarrow N(\mathcal{C})$, which is, in fact, a symmetric monoidal object in $N(\mathcal{C})$. Of course, this map is unique only up to a contractible space of choices. That explains the difficulty of the direct construction. We will write any one of the maps obtained with this procedure as $N(X^\otimes)$.

We would like to introduce a couple of construction on dendroidal sets that will help us in defining the symmetric monoidal structure on Bord_n . The problem of going yet again in a roundabout fashion in defining such structures is the fact that Bord_n does not have a strict action of an E_∞ -operad on it. The problem is that reparametrization of cuts. We will discuss this in detail in section 2.1.2.

2 Integration Pairing

2.1 Introduction

In this introduction we will give a heuristic description of the notions of bordism categories, ordinary and extended topological field theories and of various other constructions. We begin by discussing the notion of topological field theories and their higher categorical generalizations. The discussion is mostly borrowed from [Lu4], so the readers familiar with the topic may skip it. Then we proceed to describe a construction of topological field theories in dimension 1, which comes from the work of Kostant and Weil, and in dimension 2, following Brylinski and McLaughlin, [BM1] and [BM2]. After that we will describe the generalization of these constructions by Lipsky. We will conclude by stating the main theorem this paper.

WARNING. The definitions given in this section should not be assumed for the rest of the work. Here we mostly present some aspects of the classical theory of topological field theories—the ones dealing with ordinary categories. The rest of the paper will concentrate on extended topological field theories, which are the higher categorical analogues of ordinary topological field theories.

Topological (quantum) field theories were first defined by Atiyah in [A]. The definition was inspired by Segal's definition of conformal field theories, [S]. They are important to mathematicians because they can be used to capture invariants of manifolds. To get a flavor of how this works we need to give a definition of topological field theories. In order to do that we first need to give the definitions of bordism categories.

For each integer $n \geq 1$, there is an n -dimensional bordism category denoted as $\text{Bord}_n^{\text{or}}$, and defined as follows. The objects of the category are the $(n-1)$ -dimensional oriented compact manifolds without boundary. A morphism from oriented manifold Σ_1 to oriented manifold Σ_2 , is an equivalence class of compact oriented n -dimensional manifolds M along with an orientation preserving diffeomorphism $\partial M \simeq \overline{\Sigma_1} \amalg \Sigma_2$, where $\overline{\Sigma}$ denotes the same manifold as Σ , except with the opposite orientation, and the orientation on the boundary is given by contracting with an inward point normal vector field. Two such oriented manifolds M and N are identified if they are diffeomorphic (in the oriented sense), so that the induced diffeomorphism on the boundaries $\partial M \simeq \partial N$ commutes with the diffeomorphisms to $\overline{\Sigma_1} \amalg \Sigma_2$. Whenever specifying a morphism we will simply write the manifold with boundary: we will omit writing the specified diffeomorphism on the boundary and the fact that it is really a representative of an equivalence class. Given morphisms $M : \Sigma_1 \longrightarrow \Sigma_2$ and $N : \Sigma_2 \longrightarrow \Sigma_3$, the composition, $N \circ M$, is given by the manifold $M \cup_{\Sigma_2} N$.² This composition is associative. The identity on Σ is given by the cylinder $\Sigma \times I$, where I denotes the closed unit interval. Finally, we note that the disjoint union, \amalg , specifies a symmetric monoidal product on $\text{Bord}_n^{\text{or}}$, and the empty

²We are omitting a technical point on specification of a smooth structure on the glued space. The simplest strategy to overcome this issue would be taking a smooth collar for Σ_2 in both manifold and then defining $M \cup_{\Sigma_2} N$ as the pushout of $M \longleftarrow \Sigma_2 \times (0, 1) \longrightarrow N$. After that, one has to show that different choices of the collars give manifolds representing the same morphism. Later we will give a more enhanced definition, which will take care of this issue.

manifold, \emptyset , is the unit of this product.

REMARK 12. As the reader might have already guessed the superscript “or” on top of $\text{Bord}_n^{\text{or}}$, stands for “oriented”. There are topological field theories that implement other structures on manifolds, such as spin structure, framing. We will label these categories $\text{Bord}_n^{\text{spin}}$, $\text{Bord}_n^{\text{fr}}$, respectively. In addition to this, we may put no structure at all, in which case we obtain the unoriented bordism category, Bord_n .

REMARK 13. One can define a general notion of the tangential structure. Let $X \rightarrow \text{BO}(n)$ be a fibration classifying a vector bundle ζ on X . A (tangential) (X, ζ) -structure on a manifold M of dimension $m \leq n$, is a choice of the dotted arrow in the diagram

$$\begin{array}{ccc} & & X \\ & \nearrow \text{dotted} & \downarrow \\ M & \longrightarrow & \text{BO}(n) \end{array}$$

where the horizontal map classifies $TM \oplus (n - m)$. Note that an (X, ζ) -structure on M induces an (X, ζ) -structure on ∂M . This comes from the fact that normal bundle of ∂M is trivial in M , since that means that the composite $\partial M \rightarrow M \rightarrow \text{BO}(n)$ classifies $T(\partial M) \oplus (n - m + 1)$. We can define $\text{Bord}_n^{(X, \zeta)}$ by just considering manifolds with (X, ζ) -structure. This definition generalizes the bordism categories in the previous remark. The spaces $\text{BSO}(n)$, $\text{BSpin}(n)$ and $\text{EO}(n)$ come equipped with maps to $\text{BO}(n)$ ³, which give a vector bundle of rank n over each one of these spaces—we label these vector bundles by ζ_n^{or} , ζ_n^{spin} and ζ_n^{fr} , respectively. Then a $(\text{BSO}(n), \zeta_n^{\text{or}})$ -structure is equivalent to orientation, $(\text{BSpin}(n), \zeta_n^{\text{spin}})$ -structure is equivalent to a spin structure, and $(\text{EO}(n), \zeta_n^{\text{fr}})$ -structure is equivalent to framing.

REMARK 14. These categories were implicitly of central interest to topologists and geometers long before topological field theories came around. This interest dates at least as far back as Pontryagin’s work on framed bordisms. Thom in his thesis, [T], showed that $\pi_0(|\text{Bord}_n|)$ is isomorphic to $\pi_{n-1}(\text{MO})$. Here MO denotes the unoriented bordism spectrum, and $|\mathcal{C}|$ denotes the geometric realization of the nerve of \mathcal{C} . Note that $\pi_0(|\text{Bord}_n|)$ has a group structure coming from the disjoint union. There is a map of form $\pi_0(\text{Bord}_{n+1}) \times \pi_0(\text{Bord}_{m+1}) \rightarrow \pi_0(\text{Bord}_{n+m+1})$ that takes a pair of manifolds (Σ, Γ) sends it to $\Sigma \times \Gamma$. This gives a ring structure on the graded ring $\bigoplus_{n=0}^{\infty} \pi_0(|\text{Bord}_{n+1}|)$, and this ring structure coincides with that of $\pi_*(\text{MO})$.⁴

Even more impressive was the realization of Pontryagin, that $\pi_0(|\text{Bord}_n^{\text{fr}}|)$ is isomorphic to $\pi_{n-1}(\text{S})$, where S denotes the sphere spectrum. This implies that the stable homotopy groups of the sphere spectrum are in bijective correspondence with the cobordism classes of framed manifolds. This is the beginning of the surgery theory, which is an essential tool in the theory of manifolds in high dimensions.

Now we are ready to phrase Atiyah’s definition of topological field theory.

³In order to make these maps fibrations, we need to choose appropriate models for these classifying spaces.

⁴The spectra MO , MSO and MSpin are all E_{∞} -ring spectra.

DEFINITION 16. An n -dimensional topological field theory is a symmetric monoidal functor from the bordism category $(\text{Bord}_n^{\text{or}}, \coprod)$ to a symmetric monoidal category (\mathcal{C}, \otimes) .

One can think of topological field theories as invariants of $\text{Bord}_n^{\text{or}}$, that capture essential features of the symmetric monoidal structure of this category. The slogan is that: we do not understand $\text{Bord}_n^{\text{or}}$, and we would like to understand it by understanding the maps out of it into something that we understand. Examples of target categories that are understandable are Mod_R , where R is a commutative ring, and \otimes is the regular tensor product over R . Our hope is that these functors admit some tractable algebraic description, because, in fact, we hope that $\text{Bord}_n^{\text{or}}$ itself can be given such a description. Someone may ask, why hope for such a thing? The reason stems from low dimensional examples.

EXAMPLE 20. We begin with a description of $\text{Bord}_1^{\text{or}}$. The objects of this category are the 0-dimensional manifolds with an orientation. An oriented point comes in two versions—with positive orientation, which we label as $+$, and with negative orientation, which we label as $-$. Any other compact oriented 0-manifold can be formed as a finite disjoint union of these points, i.e. $+$ and $-$ generate the objects of $\text{Bord}_1^{\text{or}}$ under \coprod . A similar statement applies to morphisms. There are two connected manifolds of dimension 1—the interval I and the circle S^1 . The interval I can be thought of as the identity morphism on $+$ or the identity morphism on $-$, since we can see that $\partial I \simeq \overline{+} \coprod + \simeq \overline{-} \coprod -$. However, one other important thing is that I can be thought of as a morphism from $+\coprod-$ to \emptyset or from \emptyset to $+\coprod-$. This means that we get a morphism $\text{cov} : \emptyset \rightarrow +\coprod-$ and $\text{ev} : +\coprod- \rightarrow \emptyset$. Note that $\partial S^1 = \emptyset$, so it can only represent a morphism $S^1 : \emptyset \rightarrow \emptyset$. Also observe that $S^1 = \text{ev} \circ \text{cov}$. One can also show that

$$(\mathbb{1}_+ \coprod \text{ev}) \circ (\text{cov} \coprod \mathbb{1}_+) = \mathbb{1}_+ \text{ and } (\text{ev} \coprod \mathbb{1}_-) \circ (\mathbb{1}_- \coprod \text{cov}) = \mathbb{1}_-.$$

These two identities are the categorical formulation of duality. Here is a definition of duality in a general setting.

DEFINITION 17. Let (\mathcal{C}, \otimes) be a symmetric monoidal category with unit S . An object A^\vee is dual to A if there exist maps $\text{cov} : S \rightarrow A \otimes A^\vee$ and $\text{ev} : A \otimes A^\vee \rightarrow S$, such that

$$(\mathbb{1}_A \otimes \text{ev}) \circ (\text{cov} \otimes \mathbb{1}_A) = \mathbb{1}_A \text{ and } (\text{ev} \otimes \mathbb{1}_{A^\vee}) \circ (\mathbb{1}_{A^\vee} \otimes \text{cov}) = \mathbb{1}_{A^\vee}.$$

If A admits a dual, then we call it dualizable. The category \mathcal{C} is said to be with duals if every object in it is dualizable.

Note that if the dual exists it is unique up to natural isomorphism. If A is dualizable, then so is A^\vee , and $(A^\vee)^\vee \simeq A$ in a natural way. If $f : A \rightarrow B$ is a morphism between dualizable objects, then we can naturally

define $f^\vee : B^\vee \rightarrow A^\vee$ as the following composite:

$$B^\vee \rightarrow A^\vee \otimes A \otimes B^\vee \rightarrow A^\vee \otimes B \otimes B^\vee \rightarrow A^\vee.$$

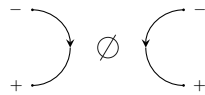
We will call f^\vee the *dual* of f . In fact, if \mathcal{C} is a category with duals then $-^\vee : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ is a functor.

If we have a topological field theory $Z : \text{Bord}_1^{\text{or}} \rightarrow \mathcal{C}$, then in \mathcal{C} the objects $Z(+)$ and $Z(-)$ would be dual to each other. In fact, the essential datum for this functor is the value of Z on $+$: $Z(-)$ and the morphisms are determined automatically up to canonical isomorphisms. The object $Z(+)$ must be dualizable. Therefore, giving a 1-dimensional topological field theory over \mathcal{C} is equivalent to giving a dualizable object in \mathcal{C} . Being dualizable puts a restriction on the values of Z on $+$. For instance, if \mathcal{C} is Mod_k for some field k , then the dualizable objects are the finite dimensional vector spaces. The morphism $Z(S^1) : k \rightarrow k$ is the multiplication by the dimension of this vector space.

REMARK 15. An n -dimensional topological field theory gives an invariant for n -dimensional manifolds. This is based on the observation that an n -dimensional manifold N can be regarded as an endomorphism of \emptyset in $\text{Bord}_n^{\text{or}}$. Therefore, $Z(N)$ is an element of $\text{End}(S)$, where S is the unit for \otimes in \mathcal{C} . In fact, the map is multiplicative, in the sense, that $Z(N \amalg M) = Z(N) \otimes Z(M)$.

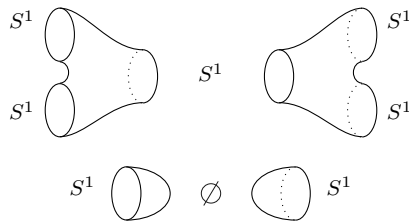
EXAMPLE 21. Omitting the details, if $Z : \text{Bord}_2^{\text{or}} \rightarrow \mathcal{C}$ is a topological field theory, then $Z(S^1)$ is a commutative Frobenius algebra in \mathcal{C} . A commutative Frobenius algebra in a symmetric monoidal category \mathcal{C} is a commutative algebra A , which is dualizable as an object of \mathcal{C} , along with an isomorphism $A \simeq A^\vee$. If $\mathcal{C} = \text{Mod}_k$, then Frobenius algebras could be characterized as finite dimensional k -algebras A with a trace map $\text{tr} : A \rightarrow k$, such that $A \otimes A \rightarrow A \rightarrow k$ defines a non-degenerate pairing on A . The element defined by the composite $k \rightarrow A \otimes A \rightarrow A$ is called the *Euler element* and one can show that $Z(S_g) = \text{tr}(\alpha^g)$, where S_g denotes the compact oriented surface of genus g .

The procedure of analyzing the bordism category is done in two steps: first—we identify some generators of objects and morphisms under the disjoint union and composition, second—we identify the relations that these generators satisfy. These generators are usually the handles that one gets from Morse theory. In the case of $n = 1$, the objects are generated by $+$ and $-$ under the tensor product, and the morphisms are generated by the *ev* and *cov* morphisms. Pictorially it looks like:



The generators are dual to one another. Any other bordism can be constructed out of these.

In the case $n = 2$, we have slightly richer collection of generators, we depict them below.



The discussion on commutative Frobenius algebras can be understood in terms of calculus of cutting and gluing of these generating handles. However, this type of an analysis becomes quite cumbersome for $n \geq 3$. The central difficulty is the fact that the axioms of topological field theories only allow one to cut and paste along codimension 1 submanifolds. We would be in a significantly better situation if we could cut along manifolds of higher codimension. This could be accomplished if we allow the consideration of manifolds with corners. Unfortunately, problems of formulation start piling up since the language of ordinary category theory is not adequate in this context. We need higher category theory to handle these types of considerations.

The language of higher category theory is not only adequate for the aforementioned task, but it also gives the bordism categories a universal characterization. Heuristically speaking, a higher category is a category with a hierarchy of morphisms, meaning, for each $n \geq 1$ we have a notion of n -morphism. The 1-morphisms should be thought of as ordinary morphisms and n -morphisms as morphisms between $(n - 1)$ -morphisms. There are, of course, compatibility conditions. Ordinary categories fall into this context. A stranger statement is that topological spaces are also higher categories! Without elaborating any further, let us just say that higher category theory is inherently topological in nature, and that the tools of algebraic topology do not only formulate precisely what higher categories are, but also give possibility of understanding them. We will review higher categories in the next section.

Now we give a very rough description of how we can enhance $\text{Bord}_n^{\text{or}}$ to a higher category. Let us write these higher categories with boldface letters, e.g. $\mathbf{Bord}_n^{\text{or}}$. The set of objects of this category are just finite sets of oriented points, i.e. objects of $\text{Bord}_1^{\text{or}}$. The 1-morphisms are the 1-dimensional manifolds with boundary, the 2-morphisms are the 2-dimensional manifolds with corners, and so on, until we get to dimension $n + 1$. Let us halt this description for a moment. This definition actually enhances our ability to cut the n -manifold into pieces of arbitrary codimension. Now if we are given an enhanced version of topological field theory, i.e. a symmetric monoidal functor from $\mathbf{Bord}_n^{\text{or}}$ into some higher category \mathcal{C} , which is also symmetric monoidal, we can hope for a more tractable combinatorial procedure that computes the value of this theory on closed n -manifolds.

The pending question is: what happens above dimension n ? One of the central philosophies of higher category theory is that one should not only know that two objects are isomorphic but also *why* they are isomorphic. That means that we are not allowed to simply state that two object are isomorphic: we need to also give an isomorphism. We have already transgressed in applying this. When we defined a morphism

in the ordinary bordism category, we stated that we consider *equivalence classes* of manifolds relative to their boundary. What we should do in the higher categorical setting is to incorporate all the possible ways that two bordisms are diffeomorphic into the structure of the higher category. With this in mind we define $(n+1)$ -morphisms to be the isotopies relative to their boundary, then the $(n+2)$ -morphisms as the isotopies between isotopies, etc. The effect of such an extension is that the higher category contains information on the diffeomorphism groups of manifolds.

It was mentioned that the higher bordism categories admit universal descriptions. We will phrase the result for framed bordism. Notice from the heuristic definition that we gave, that the m -morphisms in $\mathbf{Bord}_n^{\text{fr}}$ for $m > n$ are invertible. This type of description is characteristic of (∞, n) -categories. We will give a precise definition in the next section. $\mathbf{Bord}_n^{\text{fr}}$ is a symmetric monoidal (∞, n) -category, as a consequence of existence of the disjoint union.

DEFINITION 18. An extended n -dimensional framed topological field theory is a symmetric monoidal functor from $\mathbf{Bord}_n^{\text{fr}}$ to a symmetric monoidal (∞, n) -category \mathcal{C} .

As in the case of low dimensional examples in the classical case, extended topological field theories land in dualizable object of the target. One can formulate a higher categorical definition of dualizability. As we do not need a discussion of dualizability in this paper, we will omit the details and refer the reader to [Lu4]. The following theorem gives a universal description to $\mathbf{Bord}_n^{\text{fr}}$.

THEOREM 8. Let \mathcal{C} be a symmetric monoidal (∞, n) -category with duals, and let \mathcal{C}^\sim denote the maximal ∞ -subgroupoid in \mathcal{C} . Then the evaluation-at-the-point map

$$\text{Fun}^\otimes(\mathbf{Bord}_n^{\text{fr}}, \mathcal{C}) \longrightarrow \mathcal{C}^\sim$$

is an equivalence.

One can interpret this theorem as follows: giving an extend n -dimensional framed topological field theory with target \mathcal{C} is equivalent to giving a dualizable object in \mathcal{C} . Specifically, it is the image of the positively framed point in $\mathbf{Bord}_n^{\text{fr}}$. This is the celebrated Baez-Dolan cobordism hypothesis, which was described in [BD]. In [Lu4], Lurie formulated the theorem in the form above and also sketched the proof of it. The theorem can be rephrased as saying that $\mathbf{Bord}_n^{\text{fr}}$ is the free symmetric monoidal (∞, n) -category with duals generated by a point. For details on how to state this theorem and all the underlying notions precisely, we again refer reader to [Lu4]. Furthermore, there are other versions of the cobordism hypothesis for structures other than framing. Even though, the cobordism hypothesis demonstrates that from any dualizable object one can produce a topological field theory, it is not obvious how one ought to construct one. In certain situation one can give explicit constructions. For instance, there is the notion of topological chiral homology, which can be

used to give an explicit construction of an n -dimensional topological field theory taking values in \mathbb{E}_n -algebras over some well-behaved $(\infty, 1)$ -category. When we also incorporate a background manifold for the bordisms, we can use the differential geometric structure of the manifold to construct topological field theories over the manifold.

To describe the construction heuristically, let us go back to ordinary categories. We need to define the notion of a topological field theory over a manifold. Suppose that X is a manifold. We can then produce a category of oriented bordisms over X , $\text{Bord}_n^{\text{or}}(X)$. It is exactly the same definition as that of $\text{Bord}_n^{\text{or}}$, except all the manifolds and bordisms come equipped with a smooth map to X . One should think of X as a background manifold for bordisms. Using a similar construction, we can define the categories $\text{Bord}_n^{\text{spin}}(X)$, $\text{Bord}_n^{\text{fr}}(X)$, $\text{Bord}_n(X)$.

This category is “almost” equivalent to the bordism category of manifolds with $(X \times \text{BSO}(n), (0) \times \zeta_n^{\text{or}})$ -structure. The reason why it is “almost” the same is because the maps defining the structure, e.g. $M \rightarrow X \times \text{BSO}(n)$, should be chosen so that the projections onto X are smooth. With this point of view we can see that we do not even need X to be a manifold. However, we will assume that it is for the remainder of the paper.

EXAMPLE 22. The following example of 1-dimensional topological field theory comes from Kostant-Weil theory. Suppose that we are given a hermitian line bundle $\mathcal{L} \rightarrow X$, and a hermitian connection on it $\nabla \in \Omega^1(X) \otimes \text{End}(\mathcal{L})$. For each positively oriented point of X , $x : + \rightarrow X$, set $Z_{(\mathcal{L}, \nabla)}(x)$ to equal to \mathcal{L}_x , and for each negatively oriented point of X , $x : - \rightarrow X$, set $Z_{(\mathcal{L}, \nabla)}(x)$ to equal to \mathcal{L}_x^\vee . If we are given a smooth path $f : I \rightarrow M$, we can use the parallel transport provided by the connection ∇ to define a linear isomorphism $Z_{(\mathcal{L}, \nabla)}(f) : \mathcal{L}_{f(0)} \rightarrow \mathcal{L}_{f(1)}$. One can check that this prescription can be extended to a topological field theory with target category of complex lines and isomorphisms, or equivalently, $U(1)$ -torsors,

$$Z_{(\mathcal{L}, \nabla)} : \text{Bord}_1^{\text{or}}(X) \rightarrow \mathcal{Tors}_{U(1)}.$$

Note that if we are given a loop $\alpha : S^1 \rightarrow X$, then $Z(\alpha) \in U(1)$ coincides with the holonomy of α .

The same construction can be done differently using the language of stacks. Let Man denote the Grothendieck site of smooth manifolds and smooth maps. Recall that if one is given a sheaf of groups on Man , then one can form the classifying stack BG , which classifies the principal G -bundles and their isomorphism. One also can form the stack EG , which classifies the principal G -bundles along with a trivialization of the bundle. In this case we are interested in the group $U(1)$, the sheaf represented by non-zero complex numbers under multiplication, and \mathcal{A}^1 the sheaf of 1-forms. These groups are connected by a homomorphism $d \log : U(1) \rightarrow \mathcal{A}^1$, which classifies the Maurer-Cartan form on $U(1)$ using an identification of $\mathfrak{u}(1) \simeq \mathbb{R}$ from here on fixed. One can show that the homotopy (or categorical) fiber of the stack map $B(d \log) : BU(1) \rightarrow B\mathcal{A}^1$

is equivalent to the stack classifying the hermitian line bundles with connections. We will write this stack as $F_{d\log}$ to emphasize this equivalence.

REMARK 16. The Maurer-Cartan form is defined for any Lie group. Thus, we have a general morphism $d\log : G \rightarrow \mathcal{A}_{\mathfrak{g}}^1$, where $\mathcal{A}_{\mathfrak{g}}^1$ is the sheaf of forms taking values in \mathfrak{g} —the Lie algebra of G . If G is connected and not abelian, then this map is not a homomorphism of sheaves of groups, so we will not be able to apply the classifying stack construction. We do know that there is stack of principal G -bundles and connections, which we will write as $F_{d\log}$. We have a fiber sequence $\mathcal{A}_{\mathfrak{g}}^1 \rightarrow F_{d\log} \rightarrow BG$. The fiber of the map $\mathcal{A}_{\mathfrak{g}}^1 \rightarrow F_{d\log}$ is G and the map $G \rightarrow \mathcal{A}_{\mathfrak{g}}^1$ is nothing but the Maurer-Cartan form.

Let Γ be a 1-dimensional compact oriented manifold. Then there is a group homomorphism

$$\exp\left(\int_{\Gamma}\right) : (\mathcal{A}^1)^{\Gamma} \rightarrow U(1),$$

which when evaluated on a manifold X sends a form on $X \times \Gamma$ to the function which maps $x \in X$ to the complex number $\exp(\int_{\{x\} \times \Gamma} \omega)$. One can write Stoke's theorem in the following diagrammatic form:

$$\begin{array}{ccc} U(1)^{\Gamma} & \longrightarrow & U(1)^{\partial\Gamma} \\ (d\log)^{\Gamma} \downarrow & & \downarrow \\ (\mathcal{A}^1)^{\Gamma} & \xrightarrow{\exp(f_{\Gamma})} & U(1) \end{array}$$

All the maps are group homomorphisms. Something interesting happens, when we apply the classifying stack construction—we get the following diagram

$$\begin{array}{ccccc} (F_{d\log})^{\Gamma} & \cdots\cdots\cdots & & & EU(1) \\ \downarrow & & & & \downarrow \\ (BU(1))^{\Gamma} \simeq B(U(1)^{\Gamma}) & \longrightarrow & B(U(1)^{\partial\Gamma}) & \longrightarrow & BU(1) \\ B(d\log)^{\Gamma} \downarrow & & & & \parallel \\ (B\mathcal{A}^1)^{\Gamma} \simeq B((\mathcal{A}^1)^{\Gamma}) & \xrightarrow{B\exp(f_{\Gamma})} & & & BU(1) \end{array}$$

The dotted arrow is the naturally induced map between the homotopy fibers. Suppose that $X \rightarrow F_{d\log}$ classifies the line bundle \mathcal{L} with connection ∇ . The composition $X^{\Gamma} \rightarrow (F_{d\log})^{\Gamma} \rightarrow EU(1)$, gives a trivialization of the line bundles, which is signed tensor product of line bundles over the points of Γ obtained by pulling back \mathcal{L} . This is precisely the data for a 1-dimensional topological field theory. Notice that if Γ has no boundary, then the map $(F_{d\log})^{\Gamma} \rightarrow EU(1)$ trivializes the trivial line bundle, so can be naturally lifted to the map $H : (F_{d\log})^{\Gamma} \rightarrow U(1)$ called the *holonomy* of Γ .

REMARK 17. The equivalences $B(U(1)^{\Gamma}) \simeq (BU(1))^{\Gamma}$ and $(B\mathcal{A}^1)^{\Gamma} \simeq B((\mathcal{A}^1)^{\Gamma})$ need some justification.

The first equivalence follows from the fact that the only principal $U(1)$ -bundle on Γ is the trivial one; the second equivalence follows from the fact that \mathcal{A}^1 is a fine sheaf. In more detail, there is morphism $B(U(1)^\Gamma) \rightarrow (BU(1))^\Gamma$. Indeed giving a morphism $X \rightarrow B(U(1)^\Gamma)$ is equivalent to giving a covering of $\{U_\alpha\}$ of X and a collection of smooth maps $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \Gamma \rightarrow U(1)$ satisfying the cocycle condition. This gives a principal $U(1)$ -bundle on $X \times \Gamma$, i.e. a map $X \times \Gamma \rightarrow BU(1)$ or a map $X \rightarrow (BU(1))^\Gamma$. Thus, we obtain the desired map $B(U(1)^\Gamma) \rightarrow (BU(1))^\Gamma$. To see that this map is an equivalence note that a principal $U(1)$ -bundle on $X \times \Gamma$ can be trivialized using the cover $\{U_\alpha \times \Gamma\}$.

As in the previous case there is a morphism $B((\mathcal{A}^1)^\Gamma) \rightarrow (BA^1)^\Gamma$. Note that a morphism $X \rightarrow B((\mathcal{A}^1)^\Gamma)$ is equivalent to picking a covering $\{U_\alpha\}$ of X and specifying 1-forms $\omega_{\alpha\beta} \in (U_\alpha \cap U_\beta) \times \Gamma$. A morphism $X \rightarrow (BA^1)^\Gamma$ is equivalent to giving a principal \mathcal{A}^1 -bundle on $X \times \Gamma$, i.e. a collection of 1-forms $\nu_{\alpha\beta} \in V_\alpha \cap V_\beta$ for $\{V_\alpha\}$ a covering of $X \times \Gamma$. The main observation, which results in equivalence is that all principal \mathcal{A}^1 -bundle are trivial on $U_\alpha \times \Gamma$. A stronger statement is true, all principal \mathcal{A}^1 -bundles are trivial, though the morphisms are not trivial.

In [BM1], Brylinski and McLaughlin extend the Kostant-Weil theory to produce 2-dimensional topological field theories. The idea is to use gerbes bound by $U(1)$ as the higher analogues of the hermitian line bundles. These types of gerbes can be given a connective structure and curving. We will present their construction from a point of view different from the original paper.

There is a curvature homomorphism given by the following diagram

$$\begin{array}{ccc}
 F_{d\log} & \xrightarrow{\quad K \quad} & \mathcal{A}^2 \\
 \downarrow & & \downarrow \\
 BU(1) & \longrightarrow & * \\
 \downarrow \text{B}(d\log) & & \downarrow \\
 BA^1 & \xrightarrow{\quad Bd \quad} & BA^2
 \end{array}$$

The map K is the natural map between the fibers and is called the *curvature*. This map is symmetric monoidal, in that it produces a symmetric monoidal functor when evaluated on any object of $\mathcal{M}\text{an}$. This implies that we can apply the bar construction to get 2-stack map $BK : B(F_{d\log}) \rightarrow BA^2$, whose fiber will be denoted by F_K . The 2-stack F_K is the stack classifying all the gerbes bounded by $U(1)$, along with connective structure and curving.

Given a 1-dimensional closed compact curve Γ , we can consider the map $(F_K)^\Gamma \rightarrow B(F_{d\log})^\Gamma \rightarrow BU(1)$, where the second map is the bar construction on H . This map is referred to as the *holonomy line bundle*. Using a similar analysis as in the previous example, we can construct a map $(F_K)^\Sigma \rightarrow EU(1)$ for any compact 2-dimensional manifold Σ , which trivializes the holonomy line bundle on the boundary of Σ .

In [Li], Lipsky generalizes this procedure for all n . We denote by \mathcal{D}_n^* , the following complex of sheaves on X

$$U(1) \longrightarrow \mathcal{A}^1 \longrightarrow \dots \longrightarrow \mathcal{A}^n,$$

where $U(1)$ denotes the sheaf of smooth maps into $U(1)$ and \mathcal{A}^k is the sheaf of complex valued differential k -forms. All the maps are the deRham differentials except for the first map, which is the $d \log$ map. The hypercohomology of this complex bears the name of smooth Deligne cohomology. The way one can compute it is by first taking a good cover \mathcal{U} , then forming the totalization of the Deligne-Čech double-complex with respect to the cover, $C^*(\mathcal{U}, \mathcal{D}_n^*)$, and then computing the cohomology of the double complex. Complex line bundles with connection arise as 1-cocycles in a complex $C^1(\mathcal{U}; \mathcal{D}_1^*)$, and gerbes with connective structure and curving as 2-cocycles in $C^2(\mathcal{U}; \mathcal{D}_2^*)$. Because of this observation, the definition of an n -gerbe in [Li] is given as a cocycle in $C^{n+1}(\mathcal{U}; \mathcal{D}_{n+1}^*)$. The main result of Lipsky's work is that there are maps of form

$$\tau_\Sigma : C^{n+m}(\mathcal{U}; \mathcal{D}_{n+m}^*) \longrightarrow C^n(\mathcal{U}^\Sigma; \mathcal{D}_n^*),$$

where Σ is an m -dimensional manifold and \mathcal{U}^Σ is a good cover on the smooth mapping spaces X^Σ . These maps satisfy various compatibility conditions. The definition of τ_Σ is quite involved requiring triangulation of Σ , choices of indexing, and bulky integrations. Similar integrals were described also in [GT]. The way one gets ordinary topological field theories out of this is simple: set $n = 1$. In this case τ_Σ maps $C^{m+1}(\mathcal{U}; \mathcal{D}_{m+1}^*)$ to $C^1(\mathcal{U}^\Sigma; \mathcal{D}_1^*)$. One can deduce from the formal properties of τ_Σ , that if Σ is closed then it maps m -gerbes to complex line bundles over X^Σ . This means that provided an m -gerbe, we get a complex line for any map $\Sigma \longrightarrow X$. Furthermore, if we use the formal properties of τ , we will discover that the above prescription can be extended to topological field theory of form $\text{Bord}_m^{\text{or}}(X) \longrightarrow \mathcal{Tors}_{U(1)}$. In case $m = 1$ and $m = 2$, we recover the theories of Kostant-Weil and Brylinski-McLaughlin, respectively.

Let us also acknowledge the work of Schreiber, [Sch], where a similar construction along with many others are present. One of the central topics in the manuscript is the construction of the so-called ∞ -Chern-Simons theory and ∞ -Wess-Zumino-Witten term. In the present work, we will show how the classical Chern-Simons theory and Wess-Zumino-Witten term can be interpreted in our framework and it is not very difficult to see how we can extend it to higher dimensional versions.

For a manifold X , in section , we construct an (∞, n) -category $\mathfrak{D}_n(X)$, which is derived from the Deligne complex using a version of the Dold-Kan construction. We will also construct the space of extended topological field theories over X taking values in $\mathbf{B}^n U(1)$ —the n -fold bar construction on $U(1)$, which we again obtain by using the Dold-Kan construction. We write the space of these extended topological field theories as $\text{TFT}_n(X)$. In addition to this, we consider the structured versions of extended topological field theories, such

as $\text{TFT}_n^{\text{or}}(X)$, $\text{TFT}_n^{\text{spin}}(X)$, $\text{TFT}_n^{\text{fr}}(X)$. The main theorem is going to be phrased in terms of the framed extended topological field theories. The assignments TFT_n^{fr} can be regarded as homotopy sheaves of Kan complexes on site of smooth manifolds. The assignment \mathfrak{D}_n is a presheaf of (∞, n) -categories. We can convert it to a sheaf of Kan complexes by first taking the geometric realization, $|\mathfrak{D}_n|$, and then sheafifying the resulting presheaf of Kan complexes, $|\mathfrak{D}_n|^+$. Thus, both TFT_n^{fr} and $|\mathfrak{D}_n|^+$ can be regarded as objects in the ∞ -topos $\text{Sh}(\text{Man})$. We will construct the *adjoint integration pairing* morphism in $\text{Sh}(\text{Man})$

$$\int^{\vee} : |\mathfrak{D}_n|^+ \longrightarrow \text{TFT}_n^{\text{fr}}.$$

The main theorem is the following.

THEOREM 9. *The adjoint integration pairing, \int^{\vee} , is an equivalence in $\text{Sh}(\text{Man})$.*

2.2 Construction and Main Theorem

2.2.1 The Dold-Kan Construction

Recall that the classical Dold-Kan correspondence gives an equivalence between connective chain complexes and simplicial abelian groups. The correspondence is valid if we consider chain complexes of topological abelian groups and simplicial abelian spaces. In this subsection we are going to present a version of the Dold-Kan construction for finite length chain complexes of topological abelian groups. This construction is going to work well for our goals as it naturally produces a simplicial presheaf on $\Delta^{\times n}$, instead of a simplicial space.

Let \mathcal{C} be an arbitrary category with products. Let $\text{Chain}_n(\mathcal{C})$ denote the category of connective chain complexes of abelian group objects in \mathcal{C} of length $n + 1$ and degree 0 maps. In other words, an object in $\text{Chain}_n(\mathcal{C})$ is a sequence

$$A_0 \xleftarrow{\partial_0} A_1 \xleftarrow{\partial_1} A_2 \xleftarrow{\partial_2} \dots \xleftarrow{\partial_{n-1}} A_n,$$

where A_i is an abelian group object in \mathcal{C} , ∂_i is a group homomorphism, and $\partial_i \partial_{i+1} : A_{i+2} \longrightarrow A_i$ factors through the unit morphism $0 : * \longrightarrow A_i$. We will write such a chain complex as A_* . A morphism $f_* : A_* \longrightarrow B_*$ in $\text{Chain}_n(\mathcal{C})$ is a collection of homomorphisms $f_i : A_i \longrightarrow B_i$, such that $f_i \partial_i = \partial_i f_{i+1}$. Note that $\text{Chain}_0(\mathcal{C})$ is that category of abelian group objects in \mathcal{C} .

To define our version of the Dold-Kan construction we need to use a simple case of the classical Dold-Kan construction. We define the following functor

$$\mathfrak{d}_1 : \text{Chain}_1(\mathcal{C}) \longrightarrow \mathcal{C}^{\Delta^{\text{op}}}$$

as follows,

- the simplicial object $\mathfrak{d}_1(A_*)$ is specified by the formula

$$\mathfrak{d}_1(A_*)([n]) = A_0 \times (A_1)^n$$

on the level of objects of Δ^{op} ,

- and on the level of morphisms of Δ^{op} , if given an order preserving map $f : [m] \rightarrow [n]$, we define $\mathfrak{d}_1(A_*)(f)$ as the following composite

$$\begin{array}{ccc} A_0 \times (A_1)^n & \xrightarrow{\mathfrak{d}_1(A_*)(f)} & A_0 \times (A_1)^m \\ \cong \downarrow & & \uparrow \\ A_0 \times (A_1)^{f(0)+1} \times \prod_{i=0}^{m-1} (A_1)^{f(i+1)-f(i)} & \longrightarrow & A_0 \times (A_1)^{f(0)+1} \times \prod_{i=0}^{m-1} (A_1) \end{array}$$

- what \mathfrak{d}_1 does on the level of morphisms of $\text{Chain}_1(\mathcal{C})$ is obvious.

We define the rest of the Dold-Kan functors inductively. To do this we need the following functor,

$$\tau : \text{Chain}_n(\mathcal{C}) \longrightarrow \text{Chain}_1(\text{Chain}_{n-1}(\mathcal{C})).$$

This functor takes a chain complex of length $n + 1$,

$$A_0 \longleftarrow A_1 \longleftarrow \dots \longleftarrow A_n,$$

and converts it to a length 2 chain complex of chain complexes of length n ,

$$\begin{array}{ccccccc} 0 & \longleftarrow & 0 & \longleftarrow & \dots & \longleftarrow & 0 & \longleftarrow & A_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ A_0 & \longleftarrow & A_1 & \longleftarrow & \dots & \longleftarrow & A_{n-2} & \longleftarrow & A_{n-1}. \end{array}$$

Then we define the Dold-Kan functor as the following composite,

$$\begin{array}{ccc} \text{Chain}_n(\mathcal{C}) & \xrightarrow{\mathfrak{d}_n} & \mathcal{C}^{(\Delta^{\times n})^{\text{op}}} \\ \tau \downarrow & & \uparrow \cong \\ \text{Chain}_1(\text{Chain}_{n-1}(\mathcal{C})) & & (\mathcal{C}^{(\Delta^{\times (n-1)})^{\text{op}}})^{\Delta^{\text{op}}} \\ & \searrow \mathfrak{d}_1 & \nearrow (\mathfrak{d}_{n-1})^{\Delta^{\text{op}}} \\ & \text{Chain}_{n-1}(\mathcal{C})^{\Delta^{\text{op}}} & \end{array}$$

We also adopt the convention that $\mathfrak{d}_0 : \text{Chain}_0(\mathcal{C}) \rightarrow \mathcal{C}$ is the functor that forgets the group structure. We will be primarily interested in the case, where \mathcal{C} is the category of topological spaces $\mathcal{T}\text{op}$.

This construction yields a very manageable output, which can be described explicitly as in the following proposition.

PROPOSITION 14. Let \mathcal{C} be a category with products and let A_* be an object in $\text{Chain}_n(\mathcal{C})$. Then $\mathfrak{d}_n(A_*)([m_1], \dots, [m_n]) \simeq A_0 \times (A_1)^{m_1} \times (A_2)^{m_1 m_2} \times (A_n)^{m_1 \dots m_n}$.

Proof. We will prove the proposition by using induction on n . For $n = 1$ the statement is in the definition. Using the induction we can write,

$$\begin{aligned} \mathfrak{d}_n(A_*)([m_1], \dots, [m_n]) &\simeq (\mathfrak{d}_{n-1}(\mathfrak{t}(A_*)_0) \times \mathfrak{d}_{n-1}(\mathfrak{t}(A_*)_1)^{m_n})([m_1], \dots, [m_{n-1}]) \simeq \\ &\simeq \mathfrak{d}_{n-1}(\mathfrak{t}(A_*)_0)([m_1], \dots, [m_{n-1}]) \times \mathfrak{d}_{n-1}(\mathfrak{t}(A_*)_1)([m_1], \dots, [m_{n-1}])^{m_n} \simeq \\ &\simeq A_0 \times (A_1)^{m_1} \times \dots \times (A_{n-1})^{m_1 \dots m_{n-1}} \times (A_n)^{m_1 \dots m_n}. \end{aligned}$$

This completes the proof of the proposition. \square

EXAMPLE 23. Let A be an abelian group object in a category with products, \mathcal{C} . Let us denote by $\Sigma^n A$, the chain complex,

$$\underbrace{0 \leftarrow 0 \leftarrow \dots \leftarrow 0}_{n \text{ times}} \leftarrow A.$$

Using the Dold-Kan construction we obtain a functor $B^n A = \mathfrak{d}_n(\Sigma^n A) : (\Delta^{\times n})^{\text{op}} \rightarrow \mathcal{C}$. One should interpret this object as some sort of an n -fold bar construction on A —an n -category with trivial morphisms up to level $n - 1$ and A -worth of morphisms on level n .

Previous example is interesting, since maps into $B^n A$ are easy to describe and will be of great use later in the paper. Let F be an object in $\mathcal{C}^{(\Delta^{\times n})^{\text{op}}}$. Let us denote by $[F]$ the object $F([1], [1], \dots, [1])$. Note that if A_* is a chain complex in $\text{Chain}_n(A_*)$, then

$$[\mathfrak{d}_n(A_*)] \simeq A_0 \times A_1 \times \dots \times A_n.$$

In particular, $[B^n A] \simeq A$. Thus, given a natural transformation $\alpha : F \rightarrow B^n A$, one gets a morphism $[\alpha] : [F] \rightarrow [B^n A] \simeq A$. However, not all morphisms from $[F]$ to A come from natural transformations. Let $[F]_k$ denote $F([1], \dots, [2], \dots, [1])$, where $[2]$ occurs at the i -th spot and the rest of entries are $[1]$. Note that $[B^n A]_k \simeq A^2$. Note that there are three maps from $[F]_k$ to $[F]$ that come from the face maps d^0 , d^1 and d^2 on the k -th component of $\Delta^{\times n}$. If α is a natural transformation as above, then we have the following commuting diagram,

$$\begin{array}{ccc} [F]_k & \xrightarrow{d^0 \times d^2} & [F]^2 & \xrightarrow{[\alpha]^2} & A^2 \\ d^1 \downarrow & & & & \downarrow \cdot \\ [F] & \xrightarrow{[\alpha]} & & & A. \end{array}$$

Clearly, $[\alpha]$ has to make the diagram above commute in order to form a natural transformation of functors $\alpha : F \rightarrow \mathbf{B}^n A$. Using the explicit description of the proposition above, one can prove that the converse also holds.

LEMMA 2. *Let F be a functor from $(\Delta^{\times n})^{\text{op}}$ to \mathcal{C} , A an abelian group object in \mathcal{C} , and $\beta : [F] \rightarrow A$ a morphism. If β makes the diagram above commute (when $[\alpha]$ is replaced with β), then there exists a unique natural transformation $\alpha : F \rightarrow \mathbf{B}^n A$, such that $\beta = [\alpha]$.*

After these general considerations we would like to fix \mathcal{C} to be $\mathcal{J}\text{op}$. We will denote by $\mathcal{C}\text{hain}_n$ the category $\mathcal{C}\text{hain}_n(\mathcal{J}\text{op})$. In this case, \mathfrak{d}_n is a functor from $\mathcal{C}\text{hain}_n$ to $\mathcal{J}\text{op}^{(\Delta^{\times n})^{\text{op}}}$. Let us denote by \mathfrak{b}_n the composite,

$$\mathcal{C}\text{hain}_n \rightarrow \mathcal{J}\text{op}^{(\Delta^{\times n})^{\text{op}}} \rightarrow (\text{Set}_{\Delta}^{\circ})^{(\Delta^{\times n})^{\text{op}}},$$

where the second functor is $(\text{Sing})^{(\Delta^{\times n})^{\text{op}}}$. If we apply the nerve construction to $\mathfrak{b}_n(A_*)$ we obtain an object in $\mathcal{P}\text{sh}(\mathbf{N}(\Delta^{\times n}))$. The nerve of $\mathfrak{b}_n(A_*)$ is a Glob_n and Segal_n local object, therefore, $\mathfrak{b}_n(A_*)$ can be interpreted as an (∞, n) -precategory. Using the explicit description of $\mathfrak{d}_n(A_*)$ one can show that it is local with respect to the morphisms $|\text{Glob}_n|$ and $|\text{Segal}_n|$. Thus, $\mathfrak{b}_n(A_*) = \text{Sing}(\mathfrak{d}_n(A_*))$ is local with respect to Glob_n and Segal_n . Unfortunately, $\mathbf{N}(\mathfrak{b}_n(A_*))$ is not in general an (∞, n) -category. Therefore, in order to obtain an (∞, n) -category we need to apply Rezk completion.

REMARK 18. This construction works for \mathcal{C} a topological category that is closed under products.

EXAMPLE 24. Let A be a topological abelian group. We will denote by $\mathfrak{B}^n A$ the (∞, n) -precategory $\mathbf{N}(\mathbf{B}^n A)$. We will write $\mathbf{B}^n A$ for the (∞, n) -category $(\mathbf{B}^n A)^{\wedge}$. We will focus on the case $A = U(1)$.

EXAMPLE 25. Let X be a smooth manifold. Then we can consider the so-called *smooth Deligne complex* of X :

$$\mathcal{A}^n(X) \leftarrow \mathcal{A}^{n-1}(X) \leftarrow \dots \leftarrow \mathcal{A}^1(X) \leftarrow C^{\infty}(X, U(1)),$$

where $\mathcal{A}^k(X)$ is the topological abelian group of complex differential k -forms on X given the C^{∞} -topology, the differentials are the de Rham differentials except for the map $C^{\infty}(X, U(1)) \rightarrow \mathcal{A}^1(X)$, which is the $d \log$ map. Let us denote this chain complex by $\mathbf{D}_*^n(X)$. Again we apply the Dold-Kan functor to obtain a functor $\mathfrak{d}_n(\mathbf{D}_*^n(X)) : (\Delta^{\times n})^{\text{op}} \rightarrow \mathcal{J}\text{op}$. If we further apply the singularization functor we obtain $\mathfrak{b}_n(\mathbf{D}_*^n(X)) : (\Delta^{\times n})^{\text{op}} \rightarrow \text{Set}_{\Delta}^{\circ}$, which after the application of the simplicial nerve construction can be interpreted as an object in $\mathfrak{D}_n(X) \equiv \mathbf{N}(\mathfrak{b}_n(\mathbf{D}_*^n(X))) \in \mathcal{P}\text{sh}(\mathbf{N}(\Delta^{\times n}))$.

EXAMPLE 26. We will also use the notion of *transposed Deligne complex*, which is defined as follows. Notice that the category $\Delta^{\times n}$ has an involution, τ , which sends $[m_1, \dots, m_n]$ to $[m_n, \dots, m_1]$. We define $\text{TD}_n(X) (\mathfrak{T}\mathfrak{D}_n(X))$ as the composite of $\mathbf{D}_n(X) \circ \tau (\mathfrak{D}_n(X) \circ \tau)$. The resulting n -simplicial spaces are not

going to produce (∞, n) -categories, but will be useful in the construction of the integration pairing.

Next we would like to investigate the interaction of the Dold-Kan construction with the geometric realization functor.

PROPOSITION 15. *Let A_* be a chain complex of topological abelian groups. There exists a spectral sequence with \mathcal{E}^2 -page given by the formula $\mathcal{E}_{p,q}^2 \simeq \mathbb{H}_p(\pi_q(A_*))$, converging to $\pi_{p+q}(|\mathbf{b}_n(A_*)|)$. The differentials on \mathcal{E}^k -page are of the form $d^r : \mathcal{E}_{p,q}^r \rightarrow \mathcal{E}_{p-r+1,q}^r$.*

COROLLARY 4. *There is an equivalence of the form $|\mathbf{B}^n U(1)| \simeq \mathbb{K}(\mathbb{Z}, n+1)$.*

COROLLARY 5. *There is an equivalence $|\mathbf{b}_n(\mathbf{D}_*^n(X))| \simeq |\mathbf{B}^n(\text{Map}(X, U(1)))| \simeq \text{Map}(X, |\mathbf{B}^n U(1)|)$.*

Proof. There is a map of chain complexes

$$\begin{array}{ccccccc} 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & 0 \longleftarrow C^\infty(X, U(1)) \\ \downarrow & & \downarrow & & & & \downarrow & \downarrow \\ \mathcal{A}^n(X) & \longleftarrow & \mathcal{A}^{n-1}(X) & \longleftarrow & \cdots & \longleftarrow & \mathcal{A}^1(X) \longleftarrow C^\infty(X, U(1)). \end{array}$$

Using the comparison of spectral sequences and that $C^\infty(X, U(1)) \simeq \text{Map}(X, U(1))$ the weak equivalence follows right away. \square

COROLLARY 6. *The presheaf $|\mathfrak{D}_n|$ is homotopy invariant, in the sense, that if $f, g : X \rightarrow Y$ are homotopic, then so are $|\mathfrak{D}_n(f)|$ and $|\mathfrak{D}_n(g)|$ are also homotopic.*

The proof of the proposition is based on the following lemmas.

LEMMA 3. *Let A_* be an object in Chain_1 . Then the natural map $|\mathfrak{d}_1(A_*)| \rightarrow |\mathfrak{d}_1(\Sigma^1 A_1)|$ can be given a structure of a principal A_0 -bundle.*

Proof. Notice first of all that A_0 acts on $\mathfrak{d}_1(A_*)$ in such a way that $a \in A_0$ sends $(a_0, a_1^{(1)}, \dots, a_1^{(m)}) \in \mathfrak{d}_1(A_*)([m])$ to $(a+a_0, a_1^{(1)}, \dots, a_1^{(m)})$. This action is compatible with the face and degeneracy maps. Therefore, this action can be promoted to one on $|\mathfrak{d}_1(A_*)|$. The action can be checked to be free, and that the quotient $|\mathfrak{d}_1(A_*)|/A_0$ is homeomorphic to $|\mathfrak{d}_1(\Sigma^1 A_1)|$. One can show that the projection is a principal A_0 -bundle using the natural filtration on $|\mathfrak{d}_1(\Sigma^1 A_1)|$. \square

The following lemma follows from the inductive definition of the Dold-Kan construction and the fact that geometric realization commutes with products.

LEMMA 4. *Let A_* be an object in Chain_n . Let us denote by $\mathfrak{s}(A_*)$ the following chain complex $|\mathfrak{d}_{n-1}(\mathfrak{r}(A_*)_0)| \leftarrow |\mathfrak{d}_{n-1}(\mathfrak{r}(A_*))|$. Then there is an equivalence $|\mathfrak{d}_n(A_*)| \simeq |\mathfrak{d}_1(\mathfrak{s}(A_*))|$.*

LEMMA 5. *The n -fold loop space $\Omega^n |\mathbf{B}^n A|$ is weakly homotopy equivalent to A .*

Proof. Note that $|\mathfrak{d}_1(\Sigma^1 A)|$ is the classical bar construction on A , for which we know that $\Omega|\mathfrak{d}_1(\Sigma^1 A)| \simeq A$. Suppose that $|\mathfrak{d}_{n-1}(\Sigma^{n-1} A)|$ is homeomorphic to the classical $(n-1)$ -fold bar construction. Note that $\mathfrak{s}(A_*)$ is the chain complex $0 \leftarrow |\mathfrak{d}_{n-1}(\Sigma^{n-1} A)|$, so $|\mathfrak{d}_1(\mathfrak{s}(A_*))|$ is the classical bar construction on $|\mathfrak{d}_{n-1}(\Sigma^{n-1} A)|$. However, $|\mathfrak{d}_1(\mathfrak{s}(A_*))|$ is homeomorphic to $|\mathfrak{d}_n(A_*)|$, which completes the inductive step. \square

Proof of Proposition. Let $A_*^{\leq k}$ denote the complex

$$A_0 \leftarrow A_1 \leftarrow \dots \leftarrow A_k.$$

Then from lemmas it follows that there is a fiber sequence

$$|\mathfrak{d}_q(A_*^{\leq q})| \xrightarrow{i} |\mathfrak{d}_{q+1}(A_*^{\leq q+1})| \xrightarrow{k} |\mathfrak{d}_{q+1}(\Sigma^{q+1} A_{q+1})|.$$

This allows us to set up the following exact couple,

$$\begin{array}{ccc} \bigoplus_{p,q} \pi_p(|\mathfrak{d}_q(A_*^{\leq q})|) & \xrightarrow{i} & \bigoplus_{p,q} \pi_p(|\mathfrak{d}_q(A_*^{\leq q})|) \\ & \swarrow & \searrow k \\ & \bigoplus_{p,q} \pi_{p-q}(A_q), & \end{array}$$

which produces a spectral sequence with the desired properties. \square

Lastly, let us discuss the symmetric monoidal structures. The object $\mathfrak{d}_n(A_*)$ is an abelian group object in $\mathcal{C}^{(\Delta^{\times n})^{\text{op}}}$, since \mathfrak{d}_n commutes with products. Therefore, we can construct a functor $\mathfrak{d}_n(A_*)^{\otimes} : \Upsilon \rightarrow \mathcal{C}^{(\Delta^{\times n})^{\text{op}}}$ as in Ex. 1.3.4.3. If \mathcal{C} is topologically enriched, then we can convert this into a map between ∞ -categories $\mathbf{N}(\mathfrak{d}_n(A_*)^{\otimes}) : \mathbf{N}(\Upsilon) \rightarrow \mathbf{N}(\mathcal{C}^{(\Delta^{\times n})^{\text{op}}})$. In the case, when $A_* = \Sigma^n A$, we obtain $\mathfrak{d}_n(\Sigma^n A)^{\otimes}$, which we will write as $\mathbf{B}^n A^{\otimes}$.

EXAMPLE 27. We can apply this construction to the case $\Sigma^n U(1)$ to get the functor $\mathbf{B}^n U(1)^{\otimes} : \Upsilon \rightarrow \mathcal{J}\text{op}^{(\Delta^{\times n})^{\text{op}}}$. After applying the nerve construction and completion functors, we get $\mathbf{N}(\mathbf{B}^n U(1)^{\otimes})^\wedge : \mathbf{N}(\Upsilon) \rightarrow \mathcal{C}\text{at}_{(\infty, n)}$, which gives a symmetric monoidal structure on $\mathbf{B}^n U(1)$. We will write $\mathbf{B}^n U(1)^{\otimes}$ as a shorthand for $\mathbf{N}(\mathbf{B}^n U(1)^{\otimes})^\wedge$. Note that we can evaluate on $\mathbf{B}^n U(1)$ on trees using the functor $L : \Omega^{\text{op}} \rightarrow \Upsilon$.

2.2.2 Bordism Categories and Topological Field Theories

In this section we construct the bordism (∞, n) -categories. As in the previous section, we will construct functors from $(\Delta^{\times n})^{\text{op}}$ to $\mathcal{J}\text{op}$, and then we will apply successively the singularization, the nerve construction and then the localization. The basic premise of the construction is the observation that a bordism can be thought of as a manifold sliced along a smooth function. More specifically, we are going to construct a space

of (possibly non-compact) manifolds with a choice of parametrizing proper smooth functions, and using this space we are going to define the space of compact manifolds by choosing a regular point for this parametrizing functions. The bordisms are going to be given by choices of two regular points, the compositions of bordisms by three regular values, and so on. The resulting bordism (∞, n) -category is precisely the one defined in [Lu1]. The main result of this section is that the geometric realization of this category is an n -fold delooping of a certain tangential Thom spectrum. We are going to prove this result in the next subsection.

Our first task is to construct the space of all manifolds. The definitions follow closely the ones given by Galatius in [G].

DEFINITION 19. *Let $\Psi_{n,N}$ denote the topological space, whose points are given by closed n -dimensional submanifolds of \mathbb{R}^{n+N} . This set is topologized according to section 6 of [G].*

REMARK 19. Notice that the empty set, \emptyset , is a point in all of $\Psi_{n,N}$. We will think of this point as the basepoint of these spaces. The topology on $\Psi_{n,N}$ is designed so that if a submanifold moves further and further away from the origin it gets closer to \emptyset . Let $\text{Gr}_{n,N}$ denote the Grassmannian of n dimensional linear subspaces of \mathbb{R}^{n+N} , and let $\zeta_{n,N}^\perp$ the vector bundle of orthogonal vectors. More specifically, a point in $\zeta_{n,N}^\perp$ is a pair (V, v) , where V is an n -dimensional subspace of \mathbb{R}^{n+N} and v is a vector orthogonal to V . One can regard the affine plane $V - v$ as an n -dimensional submanifold of \mathbb{R}^{n+N} . Because of the topology on $\Psi_{n,N}$ we obtain a *continuous* map

$$q : \text{Th}(\zeta_{n,N}^\perp) \longrightarrow \Psi_{n,N},$$

where the basepoint of the Thom space, $\text{Th}(\zeta_{n,N}^\perp)$, goes to \emptyset . The following result is proven in section 6 of [G].

THEOREM 10. *The map $q : \text{Th}(\zeta_{n,N}^\perp) \longrightarrow \Psi_{n,N}$ is a weak homotopy equivalence.*

We would like to generalize the definition above to submanifolds with structure. Notice that if we are given an embedding of an n -dimensional manifold $i : M \hookrightarrow \mathbb{R}^{n+N}$, then the tangent space at any point of M can be identified with a linear subspace of \mathbb{R}^{n+N} . This gives us a natural map $M \longrightarrow \text{Gr}_{n,N}$. The standard inclusion of \mathbb{R}^{n+N} into \mathbb{R}^{n+N+1} gives an inclusion of Grassmannians $\text{Gr}_{n,N} \longrightarrow \text{Gr}_{n,N+1}$. The colimit (union) of these inclusions is a model for $\text{BO}(n)$ and we will identify $\text{BO}(n)$ with $\text{colim } \text{Gr}_{n,N}$. This gives a map of topological spaces $\text{Emb}(M, \mathbb{R}^{n+N}) \longrightarrow \text{Map}(M, \text{BO}(n))$.

Suppose now that $m \geq n$ and $\xi : Y \longrightarrow \text{BO}(m)$ is a fibration, then we can define the following pullback diagram for all M

$$\begin{array}{ccc} \text{Emb}^\xi(M, \mathbb{R}^{n+N}) & \rightarrow & \text{Emb}(M, \mathbb{R}^{n+N}) \\ \downarrow & & \downarrow \tau \\ \text{Map}(M, Y) & \longrightarrow & \text{Map}(M, \text{BO}(m)) \end{array}$$

Here map τ is the one above composed with the standard inclusion $\text{BO}(n) \rightarrow \text{BO}(m)$. Notice that this is a homotopy pullback, since the horizontal arrow on the bottom is a fibration. As a consequence, of the same fact, we deduce that the horizontal arrow on the top is also a fibration. A point of $\text{Emb}^\xi(M, \mathbb{R}^{n+N})$ is an embedding of M into \mathbb{R}^{n+N} along with a ξ -structure on M . Since we are interested in having a background manifold X , we can also define $\text{Emb}_X^\xi(M, \mathbb{R}^{n+N})$, which is simply $\text{Emb}^\xi(M, \mathbb{R}^{n+N}) \times C^\infty(M, X)$.

DEFINITION 20. Let $\Psi_{n,N}^\xi$ be the topological space, whose points are given by an n -dimensional submanifold M in \mathbb{R}^{n+N} along with a point in $\text{Emb}^\xi(M, \mathbb{R}^{n+N})$ lifting the inclusion under the map above. The topology on $\Psi_{n,N}^\xi$ is given as in section 6 of [G]. If X is a smooth manifold, we define $\Psi_{n,N}^{\xi,X}$ to be the topological space, whose points are the n -dimensional submanifolds of \mathbb{R}^{n+N} along with a point in $\text{Emb}_X^\xi(M, \mathbb{R}^{n+N})$ lifting the inclusion.

REMARK 20. If (Y, ξ) is one of the pairs $(\text{BSO}(m), \zeta_m^{\text{or}})$, $(\text{BSpin}(m), \zeta_m^{\text{spin}})$, $(\text{EO}(m), \zeta_m^{\text{fr}})$, then we will write $\Psi_{n,N}^{\text{or}}$, $\Psi_{n,N}^{\text{spin}}$, $\Psi_{n,N}^{\text{fr}}$, respectively, for the corresponding spaces of structured manifolds. Notice that $\Psi_{n,N}^{\zeta_m} \simeq \Psi_{n,N}$. Also, it is clear that $\Psi_{n,N}^{\xi,*} \simeq \Psi_{n,N}^\xi$.

REMARK 21. An analogue of the q map can be constructed in the structured setting as well. For each fibration $\xi : Y \rightarrow \text{BO}(m)$, we can consider the pullback to the Grassmannian $\xi_{n,N} : Y_{n,N} \rightarrow \text{Gr}_{n,N}$. We can consider the vector bundle $\xi_{n,N}^\perp$ on $Y_{n,N}$, which is the pullback of $\zeta_{n,N}^\perp$. A point in $\xi_{n,N}^\perp$ is given by a pair (U, u) , where U is a point $Y_{n,N}$ and u is a vector in \mathbb{R}^{n+N} orthogonal to $\xi_{n,N}(U)$. We can regard $\xi_{n,N}(U) - u$ as a submanifold of dimension n in \mathbb{R}^{n+N} . We give it a ξ -structure given by the constant map into Y with value U , since we can regard $Y_{n,N}$ as a subset of Y . This map extends to a map of the following form,

$$q^\xi : \text{Th}(\xi_{n,N}^\perp) \rightarrow \Psi_{n,N}^\xi.$$

This map is a weak homotopy equivalence, and the proof is analogous to the one for the theorem above.

Now we turn to the definition of the bordism categories. Let E^k denote the subspace of \mathbb{R}^{k+1} consisting of points (e_0, \dots, e_k) , such that $e_0 \leq e_1 \leq \dots \leq e_k$. Notice that E^k comes equipped with $k+1$ projections to \mathbb{R} . Another way of interpreting E^k is as the space of order preserving maps from $\{0, \dots, k\}$ to \mathbb{R} . With this interpretation it becomes clear that E^\bullet is a simplicial space.

DEFINITION 21. We define $\text{Bord}_{n,N}^\xi(X) : (\Delta^{\times n})^{\text{op}} \rightarrow \text{Top}$ by setting $\text{Bord}_{n,N}^{\xi,\perp}(X)([m_1], \dots, [m_n])$ to be the subspace of points of $\Psi_{n,N}^{\xi,X} \times E^{m_1} \times \dots \times E^{m_n}$, consisting of points, n -dimensional submanifold M of \mathbb{R}^{n+N} , an element $f \in C^\infty(M, X)$ and $(e^1, \dots, e^n) \in E^{m_1} \times \dots \times E^{m_n}$, such that

- 1) the composition $M \hookrightarrow \mathbb{R}^{n+N} \rightarrow \mathbb{R}^n$ is proper,
- 2) if $S \subset \{1, \dots, n\}$, then the set of all possible projections of $(e^s)_{s \in S} \in \prod_{s \in S} E^{m_s}$ onto \mathbb{R}^S does not contain any critical values of $M \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^S$

3) the projections $M \longrightarrow \mathbb{R}^{\{i+1, \dots, n\}}$ is submersive for all points $x \in M$ that get sent to one of the projections of $e^i \in E^i$ under the map $M \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^{\{i\}}$; this implies that if $P_{k,i}$ is the preimage of the point $\pi_k(e_i)$ under the map $M \longrightarrow \mathbb{R}^{\{i\}}$, then $P_{k,i} \simeq L_{k,i} \times \mathbb{R}^{\{i+1, \dots, n\}}$ in such a way that the restriction of the map $M \longrightarrow \mathbb{R}^{\{k+1, \dots, n\}}$ to $P_{k,i}$ is the projection onto the second component;

4) the background map f when restricted to $P_{k,i}$ factors through $L_{k,i}$.

There is a natural inclusion $\text{Bord}_{n,N}^{\xi}(X)$ into $\text{Bord}_{n,N+1}^{\xi}(X)$ obtained from the inclusion $\mathbb{R}^{n+N} \longrightarrow \mathbb{R}^{n+N+1}$. We denote the colimit of these inclusion by $\text{Bord}_n^{\xi}(X)$.

This is a natural covariant construction in variable X , therefore, we can write that Bord_n^{ξ} is a functor from Man to $\text{Top}^{(\Delta^{\times n})^{\text{op}}}$. We can further apply the singularization functor to get a map $\text{Sing}(\text{Bord}_n^{\xi}) : \text{Man} \longrightarrow (\text{Set}_{\Delta}^{\circ})^{(\Delta^{\times n})^{\text{op}}}$. Now apply the simplicial nerve construction to get a functor

$$\text{N}(\text{Man}) \longrightarrow \text{N}((\text{Set}_{\Delta}^{\circ})^{(\Delta^{\times n})^{\text{op}}}) \simeq \mathfrak{S}^{\text{N}((\Delta^{\times n})^{\text{op}})}.$$

This composite factors through $\mathbf{Bord}_n^{\xi} : \text{N}(\text{Man}) \longrightarrow \mathcal{PCat}_{(\infty, n)}$. We can apply the completion functor to obtain the copresheaf of (∞, n) -categories $(\mathbf{Bord}_n^{\xi})^{\wedge} : \text{N}(\text{Man}) \longrightarrow \text{Cat}_{(\infty, n)}$.

Now let us discuss the symmetric monoidal structure on Bord_n . We consider this simple case out of notational convenience. To do that we need to consider a modified version of the bordism (∞, n) -category that is equivalent to the original one.

DEFINITION 22. We define $\text{Bord}_{n,N}^{\xi, \perp}(X) : (\Delta^{\times n})^{\text{op}} \longrightarrow \text{Top}$ by setting $\text{Bord}_{n,N}^{\xi, \perp}(X)([m_1], \dots, [m_n])$ to be the subspace of points of $\Psi_{n,N}^{\xi, X} \times E^{m_1} \times \dots \times E^{m_n}$, consisting of points, n -dimensional submanifold M of \mathbb{R}^{n+N} , an element $f \in C^{\infty}(M, X)$ and $(e^1, \dots, e^n) \in E^{m_1} \times \dots \times E^{m_n}$, such that

1) the composition $M \hookrightarrow \mathbb{R}^{n+N} \longrightarrow \mathbb{R}^n$ is proper,

2) if $S \subset \{1, \dots, n\}$, then the set of all possible projections of $(e^s)_{s \in S} \in \prod_{s \in S} E^{m_s}$ onto \mathbb{R}^S does not contain any critical values of $M \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^S$, and near any such projection $\pi \in \mathbb{R}^S$, there is a neighborhood U_{π} and submanifold $N_{S,\pi} \subset \mathbb{R}^{n-S+N}$, such that the preimage of U_{π} under the projection is equal to $N_{S,\pi} \times U_{\pi}$,

3) the projections $M \longrightarrow \mathbb{R}^{\{i+1, \dots, n\}}$ is submersive for all points $x \in M$ that get sent to one of the projections of $e^i \in E^i$ under the map $M \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^{\{i\}}$; this implies that if $P_{k,i}$ is the preimage of the point $\pi_k(e_i)$ under the map $M \longrightarrow \mathbb{R}^{\{i\}}$, then $P_{k,i} = L_{k,i} \times \mathbb{R}^{\{i+1, \dots, n\}}$ in such a way that the restriction of the map $M \longrightarrow \mathbb{R}^{\{k+1, \dots, n\}}$ to $P_{k,i}$ is the projection onto the second component;

4) the background map f when restricted to $P_{k,i}$ factors through $L_{k,i}$.

There is a natural inclusion $\text{Bord}_{n,N}^{\xi, \perp}(X)$ into $\text{Bord}_{n,N+1}^{\xi, \perp}(X)$ obtained from the inclusion $\mathbb{R}^{n+N} \longrightarrow \mathbb{R}^{n+N+1}$. We denote the colimit of these inclusion by $\text{Bord}_n^{\xi, \perp}(X)$.

Notice that there is a natural levelwise inclusion $\text{Bord}_{n,N}^{\xi, \perp} \longrightarrow \text{Bord}_{n,N}^{\xi}$.

PROPOSITION 16. *The natural map $\text{Bord}_{n,N}^{\xi, \perp} \longrightarrow \text{Bord}_{n,N}^{\xi}$ is a levelwise deformation retraction.*

Proof Sketch. The proof relies on the following observation. If $M \subset \mathbb{R} \times \mathbb{R}^k$ is a submanifold, such that the projection onto the first component, $p : M \longrightarrow \mathbb{R}$, is a submersion, there is a continuous family of submanifold $M_t \subset \mathbb{R} \times \mathbb{R}^k$, such that $M_0 = M$, the projections $p_t : M_t \longrightarrow \mathbb{R}$ are all submersions, $p_t^{-1}(\mathbb{R} - [-2, 2]) \cap M_t$ is constant, and $M_1 \cap p_1^{-1}([-1, 1])$ is equal to $[-1, 1] \times N$ for some $N \subset \mathbb{R}^k$. \square

We will define a functor $\text{Bord}_n^{\otimes} : \Omega^{\text{op}} \longrightarrow \mathcal{T}\text{op}^{(\Delta^{\times n})^{\text{op}}}$ as follows. Let M be a closed submanifold of $\mathbb{R}^{n+\infty}$. Define P , to be the subset of points $(x_1, \dots, x_n) \in \mathbb{R}^n$, such that for any subset $S \subset \{1, \dots, n\}$, there exists a neighborhood of $x_S \in U$ and a closed submanifold $N \subset \mathbb{R}^{n-S+\infty}$, such that $\pi^{-1}(U) = U \times N$, where π is the projection of M onto \mathbb{R}^n . Let us also denote by \mathfrak{C} the subset of $C^\infty(\mathbb{R}, \mathbb{R})^n$ consisting of functions that are non-decreasing. We will say that $f \in \mathfrak{C}$ is perpendicular to $M \subset \mathbb{R}^{n+\infty}$, if for any subset $S \in \{1, \dots, n\}$ and $x_S \in \mathbb{R}^S$, such that $d(f^S)(x_S) = 0$, the point $f(x_S)$ is in P . Now let T be a rooted tree. We define $\text{Bord}_n^{\otimes}(T)([m_1, \dots, m_n])$ to be the subspace of the product

$$\prod_{v \in V_T} \mathbb{I}(\text{val}(v)) \times \prod_{L_T} \text{Bord}_n([m_1, \dots, m_n]) \times \prod_{s \in E_T} \mathfrak{C} \times \prod_{v \in V_T} E^{m_1, \dots, m_n},$$

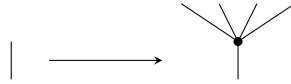
consisting of points $(\iota_v)_{v \in V_T} \times (M_l \subset \mathbb{R}^{n+\infty}, e_l^1, \dots, e_l^n)_{l \in L_T} \times (f_s^1, \dots, f_s^n)_{s \in E_T} \times (e_v^1, \dots, e_v^n)_{v \in V_T}$, that satisfy the following conditions

1) if s is an incoming edge for v and outgoing for u , then $f_s^k(e_v^k) = e_u^k$;

2) if l is an incoming leaf for v , then $f_l^k(e_v^k) = e_l^k$;

3) if $(s_1 = r, \dots, s_j = l)$ defines the unique path from the root to a leaf l , then the composite $f_{s_j} \circ \dots \circ f_{s_1}$ is perpendicular to $M \subset \mathbb{R}^{n+\infty}$.

To demonstrate how Bord_n^{\otimes} works on maps of trees let us consider an example. Let us consider the map



Let us pick a point $(\iota) \times (M_i \subset \mathbb{R}^{n+\infty}, e_i^1, \dots, e_i^n)_{i=1}^m \times (f_i)_{i=1}^{m+1} \times (e^1, \dots, e^n)$ in $\text{Bord}_n^{\otimes}(\langle m \rangle) = \mathbb{I}(m) \times \text{Bord}_n([m_1, \dots, m_n])^m \times \mathfrak{C}^{m+1} \times E^{m_1, \dots, m_n}$. Note that f_i is perpendicular to M_i , so it makes sense to talk about $f_i^*(M_i) \subset \mathbb{R}^{n+\infty}$. Then, we can take the coproduct and compose it with ι , $\coprod f_i^*(M) \subset \coprod \mathbb{R}^{n+\infty} = \mathbb{R}^n \times (\coprod \mathbb{R}^\infty) \hookrightarrow \mathbb{R}^n \times \mathbb{R}^\infty$. The point in $\text{Bord}_n^{\otimes}(\langle 1 \rangle) = \text{Bord}_n([m_1, \dots, m_n]) \times \mathfrak{C}$ that we get is $(\coprod f_i^*(M) \subset \mathbb{R}^{n+\infty}, e^1, \dots, e^n) \times f_{m+1}$.

LEMMA 6. *If $T \longrightarrow W$ induces an isomorphism on leaves, then $\text{Bord}_n^{\otimes}(W) \longrightarrow \text{Bord}_n^{\otimes}(T)$ is an equivalence of spaces, and these two n -simplicial spaces are equivalent to $(\text{Bord}_n)^m$, where m is the number of leaves of T or W .*

COROLLARY 7. *There is a factorization $N(\Omega^{\text{op}}) \rightarrow N(\Upsilon) \rightarrow \mathcal{PCat}_{(\infty, n)}$ of $\mathbf{Bord}_n^{\otimes}$.*

We will denote the induced map by $\mathbf{Bord}_n^{\otimes} : N(\Upsilon) \rightarrow \mathcal{PCat}_{(\infty, n)}$ by the abuse of notation. Similarly, we can define $(\mathbf{Bord}_n^{\otimes})^{\wedge} : N(\Upsilon) \rightarrow \mathcal{Cat}_{(\infty, n)}$.

DEFINITION 23. *Let X be a manifold. We define*

$$\text{TFT}_n^{\xi}(X) = \text{Fun}^{\otimes}(\mathbf{Bord}_n^{\xi, \otimes}(X), \mathbf{B}^n U(1)^{\otimes}),$$

the space of topological field theories on X .

The symmetric monoidal (∞, n) -category $\mathbf{B}^n U(1)^{\otimes}$ is, in fact, a symmetric monoidal ∞ -groupoid, which means that $\text{TFT}_n^{\xi}(X) \simeq \text{Fun}^{\otimes}(|\mathbf{Bord}_n^{\xi, \otimes}(X)|, \mathbf{B}^n U(1)^{\otimes})$.

Recall that there is a geometric realization functor $\mathcal{Jop}^{(\Delta^{\times n})^{\text{op}}} \rightarrow \mathcal{Jop}$. The geometric realization of \mathbf{Bord}_n^{ξ} has an alternate description. In order to state the theorem let us recall the notion of the tangential Thom spectrum. Recall the construction of the bundle $\xi_{n, N}^{\perp}$ above over the space $Y_{n, N}$. There is a natural inclusion of $i : Y_{n, N} \rightarrow Y_{n, N+1}$, and it is easy to see that $i^* \xi_{n, N+1}^{\perp} = \xi_{n, N}^{\perp} \oplus 1$, where 1 denotes the trivial line bundle. Thus, there is a vector bundle map $i_{\#} : \xi_{n, N}^{\perp} \oplus 1 \rightarrow \xi_{n, N+1}^{\perp}$. Applying the Thom space construction gives a map $\text{Th}(i_{\#}) : \Sigma \text{Th}(\xi_{n, N}^{\perp}) \simeq \text{Th}(\xi_{n, N}^{\perp} \oplus 1) \rightarrow \text{Th}(\xi_{n, N+1}^{\perp})$. We define the *tangential Thom spectrum* $\text{MT}\xi$ to be specified as follows $(\text{MT}\xi)_N = \text{Th}(\xi_{n, N}^{\perp})$.

THEOREM 11. *The geometric realization $|\mathbf{Bord}_n^{\xi}(X)|$ is weakly equivalent to $\Omega^{\infty-n}(\text{MT}\xi \wedge \Sigma_+^{\infty} X)$ as an infinite loop space.*

Proof. The proof will be given in 2.2.4. \square

COROLLARY 8. *$|\mathbf{Bord}_n^{\xi}|$ is a homotopy invariant cosheaf of infinite loop spaces.*

Proof. The homotopy invariance is clear enough. To show that the functor is cosheaf, we have to notice that $\Sigma^n \text{MT}\xi \wedge \Sigma_+^{\infty} X$ is a connective spectrum, and the category of connective spectra is equivalent to that of infinite loop spaces, and since Σ_+^{∞} preserves pushouts the corollary follows. \square

In this paper we will use this theorem in the case, where $\xi = \zeta_n^{\text{fr}}$. In this case, the tangential Thom spectrum can be explicitly computed. Let $E_{n, N}$ denote the pullback of $\text{EO}(n)$ to $\text{Gr}_{n, N}$. We have two vector bundles on $E_{n, N}$, $\zeta_{n, N}^{\text{fr}}$ and $\zeta_{n, N}^{\text{fr}, \perp}$, such that their sum is the trivial bundle of rank $n + N$. In fact, $\zeta_{n, N}^{\text{fr}}$ is the trivial bundle of rank n , since the classifying map factors through $\text{EO}(n)$, which is contractible. This implies that $\Sigma^n \text{Th}(\zeta_{n, N}^{\text{fr}, \perp}) \simeq \text{Th}(n \oplus \zeta_{n, N}^{\text{fr}, \perp}) \simeq \Sigma_+^{n+N} E_{n, N}$. Notice that $E_{n, N}$ is the homotopy fiber of the inclusion $\text{Gr}_{n, N} \rightarrow \text{BO}(n)$, which implies that it is getting highly connective as N gets larger. Therefore, $\text{MT}\zeta_n^{\text{fr}}$ is weakly equivalent to $\Sigma^{-n} \mathbb{S}$, where \mathbb{S} is the sphere spectrum. Therefore, we conclude the following

weak equivalence,

$$|\mathrm{Bord}_n^{\mathrm{fr}}(X)| \simeq \Omega^{\infty-n}(\Sigma^{-n}S \wedge \Sigma_+^{\infty}X) \simeq \Omega^{\infty}(\Sigma_+^{\infty}X).$$

This equivalence can be phrased by saying that $|\mathrm{Bord}_n^{\mathrm{fr}}(X)|$ is the free infinite loop space generated by X , which implies that infinite loop space maps out of are equivalent to ordinary maps out of X .

2.2.3 The Construction of The Integration Pairing and The Main Theorem

This section is devoted to the construction of the *adjoint integration pairing* functor mentioned in the introduction

$$\int^{\vee} : |\mathfrak{D}_n|^+ \longrightarrow \mathrm{TFT}_n^{\mathrm{or}}.$$

We are going to construct the integration pairing construction inductively using maps that interpolate between bordism categories, and the adjoint integration pairing will be obtained via adjunction manipulations on the closed Cartesian structure of $\mathrm{Fun}((\Delta^{\times n})^{\mathrm{op}}, \mathcal{T}\mathrm{op})$. The natural thing to expect in this situation would have been that the integration pairing map is of the form

$$\mathrm{Bord}_n^{\mathrm{or}}(X) \times \mathrm{D}_n(X) \longrightarrow \mathrm{B}^n U(1).$$

Strangely enough this does not work! Instead of $\mathrm{D}_n(X)$ one has to use $\mathrm{TD}_n(X)$, i.e. the transposed Deligne complex mentioned previously. The adjunction manipulations will result in a functor of form

$$|\mathfrak{I}\mathfrak{D}_n|^+ \longrightarrow \mathrm{TFT}_n^{\mathrm{or}},$$

however, $|\mathfrak{I}\mathfrak{D}_n| \simeq |\mathfrak{D}_n|$. The main theorem will follow easily from the following observations:

- the sheaves $|\mathfrak{D}_n|^+$ and $\mathrm{TFT}_n^{\mathrm{fr}}$ are homotopy invariant,
- the adjoint integration pairing is an equivalence when evaluated at the point.

As mentioned in the overview the construction is going to implement the transposed Deligne complex, which is simply a functor in $\mathcal{T}\mathrm{op}^{(\Delta^{\times n})^{\mathrm{op}}}$. We would like to construct a natural transformation of the form,

$$\begin{array}{ccc} & \mathrm{TD}_n(-) & \\ & \curvearrowright & \\ \mathrm{Man}^{\mathrm{op}} & & \mathcal{T}\mathrm{op}^{(\Delta^{\times n})^{\mathrm{op}}}, \\ & \Downarrow \int^{\vee} & \\ & \mathrm{Map}(\mathrm{Bord}_n^{\mathrm{or}}(-), \mathrm{B}^n U(1)) & \end{array}$$

where $\mathrm{Map}(-, -)$ denotes the mapping object in the category $\mathcal{T}\mathrm{op}^{(\Delta^{\times n})^{\mathrm{op}}}$. Our strategy would be to construct

the adjoint. More explicitly, for any topological space X , we will construct a pairing morphism in $\mathcal{T}\text{op}^{(\Delta^{\times n})^{\text{op}}}$,

$$\int_X : \text{Bord}_n^{\text{or}}(X) \times \text{TD}_n(X) \longrightarrow \text{B}^n U(1).$$

One can easily show that the adjoints of these morphisms will form a natural transformation if the following diagram commutes

$$\begin{array}{ccc} \text{Bord}_n^{\text{or}}(X) \times \text{TD}_n(Y) & \xrightarrow{\mathbb{1} \times f^*} & \text{Bord}_n^{\text{or}}(X) \times \text{TD}_n(X) \\ f_* \times \mathbb{1} \downarrow & & \downarrow f \\ \text{Bord}_n^{\text{or}}(Y) \times \text{TD}_n(Y) & \xrightarrow{f} & \text{B}^n U(1) \end{array}$$

for all smooth maps $f : X \longrightarrow Y$. These morphisms will be called *integration pairings*.

As it was mentioned above that the construction of the integration pairings is inductive. To facilitate the inductive step we introduce maps that interpolate between spaces associated with the bordism categories. Recall that if $\mathcal{C} : (\Delta^{\times n})^{\text{op}} \longrightarrow \mathcal{T}\text{op}$, then $[\mathcal{C}]$ denotes the space $\mathcal{C}([1], \dots, [1])$. We construct a map

$$-_{+} : [\text{Bord}_n^{\text{or}}(X)] \longrightarrow [\text{Bord}_{n-1}^{\text{or}}(X)].$$

A point μ in $[\text{Bord}_n^{\text{or}}(X)]$ is characterized by the following data: an n -dimensional oriented submanifold M of \mathbb{R}^{n+N} , a background map $f : M \longrightarrow X$ and set of points $(e^1, \dots, e^n) \in (E^1)^n$. Let $p = (p_1, \dots, p_n)$ denote the proper map $M \hookrightarrow \mathbb{R}^{n+N} \longrightarrow \mathbb{R}^n$ (due to the definition). The point μ_{+} in $[\text{Bord}_{n-1}^{\text{or}}(X)]$ is characterized as follows: the underlying $(n-1)$ -manifold is $N = p_n^{-1}(\pi_1(e^n))$ considered as a submanifold of $\mathbb{R}^{n-1} \times \{\pi_1(e^n)\} \times \mathbb{R}^N \simeq \mathbb{R}^{n-1+N}$, the orientation on N is given by contracting $\partial/\partial x_n$ with orientation form on M , and the set of point $(e^1, \dots, e^{n-1}) \in (E^1)^{n-1}$.

PROPOSITION 17. *The map $-_{+} : [\text{Bord}_n^{\text{or}}(X)] \longrightarrow [\text{Bord}_{n-1}^{\text{or}}(X)]$ is continuous.*

Proof. In [BM], Bökstedt and Madsen define functions $d_r : \Psi_{n,N} \times \Psi_{n,N} \longrightarrow \mathbb{R}_{>0}$, such that the subsets $U_{r,\epsilon}(M) = \{N \mid d_r(M, N) < \epsilon\}$ define neighborhood basis for M . For small values of ϵ (depending on M and r) one can show that $d_r(M, N) > 3d_r(M_+, N_+)$. \square

To begin the induction we need to construct the base case, when $n = 0$. The space $\text{TD}_0(X)$ is homeomorphic to $C^\infty(X, U(1))$. The space $\text{Bord}_0^{\text{or}}(X)$ admits a map to the space $\text{Sym}_+(X) \times \text{Sym}_-(X)$, where $\text{Sym}_\pm(X)$ are homeomorphic to the free commutative monoid on X . The map corresponds to the image of the 0-dimensional manifolds mapping into X . The integration pairing on X is defined as the following composition,

$$\text{Bord}_0^{\text{or}}(X) \times \text{TD}_0(X) \longrightarrow \text{Sym}_+(X) \times \text{Sym}_-(X) \times C^\infty(X, U(1)) \longrightarrow \text{Sym}_+(U(1)) \times \text{Sym}_-(U(1)) \longrightarrow U(1).$$

Basically, we are multiplying the values of the function into $U(1)$ at positively oriented points and the inverses of the values of the function at negatively oriented points.

Now suppose that we have constructed the integration pairing up to dimension $n - 1$. From Lemma 2 we know that to specify a morphism from \mathcal{C} into $B^n U(1)$, one ought to specify a map $[\mathcal{C}] \rightarrow U(1)$ satisfying certain condition. Thus, we need to construct a map

$$\iota_X = \left[\int_X \right] : [\text{Bord}_n^{\text{or}}(X)] \times [\text{TD}_n(X)] \rightarrow U(1).$$

Notice that $[\text{TD}_n(X)] \simeq \mathcal{A}^n(X) \times \mathcal{A}^{n-1}(X) \times \cdots \times U(1)(X)$, which can be rewritten as $\mathcal{A}^n(X) \times [\text{TD}_{n-1}(X)]$.

We can define the following inductive formula:

$$\iota_X(\mu, \omega \times \nu) = e^{\int_{\mu^\circ} f^* \omega} \cdot \iota_X(\mu_+, \nu),$$

where μ° is the compact submanifold with corners $\bigcap_{k=1}^n p_k^{-1}[\pi_0(e^k), \pi_1(e^k)]$ of M , f is the background map of M . Note that $[\text{TD}_n(X)]$ is an abelian group and it is easy to see that the map ι_X is group homomorphism in the second variable. To show that ι_X satisfies the condition in the Lemma 2 we need the following facts.

LEMMA 7. *Suppose that $\mu \in [\text{Bord}_n^{\text{or}}(X)]$ and $\omega \times \nu \in [\text{TD}_{n-1}(X)] \simeq \mathcal{A}^{n-1}(X) \times [\text{TD}_{n-2}(X)]$. Then the following equality holds*

$$e^{\int_{\mu_+^\circ} f^* \omega} \iota_X(\mu_-, \omega \times \nu) = e^{\int_{\mu_-^\circ} f^* \omega} \iota_X(\mu_+, \omega \times \nu).$$

Proof. This follows easily from the inductive definition and the observation that $\mu_{-+} = \mu_{++}$. \square

LEMMA 8. *Suppose $\mu \in [\text{Bord}_n^{\text{or}}(X)]$ and $v \in \mathcal{A}^{n-1}(X)$. Then $\int_{\mu^\circ} f^*(dv) = \int_{\mu_+^\circ} f^*v - \int_{\mu_-^\circ} f^*v$.*

Proof. It is not true that $\partial\mu^\circ = \mu_+^\circ \cup \mu_-^\circ$, however, it is true that the latter is contained in the former. The lemma follows from the fact that the background map f is constant (part 4 of the Definition 3.18) on the complement of $\mu_+^\circ \cup \mu_-^\circ$ in $\partial\mu^\circ$ and the Stokes' theorem. \square

PROPOSITION 18. *The map ι_X satisfies the condition in Lemma 2.*

Proof. We prove the proposition inductively on n . If $n = 0$ there is nothing to verify. We consider two cases for k . The first case is $k \neq n$. In this case it is easy to verify that $[\text{TD}_n(X)]_k \simeq \mathcal{A}^n(X) \times [\text{TD}_{n-1}(X)]_k$ and the differentials d^0 , d^1 and d^2 do not do anything on $\mathcal{A}^n(X)$ and simply act on $[\text{TD}_{n-1}(X)]$. Suppose now that we are given a point in $(\mu, \omega, \nu) \in [\text{Bord}_n^{\text{or}}(X)]_k \times \mathcal{A}^n(X) \times [\text{TD}_{n-1}(X)]_k$. Suppose that after applying d^i we obtain the point (μ_i, ω, ν_i) . There is a version of the map $-_+$ for $[\text{Bord}_n^{\text{or}}(X)]_k$, and it can be shown that $(\mu_+)_i = (\mu_i)_+$. One can also observe that $\mu_1^\circ = \mu_0^\circ \cup \mu_2^\circ$. Now the condition of the lemma follows from

the following chain of equalities

$$\begin{aligned} \iota_X(\mu_0, \omega \times \nu_0) \cdot \iota_X(\mu_2, \omega \times \nu_2) &= e^{\int_{\mu_0^\circ} f^* \omega} \iota_X(\mu_{0+}, \nu_0) e^{\int_{\mu_2^\circ} f^* \omega} \iota_X(\mu_{2+}, \nu_2) \\ &= e^{\int_{\mu_1^\circ} f^* \omega} \iota_X(\mu_{1+}, \nu_1) = \iota(\mu_1, \omega \times \nu_1). \end{aligned}$$

Now suppose $k = n$. First thing to note is that $[\mathrm{TD}_n(X)]_n \simeq \mathcal{A}^n(X) \times [\mathrm{TD}_{n-1}(X)]^2$. The differentials are given as follows: $d^0(\omega \times (\nu_1, \nu_2)) = (\omega + dv) \times \nu_2$, $d^1(\omega \times (\nu_1, \nu_2)) = \omega \times (\nu_1 + \nu_2)$, and $d^2(\omega \times (\nu_1, \nu_2)) = \omega \times \nu_1$, here v is the $\mathcal{A}^{n-1}(X)$ component of ν_1 . Now suppose that $\mu \in [\mathrm{Bord}_n^{\mathrm{or}}(X)]_n$ and μ_i are given as above. As above $\mu_1^\circ = \mu_0^\circ \cup \mu_2^\circ$, however, more is true. One can also see that $\mu_{2+} = \mu_{0-}$ and $\mu_{1+} = \mu_{0+}$. We have the following chain of equalities:

$$\begin{aligned} \iota_X(\mu_0, (\omega + dv) \times \nu_2) \cdot \iota_X(\mu_2, \omega \times \nu_1) &= e^{\int_{\mu_0^\circ} f^*(\omega + dv)} \iota_X(\mu_{0+}, \nu_2) e^{\int_{\mu_2^\circ} f^* \omega} \iota_X(\mu_{2+}, \nu_1) \\ &= e^{\int_{\mu_1^\circ} f^* \omega} e^{\int_{\mu_0^\circ} f^*(dv)} \iota_X(\mu_{1+}, \nu_2) \iota_X(\mu_{0-}, \nu_1) \\ &= e^{\int_{\mu_1^\circ} f^* \omega} e^{\int_{\mu_{0+}^\circ} f^* v} e^{-\int_{\mu_{0-}^\circ} f^* v} \iota_X(\mu_{1+}, \nu_2) \iota_X(\mu_{0-}, \nu_1) \\ &= e^{\int_{\mu_1^\circ} f^* \omega} \iota_X(\mu_{1+}, \nu_2) \iota_X(\mu_{0+}, \nu_1) \\ &= e^{\int_{\mu_1^\circ} f^* \omega} \iota_X(\mu_{1+}, \nu_1 + \nu_2) = \iota_X(\mu_1, \omega \times (\nu_1 + \nu_2)). \end{aligned}$$

This completes the inductive step for $k = n$ case, and the proof is complete. \square

LEMMA 9. *The integration pairing is natural in X .*

Proof. The naturality is clear in $n = 0$ case. Suppose that $g : X \rightarrow Y$ is a smooth map of manifolds and $(\mu, \omega \times \nu)$ is an element in $[\mathrm{Bord}_n^{\mathrm{or}}(X)] \times [\mathrm{TD}_n(Y)]$. Then the following equalities hold:

$$\iota_X(\mu, (g^* \omega) \times (g^* \nu)) = e^{\int_{\mu^\circ} f^*(g^* \omega)} \iota_X(\mu, g^* \nu) = e^{\int_{\mu^\circ} (fg)^* \omega} \iota_X(g_* \mu, \nu) = \iota_X(g_* \mu, \omega \times \nu).$$

This uniqueness part in Lemma 2 shows the naturality on the level of functors. \square

Therefore, we obtain the following natural transformation between functors

$$\int_X : \mathrm{Bord}_n^{\mathrm{or}}(X) \times \mathrm{TD}_n(X) \rightarrow \mathrm{B}^n U(1).$$

Now we enhance the pairing to symmetric monoidal setting. Suppose that T is a tree in Ω^{op} . Then we can construct the following composite:

$$\begin{array}{ccc}
\mathrm{Bord}_n^{\mathrm{or},\perp}(X)(T) \times \mathrm{TD}_n(X) & \cdots \cdots \cdots \longrightarrow & \mathrm{B}^n U(1)(T) \\
\downarrow & & \downarrow \\
\prod_{\ell \in L_T} \mathrm{Bord}_n^{\mathrm{or},\perp}(X) \times \mathrm{TD}_n(X) & \longrightarrow & \prod_{\ell \in L_T} \mathrm{B}^n U(1)
\end{array}$$

This map is natural in T , since if μ is a submanifold of $\mathbb{R}^{n+\infty}$ with cuts and a map $f : M \rightarrow X$, which is a disjoint union of $\mu_1, \dots, \mu_n \subset \mathbb{R}^{n+\infty}$ with inherited cuts, then $\iota_X(\mu, \omega) = \prod_{i=1}^n \iota_X(\mu_i, \omega)$. This demonstrates that there is a map:

$$\int_X^\vee : \mathfrak{D}_n(X) \simeq |\mathrm{TD}_n(X)| \longrightarrow \mathrm{TFT}_n^{\mathrm{or}}(X).$$

This map is natural in X . Recall that the functor $\mathrm{TFT}_n^{\mathrm{or}}$ is a sheaf, therefore we obtain the natural factorization:

$$\int_X^\vee : \mathfrak{D}_n^+ \longrightarrow \mathrm{TFT}_n^{\mathrm{or}}.$$

We call this functor the *adjoint integration pairing*.

THEOREM 12. *The adjoint integration pairing, \int_X^\vee , is an equivalence in $\mathrm{Sh}(\mathrm{Man})$.*

Proof. Since both the source and the target are homotopy invariant sheaves, then it suffices to show that the adjoint integration pairing is an equivalence on a point, $*$. Note that $|\mathrm{Bord}_n^{\mathrm{fr}}(*)| \simeq \Omega^\infty \Sigma_+^\infty(*)$. The inclusion of the point $*$ into $\Omega^\infty \Sigma_+^\infty(*)$ gives a homotopy equivalence,

$$\mathrm{Map}^\otimes(\Omega^\infty \Sigma_+^\infty(*), \mathrm{B}^n U(1)) \longrightarrow \mathrm{Map}(*, \mathrm{B}^n U(1)) \simeq \mathrm{B}^n U(1).$$

The inclusion of $*$ into $\Omega^\infty \Sigma_+^\infty(*)$ is induced by the map $*$ into $\mathrm{Bord}_n^{\mathrm{fr}}(*)$ is determined by \mathbb{R}^n with standard framing in \mathbb{R}^{n+N} and $(0, \dots, 0) \in (E^0)^n$ in $\mathrm{Bord}_n^{\mathrm{fr}}(*)([0], \dots, [0])$. One can easily verify from the definition that the composite

$$\mathrm{B}^n U(1) \simeq \mathfrak{D}_n^+(*) \longrightarrow \mathfrak{D}_n^+(*) \times \mathrm{Bord}_n^{\mathrm{fr}}(X) \longrightarrow \mathrm{B}^n U(1),$$

is an isomorphism. This shows that after taking adjoints we indeed obtain an equivalence. \square

2.3 The Geometric Realization of Bordism (∞, n) -Category

2.3.1 Model Structures on Diagram Categories

The purpose of this section is to establish connection between various notions of homotopy colimits via the language of model categories. The theory of model categories is a powerful tool for producing homotopy invariant constructions. In this section homotopy colimits, in particular, the geometric realizations of multisimplicial spaces, are of great importance. In model categorical setting homotopy colimits have an elegant interpretation as left derived Quillen functors of the colimit functor, [Lu1]. As such they are manifestly homotopy invariant. The downside of the definition is that it does not yield itself to direct computations as it involves a mysterious cofibrant-fibrant replacement functor. There are other definitions of homotopy colimits, which involve extra structures (e.g. simplicial structures), [Hi]. These models tend to be more accessible and are of greater use to us. No claims of originality are made for the material in this subsection—it is simply the author’s attempt to document the connections between seemingly disparate notions in the literature concerning homotopy colimits. Most of the facts, can be found in [Lu1], [Ho], and [Hi].

Recall that a model category \mathbf{M} is a complete and cocomplete category with three classes of distinguished morphisms called *cofibrations*, *fibrations*, and *weak equivalences*, which satisfy various axioms, [Ho]. The definition of model categories is redundant in the sense that cofibrations and weak equivalences determine fibrations, and fibrations and weak equivalences determine cofibrations. As we will see later the model structure can be specified with small amount of data. If a (co)fibration happens to be a weak equivalence then we will call it a *trivial* (co)fibration. Let \emptyset and $*$ denote the initial and final objects of \mathbf{M} , respectively. An object $X \in \mathbf{M}$ is *cofibrant* if the map $\emptyset \rightarrow X$ is a cofibration and it is called *fibrant* if $X \rightarrow *$ is a fibration. One can think of cofibrant objects as the homotopically well-behaved objects to *map out of*, and the fibrant objects as the homotopically well-behaved objects to *map into*. The cofibrant-fibrant objects combine the best of both settings. In model categories any object X has *cofibrant and fibrant replacements*, X^c and X^f . These are objects of \mathbf{M} equipped with maps $X^c \rightarrow X \rightarrow X^f$ both of which are weak equivalences, such that X^c is cofibrant and X^f is fibrant. As in [Ho], we will require existence of functorial cofibrant and fibrant replacements. For more detail on the definitions, see [Ho].

EXAMPLE 28. Let \mathbf{Top} denote the category of compactly generated weakly Hausdorff spaces. This category can be endowed with a model structure with the following specifications:

- the cofibrations are retracts of relative cell-complexes;
- fibrations are the Serre fibrations;
- the weak equivalences are the weak homotopy equivalences.

This model structure is called the Serre model structure. This model category is going to be of great impor-

tance to us.

EXAMPLE 29. There is another model structure on $\mathcal{J}\text{op}$ called the Hurewicz model structure, where the classical notions of cofibration and fibrations are used and the weak equivalences are homotopy equivalences. We will not use the Hurewicz model structure as it is not a well-behaved model structure, and because in homotopy theory we are interested in constructions that are invariant under *weak* homotopy equivalences.

EXAMPLE 30. Let Δ denote the simplex category, i.e. the category whose objects are the finite ordered sets of form $[k] = \{0, 1, \dots, k\}$ and whose morphisms are the order preserving maps. Let Set_Δ denote the category of simplicial sets, i.e. the functor category $\text{Set}^{\Delta^{\text{op}}}$. This category can be given a model structure as follows:

- cofibrations are objectwise monomorphism;
- fibrations are the Kan fibrations;
- weak equivalence are the morphisms that turn into weak homotopy equivalences when geometrically realized.

In the last statement, geometric realization is a functor $|-| : \text{Set}_\Delta \rightarrow \mathcal{J}\text{op}$, which is specified by the following coend

$$|X_\bullet| = \int_{\Delta^{\text{op}}} X_\bullet \times \Delta^\bullet,$$

where Δ^n is the standard n -simplex in \mathbb{R}^{n+1} .

Arguably the most important invariant of a model category is its *homotopy category*—the category obtained by localizing with respect to weak equivalences. The homotopy category is analogous to homotopy groups for topological spaces. The homotopy category of \mathbf{M} will be denoted by $\text{Ho}(\mathbf{M})$. The notion of a homotopy category can be defined for any category with distinguished class of morphisms. The model category structure on \mathbf{M} allows to get a description of $\text{Ho}(\mathbf{M})$ via the cofibrant and fibrant replacement functors. For more details consult [Ho].

The purpose of model categories is to provide a setting, where one can mimic homotopy theoretic techniques. Changing the settings is an extremely important paradigm, since certain model categories may be convenient than the others. A *Quillen adjunction* is an adjunction between model categories $F : \mathbf{M} \rightleftarrows \mathbf{N} : U$, such that one of the equivalent conditions holds:

- F preserves cofibrations and trivial cofibrations;
- U preserves fibrations and trivial fibrations.

Attached to a Quillen adjunction is the notion of left and right derived functors. The left derived functor of (F, U) , written as $\mathbb{L}F$ is the composite of the cofibrant replacement functor $\mathbf{M} \rightarrow \mathbf{M}$ and $F : \mathbf{M} \rightarrow \mathbf{N}$. Dually, one defines the right derived functor $\mathbb{R}U$ as the composite of the fibrant replacement and U . In fact,

these functor descend to functors on homotopy categories $\mathbb{L}F : \text{Ho}(\mathbf{M}) \rightleftarrows \text{Ho}(\mathbf{N}) : \mathbb{R}U$. This pair is an adjunction.

A Quillen adjunction (F, U) as above is called a *Quillen equivalence* if for any cofibrant $X \in \mathbf{M}$ and fibrant $Y \in \mathbf{N}$, $F(X) \rightarrow Y$ is a weak equivalence if and only if its adjoint $X \rightarrow U(Y)$ is a weak equivalence. This is equivalent to saying that the pair $(\mathbb{L}F, \mathbb{R}U)$ on the homotopy categories is an equivalence. If there is a Quillen equivalence between two model categories—the homotopy invariant constructions in both model categories produce equivalent results.

EXAMPLE 31. There is a Quillen equivalence $\| : \text{Set}_\Delta \rightleftarrows \text{Top} : \text{Sing}$, where $\|$ is the geometric realization functor mentioned earlier and Sing is the singularization functor, namely $\text{Sing}(X)_\bullet = \text{Hom}_{\text{Top}}(\Delta^\bullet, X)$, [Ho]. These are two models for the homotopy theory of spaces or the theory of homotopy types. We need to work with both model categories. The bordism categories are defined using the category of topological spaces and some of the initial manipulations will be done using this category. In particular, central to our argument is the result of Bökstedt and Madsen on abstract transversality. However, for later manipulations and for making connection with (∞, n) -categories we need to convert topological spaces to simplicial sets as they are more convenient for ∞ -categorical manipulations.

In the theory of model categories the notion of cofibrant generation plays a very important role. This notion allows to reduce a great number of questions pertaining to the model category to a small collection of cofibrations and trivial cofibrations. There is a more restrictive notion of combinatorial model category. The latter model categories are very convenient in terms of their general properties, and more specifically, in terms of their interactions with diagram categories.

Let I be a collection of morphisms in \mathbf{M} . Let $I\text{-inj}$ denote the collection of all morphisms that have the right lifting property with respect to all morphisms in I . The collection $I\text{-cell}$ will denote the smallest class of morphisms containing I and which is closed under pushouts and transfinite compositions. An object A is called κ -small relative to I for some cardinal κ , if $\text{Hom}(A, -)$ commutes with κ -filtered colimits, where the images of the successor maps land in I . An object is called small if it is κ -small for some cardinal κ .

DEFINITION 24. We say that a model category \mathbf{M} is cofibrantly generated if there are sets I and J of morphisms, such that

- 1) the domains of maps of I are small relative to $I\text{-cell}$;
- 2) the domains of maps of J are small relative to $J\text{-cell}$;
- 3) the class of fibrations is $J\text{-inj}$;
- 4) the class of trivial fibrations is $I\text{-inj}$.

We will call I generating cofibrations and J generating trivial cofibrations. If furthermore, \mathbf{M} is presentable, then we call \mathbf{M} combinatorial.

EXAMPLE 32. The category of simplicial sets Set_Δ under the Kan model structure is cofibrantly generated. The generating cofibrations are $\partial\Delta^n \longrightarrow \Delta^n$ and the generating trivial cofibrations are $\Lambda_i^n \longrightarrow \Delta^n$. Notice that $\partial\Delta^n$ and Λ_i^n are small relative to Set_Δ itself. In fact, all objects in Set_Δ are small. Thus, Set_Δ is a combinatorial model category.

EXAMPLE 33. The category of $\mathcal{T}\text{op}$ under the Serre model structure is cofibrantly generated. In this case the generating cofibrations are the disk boundary inclusions $\partial D^n \longrightarrow D^n$ and the generating trivial cofibrations are $D^n \times \{0\} \longrightarrow D^n \times I$. Unlike the previous example, not all objects in $\mathcal{T}\text{op}$ are small. The Sierpinski space is an example of pathological space, [Ho]. Compact spaces are small relative to I -cell and J -cell. $\mathcal{T}\text{op}$ is not a presentable category, and hence is not combinatorial.

We would like to endow diagram categories with model structures. The following theorem, [Hi], guarantees that there is at least one model structure on $\mathbf{M}^{\mathcal{C}}$ if \mathcal{C} is a small category and \mathbf{M} is a cofibrantly generated model category.

THEOREM 13. *Suppose \mathbf{M} is a cofibrantly generated model category and \mathcal{C} is a small category. Then the functor category $\mathbf{M}^{\mathcal{C}}$ admits a cofibrantly generated model structure, called the projective model structure, where the fibrations and weak equivalences are determined objectwise.*

There is a description of the generating cofibrations and generating acyclic cofibrations. Since \mathbf{M} is cocomplete, then we can tensor it with sets. More specifically, there is a functor $\otimes : \text{Set} \times \mathbf{M} \longrightarrow \mathbf{M}$, sending (S, M) to the S -fold coproduct of M with itself. Using the homomorphism functor on \mathcal{C} , we can define $\mathcal{C}^{\text{op}} \times \mathcal{C} \times \mathbf{M} \longrightarrow \text{Set} \times \mathbf{M} \longrightarrow \mathbf{M}$ and taking its adjoint with respect to \mathcal{C} , we obtain the functor $\mathcal{C}^{\text{op}} \times \mathbf{M} \longrightarrow \mathbf{M}^{\mathcal{C}}$, where we will write its value on (C, M) as \mathcal{F}_M^C . The generating (trivial) cofibrations are $\mathcal{F}_M^C \longrightarrow \mathcal{F}_{M'}^C$, where C varies over all objects of \mathcal{C} and $M \longrightarrow M'$ varies over all generating (trivial) cofibrations of \mathbf{M} . We can conclude from this observation that a natural transformation in $\mathbf{M}^{\mathcal{C}}$ is a cofibration then it is a cofibration objectwise. The converse does not hold in general.

REMARK 22. If \mathbf{M} is combinatorial, then $\mathbf{M}^{\mathcal{C}}$ is also combinatorial. In addition to this, we can specify another model structure, called the injective model structure on $\mathbf{M}^{\mathcal{C}}$, where this time it is the cofibration and weak equivalences are determined by objectwise. This is quite convenient for the purposes of considering homotopy limits from model categorical point of view, but we will not have a use for the injective model structure.

There always exists an adjunction $\text{colim} : \mathbf{M}^{\mathcal{C}} \rightleftarrows \mathbf{M} : \text{const}$ for cocomplete categories \mathbf{M} . With the projective model structure on $\mathbf{M}^{\mathcal{C}}$, this adjunction is clearly a Quillen adjunction. We define hocolim to be the left derived functor of colim . More specifically, if X is a functor from \mathcal{C} to \mathbf{M} , then $\text{Lcolim } X$ is defined as $\text{colim } X^c$, where $X^c \longrightarrow X$ is the cofibrant replacement of X in the diagram category $\mathbf{M}^{\mathcal{C}}$.

Earlier we mentioned that the homotopy invariant constructions in Quillen equivalent model categories. We would like to demonstrate this feature of Quillen equivalences in the setting of homotopy colimits. This requires certain amount of care, since there are two functors involved in a Quillen equivalence and their manipulations differ from each another. Suppose now that $F : \mathbf{M} \rightleftarrows \mathbf{N} : U$ is a Quillen adjunction. Then the induced adjunction $F^c : \mathbf{M}^c \rightleftarrows \mathbf{N}^c : U^c$ is also a Quillen adjunction. In fact, it is clear that if (F, U) is a Quillen equivalence, then so is (F^c, U^c) . It follows that $\mathbb{L}\text{colim}(\mathbb{L}(F^c)(X)) \simeq \mathbb{L}F(\mathbb{L}\text{colim}(X))$ for any diagram $X \in \mathbf{M}^c$. Indeed, to see this one has note that to compute either side of the equivalence one needs to replace X with its cofibrant replacement $X^c \rightarrow X$ and then apply colim and F on one side, and F and colim on the other side, and since F commutes with colimits the results are going to be weakly equivalent. The following claim is a bit more involved, but it demonstrates the care that needs to be taken, when mixing different sides of the adjunction.

PROPOSITION 19. *There is a zigzag of weak equivalence $\mathbb{L}\text{colim}(\mathbb{R}U(X)) \simeq \mathbb{R}U(\mathbb{L}\text{colim}(X))$ for any diagram X in \mathbf{N}^c .*

Proof. Let us denote by X° the object $(X^c)^f$. Note that there is a weak equivalence $U(X^\circ)^c \rightarrow U(X^\circ)$. Then it follows that $F(U(X^\circ)^c) \rightarrow X^\circ$ is a weak equivalence, since X° is fibrant and $U(X^\circ)^c$ is cofibrant. Since X° is also cofibrant, we conclude that $F\text{colim}(U(X^\circ)^c) \cong \text{colim} F(U(X^\circ)^c) \rightarrow \text{colim} X^\circ$ is a weak equivalence. Note that $X^c \rightarrow X^\circ$ is a trivial cofibration, thus so is $\text{colim} X^c \rightarrow \text{colim} X^\circ$ and since $(\text{colim} X^c)^f$ is fibrant, we conclude that there is a weak equivalence $\text{colim} X^\circ \rightarrow (\text{colim} X^c)^f$. Thus, we obtained a weak equivalence $F\text{colim}(U(X^\circ)^c) \rightarrow (\text{colim} X^c)^f$, from which it follows that the adjoint $\text{colim}(U(X^\circ)^c) \rightarrow U((\text{colim} X^c)^f) = \mathbb{R}U(\mathbb{L}\text{colim}(X))$. There is a weak equivalence $X^\circ \rightarrow X^f$ between fibrant objects. Then we see that $U(X^\circ) \rightarrow U(X^f)$ is an equivalence and so is $U(X^\circ)^c \rightarrow U(X^f)^c$. Since the latter two objects are cofibrant we get an equivalence $\text{colim}(U(X^\circ)^c) \rightarrow \text{colim}(U(X^f)^c) = \mathbb{L}\text{colim}(\mathbb{R}U(X))$. This establishes a zigzag of weak equivalences between the desired objects. \square

There are other ways also of putting model structures on the model categories. In particular, if the diagram category \mathcal{J} is a Reedy category, then the diagram categories $\mathbf{M}^{\mathcal{J}}$ admit a model structure, called the *Reedy model structure*, for all model categories \mathbf{M} . Recall that Reedy structure on a category \mathcal{J} consists of the following data:

- 1) two subcategories \mathcal{J}_+ and \mathcal{J}_- ;
- 2) a degree function $d : \text{Ob}(\mathcal{J}) \rightarrow \lambda$, where λ is an ordinal.

The subcategories $(\mathcal{J}_-, \mathcal{J}_+)$ form a factorization system, meaning that any morphism f in \mathcal{J} factors uniquely as $f = gh$, where h is a morphism in \mathcal{J}_- and g is a morphism in \mathcal{J}_+ . The final piece of Reedy structure is the requirement that if $h : I \rightarrow J$ is a non-identity morphism in \mathcal{J}_- , then $d(J) \prec d(I)$, and if h is a non-identity morphism in \mathcal{J}_+ , then $d(I) \prec d(J)$. Note that $(\mathcal{J}^{\text{op}})_+ = \mathcal{J}_-$, $(\mathcal{J}^{\text{op}})_- = \mathcal{J}_+$ and $d : \text{Ob}(\mathcal{J}^{\text{op}}) = \text{Ob}(\mathcal{J}) \rightarrow \lambda$

define a Reedy structure on \mathcal{J}^{op} .

In order to define the Reedy model structure we need to introduce the notions of latching and matching functors. Suppose that I is an object in \mathcal{J} . Then let us denote by \mathcal{J}_+^I the full subcategory of \mathcal{J}_+ of objects that admit a map to I and are not equal to I . Then we define the I -latching functor

$$L_I : \mathbf{M}^{\mathcal{J}} \longrightarrow \mathbf{M}^{\mathcal{J}_+^I} \longrightarrow \mathbf{M},$$

where the second functor is the colimit functor. Dually, we can define \mathcal{J}_-^I as the full subcategory of \mathcal{J}_- of objects that admit a map from I and that are not equal to I . The I -matching functor is defined as

$$M_I : \mathbf{M}^{\mathcal{J}} \longrightarrow \mathbf{M}^{\mathcal{J}_-^I} \longrightarrow \mathbf{M},$$

where the second map is the limit functor. One can observe that there is a natural sequence of maps $L_I(X) \longrightarrow X(I) \longrightarrow M_I(X)$ for all $X \in \mathbf{M}^{\mathcal{J}}$ and $I \in \mathcal{J}$. The proof can be found in [Ho].

PROPOSITION 20. *Suppose that \mathcal{J} is a Reedy category and \mathbf{M} is a model category. Then $\mathbf{M}^{\mathcal{J}}$ admits a model structure via the following specifications:*

- 1) $X \longrightarrow Y$ is a (trivial) cofibration if the map $X(I) \coprod_{L_I(X)} L_I(Y) \longrightarrow Y(I)$ is a (trivial) cofibration for all objects I in \mathcal{J} ;
- 2) $X \longrightarrow Y$ is a (trivial) fibration if the map $X(I) \longrightarrow Y(I) \times_{M_I(Y)} M_I(X)$ is a (trivial) fibration for all objects I in \mathcal{J} ;
- 3) weak equivalences are determined objectwise;

This model structure is called the Reedy model structure.

REMARK 23. If \mathbf{M} is a combinatorial model category and \mathcal{J} is a Reedy category, then we have three model structures on $\mathbf{M}^{\mathcal{J}}$. The identity functors form the following chain of Quillen equivalences:

$$\mathbf{M}_{\text{proj}}^{\mathcal{J}} \rightleftarrows \mathbf{M}_{\text{Reedy}}^{\mathcal{J}} \rightleftarrows \mathbf{M}_{\text{inj}}^{\mathcal{J}}.$$

Reedy model structure is useful because of its technically advantageous interaction with the Quillen bifunctors. Recall that a functor

$$\otimes : \mathbf{M} \times \mathbf{N} \longrightarrow \mathbf{K},$$

is a left Quillen bifunctor if it admits adjoints in both variables and satisfies the pushout product axiom, [Ho]. The first condition is equivalent to saying that \otimes preserves colimits in each variable. We can consider

the coend functor

$$\int_{\mathcal{J}} : \mathbf{M}^{\mathcal{J}} \times \mathbf{N}^{\mathcal{J}^{\text{op}}} \longrightarrow \mathbf{K}.$$

Then we have the following result from [Lu1], A.2.9.26.

PROPOSITION 21. *If we endow $\mathbf{M}^{\mathcal{J}}$ and $\mathbf{N}^{\mathcal{J}^{\text{op}}}$ with Reedy model structures then the coend functor is a left Quillen bifunctor. If \mathbf{M} is combinatorial, then the same conclusion holds if we endow $\mathbf{M}^{\mathcal{J}}$ with the projective model structure and $\mathbf{N}^{\mathcal{J}^{\text{op}}}$ with the injective model structure.*

Recall that the Cartesian product on Set_{Δ} is left Quillen bifunctor $\times : \text{Set}_{\Delta} \times \text{Set}_{\Delta} \longrightarrow \text{Set}_{\Delta}$. In fact, this functor can be enhanced to a symmetric monoidal product on Set_{Δ} , which is compatible with the Kan model structure, thus, turning Set_{Δ} into a symmetric monoidal model category. We can define symmetric monoidal model categories as the model categories on which Set_{Δ} has an action, i.e. there is left Quillen bifunctor

$$\otimes : \mathbf{M} \times \text{Set}_{\Delta} \longrightarrow \mathbf{M},$$

which satisfies various axioms, [Ho]. The coend functor produces a Quillen bifunctor for

$$\int_{\mathcal{J}} : \mathbf{M}^{\mathcal{J}} \times \text{Set}_{\Delta}^{\mathcal{J}^{\text{op}}} \longrightarrow \mathbf{M}.$$

In $\text{Set}_{\Delta}^{\mathcal{J}^{\text{op}}}$ there is the terminal functor $*$ that evaluates to $\mathbf{\Delta}^0$ on each object $J \in \mathcal{J}^{\text{op}}$. The important feature of this functor is that if X is a diagram in $\mathbf{M}^{\mathcal{J}}$, then $\int_{\mathcal{J}} X \otimes *$ is the colimit of X . There is another functor $\mathcal{J}^{\bullet} : \mathcal{J}^{\text{op}} \longrightarrow \text{Set}_{\Delta}$, which is defined by sending an object J to $\mathbf{N}(\mathcal{J}_J)$, where \mathbf{N} is the nerve construction. Note that since \mathcal{J}^J has an initial point, then the map to $\mathcal{J}^{\bullet} \longrightarrow *$ is weak equivalence.

LEMMA 10. *The diagram \mathcal{J}^{\bullet} is a Reedy cofibrant diagram.*

Proof. To show the lemma we need to demonstrate that $L_I(\mathcal{J}^{\bullet})_n$ maps injectively into $\mathcal{J}^{\bullet}(I)_n$. This follows from the fact that \mathcal{J}_- maps inverted category. \square

From the lemma we can conclude the following proposition, which gives an effective way of describing homotopy colimits.

PROPOSITION 22. *Let \mathbf{M} be a cofibrantly generated model category. Then there is natural morphism*

$$\mathbb{L}\text{colim } X \longrightarrow \int_{\mathcal{J}} X \otimes \mathcal{J}^{\bullet},$$

with X a diagram in $\mathbf{M}^{\mathcal{J}}$, which is a weak equivalence if X is Reedy cofibrant.

This result can be applied to the cases, where \mathbf{M} is either Set_{Δ} or Top . In case of Set_{Δ} any diagram is

Reedy cofibrant, therefore, the natural morphism above is always a weak equivalence. Not every diagram in $\mathcal{Top}^{\mathcal{J}}$ is Reedy cofibrant, and in fact, the diagrams of interest to us will not be Reedy cofibrant in Serre model structure. Fortunately, we can work around this issue using the Hurewicz Reedy cofibrant diagrams. The following proposition is the essential ingredient in our argument.

PROPOSITION 23. *Let X be a Hurewicz Reedy cofibrant diagram in $\mathcal{Top}^{\mathcal{J}}$. Then there is a strongly convergent spectral sequence*

$$L^q \text{colim} (\mathbb{H}_p(X(-), \mathbb{Z})) \implies \mathbb{H}_{p+q}(\int_{\mathcal{J}} X \times \mathcal{J}^{\bullet}, \mathbb{Z}),$$

where $L^q \text{colim}$ denotes the left derived functor of the colimit functor $\mathcal{Ab}^{\mathcal{J}} \rightarrow \mathcal{Ab}$.

Proof. The space $\int_{\mathcal{J}} X \times \mathcal{J}^{\bullet}$ is the geometric realization of the following simplicial space

$$\coprod_{J_0} X(J_0) \longleftarrow \coprod_{J_0 \rightarrow J_1} X(J_0) \rightleftarrows \coprod_{J_0 \rightarrow J_1 \rightarrow J_2} X(J_0) \rightleftarrows \dots$$

Each such geometric realization comes with a natural simplicial filtration, which produces the spectral sequence above. \square

COROLLARY 9. *Suppose the $F : X \rightarrow Y$ is a natural transformation between Hurewicz Reedy cofibrant diagrams in $\mathcal{Top}^{\mathcal{J}}$ that evaluates to a weak homotopy equivalence on each object of \mathcal{J} . Then the induced map*

$$\int_{\mathcal{J}} X \times \mathcal{J}^{\bullet} \rightarrow \int_{\mathcal{J}} Y \times \mathcal{J}^{\bullet},$$

is also a weak homotopy equivalence.

Proof. From the previous proposition, we see that the induced map is homology equivalence. It can be demonstrated easily that the map also induces an equivalence in π_0 and π_1 . \square

The bordism (∞, n) -categories $\text{Bord}_{n,N}^{\xi, \perp}$ and $\text{Bord}_{n,N}^{\xi}$ are Hurewicz Reedy cofibrant, so Claim 2.1.24 shows that these two categories are equivalent. In addition to this, we see that if we consider a simplicial topological space X_{\bullet} , which is Hurewicz Reedy cofibrant, then if we consider the simplicial space $|\text{Sing}(X)|_{\bullet}$ obtained by applying the singularization and geometric realization functor levelwise, then $|X_{\bullet}|$ is weakly equivalent to the geometric realization of $|\text{Sing}(X)|_{\bullet}$. The point is that so long as we consider Hurewicz Reedy cofibrant topological diagram their geometric realizations done by using the coend construction are the same as that of the one done in the simplicial setting.

2.3.2 Abstract Transversality and Other Constructions

In this section we review Bökstedt-Madsen's streamlined abstract transversality theorem for simplicial spaces as written in [BoMa]. They prove a general theorem for multisimplicial spaces. Their theorem in its full generality is not applicable to this paper. We will only need the case of simplicial spaces, which is simpler to explain.

We will proceed to describe the category of topological spaces with critical locus, Crit . The objects of this category are pairs (X, Z) , where X is topological spaces in Top , and Z is a closed subset of $X \times \mathbb{R}$. A morphism from (X, Z) to (Y, W) is a continuous map $f : X \rightarrow Y$, such that $(f \times \mathbb{1}_{\mathbb{R}})^{-1}(W)$ is a subset of Z . There is functor

$$K_{\bullet} : \text{Crit} \rightarrow \text{Top}^{\Delta^{\text{op}}},$$

which is characterized by the following properties,

- $K_n(X; Z)$ of points in $(x, r_0, \dots, r_n) \in X \times \mathbb{R}^{n+1}$, so that $(x, r_m) \notin Z \cap (\{x\} \times \mathbb{R})$ for all m in $\{0, \dots, n\}$;
- the face and degeneracy maps are given by forgetting and inserting r_i 's at appropriate spots.

Note that $K_{\bullet}(X; Z)$ admits a map to the constant diagram at X . The abstract transversality theorem states the following.

THEOREM 14. *Suppose that X is a metrizable topological space and $Z \subset X \times \mathbb{R}$ satisfies the property that $Z \cap (\{x\} \times \mathbb{R})$ has measure 0 for all $x \in X$. The natural map $|K_{\bullet}(X; Z)| \rightarrow X$ is a weak homotopy equivalence.*

EXAMPLE 34. Let us consider $X = \Psi_n(\mathbb{R} \times I^N, \mathbb{R} \times \partial I^N)$ the space of closed n -dimensional submanifolds of $\mathbb{R} \times I^N$ with no intersection with the boundary, and let $Z \subset X \times \mathbb{R}$, be the union of $\{M\} \times \{\text{critical locus of the projection } M \rightarrow \mathbb{R}\}$. The simplicial space $K_{\bullet}(X; Z)$ is a model of bordism ∞ -category. By Saard's theorem and the abstract transversality theorem demonstrates that $|K_{\bullet}(X; Z)| \rightarrow X$ is homotopy equivalence.

We also need to introduce a couple of constructions on multisimplicial diagrams. We will make use of the results in the Rezk, [Re3]. We begin by introducing the notion of realization fibration. From this point on we will assume that $\text{Set}_{\Delta}^{(\Delta^{\times n})^{\text{op}}}$ is given the Reedy model structure.

DEFINITION 25. *A morphism $X \rightarrow Y$ in $\text{Set}_{\Delta}^{(\Delta^{\times n})^{\text{op}}}$ is called a realization fibration if any homotopy pullback of the form*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

stays a homotopy pullback after geometric realization.

There is another notion of local realization fibration, which is a simpler condition to test. Note that $[\mathbf{m}]$ be an element in $\Delta^{\times n}$. By Yoneda's lemma there is an embedding $i : \Delta^{\times n} \longrightarrow \text{Set}_{\Delta}^{(\Delta^{\times n})^{\text{op}}}$. The n -simplicial space representing $[\mathbf{m}]$ is $i([\mathbf{m}])$.

DEFINITION 26. Let $f : X \longrightarrow Y$ be a morphism in $\text{Set}_{\Delta}^{(\Delta^{\times n})^{\text{op}}}$ and let $\phi : i([\mathbf{m}]) \longrightarrow Y$ be another morphism. Let $F(f, \phi)$ denote the pullback of the resulting diagram. The morphism f will be called a local realization fibration if for any map $\psi : [\mathbf{r}] \longrightarrow [\mathbf{m}]$ the map $|F(f, i(\psi) \circ \phi)| \longrightarrow |F(f, \phi)|$ is a weak equivalence.

The following theorem is Theorem 2.13 in [Re3].

THEOREM 15. A morphism $f : X \longrightarrow Y$ in $\text{Set}_{\Delta}^{(\Delta^{\times n})^{\text{op}}}$ is a realization fibration if and only if it is a local realization fibration.

Let us first define the following n -cosimplicial space. Recall that $\Delta^{\bullet} : \Delta \longrightarrow \text{Set}_{\Delta}$ corresponds to the Yoneda embedding. Let us denote by $\hat{\Delta}^{\bullet} : \Delta \longrightarrow \text{Set}_{\Delta}$, which sends $[n]$ to the discrete simplicial set corresponding to $\{0, 1, \dots, n\}$. For any functor $G : \Delta^{\times n} \longrightarrow \text{Set}_{\Delta}$ we can define $\vartheta(G)$ as follows

$$\begin{array}{ccc} \prod_{k=1}^{n-1} \prod_{\Delta^0 \longrightarrow \Delta^{m_k}} G([m_1], \dots, [0], \dots, [m_n]) & \longrightarrow & G([m_1], \dots, [m_n]) \\ \downarrow & & \downarrow \\ \prod_{k=1}^{n-1} \prod_{\Delta^0 \longrightarrow \Delta^{m_k}} G([m_1], \dots, [0], \dots, [0]) & \longrightarrow & \vartheta(G)([m_1], \dots, [m_n]) \end{array}$$

Using this construction we make the following definitions:

$$\begin{aligned} \vartheta_n &= \vartheta(\underbrace{\Delta^{\bullet} \times \dots \times \Delta^{\bullet}}_{n \text{ times}}) \\ \vartheta_n^{\leq k} &= \vartheta(\underbrace{\Delta^{\bullet} \times \dots \times \Delta^{\bullet}}_{k \text{ times}} \times \underbrace{\hat{\Delta}^{\bullet} \times \dots \times \hat{\Delta}^{\bullet}}_{n-k \text{ times}}) \text{ where } 0 \leq k < n, \end{aligned}$$

This gives us a sequence of inclusions $\vartheta_n^{\leq 0} \hookrightarrow \vartheta_n^{\leq 1} \hookrightarrow \dots \hookrightarrow \vartheta_n^{\leq n-1} \hookrightarrow \vartheta_n$.

Recall that Set_{Δ} is Cartesian closed, which means that there is a functor $\text{Map} : \text{Set}_{\Delta}^{\text{op}} \times \text{Set}_{\Delta} \longrightarrow \text{Set}_{\Delta}$ adjoint to the product. We can, therefore, take a composites $T_n(X) = \text{Map} \circ (\vartheta_n \times X) : (\Delta^{\times n})^{\text{op}} \longrightarrow \text{Set}_{\Delta}$, and also, $T_n^{\leq k}(X) = \text{Map} \circ (\vartheta_n^{\leq k} \times X) : (\Delta^{\times n})^{\text{op}} \longrightarrow \text{Set}_{\Delta}$. Thus, we have a tower of fibrations

$$T_n(X) \longrightarrow T_n^{\leq n-1}(X) \longrightarrow \dots \longrightarrow T_n^{\leq 0}(X).$$

These constructions have a few interesting properties.

LEMMA 11. *If $0 \leq k < n$ and X is non-empty Kan complex, then the geometric realization of the n -simplicial space $T_n^{\leq k}(X)$ is equivalent to $\tau_{\leq k-1}(X)$, where $\tau_{\leq k-1}$ denotes the k -truncation functor.*

Proof. The proof is done inductively. Notice that $T_1^{\leq 0}(X)([n]) = X^n$ with face maps defined by forgetting and the degeneracy maps are defined by repetition. This simplicial space is known to have contractible resolution.

Now let us assume that $k \geq 1$ and $n \geq 1$. Let us denote by $t_n^{\leq k}(X)$ the simplicial set $\text{Hom} \circ (\vartheta_n^{\leq k} \times X)$. We will demonstrate that $|t_n^{\leq k}(X)| \simeq \tau_{\leq k-1}(X)$. Note that $t_n^{\leq k}(X^Y)$ is isomorphic to $t_n^{\leq k}(X)^Y$. If we take Y to be Δ^1 , then we see that $t_n^{\leq k}$ preserves homotopy invariance. We can always choose a subcomplex X' of X , which contains a single vertex for each path component of X , and such that the inclusion $X' \hookrightarrow X$ is a deformation retract of X . Thus, without loss of generality we can assume that X has a single vertex in each path component. Notice that in this case $t_n^{\leq k}(X)$ is isomorphic to $\coprod t_n^{\leq k}(X_\alpha)$, where X_α 's denote the path components of X . This means that we can assume that X is connected with a single vertex, $*$, without loss of generality. This where the inductive assumption kicks in, $t_n^{\leq k}(X)$ is isomorphic to

$$* \rightleftarrows t_{n-1}^{\leq k-1}(\Omega X) \rightleftarrows t_{n-1}^{\leq k-1}(\Omega X \times \Omega X) \rightleftarrows \dots$$

Note that $|t_{n-1}^{\leq k-1}(\Omega X^n)| \simeq \tau_{k-2}(\Omega X^n) \simeq \Omega \tau_{k-1}(X^n) \simeq \Omega(\tau_{k-1}(X))^n$. Thus, the realization simplicial $n-1$ -simplicial set above is equivalent to realization of the following simplicial space

$$* \rightleftarrows \Omega \tau_{k-1}(X) \rightleftarrows \Omega \tau_{k-1}(X) \times \Omega \tau_{k-1}(X) \rightleftarrows \dots,$$

whose realization is $\tau_{k-1}(X)$. \square

LEMMA 12. *If X is a Kan complex then the geometric realization of $T_n(X)$ is equivalent to X .*

Proof. Note that X defines a functor $(\text{Set}_\Delta^{\Delta^{\times n}})^{\text{op}} \longrightarrow \text{Set}_\Delta^{(\Delta^{\times n})^{\text{op}}}$ by sending a diagram $\alpha : \Delta^{\times n} \longrightarrow \text{Set}_\Delta$ to $\text{Map} \circ (\alpha \times X)$. Since X is a Kan complex, then this functor sends trivial cofibrations to trivial fibrations. By Ken Brown's lemma, we conclude that it sends weak equivalences between cofibrations to weak equivalences. In particular, consider the map $* \longrightarrow \vartheta_n$, which is an equivalence between cofibrant diagrams, since all $\vartheta_n([m_1], \dots, [m_n])$'s are contractible simplicial sets. Thus, $T_n(X) \longrightarrow \text{Map}(*, X)$ is a weak equivalence. Since, the geometric realization also preserves weak equivalences between cofibrant objects, the lemma follows. \square

LEMMA 13. *Suppose that $X \longrightarrow Y$ is a map of simplicial sets that is k -connected. Then the morphism of n -simplicial space $T_n^{\leq k}(X) \longrightarrow T_n^{\leq k}(Y)$ is a realization fibration.*

Proof Sketch. We can assume that $f : X \longrightarrow Y$ is a fibration. Therefore, $T_n^{\leq k}(f) : T_n^{\leq k}(X) \longrightarrow T_n^{\leq k}(Y)$ is

also a fibration, so homotopy pullbacks match the pullbacks.

The proof will be performed inductively. We first demonstrate that if the map $X \rightarrow Y$ is surjective on path-components, then the map $T_1^{\leq 0}(X) \rightarrow T_1^{\leq 0}(Y)$ is a local realization-fibration. Suppose that we have map $\phi : i([m]) \rightarrow T_1^{\leq 0}(Y)$ that corresponds to the point $(y_0, \dots, y_m) \in Y_0^m$. Let F_k denote the fiber of the map $X \rightarrow Y$ at the point y_k . Then

$$F(T_n^{\leq k}(f), \phi)([k]) = \coprod_{g:[k] \rightarrow [m]} F_{g(0)} \times \dots \times F_{g(k)}.$$

One can show that if one of F_0, \dots, F_m are non-empty, then $|F(T_n^{\leq k}(f), \phi)|$ is contractible. This can be the case if $X \rightarrow Y$ is surjective on vertices. This is the case, since $X \rightarrow Y$ is fibration and is surjective on components.

Now assume that $k \geq 1$. We will demonstrate that $t_n^{\leq k}(X) \rightarrow t_n^{\leq k}(Y)$ is a realization fibration. Note that $\Omega X^r \rightarrow \Omega Y^r$ is $(k-1)$ -connected for any r . Since geometric realization commutes with disjoint unions we can assume that both X and Y have a single vertex. Note that if we have map $Z \rightarrow t_n^{\leq k}(Y)$, and we consider the pullback W , then $W([r])$ is given as the pullback of the following diagram

$$\begin{array}{ccc} W([r]) & \longrightarrow & t_{n-1}^{\leq k-1}(\Omega X^r) \\ \downarrow & & \downarrow \\ Z([r]) & \longrightarrow & t_{n-1}^{\leq k-1}(\Omega Y^r) \end{array}$$

Since $\Omega X^r \rightarrow \Omega Y^r$ is $(k-1)$ -connected, then the diagram above remains a homotopy pullback after we apply realization fibration. If we take the homotopy colimit with respect to r of all the geometrically realized diagrams, we will still have a homotopy pullback diagram, because $t_{n-1}^{\leq k-1}(\Omega X^{\times \bullet}) \rightarrow t_{n-1}^{\leq k-1}(\Omega Y^{\times \bullet})$ is an equifibered natural transformation of functors from Δ^{op} to $\text{Fun}((\Delta^{\times n-1})^{\text{op}}, \text{Set})$. The resulting diagram is

$$\begin{array}{ccc} |W| & \longrightarrow & |t_n^{\leq k}(X)| \\ \downarrow & & \downarrow \\ |Z| & \longrightarrow & |t_n^{\leq k}(Y)| \end{array}$$

which demonstrates that $t_n^{\leq k}(f)$ is a realization fibration.

To show that $T_n^{\leq k}(f)$ is a realization fibration, we notice that $T_n^{\leq k}(f)([r]) = t_n^{\leq k}(f^{\Delta^r})$ from which it can be easily show that the realization fibration. \square

Now suppose that we are given a sequence of maps of Kan complexes, X_* ,

$$X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n.$$

Suppose that map $X_i \rightarrow X_{i+1}$ is i -connected. We can form the following zigzag of map of n -simplicial spaces:

$$\begin{array}{ccccccc}
T_n^{\leq 0}(X_0) & & T_n^{\leq 1}(X_1) & & T_n^{\leq 2}(X_2) & & \dots & & T_n^{\leq n-1}(X_{n-1}) & & T_n(X_n) \\
& \searrow & & \swarrow & \searrow & & & & \swarrow & & \searrow \\
& & T_n^{\leq 0}(X_1) & & T_n^{\leq 1}(X_2) & & & & & & T_n^{\leq n-1}(X_n)
\end{array}$$

Let us denote by $P(X_*)$ the pullback of this diagram. This pullback is, in fact, a homotopy pullback, since the maps $T_n^{\leq k}(X_k) \rightarrow T_n^{\leq k-1}(X_k)$ are fibrations. Due to the connectedness condition, we see that all the maps $T_n^{\leq k}(X_k) \rightarrow T_n^{\leq k}(X_{k+1})$ are realization fibrations, which become equivalences after realization. Therefore, we conclude the following proposition.

PROPOSITION 24. *The geometric realization of $P(X_*)$ is equivalent to X_n .*

2.3.3 Flexible and Microflexible Sheaves

The homotopy principle (or h-principle) of Gromov is a powerful tool in differential geometry for resolving questions of existence of solutions to differential equations. One of the formulation of the h-principle uses the notion of continuous sheaves on manifolds. Differential relations can be modeled using this notion and the theorems can be phrased using the sheaf language. From this point of view the h-principle is equivalence between the ordinary notion of sheaf and the notion of homotopy sheaf. The ordinary sheaves are not always homotopy sheaves. It is possible to talk about some sort of obstruction theory measuring the failure of being a homotopy presheaf, but the h-principle takes a different route. If one considers the flexible sheaves, which can be thought of generalizations flasque sheaves, then it does follow these sheaves are homotopy sheaves. The condition is restrictive, but very desirable, since it not only gives a means of computing the values of the sheaf on specific open and closed subsets, but also, the relative inclusions. Unfortunately, the sheaves of interest for us are not going to be flexible right away. Since we are only interested in the geometric realization to prove the main theorem we can use the weaker notion of microflexibility.

DEFINITION 27. *Let M be a smooth manifold and let $\text{Open}(M)$ denote the category of open subsets of M and inclusions. A functor $\mathcal{F} : \text{Open}(M)^{\text{op}} \rightarrow \mathcal{J}\text{op}$ is called a (rigid) presheaf of spaces on M . If furthermore the diagram*

$$\mathcal{F}(U) \rightarrow \prod \mathcal{F}(U_\alpha) \rightrightarrows \prod \mathcal{F}(U_\alpha \cap U_\beta)$$

is an equalizer diagram for any open covering $U = \bigcup U_\alpha$, then we call \mathcal{F} a (rigid) sheaf of spaces on M .

The sheaves of spaces have various notions attached to them, which can be used to detect isomorphisms between them. Given a point $x \in M$, we can define the *stalk of \mathcal{F} at x* , \mathcal{F}_x , which the topological space $\text{colim}_{x \in U} \mathcal{F}(U)$. In fact, the same construction can be done for any subset of M . More specifically, if $X \subset M$, then $\mathcal{F}(X) = \text{colim}_{X \subset U} \mathcal{F}(U)$. In particular, $\mathcal{F}_x = \mathcal{F}(\{x\})$.

EXAMPLE 35. The sheaves of sets can be interpreted as sheaves of spaces if think of sets as discrete topological spaces.

EXAMPLE 36. Given any manifold X we can construct a sheaf of spaces from this manifold. There is a sheaf $C^k(-, X) : \text{Open}(M)^{\text{op}} \rightarrow \mathcal{T}\text{op}$ for each natural $0 \leq k \leq \infty$ sending U to the space of all of the C^k -maps from U to X , where the space is endowed with Whitney C^k -topology. We will be interested in the case $k = \infty$.

EXAMPLE 37. Given a smooth fiber bundle $\pi : E \rightarrow M$, then the sections of this fiber bundle form a sheaf of topological spaces. Depending on the differentiability constraints we get different sheaves. For each natural $0 \leq k \leq \infty$, we can consider $\Gamma^k(-, E)$ the space of C^k -sections of E . For instance, in this way we get that $\Omega^n(-)$ the sheaf of smooth n -forms on M is a sheaf of spaces. This example subsumes the previous one, since the sections of the trivial fibration $M \times X \rightarrow M$ are the same as the functions from M to X .

EXAMPLE 38. Given a manifold X we can also pick any local condition on the space of differentiable maps. In particular, we can define the sheaf $\text{Imm}(-, X)$, which on U evaluates to the space of all immersions from U to X . Similarly, we can consider the space of all submersions $\text{Sub}(-, X)$.

EXAMPLE 39. Let M be an m -dimensional smooth manifold. Then we can define the sheaf Ψ_n on M , which gives the space of all closed n -dimensional submanifolds of U , when evaluated on U . Giving the topology of this space is a little tricky, but it has the characteristics of compact-open topology. This example will be of central interest to us. For details, see [G].

By design, the sheaves are local-to-global gadgets in the sense that they allow one to compute the global sections using the notion of local sections. This allows one to reconstruct the topological space of global sections from the information of local sections. However, in a lot of cases, we are interested in type of sheaves that will allow efficient extraction of the homotopy type of the global sections. The notion of sheaf that we have defined does not preserve under homotopy.

Luckily, there is a notion analogous that of a sheaf that is designed to work well with weak equivalences. To define this notion let us first recall a certain construction. Suppose that we are given a covering $\{U_\alpha\}$ of an open set U , and \mathcal{F} is a presheaf of spaces on X . Then we can form the Čech cosimplicial set $\mathcal{F}(U_{\alpha_\bullet})$

$$\coprod \mathcal{F}(U_{\alpha_0}) \rightrightarrows \coprod \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1}) \rightrightarrows \coprod \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}) \rightrightarrows \dots$$

In ordinary sheaves we would take the limit of small portion of this diagram. However, the correct thing to do from the homotopy theoretic point of view is to take the homotopy limit of this cosimplicial set. The space $\mathcal{F}(U)$ maps naturally (up to a contractible space of choices) to this homotopy limit, $\text{holim } \mathcal{F}(U_{\alpha_\bullet})$.

DEFINITION 28. A presheaf of spaces $\mathcal{F} : \text{Open}(M)^{\text{op}} \rightarrow \mathcal{J}\text{op}$ is called a homotopy sheaf on M if the natural map $\mathcal{F}(U) \rightarrow \text{holim } \mathcal{F}(U_{\alpha_{\bullet}})$ is a weak equivalence for all open sets U and all open coverings $\{U_{\alpha}\}$ of U .

The notions attached to ordinary sheaves can also be attached to homotopy sheaves with the caveat that all the construction should be homotopy invariant. For instance, the evaluation of a homotopy sheaf \mathcal{F} on a subset X of M is done by taking a homotopy colimit: $\mathcal{F}(X) = \text{hocolim}_{X \subset U} \mathcal{F}(U)$. In particular, $\mathcal{F}_x = \text{hocolim}_{x \in U} \mathcal{F}(U)$.

EXAMPLE 40. A sheaf of sets is automatically a homotopy sheaf of spaces. This follows from the fact that all the paths in discrete spaces are constants, so the homotopy limits reduce to ordinary limits.

EXAMPLE 41. The sheaves $C^k(-, X)$ are homotopy sheaves. The case $C^0(-, X)$ is a sheaf follows from the fact that if U is covered by $\{U_{\alpha}\}$, then U is weakly equivalent to the geometric realization of an appropriate simplicial diagram, $U_{\alpha_{\bullet}}$. The functor $C^0(-, X)$ converts homotopy colimits into homotopy limits, therefore, we get that $C^0(-, X)$ is a homotopy sheaf. The rest of the functors are weakly equivalent to it via the inclusion maps $C^k(-, X) \rightarrow C^0(-, X)$. The equivalence can be demonstrated using the local-to-global principle, once we establish that all the sheaves are flexible.

The notion of a homotopy sheaf does not have a relation to the notion of a sheaf of spaces. We now introduce conditions under which a sheaf of spaces is a homotopy sheaf.

Recall that $p : X \rightarrow Y$ is (*Serre*) *fibration* if for any diagram

$$\begin{array}{ccc} K \times \{0\} & \rightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ K \times I & \rightarrow & Y \end{array}$$

with K a compact simplicial complex, there exists a dotted lift that makes the diagram commute. A variation of this notion is the notion of microfibration. The map p is a *microfibration* if for any diagram

$$\begin{array}{ccccc} & & K \times \{0\} & \rightarrow & X \\ & \nearrow & \downarrow & \nearrow & \downarrow p \\ K \times [0, \epsilon] & \hookrightarrow & K \times I & \rightarrow & Y \end{array}$$

with K a simplicial complex, there exists a positive ϵ not greater than 1, and a dotted lift that makes the diagram commute.

DEFINITION 29. A rigid sheaf of spaces, \mathcal{F} , is called (micro)flexible if each inclusion of compact subsets C into D induces a (micro)fibration $\mathcal{F}(D) \rightarrow \mathcal{F}(C)$.

EXAMPLE 42. $\Gamma^k(-, E)$ is flexible. If E is classified by a map $\phi : X \rightarrow B$, where B is the classifying

space of the bundle and $F \rightarrow B$ is the universal bundle, then $\Gamma^k(X, E)$ is equivalent to the space of sections $X \rightarrow F$.

EXAMPLE 43. The sheaves Ψ_n are microflexible. This can be proven by adapting Prop. 4.14 in [G] to the sheaf Ψ_n .

The following theorem holds.

THEOREM 16. *Flexible sheaves are homotopy sheaves.*

Proof Sketch. The global sections can be recovered as a sequence of pullback diagrams of local sections. Due to flexibility these diagrams are also homotopy pullbacks, which basically demonstrates that the flexible sheaves are homotopy sheaves. \square

Flexible sheaves are advantageous from another point of view: they interact well with relative evaluations. Suppose that Φ is sheaf taking value in the category of pointed topological spaces, \mathcal{Top}_* . Then we can define $\Phi(U, X)$ to be the preimage of the basepoint of $\Phi(X)$ under the natural restriction maps. If X is compact and Φ is flexible, then we can see that $\Phi(U, X)$ is the homotopy fiber.

Unfortunately, the sheaf of our primary interest Ψ_n is not flexible, but as we saw earlier it is microflexible. One of the central theorems in h-principle is the Main Flexibility Theorem, which allows the construction of flexible sheaves from microflexible sheaves. In order to state the theorem let us introduce the notion of diffeomorphism sheaves. There is refinement of the category $\text{Open}(M)$ to a topological category $\text{Open}_{\text{Diff}}(M)$. The objects of this category are the open subsets of M and the morphisms are smooth embeddings with C^∞ -topology. A $\text{Diff}(M)$ -invariant sheaf is topologically enriched functor $\Phi : \text{Open}_{\text{Diff}}(M)^{\text{op}} \rightarrow \mathcal{Top}$, which when restricted to $\text{Open}(M)$ gives a sheaf.

THEOREM 17. *Let $V = V_0 \times \mathbb{R}$, and let $\pi : V \rightarrow V_0$ denote the projection on the first factor. Let Φ be a microflexible sheaf over V , invariant under $\text{Diff}(V, \pi)$. Then the restriction $\Phi|_{V_0=V_0 \times 0}$ is a flexible sheaf over V_0 .*

REMARK 24. Microflexible sheaves are not always homotopy sheaves even if they are diffeomorphism invariant.

2.3.4 Sheaves and Bordism Categories

In this section we will describe a general procedure of constructing n -simplicial spaces out of sheaves on \mathbb{R}^n . As a particular instance of this construction we will recover the bordism categories. In addition to this we will have a machinery that will allow us to construct various comparison maps that will be crucial in proving the main theorem.

We begin with defining the extended topological cosimplices, which we will denote as $E^\bullet : \Delta^{\text{op}} \rightarrow \mathcal{Top}$.

This functor assigns to $[n]$ the space of order preserving maps from $[n]$ to \mathbb{R} , topologized with the subspace of topology of \mathbb{R}^{n+1} . Notice that E^n is a closed subspace of \mathbb{R}^{n+1} ; we will denote its interior as E^n_\circ . These spaces no longer assemble into a simplicial space, but we will see that they form a semisimplicial space. There is a continuous map $i : E^m \rightarrow \Psi_0(\mathbb{R})$, which sends the map $f : [m] \rightarrow \mathbb{R}$ to its image. Finally, for each natural number m , we consider the element $\langle m \rangle \in E^m$, which sends $i \in [m]$ to $i \in \mathbb{R}$. In particular, $i(\langle m \rangle) = \{0, 1, \dots, m\}$.

Suppose that \mathcal{F} is a sheaf of spaces on \mathbb{R}^n , and suppose that we have a sequence of sheaves on \mathbb{R}^n :

$$\mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \dots \longrightarrow \mathcal{F}^{n-1} \longrightarrow \mathcal{F}^n = \mathcal{F},$$

which we will write as \mathcal{F}^* for brevity. We define an n -simplicial space $\widehat{\mathcal{F}^*} : (\Delta^{\times n})^{\text{op}} \rightarrow \mathcal{J}\text{op}$ as follows:

- $\widehat{\mathcal{F}^*}([m_1], \dots, [m_n])$ is the space of points $(e_1, \dots, e_n; x_0, \dots, x_n)$ where $e_i \in E^{m_i}$ and x_s is a point in $\mathcal{F}^s(\cap_{k=1}^s \mathbb{R}^{k-1} \times i(e_k) \times \mathbb{R}^{n-k})$, such that for if $r > s$, then x_s maps onto x_r on $\mathcal{F}^r(\cap_{k=1}^r \mathbb{R}^{k-1} \times i(e_k) \times \mathbb{R}^{n-k})$;
- the face and degeneracy maps are given by forgetting and double inserting E^{m_k} coordinates.

We will use this construction to create variants of the bordism n -simplicial spaces.

EXAMPLE 44. We define a sequence of sheaves \mathcal{B}^* , on \mathbb{R}^n as follows. Let U be an open set of \mathbb{R}^n , then $\mathcal{B}_N^k(U)$ is the subspace of $\Psi_n(U \times I^N, U \times \partial I^N)$ consisting of submanifolds $M \subset U \times I^N$ for which the projection onto the last $n - k$ coordinates $M \rightarrow U \times I^N \hookrightarrow \mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^{\{k+1, \dots, n\}}$ is submersive. Clearly, there are natural maps $\mathcal{B}_N^k \rightarrow \mathcal{B}_N^{k+1}$, which give us a sequence of sheaves on \mathbb{R}^n . Using the construction above we create an n -simplicial space $\widehat{\mathcal{B}_N^*}$. Letting N go to infinity, we obtain an n -simplicial space $\widehat{\mathcal{B}_\infty^*}$, which is the bordism n -simplicial set that we are aiming to investigate. More specifically, we are interested in $\widehat{\mathcal{B}_\infty^*}$.

An alternative approach would be defining sheaves, Ψ_n^k on $\mathbb{R}^n \times \mathbb{R}^N$. Namely, $\Psi_n^k(V)$ is the subspace of $\Psi_n(V)$ consisting of submanifolds M of U , such that the projection onto the last $n - k$ coordinates $M \rightarrow V \hookrightarrow \mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^{\{k+1, \dots, n\}}$ is submersive. Then almost tautologically we get that $\mathcal{B}_N^k(U) = \Psi_n^k(U \times I^N, U \times \partial I^N)$. From the previous section, if $k \geq 1$ we know that Ψ_n^k is a microflexible sheaf on $\mathbb{R}^n \times \mathbb{R}^N$, therefore, \mathcal{B}_N^k is a microflexible sheaf on \mathbb{R}^n . If $k = 0$, then one can show directly that \mathcal{B}_N^0 is a microflexible sheaf on \mathbb{R}^n .

EXAMPLE 45. The second example is a sequence of sheaves on \mathbb{R}^{n-1} . We define the following sequence of sheaves

$$\mathcal{S}_N^k(U) = \begin{cases} \mathcal{B}_N^k(U \times \mathbb{R}) & \text{if } k < n - 1 \\ \mathcal{B}_N^n(U \times \mathbb{R}) & \text{if } k = n - 1. \end{cases}$$

Thus, we obtain an $(n - 1)$ -simplicial space $\widehat{\mathcal{S}_N^*}$, i.e. a functor $(\Delta^{\times (n-1)})^{\text{op}} \rightarrow \mathcal{J}\text{op}$. This functor factors as a

composite $(\Delta^{\times n-1})^{\text{op}} \longrightarrow \text{Crit} \longrightarrow \mathcal{T}\text{op}$, where the second functor is the functor that forgets the critical locus. To define this factorization, we have to define a critical locus for each point in $X \in \widehat{\mathcal{S}}_N^*([m_1], \dots, [m_n])$. Note that the latter maps to X' in $\mathcal{S}^{n-1}(\mathbb{R}^{n-1}) = \mathcal{B}^n(\mathbb{R}^n)$. Note that X' is a submanifold of \mathbb{R}^n and the we define the critical locus of X to be the set of critical values of X' with respect to the projection unto to the last coordinate, $X' \hookrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^{\{n\}}$. By the abuse of notation we will denote this functor again as $\widehat{\mathcal{S}}_N^* : (\Delta^{\times n-1})^{\text{op}} \longrightarrow \text{Crit}$. Again, in this case we can set $N = \infty$ and obtain the $(n-1)$ -simplicial space with critical locus $\widehat{\mathcal{S}}_\infty^*$.

REMARK 25. The previous example can be generalized to the following construction. Let us say that \mathcal{F}^* is a *short* sequence of sheaves of length on \mathbb{R}^n if it has length $n-1$. Then we can define a sequence of sheaves on \mathbb{R}^{n-1} , $\overline{\mathcal{F}}^*$, so that $\overline{\mathcal{F}}^k(U) = \mathcal{F}^k(U \times \mathbb{R})$. We will denote by $\widehat{\mathcal{F}}^*$ the n -simplicial space $\widehat{\mathcal{F}}^*$.

The following lemma essentially follows from the definitions.

LEMMA 14. *The adjoint of the composite $K \circ \widehat{\mathcal{S}}_N^* : (\Delta^{\times n-1})^{\text{op}} \longrightarrow \mathcal{T}\text{op}_\Delta$ is isomorphic to $\widehat{\mathcal{B}}_N^*$.*

Thus, in order to understand $|\widehat{\mathcal{B}}_N^*|$, we can understand $\|K \circ \widehat{\mathcal{S}}_N^*\|$. However, note that the critical locus in our case is always of measure 0; therefore, $\|K \circ \widehat{\mathcal{S}}_N^*\|$ is equivalent to $|\widehat{\mathcal{S}}_N^*|$.

The construction is natural in sequences of sheaves. We would like to establish a criterion for showing that two functors that arise from this construction are equivalent. Ideally we would expect that given a map of sequences:

$$\begin{array}{ccccccc} \mathcal{F}^0 & \longrightarrow & \mathcal{F}^1 & \longrightarrow & \dots & \longrightarrow & \mathcal{F}^n = \mathcal{F} \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathcal{G}^0 & \longrightarrow & \mathcal{G}^1 & \longrightarrow & \dots & \longrightarrow & \mathcal{G}^n = \mathcal{G} \end{array}$$

such that the vertical maps are weak equivalences, then the natural map $\widehat{\mathcal{F}}^* \longrightarrow \widehat{\mathcal{G}}^*$ is also a weak equivalence. Unfortunately, this does not hold for general sheaves. To overcome this difficulty we will introduce the weaker notion of n -semisimplicial spaces. Under relative mild assumptions the conclusion of the statement above will hold.

Let Δ_+ denote the subcategory of Δ , which contains all the objects, but has only the injective maps. An n -semisimplicial space is a function of the form $(\Delta_+^{\text{op}})^{\times n} \longrightarrow \mathcal{T}\text{op}$. One can check for instance that E_\circ^\bullet is semisimplicial space. A n -semisimplicial space is a n -simplicial set with only face maps in each variable. Thus, any simplicial set can be considered as a semisimplicial set. Analogous to the construction above we can construct a n -semisimplicial set $\widehat{\mathcal{F}}_\circ^*$ defined as follows:

- $\widehat{\mathcal{F}}_\circ^*([m_1], \dots, [m_n])$ is the space of points $(e_1, \dots, e_n; x_0, \dots, x_n)$ where $e_i \in E_\circ^{m_i}$ and x_s is a point in $\mathcal{F}^s(\cap_{k=1}^s \mathbb{R}^{k-1} \times i(e_k) \times \mathbb{R}^{n-k})$, such that for if $r > s$, then x_s maps onto x_r on $\mathcal{F}^r(\cap_{k=1}^r \mathbb{R}^{k-1} \times i(e_k) \times \mathbb{R}^{n-k})$;
- the face are given by forgetting $E_\circ^{m_k}$ coordinates.

Again this construction is natural in sequences of sheaves. Also notice that there is a natural inclusion map

$$\widehat{\mathcal{F}}_{\circ}^* \longrightarrow \widehat{\mathcal{F}}^*.$$

In order to state the next proposition, we need to introduce the notion of diffeomorphism invariance of sequences of sheaves. Essentially means that the sheaf in the sequence is preserved under the action of $\text{Diff}(\mathbb{R}^n) \subset \text{Diff}(\mathbb{R}^n)$.

PROPOSITION 25. *Let $\mathcal{F}^* \rightarrow \mathcal{G}^*$ be a map of diffeomorphism invariant short sequences of microflexible sheaves over \mathbb{R}^n , such that on each open set the map is degreewise a weak equivalence. Then the induced map of n -semisimplicial spaces $\widehat{\mathcal{F}}_{\circ}^* \rightarrow \widehat{\mathcal{G}}_{\circ}^*$ is an equivalence.*

Proof. The idea behind the proof is that under the hypothesis of the proposition, $\widehat{\mathcal{F}}_{\circ}^*([m_1], \dots, [m_{n-1}])$ can be constructed as a homotopy limit of diagrams which are natural and homotopy invariant in \mathcal{F}^* . As a consequence, if we have a map between two sequences of sheaves $\mathcal{F}^* \rightarrow \mathcal{G}^*$, we would obtain a morphism of diagrams, such that at each spot the morphism is a weak equivalence, and hence the homotopy limits are also weak equivalences. In particular, the map $\widehat{\mathcal{F}}_{\circ}^*([m_1], \dots, [m_{n-1}]) \rightarrow \widehat{\mathcal{G}}_{\circ}^*([m_1], \dots, [m_{n-1}])$ would be a weak equivalence.

To make this precise define $L_k = \mathbb{R}^k \times \{0, \dots, m_{k+1}\} \times \mathbb{R}^{n-k-1} \subset \mathbb{R}^n$ for $k \in \{0, \dots, n-2\}$. We will define via backwards induction flexible sheaves $\mathcal{F}_{\geq k}$ for $k \in \{0, \dots, n-1\}$. From the construction we will see that these spaces are natural in \mathcal{F}^* and are homotopy invariant. To begin the induction we define $\mathcal{F}_{\geq n-1}$ to be \mathcal{F}^{n-1} . Now suppose that we have constructed a flexible sheaf $\mathcal{F}_{\geq k}$ along with a map $\mathcal{F}^k \rightarrow \mathcal{F}_{\geq k}$. We define the sheaf $\mathcal{F}_{\geq k-1}$ on an open set U to satisfy the following pullback diagram

$$\begin{array}{ccc} \mathcal{F}_{\geq k-1}(U) & \longrightarrow & \mathcal{F}^{k-1}(U) \times \mathcal{F}_{\geq k}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}^{k-1}(L_{k-1} \cap U) & \longrightarrow & \mathcal{F}^{k-1}(L_{k-1} \cap U) \times \mathcal{F}_{\geq k}(L_{k-1} \cap U). \end{array}$$

It can be easily checked that $\mathcal{F}_{\geq k}$ is microflexible and clearly admits a map from \mathcal{F}^k . One can show inductively that $\mathcal{F}_{\geq k}$ is $\text{Diff}(\mathbb{R}^n, \pi)$ -invariant, where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is the projection onto the first $n-1$ components. This means that $\mathcal{F}_{\geq k}|_{\mathbb{R}^{n-1}}$ is flexible. If V is an open subset of \mathbb{R}^n , then we can show that $\mathcal{F}_{\geq k}|_{\mathbb{R}^{n-1}}(V)$ is homotopy equivalent to $\mathcal{F}_{\geq k}(V \times \mathbb{R})$, since there is a basis of open sets for $V \times \{0\}$ all of them isotopic to $V \times \mathbb{R}$ via isotopies that preserve the projection onto V . Note that the $\mathcal{F}_{\geq k}|_{\mathbb{R}^{n-1}}$ can be defined using a pullback diagram as above, where $\mathcal{F}_{\geq k}$ and \mathcal{F}^{k-1} are replaced by $\mathcal{F}_{\geq k}|_{\mathbb{R}^{n-1}}$ and $\mathcal{F}^{k-1}|_{\mathbb{R}^{n-1}}$, respectively. The homotopy invariance is manifest from the fact that the restricted sheaves are flexible, and the projection on the right is a fibration. Thus, the spaces $\mathcal{F}_{\geq k}(V \times \mathbb{R}) \simeq \mathcal{F}_{\geq k}|_{\mathbb{R}^{n-1}}(V)$ is also homotopy invariant. The proposition follows from the fact that $\mathcal{F}_{\geq 0}(\mathbb{R}^n)$ is homeomorphic to the subset of $\widehat{\mathcal{F}}_{\circ}^*([m_1], \dots, [m_{n-1}])$, consisting of manifolds with cuts, such that cuts are done on a standard subsets $\langle m_1 \rangle, \langle m_2 \rangle, \dots, \langle m_{n-1} \rangle$. This subspace is a deformation retraction of the total space. \square

The following corollary follows from the fact the flexible sheaves are homotopy sheaves.

COROLLARY 10. *Let \mathcal{F}^* be a short sequence of diffeomorphism invariant microflexible sheaves on \mathbb{R}^n , and let $\mathcal{F}^{*,\text{flex}}$ denote sequence*

$$\mathcal{F}^{0,\text{flex}} \longrightarrow \mathcal{F}^{1,\text{flex}} \longrightarrow \dots \longrightarrow \mathcal{F}^{n,\text{flex}}.$$

Then the natural map $\widehat{\mathcal{F}}_\circ^ \longrightarrow \widehat{\mathcal{F}}_\circ^{*,\text{flex}}$ is an equivalence of n -semisimplicial spaces.*

COROLLARY 11. *The natural map of n -simplicial spaces $\widehat{\mathcal{S}}_{N,\circ}^* \longrightarrow \widehat{\mathcal{S}}_{N,\circ}^{*,\text{flex}}$ is an equivalence of n -semisimplicial spaces.*

We are interested in the n -simplicial spaces $\widehat{\mathcal{S}}_N^*$ and $\widehat{\mathcal{S}}_N^{*,\text{flex}}$, and we would like to establish an equivalence between these two n -simplicial spaces. We can use a variant of *widehat* $\mathcal{S}_N^{*,\perp}$, where instead of using submersions we require orthogonality of map as we did when we constructed the bordism category $\text{Bord}_n^{\xi,\perp}$. This construction will result in an equivalent n -simplicial and n -semisimplicial spaces. The following proposition is the first step towards establishing this equivalence.

PROPOSITION 26. *The natural map $\widehat{\mathcal{S}}_{N,\circ}^{*,\perp} \longrightarrow \widehat{\mathcal{S}}_N^{*,\perp}$ is an equivalence of n -semisimplicial spaces.*

Proof. Notice that $\widehat{\mathcal{S}}_{N,\circ}^{*,\perp}([m_1], \dots, [m_n])$ is a subspace of $\widehat{\mathcal{S}}_N^{*,\perp}([m_1], \dots, [m_n])$. We will construct a deformation retraction onto this subspace. To do this we note that there is a map $\rho : I \times E^m \longrightarrow C^\infty(\mathbb{R}, \mathbb{R})$, so that $\rho(t, e_0, \dots, e_m)$ is a non-decreasing function, which is strictly increasing away from the points $e_i + it$, and its value at $e_i + it$ is e_i . By using the diagonal on I we can define a map $\varrho : I \times E^{m_1} \times \dots \times E^{m_n} \longrightarrow C^\infty(\mathbb{R}, \mathbb{R})^n$.

A point in $\widehat{\mathcal{S}}_N^{*,\perp}([m_1], \dots, [m_n])$ is a submanifold X of $\mathbb{R}^n \times \mathbb{R}^N$, and a collection of cut data $e^i \in E^{m_i}$. We define a map $P : I \times \widehat{\mathcal{S}}_{N,\circ}^{*,\perp}([m_1], \dots, [m_n]) \longrightarrow \widehat{\mathcal{S}}_N^{*,\perp}([m_1], \dots, [m_n])$, that sends X to $\rho(t, e^1, \dots, e^n)^*(X)$. This is a weak homotopy retraction onto $\widehat{\mathcal{S}}_{N,\circ}^{*,\perp}([m_1], \dots, [m_n])$. \square

The analogue of the proposition above holds for $\widehat{\mathcal{S}}_N^{*,\text{flex}}$. In fact, it holds in greater generality. Suppose that we have a sequence of topological spaces

$$X^0 \longrightarrow X^1 \longrightarrow \dots \longrightarrow X^n.$$

We can consider the sequence of sheaves $C^0(-, X^*)$ on \mathbb{R}^n , for brevity we will write it as $C(X^*)$. Note that $\widehat{\mathcal{S}}_N^{*,\text{flex}} \simeq C(\Psi_N^*(\mathbb{R}^n \times I^N, \mathbb{R}^n \times \partial I^N))$.

PROPOSITION 27. *The natural map $\widehat{C}(X^*)_\circ \longrightarrow \widehat{C}(X^*)$ is a weak equivalence.*

Proof. For each set of indices we note that the map $\widehat{C}(X^*)_\circ([m_1], \dots, [m_n]) \longrightarrow \widehat{C}(X^*)([m_1], \dots, [m_n])$ is an inclusion. We will show that it is a weak deformation retract. To prove this we will make a use of two

function. The first one is a deformation of E^m onto E^m_\circ , $\rho : I \times E^m \longrightarrow E^m$, defined by the formula

$$\rho(t; x_0, x_1, \dots, x_m) = (x_0, x_1 + t, \dots, x_m + mt).$$

We can enhance this functor to $\rho : I \times E^{m_1} \times \dots \times E^{m_n} \longrightarrow E^{m_1} \times \dots \times E^{m_n}$, where we use the n -fold diagonal on I , and then the product of ρ with itself n times. Clearly, this homotopy is a weak deformation retraction onto $E^{m_1}_\circ \times \dots \times E^{m_n}_\circ$.

Recall that besides E^{m_k} 's, $\widehat{C(X^*)}([m_1], \dots, [m_n])$ projects onto $C^0(\mathbb{R}^n, X^k)$ —the purpose of the second function is to rectify the potential problems with the compatibility conditions in $\widehat{C(X^*)}([m_1], \dots, [m_n])$. We define $\varrho : I \times E^m \times \mathbb{R} \longrightarrow \mathbb{R}$:

$$\varrho(t; x_0, \dots, x_m; r) = \begin{cases} r & \text{if } r \leq x_0 \\ \frac{(r-kt)(x_{k+1}-x_k)+tx_k}{x_{k+1}-x_k+t} & \text{if } x_k \leq r - kt \leq x_{k+1} + t \text{ and } x_{k+1} \neq x_k \\ x_k & \text{if } x_k \leq r - kt \leq x_{k+1} + t \text{ and } x_{k+1} = x_k \\ r + mt & \text{if } r \geq x_m + mt \end{cases}$$

Just as above we enhance this functor $\varrho : I \times E^{m_1} \times \dots \times E^{m_n} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. We can also rewrite this function as $\varrho : I \times E^{m_1} \times \dots \times E^{m_n} \longrightarrow C^0(\mathbb{R}^n, \mathbb{R}^n)$.

Finally we define a deformation retraction, $I \times E^{m_1} \times \dots \times E^{m_n} \times C^0(\mathbb{R}^n, \prod_{k=0}^n X^k) \longrightarrow E^{m_1} \times \dots \times E^{m_n} \times C^0(\mathbb{R}^n, \prod_{k=0}^n X^k)$, by sending $(t; e^1, \dots, e^n; f)$ to $(\rho(t; e^1, \dots, e^n); f \circ \varrho(t; e^1, \dots, e^n))$. This homotopy restricts to $\widehat{C(X^*)}([m_1], \dots, [m_n])$ and gives the desired deformation retraction. \square

Thus, we have demonstrated the following corollary.

COROLLARY 12. *The natural map $\widehat{\mathcal{S}}_N^* \longrightarrow \widehat{\mathcal{S}}_N^{*, \text{flex}}$ is an equivalence of n -simplicial spaces.*

To finally prove the main theorem, we need to understand the geometric realization of $\widehat{\mathcal{S}}_N^{*, \text{flex}}$. To do this we use the following lemma.

LEMMA 15. *There is a weak equivalence between $|\widehat{C(X^*)}|$ and $|P(X_*)|$.*

Proof Sketch. If we unwind the definition the construction of the n -simplicial space $C(X^\bullet)$ is the same as that of $P(X^*)$ except instead of using regular simplices we use extended topological simplices. \square

COROLLARY 13. *There is an equivalence $|\widehat{C(X^*)}| \longrightarrow X^n$ if the maps $X^i \longrightarrow X^{i+1}$ are i -connected.*

We immediately obtain the main theorem as a corollary.

COROLLARY 14. *There is a weak equivalence $|\widehat{\mathcal{S}}_N^*| \simeq \Psi_n(\mathbb{R}^n \times I^N, \mathbb{R}^n \times \partial I^N) \simeq \Psi_n(\mathbb{R}^n \times \mathbb{R}^N)^{S^N} \simeq \text{Th}(\zeta_{n,N}^\perp)^{S^N}$. When $N = \infty$, we obtain $|\widehat{\mathcal{S}}_\infty^*| \simeq \Omega^{\infty-n} \text{MTO}(n)$.*

References

- [A] M. Atiyah, [Topological Quantum Field Theories](#), Publications Mathématiques de l’IHÉS 68, 175-186, 1989
- [BaSPr] C. Barwick, C. Schommer-Pries, [On the Unicity of the Homotopy Theory of Higher Categories](#), 2013
- [BD] J. Baez, J. Dolan. Higher-Dimensional Algebra and Topological Quantum Field Theory. J. Math. Phys. 36 (11), 6073-6105, 1995
- [BoMa] M. Bökstedt, I. Madsen, [The Cobordism Category and Waldhausen’s K-Theory](#), 2011
- [BM1] J.-L. Brylinski, D. A. McLaughlin, The Geometry of Degree-Four Characteristic Classes and of Line Bundles on Loop Spaces I, Duke Mathematical Journal, Vol. 75, No. 3, 603-638, 1994
- [BM2] J.-L. Brylinski, D. A. McLaughlin, The Geometry of Degree-4 Characteristic Classes and of Line Bundles on Loop Spaces II, Duke Mathematical Journal, Vol. 83, No. 3, 105-139, 1996
- [De] P. Deligne, Séminaire de Géométrie Algébrique du Bois Marie - Cohomologie Étale - (SGA 4^{1/2}). Lecture Notes in Mathematics (in French) 569, Springer-Verlag, 1977
- [Du] D. Dugger, [Combinatorial Model Categories Have Presentations](#), 2000
- [G] S. Galatius, Stable Homology of Automorphism Groups of Free Groups, Annals of Mathematics 173, 705-768, 2011
- [GMTW] S. Galatius, I. Madsen, U. Tillmann, M. Weiss, [The Homotopy Type of the Cobordism Category](#), Acta Mathematica 202, 195–239, 2009
- [Gr] M. Gromov, Partial Differential Relations, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 9, Springer-Verlag, Berlin, 1986
- [Ha] R. Hartshorne, Algebraic Geometry, GTM 52, Springer-Verlag, 1977
- [HHM] G. Heuts, V. Hinich, I. Moerdijk, [On the Equivalence Between Lurie’s Model and the Dendroidal Model for \$\infty\$ -Operads](#), 2015
- [Hi] P. S. Hirschhorn, Model Categories and Their Localizations, Mathematical Surveys and Monographs, Vol. 99, 2002
- [Ho] M. Hovey, Model Categories, SURV 63, AMS, 1999
- [Jo1] A. Joyal, [Disks, Duality and \$\Theta\$ -Categories](#), Unpublished Manuscript, 1997
- [KM] N. Katz, B. Mazur, Arithmetic Moduli of Elliptic Curves, Annals of Mathematics Studies 108, 1985
- [Lu1] J. Lurie, [Higher Topos Theory](#), Annals of Mathematics Studies 170, 2012
- [Lu2] J. Lurie, [Higher Algebra](#), Online Notes, 2012
- [Lu3] J. Lurie, [\(\$\infty, 2\$ \)-Categories and the Goodwillie Calculus I](#), 2009
- [Lu4] J. Lurie, [On the Classification of Topological Field Theories](#), 2010
- [M] J. May, The Geometry of Iterated Loop Spaces, Springer Lecture Notes in Mathematics 271, 1972
- [Re1] C. Rezk, [A Model for the Homotopy Theory of Homotopy Theory](#), Transactions of the AMS v. 353, 2001
- [Re2] C. Rezk, [A Cartesian Presentation of Weak \$n\$ -Categories](#), 2010
- [Re3] C. Rezk, [When Are Homotopy Colimits Compatible With Homotopy Base Change?](#)
- [S] G. Segal, The Definition of Conformal Field Theory, Differential Geometrical Methods in Theoretical Physics,

NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 250, Kluwer Acad. Publ., Dordrecht, 165-171, 1988

[Sch] U. Schreiber, [Differential Cohomology in a Cohesive \$\infty\$ -Topos](#), 21st Century

[T] R. Thom, Quelques Propriétés Globales des Variétés Différentiables, Commentarii Mathematici Helvetici 28, 1786, 1954

[TT] R. Thomason, T. Trobaugh, Higher Algebraic K-Theory of Schemes and of Derived Categories, The Grothendieck Festschrift, Birkhäuser Boston, 1990