SUFFICIENT CONDITIONS FOR THE EXISTENCE OF SPECIFIED SUBGRAPHS IN GRAPHS

BY

ANDREW McCONVEY

DISSERTATION

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Doctoral Committee:

Professor József Balogh, Chair
Professor Alexandr Kostochka, Director of Research
Associate Professor Kay Kirkpatrick
J.L. Doob Research Assistant Professor Theodore Molla

Abstract

In this thesis, we will consider problems regarding the existence of specified subgraphs in a graph. Given two graphs, G and H, we say that G is a subgraph of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. A classical problem in combinatorics is, given G and H, to determine if H is a subgraph of G. It is usually computationally complex to determine if H is a subgraph of G. Therefore, we often prove conditions that are sufficient to guarantee that a graph G contains H as a subgraph.

In Chapter 2, we consider a theorem of Dirac and Erdős from 1963 that considers when a graph contains many disjoint cycles. Generalizing the seminal result of Corrádi and Hajnal, they prove that if a graph G contains at least k^2+2k-4 more vertices of degree at least $2k$ than vertices of degree at most $2k-2$, then G contains k vertex-disjoint cycles. We strengthen their result, proving that a difference of only $3k$ is sufficient to guarantee the existence of k disjoint cycles and that this bound is sharp. Moreover, when G has many vertices, G is planar, or G contains few triangles, this bound can be improved to $2k$. The bound of $2k$ is the best possible, as shown by examples of Dirac and Erdős.

In Chapter 3, we rephrase the problem of subgraphs in the language of graph packing. Two graphs G_1 and G_2 pack if G_1 is a subgraph of the complement of G_2 or, equivalently, if G_2 is a subgraph of the complement of G_1. Graph packing is a restatement of the subgraph problem that does not require one graph to be specified as the underlying graph and the other as the subgraph. Theorems of Sauer and Spencer and, independently, Bollobás and Eldridge prove that if G_1 and G_2 together have few edges or if $\Delta(G_1)$ and $\Delta(G_2)$ are small, then G_1 and G_2 pack. We explore two results that combine bounds on the maximum degrees and number of edges in G_1 and G_2.

Recently, Alon and Yuster proved that if G_1 and G_2 are graphs on n vertices with $|E(G_1)| \leq n-\delta(G_2)-1$ and $\Delta(G_2) \leq \sqrt{n}/200$, then G_1 and G_2 pack. We characterize the pairs of graphs for which their theorem is sharp. In particular, we prove the stronger result that for sufficiently large n, if $|E(G_1)| \leq n$, $\Delta(G_2) \leq \sqrt{n}/60$, and $\Delta(G_1) + \delta(G_2) \leq n-1$, then G_1 and G_2 pack whenever there is a vertex $v_1 \in V(G_1)$ such that $d(v_1) = \Delta(G_1)$ and $\alpha(G_1 - N[v_1]) \geq \delta(G_2)$.

We also consider a conjecture of Żak that states if $\Delta(G_1), \Delta(G_2) \leq n-2$, then $\|G_1\| + \|G_2\| +$

$\max\{\Delta(G_1), \Delta(G_2)\} \leq 3n - 7$ is sufficient for G_1 and G_2 to pack. We prove that, up to an additive constant, this conjecture is correct. Using the notion of list packing, we prove that there is a constant C such that if $\Delta(G_1), \Delta(G_2) \leq n-2$ and $\|G_1\| + \|G_2\| + \max\{\Delta(G_1), \Delta(G_2)\} \leq 3n - C$, then G_1 and G_2 pack. This improves a theorem of Żak from 2014.

Finally, we consider a generalization of finding a matching in a graph. The stable marriage problem was introduced by Gale and Shapley in 1962 and the generalization to multiple dimensions was first mentioned by Knuth in 1976. We consider a generalization of the Stable Marriage problem with s-dimensions and purely cyclic preferences (cyclic s-DSM). In 2004, Boros et al. showed that if there are at most s agents of each gender, then every instance of cyclic s-DSM admits a stable matching. In 2006, Eriksson et al. showed this is also true when $s = 3$ and there are 4 agents of each gender. We extend their result, proving that when there are $s + 1$ agents of each gender, each instance of s-DSM admits a stable matching. We also provide a minimal example of an instance of s-DSM which admits no strongly stable matching.

To my wife, Barbara.

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List of Symbols

$\mathbb{N}$	Set of natural numbers, not including 0.
$[n]$	Set of integers $\{1, \ldots, n\}$.
$V(G)$	Vertex set of a graph G.
$\lvert G \rvert$	Number of vertices in a graph G.
$E(G)$	Edge set of a graph G.
$\lVert G \rVert$	Number of edges in a graph G.
$d(v)$	Degree of v.
$N(v)$	Neighborhood of v.
$N[v]$	Closed neighborhood of v, $N(v) \cup \{v\}$
$\delta(G)$	Minimum degree of a vertex in a graph G.
$\Delta(G)$	Maximum degree of a vertex in graph G.
$\lVert U, U' \rVert$	Number of edges with one endpoint in U and the other endpoint U'.
$H \subseteq G$	H is a subgraph of a graph G.
$G[S]$	Subgraph of a graph G induced by the set $S \subseteq V(G)$.
$\alpha(G)$	Independence number of a graph G.
K_n	Complete graph on n vertices.
$K_{n,m}$	Complete bipartite graph with bipartite sets of size n and m.
$\overline{G}$	Complement of a graph G.
$k\,G$	The disjoint union of k copies of a graph G.
C_n	Cycle on n vertices.
SK_m	Graph obtained by subdividing one edge of K_m.
G/xy	Graph obtained from G by contracting the edge xy.

Chapter 1

Introduction

Given graphs G and H, it is computationally difficult, in general, to determine if G contains H as a subgraph. It is therefore a natural question to ask what conditions on G are sufficient to guarantee that it contains H. Determining such sufficient conditions is a classical area of study and particular interest has been given to conditions on the number of edges in G and, alternatively, the degrees of vertices in G. This thesis explores several variations of this problem.

A well known property of graphs is that every graph G with minimum degree at least 2 contains at least one cycle. It is natural to think that if the minimum degree of G is increased, then the conclusion can be strengthened. Indeed, increasing the bound on the minimum degree is sufficient for guaranteeing the existence of long cycles or, alternatively, guaranteeing the existence of many cycles. For example, a theorem of Dirac from 1952 proves that a graph G is Hamiltonian if $\delta(G) \geq \frac{1}{2}|G|$ [13] and the Corrádi-Hajnal Theorem from 1963 proves that if $|G| \geq 3k$ and $\delta(G) \geq 2k$ then G contain k vertex-disjoint cycles [9]. In Chapter 2, we will consider what other restrictions on the degrees of vertices in G are sufficient for guaranteeing the existence of k disjoint cycles. In particular, we show that not all vertices need to have high degree; rather it is enough for the vertices of high degree to appropriately outnumber the vertices of low degree.

Similarly, if a graph on n vertices contains enough edges, then it must contain a particular subgraph H. This is evident since the complete graph K_n contains every graph on at most n vertices. For a graph H, it is then natural to ask what is the minimum number of edges needed to guarantee that an n vertex graph contains a copy of H. Equivalently, the Turán Number of a graph H, denoted $\text{ex}(n, H)$ is the maximum number of edges in an n vertex graph not containing H. It is named for Pál Turán, who determined that every graph on n vertices with more than $\frac{1}{2}\left(1 - \frac{1}{r}\right)n^2$ edges contains a complete graph on r vertices [49], though the case for $r = 3$ was first proved by Mantel [42]. In Chapter 3, we consider a theorem of Alon and Yuster that determines $\text{ex}(n, H)$ for H with bounded maximum degree. We refine their result and are also able to classify the sharpness examples to their theorem. Additionally, we further study how additional bounds on both the number of edges in the graphs and on the maximum degrees can guarantee the existence of a specified subgraph. Specifically, we provide an approximate solution to a conjecture of Żak phrased in the language of graph packing.

Finally, in Chapter 4, we study a generalization of the Stable Matching problem. The problem of finding a large matching in a graph is another classical problem in graph theory. Hall's Theorem gives necessary and sufficient conditions for the existence of a matching covering a partite set in a bipartite graph [27]. The Kőnig-Egerváry Theorem gives the size of a maximum matching in a bipartite graph [37, 17] and these results were used by Kuhn to efficiently find such a matching [40]. Similarly, Tutte characterized when a (not necessarily bipartite) graph contains a perfect matching [50] and Edmonds gave an algorithm to efficiently find such a matching [16]. The notion of a Stable Matching, introduced by Gale and Shapley in 1962 [21], considers a situation where vertices have preferences for a potential matching. While Gale and Shapley were able to solve the problem in two dimensions, we will study an extension of the Stable Matching problem to higher dimensions first proposed by Knuth [36].

1.1 Notation

We will mostly use standard notation from graph theory. In this section, we present some of the notation and definitions used throughout this thesis. All logarithms are base e.

Definition 1.1.1. For a graph G, $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. The *order* of G, denoted $|G|$, is the number of vertices in G. The *size* of G, denoted $\|G\|$ is the number of edges in G.

Definition 1.1.2. For a vertex $v \in V(G)$, the *neighborhood* of v, denoted $N_G(v)$ is the set of vertices adjacent to v. The *degree* of v, $d_G(v)$ is $|N_G(v)|$. If $u \in N_G(v)$, we say that u is a *neighbor* of v. The *closed neighborhood* of v, $N_G[v]$ is $N_G(v) \cup \{v\}$. Similarly, for a set $W \subseteq V(G)$, we write $N_G(W)$ for the set of vertices with at least one neighbor in W and $N[W]$ for $N(W) \cup W$. When the graph G is clear from context, we will write $N(v)$, $d(v)$, and $N[v]$ instead of $N_G(v)$, $d_G(v)$, and $N_G[v]$, respectively.

Definition 1.1.3. A *leaf* is a vertex of degree 1 in G. A vertex is *isolated* if it has degree 0.

Definition 1.1.4. For a graph G, $\delta(G)$ is the minimum degree of a vertex in G and $\Delta(G)$ is maximum degree of a vertex in G.

Definition 1.1.5. For disjoint sets $U, U' \subseteq V(G)$, we write $\|U, U'\|_G$ for the number of edges from U to U'. If $U = \{u\}$, then we will write $\|u, U'\|_G$ instead of $\|\{u\}, U'\|_G$. When the graph G is clear from context, we will simplify the notation to $\|U, U'\|$ and $\|u, U'\|$, respectively.

Definition 1.1.6. For graphs H and G, we say that H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(H)$.

Definition 1.1.7. Let G be a graph and $S \subseteq V(G)$. The *subgraph induced by* S, denoted $G[S]$, has vertex set S and $E(G[S]) = \left\{e \in \binom{S}{2} : e \in E(G)\right\}$.

Definition 1.1.8. For a graph G, a set $S \subseteq V(G)$ is *independent* if $G[S]$ contains no edges. The *independence number* of G, denoted $\alpha(G)$ is the size of the largest independent set in G.

Definition 1.1.9. The *complete graph* on n vertices, denoted K_n, is a graph with n vertices and $E(K_n) = \binom{V(K_n)}{2}$.

Definition 1.1.10. The *complete bipartite graph* $K_{a,b}$ has vertex set $A \cup B$, with $|A| = a$ and $|B| = b$, and edge set $\{ab : a \in A \text{ and } b \in B\}$.

Definition 1.1.11. The *complement* of a graph G, denoted $\overline{G}$, is a graph with vertex set $V(G)$ and edge set $\left\{e \in \binom{V(G)}{2} : e \notin E(G)\right\}$.

Definition 1.1.12. A *matching* M is a graph with $\Delta(M) \leq 1$. A graph M is a *perfect matching* if $d(v) = 1$ for every vertex $v \in V(M)$.

Definition 1.1.13. A *cycle* on n vertices, denoted C_n, is a connected graph with $d(v) = 2$ for each $v \in V(C_n)$.

Definition 1.1.14. The graph SK_m is obtained from K_m by subdividing one edge.

Definition 1.1.15. Two graphs are *disjoint* if they have no vertices in common.

Definition 1.1.16. The *union* of two graphs G and G', denoted $G \cup G'$, is a graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. For an integer k, the disjoint union of k copies of G is denoted by $k\,G$.

Definition 1.1.17. The *join* of two graphs G and G', denoted $G \vee G'$, is a graph with vertex set $V(G) \cup V(G')$ and edge set $E(G_1) \cup E(G_2) \cup \{xx' : x \in V(G) \text{ and } x' \in V(G')\}$.

Definition 1.1.18. A subset $W \subseteq V(G)$ of vertices in G is a *clique* if $G[W]$ is a complete graph.

Definition 1.1.19. For an edge $xy \in E(G)$, $G \diagup xy$ denotes the graph obtained from G by contracting xy, and v_{xy} denotes the vertex resulting from contracting xy.

Definition 1.1.20. For a vertex $v \in V(G)$, $G - v$ denotes the graph obtained from G by removing the vertex v and all edges incident to v.

Definition 1.1.21. For a positive integer d, a graph G is *d-degenerate* if, for every subgraph H of G, $\delta(H) \leq d$.

Definition 1.1.22. For two graphs G_1 and G_2 with $|G_1| = |G_2|$, a *packing* of G_1 and G_2 is a bijection $f : V(G_1) \to V(G_2)$ such that if $xy \in E(G_1)$, then $f(x)f(y) \notin E(G_2)$. We say G_1 and G_2 *pack* if there exists a packing of G_1 and G_2.

1.2 Disjoint Cycles

Every graph G with $|G|$ edges contains at least one cycle. However, it is computationally difficult to determine whether a graph contains many disjoint cycles. Consider the following problem.

Problem 1.2.1. *Given a graph G and integer k, determine if G contains k vertex disjoint cycles.*

Indeed, Problem 1.2.1 is NP-complete, as it contains as a special case the problem of determining whether G contains $n/3$ disjoint triangles [22, p. 68]. Given the complexity of Problem 1.2.1, it is natural to ask which properties of a graph guarantee that it contains k disjoint cycles.

For a positive integer k and a graph G, define $H_k(G)$ to be the subset of vertices with degree at least $2k$ and $L_k(G)$ to be the subset of vertices of degree at most $2k-2$ in G. Resolving a conjecture of Erdős, the Corrádi-Hajnal Theorem [9] generalizes the fact that every graph with minimum degree 2 contains a cycle.

Theorem 1.2.2. *[9] Let G be a graph and k a positive integer. If $|G| \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.*

Both conditions in the theorem are sharp. Since each cycle has at least 3 vertices, a graph with fewer than $3k$ vertices does not contain k disjoint cycles. Further, there are examples of graphs without k disjoint cycles but $\delta(G) = 2k-1$. For example, consider the graph $G_{n,k} = \overline{K}_{n-2k+1} \vee K_{2k-1}$ for $n \geq 3k$. Every cycle in $G_{n,k}$ must contain at least two vertices from K_{2k-1}, so $G_{n,k}$ contains at most $k-1$ disjoint cycles.

Theorem 1.2.2 prompted a series of refinements and extensions for both undirected graphs (see, e.g., [12, 14, 26, 18, 51, 34, 8, 35, 33, 32]) and directed graphs (see, e.g., [52, 11, 10, 48]). In 1963, Dirac and Erdős [12] published their only joint paper. In it, they proved the following refinement of the Corrádi-Hajnal Theorem.

Theorem 1.2.3. *[12] Let $k \geq 3$ be an integer and G be a graph with $|H_k(G)| - |L_k(G)| \geq k^2 + 2k - 4$. Then G contains k disjoint cycles.*

The bound $k^2 + 2k - 4$ is not best possible. Dirac and Erdős provided the following example of a graph G without k disjoint cycles such that $|H_k(G)| - |L_k(G)| = 2k - 1$.

Example 1.2.4. *Let $V(G) = X \cup Y \cup Z$, where $|X| = 2k-1$ and $|Y| = |Z| \geq 2k$. Let $E(G)$ consist of all possible edges between X and Y and also a perfect matching between Y and Z. Then $H_k(G) = X \cup Y$ and $L_k(G) = Z$. Since every vertex in Z has degree 1, no vertex of Z is in a cycle. Then every cycle in G contains two vertices from X and so G contains at most $k-1$ disjoint cycles.*

For general graphs, Dirac and Erdős did not prove that a bound of $2k$ guarantees the existence of k disjoint cycles. However, in the special case of planar graphs, they proved that a linear bound on the difference $|H_k(G)| - |L_k(G)|$ is indeed sufficient.

Theorem 1.2.5. *[12] Let $k \geq 3$ be an integer and G be a planar graph such that $|H_k(G)| - |L_k(G)| \geq 5k-7$. Then G contains k disjoint cycles.*

For over 50 years, there were no improvements made to the theorems of Dirac and Erdős. In discussing his work with Dirac, Erdős would later say that the paper was "undeservedly neglected" [19]. In Chapter 2, we refine and strengthen Theorems 1.2.3 and 1.2.5. The results in this chapter are from joint work with Henry Kierstead and Alexandr Kostochka found in [31] and [30]. The main result of Chapter 2 is the following theorem.

Theorem 1.2.6. *[31] Let $k \geq 2$ be an integer and G be a graph such that $|G| \geq 3k$. Let t be the maximum number of disjoint triangles contained in G. If*

$$|H_k(G)| - |L_k(G)| \geq 2k + t,$$

then G contains k disjoint cycles.

For an integer m, let SK_m denote the graph obtained by subdividing one edge of the complete m-vertex graph K_m. The graph SK_{3k-1} shows that the bound $2k+t$ in Theorem 1.2.6 is sharp. Let $u \in V(SK_{3k-1})$ be the vertex of degree 2 and observe that $L_k(SK_{3k-1}) = \{u\}$, $|H_k(SK_{3k-1})| = 3k-1$. Since u is not in a triangle and $|SK_{3k-1}| = 3k$, G contains $k-1$ disjoint triangles and $|H_k(SK_{3k-1})| - |L_k(SK_{3k-1})| = 2k+t-1$. Any set of k disjoint cycles must partition $V(SK_{3k-1})$ into triangles, so SK_{3k-1} does not contain k disjoint cycles.

As with Theorem 1.2.2, the condition $|G| \geq 3k$ is necessary since every cycle contains at least 3 vertices. However, if the bound on $|H_k(G)| - |L_k(G)|$ in Theorem 1.2.6 is slightly strengthened, then the condition $|G| \geq 3k$ holds automatically.

Corollary 1.2.7. *Let $k \geq 2$ be an integer and G be a graph. Let t be the maximum number of disjoint* *triangles contained in G. If*

$$|H_k(G)| - |L_k(G)| \geq 2k + t + 1,$$

then G contains k disjoint cycles.

Corollary 1.2.7 is sharp since K_{3k-1} contains only $k-1$ disjoint cycles and $|H_k(K_{3k-1})| - |L_k(K_{3k-1})| = 3k-1$. Corollary 1.2.7 implies the following stronger version of Theorem 1.2.3.

Corollary 1.2.8. *Let $k \geq 2$ be an integer and G be a graph with $|H_k(G)| - |L_k(G)| \geq 3k$. Then G contains k disjoint cycles.*

Indeed, consider a graph G with $|H_k(G)| - |L_k(G)| \geq 3k$. If G contains k disjoint triangles, then these triangles are the desired k cycles. Otherwise, $t \leq k-1$, so $|H_k(G)| - |L_k(G)| \geq 2k + t$ and Corollary 1.2.7 gives that G has k disjoint cycles. In the special case $H_k(G) = V(G)$ of Corollary 1.2.8 is equivalent to Theorem 1.2.2 for $k \geq 2$.

The graph K_{3k-1} demonstrates the sharpness of Corollary 1.2.8, as $|H_k(K_{3k-1})| - |L_k(K_{3k-1})| = 3k-1$, but K_{3k-1} does not contain k disjoint cycles. However, Theorem 1.2.6 implies that all sharpness examples have fewer than $3k$ vertices. As shown in Example 1.2.4, there are graphs with $|G| \geq 3k$ and $|H_k(G)| - |L_k(G)| \geq 2k$ without k disjoint cycles. The largest such graph that we can construct has $4k$ vertices and is obtained as follows.

Example 1.2.9. *Let F be a copy of K_{3k-1}. Choose $W \subset V(F)$ with $|W| = k$ and delete all edges between the vertices in W. Then add $k+1$ new vertices $x_0, x_1, \ldots, x_k$, and make x_0 adjacent to $x_1, \ldots, x_k$ and all vertices in W. In other words, let $(K_{2k-1} \cup K_1) \vee \overline{K}_k$ be the 2-core of G, and complete the construction by adding k leaves adjacent to x_0, where $V(K_1) = \{x_0\}$. Then, $H_k(G) = V(F) \cup \{x_0\}$ and $L_k(G) = \{x_1, \ldots, x_k\}$.*

This led to the the following question from [31].

Question 1.2.10. Is it true that every graph G with $|G| \geq 4k+1$ and $|H_k(G)| - |L_k(G)| \geq 2k$ has k disjoint cycles?

The answer to this question is open, though for large graphs, $|G| \geq 19k$, it is true that a difference of $2k$ is sufficient to guarantee k disjoint cycles.

Theorem 1.2.11. *Let $k \geq 2$ be an integer and G be a graph with $|G| \geq 19k$ and*

$$|H_k(G)| - |L_k(G)| \geq 2k.$$

Then G contains k disjoint cycles.

As stated above and witnessed by $K_{2k-1,n-2k+1}$ for n large, the difference of $2k$ is necessary. However, it is possible that bound $|G| \geq 19k$ can be improved. In addition to large graphs, there are other special cases when a difference of $2k$ is sufficient. For example, Theorem 1.2.6 implies that a bound of $2k$ is sufficient for triangle free graphs. In fact, if G contains no two disjoint triangles, then a difference of $2k$ is enough.

Theorem 1.2.12. *Let $k \geq 3$ be an integer and G be a graph such that G does not contain two disjoint triangles. If*

$$|H_k(G)| - |L_k(G)| \geq 2k,$$

then G contains k disjoint cycles.

Further, when G is a planar graph, a difference of $2k$ is sufficient. This gives the following stronger version of Theorem 1.2.5.

Theorem 1.2.13. *Let $k \geq 2$ be an integer and G be a planar graph. If*

$$|H_k(G)| - |L_k(G)| \geq 2k,$$

then G contains k disjoint cycles.

The condition that G be planar is necessary. Indeed, consider the non-planar graph SK_5. If u is the newly created vertex, then $H_2(SK_5) = V(SK_5) - u$ and $L_2(SK_5) = \{u\}$, but SK_5 does not have two disjoint cycles. The bound $2k$ in Theorem 1.2.13 is sharp (see, e.g. $K_5 - e$ for $k = 2$), however only for small values of k. In fact, since the average degree of every planar graph is less than 6, there are no planar graphs that satisfy $|H_k(G)| - |L_k(G)| \geq 2k$ for $k \geq 7$.

1.3 Graph Packing

In Chapter 3, we will translate the problem of finding a subgraph in a graph into the language of graph packing. The results in this chapter are from joint work with Ervin Győri, Alexandr Kostochka, and Derrek Yager found in [38] and [25].

Two graphs G_1 and G_2 with $|G_1| = |G_2|$ *pack* if there is a bijection $f : V(G_1) \to V(G_2)$ such that if $uv \in E(G_1)$, then $f(u)f(v) \notin E(G_2)$. We call the function f a *packing* of G_1 and G_2.

Notice that graph packing is simply a rephrasing of the subgraph problem since graphs G_1 and G_2 pack if and only if G_1 is a subgraph of the complement of G_2. Equivalently, G_1 and G_2 pack if G_2 is a subgraph of the complement of G_1. This symmetry highlights the advantage of using the language of graph packing, as we no longer need to specify that, for instance, G_1 is the subgraph and $\overline{G}_2$ is the underlying graph. Consider the following example.

Example 1.3.1. *Let G_1 be an n vertex graph containing at least one isolated vertex and G_2 be $K_{1,n-1}$, the star on n vertices. Then, G_1 and G_2 pack. To see this let x be the isolated vertex in G_1 and let y be the center of the star in G_2. Consider a bijection $f : V(G_1) \to V(G_2)$ such that $f(x) = y$. The function f is a packing because if $uv \in E(G_1)$, then $x \notin \{u, v\}$. Since all edges in G_2 contain $y = f(x)$, $f(u)f(v) \notin E(G_2)$. Equivalently, notice that $\overline{G}_2 \cong K_1 \cup K_{n-1}$. Since G_1 contains an isolated vertex, G_1 is a subgraph of $\overline{G}_2$.*

Graph packings generalize several important problems in graph theory. For instance, determining if an n vertex graph G is hamiltonian (a close relative of the famous Traveling Salesman Problem) is equivalent to determining if $\overline{G}$ packs with the cycle C_n. Similarly, a graph G is k-colorable if and only if G packs with a disjoint union of k cliques. In this way, packing problems include graph coloring problems.

Important results on graph packing were obtained in 1978 by Bollobás and Eldridge [5] and by Sauer and Spencer [46]. Resolving a conjecture of Milner and Welsh [43], they proved that if two n-vertex graphs together contain at most $\frac{3}{2}n - 2$ edges, they are guaranteed to pack.

Theorem 1.3.2 ([5, 46])**.** *Let G_1 and G_2 be two n-vertex graphs. If*

$$\|G_1\| + \|G_2\| \leq \frac{3}{2}n - 2, \tag{1.1}$$

then G_1 and G_2 pack.

Restriction (1.1) cannot be relaxed in view of the pair $\{G_1, G_2\}$ where G_1 is an n-vertex star and G_2 has no isolated vertices. Furthermore, Bollobás and Eldridge showed that if neither graph contains a star on n vertices, then (1.1) can be relaxed significantly.

Theorem 1.3.3 ([5])**.** *Let G_1 and G_2 be two n-vertex graphs. If $\Delta(G_1), \Delta(G_2) \leq n-2$ and $\|G_1\| + \|G_2\| \leq 2n - 3$, then either G_1 and G_2 pack, or $\{G_1, G_2\}$ is one of the following 7 pairs: $\{2K_2, K_1 \cup K_3\}$, $\{\overline{K}_2 \cup K_3, K_2 \cup K_3\}$, $\{3K_2, \overline{K}_2 \cup K_4\}$, $\{\overline{K}_3 \cup K_3, 2K_3\}$, $\{2K_2 \cup K_3, \overline{K}_3 \cup K_4\}$, $\{\overline{K}_4 \cup K_4, K_2 \cup 2K_3\}$, $\{\overline{K}_5 \cup K_4, 3K_3\}$.*

The restriction $2n - 3$ in Theorem 1.3.3 is again sharp and, unlike Theorem 1.3.2, there are multiple sharpness examples. Indeed, the pairs $\{K_{1,n-2} \cup K_1\}$, $\{K_{1,n-4} \cup K_3, K_{1,n-4} \cup K_3\}$, and (for $n \equiv 0 \pmod 3$) $\{K_{n-3,1} \cup K_2, \frac{n}{3}K_3\}$ each contain $2n - 2$ but do not pack. A theorem of Teo and Yap shows that these 3 examples are the only pairs with $n \geq 13$ vertices and $2n - 2$ edges that do not pack, and also characterize the 40 graphs with at most $n \leq 12$ vertices and $2n - 2$ edges that do not pack. [47].

Theorem 1.3.4 ([47]). *Suppose G_1 and G_2 are graphs of order $n \geq 13$ such that $\Delta(G_1), \Delta(G_2) < n-1$ and $\|G_1\| + \|G_2\| \leq 2n-2$. Then, either G_1 and G_2 pack or $\{G_1, G_2\}$ is one of $\{K_{1,n-2} \cup K_1, C_n\}$, $\{K_{1,n-4} \cup K_3, K_{1,n-4} \cup K_3\}$, or, for $n \equiv 0 \pmod 3$, $\{K_{n-3,1} \cup K_2, \frac{n}{3}K_3\}$.*

It is possible that stronger conditions on the maximum degree of G_1 and G_2 allow for the bound of $2n-2$ to be improved further. Bollobás and Eldridge conjecture the following.

Conjecture 1.3.5 ([5]). *There exists an absolute constant $c > 0$ such that if $\Delta(G_1), \Delta(G_2) < n-k$ and $\|G_1\| + \|G_2\| < ckn$, then G_1 and G_2 pack.*

In a sense, Theorem 1.3.2 and Theorem 1.3.3 describe global properties of the graphs, since there are no restrictions on how the edges are arranged in the graph. On the other hand, the following result of Sauer and Spencer shows that two graphs with many more edges will pack if their maximum degrees are not too large.

Theorem 1.3.6 ([46]). *Let G_1 and G_2 be two n-vertex graphs. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then G_1 and G_2 pack.*

Recently, Alon and Yuster [2] considered packing a graph with few edges with a graph of bounded maximum degree.

Theorem 1.3.7 ([2]). *For all n sufficiently large, let G_1 and G_2 be n-vertex graphs such that $\|G_1\| \leq n - \delta(G_2) - 1$ and $\Delta(G_2) \leq \sqrt{n}/200$. Then G_1 and G_2 pack.*

Alon and Yuster phrased their theorem in the language of Turán numbers. The *Turán number* $\mathrm{ex}(n, G)$ of a graph G is the maximum number of edges in an n-vertex graph that does not contain a subgraph isomorphic to G. A result of Ore [45] from 1961 shows that $\mathrm{ex}(n, C_n) = \binom{n-1}{2} + 1$ and that for $n \geq 5$ the only graph with n vertices and $\binom{n-1}{2} + 1$ edges that does not contain a C_n is K_n minus a star with $n-2$ edges [45]. In this language, Theorem 1.3.7 is the following stronger version of Ore's result.

Theorem 1.3.8 ([2]). *For all n sufficiently large, if G is a graph of order n with no isolated vertices and $\Delta(G) \leq \sqrt{n}/200$, then $\mathrm{ex}(n, G) = \binom{n-1}{2} + \delta(G) - 1$.*

Theorem 1.3.7 has the additional property that, unlike Ore's result, there are different sharpness examples. In particular, the following two examples are provided in [2], though we rephrase them in the language of graph packing.

Example 1.3.9. *Let G_1 be a star with $n-2$ edges and an additional vertex, that is $G_1 = K_{1,n-2} \cup K_1$. Let G_2 be a graph on n vertices in which all vertices but one have degree 3, the last vertex has degree 2 and the neighbors of this vertex are adjacent. The graph G_1 has $n - \delta(G_2)$ edges, but $\Delta(G_1) + \delta(G_2) \geq n$. The* *graphs G_1 and G_2 do not pack since there is no suitable vertex in G_2 to which we might map the vertex of maximum degree in G_1. (Figure 1.1a)*

Example 1.3.10. *Let G_1 be the disjoint union of a star with $n-3$ edges and an edge and let G_2 be as in Example 1.3.9. As in Example 1.3.9, G_1 has $n-\delta(G_2)$ edges, but now $\Delta(G_1)+\delta(G_2)=n-1$. A potential packing could (and must) map the vertex of maximum degree in G_1 to the vertex of degree 2 in G_2. However, such an attempt will eventually fail to be a packing because no set of vertices could be mapped to the neighborhood of the degree 2 vertex. (Figure 1.1b).*

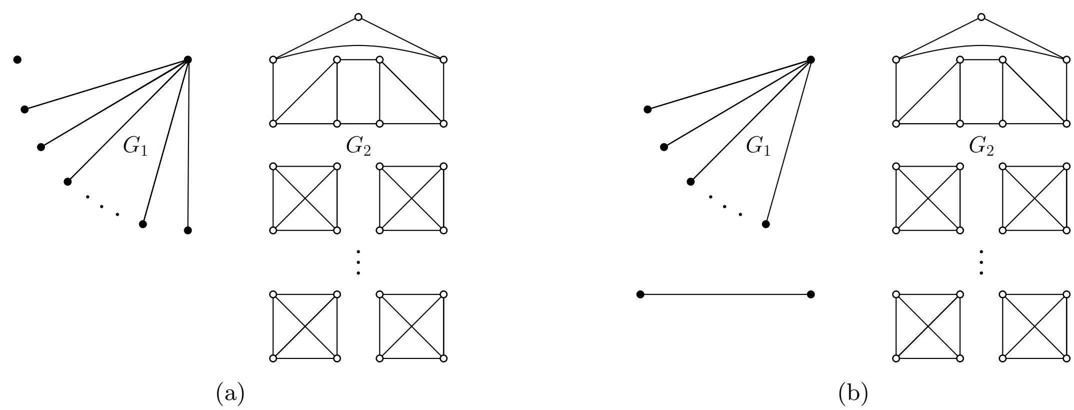

Figure 1.1: Sharpness examples for Theorem 1.3.7 [2]

By observing the reasons that G_1 and G_2 in Example 1.3.10 fail to pack, we can obtain a larger set of sharpness examples for Theorem 1.3.7.

Example 1.3.11. *Fix constants n and d with n much larger than d. Let G_2 be a d-regular graph on n vertices consisting of a disjoint union of cliques. Let G_1 be the disjoint union of $d-1$ edges, together with a star containing $n-2(d-1)-1$ edges (Figure 1.2a, here $d=6$).*

In fact, as long as there is no independent set of size d among the vertices in G_1 not in the star, we can create still more examples, e.g. Figure 1.2b.

A main result of Chapter 3 shows that if there is such an independent set of size $\delta(G_2)$, then G_1 and G_2 will pack even if G_1 contains as many as n edges.

Theorem 1.3.12. *For n sufficiently large ($n \geq 10^9$), let G_1 and G_2 be graphs of order n such that $\Delta(G_2) \leq \sqrt{n}/60$, $\|G_1\| \leq n$, and $\Delta(G_1)+\delta(G_2) \leq n-1$. If there is a vertex $v_1 \in V(G_1)$ such that*

$$d(v_1)=\Delta(G_1) \text{ and } \alpha(G_1 - N[v_1]) \geq \delta(G_2), \tag{1.2}$$

then G_1 and G_2 pack.

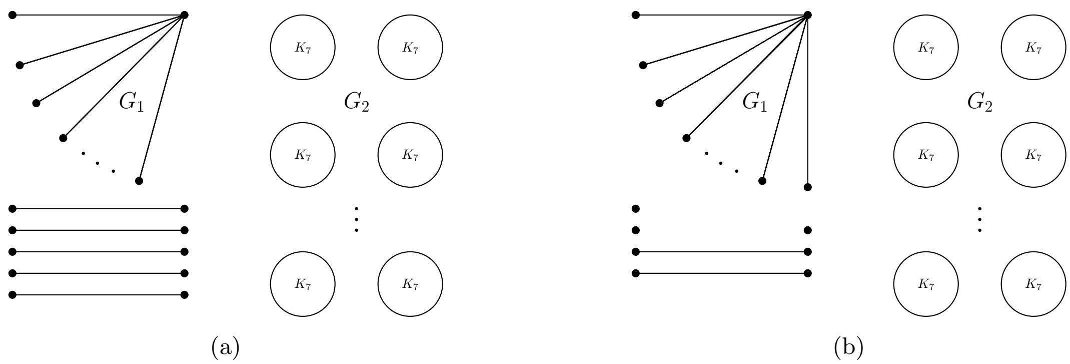

Figure 1.2: Additional sharpness examples for Theorem 1.3.7

Our theorem shows that if we are able to appropriately place the vertex of maximum degree in the sparse graph, then the remainder of the graph can also be placed. In fact, Theorem 1.3.12 is a generalization of Theorem 1.3.7. Indeed, if $\|G_1\| \leq n - \delta(G_2) - 1$, then $\Delta(G_1) + \delta(G_2) \leq n - 1$. Also, if $v_1 \in V(G_1)$ with $d(v_1) = \Delta(G_1)$, then $G - N[v_1]$ contains $n - d(v_1) - 1$ vertices and $n - d(v_1) - \delta(G_2) - 1$ edges. Hence, $G - N[v_1]$ contains at least $\delta(G_2)$ components and an independent set of size at least $\delta(G_2)$.

We also adapt the methods used in the proof of Theorem 1.3.12 to characterize the sharpness examples for Theorem 1.3.7.

Corollary 1.3.13. *For n sufficiently large ($n \geq 10^9$), let G_1 and G_2 be graphs of order n such that $\Delta(G_2) \leq \sqrt{n}/60$, $\|G_1\| \leq n - \delta(G_2)$. Then,*

1. *G_1 and G_2 pack, or*

2. *$\Delta(G_1) + \delta(G_2) = n$, or*

3. *G_1 has exactly $n - \delta(G_2)$ edges and exactly one vertex of degree greater than 1. Moreover, for each $w \in V(G_2)$ with $d(w) = \delta(G_2)$, the neighborhood of w induces a clique.*

Broadly speaking, Theorems 1.3.2 and 1.3.3 show that bounding the number of edges in both G_1 and G_2 is sufficient to show that the graphs pack, while Theorem 1.3.6 shows that bounding the maximum degrees of both G_1 and G_2 is also sufficient. In a sense, Theorems 1.3.7 and 1.3.12 refine this reasoning, showing that two graphs pack if the number of edges one graph is bounded and the maximum degree in the other graph is bounded.

Recently, Żak suggested another lens through which one can view the relationship between Theorem 1.3.2 and Theorem 1.3.3. In particular, Żak showed that by further restricting the maximum degrees of G_1 and G_2, additional edges can be permitted while still maintaining that the graphs pack [53]. Namely, he proved the following

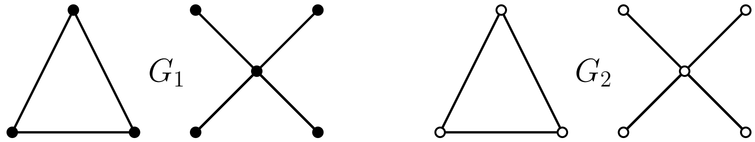

Figure 1.3: Sharpness example for Conjecture 1.3.16. In this example $n = 8$ and $|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} = 3n - 6$ but the graphs do not pack.

Theorem 1.3.14 ([53])**.** *Let G_1 and G_2 be two graphs of order $n \geq 10^{10}$. If*

$$\|G_1\| + \|G_2\| + \max\{\Delta(G_1), \Delta(G_2)\} < \frac{5}{2}n - 2,$$

then G_1 and G_2 pack.

By forbidding the star on n vertices, Żak showed that this result can also be strengthened.

Theorem 1.3.15 ([53])**.** *Let G_1 and G_2 be n-vertex graphs with $\Delta(G_1), \Delta(G_2) \leq n - 2$. If $\|G_1\| + \|G_2\| + \max\{\Delta(G_1), \Delta(G_2)\} \leq 3n - 96n^{3/4} - 65$, then G_1 and G_2 pack.*

This theorem is asymptotically sharp since $K_{1,n-2} \cup K_1$ and C_n do not pack. In the same paper Żak makes the following conjecture.

Conjecture 1.3.16 ([53])**.** *Let G_1 and G_2 be n-vertex graphs with $\Delta(G_1), \Delta(G_2) \leq n-2$. If $\|G_1\| + \|G_2\| + \max\{\Delta(G_1), \Delta(G_2)\} \leq 3n - 7$, then G_1 and G_2 pack.*

Żak also provides the following example to show that, if true, the conjecture is best possible. Let $n \geq 8$ and let G_1 and G_2 each be isomorphic to $K_3 + K_{1,n-4}$, a disjoint union of a triangle and a star (Figure 1.3). Then, $\Delta(G_1) = \Delta(G_2) = n-4$ and $\|G_1\| + \|G_2\| + \max\{\Delta(G_1), \Delta(G_2)\} = (n-1)+(n-1)+(n-4) = 3n-6$. A simple check shows that G_1 and G_2 do not pack.

However, for small values of n, we observe that Conjecture 1.3.16 fails. Consider the following example.

Example 1.3.17. *Let $G_1 = 4K_3$ and $G_2 = K_5 \cup \overline{K}_7$ (Figure 1.4). In any attempted packing, we are forced to send at least two vertices from the same component in G_1 to the clique in G_2, so the graphs do not pack. In this example, $\|G_1\| + \|G_2\| + \max\{\Delta(G_1), \Delta(G_2)\} = 12 + 10 + 4 = 26 = 3n - 10$.*

We were unable to find large counterexamples, so the conjecture may hold with a finite set of exceptions. Further, the other main result of Chapter 3 shows that, up to the choice of the additive constant, Conjecture 1.3.16 is true.

Theorem 1.3.18. *Let $C = 11(195^2) = 418,275$. Let G_1 and G_2 be n-vertex graphs with $\Delta(G_1), \Delta(G_2) \leq n - 2$. If $\|G_1\| + \|G_2\| + \max\{\Delta(G_1), \Delta(G_2)\} \leq 3n - C$, then G_1 and G_2 pack.*

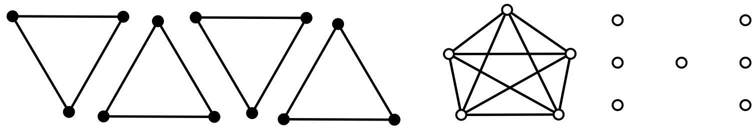

Figure 1.4: Żak's Conjecture is false for small values of n.

Our constant C is not optimal and we can somewhat decrease it by a more detailed case analysis in our proofs. However, $3n - 96n^{3/4} - 65 \le 0$ for $n \le 10^6$, so all graphs satisfying the hypothesis of Theorem 1.3.15 for $n \le 10^6$ contain zero edges and the conclusion holds trivially. Since $96n^{3/4} - 65 \ge C$ when $n \ge 300,000$, Theorem 1.3.18 is a stronger result all non-trivial values of n. Further, Theorems 1.3.18 and 1.3.2 together imply that Theorem 1.3.14 holds when n is at least $2C - 2 \approx 10^6$. To see this notice that if $\Delta(G_1) = n - 1$ or $\Delta(G_2) = n - 1$, then $\|G_1\| + \|G_2\| \le \frac{3}{2}n - 1$ and Theorem 1.3.2 applies. Alternatively, when $n \ge 2C - 2$, $\frac{5}{2}n - 2 \le 3n - C$ and Theorem 1.3.18 applies.

Our proof of Theorem 1.3.18 uses the concept of list packing introduced in [24]. A *graph triple* $\mathbf{G} = (G_1, G_2, G_3)$ consists of two disjoint n-vertex graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ and a bipartite graph $G_3 = (V_1 \cup V_2, E_3)$ with partite sets V_1 and V_2. A *list packing* of $\mathbf{G}$ is a packing of G_1 and G_2 such that $uf(u) \notin E_3$ for any $u \in V_1$. Essentially, a list packing is a packing of G_1 and G_2 with an additional set of restrictions on the bijection f.

We prove the following list version of Theorem 1.3.18.

Theorem 1.3.19. *Let* $C = 11(195^2)$*. Let* $n \ge 2$ *and* $\mathbf{G} = (G_1, G_2, G_3)$ *be a graph triple with* $|V_1| = |V_2| = n$*,* $\Delta(G_1), \Delta(G_2) \le n - 2$*, and* $\Delta(G_3) \le n - 1$*. If* $|E_1| + |E_2| + |E_3| + \max\{\Delta(G_1), \Delta(G_2)\} + \Delta(G_3) \le 3n - C$*, then* $\mathbf{G}$ *packs.*

Note that Theorem 1.3.18 is the special case of Theorem 1.3.19 in which G_3 has no edges. The pair shown in Figure 1.4 shows that, up to an additive constant, the theorem is sharp. Moreover, there are other infinite families of examples showing that, up to an additive constant, the theorem is sharp when E_3 is nonempty. Several of these examples are shown in Figure 1.5.

1.4 Cyclic Stable Matchings

Consider the problem of taking two sets of n people, say men and women, and attempting to match the women and men into n pairs. If the only restriction is that each pair must consist of a man and a woman, then finding such a pairing is equivalent to finding a perfect matching in the complete bipartite graph $K_{n,n}$. However, if the men and women have preferences, then it is not enough to simply find a pairing of the men

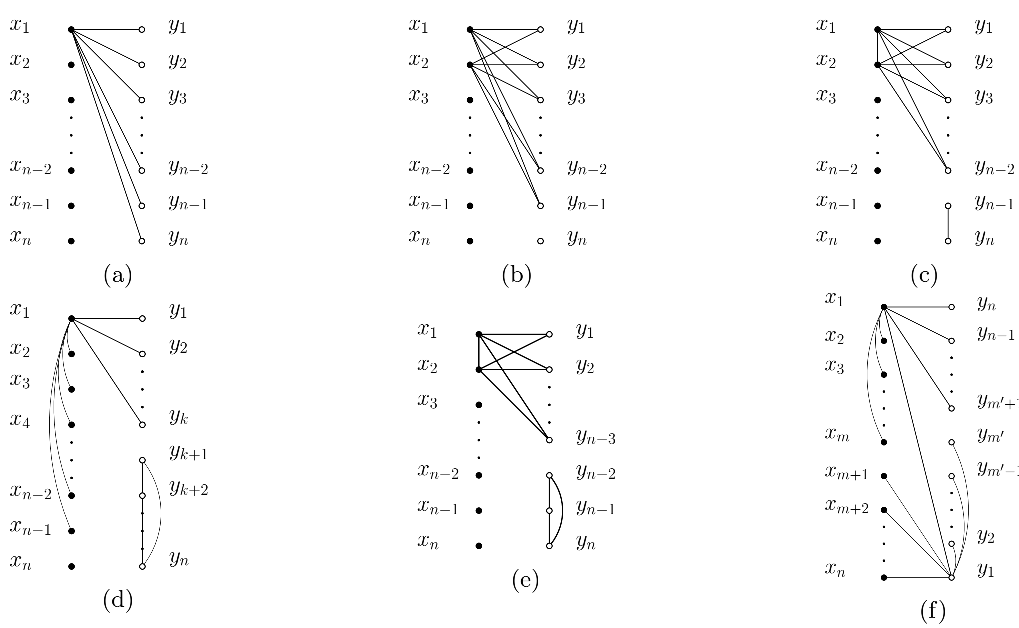

Figure 1.5: Sharpness examples for Theorem 1.3.19

and women. Rather, a desired matching should be acceptable to the $2n$ people and should not allow two people to mutually prefer the other over his/her assigned partner.

This problem, known as the *Stable Marriage* problem was introduced in 1962 by Gale and Shapley [21]. Formally, an *instance* of the problem (of size n) is a set of n men and n women together with their list of preferences. For each woman, her preference list is a ranking of the n men (with no ties) and, for each man, his preference list is a ranking of the n women. A *matching* M is a set of n man-woman pairs such that each person is in exactly one pair. A matching M is *stable* if there is no man and woman, not matched in M, who each prefer each other to their assigned partner. On the other hand, if such a pair exists, we say that it is a *blocking pair*. The following is a seminal result of Gale and Shapley.

Theorem 1.4.1. *[21] Every instance of the stable marriage problem admits a stable matching.*

Not only is it always possible to find a stable matching, but Gale and Shapley gave an algorithm, now known as the Gale-Shapley Algorithm, to find such a matching in polynomial time. This celebrated result eventually led to Shapley receiving the 2012 Nobel Prize in Economics. In the years since, stable marriages with 2 genders have been extensively studied and we refer the reader to [23, 41] for additional information.

In 1976, Knuth proposed extending the stable marriage problem to 3 dimensions [36]. To do so, we introduce a third group, say dogs. Now, there are n men, n women, and n dogs and the goal is to form a matching of n triples. Each triple contains exactly one man, woman, and dog and each person/dog is contained in exactly one triple.

However, the notion of a preference list is now more nuanced. One natural extension is to have each man rank the n^2 woman/dog pairs, each woman rank the n^2 man-dog pairs, and each dog rank the n^2 man-woman pairs. A matching M is stable if there is no triple (m, w, d), not matched in M, such that m, w, and d each prefer (m, w, d) to their assigned triple. It was proved by Alkan in 1988 that, unlike the 2-dimensional case, not all instances of the 3-dimensional stable matching problem admit a stable matching [1]. In 1991, Ng and Hirschberg proved that determining whether or not a given instance contains a stable matching is NP-complete [44]. This led to an alternate approach to defining preference lists.

In the same paper, Ng and Hirschberg introduced the notion of the *cyclic 3-dimensional stable matching problem* (cyclic 3-DSM) [44], though they credit the problem to Knuth. In this version of the problem, each man ranks the n women, each woman ranks the n dogs, and each dog ranks the n men. A matching M is stable if there is no triple (m, w, d), not matched in M, such that m, w, and d each *strictly* prefers (m, w, d) to their assigned triple. Notice that, for any positive integer s, this problem can naturally be extended to s dimensions by labeling the genders 1 through s, and having each member of gender i rank the members of gender $i + 1$ (with addition modulo s). The main open question is to determine if it is always possible to find a stable marriage.

Question 1.4.2. Does every instance of cyclic s-DSM admit a stable matching?

Notice that the strict preference is needed in the definition of stability as a person may now be indifferent when comparing two families. For instance, since the men do not have preferences for the dogs, a man m is equally happy to be in the family (m, w, d_1) as he is to be in the family (m, w, d_2). In fact, when comparing two families, an agent in gender i is indifferent if and only if both families contain the same member of gender $i + 1$. In 2004, Boros, Gurvich, Jaslar, and Krasner proved the first result for this problem:

Theorem 1.4.3. *[7] Let $n, s \in \mathbb{N}$ with $2 \leq n \leq s$, then each instance of cyclic s-DSM of size n admits a stable matching.*

In 2006, Eriksson, Sjöstrand, and Strimling [20] extended this result in the special case when $s = 3$.

Theorem 1.4.4. *[20] Let $n = 4$, then each instance of cyclic 3-DSM admits a stable matching.*

The results in Chapter 4 are joint work with Sarah Behrens and Nicholas Kosar. The main result is the following generalization of Theorem 1.4.4 to cyclic s-DSM.

Theorem 1.4.5. *For $s \geq 3$ and $n \leq s + 1$, any instance of cyclic s-DSM has a stable matching.*

This result was proved independently by Hofbauer in 2016 [28]. The result in [28] uses a similar algorithm to find a stable matching, though the analysis is somewhat different.

We will also consider one final variation of the stable marriage problem, the *cyclic s-dimensional strongly stable matching problem* (cyclic s-DSSM). The setup of this problem is the same as in cyclic s-DSM, however we now require a stronger notion of stability. A matching M is *strongly stable* if there is no triple (m, w, d), not matched in M, such that m, w, and d each either prefers (m, w, d) to their assigned triple or is indifferent.

The notion of indifference is not unique to cyclic s-DSM, and has been studied in the two dimensional case as well. However, in 2 dimensions, indifference will only occur if the definition of a preference list is altered to explicitly allow for ties. It is possible for an instance of the two dimensional stable marriage problem with ties to not admit a strongly stable matching. However, in [29], Irving provides an algorithm that, in polynomial time, determines if a strongly stable matching exists and returns a matching if one exists [29]. Unfortunately, for cyclic 3-DSSM, no such algorithm is known.

Biró and McDermid [3] showed that in cyclic 3-DSSM not all instances contain a strongly stable matching by providing an example with $n = 6$. Recently, Irving provided an example of 3-DSSM with $n = 3$ that admits no strongly stable matching [41, p. 280]. We improve on this result by providing a (different) instance of cyclic 3-DSSM with no strongly stable matching and extending it to an example of s-DSSM for $s \geq 3$.

Theorem 1.4.6. *Let $s \geq 3$.*

(i) *If $n \leq 2$, every instance of cyclic s-DSSM of size n admits a strongly stable matching, and*

(ii) *If $n \geq 3$, there exists an instance of cyclic s-DSSM of size n that admits no strongly stable matching.*

Chapter 2

Cycles

The results of this chapter are joint work with Hal Kierstead and Alexandr Kostochka; the results appear in [31] and [30].

2.1 Introduction

Every graph G with at least $|G|$ edges contains at least one cycle. However, the following attempt to generalize this fact is computationally much more difficult.

Problem 1.2.1. *Given a graph G and integer k, determine if G contains k vertex disjoint cycles.*

Indeed, Problem 1.2.1 contains, as a special case, the problem of determining whether G contains $n/3$ disjoint triangles and is NP-complete [22, p. 68]. Problem 1.2.1 is fixed parameter tractable. When the value k is fixed and not an input parameter, it can determined in linear time whether the graph contains k disjoint cycles [15, 4]. Given the complexity of Problem 1.2.1 when k is not predetermined, it is natural to ask which properties of a graph G guarantee that G contain k disjoint cycles.

For a positive integer k, let $H_k(G)$ be the subset of vertices of degree at least $2k$ and $L_k(G)$ be the subset of vertices of degree at most $2k-2$. We say a vertex in $H_k(G)$ is *high* and a vertex in $L_k(G)$ is *low*. The Corrádi-Hajnal Theorem states that if a graph contains $3k$ vertices and every vertex is high, then G contains k disjoint cycles.

Theorem 1.2.2. *[9] Let G be a graph and k a positive integer. If $|G| \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.*

In 1963, Dirac and Erdős considered the situation that a graph has many more high vertices than low vertices. In particular, they proved the following generalization of the Corrádi-Hajnal Theorem.

Theorem 1.2.3. *[12] Let $k \geq 3$ be an integer and G be a graph with $|H_k(G)| - |L_k(G)| \geq k^2 + 2k - 4$. Then G contains k disjoint cycles.*

The bound $k^2 + 2k - 4$ is not best possible. Dirac and Erdős provide an example of a graph G with $|H_k(G)| - |L_k(G)| = 2k - 1$ not containing k disjoint cycles.

Example 1.2.4. *Let $V(G) = X \cup Y \cup Z$, where $|X| = 2k - 1$ and $|Y| = |Z| \geq 2k$. Let $E(G)$ consist of all possible edges between X and Y and also a perfect matching between Y and Z. Then $H_k(G) = X \cup Y$ and $L_k(G) = Z$. Since every vertex in Z has degree 1, no vertex of Z is in a cycle. Then every cycle in G contains two vertices from X and so G contains at most $k - 1$ disjoint cycles.*

In the same paper, Dirac and Erdős also consider sufficient conditions for k disjoint cycles in special classes of graphs. In particular, they provide a bound on $|H_k(G)| - |L_k(G)|$ that is linear in k and sufficient to guarantee the existence of k disjoint cycles in planar graphs.

Theorem 1.2.5. *[12] Let $k \geq 3$ be an integer and G be a planar graph such that $|H_k(G)| - |L_k(G)| \geq 5k - 7$. Then G contains k disjoint cycles.*

The main result of this chapter is the following refinement of Theorem 1.2.3.

Theorem 1.2.6. *Let $k \geq 2$ be an integer and G be a graph such that $|G| \geq 3k$. Let t be the maximum number of disjoint triangles contained in G. If*

$$|H_k(G)| - |L_k(G)| \geq 2k + t,$$

then G contains k disjoint cycles.

Theorem 1.2.6 is sharp in both senses. As each cycle contains at least 3 vertices, a graph G with k disjoint cycles cannot have fewer than $3k$ vertices. Further, the graph SK_{3k-1} satisfies $|H_k(SK_{3k-1})| - |L_k(SK_{3k-1})| = 3k - 2 = 2k + t - 1$ but does not contain k disjoint cycles. If the bound on $|H_k(G)| - |L_k(G)|$ in Theorem 1.2.6 is slightly strengthened, then the condition $|G| \geq 3k$ holds automatically.

Corollary 1.2.7. *Let $k \geq 2$ be an integer and G be a graph. Let t be the maximum number of disjoint triangles contained in G. If*

$$|H_k(G)| - |L_k(G)| \geq 2k + t + 1,$$

then G contains k disjoint cycles.

Corollary 1.2.7 is also sharp since $|H_k(K_{3k-1})| - |L_k(K_{3k-1})| = 3k - 1 = 2k + t$, but K_{3k-1} contains only $k - 1$ disjoint cycles. Clearly, Theorem 1.2.6 implies that all sharpness examples to Corollary 1.2.7 contain fewer than $3k$ vertices. Corollary 1.2.7 requires a short proof that is given in Section 2.5. A straightforward consequence of Corollary 1.2.7 is the following stronger version of Theorem 1.2.3.

Corollary 1.2.8. *Let $k \geq 2$ be an integer and G be a graph with $|H_k(G)| - |L_k(G)| \geq 3k$. Then G contains k disjoint cycles.*

Observe that the special case $H_k(G) = V(G)$ of Corollary 1.2.8 is equivalent to Theorem 1.2.2 for $k \geq 2$. However, it is sometimes the case that a bound of only $2k$ is sufficient to guarantee the existence of k disjoint cycles. For instance, Theorem 1.2.6 shows that this is the case for triangle free graphs. However, the result also holds for graphs not containing 2 disjoint triangles.

Theorem 1.2.12. *Let $k \geq 3$ be an integer and G be a graph such that G does not contain two disjoint triangles. If*

$$|H_k(G)| - |L_k(G)| \geq 2k,$$

then G contains k disjoint cycles.

Using the techniques of Theorem 1.2.6, we also strengthen Theorem 1.2.5.

Theorem 1.2.13. *Let $k \geq 2$ be an integer and G be a planar graph. If*

$$|H_k(G)| - |L_k(G)| \geq 2k,$$

then G contains k disjoint cycles.

The condition that G be planar is necessary. Indeed, consider the non-planar graph SK_5. If u is the newly created vertex, then $H_2(SK_5) = V(SK_5) - u$ and $L_2(SK_5) = \{u\}$, but SK_5 does not have two disjoint cycles. The bound $2k$ in Theorem 1.2.13 is sharp (see, e.g. $K_5 - e$ for $k = 2$), however only for small values of k. Since the average degree of every planar graph is less than 6, for $k \geq 5$ much weaker restrictions provide existence of k disjoint cycles in planar graphs.

We are unable to find a graph G with many vertices and $|H_k(G)| - |L_k(G)| \geq 2k$, but without k disjoint cycles. Indeed, the largest example that we can find has only $4k + 1$ vertices.

Example 1.2.9. *Let F be a copy of K_{3k-1}. Choose $W \subset V(F)$ with $|W| = k$ and delete all edges in $G[W]$. Then add $k + 1$ new vertices $x_0, x_1, \ldots, x_k$, and make x_0 adjacent to $x_1, \ldots, x_k$ and all vertices in W. In other words, let $(K_{2k-1} \cup K_1) \vee \overline{K}_k$ be the 2-core of G, and complete the construction by adding k leaves adjacent to x_0, where $V(K_1) = \{x_0\}$. Then, $H_k(G) = V(F) \cup \{x_0\}$ and $L_k(G) = \{x_1, \ldots, x_k\}$.*

This led to the the following question.

Question 2.1.1. Is it true that every graph G with $|G| \geq 4k + 1$ and $|H_k(G)| - |L_k(G)| \geq 2k$ has k disjoint cycles?

While this question is still open, the following theorem shows that the conclusion holds for graphs with at least $19k$ vertices.

Theorem 1.2.11. *Let $k \geq 2$ be an integer and G be a graph with $|G| \geq 19k$ and*

$$|H_k(G)| - |L_k(G)| \geq 2k.$$

Then G contains k disjoint cycles.

Our proofs are based on the approach and ideas of Dirac and Erdős [12]. We also heavily use an extension of Theorem 1.2.2 from [33] (Theorem 2.2.1 below).

The remainder of this chapter is organized as follows. The next section outlines the notation that we will use throughout the chapter, and introduces some tools to be used in the proofs of the various theorems. In Section 2.3 we will prove several lemmas for the case $k = 2$ and in Section 2.4 we prove Theorem 1.2.6. In Sections 2.5, 2.6 and 2.7, we use Theorem 1.2.6 to prove Corollary 1.2.7, Theorem 1.2.13, and Theorem 1.2.12, respectively.

Finally, in Section 2.8, we prove Theorem 1.2.11. The proof of this theorem is separated into two parts. Section 2.8.1 introduces Theorem 2.8.1, which is a more technical version of Theorem 1.2.11, and proves a lemma that will be used in the proof. Theorem 2.8.1 is proved in Section 2.8.2. The proof builds on the techniques of Dirac and Erdős [12] and also relies on Corollary 1.2.8.

2.2 Notation and Tools

Given an integer k, we say a vertex in $H_k(G)$ is *high*, and set $h_k(G) = |H_k(G)|$. A vertex in $L_k(G)$ is *low*. Set $\ell_k(G) = |L_k(G)|$. Occasionally, it will be helpful to classify vertices more carefully than just high and low. Following the notation of Dirac and Erdős, we say a vertex v is in $V^i(G)$ if $d_G(v) = i$. Similarly, $v \in V^{\leq i}(G)$ if $d_G(v) \leq i$ and $v \in V^{\geq i}(G)$ if $d_G(v) \geq i$. In these terms, $H_k(G) = V^{\geq 2k}(G)$ and $L_k(G) = V^{\leq 2k-2}(G)$.

We say that $x, y, z \in V(G)$ *form a triangle* $T = xyz$ in G if $G[\{x, y, z\}]$ is a triangle. We say $v \in T$, if $v \in \{x, y, z\}$. A set $\mathcal{T}$ of triangles is a set of subgraphs of G such that each subgraph is a triangle and all the triangles are disjoint. For a set $\mathcal{S}$ of graphs, let $V(\mathcal{S}) = \bigcup\{V(S) : S \in \mathcal{S}\}$.

For a graph G, let $c(G)$ be the maximum number of disjoint cycles in G and $t(G)$ be the maximum number of disjoint triangles in G.

When the graph G is clear from context, we will use t instead of $t(G)$. Similarly, when the integer k is also clear, we will use H and L for $H_k(G)$ and $L_k(G)$, respectively. The sizes of H and L will be denoted by h and ℓ, respectively. (2.1)

As shown in [33], if a graph G with $|G| \geq 3k$ and $\delta(G) \geq 2k-1$ does not contain a large independent set, then with two exceptions, G contains k disjoint cycles:

Theorem 2.2.1. *[33] Let $k \geq 2$. Let G be a graph with $|G| \geq 3k$ and $\delta(G) \geq 2k-1$ such that G does not contain k disjoint cycles. Then,*

1. *$\alpha(G) \geq |G| - 2k + 1$, or*
2. *k is odd and $G = 2K_k \vee \overline{K_k}$, or*
3. *$k = 2$ and G is a wheel.*

We will use the following corollary of Theorem 2.2.1 throughout the chapter.

Corollary 2.2.2. *Let $k \geq 2$ be an integer and G be a graph with $|G| \geq 3k$. If $|H| \geq 2k$ and $\delta(G) \geq 2k-1$ (i.e. $L = \emptyset$), then G contains k disjoint cycles.*

Proof. First, if $G = 2K_k \vee \overline{K_k}$, then $|H| = k$, a contradiction. Next, if $\alpha(G) \geq |G| - 2k + 1$, then let U be an independent set of size $|G| - 2k + 1$. For each $u \in U$, $d(u) \leq 2k-1$, so $H \subseteq V(G) \setminus U$ and $|H_k(G)| \leq 2k-1$. Finally, if $k = 2$, then G is not a wheel, as the wheel has only one vertex of degree at least 4. Therefore, by Theorem 2.2.1, G contains k disjoint cycles. □

Call a graph G minimal if among graphs with a certain property, $|G|$ is minimal, and subject to this, $\|G\|$ is minimal. Dirac and Erdős [12] observed the following.

Property 2.2.3. Let $k \geq 2$ be an integer and $f : \mathbb{N} \to \mathbb{Z}$ a function. Suppose G is minimal among the graphs without k disjoint cycles satisfying $|H| - |L| \geq f(k)$. Then,

1. $\delta(G) \geq 2$, and
2. if $uv \in E(G)$, then $d(u) \in \{2k-1, 2k\}$ or $d(v) \in \{2k-1, 2k\}$.

Indeed, if such a graph G contained a vertex v with $d(v) \leq 1$, then $G - v$ is a smaller counterexample. Similarly, if *(2)* does not hold, then $G - uv$ is a smaller counterexample.

2.3 Graphs with two disjoint cycles.

In this section we prove several lemmas that will serve as the base case $k = 2$ for our various proofs. Throughout Section 2.3, we will use convention (2.1) with $k = 2$.

Lemma 2.3.1. *Every triangle-free graph G with $h \geq \ell + 4$ contains 2 disjoint cycles.*

Proof. Let G be a minimal counterexample. As G is triangle-free, and $h \geq 4$, $|G| \geq 8$. By Property 2.2.3, $\delta(G) \geq 2$, and by Corollary 2.2.2, $\delta(G) = 2$. Say $d(x) = 2$ and $N(x) = \{y, z\}$. By Property 2.2.3, $d(y), d(z) \in \{3, 4\}$. Set $G' = G/xy$. Since G is triangle-free, $d_{G'}(v) = d_G(v)$ for all $v \in V(G) \setminus \{x, y\}$. As $x \in L$, this implies $|H_2(G')| \geq |L_2(G')| + 4$. Since G is minimal, G' has a triangle, say $v_{xy}zw$. Then $C := yxzw$ is a 4-cycle in G. Let $W = V(G) \setminus C$.

As $x \in L$, $|C \cap H| - |C \cap L| \leq 2$. So, since $h - \ell \geq 4$, $|H \cap W| - |L \cap W| \geq 2$. Thus

$$\sum_{u \in W} d(u) \geq 3|W| + |H \cap W| - |L \cap W| \geq 3|W| + 2. \tag{2.2}$$

Each $v \in W$ has no two adjacent neighbors as G is triangle free, and is not adjacent to x as $N(x) \subset C$. Thus if $\|v, C\| \geq 2$ then $N(v) \cap C = \{y, z\}$. As $d(y) \leq 4$, there are at most two such v. So $\|W, C\| \leq |W| + 2$. Hence by (2.2),

$$2\|G[W]\| = \sum_{u \in W} d(u) - \|W, C\| \geq (3|W| + 2) - (|W| + 2) = 2|W|.$$

Therefore $\|G[W]\| \geq |W|$, and so $G[W]$ contains a cycle (disjoint from C). □

The *2-core* of a graph G is the union of all $G' \subseteq G$ with $\delta(G') \geq 2$. It can be obtained from G by iterative deletion of vertices of degree at most 1.

Lemma 2.3.2. *Suppose the 2-core of G contains at least 6 vertices, and it is not isomorphic to SK_5. If $h \geq \ell + 4$, then G contains 2 disjoint cycles.*

Proof. Let G be a minimal counterexample. If there exists a vertex of degree at most 1, then removing it yields a smaller counterexample. So G is its own 2-core and $\delta(G) \geq 2$. Also $|G| \geq 6$ and by Corollary 2.2.2, $L \neq \emptyset$. Thus $h \geq 5$, and $|G| \geq 7$, since G is not isomorphic to SK_5. Pick $x \in L$. Let $N(x) = \{y, z\}$.

Suppose $yz \notin E(G)$. Set $G' = G/xy$. Then $|G'| = |G| - 1 \geq 6$. Since $d(x) = 2$, all $v \in V(G')$ satisfy $d_{G'}(v) = d_G(v)$. So G' is its own 2-core, $|H_2(G')| - |L_2(G')| = h - \ell + 1 \geq 5$, and G' is not isomorphic to SK_5. As $|G'| < |G|$, by the minimality of G, G' has two disjoint cycles. But then so does G.

Otherwise $yz \in E(G)$. Now xyz is a triangle in G, so $G' := G - \{x, y, z\}$ is acyclic, and $\|G'\| < |G'|$. Since $h \geq \ell + 4$ and $x \in L$, we have $|H \cap V(G')| - |L \cap V(G')| \geq 3$. So $\sum_{v \in V(G')} d_G(v) \geq 3|G'| + 3$. As

$N(x) = \{y, z\}$,

$$\|V(G'), \{y, z\}\| = \|V(G'), V(G) \setminus V(G')\| \geq 3|G'| + 3 - 2(|G'| - 1) \geq |G'| + 5.$$

Thus $d(y), d(z) \geq 6$. Let $G^* = G - x$. Then $|G^*| \geq 6$, $d_{G^*}(y), d_{G^*}(z) \geq 5$, and $d_{G^*}(v) = d_G(v)$ for all $v \in V(G^*) \setminus \{y, z\}$. So $|H_2(G^*)| - |L_2(G^*)| \geq 5$ and G^* coincides with its 2-core. As $|G^*| < |G|$, by the minimality of G, G^* has two disjoint cycles. But then so does G. □

Lemma 2.3.3. *Every graph G containing a triangle $X = x_1x_2x_3$ has two disjoint cycles provided (a) $|H \setminus X| - |L \setminus X| \geq 2$ and (b) $\|v, X\| \leq 2$ for all $v \in V(G) \setminus X$.*

Proof. Let G be a minimal counterexample to the lemma. Then $G - X$ is acyclic. Let $Y = V(G) \setminus X$. By (a), there is $u \in H \setminus X$, and by (b), $\|u, Y\| \geq 2$. This yields $|G| \geq 6$. First, we show:

$$\text{If } v \in Y \text{ and } \|v, Y\| \leq 1, \text{ then } \|v, Y\| = 1 \text{ and } \|v, X\| = 2. \tag{2.3}$$

Indeed, by (b), $\|v, X\| \leq 2$. So if $\|v, Y\| = 0$ or $\|v, X\| \leq 1$ and $\|v, Y\| \leq 1$, then $v \in L \setminus X$. Thus $G - v \subset G$ satisfies (a) and (b). Then by the minimality of G, $G - v$ has two disjoint cycles, and hence so does G.

By (a), there are $z, z' \in H \setminus X$. If they are in the same component of $G - X$, then let Q be the interior of the unique z, z'-path in $G - X$ and put $G' = G - X - Q - zz'$; otherwise put $G' = G - X$. Pick maximum paths $P = y_1 \dots z \dots y_2$ and $P' = y'_1 \dots z' \dots y'_2$ in G'. Perhaps $z = y_1$ or $z' = y'_1$, but $z, z' \in H$ implies $|P|, |P'| \geq 2$. For $i \in \{1, 2\}$, if $y_i \neq z$ and $N(y_i) \cap Q \neq \emptyset$, then $G[P \cup Q]$ contains a cycle, a contradiction. Then,

$$\text{either } d_{G-X}(y_i) = d_{G'}(y_i) \text{ or } y_i = z. \tag{2.4}$$

So, if $y_i \neq z$, then by (2.3), $\|y_i, X\| = 2$. Otherwise $y_i = y_1 = z$, and $\|z, X\| \geq d_G(z) - d_{G-X}(z) \geq d_G(z) - d_{G'}(z) - 1 \geq 2$. So in any case, $\|y_i, X\| = 2$ and a similar argument shows $\|y'_i, X\| = 2$. Now y_1 and y_2 have a common neighbor, say x_1 in X, and $G[P + x_1]$ contains a cycle C_1. If y'_1 and y'_2 have a common neighbor $x_i \in X - x_1$, then $G[P' + x_i]$ contains a cycle disjoint from C_1. Otherwise, one of y'_1 and y'_2 is adjacent to x_2 and the other to x_3. Then $G[P' \cup \{x_2, x_3\}]$ contains a cycle disjoint from C_1. □

2.4 Proof of Theorem 1.2.6

Recall that we use convention (2.1). Let k be the smallest integer such that there exists a graph G without k disjoint cycles satisfying $|H| - |L| \geq 2k + t$ and $|G| \geq 3k$. By Lemmas 2.3.1 and 2.3.2, $k \geq 3$. Choose such

G to be minimal.

Lemma 2.4.1. *$|G| \geq 3k+1$.*

Proof. Suppose that $|G| = 3k$. Create the graph $G' \supseteq G$ by adding edges to G until, for each $x \in L$, $N_{G'}(x) = V(G') - x$. Then $\delta(G') \geq 2k-1$, so by Corollary 2.2.2, G' contains k disjoint cycles. As $|G'| = 3k$, these cycles are triangles, and at most ℓ of them contain edges from $E(G') \setminus E(G)$. Thus $t \geq k - \ell$ and so $h \geq \ell + 2k + t \geq 3k = |G|$. Hence $H = V(G)$ and by Theorem 1.2.2, G contains k disjoint cycles. $\square$

Lemma 2.4.2. *Each $x \in L$ is in a triangle in G.*

Proof. Suppose x is not in a triangle. By Property 2.2.3, $d(x) \geq 2$. Let $y \in N(x)$ and set $G' = G/xy$. Then $d_{G'}(v_{xy}) \geq d(y)$ and $d_{G'}(z) = d_G(z)$ for all $z \in V(G') - v_{xy}$. Since any triangle in G' not containing v_{xy} is also a triangle in G, $t' := t(G') \leq t+1$. Thus $H \subseteq H_k(G')$ and $L_k(G') + x \subseteq L$. So,

$$|H_k(G')| - |L_k(G')| \geq h - (\ell - 1) \geq (\ell + 2k + t) - \ell + 1 = 1 + 2k + t \geq 2k + t'.$$

By Lemma 2.4.1, $|G'| \geq 3k$. As G is minimal, G' has k disjoint cycles and so does G. $\square$

By Corollary 2.2.2, $L \neq \emptyset$. Fix an $x \in L$. Let $\mathcal{T}$ be a set of disjoint triangles in G such that (a) $x \in V(\mathcal{T})$, and (b) subject to (a), $|\mathcal{T}|$ is maximum. By Lemma 2.4.2, $|\mathcal{T}| \geq 1$. Let $T_0 = T_0(\mathcal{T})$ be the triangle in $\mathcal{T}$ containing x; say $T_0 = xyz$.

Define an auxiliary digraph $\mathcal{D} = \mathcal{D}(\mathcal{T})$ with $V(\mathcal{D}) = \mathcal{T}$ and $\overrightarrow{TU} \in E(\mathcal{D})$ if and only if $T, U \in \mathcal{T}$ and $\|v, U\| = 3$ for some $v \in T$. If $v \in T$ and $\|v, U\| = 3$, we say the vertex v *witnesses* the edge $\overrightarrow{TU}$. We say a vertex T is *reachable from a vertex S* if there exists a directed ST-path in $\mathcal{D}(\mathcal{T})$. Let $\mathcal{R} = \mathcal{R}(\mathcal{T}) \subseteq \mathcal{T}$ be the set of triangles from which T_0 is reachable in $\mathcal{D}(\mathcal{T})$. Let $r = |\mathcal{R}|$. Since $T_0 \in \mathcal{R}$, $r \geq 1$. Finally, define $B = B(\mathcal{T}) = \{v \in V(G) \setminus V(\mathcal{T}) : \|v, T_0\| = 3\}$. By the definitions of $\mathcal{R}$ and B, if $\|v, T_0\| = 3$ for a vertex v, then $v \in V(\mathcal{R}) \cup B$.

Lemma 2.4.3. *If $|B| \leq 1$, then $\|v, T\| = 3$ for some vertex $v \notin V(\mathcal{R}) \cup B$ and triangle $T \in \mathcal{R}$.*

Proof. Suppose $|B| \leq 1$ and $\|v, T\| \leq 2$ for every $v \notin V(\mathcal{R}) \cup B$ and $T \in \mathcal{R}$.

Case 1: $r \leq k-2$. Let $G' = G - V(\mathcal{R})$ and observe $t(G') \leq t - r$. We will find $k' := k - r$ disjoint cycles in G'. For each $v \notin V(\mathcal{R}) \cup B$, $\|v, V(\mathcal{R})\| \leq 2r$, so $d_{G'}(v) \geq d_G(v) - 2r$. Thus $H \setminus (V(\mathcal{R}) \cup B) \subseteq H_{k'}(G')$ and $L_{k'}(G') \subseteq (L \setminus V(\mathcal{R})) \cup B$. As $x \in L \cap V(\mathcal{R})$ and $|B| \leq 1$,

$$|H_{k'}(G')| \geq h - (3r-1) - |B| \geq h - 3r \quad \text{and} \quad |L_{k'}(G')| \leq (\ell - 1) + |B| \leq \ell.$$

Combining these inequalities yields

$$|H_{k'}(G')| - |L_{k'}(G')| \geq (h - 3r) - \ell \geq 2(k - r) + (t - r) \geq 2k' + t(G').$$

As $k' \geq 2$ and $|G'| = |G| - 3r \geq 3k'$, G' contains k' disjoint cycles by the minimality of G, and thus G has k disjoint cycles.

Case 2: $r = k - 1$. Let $\mathcal{R}^- = \mathcal{R} - T_0$ and consider $G' = G - V(\mathcal{R}^-)$. For each $v \notin V(\mathcal{R}) \cup B$, since $\|v, V(\mathcal{R}^-)\| \leq 2(r - 1)$, $d_{G'}(v) \geq d_G(v) - 2k + 4$. This implies that $H \setminus (V(\mathcal{R}) \cup B) \subseteq H_2(G') \setminus T_0$ and, since each vertex in B is adjacent to three vertices in $T_0 \subseteq G'$, $L_2(G') \setminus T_0 \subseteq L \setminus V(\mathcal{R})$. Therefore, since $x \in L \cap V(\mathcal{R})$ and $|B| \leq 1$,

$$|H_2(G') \setminus T_0| \geq h - (3r - 1) - |B| \geq h - 3k + 3 \quad \text{and} \quad |L_2(G') \setminus T_0| \leq \ell - 1.$$

Since $t = k - 1$, these inequalities give

$$|H_2(G') \setminus T_0| - |L_2(G') \setminus T_0| \geq (h - 3k + 3) - \ell + 1 \geq (\ell + 2k + (k - 1)) - 3k + 3 - \ell + 1 = 3.$$

If $\|u, T_0\| = 3$, then u is the unique vertex in B; in this case let e be an edge from u to T_0 and let $G'' = G' - e$. Otherwise, let $G'' = G'$. Since in both cases, $d_{G''}(v) = d_{G'}(v)$ for $v \in V(G') \setminus (T_0 + u)$, $|H_2(G'') \setminus T_0| - |L_2(G'') \setminus T_0| \geq 2$. By Lemma 2.3.3, G'' contains two disjoint cycles, and so G contains k disjoint cycles, a contradiction. □

Lemma 2.4.4. *If $v \notin V(\mathcal{R}) \cup B$ and $\|v, T\| = 3$ for some $T \in \mathcal{R}$, then there are a vertex $v' \in V(\mathcal{T})$ and a set $\mathcal{T}'$ of disjoint triangles such that $xyz \in \mathcal{T}'$, $|\mathcal{T}'| = |\mathcal{T}|$, $B(\mathcal{T}') = B + v'$, and $V(\mathcal{T}') = V(\mathcal{T}) + v - v'$.*

Proof. Let $T = T_j, T_{j-1}, \ldots, T_0$ be a $T \to T_0$ path in $\mathcal{D}(\mathcal{T})$ and, for each $i \in [j]$, let v_i witness the edge $\overrightarrow{T_i T_{i-1}}$. Define the triangle T'_j to be $T_j - v_j + v$ and the triangle T'_i to be $T_i - v_i + v_{i+1}$ for all $i \in [j-1]$. Then, $\mathcal{T}' = (\mathcal{T} \setminus \{T_1, \ldots, T_j\}) \cup \{T'_1, \ldots, T'_j\}$ is a set of $|\mathcal{T}|$ disjoint triangles in G, $v' := v_1 \notin V(\mathcal{T}') \cup B$, and $\|v', T_0\| = 3$. Thus $B + v' = B(\mathcal{T}')$. □

Now choose $\mathcal{T}$ subject to (a) and (b) so that B is maximum.

Lemma 2.4.5. $|B| = 2$. *Moreover, $\|v, T_0 \cup B\| \leq 2$ for all $v \notin V(\mathcal{T}) \cup B$.*

Proof. As B is maximum, Lemmas 2.4.3 and 2.4.4 imply $|B| \geq 2$. Fix a vertex $u_1 \in B$ and let T'_0 be the triangle xyu_1. Observe $\mathcal{T}' = \mathcal{T} - T_0 + T'_0$ is a set of $|\mathcal{T}|$ disjoint triangles in G. Let $\mathcal{R}' = \mathcal{R}(\mathcal{T}')$, $r' = |\mathcal{R}'|$,

$B' = B'(\mathcal{T}')$ and note $z \in B'$. If $|B'| \geq 2$, let $\mathcal{T}'' = \mathcal{T}'$. Otherwise by Lemma 2.4.3, there are $v \notin V(\mathcal{R}') \cup B'$ and $T \in \mathcal{R}'$ with $\|v, T\| = 3$. By Lemma 2.4.4, there are $z' \in V(\mathcal{T}')$ and a set $\mathcal{T}''$ of triangles satisfying $T_0' \in \mathcal{T}''$, $|\mathcal{T}''| = |\mathcal{T}'|$, and $B(\mathcal{T}'') = \{z, z'\}$.

If $|B| \geq 3$ then pick $u_2 \in B \setminus \{u_1, v\}$. As $V(\mathcal{T}'') \setminus V(\mathcal{T}) \subseteq \{u_1, v\}$, $u_2 \notin V(\mathcal{T}'')$. Thus $\mathcal{T}'' - T_0' \cup \{xu_1z', yu_2z\}$ is a set of $|\mathcal{T}| + 1$ disjoint triangles containing x, contradicting (b). So $|B| = 2$.

Lastly, if $v \notin V(\mathcal{T}) \cup B$ and $\|v, T_0 \cup B\| \geq 3$, then v has neighbors $w \in T_0$ and $u \in B$. Thus vuw and $(T_0 - w) \cup (B - u)$ are disjoint triangles in $T_0 \cup (B + v)$, contradicting (b). □

Let $T^* := G[T_0 \cup B]$. Define a second auxiliary digraph $\mathcal{D}^*(\mathcal{T})$ to have vertex set $\mathcal{T} - T_0 + T^*$ and $\overrightarrow{TU} \in E(\mathcal{D}^*(\mathcal{T}))$ if and only if $\|v, U\| \geq 3$ for some $v \in T$. Again, we say the vertex v *witnesses* the edge $\overrightarrow{TU}$. Define the set of graphs $\mathcal{R}^*$ to be T^* together with the set of triangles from which T^* is reachable in $\mathcal{D}^*(\mathcal{T})$. Let $r^* = |\mathcal{R}^*|$.

Lemma 2.4.6. *If $v \in V(G) \setminus V(\mathcal{R}^*)$, then $\|v, T\| \leq 2$ for each $T \in \mathcal{R}^*$.*

Proof. Suppose $v \in V(G) \setminus V(\mathcal{R}^*)$, $T \in \mathcal{R}^*$, and $\|v, T\| \geq 3$. Let $T = T_j, T_{j-1}, \ldots, T_1, T^*$ be a $T \to T^*$ path in $\mathcal{D}^*(\mathcal{T})$. By Lemma 2.4.5, v is adjacent to at most 2 vertices in T^*, so $j \geq 1$.

Let v_1 witness the edge $\overrightarrow{T_1T^*}$ and, for $i \in \{2, \ldots, j\}$, let v_i witness the edge $\overrightarrow{T_iT_{i-1}}$. As in the proof of Lemma 2.4.4, define the triangle T_j' to be $T_j - v_j + v$ and the triangle T_i' to be $T_i - v_i + v_{i+1}$ for all $i \in [j-1]$. If $\|v_1, T_0\| = 3$, then $\mathcal{T}' = \mathcal{T} \setminus \{T_1, \ldots, T_j\} \cup \{T_1', \ldots, T_j'\}$ is a set of $|\mathcal{T}|$ triangles in G, but $B + v_1 = B(\mathcal{T}')$, contradicting the maximality of B. Otherwise, there exist a vertex $w \in N(v_1) \cap T_0$, a vertex $u \in N(v_1) \cap B$, and a triangle $T_0' = (T_0 - w) \cup (B - u)$. Then $\mathcal{T}' = \mathcal{T} \setminus \{T_0, T_1, \ldots, T_j\} \cup \{v_1wu, T_0', \ldots, T_j'\}$ is a set of $|\mathcal{T}| + 1$ disjoint triangles in G, contradicting the maximality of $\mathcal{T}$. □

Proof of Theorem 1.2.6. Let $G' = G - V(\mathcal{R}^*)$. Set $k' = k - r^*$ and $t' = t - r^*$. Then $k' \geq 1$. By Lemma 2.4.5, B has the form $\{w_1, w_2\}$, and by Lemma 2.4.6, every $v \in V(G')$ satisfies

$$d_{G'}(v) \geq d_G(v) - 2r^*. \tag{2.5}$$

Thus $H \setminus V(\mathcal{R}^*) \subseteq H_{k'}(G')$ and $L_{k'}(G') \subseteq (L \setminus V(\mathcal{R}^*))$. As $x \in L \cap T_0$, this implies

$$|H \cap H_{k'}(G')| \geq h - (3r^* - 1) - |B| \geq h - 3r^* - 1 \quad \text{and} \quad |L_{k'}(G')| \leq |L \cap V(G')| \leq \ell - 1.$$

Combining these inequalities yields

$$|H_{k'}(G')| - |L_{k'}(G')| \geq (h - 3r^* - 1) - (\ell - 1) \geq 2(k - r^*) + (t - r^*) = 2k' + t' \tag{2.6}$$

and

$$|H \cap H_{k'}(G')| - |L \cap V(G')| \geq (h - 3r^* - 1) - (\ell - 1) \geq 2(k - r^*) + (t - r^*) = 2k' + t'. \tag{2.7}$$

Case 1: $|G'| = 3k' - 1$. As $H'_k(G') \neq \emptyset$, $\Delta(G') \geq 2k'$ and $2k' + 1 \leq |G'| = 3k' - 1$. So $k' \geq 2$. Let $G^+ = G' \vee K_1$, where $V(K_1) = \{u\}$. Then $|G^+| = 3k'$ and $t(G^+) \leq t' + 1$. So

$$|H_{k'}(G^+)| - |L_{k'}(G^+)| \geq |H_{k'}(G') + u| - |L_{k'}(G^{'})| \geq 2k' + t' + 1 \geq 2k' + t(G^+).$$

As $|G|$ is minimal, G^+ has a set $\mathcal{S}'$ of k' disjoint triangles. Since $|G^+| = 3k'$, we may assume $T' = uu_1u'_1 \in \mathcal{S}'$. Let $T^{''} = xyw_1$ and $\mathcal{S} = (\mathcal{S}' - T') \cup (\mathcal{R} - T_0 + T^{''})$. Thus $t = k - 1$, $h \geq 2k + t + \ell = 3k$ and $\ell = 1$. So $H = V(G) - x$. Let $u_2u'_2 := zw_2$, $U = \{u_1, u'_1, u_2, u'_2\}$, and note that $u_1u'_1, u_2u'_2 \in E(G - V(\mathcal{S}))$.

Since $U \subseteq H$, and $G[U]$ is acyclic, $\|U, V(G) \setminus U\| \geq 8k - 6 > 8(k - 1)$. Thus $\|U, T\| \geq 9$ for some $T = q_1q_2q_3 \in \mathcal{S}$. Say $\|q_1, U\| \leq \|q_2, U\| \leq \|q_3, U\|$. Then $q_2u_iu'_i$ is a triangle for some $i \in [2]$. Now $\|\{q_1, q_3\}, \{u_{3-i}, u'_{3-i}\}\| \geq 2$, so $\{q_1, q_3, u_{3-i}, u'_{3-i}\}$ contains a cycle. Thus G has k disjoint cycles, a contradiction.

Case 2: $|G'| \geq 3k'$. If $k' \geq 2$ then (2.5), (2.6), and the minimality of G imply G' contains k' cycles and so G contains k cycles. So assume $k' = 1$ and G' is acyclic.

By (2.7), $|H \cap H_{k'}(G')| - |L \cap V(G')| \geq 2$. Thus, there is a component G_0 of G' with

$$|H \cap H_{k'}(G_0)| - |L \cap V(G_0)| \geq 1. \tag{2.8}$$

By (2.5), $|G_0| \geq 3$. Let $W_0 = V(G_0)$ and $G'_0 = G[T^* \cup W_0]$. By Lemma 2.4.5 and the fact that G_0 has no isolated vertices,

$$d_{G'_0}(v) \geq \begin{cases} 4, & \text{if } v \in H \cap W_0; \\ 1, & \text{if } v \in L \cap W_0; \\ 3, & \text{if } v \in W_0 \setminus (L \cup H). \end{cases}$$

By this and (2.8),

$$\begin{aligned} \|W_0, T^*\| = \sum_{v \in W_0} d_{G'_0}(v) - 2\|G_0\| &\geq 2.5|W_0| + 1.5(|H \cap W_0| - |L \cap W_0|) - 2(|W_0| - 1) \\ &\geq 0.5|W_0| + 1.5 + 2 \geq 5. \end{aligned}$$

It follows that there are $w \in T_0$ and $u \in B$ such that $\|\{w, u\}, W_0\| \geq 2$. Then $G[W_0 \cup \{w, u\}]$ contains a cycle, and $(T_0 - w) \cup (B - u)$ induces a triangle. This gives k disjoint cycles. $\square$

2.5 Removing the explicit constraint $|G| \geq 3k$

Suppose an integer $k \geq 2$ and a graph G satisfy $h - \ell \geq 2k + t + 1$, and G has no k disjoint cycles. By Lemmas 2.3.1 and 2.3.2, $k \geq 3$. Let $|G| = 3k' + r$, where $k' = \lfloor |G|/3 \rfloor$ and $0 \leq r \leq 2$. By Theorem 1.2.6, $3k - 1 \geq |G| \geq h \geq 2k + 1 \geq 7$, so $k - 1 \geq k' \geq 2$. Pick $R \subset V(G)$ so that $G' := G - R$ has t disjoint triangles. Let $r = |R|$. Then $t(G') = t$, and $d_{G'}(v) \geq d_G(v) - 2$ for each $v \in V(G')$. Thus

$$|H_{k'}(G')| - |L_{k'}(G')| \geq |H \setminus R| - \ell \geq 2k + t + 1 - r \geq 2k' + t(G') + 1.$$

By Theorem 1.2.6, G' has k' disjoint triangles, so $t(G') = k'$ and $|H_{k'}(G')| \geq 3k' + 1 > |G'|$, a contradiction.

2.6 Cycles in planar graphs

In this section, we prove Theorem 1.2.13 by contradiction. Consider the smallest k such that there exists a counterexample G, and choose such G to be minimal. If $k = 2$, then $h \geq 4$, so $G = K_5$ or $|G| \geq 6$. As G is planar, G contains neither K_5 nor SK_5. Thus by Lemma 2.3.2, G has two disjoint cycles. Hence $k \geq 3$.

We first show that $L \neq \emptyset$. Since G is planar, $\|G\| \leq 3|G| - 6$ and the average degree is less than 6. If $k \geq 4$, then $L \neq \emptyset$ follows immediately. If $k = 3$ and $\delta(G) = 5$, then since $h \geq 2k = 6$, $\|G\| \geq \frac{1}{2}(36 + 5(|G| - 6))$. This implies $|G| \geq 18 = 6k$, and by Corollary 2.2.2, $L \neq \emptyset$.

Let $x \in L$. We claim that

$$\text{for every } y \in N(x), \text{ the edge } xy \text{ is contained in a triangle.} \tag{2.9}$$

Indeed, if xy is not in a triangle, then consider the graph $G^* = G/xy$. The degree of every vertex other than x and y remains unchanged and the degree of v_{xy} is at least the degree of y. Therefore, $|H_k(G^*)| \geq |L_k(G^*)| + 2k$ and by the minimality of G, G^* contains k disjoint cycles. Expanding the edge xy yields k-disjoint cycles in G. This proves (2.9).

Fix a plane drawing of G. Every triangle T separates the plane into the exterior region $R_1(T)$ and interior region $R_2(T)$. Among all triangles containing x, choose T' so that $R_2(T')$ contains the fewest vertices. Let $T' = xyz$, $R_1 = R_1(T')$ and $R_2 = R_2(T')$. By (2.9), R_2 contains no neighbors of x.

Suppose G has two vertices v_1 and v_2 adjacent to all three vertices of T'. By the choice of T' and R_2, both v_1 and v_2 are in R_1. The planar drawing induced by $T' \cup \{v_1, v_2\}$ contains no edges in the interior of R_2. Adding a vertex v in R_2 adjacent to all three vertices of T' gives a planar embedding of $K_{3,3}$, a contradiction. So G has at most one vertex v_1 adjacent to all 3 vertices of T'.

Let $G' = G - T'$, $k' = k-1$. Then for each $u \in V(G) - v_1$, $d_{G'}(u) \geq d_G(u) - 2$ and $d_{G'}(v_1) = d_G(v_1) - 3$. It follows that $|H \cap \{v_1\}| + |L_{k'}(G') \cap \{v_1\}| \leq 1$. Hence

$$|H_{k'}(G')| - |L_{k'}(G')| \geq (h - 2 - |H \cap \{v_1\}|) - (\ell - 1 + |L_{k'}(G') \cap \{v_1\}|) \geq 2k - 2 = 2k'.$$

By the minimality of G, G' contains $k-1$ disjoint cycles, and so G contains k disjoint cycles.

2.7 Graphs with at most one triangle

Following Dirac and Erdős [12], let $V^{\geq s}(G)$ (respectively, $V^{\leq s}(G)$) denote the set of vertices of G of degree at least s (respectively, at most s). In these terms, $H = H_k(G) = V^{\geq 2k}(G)$ and $L = L_k(G) = V^{\leq 2k-2}(G)$. The following lemma may be of interest on its own.

Lemma 2.7.1. *Let G be a triangle-free graph with $V(G) \neq \emptyset$. If*

$$|V^{\geq 2k+1}(G)| - |V^{\leq 2k-1}(G)| \geq 2k - 2, \tag{2.10}$$

then G has k disjoint cycles.

Proof. Suppose the lemma does not hold and consider the smallest k such that there exists a counterexample. Among all such counterexamples, choose the graph G to be minimal. First consider $k = 1$. Since $|V^{\geq 3}(G)| \geq |V^{\leq 1}(G)|$, G contains a component with average degree at least 2. Therefore, G contains a cycle and the claim holds. Now, let $k \geq 2$.

By (2.10), the sum of degrees of the vertices in $V^{\geq 2k}(G)$ is greater than the sum of degrees of the vertices in $V^{\leq 2k-1}(G)$. Thus there are vertices $u, v \in V^{\geq 2k}(G)$ such that $uv \in E(G)$. Since G is triangle-free, $N(v) \cap N(u) = \emptyset$ and so $|G| \geq 4k$. Since G has no k disjoint cycles, by Theorem 1.2.2, G has a vertex $x \in V^{\leq 2k-1}(G)$.

As in Property 2.2.3, if $d(x) \leq 1$, then $G - x$ is a smaller counterexample, so $d(x) \geq 2$. Let $y \in N(x)$. Since G is triangle-free, contracting the edge xy does not change the degree of any vertex distinct from x, y. By the minimality of G, G/xy contains either k disjoint cycles or a triangle. If G/xy contains k disjoint cycles, then G does as well. Otherwise, let $v_{xy}zw$ be a triangle in G/xy. Then by symmetry we may assume $xyzw$ is a 4-cycle in G. Every vertex in $G - \{w, x, y, z\}$ is adjacent to at most 2 vertices in $\{w, x, y, z\}$.

Let $k' = k-1$ and $G' = G - \{w, x, y, z\}$. Then, for each $v \in V(G')$, $d_{G'}(v) \geq d_G(v) - 2$, so $|V^{\geq 2k'+1}(G')| \geq$

$|V^{\geq 2k+1}(G)| - 3$ and $|V^{\leq 2k'-1}(G')| \leq |V^{\leq 2k-1}(G)| - 1$. Therefore,

$$|V^{\geq 2k'+1}(G')| - |V^{\leq 2k'-1}(G')| \geq |V^{\geq 2k+1}(G)| - 3 - (|V^{\leq 2k-1}(G)| - 1) \geq 2k - 2 - 2 = 2k' - 2.$$

By the minimality of G, G' contains k' disjoint cycles. Hence G contains k disjoint cycles. □

Suppose that Theorem 1.2.12 is false and let k be the smallest integer such that there exists a counterexample. Among all counterexamples, choose G to be minimal.

Lemma 2.7.2. *$|G| \geq 4k - 1$ and $L \neq \emptyset$.*

Proof. Suppose $|G| \leq 4k - 2$. For all $u \in H$, $|N(u) \cap H| \geq 2$ and if also $w \in H$ then $|N(w) \cap N(u)| \geq 2$. It suffices to show that G has two disjoint triangles. As $h \geq 2k \geq 6$, if $G[H]$ is a complete graph, then we are done, so assume there are $x, y \in H$ with $xy \notin E(G)$.

Choose $w \in N(x) \cap H$, $z \in N(y) \cap H - w$, and $v \in N(w) \cap N(x) - z$. If $N(y) \cap N(z) \neq \{v, w\}$, then there are two triangles in G; else put $Q = \{v, w, y, z\}$ and $P = N(x) \setminus Q$. Now $|P| \geq 2k - 3 \geq k$. If there is $u \in P$ with $d(u) \geq 2k - 1$, then there is $t \in N(x) \cap N(u)$. Thus txu is a triangle, and $Q - t$ contains another triangle. So $V(P) \subseteq L$ and $|L| \geq k$. Therefore, $|G| \geq h + \ell \geq 2\ell + 2k \geq 4k$. □

Lemma 2.7.3. *If $x \in L$, then x is not contained in a triangle.*

Proof. Let $x \in L$ and suppose T_0 is a triangle in G containing x. Let $B = B(T_0) = \{v \in V(G) : \|v, T_0\| = 3\}$ and fix $T_0 = xyz$ so that $|B|$ is minimized. Let $k' = k - 1$, $G' = G - T_0$. For each $v \in V(G') \setminus B$, $d_{G'}(v) \geq d_G(v) - 2$, so $|H_{k'}(G')| \geq |H \setminus (B \cup T_0)|$ and $|L_{k'}(G')| \leq |L \setminus (B \cup T_0)| \leq \ell - 1$.

If $|B| \leq 1$, then $|H_{k'}(G')| - |L_{k'}(G')| \geq (h - 3) - (\ell - 1) \geq 2k - 2 = 2k'$. Since G' is triangle-free, by Theorem 1.2.6, G' contains $k - 1$ disjoint cycles. Then G contains k disjoint cycles. Similarly, if $|B| = 2$ and $B \cup T_0$ contains at most 3 vertices in H, then G' contains k disjoint cycles. So we may assume that $|B| \geq 2$ and $B \cup T_0$ contains at least 4 vertices in H. We complete the proof in 3 cases.

Case 1: *B is an independent set.* Let $u_1, u_2 \in B$ and $T_1 = xyu_1$. If $v \notin B \cup T_0$ and $\|v, T_1\| = 3$, then xu_1v and yzu_2 are two disjoint triangles in G. Let $k' = k - 1$, $G'' = G - T_1$. For each $v \in V(G'') - z$, $d_{G''}(v) \geq d_G(v) - 2$ and $d_{G''}(z) = d_G(z) - 3$. So possibly $z \in H \setminus H_{k'}(G'')$ or $z \in L_{k'}(G'') \setminus L$, but not both, i.e., $|\{z\} \cap H| + |\{z\} \cap L''| \leq 1$. Therefore,

$$\begin{aligned} |H_{k'}(G'')| - |L_{k'}(G'')| &\geq (h - 2 - |\{z\} \cap H|) - (\ell - 1 + |\{z\} \cap L_{k'}(G'')| \\ &\geq (h - \ell) - 1 - (|\{z\} \cap H| + |\{z\} \cap L_{k'}(G'')|) \qquad (2.11) \\ &\geq 2k - 2 = 2k'. \end{aligned}$$

By Theorem 1.2.6, G'' contains $k-1$ disjoint cycles. Then G contains k disjoint cycles.

Case 2: $|B| \geq 3$. Let $u_1, u_2, u_3 \in B$ and, by Case 1 assume $u_1u_2 \in E(G)$. Then xu_1u_2 and yzu_3 are two triangles in G, a contradiction.

Case 3: $|B| = 2$. Let $u_1, u_2 \in B$ and, by Case 1, assume $u_1u_2 \in E(G)$. In particular $B \cup T_0 \cong K_5$ and every vertex in $B \cup T_0$ apart from x is in H. If $v \notin B \cup T_0$ is adjacent to 2 vertices in $B \cup T_0$, then G contains 2 disjoint triangles, a contradiction. Let $k' = k-1$ and $G' = G - (B \cup T_0)$. For each $v \in V(G')$, $d_{G'}(v) \geq d_G(v) - 1$. In particular, $|V^{2k'+1}(G')| \geq h-4$ and $|V^{2k'-1}(G')| \leq \ell - 1$. Therefore,

$$|V^{2k'+1}(G')| - |V^{2k'-1}(G')| \geq (h-4) - (\ell-1) \geq 2k-3 = 2k'-1. \tag{2.12}$$

The graph G' is triangle-free, so by Lemma 2.7.1, G' contains $k-1$ disjoint cycles. Then G contains k disjoint cycles. □

Lemma 2.7.4. *If $x, z \in L$, then $|N_G(x) \cap N_G(y)| \leq 1$.*

Proof. Suppose $w, y \in N_G(x) \cap N_G(z)$. Then $X = wxyz$ is a copy of C_4 in G. If $v \notin X$ is adjacent to at least 3 vertices in X, then either x or z is contained in a triangle, contradicting Lemma 2.7.3. Let $G' = G - X$. For each $v \in V(G')$, $d_{G'}(v) \geq d_G(v) - 2$. Therefore,

$$|H_{k'}(G')| - |L_{k'}(G')| \geq (h-2) - (\ell-2) \geq 2k = 2k'+2. \tag{2.13}$$

Since G' contains at most 1 triangle, by Theorem 1.2.6, G' contains $k-1$ disjoint cycles. Then G contains k disjoint cycles. □

Let $L = \{x_1, \ldots, x_\ell\}$ and, for each i, let $y_i \in N_G(x_i)$. Starting with the graph $G = G_0$, we construct a sequence of graphs by defining $G_i = G_{i-1} / x_iy_i$. For simplicity, if we contract the edge x_iy_i, we label the contracted vertex in G_i as y_i. We terminate this process if G_i contains k cycles or when $i = \min\{\ell, k-1\}$. Suppose, after terminating the process, we have defined graphs $G_0, \ldots, G_r$ for some non-negative integer r.

Lemma 2.7.5. *For the graphs $G_0, \ldots, G_r$ and $i \in \{0, \ldots, r\}$, all of the following hold:*

1. $|G_i| = |G_0| - i \geq 3k$;
2. *if $i < r$, then G_i contains $i+1$ disjoint triangles;*
3. *L_i is an independent set;*
4. *if $x \in L_k(G_i)$, then $N_{G_i}(x)$ is an independent set;*

5. *if $x, x' \in L_k(G_i)$, then $|N_{G_i}(x) \cap N_{G_i}(x')| \leq 1$;*

6. *$L_k(G_i) = L_0 - \{x_1, \dots, x_i\}$ and $H_k(G_i) \supseteq H_k(G_0)$;*

7. *if $i \geq 1$ and G_i contains k disjoint cycles, then G_{i-1} does as well.*

Proof of Theorem 1.2.12. For all i, *(1)* holds by Lemma 2.7.2 and *(7)* holds since a contraction cannot increase the number of disjoint cycles.

The proof of *(2)*–*(6)* will be by induction on i. By assumption, G_0 contains at most 1 triangle. If G is triangle-free, then by Theorem 1.2.6, G_0 contains k disjoint cycles, so *(2)* holds for $i = 0$. Since G_0 is a minimum counterexample, *(3)* holds for $i = 0$ by Property 2.2.3. Further, *(4)* and *(5)* hold for $i = 0$ by Lemma 2.7.3 and Lemma 2.7.4, respectively. And *(6)* is trivial for $i = 0$.

Suppose that $r \geq 1$ and consider $i \in \{1, \dots, r\}$. Assume that *(2)* – *(6)* hold for all $j < i$. Recall that $G_i = G_{i-1} / x_i y_i$. By *(4)* for $i-1$, $d_{G_i}(y_i) \geq d_{G_{i-1}}(y_i)$ and no other vertex v is adjacent to both x_i and y_i, so $d_{G_i}(v) = d_{G_{i-1}}(v)$. Thus, *(6)* holds.

To see that *(3)* holds, observe if $uv \notin E(G_{i-1})$, then $uv \in E(G_i)$ only if $u, v \in N_{G_{i-1}}(x_i)$. Since L_{i-1} is an independent set and $L_k(G_i) \supseteq L_k(G_{i-1})$ by *(6)*, $L_k(G_i)$ is also an independent set.

If $x \in L_k(G_i)$, then by *(6)*, $x \in L_k(G_{i-1})$ also and $x \neq x_i$. Let $y, y' \in N_{G_i}(x)$ and note that since *(4)* holds for G_{i-1}, $yy' \notin E(G_{i-1})$. Edges are only added to G_i between pairs of vertices in $N_{G_{i-1}}(x_i)$. Since *(5)* holds for $i-1$, $|N_{G_{i-1}}(x_i) \cap N_{G_{i-1}}(x)| \leq 1$, so y and y' cannot both be in $N_{G_{i-1}}(x_i) \cap N_{G_{i-1}}(x)$. Thus, $yy' \notin E(G)$ and *(4)* holds for i.

If $x, x' \in L_k(G_i)$, then by *(6)*, $x, x' \in L_k(G_{i-1})$ and $|N_{G_{i-1}}(x) \cap N_{G_{i-1}}(x')| \leq 1$. Since $L_k(G_{i-1})$ is an independent set, $N_{G_i}(x) = N_{G_{i-1}}(x)$ and $N_{G_i}(x') = N_{G_{i-1}}(x')$, so $|N_{G_i}(x) \cap N_{G_i}(x')| \leq 1$ and *(5)* holds.

Finally, by *(2)*, G_{i-1} contains exactly i disjoint triangles. Contracting an edge introduces increases the number of disjoint triangles by at most 1, so G_i contains at most $i+1$ disjoint triangles. By *(6)*,

$$|H_k(G_i)| - |L_k(G_i)| \geq h - (\ell - i) \geq 2k + i. \tag{2.14}$$

Since $|G_i| \geq 3k$, if G_i contains i disjoint triangles, by Theorem 1.2.6, G_i contains k disjoint cycles and $i = r$. Therefore, if $i < r$ then G contains exactly $i+1$ disjoint triangles and *(2)* holds. □

We are now ready to complete the proof. If $r < \min\{\ell, k-1\}$, then we stopped the process because G_r contains k disjoint cycles. If $r = k-1 = \min\{\ell, k-1\}$, then G_{k-2} contains $k-1$ disjoint triangles and G_{k-1} contains at least this many disjoint triangles. If G_{k-1} contains only $k-1$ disjoint triangles, then by

Lemma 2.7.5 *(6)*,

$$|H_k(G_{k-1})| - |L_k(G_{k-1})| \geq h - (\ell - (k-1)) \geq 2k + (k-1) = 3k - 1. \tag{2.15}$$

Lemma 2.7.2 implies that G_{k-1} contains $3k$ vertices and by Theorem 1.2.6, $G_r = G_{k-1}$ contains k disjoint cycles. Finally if $r = \ell = \min\{\ell, k-1\}$, then $L_r = \emptyset$ and $|H_k(G_r)| \geq 2k$. Corollary 2.2.2 implies $G_r = G_\ell$ contains k disjoint cycles. Therefore, in any case G_r contains k disjoint cycles and by Lemma 2.7.5 *(7)*, G contains k disjoint cycles as well.

2.8 Cycles in large graphs

2.8.1 Preliminaries

The goal of Section 2.8 is to provide a proof of Theorem 1.2.11. We prove the following technical statement that implies Theorem 1.2.11, but is more amenable to induction.

Theorem 2.8.1. *Let k and i be integers with $k \geq 2$ and $i \leq k$. Let $\alpha = 16$ be a constant. If G is a graph with $|G| \geq \alpha k + 3i$ and $h \geq \ell + 3k - i$, then $c(G) \geq k$.*

Theorem 1.2.11 is the special case of Theorem 2.8.1 for $i = k$. The heart of Section 2.8 will be a proof of Theorem 2.8.1. In the remainder of this section we organize the induction and establish some preliminary results.

The proof of Theorem 2.8.1 is by induction on i. The base case $i \leq 0$ follows from Corollary 1.2.8. Suppose $i \geq 1$. The equations $|G| \geq h + \ell$ and $h - \ell \geq 2k$ give

$$\ell \leq \frac{|G|}{2} - k. \tag{2.16}$$

Now, we prove a result regarding minimal counterexamples to Theorem 2.8.1. Call a triangle T *good* if $T \cap L_k(G) \neq \emptyset$.

Lemma 2.8.2. *Suppose $k \geq 2$ and $i \in \{0, \ldots, k\}$. Let $\alpha = 16$. If a graph G satisfies all of:*

(a) $|G| \geq \alpha k + 3i$,

(b) $h \geq \ell + 3k - i$,

(c) $c(G) < k$, *and*

(d) subject to (a–c), $\sigma := (k, i, |G| + \|G\|)$ is lexicographically minimum,

then all of the following hold:

(i) G has no isolated vertices;

(ii) $k \geq 3$;

(iii) $L(G) \cup V^{\geq 2k+1}(G)$ is independent;

(iv) if $x \in L(G)$, $d(x) \geq 2$, and $xy \in E$, then xy is in a triangle; and

(v) if $\mathcal{T}$ is a set of disjoint good triangles in G with $X := V(\mathcal{T})$, then $\|v, X\| \geq 2|\mathcal{T}| + 1$ for at least two vertices $v \in V \smallsetminus X$.

Proof. Assume (a–d) hold. Using Corollary 1.2.8, (a–c) imply $i \geq 1$. If (i) fails, then let v be an isolated vertex in G. Now $G' := G - v$ and $i' := i - 1$ satisfy conditions (a–c), contradicting (d). Hence, (i) holds.

For (ii), suppose $k = 2$. Then $t(G) \leq c(G) \leq 1$. If $i = 1$ then $h - \ell \geq 3k - i \geq 2k + t(G)$, so $c(G) \geq 2$ by Theorem 1.2.6. Thus $i = 2$ and $h - \ell = 4$. Using (2.16) and (i),

$$\begin{aligned}\|G\| &\geq \frac{1}{2}(\ell + 3(|G| - \ell) + h) = \frac{1}{2}(3|G| + h - 2\ell) \\ &= \frac{1}{2}(3|G| - \ell + 4) \geq \frac{1}{2}\left(3|G| - \left(\frac{|G|}{2} - 2\right) + 4\right) \\ &= |G| + \frac{|G|}{4} + 3 \geq |G| + \frac{\alpha}{2} + \frac{3i}{4} + 3 = |G| + \frac{\alpha}{2} + \frac{9}{2}.\end{aligned}$$

If G' is the 2-core of G, then $\|G'\| - |G'| = \|G\| - |G|$. Since $\alpha > 1$, $\|G'\| > |G'| + 5$, and so $|G'| > 5$ and $G' \ncong SK_5$. By Lemma 2.3.2, $c(G) \geq 2$, contradicting (c).

For (iii), suppose $e \in E(G[L \cup V^{\geq 2k+1}(G)])$, and set $G' := G - e$. Since G' is a spanning subgraph of G, it satisfies (a) and (c). Moreover, $h = h_k(G')$ and $\ell = \ell_k(G')$, so (d) fails.

If (iv) fails, then let $G' = G / xy$ and $i' = i - 1$. Since $d_{G'}(v_{xy}) \geq d(y)$ and the degrees of all other vertices in G' are unchanged, G' and i' satisfy (a–c), contradicting (d).

Finally, suppose (v) fails, and let $u \in V \smallsetminus X$ with $\|u, X\|$ maximum. Then $\|v, X\| \leq 2|\mathcal{T}|$ for all $v \in V \smallsetminus (X + u)$. Set $G' = G - X$, $k' = k - |\mathcal{T}|$, and $i' = i - |\mathcal{T}| \leq k'$. Then $H \cap V(G') - u \subseteq H_{k'}(G')$ and $L_{k'}(G') - u \subseteq L \cap V(G')$. Since $\alpha \geq 3$ and $|G'| \geq \alpha k' + 3i$, G' satisfies (a). Let $\beta_1 = 1$ if $u \in H \smallsetminus H_{k'}(G')$; else $\beta_1 = 0$. Let $\beta_2 = 1$ if $u \in L_{k'}(G) \smallsetminus L$; else $\beta_2 = 0$. Then $\beta_1 + \beta_2 \leq |\mathcal{T}|$. Since

$$h_{k'}(G') \geq h - 2|\mathcal{T}| - \beta_1 \geq \ell + 3k - i - 2|\mathcal{T}| - \beta_1$$

$$\geq (\ell_{k'}(G') + |\mathcal{T}|) + 3k' - i + |\mathcal{T}| - \beta_1 - \beta_2 \geq \ell_{k'}(G') + 3k' - i',$$

G' satisfies (b). As $c(G') + |\mathcal{T}| \leq c(G) < k$, $c(G') < k'$. Thus G' satisfies (c). If $k' \geq 2$, then this contradicts the choice of k in (d), so (v) holds.

Otherwise, $|\mathcal{T}| = k - 1$ and so $|X| = 3k - 3$. Each triangle in $\mathcal{T}$ has a low vertex, and so by (iii), it has no vertex with degree greater than $2k$. Thus

$$\|X, V(G')\| < 2k|X| < 6k^2.$$

Since $|H \cap V(G')| = h - |H \cap X| \geq 3k - i + \ell - |H \cap X| \geq 2k - i$,

$$2\|G'\| \geq k(|G'| + |H \cap V(G')|) - \|X, V(G')\| \geq k(|G'| - 4k - i). \tag{2.17}$$

By (c), $c(G) \leq k - 1$, so G' has no cycle. Thus by (2.17),

$$2|G'| > 2\|G'\| \geq k(|G'| - 4k - i).$$

By (a), $|G'| \geq |G| - 3k \geq (\alpha - 3)k + 3i = 13k + 3i$. Solving yields

$$k(4k + i) > (k - 2)|G'| \geq (k - 2)(13k + 3i)$$

$$26k > 9k^2 + i(2k - 6).$$

As $i \geq 0$, and $k \geq 3$ by (ii), this is a contradiction. □

2.8.2 Proof of Theorem 2.8.1

Fix k, i, and $G = (V, E)$ satisfying the hypotheses of Lemma 2.8.2. First choose a set $\mathcal{S}$ of disjoint good triangles with $s := |\mathcal{S}|$ maximum, and put $S = V(\mathcal{S})$. Next choose a set $\mathcal{S}'$ of disjoint triangles, each contained in $V^{\leq 2k}(G) \smallsetminus S$, with $s' := |\mathcal{S}'|$ maximum, and put $S' = V(\mathcal{S}')$. Say $\mathcal{S} = \{T_1, \dots, T_s\}$ and $\mathcal{S}' = \{T_{s+1}, \dots, T_{s+s'}\}$.

Let $\mathcal{H}$ be the directed graph defined on vertex set $\mathcal{S}$ by $CD \in E(\mathcal{H})$ if and only if there is $v \in C$ with $\|v, D\| = 3$. A vertex C' is *reachable* from a vertex C if $\mathcal{H}$ contains a directed CC'-path.

Fact 2.8.3. *If $x \in L \setminus S$ and $d(x) \geq 2$ then $N(x) \subseteq S$.*

Proof. Suppose $y \in N(x) \setminus S$. As x is low, $x \notin S'$. By Lemma 2.8.2(iv), xy is in a triangle xyz. As $\mathcal{S}$ is maximal, $z \in S$, so $z \in C$ for some $C \in \mathcal{S}$. Let

$$\mathcal{S}_0 = \{C' \in \mathcal{S} : C \text{ is reachable from } C' \text{ in } \mathcal{H}\}.$$

By Lemma 2.8.2(v), there is $w \in (V \setminus V(\mathcal{S}_0)) - y$ with $\|w, V(\mathcal{S}_0)\| \geq 2|\mathcal{S}_0| + 1$. Then $\|w, D\| = 3$ for some $D \in \mathcal{S}_0$. By Lemma 2.8.2(iii), $w \neq x$. Further, $w \notin S$ as otherwise the triangle in $\mathcal{S}$ containing w is in $\mathcal{S}_0$, contradicting that $w \notin V(\mathcal{S}_0)$.

Let $D = C_1, \ldots, C_j = C$ be a D, C-path in $\mathcal{H}$, and for $i \in [j-1]$ let $x_i \in C_i$ with $\|x_i, C_{i+1}\| = 3$. If $C'_1 = C_1 - x_1 + w$, $C'_j = C_j - z + x_j$ and $C'_i = C_i - x_i + x_{i-1}$ for $i \in \{2, \ldots, j-1\}$, then $\left(\mathcal{S} \setminus \bigcup_{i=1}^{j} C_i\right) \cup \bigcup_{i=1}^{j} C'_i \cup \{xyzx\}$ is a set of $s+1$ disjoint good triangles. This contradicts the maximality of $\mathcal{S}$. □

Fact 2.8.4. *Each $v \in V$ is adjacent to at most 2 leaves. Moreover, if v is adjacent to 2 leaves, then $v \in V^{2k}$.*

Proof. Let v be adjacent to a leaf. By Lemma 2.8.2(iii), $v \in V^{2k-1} \cup V^{2k}$. Let X be the set of leaves adjacent to v, and put $G' = G - X$. Let $i' = i - (|X| - 1 - |\{v\} \cap V^{2k}|)$. Now (a) holds for G', k and i' since $|G'| \geq \alpha k + 3i - |X| \geq \alpha k + 3i'$. Observe

$$\begin{aligned} h_k(G') - \ell_k(G') &\geq (h - |\{v\} \cap V^{2k}|) - (\ell + 1 - |X|) \\ &= h - \ell - |\{v\} \cap V^{2k}| + |X| - 1 \\ &\geq 3k - i - |\{v\} \cap V^{2k}| + |X| - 1. \end{aligned}$$

If $|X| \geq 3$, then $h_k(G') - \ell_k(G') \geq 3k - i'$, and (b) holds for G', k and i'. So, as $G' \subset G$, (d) does not hold for G, k, and i, a contradiction. Similarly, if $v \in V^{2k-1}$ and $|X| = 2$, then G' also contradicts the choice of G, k and i. □

Let $G_1 = G - V^1$. Let $H^1 = V^{\geq 2k}(G_1)$, $R^1 = V^{2k-1}(G_1)$, $L^1 = L_k(G_1) \cap L$, and $M = L_k(G_1) \setminus L^1$. Then $G_1 = G[H^1 \cup R^1 \cup M \cup L^1]$ and $V^{\geq 2k-1}(G) = H^1 \cup R^1 \cup M$. Since deleting a leaf does not decrease the difference $h - \ell$,

$$h_k(G_1) - \ell_k(G_1) \geq 3k - i. \tag{2.18}$$

Fact 2.8.5. *If $x \in M$, then x is in a triangle xyz in G with $d(x), d(y), d(z) \leq 2k$.*

Proof. Suppose $x \in M$. By Fact 2.8.4, either (i) $x \in V^{2k-1}$ and is adjacent to one leaf or (ii) $x \in V^{2k}$ and is adjacent to two leaves. Thus $d(x) \leq 2k$. We first claim:

$$x \text{ has a neighbor } y \text{ such that } 2 \leq d(y) \leq 2k. \tag{2.19}$$

Suppose not. Let X be the set consisting of x and the leaves adjacent to x. For each vertex $v \notin X$, $d_{G-X}(v) \geq d(v) - 1$, with equality if $v \in N(x)$. Moreover, if $v \in N(x)$, then $d_{G-X}(v) \geq 2k$. Therefore, $h_k(G - X) = h - |\{x\} \cap V^{2k}|$ and $\ell_k(G - X) = \ell - (|X| - 1)$. So

$$h_k(G - X) - \ell_k(G - X) = h - \ell + 1 \geq 3k - (i - 1)$$

and $|G - X| \geq |G| - 3 \geq \alpha k + 3(i - 1)$, contradicting the minimality of i. So (2.19) holds.

Now, suppose xy is not in a triangle. Let G' be formed from G by removing the leaves adjacent to x and contracting xy. By Fact 2.8.4, $|G'| \geq |G| - 3$. Since $d(x) \geq 2k - 1$ and x does not share neighbors with y, $d_{G'}(v_{xy}) \geq d(y)$. Similarly, $d_{G'}(v) = d(v)$ for all $v \in V(G') - v_{xy}$. Now, $h_k(G') - \ell_k(G') = h - \ell + 1 \geq 3k - (i - 1)$, contradicting the choice of i.

Let xyz be a triangle containing xy. If $d(z) \leq 2k$, we are done. Otherwise, let G'' be the graph obtained from G by removing the leaves adjacent to x and deleting the vertices x, y, and z. Observe $|G''| \geq |G| - 5 \geq \alpha(k - 1) + 3(i - 1)$. If there exists a vertex $u \in H \setminus H_{k-1}(G'')$, then $N(u) \supseteq \{x, y, z\}$, and $d(u) \leq 2k$, since $d(z) \geq 2k + 1$. In this case xyu is the desired triangle. Similarly, if $v \in L_{k-1}(G'') \setminus L$, then xyv is the desired triangle. Thus $h - h_{k-1}(G'') \leq 2 + |\{x\} \cap V^{2k}|$ and $\ell - \ell_{k-1}(G'') \geq 1 + |\{x\} \cap V^{2k}|$. Now,

$$h_{k-1}(G'') - \ell_{k-1}(G'') \leq h - \ell - 1 \geq 3k - i - 1 = 3(k - 1) - (i - 2).$$

By the minimality of G, $c(G'') \geq k - 1$. Hence $c(G) \geq k$, a contradiction. We conclude that $xyzx$ is a triangle with $d(x), d(y), d(z) \leq 2k$. □

Fact 2.8.6. $s + s' \geq 1$.

Proof. Suppose $s + s' = 0$. In this case, $M = \emptyset$. Indeed, if $v \in M$, Fact 2.8.5 implies the existence of a triangle vuw with $d(v), d(u), d(w) \leq 2k$, contradicting the choice of $\mathcal{S}'$. By Fact 2.8.3 and since $\mathcal{S} = \emptyset$, all vertices in L have degree at most 1. By Lemma 2.8.2(i), all vertices in L are leaves in G and $L^1 = \emptyset$.

Now, for every $x \in H - H_k(G_1)$, there is a leaf $y \in L - L_k(G_1)$ such that $xy \in E(G)$. Hence,

$$h_k(G_1) \geq h_k(G_1) - \ell_k(G_1) \geq h - \ell \geq 2k.$$

By (2.16) and since $\alpha \geq 4$, $|G_1| \geq |G| - \ell \geq |G|/2 + k \geq \alpha k/2 + k \geq 3k$. Finally, $L_k(G_1) = L^1 \cup M = \emptyset$, so Corollary 2.2.2 implies G_1 (and also G) contains k disjoint cycles. $\square$

Let $G_2 = G - (L \setminus S)$. By (2.16),

$$|G_2| \geq \frac{\alpha + 2}{2}k + \frac{3i}{2}. \tag{2.20}$$

Proof of Theorem 2.8.1. Define $s^* = \max\{1, s\}$. Let $\mathcal{S}^* = \{T_1, \ldots, T_{s^*}\}$; by Fact 2.8.6, T_{s^*} exists. Put $S^* = V(\mathcal{S}^*)$. Let $W = V(G_2) \setminus S^*$, $F = G[W]$ and $k' = k - s^*$. It suffices to prove $c(F) \geq k'$.

Case 1: $s^* = k - 1$. Since $k \geq 3$, $s^* \geq 2$. Thus, $s = s^* = k - 1$. By Fact 2.8.4, all vertices in M have degree $2k - 2$. Let $M' = M \cap W$ and $H' = H(G_2) \cap W$. Fact 2.8.3 implies that if $v \in W$, then $d_{G_1}(v) = d_{G_2}(v)$. Thus

$$H' = H^1 \cap W \text{ and } L(G_1) \cap W = L(G_2) \cap W.$$

Hence, by (2.18),

$$\begin{aligned} 2k \leq h(G_1) - \ell(G_1) &\leq (|H(G_1) \cap S| + |H'|) - (|L(G_1) \cap S| + |M \cap W| + |L^1 \setminus S|) \\ &= (|H(G_1) \cap S| - |L(G_1) \cap S|) + |H'| - |M'| - |L^1 \setminus S| \\ &\leq (k-1) + |H'| - |M'|. \end{aligned} \tag{2.21}$$

Here, the last inequality holds because S contains $s = k - 1$ low vertices and at most $2s = 2k - 2$ high vertices. Equation (2.21) implies $|H'| - |M'| \geq k + 1$. Further, if W does not contain a cycle, then

$$\begin{aligned} \|W, S\|_{G_2} &\geq \sum_{v \in W} d_{G_2}(v) - 2(|W| - 1) \\ &\geq ((2k-1)|W| + |H'| - |M'|) - 2(|W| - 1) \\ &\geq ((2k-1)|W| + k + 1) - 2(|W| - 1) \\ &\geq (2k-3)|W| + k + 3. \end{aligned} \tag{2.22}$$

On the other hand,

$$\|W, S\|_{G_2} \leq \sum_{w \in S} (d_{G_2}(w) - 2) \leq (k-1)(6k-8). \tag{2.23}$$

Therefore, combining (2.22) and (2.23), $|W| \leq 3(k-1) - \frac{4}{2k-3}$. Since $|S| = 3(k-1)$ and $|G_2| = |S| + |W|$, this contradicts (2.20) when $\alpha \geq 10$.

Case 2: $s^* \le k-2$. Consider a vertex v in $V^{\le 2k'-2}(F)$. Since every vertex in F has degree at least $2k-2$ in G_2, v must be adjacent to at least $2s^*$ vertices in S^*. Further, every vertex in S^* is adjacent to at most $2k-2$ vertices outside of S^*. Therefore,

$$2s^*|V^{\le 2k'-2}(F)| \le \|V^{\le 2k'-2}(F), S^*\| \le 3s^*(2k-2), \tag{2.24}$$

and so

$$|V^{\le 2k'-2}(F)| \le 3k-3. \tag{2.25}$$

Similarly, if $u \in V^{2k'-1}(F)$, then u is adjacent to at least $2s^*-1$ vertices in S^*. Moreover, there are at most $3s^*(2k-2) - \|V^{\le 2k-2}(F), S^*\|$ edges from $V^{2k-1}(F)$ to S^*. So,

$$(2s^*-1)|V^{2k'-1}(F)| \le \|V^{2k'-1}(F), S^*\| \le 3s^*(2k-2) - \|V^{\le 2k'-2}(F), S^*\|,$$

and, combining with (2.24) gives,

$$\begin{aligned} |V^{2k'-1}| &\le \frac{2s^*(3k-3)}{2s^*-1} - \frac{2s^*|V^{\le 2k'-2}(F)|}{2s^*-1} \\ &= 3k-3+\frac{3k-3}{2s^*-1} - \frac{2s^*|V^{\le 2k'-2}(F)|}{2s^*-1}. \end{aligned} \tag{2.26}$$

Using (2.25) and (2.26), we see that

$$\begin{aligned} h_{k'}(F) - \ell_{k'}(F) &= |W| - 2|V^{\le 2k'-2}(F)| - |V^{2k'-1}(F)| \\ &\ge |W| - 2|V^{\le 2k'-2}(F)| - \left(3k-3+\frac{3k-3}{2s^*-1} - \frac{2s^*|V^{\le 2k'-2}(F)|}{2s^*-1}\right) \\ &= |W| - \frac{(2s^*-2)|V^{\le 2k'-2}(F)|}{2s^*-1} - 3k+3 - \frac{3k-3}{2s^*-1} \\ &\ge |W| - \frac{(2s^*-2)(3k-3)}{2s^*-1} - 3k+3 - \frac{3k-3}{2s^*-1} \\ &= |W| + \left(-(3k-3)+\frac{3k-3}{2s^*-1}\right) - 3k+3 - \frac{3k-3}{2s^*-1} \\ &= |W| - 6k+6 \\ &\ge \left(\frac{\alpha+2}{2}k + \frac{3i}{2} - 3s^*\right) - 6k+6 \\ &\ge \frac{\alpha+2}{2}k + \frac{3i}{2} - 9k+6+3k'. \end{aligned}$$

When $\alpha \geq 16$, this is at least $3k'$ and F contains k' disjoint cycles by Corollary 1.2.8. □

2.9 Concluding remarks

Remark 2.9.1. As mentioned earlier, there are graphs G with $|G| \geq 3k$ and $|H_k(G)| - |L_k(G)| \geq 2k$ that have no k disjoint cycles, but all examples that we know have rather few vertices. The largest such graph G that we can construct has $4k$ vertices and is described in Example 1.2.9.

Is it true that every graph G with $|G| \geq 4k+1$ and $|H_k(G)| - |L_k(G)| \geq 2k$ has k disjoint cycles?

Remark 2.9.2. Lemma 2.7.1 suggests that considering $|V^{\geq 2k+1}(G)| - |V^{\leq 2k-1}(G)|$ instead of $|H_k(G)| - |L_k(G)|$ may result in different bounds providing the existence of k disjoint cycles. It could be that the claim of Lemma 2.7.1 holds not only for triangle-free graphs. That is, it could be that *for any non-empty graph G with $|V^{\geq 2k+1}(G)| - |V^{\leq 2k-1}(G)| \geq 2k-2$, G contains k disjoint cycles.* This is trivially true for $k = 1$.

Chapter 3

Packing

The results in this chapter are joint work with Ervin Győri, Alexandr Kostochka, and Derrek Yager found in [38] and [25].

3.1 Introduction

In this chapter, we will translate the problem of finding a subgraph in a graph into the language of graph packing. Two graphs G_1 and G_2 with $|G_1| = |G_2|$ *pack* if there is a bijection $f : V(G_1) \to V(G_2)$ such that if $uv \in E(G_1)$, then $f(u)f(v) \notin E(G_2)$. We call the function f a *packing* of G_1 and G_2.

In 1978, Bollobás and Eldridge [5] and Sauer and Spencer [46] proved significant results related to graph packings.

Theorem 1.3.2. *[5, 46] Let G_1 and G_2 be two n-vertex graphs. If*

$$|E(G_1)| + |E(G_2)| \leq \frac{3}{2}n - 2, \tag{3.1}$$

then G_1 and G_2 pack.

If G_1 is a star on n vertices and G_2 is a perfect matching, then G_1 and G_2 do not pack, as the center of the star can be mapped to no vertex in G_2. Hence, restriction (3.1) is sharp. However, this is the only sharpness example. If neither graph contains a star on n vertices, Bollobás and Eldridge showed that (3.1) can be relaxed.

Theorem 1.3.3. *[5] Let G_1 and G_2 be two n-vertex graphs. If $\Delta(G_1), \Delta(G_2) \leq n - 2$ and*

$$|E(G_1)| + |E(G_2)| \leq 2n - 3, \tag{3.2}$$

then either G_1 and G_2 pack, or $\{G_1, G_2\}$ is one of the following 7 pairs: $\{2K_2, K_1 \cup K_3\}, \{\overline{K}_2 \cup K_3, K_2 \cup K_3\}, \{3K_2, \overline{K}_2 \cup K_4\}, \{\overline{K}_3 \cup K_3, 2K_3\}, \{2K_2 \cup K_3, \overline{K}_3 \cup K_4\}, \{\overline{K}_4 \cup K_4, K_2 \cup 2K_3\}, \{\overline{K}_5 \cup K_4, 3K_3\}$ (Figure 3.1).

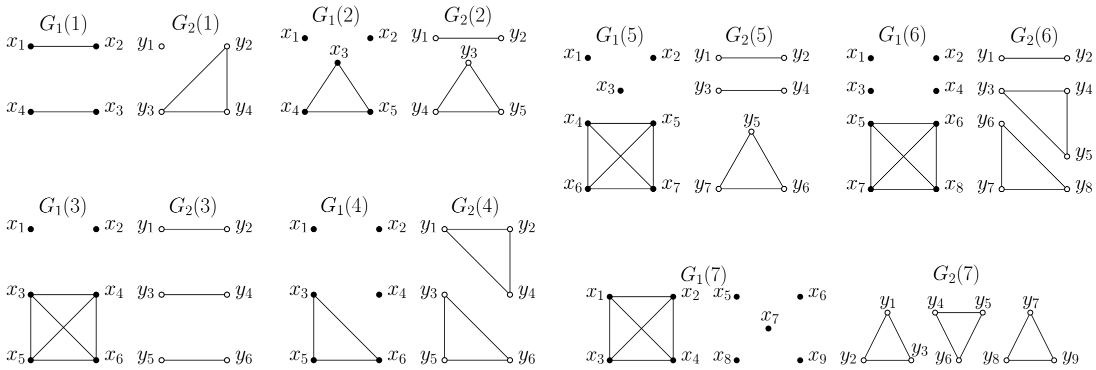

Figure 3.1: Bad pairs in Theorems 1.3.3 and 3.4.2.

Restriction (3.2) is best possible, as the cycle C_n does not pack with $K_{1,n-2} \cup K_1$ and, together, they have $2n-2$ edges. It is clear that C_n and $K_{1,n-2} \cup K_1$ do not pack since $\Delta(K_{1,n-2} \cup K_1) = n-2$. Since C_n does not contain a vertex of degree 1, no vertex in C_n may be mapped to a vertex of degree $n-2$. In fact, it is a similar reason that prevents the packing of the only sharpness example of Theorem 1.3.2. While Theorems 1.3.2 and 1.3.3 suggest that graphs with many edges are difficult to pack, their sharpness examples suggest two graphs may pack if their maximums degrees are not too large. Indeed, the following result of Sauer and Spencer shows that graphs with low maximum degree pack.

Theorem 1.3.6. *[46] Let G_1 and G_2 be two n-vertex graphs. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then G_1 and G_2 pack.*

In this chapter, we consider what restrictions on the number of edges *and* the maximum degree will guarantee the packing of two graphs. In particular, we focus on two recent theorems, one of Alon and Yuster [2] and another of Żak [53], that combine these two parameters.

We first consider the result of Alon and Yuster [2], in which they prove that two graphs G_1 and G_2 pack if the number of edges in G_1 and also the maximum degree of G_2 are both restricted.

Theorem 1.3.7. *[2] For all n sufficiently large, let G_1 and G_2 be n-vertex graphs such that $|E(G_1)| \leq n - \delta(G_2) - 1$ and $\Delta(G_2) \leq \sqrt{n}/200$. Then G_1 and G_2 pack.*

Alon and Yuster phrased their theorem in the language of Turán numbers. Introduced by Turán in 1941, the *Turán number* $\mathrm{ex}(n, G)$ of a graph G is the maximum number of edges in an n-vertex graph that does not contain a subgraph isomorphic to G [49]. Since their introduction, Turán numbers have become a widely studied area within extremal graph theory and have produced a number of theorems. One such theorem, due to Ore, gives the maximum number of edges in an n vertex graph without a hamilton cycle [45].

Theorem 3.1.1 ([45]). *Let $n \geq 2$, $\mathrm{ex}(n, C_n) = \binom{n-1}{2} + 1$.*

Moreover, for $n \geq 5$ the only graph with n vertices and $\binom{n-1}{2}+1$ edges that does not contain a hamilton cycle is K_n minus a star with $n-2$ edges [45]. In this language, Theorem 1.3.7 is the following stronger version of Ore's result.

Theorem 1.3.8. *[2] For all n sufficiently large, if G is a graph of order n with no isolated vertices and $\Delta(G) \leq \sqrt{n}/200$, then $\text{ex}(n, G) = \binom{n-1}{2} + \delta(G) - 1$.*

Unlike Ore's result, Theorem 1.3.7 has several different sharpness examples. Alon and Yuster provide the following two examples in [2], though we rephrase them in the language of graph packing.

Example 1.3.9. *Let G_1 be a star with $n-2$ edges and an additional vertex, that is $G_1 = K_{1,n-2} \cup K_1$. Let G_2 be a graph on n vertices in which all vertices but one have degree 3, the last vertex has degree 2 and the neighbors of this vertex are adjacent (Figure 3.2a).*

Example 1.3.10. *Let G_1 be the disjoint union of a star with $n-3$ vertices and an edge and G_2 as in Example 1.3.9 (Figure 3.2b).*

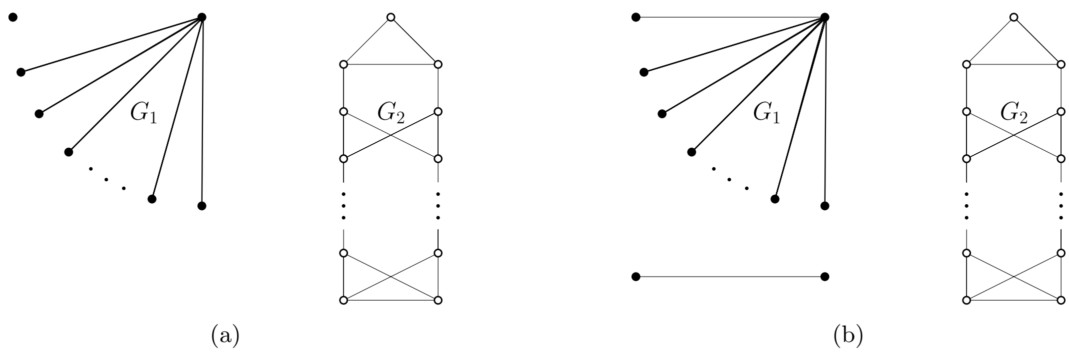

Figure 3.2: Sharpness examples for Theorem 1.3.7 [2]

In Example 1.3.9, G_1 contains exactly $\binom{n-1}{2} + \Delta(G_2) - 1$ edges, but $\Delta(G_1) + \delta(G_2) \geq n$. So G_1 and G_2 cannot pack since there is no suitable vertex in G_2 to which we might map the vertex of maximum degree in G_1. In Example 1.3.10, G_1 still contains exactly $\binom{n-1}{2} + \Delta(G_2) - 1$ edges. In this case, $\Delta(G_1) + \delta(G_2) = n-1$, so a potential packing could (and must) map the vertex of maximum degree in G_1 to the vertex of degree 2 in G_2. However, such an attempt will eventually fail to be a packing because no set of vertices could be mapped to the neighborhood of the degree 2 vertex.

With this observation, we can obtain a larger set of sharpness examples for Theorem 1.3.7. For example, fix constants n and d with n much larger than d. Let G_2 be a d-regular graph on n vertices consisting of a disjoint union of cliques. Let G_1 be the disjoint union of $d-1$ edges, together with a star containing $n - 2(d-1) - 1$ edges (Figure 1.2a, here $d = 6$). Indeed, though we can find a vertex $w \in V(G_2)$ to which we may map the center of the star, there is no set of $d-1$ vertices that may be mapped to $N_{G_2}(w)$. As long as there is no independent set of size d among the vertices in G_1 not in the star, we can create even more sharpness examples, e.g. Figure 1.2b.

In each sharpness example described above, either the vertex of maximum degree cannot be placed in any packing or placing the vertex of maximum degree prevents the placement of some remaining vertex. The first result of this chapter shows that all sharpness examples to Theorem 1.3.7 arise in this way.

Theorem 1.3.12. *For n sufficiently large ($n \geq 10^9$), let G_1 and G_2 be graphs of order n such that $\Delta(G_2) \leq \sqrt{n}/60$, $|E(G_1)| \leq n$, and $\Delta(G_1) + \delta(G_2) \leq n-1$. If there is a vertex $v_1 \in V(G_1)$ such that*

$$d(v_1) = \Delta(G_1) \text{ and } \alpha(G_1 - N[v_1]) \geq \delta(G_2), \tag{3.3}$$

then G_1 and G_2 pack.

Theorem 1.3.12 actually implies Theorem 1.3.7. Indeed, if $|E(G_1)| \leq n-\delta(G_2)-1$, then $\Delta(G_1)+\delta(G_2) \leq n-1$. Also, if $v_1 \in V(G_1)$ with $d(v_1) = \Delta(G_1)$, then $G - N[v_1]$ contains $n - d(v_1) - 1$ vertices and $n - d(v_1) - \delta(G_2) - 1$ edges. Hence, $G - N[v_1]$ contains at least $\delta(G_2)$ components and an independent set of size at least $\delta(G_2)$. We can adapt the methods used in the proof of Theorem 1.3.12 to characterize the sharpness examples for Theorem 1.3.7.

Corollary 1.3.13. *For n sufficiently large ($n \geq 10^9$), let G_1 and G_2 be graphs of order n such that $\Delta(G_2) \leq \sqrt{n}/60$, $|E(G_1)| \leq n - \delta(G_2)$. Then,*

1. *G_1 and G_2 pack, or*
2. *$\Delta(G_1) + \delta(G_2) = n$, or*
3. *G_1 has exactly $n - \delta(G_2)$ edges and exactly one vertex of degree greater than 1. Moreover, for each $w \in V(G_2)$ with $d(w) = \delta(G_2)$, the neighborhood of w induces a clique.*

Unlike Theorems 1.3.2 and 1.3.3, Theorem 1.3.7 guarantees a packing of G_1 and G_2 even though one of the graphs may have not have a bounded number of edges. Similarly, unlike Theorem 1.3.6, it does not require that both graphs have bounded maximum degree. Recently, Żak proved that there is more direct interaction between the edges in G_1 and G_2 and the maximum degrees of G_1 and G_2. He showed that if the number of edges in one graph, say G_1, is reduced by some amount, then an equivalent increase in the maximum degree of G_1 or G_2 will not prevent a packing.

Theorem 1.3.14. *[53] Let G_1 and G_2 be two graphs of order $n \geq 10^{10}$. If*

$$|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} < \frac{5}{2}n - 2,$$

then G_1 and G_2 pack.

As in Theorem 1.3.3, forbidding vertices of degree $n-1$ allows for a significant improvement.

Theorem 1.3.15. *[53] Let G_1 and G_2 be n-vertex graphs with $\Delta(G_1), \Delta(G_2) \leq n-2$. If*

$$|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} \leq 3n - 96n^{3/4} - 65, \tag{3.4}$$

then G_1 and G_2 pack.

The sharpness example for Theorem 1.3.3, namely the pair $\{K_{1,n-2} \cup K_1, C_n\}$, shows that Theorem 1.3.15 is asymptotically sharp. Żak suggests that the lower order terms in restriction (3.4) can be replaced with a constant and makes the following conjecture.

Conjecture 1.3.16. *[53] Let G_1 and G_2 be n-vertex graphs with $\Delta(G_1), \Delta(G_2) \leq n-2$. If*

$$|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} \leq 3n - 7, \tag{3.5}$$

then G_1 and G_2 pack.

The following example from [53] shows that, if true, then Conjecture 1.3.16 is the best possible.

Example 3.1.2. *Let $n \geq 8$ and let $G_1 \cong G_2 \cong K_3 \cup K_{1,n-4}$ (Figure 3.3). Then, $\Delta(G_1) = \Delta(G_2) = n-4$ and $|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} = (n-1) + (n-1) + (n-4) = 3n-6$. However, G_1 and G_2 do not pack.*

To see that G_1 and G_2 do not pack, let v be the center of the star in G_1 and x be the center fo the star in G_2. In any packing, the vertex v cannot be mapped to x, as otherwise the four neighbors of v must all be mapped to the triangle in G_2. Similarly, since at most one vertex from the triangle in G_1 can be mapped to the triangle in G_2, v cannot be mapped to the triangle in G_2. So we see that v must be mapped to a degree one vertex in G_2. However, since $G_1 \cong G_2$, the same reasoning also shows that a degree 1 vertex in G_1 must be mapped to x, a contradiction.

However, for small values of n, Conjecture 1.3.16 fails, as there exist graphs G_1 and G_2 that do not pack but satisy $|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} \leq 3n - 7$. Consider the following example.

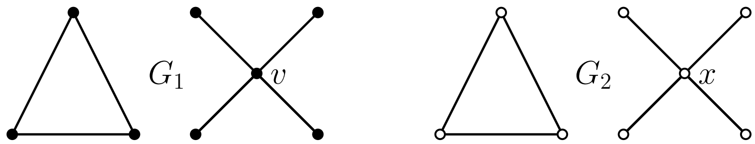

Figure 3.3: Sharpness example for Conjecture 1.3.16. In this example $n = 8$ and $|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} = 3n - 6$ but the graphs do not pack.

Example 1.3.17. *Let $G_1 = 4K_3$ and $G_2 = K_5 \cup \overline{K}_7$ (Figure 1.4). In any attempted packing, we are forced to send at least two vertices from the same component in G_1 to the clique in G_2, so the graphs do not pack. In this example, $|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} = 12 + 10 + 4 = 26 = 3n - 10$.*

Though the conjecture is false in general, we are unable to find counterexamples for large values of n and it is possible that Conjecture 1.3.16 is true for large graphs. The second main result of this chapter shows that, up to an additive constant, Conjecture 1.3.16 is true.

Theorem 1.3.18. *Let $C = 11(195^2) = 418,275$. Let G_1 and G_2 be n-vertex graphs with $\Delta(G_1), \Delta(G_2) \leq n - 2$. If*

$$|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} \leq 3n - C, \tag{3.6}$$

then G_1 and G_2 pack.

Our constant C is not optimal. However, $3n - 96n^{3/4} - 65 \leq 0$ for $n \leq 10^6$, so all graphs satisfying the hypothesis of Theorem 1.3.15 for $n \leq 10^6$ are empty and trivially pack. Since $96n^{3/4} - 65 \geq C$ when $n \geq 300,000$, Theorem 1.3.18 is a stronger result all non-trivial values of n. By combining this result with Theorem 1.3.2, we can conclude that Theorem 1.3.14 holds for values of $n \geq 2C - 2 \approx 10^6$. Observe that if $\Delta(G_1) = n - 1$ or $\Delta(G_2) = n - 1$, then $|E(G_1)| + |E(G_2)| \leq \frac{3}{2}n - 1$ and Theorem 1.3.2 guarantees that G_1 and G_2 pack. Alternatively, when $n \geq 2C - 2$, $\frac{5}{2}n - 2 \leq 3n - C$ and Theorem 1.3.18 guarantees a packing.

In order to facilitate and inductive proof of Theorem 1.3.18, we actually prove a more general result that uses the concept of list packing introduced in [24]. A *graph triple* $\mathbf{G} = (G_1, G_2, G_3)$ consists of two disjoint n-vertex graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ and a bipartite graph $G_3 = (V_1 \cup V_2, E_3)$ with partite sets V_1 and V_2. A *list packing* of $\mathbf{G}$ is a packing of G_1 and G_2 such that $uf(u) \notin E_3$ for any $u \in V_1$. One can view a list packing as a packing of G_1 and G_2, where each vertex in G_1 is given a "list" of forbidden vertices in G_2. We prove the following list version of Theorem 1.3.18.

Theorem 1.3.19. *Let $C = 11(195^2)$. Let $n \geq 2$ and $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple with $|V_1| = |V_2| = n$, $\Delta(G_1), \Delta(G_2) \leq n - 2$, and $\Delta(G_3) \leq n - 1$. If $|E_1| + |E_2| + |E_3| + \max\{\Delta(G_1), \Delta(G_2)\} + \Delta(G_3) \leq 3n - C$, then $\mathbf{G}$ packs.*

Theorem 1.3.18 is the special case of Theorem 1.3.19 in which G_3 has no edges. The pair shown in Figure 1.4 shows that, up to an additive constant, the theorem is sharp. However, there are other infinite families of examples that show, up to an additive constant, the theorem is sharp when E_3 is nonempty, e.g. Figure 1.5.

The remainder of the chapter is organized as follows. Sections 3.2, 3.3, and 3.4 contain the proofs of Theorem 1.3.12, Corollary 1.3.13, and Theorem 1.3.18, respectively.

The proof of Theorem 1.3.12 is split into three parts. In Section 3.2.1, we provide some notation and preliminary results that will be used in the later sections. Section 3.2.2 introduces the framework of the proof and includes two lemmas that will be used in the proof of Theorem 1.3.12. In Section 3.2.3, we prove Theorem 1.3.12 by providing a packing of G_1 and G_2 in a 4-stage process.

Similarly, the proof of Theorem 1.3.18 is divided into several parts. In Section 3.4.1, we state definitions, some useful preliminary results, and the main technical result, Theorem 3.4.3. The proof of Theorem 3.4.3 will be by contradiction. In Section 3.4.2 we prove several lemmas regarding the degree requirements of a minimal counterexample $\mathbf{G} = (G_1, G_2, G_3)$. We then use these properties in Section 3.4.3 to show that a minimal counterexample has at most one vertex with at least two neighbors of degree 1. Next, in Section 3.4.4, we introduce the notion of supersponsors and show that each of G_1 and G_2 contains at least two supersponsors. Finally, in Section 3.4.5, we arrive at a contradiction by using the structure of a minimal counterexample to construct a packing.

3.2 Refinement of a Theorem of Alon and Yuster

3.2.1 Notation and previous results

Throughout Section 3.2, we will consider two n-vertex graphs G_1 and G_2 that satisfy the conditions of Theorem 1.3.12. For $i \in \{1, 2\}$, we let $V_i = V(G_i)$ and $E_i = E(G_i)$. Similarly, let Δ_i denote the maximum degree of G_i and δ_i denote the minimum degree of G_i. The core of the proof is Section 3.2.3, in which we explicitly construct the packing $f : V_1 \to V_2$.

We will construct this packing iteratively. For subsets $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$, we say that $f' : W_1 \to W_2$ is a *partial packing* of G_1 and G_2 if f' is a packing of $G_1[W_1]$ and $G_2[W_2]$. Throughout the proof, we will have a partial packing f of G_1 and G_2 and enlarge the domain of f at each step.

Recall that a graph G is d-degenerate if $\delta(H) \leq d$ for every $H \subseteq G$ (Definition 1.1.21). A *degenerate ordering* $v_1, \ldots, v_n$ of a graph G on n vertices is defined inductively. Let $G_1 = G$ and define v_1 to be a vertex of minimum degree in G_1. For $i \in \{1, \ldots, n-1\}$, let $G_{i+1} = G_i - v_i$ and define v_{i+1} be a vertex of minimum degree in G_{i+1}. A *greedy ordering* of $V(G)$ is defined similarly, with the only difference that we always choose a vertex of the maximum (and not minimum) degree.

We will use the following result from [6] on packing a d-degenerate graph with a graph with a small maximum degree.

Theorem 3.2.1 ([6]). *Let $d \geq 2$. Let G_1 be a d-degenerate graph of order n and maximum degree Δ_1 and G_2 a graph of order n and maximum degree at most Δ_2. If $40\Delta_1 \log \Delta_2 < n$ and $40d\Delta_2 < n$, then G_1 and G_2 pack.*

We use Theorem 3.2.1 only for $d = \lceil \sqrt{2n} \rceil - 1$. The proof of it uses the following lemma that also will be helpful for us.

Lemma 3.2.2 ([6]). *Fix $\Delta \geq 90$ (hence $\Delta \geq 20 \log \Delta$) and let $m = \left\lceil \frac{\Delta}{\log \Delta} \right\rceil$. Let G be a graph with maximum degree at most Δ. Then, for every $V' \subseteq V(G)$, there exists a partition $\left(V^{(1)}, \ldots, V^{(m)}\right)$ of V' such that for each vertex v of G, the neighborhood $N(v)$ has the following properties:*

1. *for each i, $|N(v) \cap V^{(i)}| \leq 5 \log \Delta$,*
2. *for each i_1 and i_2, $|N(v) \cap (V^{(i_1)} \cup V^{(i_2)})| \leq 8.7 \log \Delta$, and*
3. *for each i_1, i_2, and i_3, $|N(v) \cap (V^{(i_1)} \cup V^{(i_2)} \cup V^{(i_3)})| \leq 12.3 \log \Delta$.*

3.2.2 Preliminary Results

We will construct a packing $f : V_1 \to V_2$ in four stages. In the first two stages, we consider each vertex $v \in V_1$ of large degree and for each such vertex, we find a permissible vertex in V_2 for its image. Then, we use a technique of Alon and Yuster in [2] to find a set $X \subseteq V_1$ such that an assignment $f(X) = N(f(v))$ keeps f a partial packing. Lemma 3.2.3 and Lemma 3.2.4 show that, for each vertex $v \in V_1$ with large degree, we can find a permissible set X to map to $N(f(v))$. Lemma 3.2.2 will guarantee $N(f(v))$ is evenly distributed and we will then use a method similar to [6] to construct the packing of the remaining vertices.

First, observe that

$$G_1 \text{ is } d\text{-degenerate for } \mathcal{D}d = \lceil \sqrt{2n} \rceil - 1. \tag{3.7}$$

Indeed, if there is a subgraph $H \subseteq G_1$ such that $\delta(H) > d$, then $\delta(H) \geq d + 1$ and $|V(H)| > d + 2$. So

$$2|E(H)| = \sum_{v \in H} d(v) \geq |V(H)| \cdot \delta(H) \geq (d+2)(d+1) > 2n,$$

a contradiction to $|E(G_1)| \leq n$. Thus, (3.7) holds.

Since $\Delta_2 \leq \frac{\sqrt{n}}{60}$, we obtain $40d\Delta_2 < 40\sqrt{2n}\frac{\sqrt{n}}{60} < n$. If also $40\Delta_1 \log \Delta_2 < n$, then G_1 and G_2 pack by Theorem 3.2.1. Thus we assume that $40\Delta_1 \log \Delta_2 \geq n$. Then, since $\Delta_2 < \sqrt{n}$,

$$\Delta_1 > \frac{n}{20 \log n}. \tag{3.8}$$

Let $V_1 = \{v_1, \ldots, v_n\}$ and $d(v_1) \geq \cdots \geq d(v_n)$. We also may assume that (3.3) holds. Let $k \in [n]$ be the largest integer such that $d(v_k) \geq \frac{n}{50 \log n}$. Since $e_1 \leq n$, we have $2n \geq \sum_{i=1}^{k} d(v_i) \geq k\left(\frac{n}{50 \log n}\right)$ and so $k \leq 100 \log n$.

Lemma 3.2.3. *G_1 has an independent set $B_1 \subseteq V_1 - N[v_1]$ with $|B_1| = \delta_2$. Moreover, if $k > 1$, then such a set B_1 can be chosen so that each vertex in it has degree at most 2 in G_1.*

Proof. By (3.3), G_1 has an independent set $B_1 \subseteq V_1 - N[v_1]$ such that $|B_1| \geq \delta(G_2)$. This proves the first part. If $k \geq 2$, then $d(v_2) \geq \frac{n}{50 \log n}$.

The subgraph $G' = G_1[V_1 - v_1 - v_2]$ has $n - 2$ vertices and at most $n - d(v_1) - d(v_2) + \|v_1, v_2\|$ edges. Then G' has at least $d(v_1) + d(v_2) - 2 - \|v_1, v_2\|$ tree components and therefore contains an independent set of size at least $d(v_1) + d(v_2) - 2 - \|v_1, v_2\|$. Moreover, we form this independent set using only vertices of degree at most one in G'. Let B_1' denote the set of these vertices that are contained in $V_1 - N[v_1] - v_2$. By the above, $|B_1'| \geq d(v_2) - 2$. Since $n \geq 10^9$ and $d(v_2) \geq \frac{n}{50 \log n}$,

$$|B_1'| \geq \frac{n}{50 \log n} - 2 \geq \delta_2.$$

Since each vertex in B_1' has degree at most 1 in $G_1 - v_1 - v_2$ and $B_1 \cap N[v_1] = \emptyset$, every vertex in B_1' has degree at most 2 in G_1. So we let B_1 be a subset of B_1' of cardinality δ_2. $\square$

When $k \geq 2$, we also wish to find, for each $i \in \{2, \ldots, k\}$, an independent set $B_i \subseteq V_1 - N[v_i]$ such that we can map the vertices of B_i to the neighborhood of $f(v_i)$.

Lemma 3.2.4. *Let $k \geq 2$ and B_1 satisfy Lemma 3.2.3. There exist disjoint sets $B_2, \ldots, B_k$ such that*

(a) $|B_i| \geq \Delta_2$ for each $i \in \{2, \ldots, k\}$,

(b) $B_j \cap B_i = \emptyset$ for all $j \neq i$,

(c) each vertex in $\bigcup_{i=1}^{k} B_i$ has degree at most 2 in G_1,

(d) the set $\bigcup_{i=1}^{k} B_i$ is independent in G_1,

(e) each vertex in $V_1 - v_1$ is adjacent in G_1 to at most one vertex in $\bigcup_{j=2}^{k} B_j$.

Proof. Let $W \subseteq V_1$ be the set of all vertices reachable in $G_1 - v_1$ from $\{v_2, \ldots, v_k\}$. In particular, $\{v_2, \ldots, v_k\} \subseteq W$. By defiinition, $G_1[W]$ has at least $|W| - (k-1)$ edges. Let $X = V_1 - W - v_1$. Then $|X| = n - 1 - |W|$ and, since G_1 has at most n edges, $|E(G_1[X])| \leq n - [|W| - (k-1)] - d(v_1)$. Therefore, the number of tree components in $G_1[X]$ is at least $d(v_1) - k$. We form an independent set B by choosing one leaf or isolated vertex from each tree component in $G[X]$ and then removing all vertices in $N[B_1]$. Since each vertex in B_1 has degree at most 2 by Lemma 3.2.3, we have

$$|B| \geq (d(v_1) - k) - 3\delta_2. \tag{3.9}$$

Suppose that

$$d(v_1) - k - 3\delta_2 \geq (k-1)\Delta_2. \tag{3.10}$$

Then by (3.9), B can be partitioned into $k-1$ disjoint sets $B_2, \ldots, B_k$, each of size at least Δ_2. Since all vertices $u \in B$ are leaves or isolated vertices in distinct components of $G_1 - v_1$, the claims (c) and (e) of the lemma hold. Since the sets $B_2, \ldots, B_k$ are formed by partitioning an independent set that is disjoint from $N[B_1]$, claims (b) and (d) also hold. So, to prove the lemma, it is enough to check that (3.10) holds. Now,

$$\begin{aligned}
(d(v_1) - k) - 3\delta_2 &\geq (k-1)\Delta_2 && \text{if} \\
(d(v_1) - k) - 3\Delta_2 &\geq (k-1)\Delta_2 && \text{if} \\
d(v_1) - 1 - 3\Delta_2 &\geq (k-1)(\Delta_2 + 1) && \text{if} \\
d(v_1) &\geq (k+2)(\Delta_2 + 1) - 2.
\end{aligned}$$

Since $k \leq 100 \log n$, $d(v_1) \geq n/(20 \log n)$, and $\Delta_2 \leq \sqrt{n}/60$, the last inequality follows from

$$\frac{n}{20 \log n} \geq (100 \log n + 2)\left(\frac{\sqrt{n}}{60} + 1\right),$$

which holds for $n \geq 10^9$. This proves (3.10) and thus the lemma. ☐

3.2.3 Packing of G_1 and G_2

Let $B_1, B_2, \ldots, B_k$ be as stipulated in Lemmas 3.2.3 and 3.2.4. Note that by 3.2.4(c), $\{v_1, \ldots, v_k\} \cap (B_1 \cup \ldots \cup B_k) = \emptyset$. Let $m = \left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil$ and $(V^{(1)}, \ldots, V^{(m)})$ be a partition of V_2 with the properties guaranteed by Lemma 3.2.2. Order the parts of the partition so that $|V^{(1)}| \geq \cdots \geq |V^{(m)}|$. For $i \in \{1, \ldots, k\}$, let $\hat{V}^{(i)} = \bigcup_{j=1}^{i} V^{(j)}$. We will construct a packing $f : V_1 \to V_2$ in 4 stages. At each step in the proof, we ensure that f remains a partial packing.

Stage 1. Let $w_1 \in V_2$ be a vertex of minimum degree in G_2. Define $f(v_1) = w_1$. For each $w' \in N_{G_2}(w_1)$, we can choose an element $u \in B_1$ and assign $f(u) = w'$. In this way, all neighbors of w_1 are matched and, since $B_1 \cup \{v_1\}$ is an independent set, after this assignment f remains a partial packing.

Stage 2. If $k = 1$, then proceed to Stage 3. Otherwise, we will iteratively match $v_2, \dots, v_k$ with vertices of G_2. During iteration i, we will match v_i to some vertex $f(v_i)$ in $V^{(1)}$. We will then proceed to match an unmatched subset of B_i to $N_{G_2}(f(v_i))$. Notice that after iteration i, the function f will remain a partial packing, the only matched vertices of V_1 will be $v_1, \dots, v_i$, and vertices from $\bigcup_{j=1}^{i} B_j$, and at most $i(\Delta_2 + 1)$ vertices of G_1 (and, respectively, G_2) will be matched.

Consider the i^{th} iteration. At this point, we have matched vertices $v_1, \dots, v_{i-1} \in V_1$ to vertices $w_1, \dots, w_{i-1} \in V_2$, respectively. Since $w_2, \dots, w_{i-1}$ were chosen to be in $V^{(1)}$ and w_1 may also have been in $V^{(1)}$, at most $i - 1 < k$ of these vertices are in $V^{(1)}$. The only other matched vertices in G_2 are in $\bigcup_{j=1}^{i-1} N_{G_2}(w_j)$. By Lemma 3.2.2, $|N_{G_2}(w_j) \cap V^{(1)}| \leq 5 \log \Delta_2$. So there are at most $k(1 + 5\log \Delta_2)$ vertices in $V^{(1)}$ that have already been matched. There are at least $\lceil \frac{n}{m} \rceil - k(1 + 5\log \Delta_2)$ remaining vertices in $V^{(1)}$. From these remaining vertices, we will choose a vertex w_i such that after assigning $f(v_i) = w_i$, the function f remains a partial packing. If a vertex $x \in N(v_i)$ is already matched, then either $x \in \{v_1, \dots, v_{i-1}\}$ or $x \in \bigcup_{j=1}^{i-1} B_j$. However, for each $j < i$, $N(f(v_j))$ is already matched, so v_i will not be matched to a neighbor of $f(v_j)$. Further, by Lemma 3.2.4, no vertex adjacent to $\{v_1, \dots, v_{i-1}\}$ was chosen to be in B_j. So any available choice for w_i will allow f to remain a partial packing. Since there were $\lceil \frac{n}{m} \rceil - k(1 + 5\log \Delta_2) > 0$ vertices to choose from, there is a permissible choice of w_i.

To complete the iteration, we must map some subset of B_i to the unmatched neighbors of w_i. However, by Lemma 3.2.4(d), B_j and B_i were chosen to be disjoint for each j, so no vertex in B_i is already matched. Further, if sending a vertex $x \in B_i$ to an unmatched vertex in $y \in N(w_i)$ causes f to no longer be a partial packing, then x has a neighbor u such that $y \in N(f(u))$. Notice that if this is the case, then $u \notin \{v_1, \dots, v_{i-1}\}$, since y is unmatched and $N(f(\{v_1, \dots, v_{i-1}\}))$ contains only matched vertices. Therefore, if there is an $x \in B_i$ such that sending x to a vertex $y \in N_{G_2}(w_i)$ forces f to not be a partial packing, then $x \in \bigcup_{j=1}^{i-1} N(B_j)$. Again by Lemma 3.2.4, B_i does not contain any such vertex, so any vertex $x \in B_i$ can be mapped to any unmatched vertex in $N_{G_2}(w_i)$. By Lemma 3.2.4, $|B_i| \geq \Delta_2$, so we can match a subset of B_i to the neighborhood of w_i.

Stage 3. Let $W_1 \subset V_1$ be the set of vertices that have been matched before the start of Stage 3 and let V_2' be their matches. Recall that G_1 is d-degenerate for some $d \leq \sqrt{2n}$. Since $G_1[V_1 - W_1] \subseteq G_1$, it must also be d-degenerate. We will define disjoint subsets $W_2, \dots, W_m$ with the goal of sending W_i into $\hat{V}^{(i)}$ for each i.

Let $X_1 = Y_1 = \emptyset$ and $z := \lceil \frac{n}{15m} \rceil \leq \left\lceil \frac{n \log \Delta_2}{15\Delta_2} \right\rceil \leq 2\sqrt{n} \log n$. We now inductively construct sets X_i, Y_i and W_i for $i \in 2, \dots, m$. Let $\hat{X}_i = \bigcup_{j=1}^{i} X_j$, $\hat{Y}_i = \bigcup_{j=1}^{i} Y_j$, and $\hat{W}_i = \bigcup_{j=1}^{i} W_j$ and then consider a greedy ordering of $V_1 - \hat{W}_{i-1}$. Define X_i to be the first z vertices in this ordering, so $|\hat{X}_i| = (i-1)z$. Add to Y_i any vertex in $y \in V_1 - \hat{W}_{i-1} - X_i$ such that y has at least $4d$ neighbors in $\{\hat{W}_{i-1} \cup X_i \cup Y_i\} - W_1$. Continue to add vertices to Y_i until every remaining vertex has at most $4d$ neighbors in $\hat{W}_{i-1} \cup X_i \cup Y_i$. Finally, let $W_i = X_i \cup Y_i$.

We next show that $|\hat{W}_i|$ is not too large. We have $e(G_1[\hat{W}_i - W_1]) \geq 4d|\hat{Y}_i|$, since each vertex in $\hat{Y}_i$ has at least $4d$ edges to previously matched vertices and at most $k+1$ of them are incident to vertices mapped in Stage 1. However, since $G[\hat{W}_i - W_1]$ is d-degenerate and has $|\hat{X}_i| + |\hat{Y}_i|$ vertices, it has less than $(|\hat{X}_i| + |\hat{Y}_i|)d$ edges. This implies that $4d|\hat{Y}_i| - (k+1) < d(|\hat{X}_i| + |\hat{Y}_i|)$. Since $d \geq 1$, solving for $|\hat{Y}_i|$ yields $|\hat{Y}_i| < \frac{(i-1)z}{3} + \frac{1}{3}(k+1)$. Finally, since $|\hat{W}_i| = |W_1| + |\hat{X}_i| + |\hat{Y}_i|$ and $W_1 \leq k(\Delta_2 + 1)$, we have

$$
\begin{aligned}
|\hat{W}_i| &< \frac{4(i-1)}{3} z + k\left(\Delta_2 + \frac{4}{3}\right) + \frac{1}{3} \\
&\leq \frac{4(i-1)}{3} \left\lceil \frac{n}{15m} \right\rceil + k\left(\Delta_2 + \frac{4}{3}\right) + \frac{1}{3} \\
&\leq \frac{4(i-1)}{3} \frac{n}{15m} + \frac{4(i-1)}{3} + k\left(\Delta_2 + \frac{4}{3}\right) + \frac{1}{3} \\
&\leq \frac{4(i-1)}{3} \frac{n}{15m} + \frac{4i}{3} + k\left(\Delta_2 + \frac{4}{3}\right) - 1 \\
&\leq \left(4(i-1) + \frac{60im}{n} + \frac{km(45\Delta_2 + 60)}{n}\right) \frac{n}{45m} - 1 \\
&\leq \left(4 + \frac{60m}{n} + \frac{km(45\Delta_2 + 60)}{n}\right) \frac{in}{45m} - 1 \\
&\leq \left(4 + \frac{60}{n} \left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil + \frac{k(45\Delta_2 + 60) \left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil}{n}\right) \frac{in}{45m} - 1.
\end{aligned}
\tag{3.11}
$$

Finally, recall that $\Delta_2 / \log(\Delta_2) \leq \sqrt{n}/(60 \log(\sqrt{n}/60))$ and $k \leq 100 \log n$. We can substitute these upper bounds into (3.11) and calculate that for $n \geq 10^9$,

$$
4 + \frac{60}{n} \left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil + \frac{k(45\Delta_2 + 60) \left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil}{n} < 9. \tag{3.12}
$$

Therefore, by (3.11) and (3.12),

$$|\hat{W}_i| < \frac{in}{5m}. \tag{3.13}$$

Now, we place W_i in $\hat{V}^{(i)}$ for each $i \in \{2, \ldots, m\}$. Consider a degenerate ordering of the vertices in W_i. We pack the vertices into $V^{(i)}$ in this order. Suppose it is the turn of vertex w to be packed. In particular, we have placed at most $|\hat{W}_i|$ vertices so far, so there are at least $\frac{in}{m} - |\hat{W}_i| \geq \frac{4in}{5m}$ free vertices left in $\hat{V}^{(i)}$. Suppose we send w to some unmatched vertex $v \in \hat{V}^{(i)}$. If w has a neighbor w' already matched to a neighbor of v, then f is not a partial packing. We show that the number of such bad vertices v is at most $\frac{4in}{5m}$.

Let w' be a matched neighbor of w. Then either $w' \in W_1$ or $w' \in \hat{X}_j \cup \hat{Y}_j$ for some j. If $w' \in \{v_1, \ldots, v_k\}$, by Stage 1 and 2, all neighbors of the images of $\{v_1, \ldots, v_k\}$ are already matched and are therefore not adjacent to v. On the other hand, by Lemma 3.2.4, w is only adjacent to at most one vertex of $W_1 - \{v_1, \ldots, v_k\}$ (since vertices in B_i were chosen from distinct components of $V_1 - v_1$).

Next, since the vertices of W_i are placed using a degenerate ordering of W_i and each vertex in W_i has fewer than $4d$ vertices in $\hat{W}_{i-1} - W_1$, vertex w has at most $5d$ neighbors in $\hat{W}_i - W_1$. We conclude that w has at most $1 + 5\sqrt{2n} \leq 8\sqrt{n}$ previously matched neighbors adjacent to unmatched neighbors in V_2. Further, by Lemma 3.2.2 the image of each of these neighbors has at most $5i \log \Delta_2$ neighbors in $\hat{V}^{(i)}$. Thus, there are at most $40i\sqrt{n}\log \Delta_2$ choices for v that cause f to not be a partial packing. Since we have $|W_i| > \frac{4in}{5m}$ vertices to choose from and $\frac{4n}{5m} - 40\sqrt{n}\log \Delta_2 > 0$, there is a vertex to which we can send w.

Stage 4. We now place the remaining vertices, i.e. those in $V_1 - \hat{W}_m$. Consider a degenerate ordering of $V_1 - \hat{W}_m$ and place these vertices in the reverse order. Suppose it is the turn of vertex w to be packed. Then, there is some unmatched vertex $v \in V_2$. We show that either we can send w to v or that there is another previously matched vertex $w' \in V_1 - \hat{W}_m$ such that w can be matched to the image of w', let us call it $v' \in V_2$, and w' can be matched to v.

Notice that for any $x \in N(w)$, we are unable to match w to an unmatched vertex $v \in V_2$ that is a neighbor of the image of x. Let us call such vertices *red/blue neighbors*, since they can be reached from w via a 2-edge path with the first edge being $wx \in E_1$ (i.e. red) and the second edge being $f(x)v \in E_2$ (i.e. blue). As in Stage 3, we notice that when it is the turn of w to be packed, it has at most $4d$ neighbors in $\hat{W}_m - W_1$ and, since we are placing $V_1 - \hat{V}_m$ in the reverse of a degenerate order, it has at most d neighbors in $V_1 - \hat{V}_m$ previously matched during Stage 4. In total, w has at most $5d$ previously matched neighbors in $V_1 - W_1$. By Lemma 3.2.4, w has at most 1 neighbor in $W_1 - \{v_1, \ldots, v_k\}$. The vertex w may be adjacent to many vertices in $\{v_1, \ldots, v_k\}$ but, by Stage 1 and Stage 2, the images of $\{v_1, \ldots, v_k\}$ have no unmatched neighbors so no red/blue neighbors may arise from these vertices. We conclude that, apart from $\{v_1, \ldots, v_k\}$, the vertex w has at most $2+5d \leq 8\sqrt{n}$ previously matched neighbors. The image of each of these neighbors has at most Δ_2 blue neighbors, so there are at most $8\sqrt{n}\Delta_2 \leq \frac{8n}{60}$ red/blue neighbors of w.

On the other hand, for each $v_i \in \{v_1, \ldots, v_k\}$, the neighbors of $f(v_i)$ are matched to vertices in W_1. So v has no neighbors in $f(W_1)$. We now count the number of vertices $x \in V_1$ such that x has a neighbor in V_1 matched to a neighbor of v in V_2. We call this set of vertices the *blue/red neighbors* of v. In particular, we only concern ourselves with blue/red neighbors x such that $x \notin \hat{W}_m$. We will use the method used in [39] to bound the number of such neighbors.

Let $\mathrm{br}(v)$ be the number of blue/red neighbors in $V_1 - \hat{W}_m$ and let $n_i = |N_{G_2}(v) \cap V^{(i)}|$. Recall that, in Stage 3, we considered a greedy ordering of $V_1 - \hat{W}_{i-1}$. Let D_i be the maximum degree of a vertex in $G[V_1 - W_{i-1}]$. In particular, if $x \in X_j$ is matched to a vertex in $\hat{V}^{(j)}$ for $j \geq 2$, then x has at most D_j neighbors in $V_1 - \hat{W}_i$ and at least D_{j+1} such neighbors. This implies $|X_2|D_3 + \cdots + |X_{m-1}|D_m = z(D_3 + \cdots + D_m) < n$, since there are at most n edges in G_1.

Further, we know that if a vertex $v_i \in V_1$ has more than $n/(50 \log n)$ neighbors in G_1, then not only was it matched in Stage 1, but all vertices of $N_{G_2}(f(v_i))$ are matched in Stage 1 as well. So if a vertex x is matched to a neighbor of v, then $d(x) \leq n/(100 \log n)$. In particular,

$$\begin{aligned}
\mathrm{br}(v) &\leq n_1 \frac{n}{50 \log n} + n_2 \frac{n}{50 \log n} + \sum_{k=3}^{m} n_k D_k \\
&\leq \frac{n}{50 \log n}(n_1 + n_2) + (D_3 + \cdots + D_m) 5 \log \Delta_2 \\
&\leq \frac{n}{50 \log n}(8.7 \log \Delta_2) + \frac{n}{z} 5 \log \Delta_2 \\
&\leq \frac{4.35n}{50} + (15m) 5 \log \Delta_2 \\
&\leq \frac{4.35n}{50} + 75 \left\lceil \frac{\Delta_2}{\log \Delta_2} \right\rceil \log \Delta_2 \\
&\leq \frac{4.35n}{50} + 75\Delta_2 + 75 \log \Delta_2 \\
&\leq \frac{4.35n}{50} + \frac{5\sqrt{n}}{4} + 75 \log\left(\sqrt{n}/60\right) < \frac{n}{10}.
\end{aligned}$$

We know that there are fewer than $\frac{n}{10}$ blue/red neighbors of v, at most $\frac{8n}{60}$ red/blue neighbors of w, and at most $\frac{n}{5}$ vertices in $\hat{W}_m$. This means that either we can send w to v and maintain that f is a partial packing or there is a vertex w' in $V_1 - \hat{W}_m$ placed on a vertex $v' \in V_2$ such that w' is not a blue/red neighbor of v and also that v' is not a red/blue neighbor of w. This implies that we can send v to w' and w to v' and maintain that f is a partial packing. Repeating this process for each unmatched vertex in V_1 yields a packing of G_1 and G_2.

3.3 Sharpness examples of Theorem 1.3.7

Let G_1 and G_2 be graphs on n vertices such that $|E(G_1)| \leq n - \delta(G_2)$ and $\Delta(G_2) \leq \sqrt{n}/60$. We will show that if G_1 and G_2 do not satisfy conclusion (2) nor conclusion (3) of Corollary 1.3.13, then they pack. As before, for $i \in \{1, 2\}$, let $G_i = (V_i, E_i)$, $\Delta_i = \Delta(G_i)$, and $\delta_i = \delta(G_i)$.

Let $v_1 \in V_1$ be a vertex of maximum degree in G_1. If $\Delta_1 = n - \delta_2$, then part 2 of the theorem holds and the proof is complete. So we assume that $\Delta_1 \leq n - \delta_2 - 1$.

Let $v_1, \ldots, v_n$ be an ordering of V_1 such that $d(v_1) \geq \cdots \geq d(v_n)$ and let $X \subseteq E(G_1 - v_1)$ be the set of edges incident to $N(v_1)$. The subgraph $G_1 - N[v_1]$ has $n - d(v_1) - 1$ vertices and $|E_1| - d(v_1) - |X|$ edges. In particular, the number of tree components in $G_1 - N[v_1]$ is at least

$$(n - |E_1|) + |X| - 1. \tag{3.14}$$

By Theorem 1.3.12, if there exists an independent set S in $G_1 - N[v_1]$ of size δ_2, then G_1 and G_2 pack. We form an independent set S by taking one vertex from each component in $G_1 - N[v_1]$.

If $|E_1| \leq n - \delta_2 - 1$ or $|X| \geq 1$, then by (3.14) there are at least δ_2 tree components in $G_1 - N[v_1]$ and so $|S| \geq \delta_2$. Therefore, we assume that $|E_1| = n - \delta_2$ and $|X| = 0$. Hence, the number of tree components in $G_1 - N[v_1]$ is exactly $\delta_2 - 1$. Moreover, if any tree component contains at least three vertices, two vertices from the same component could be selected to be in S and we would obtain an independent set of size δ_2. Finally, we assume that $G_1 - N[v_1]$ contains no other components, as otherwise we could add an additional vertex to S and obtain an independent set of size δ_2. After these assumptions, we are now in the case that G_1 is a forest with exactly $n - \delta_2$ edges and v_1 is the only vertex with degree greater than 1. Further, since $d(v_1) \leq n - 1 - \delta_2$, some component of $G_1 - N[v_1]$ contains an edge.

We finally show that if conclusion (3) of Corollary 1.3.13 is not satisfied, then G_1 and G_2 pack. In this case, there is a vertex w_1 of degree δ_2 such that $N(w_1)$ does not induce a clique. By our assumptions on G_1, we can find an independent set S of size $\delta_2 - 1$ by selecting one vertex from each component. Recall that $G_1 - N[v_1]$ has some component that contains exactly one edge and let x be the vertex in that component not chosen to be in S. Let $B_1' \subseteq V_1 - N[v_1]$ be the set of vertices obtained by adding x to S and note that $G[B_1']$ contains exactly δ_2 vertices and exactly one edge.

We can now construct a packing of G_1 and G_2 almost exactly as we did in the proof of Theorem 1.3.12. In *Stage 1*, we define $f(v_1) = w_1$ and wish to map the set B_1' to the neighborhood of w_1 so that f remains a partial packing. Since $N(w_1)$ does not induce a clique and B_1' contains only one edge, such a mapping is possible. Now, since v_1 is the only vertex in G_1 with degree greater than 1, we proceed directly to *Stage 3*.

However, *Stage 3* and *Stage 4* follow exactly as they did in Section 3.2.3, resulting in the desired packing of G_1 and G_2.

3.4 Approximate answer to a conjecture of Żak

3.4.1 The setup

We begin by outlining some notation to be used throughout Section 3.4. A graph triple $\mathbf{G} = (G_1, G_2, G_3)$ of order n consists of a pair of n-vertex graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_2)$ together with a bipartite graph $G_3 = (V_1 \cup V_2, E_3)$. Let $V(\mathbf{G}) := V_1 \cup V_2$ be the vertex set of the graph triple, $E(\mathbf{G}) = E_1 \cup E_2 \cup E_3$ be the edge set of the graph triple, and $e(\mathbf{G}) = |E(\mathbf{G})|$. We omit $\mathbf{G}$ when it is clear. The triple $\mathbf{G}$ *packs* if there is a bijection $f : V_1 \to V_2$ such that $vf(v) \notin E_3$ for any $v \in V_1$ and $uv \in E_1$ implies $f(u)f(v) \notin E_2$. An edge in $E_1 \cup E_2$ is a *white* edge, while an edge in E_3 is a *yellow* edge.

For $v \in V_i$ ($i \in \{1, 2\}$), the *white neighborhood* of v, denoted $N_i(v) \subseteq V_i$, is the set of neighbors of v in G_i and $d_i(v) = |N_i(v)|$. For convenience, when $w \in V_{3-i}$, we say that $N_i(w) = \emptyset$ (and hence $d_i(w) = 0$). The *yellow neighborhood* of $v \in V_i$, denoted $N_3(v) \subseteq V_{3-i}$ is the set of neighbors of v in G_3 and $d_3(v) = |N_3(v)|$. Vertices in the white (respectively, yellow) neighborhood of v are called *white neighbors* (respectively, *yellow neighbors*). For $v \in V_i$, the *neighborhood* of v, denoted $N(v)$ is the disjoint union $N_i(v) + N_3(v)$ and the *degree* of v is $d_i(v) + d_3(v)$ and is denoted $d(v)$. Also, we use $N[v]$ to denote the *closed neighborhood* of v, i.e. $N[v] = N(v) \cup \{v\}$. For disjoint vertex sets X and Y in a graph triple, $\|X, Y\|$ denotes the number of edges in $E(\mathbf{G})$ connecting X and Y. For brevity, if $X = \{x\}$ and $Y = \{y\}$, then we will write $\|x, y\|$ instead of $\|\{x\}, \{y\}\|$.

When considering a specific graph triple $\mathbf{G}$, we will let $e_i = |E_i|$ and define $\Delta_i = \max_{v \in V} d_i(v)$ for $i \in \{1, 2, 3\}$. In [24], Theorem 1.3.6 and Theorem 1.3.3 to extended to list packing. The following two theorems will be used throughout this section.

Theorem 3.4.1 ([24]). *Let $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple with $|V_1| = |V_2| = n$. If $\Delta_1\Delta_2 + \Delta_3 \le n/2$, then $\mathbf{G}$ does not pack if and only if $\Delta_3 = 0$ and one of G_1 or G_2 is a perfect matching and the other is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$. Consequently, if $\Delta_1\Delta_2 + \Delta_3 < n/2$, then $\mathbf{G}$ packs.*

Theorem 3.4.2 ([24]). *Let $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple with $|V_1| = |V_2| = n$. If $\Delta_1, \Delta_2 \le n - 2$, $\Delta_3 \le n - 1$, $|E_1| + |E_2| + |E_3| \le 2n - 3$ and the pair (G_1, G_2) is none of the 7 pairs in Theorem 1.3.3, then $\mathbf{G}$ packs.*

For a graph triple $\mathbf{G} = (G_1, G_2, G_3)$, let $\Delta_{3|i} = \max_{v \in V_i} d_3(v)$, $D_i = \max\{\Delta_i, \Delta_{3|i}\}$, and

$$\mathcal{D} = \max\{\Delta_1 + \max\{\Delta_{3|2} - 4, 0\}, \Delta_2 + \max\{\Delta_{3|1} - 4, 0\}\}.$$

Instead of Theorem 1.3.19, it is more convenient to prove the following.

Theorem 3.4.3. *Let $C := 11(195^2) + 4$. Let $n \geq 2$ and $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple of order n. If*

$$\Delta_1, \Delta_2 \leq n - 2, \; \Delta_3 \leq n - 1 \tag{3.15}$$

and

$$F(\mathbf{G}) := e_1 + e_2 + e_3 + \mathcal{D} \leq 3n - C, \tag{3.16}$$

then $\mathbf{G}$ packs.

Note that Theorem 3.4.3 implies Theorem 1.3.19 since $\Delta_3 \geq \Delta_{3|1}, \Delta_{3|2}$ and $F(\mathbf{G}) + 4 \leq e_1 + e_2 + e_3 + \max\{\Delta_1, \Delta_2\} + \Delta_3$. In proving this theorem, we will often consider two graph triples, $\mathbf{G}$ and $\mathbf{G}'$, and will compare $F(\mathbf{G})$ and $F(\mathbf{G}')$. Define $\partial(\mathbf{G}, \mathbf{G}') = F(\mathbf{G}) - F(\mathbf{G}')$. The rest of the section will be a proof of Theorem 3.4.3.

3.4.2 Maximum and Minimum Degrees in a Minimal Counterexample

Fix $C := 11(195^2) + 4$ and let $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple of the smallest order n such that $\mathbf{G}$ satisfies (3.15) and (3.16) but $\mathbf{G}$ does not pack. By Theorem 3.4.2 and (3.16),

$$\mathcal{D} \leq n + 2 - C. \tag{3.17}$$

This yields $n \geq C - 2$. Moreover, since $n \geq C - 2$, Theorem 3.4.1 implies $\mathcal{D} \geq 2$, and thus, by (3.17), $n \geq C$.

Lemma 3.4.4. *Every vertex of $\mathbf{G}$ has a white neighbor.*

Proof. Suppose $v \in V$ has no white neighbor. Without loss of generality, let $v \in V_1$.

Case 1: *The vertex v is isolated in $\mathbf{G}$.* If any $w \in V_2$ has degree at least 3 in $\mathbf{G}$ then taking $\mathbf{G}' = (G_1 - v, G_2 - w, G_3 - v - w)$ and $n' = n - 1$ gives $\partial(\mathbf{G}, \mathbf{G}') \geq 3$ and thus $F(\mathbf{G}') \leq 3n' - C$. Also by (3.17), for $i \in \{1, 2\}$,

$$\Delta'_i \leq \Delta_i \leq \mathcal{D} + 4 \leq n + 6 - C \leq (n - 1) - 2 = n' - 2.$$

By the minimality of $\mathbf{G}$, the new triple $\mathbf{G}'$ packs. This packing extends to a packing of $\mathbf{G}$ by sending v to

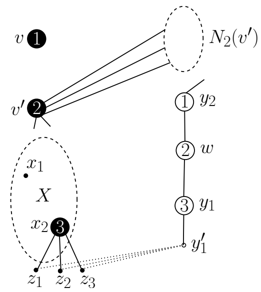

Figure 3.4: Packing used at the end of Case 1

w, contradicting the choice of $\mathbf{G}$. So suppose the degree of each $w \in V_2$ is at most 2. By Theorem 3.4.1, there is a vertex $v' \in V_1$ with $d(v') > n/6$. By (3.15), there is a non-neighbor w of v' in V_2. If w has a white neighbor, say $y \in V_2$, then let $\mathbf{G}'' = (G_1 - v - v', G_2 - w - y, G_3 - v - v' - w - y)$ with $n'' = n - 2$; otherwise, let $\mathbf{G}'' = (G_1 - v', G_2 - w, G_3 - v' - w)$ with $n'' = n - 1$. Then $\partial(\mathbf{G}, \mathbf{G}'') > d(v') = n/6 > 6$ and so $F(\mathbf{G}'') \le 3n'' - C$ which by (3.17) implies $\Delta''_i \le n + 6 - C \le n'' - 2$ for $i = 1, 2,$. Thus, again by the minimality of $\mathbf{G}$, the triple $\mathbf{G}''$ packs. Then, we extend this packing of $\mathbf{G}''$ to a packing of $\mathbf{G}$ by sending v' to w (and v to y if y exists), contradicting the choice of $\mathbf{G}$.

The last subcase of Case 1 is that $d_2(w) = 2$ for every non-neighbor w of v' in V_2. In particular, $e_2 + e_3 \ge e_2 + d_3(v') \ge n$. So, if $X = V_1 - N[v'] - v$, then by (3.16)

$$\sum_{x \in X} d_1(x) \le 2e_1 - 2d_1(v') \le 2\left[3n - C - \mathcal{D} - (e_2 + d_3(v')) - d_1(v')\right].$$

Since $d_1(v') + |X| = n - 2$, $e_3 \ge d_3(v')$, and $\mathcal{D} \ge \Delta_1 \ge d_1(v')$, we get

$$\sum_{x \in X} d_1(x) \le 2\left(3n - C - 2d_1(v') - n\right) \le 2(2|X| + 4 - C) < 4|X| - 8.$$

So, there are nonadjacent $x_1, x_2 \in X \subset V_1$ with $d_1(x_1), d_1(x_2) \le 3$.

Let w be a non-neighbor of v' in V_2 and let y_1 and y_2 be the white neighbors of w. Since $y_1 w \in E_2$ and $d(y_1) \le 2$, we may assume $y_1 x_2 \notin E_3$. Choose $z_1, z_2, z_3 \in V_1$ so that $N_1(x_2) \subset \{z_1, z_2, z_3\}$. Let y'_1 be the white neighbor of y_1 distinct from w, if exists. Then we place v' on w, v on y_2, x_2 on y_1, and add yellow edges from y'_1 to $N_1(x_2)$ (Figure 3.4). Since this decreases $e_1 + e_2 + e_3$ by at least $n/6 + 2 \ge C/6 + 2 \ge 12$ and increases $\mathcal{D}$ by at most 3, we are left with a graph triple $\mathbf{G}'$ of order at least $n - 3$ and $F(\mathbf{G}') \le 3(n-3) - C$. Also by (3.17), both inequalities in (3.15) hold. So by the minimality of $\mathbf{G}$, there is a packing of $\mathbf{G}'$, and this packing extends of a packing of $\mathbf{G}$.

Case 2: *The vertex $v \in V_1$ is incident to yellow edges.* Let $A := N_3(v)$. By the case, $|A| \ge 1$. Since

$V_2 - A \neq \emptyset$ by (3.17), there is some $w \in V_2 - A$. Since Case 1 does not hold, $d(w) \geq 1$. If $d(v) + d(w) \geq 3$, then we can construct a packing by sending v to w and creating a new graph triple $\mathbf{G}'$ by removing these two vertices. In creating $\mathbf{G}'$, we have removed 3 edges, and observe that by (3.17), the inequalities in (3.15) holds for $\mathbf{G}'$. So $\mathbf{G}'$ packs by the minimality of $\mathbf{G}$, and this packing extends to a packing of the original triple, a contradiction. Thus, $d(v) = 1$ (say $A = \{w'\}$) and $d(w) = 1$ for each $w \in V_2 - w'$.

Let $Y = V_2 - N[w']$. Since $d_2(w') \leq \Delta_2 \leq \mathcal{D} \leq n + 2 - C$, we have $|Y| \geq C - 3$. If $d(w') = 1$, then by switching the roles of v and w', we conclude that $d(v') = 1$ for each $v' \in V_1 - v$; so $\mathbf{G}$ packs by Theorem 3.4.1. Hence, $d(w') \geq 2$. There are two cases.

Case 2.1: *$G_2[Y]$ has no edges.* Since the white neighbors of w' cannot have other neighbors, every $y \in Y$ has no white neighbors. If also every vertex in V_1 has degree 1, then by (3.17),

$$e_1 + e_2 + e_3 = \frac{(2n-1) + d(w')}{2} \leq n - \frac{1}{2} + \mathcal{D} + 4 \leq n - \frac{1}{2} + (n + 6 - C) < 2n - 3.$$

In this case, $\mathbf{G}$ packs by Theorem 3.4.2, a contradiction. So we conclude that there is a vertex $x \in V_1$ of degree at least 2.

Next, assume that two vertices $y_1, y_2 \in Y$ have distinct neighbors in V_1. Then we may assume that x is not adjacent to one of these vertices, say y_1, and let $\mathbf{G}' = (G_1 - x, G_2 - y_1, G_3 - x - y_1)$ and $n' = n - 1$. Since $\partial(\mathbf{G}, \mathbf{G}') \geq 3$ and (3.15) holds for $\mathbf{G}'$ by (3.17), $\mathbf{G}'$ packs by the minimality of $\mathbf{G}$, and this packing extends to a packing of $\mathbf{G}$ by placing x on y_1.

Hence, each vertex in Y is adjacent to the same vertex $x' \in V_1$. This implies $\mathcal{D} \geq d_2(w') + d_3(x') - 4 \geq n - 5$, a contradiction to (3.17).

Case 2.2: *There is an edge $y_1 y_2 \in E(G_2[Y])$.* Then

$$\text{for every non-adjacent } x_1, x_2 \in V_1,\ d(x_1) + d(x_2) \leq 4, \tag{3.18}$$

since otherwise we could send x_1 to y_1 and x_2 to y_2 and consider $\mathbf{G}'' = (G_1 - x_1 - x_2, G_2 - y_1 - y_2, G_3 - x_1 - x_2 - y_1 - y_2)$. We have $\partial(\mathbf{G}, \mathbf{G}'') \geq 6$ and (3.15) holds for $\mathbf{G}''$ by (3.17), so $\mathbf{G}''$ packs by the minimality of $\mathbf{G}$, and this packing extends to a packing of $\mathbf{G}$.

Since none of $x \in V_1 - v$ is adjacent to v, by (3.18), $d(x) \leq 3$ for every $x \in V_1$, In particular, this yields $\Delta_1 \leq 3$, $\Delta_2 = \max\{1, d_2(w')\} \leq 1 + d_2(w')$, and $\Delta_3 \leq \max\{3, d_3(w')\} \leq 3 + d_3(w')$. Then,

$$\Delta_1 \Delta_2 + \Delta_3 \leq 3(d_2(w') + 1) + (3 + d_3(w')) \leq 3(d(w') + 2).$$

Since $\mathbf{G}$ does not pack, Theorem 3.4.1 implies that $\Delta_1\Delta_2 + \Delta_3 \geq n/2$, so $d(w') \geq \frac{n}{6} - 2$.

By (3.17), $n + 2 - C \geq \mathcal{D} \geq d_3(w') - 4$, so there are at least $C - 6$ non-neighbors of w' in V_1. By (3.18), at most 4 vertices in V_1 have degree 3. Thus there exists a non-neighbor x_0 of w' such that $d(x_0) \leq 2$ and the degrees of the white neighbors of x_0, which could be neighbors of w', as well, also do not exceed 2. If $N_1(x_0) = \emptyset$, then send x_0 to w'. If $N_1(x_0) = \{z_1\}$, then send x_0 to w', z_1 to y_1 and v to y_2. If $N_1(x_0) = \{z_1, z_2\}$ and $z_1z_2 \notin E_1$, then send x_0 to w', z_1 to y_1 and z_2 to y_2. Finally, if $N_1(x_0) = \{z_1, z_2\}$ and $z_1z_2 \in E_1$, then by the choice of x_0, z_1, z_2, these 3 vertices induce a component in $\mathbf{G}$; so we can send x_0 to w', z_1 to y_1 and z_2 to any $y_0 \in Y - y_2$. In all cases, we have deleted at least $\frac{n}{6} - 2$ edges. Since by (3.17), (3.15) also will hold in all cases, we can pack the resulting graph triple, and then extend this to a packing of $\mathbf{G}$, a contradiction. □

Lemma 3.4.5. *If a vertex in V_1 has degree 1, then no vertex in V_2 has degree 1.*

Proof. Suppose $v \in V_1, w \in V_2$ and $d(v) = d(w) = 1$. Then by Lemma 3.4.4, the edges incident to v and w are white. Let $vv' \in E_1$ and $ww' \in E_2$. Let $A_1 = N_1(v') - v$, $A_2 = N_3(v') = N(v') \cap V_2$, $B_1 = N_3(w') = N(w') \cap V_1$, $B_2 = N_2(w') - w$. Let x_0 (respectively, y_0) be a vertex of maximum degree among the vertices in $V_1 - v - v'$ (respectively, in $V_2 - w - w'$).

We obtain graph triple $\mathbf{G}' = (G'_1, G'_2, G'_3)$ by first placing v' on w, v on y_0, deleting the matched pairs, and then adding yellow edges from w' to the vertices in $A_1 \setminus B_1$. If $\mathbf{G}'$ packs, then together with our placement of v' on w and v on y_0 we will have a packing of $\mathbf{G}$. If it does not pack, then by the minimality of $\mathbf{G}$, either (3.15) or (3.16) does not hold for $\mathbf{G}'$. Since $\Delta_1, \Delta_2 \leq \mathcal{D} \leq n - C + 2$ and the white degrees of vertices did not increase, if (3.15) is violated in $\mathbf{G}'$, then by (3.17), $\mathbf{G}'$ has a vertex u with $d'_3(u) = n - 2$. Since $\Delta_3 = \max\{\Delta_{3|1}, \Delta_{3|2}\} \leq \mathcal{D} + 4$, (3.17) implies that $u = w'$. However, $n - 2 \leq d'_3(w') \leq d_1(v') + d_3(w') \leq \Delta_1 + \Delta_{3|2} \leq \mathcal{D} + 4$, a contradiction to (3.17). Thus (3.16) must be violated in $\mathbf{G}'$:

$$F(\mathbf{G}') = e(G'_1) + e(G'_2) + e(G'_3) + \mathcal{D}' \geq 3(n-2) - C + 1. \tag{3.19}$$

Symmetrically, we obtain graph triple $\mathbf{G}'' = (G''_1, G''_2, G''_3)$ by first placing v on w' and x_0 on w, deleting the matched pairs, and then adding yellow edges from v' to the vertices in $B_2 \setminus A_2$. Similarly to (3.19), we derive

$$F(\mathbf{G}'') = e(G''_1) + e(G''_2) + e(G''_3) + \mathcal{D}'' \geq 3(n-2) - C + 1. \tag{3.20}$$

The proof also will require the following claim.

Claim 3.4.6. *If there exist constants a, b such that $d(x_0) \leq a$, $d(y_0) \leq b$, and $C - 3 \geq \max\{2a(b+2), 2(a+2)b\}$, then* $\mathbf{G}$ *packs.*

Proof of Claim 3.4.6. By symmetry, we will assume that $a \geq b$ so that $C-3 \geq 2a(b+2)$. We will construct a packing of $\mathbf{G}$ that maps v to y_0, v' to w. Observe that since $|A_1|+|B_1| \leq (\Delta_1 - 1)+\Delta_{3|2} \leq \mathcal{D}+3 \leq n-C+5$, we may choose a vertex $x \in V_1 - N_1[v'] - N_3[w']$ that we may map to w'. In order to preserve the packing property, we must ensure that white neighbors of x are not mapped to white neighbors of w'. Again, by (3.17), we see that there are at least $C-3$ vertices of $V_2 - N_2[w']$. Since y_0 has maximum degree among all vertices in $V_2 - w'$, the average degree of the vertices in this set is at most b. By Turan's Theorem, we may find an independent set of vertices in $V_2 - N_2[w']$ of size at least $(C-3)/(b+1) \geq 2a$.

Now, let $\{x_1, \ldots, x_{a'}\} = N_1(x)$ be the white neighborhood of x and notice that $a' = d_1(x) \leq d(x_0) \leq a$. Since x_0 was maximal, $d_3(x_i) \leq a-1$, for each $i = 1, \ldots, a'$. Thus, we may successively map each x_i on a non-neighbor y_i chosen from the independent set in $V_2 - N_2[w']$. After each such mapping, we add yellow edges between the white neighbors of x_i and the white neighbors of y_i. This yields a new graph triple $\mathbf{G}^*$ of order $n - a' - 3$. In this new triple, we see that $\Delta_1^* \leq a, \Delta_2^* \leq b$ and, due to the added yellow edges, $\Delta_3^* \leq a + b - 2$. However, this gives

$$2\Delta_1^*\Delta_2^* + 2\Delta_3^* \leq 2ab + 2(a+b-2) \leq 2ab + 4a \leq C \leq n - a' - 3.$$

By Theorem 3.4.1, $\mathbf{G}^*$ packs and this packing extends to a packing of $\mathbf{G}$. This proves the claim. □

Along with this Claim, we will use (3.19) and (3.20) to prove the lemma. Observe that to obtain $\mathbf{G}'$, we deleted $|A_1| + |A_2| + 1$ edges adjacent to v', one edge adjacent to w, $d(y_0)$ edges adjacent to y_0 (though we may have double counted the edge $v'y_0$), and added $|A_1 \setminus B_1|$ new yellow edges adjacent to w'. Thus, by (3.19) and similarly by (3.20),

$$5 \geq \partial(\mathbf{G}, \mathbf{G}') \geq |A_1 \cap B_1| + |A_2| + d(y_0) + 1 + \mathcal{D} - \mathcal{D}'. \tag{3.21}$$

$$5 \geq \partial(\mathbf{G}, \mathbf{G}'') \geq |A_2 \cap B_2| + |B_1| + d(x_0) + 1 + \mathcal{D} - \mathcal{D}''. \tag{3.22}$$

If $\mathcal{D} - \mathcal{D}' \geq -1$ and $\mathcal{D} - \mathcal{D}'' \geq -1$, then $d(x_0), d(y_0) \leq 5$ and we are done by Claim 3.4.6. So by symmetry, we may assume that $\mathcal{D} - \mathcal{D}'' \leq -2$. In particular, since the only vertex in $\mathbf{G}''$ that has increased its degree by more than 1 is v', we have $\mathcal{D}'' = \Delta_2'' + d_3''(v') - 4$. There are two cases.

Case 1: $\mathcal{D} - \mathcal{D}' \leq -2$. In creating $\mathbf{G}'$, the only vertex that has increased its degree by at least 2 is w', so $\mathcal{D}' = \Delta_1' + d_3'(w') - 4$. Observing that $d_3'(w') = |A_1 \cup B_1|$ and plugging this in for $\mathcal{D}'$ and $\mathcal{D}''$, we can sum together (3.21) and (3.22) to get

$$10 \geq 2|A_1 \cap B_1| + 2|A_2 \cap B_2| + d(y_0) + d(x_0) + 2\mathcal{D} - \Delta_1' - \Delta_2'' - |A_1| - |B_2| + 10. \tag{3.23}$$

Since $\mathcal{D} \geq \Delta_1, \Delta_2$, we have $\mathcal{D} \geq |A_1| + 1$ and $\mathcal{D} \geq |B_2| + 1$. Furthermore, since x_0 was a maximum degree vertex in $V_1 - v'$, we have $d(x_0) \geq \Delta_1'$. Similarly, $d(y_0) \geq \Delta_2''$. Inserting these inequalities into (3.23), we get

$$10 \geq 2|A_1 \cap B_1| + 2|A_2 \cap B_2| + 12.$$

This is a contradiction, so the case is proved.

Case 2: $\mathcal{D} - \mathcal{D}' \geq -1$. We see from (3.21) that $5 \geq |A_1 \cap B_1| + |A_2| + d(y_0)$. Note also, that since w' is a vertex in $\mathbf{G}'$, $|B_2| \leq d_2'(w') + 1 \leq \mathcal{D}' - \Delta_{3|1}' + 5 \leq \mathcal{D} - \Delta_{3|1}' + 6$. Next, observe that $d_3''(v') \leq |A_2 \cup B_2|$, so we have

$$\mathcal{D}'' \leq \Delta_2'' + |B_2| + |A_2 \setminus B_2| - 4 \leq \Delta_2'' + \mathcal{D} + |A_2 \setminus B_2| - \Delta_{3|1}' + 2.$$

We now substitute these inequalities into (3.22),

$$\begin{aligned} 5 &\geq |A_2 \cap B_2| + |B_1| + d(x_0) + 1 + \mathcal{D} - \Delta_2'' - \mathcal{D} - |A_2 \setminus B_2| + \Delta_{3|1}' - 2 \\ &\geq 2|A_2 \cap B_2| + |B_1| + d(x_0) - \Delta_2'' - |A_2| + \Delta_{3|1}' - 1. \end{aligned}$$

However, y_0 is a vertex in $\mathbf{G}''$, so $\Delta_2'' \leq d(y_0) + 1$. In particular,

$$d(y_0) + |A_2| + 7 \geq 2|A_2 \cap B_2| + |B_1| + d(x_0) + \Delta_{3|1}'.$$

Finally, recall that $\mathcal{D} - \mathcal{D}' \geq -1$ implies by (3.21) that $5 \geq |A_1 \cap B_1| + |A_2| + d(y_0)$. This gives that $d(y_0) \leq 5$, and when combined with the last inequality, that $d(x_0) \leq 12$. Since $C > 1{,}000$, by Claim 3.4.6, $\mathbf{G}$ packs, a contradiction. □

From now on, by Lemma 3.4.5, we will assume that

$$d(w) \geq 2 \text{ for every } w \in V_2. \tag{3.24}$$

Lemma 3.4.7. $D_1, D_2 \geq 3$.

Proof. Suppose $D_2 \leq 2$, the case where $D_1 \leq 2$ follows similarly. The white components of G_2 are paths and cycles. By Theorem 3.4.1, $D_1 \geq n/6$. Also, by (3.16),

$$\sum_{v \in V_1} d(v) + 2\mathcal{D} \leq 6n - 2C - \sum_{w \in V_2} d(w) < 5n - 2C.$$

Let $v' \in V_1$ have maximum degree in V_1, so that $d(v') \geq n/6$. Since $\mathcal{D} \geq D_1 - 4$, this implies

$$\sum_{v \in V_1 - \{v'\}} d(v) \leq 5n - 2C - d(v') - 2\mathcal{D} \leq 5n - 2C - n/6 - 2(n/6 - 4) < 9n/2 - 2C + 8. \tag{3.25}$$

Consider a vertex $w_0 \in V_2 - N_3(v')$. There are two cases.

Case 1: *The white component containing w_0 is not a triangle.* In this case, w_0 has at most two white neighbors, $w_1, w_2 \in V_2$. (Notice w_2 may not exist). Since $D_2 \leq 2$, there are at most 4 vertices of $V_1 - N_1[v']$ adjacent to $N_2(w_0)$. By (3.25), there are at most 60 vertices of degree at least $n/12 - 6$ in $V_1 - N[v']$. So, there are at least two vertices in $V_1 - N[v']$ that have degree less than $n/12 - 6$ and are not adjacent to $N(w_0)$, call them v_1, v_2. We will map v' to w_0, v_1 to w_1, and (if w_2 exists) v_2 to w_2. Create a new triple $\mathbf{G}' = (G'_1, G'_2, G'_3)$ by deleting these matched pairs and adding new yellow edges from $(N_1(v_1) - v_2)$ to $(N_2(w_1) - w')$ and $(N_1(v_2) - v_1)$ to $(N_2(w_2) - w')$. Since $\mathbf{G}'$ has order at least $n - 3$ and $\mathcal{D} \leq n - C + 2$, we see that (3.15) holds for $\mathbf{G}'$. Notice that w_i has at most one white neighbor other than w', so we have added at most $d_1(v_1) + d_1(v_2)$ new yellow edges. Thus, $\mathbf{G}'$ has at most $e_1 + e_2 + e_3 - d(v') - d(v_1) - d(v_2) + d_1(v_1) + d_1(v_2)$ edges and $\mathcal{D}' \leq \mathcal{D} + d_1(v_1) + d_1(v_2)$. Finally, since $d(v_i) \geq d_1(v_i)$, we have

$$e'_1 + e'_2 + e'_3 + \mathcal{D}' \leq e_1 + e_2 + e_3 + \mathcal{D} - (d(v') - d_1(v_1) - d_1(v_2)). \tag{3.26}$$

If $e'_1 + e'_2 + e'_3 + \mathcal{D}' \leq 3(n-3) - C$, then $\mathbf{G}'$ packs by the minimality of $\mathbf{G}$ and this packing extends to a packing of $\mathbf{G}$. But we have chosen v_1 and v_2 so that $d(v_1), d(v_2) < n/12 - 6$. Since $d(v') \geq n/6$, we have $d(v') - d_1(v_1) - d_1(v_2) \geq 9$ and, by (3.26), $\mathbf{G}'$ packs and this extends to a packing of $\mathbf{G}$, a contradiction.

Case 2: *The white component containing w_0 is a triangle.* Let $w_0w_1w_2$ be a triangle in G_2 and let $d = d_1(v')$. Note that $d \leq \mathcal{D} < n - C + 2$. As before, there are at most 4 vertices in $V_1 - N_1[v']$ adjacent to $\{w_1, w_2\}$. Let $X = V_1 - N_1[v'] - N_3(\{w_1, w_2\})$ and notice that $|X| \geq n - d - 5 \geq C - 7$. If there are nonadjacent vertices $x_1, x_2 \in X$, then we can match v' to w_0, x_1 to w_1, and x_2 to w_2. Since $d(v') \geq n/6$, removing these vertices leaves a smaller graph triple which we can pack by the minimality of $\mathbf{G}$. This packing extends to a packing of $\mathbf{G}$, a contradiction.

On the other hand, if all vertices of X are adjacent to each other, then there are at least $\binom{|X|}{2} \geq 2|X|$ edges in $G_1[X]$. Since v' has d white neighbors, we see that $e_1 + \mathcal{D} \geq 2|X| + 2d \geq 2n - 10$. Finally,

$e_2 + e_3 \geq \frac{1}{2}\sum_{w\in V_2} d(w) \geq n$. So, $e_1 + e_2 + e_3 + \mathcal{D} \geq 3n - 10$, a contradiction. □

Lemma 3.4.8. $\mathcal{D} + \sum_{v\in V_1} d(v) \geq 2n - 12$.

Proof. The sum of degrees of vertices in a component M of G_1 containing a cycle is at least $2|V(M)|$. Thus if $\sum_{v\in V_1} d(v) < 2n - 12$, then G_1 has at least six tree-components, each adjacent to at most one yellow edge. Let H be a smallest such component and vw be the yellow edge incident to $V(H)$, if it exists. Then $s := |V(H)| \leq n/6$. Let $w_1 \in V_2$ with the maximum white degree and begin by finding a permissible vertex v_1 to send to w_1. If vw does not exist, then choose v_1 to be any vertex in $V(H)$. If vw exists and $w_1 \neq w$, then choose $v_1 = v$. Finally, if vw exists and $w_1 = w$, then choose v_1 to be any vertex in $V(H) - v$. Consider H as a rooted tree with root v_1, so that each $x \in V(H) - v_1$ has a unique parent in H. Order the vertices of H: $v_1, \ldots, v_s$ in the Breadth-First order. We now will consecutively place all vertices of H on vertices in V_2. We start by placing v_1 on w_1. Then for every $i = 2, \ldots, s$, if possible, we place v_i on a vertex $w_i \in V_2$ not adjacent to the image $w_{i'}$ of any $v_{i'}$ with $i' < i$, and if not possible, then just on any non-occupied non-neighbor of the image w_j of its parent v_j.

First, we show that we always can choose a vertex to place each v_i. Indeed, otherwise for some $2 \leq i \leq s$, we cannot place v_i and let's call its parent v_j. Then, each vertex of V_2 either is adjacent to w_j or is occupied by one of $v_1, \ldots, v_{i-1}$. If $j = 1$, then because H is a tree obtained via Breadth-First search, $i \leq d_1(v_1) + 1$. Thus in this case, $d_2(w_1) + d_1(v_1) \geq n - 1$ and since $v_1 \in H$, $d_2(w_1) \geq \frac{3}{4}n$. But then

$$\mathcal{D} + \sum_{v\in V_1} d(v) \geq d_2(w_1) + \left(d_1(v_1) + \sum_{v\in V_1 - v_1} d(v)\right) \geq 2n - 2,$$

contradicting our assumption. Otherwise, the host, say $w_j \neq w_1$, of the parent v_j of v_i has at least $n - i + 1$ neighbors in V_2. Then by the choice of w_1, also $\mathcal{D} \geq d_2(w_1) \geq n - i + 1$. Thus the total number of edges incident to w_1 and w_j is at least $d(w_1) + d(w_j) - 1 \geq 2n - 2i + 1$. By Lemma 3.4.4, $e_1 \geq n/2$. So, $\mathcal{D} + (d(w_1) + d(w_2) - 1) + e_1 \geq 3n - 3i + 2 + n/2 \geq 3n$, a contradiction to (3.16). Thus we can place all $v_1, \ldots, v_s$ on the corresponding $w_1, \ldots, w_s$.

Next, we show that for every $i \in \{1, \ldots, s\}$,

$$\text{the number of edges incident to vertices in } W_i = \{w_1, \ldots, w_i\} \text{ is at least } 2i + 1. \tag{3.27}$$

By Lemma 3.4.7, (3.27) holds for $i = 1$. Suppose (3.27) holds for some $i \leq s - 1$. If w_{i+1} is not adjacent to W_i, then (3.27) holds for $i' = i + 1$. Otherwise, by the rules, $W_i \cup N(W_i) \supseteq V_2$ and the total number of edges incident to at least one vertex in W_{i+1} is at least $n - (i + 1) \geq n - s \geq 5n/6 \geq 2(i + 1) + 1$. This proves (3.27).

By (3.27), for $\mathbf{G}' = \mathbf{G} - H - W_s$, $|E(\mathbf{G}')| \le |E(\mathbf{G})| - (s-1) - (2s+1) = |E(\mathbf{G})| - 3s$. Then, $\mathbf{G}'$ does not pack, because $\mathbf{G}$ does not pack, and a packing of $\mathbf{G}'$ would extend to $\mathbf{G}$. By the minimality of $\mathbf{G}$, this yields (3.15) does not hold. Then there exists some vertex x such that $d_j(x) \ge n-s-1$ for some $j = 1, 2, 3$. Hence $\mathcal{D} \ge n - s - 5$.

Now, we wish to say more about H. First, H cannot be a single vertex by Lemma 3.4.4. Suppose $H = K_2$. By Lemma 3.4.7, $d(w_1) \ge 3$. By (3.24), $d(w_2) \ge 2$. In this case, the triple $\mathbf{G}' = \mathbf{G} - H - w_1 - w_2$ has at most $e_1 + e_2 + e_3 - 6$ edges. So by (3.17) and the minimality of $\mathbf{G}$, triple $\mathbf{G}'$ packs, and this packing extends to $\mathbf{G}$ by placing v_1 on w_1 and v_2 on w_2. Therefore, $s \ge 3$ and the average degree of H is at least $\frac{4}{3}$. In fact, since H was the smallest tree component, all of G_1 has average degree at least $4/3$. Thus,

$$\mathcal{D} + \sum_{v \in V_1} d(v) \ge (n-s-5) + \frac{4}{3}n = 2n + \frac{n}{3} - s - 5 \ge 2n + \frac{n}{3} - \frac{n}{6} - 1 > 2n,$$

contradicting our assumption. □

The next lemma uses Lemma 3.4.8 and its proof is similar.

Lemma 3.4.9. *Every white tree-component in G_1 has at least $C/3$ vertices.*

Proof. Suppose T is a smallest white tree-component in G_1 and $s := |V(T)| \le C/3$. By Lemma 3.4.7, G_2 has a vertex w of degree at least 3. If T contains a vertex $v \notin N(w)$, then let $v_1 = v$ and $w_1 = w$. Otherwise, let v_1 be any vertex of T and w_1 be any non-neighbor of v_1 in G_2 (such w_1 exists by (3.17)). Now we repeat some arguments from the proof of Lemma 3.4.8.

Consider T as a rooted tree with root v_1, so that each $x \in V(T) - v_1$ has a unique parent in T. Order the vertices of T: $v_1, \ldots, v_s$ in the Breadth-First-Order. We will consecutively place all vertices of T on vertices in V_2. We start by sending v_1 to w_1. For every $i = 2, \ldots, s$, if possible, we send v_i to a vertex $w_i \in V_2$ not adjacent to the image $w_{i'}$ of any $v_{i'}$ with $i' < i$. If this is not possible, then just send v_i to any nonoccupied non-neighbor of the image w_j of its parent v_j.

If we cannot choose a vertex to place some v_i, then each vertex of V_2 either is a neighbor of both v_i and w_j, where v_j is the parent of v_i, or is occupied by one of $v_1, \ldots, v_{i-1}$. Thus $d_2(w_j) + d_3(v_i) + i - 1 \ge n$. Since $d_2(w_j) + d_3(v_i) + i - 1 \le \mathcal{D} + 4 + C/3 - 1$, this contradicts (3.17). Thus we can place all $v_1, \ldots, v_s$ on some $w_1, \ldots, w_s$.

Let $W_i = \{w_1, \ldots, w_i\}$. If $d(w_1) \ge 3$, then (3.27) holds for $i = 1$. So we show that (3.27) holds for each $i \le s$ exactly as in the proof of Lemma 3.4.8. In this case, for $\mathbf{G}' = \mathbf{G} - T - W_s$, $|E(\mathbf{G}')| \le |E(\mathbf{G})| - (s-1) - (2s+1) = |E(\mathbf{G})| - 3s$. If $d(w_1) = 2$, then w (and each vertex of degree at least 3 in V_2)

is adjacent to each vertex in T and, in addition, we have an analog of (3.27) with $2i$ in place of $2i+1$. So again, $|E(\mathbf{G}')| \leq |E(\mathbf{G})| - 3s$. By the choice of $\mathbf{G}$, the triple $\mathbf{G}'$ does not pack. By the minimality of $\mathbf{G}$, this yields that (3.15) does not hold. Then $\mathcal{D} \geq n - s - 5$, contradicting (3.17). □

Claim 3.4.10. *For $i \in \{1,2\}$ and $u \in V_i$ there are at least $\frac{2C-16}{3}$ vertices in $V_i - N_i[u]$ of degree at most 3.*

Proof. We will use two cases.

Case 1: $i = 1$. By (3.24), $\sum_{w \in V_2} d(w) \geq 2n$. So since $\mathcal{D} \geq d_1(u)$, we have

$$\sum_{v \in V_1 - N_1[u]} d(v) + 4d_1(u) \leq \sum_{v \in V_1 - N_1[u]} d(v) + \sum_{v \in N_1[u]} d(v) + 2d_1(u) \leq 4n - 2C.$$

Therefore, $\sum_{v \in V_1 - N_1[u]} d(v) \leq 4(|V_1| - |N_1[u]|) + 4 - 2C$.

Case 2: $i = 2$. Since $\mathcal{D} \geq d_2(u)$,

$$\begin{aligned}
\sum_{v \in V_2 - N_2[u]} d(v) + 4d_2(u) &\leq \sum_{v \in V_2 - N_2[u]} d(v) + 3d(u) + d_2(u) \\
&\leq \sum_{v \in V_2 - N_2[u]} d(v) + \sum_{v \in N_2[u]} d(v) + d_2(u) \\
&\leq 4n + 12 - 2C,
\end{aligned}$$

where $\mathcal{D} + \sum_{v \in V_2} d(v) \leq 4n + 12 - 2C$ by Lemma 3.4.8. Hence,

$$\sum_{v \in V_2 - N_2[u]} d(v) \leq 4\left(|V_2| - |N_2[u]|\right) + 16 - 2C.$$

Thus, in both cases,

$$\sum_{v \in V_i - N_i[u]} d(v) \leq 4(|V_i| - |N_i[u]|) + 16 - 2C,$$

and the average degree of vertices in $V_i - N_i[u]$ is less than four. Since every vertex has positive degree, $V_i - N_i[u]$ contains at least $\frac{2C-16}{3}$ vertices of degree strictly less than 4. □

For $i \in \{1,2\}$ and every $v \in V_i$, define the *shared degree* of v, $\mathrm{sd}(v)$, as follows. If $d_i(v) < 15$, then $\mathrm{sd}_i(v) := d_i(v) + \frac{2}{3}|\{x \in N_i(v) : d_i(x) \geq 15\}|$ and $\mathrm{sd}(v) := \mathrm{sd}_i(v) + d_3(v)$. If $d_i(v) \geq 15$, then $\mathrm{sd}_i(v) := d_i(v) - \frac{2}{3}|\{x \in N_i(v) : d_i(x) < 15\}|$ and $\mathrm{sd}(v) := \mathrm{sd}_i(v) + d_3(v)$. By definition, (a) $\sum_{v \in V_i} \mathrm{sd}_i(v) = 2e_i$ and $\sum_{v \in V_i} \mathrm{sd}(v) = 2e_i + e_3$, (b) $\mathrm{sd}(v) \geq d(v)$ if $d_i(v) < 15$, (c) $\mathrm{sd}(v) \geq d(v)/3 \geq 5$ if $d_i(v) \geq 15$, and (d) $3\,\mathrm{sd}(v)$ is an integer for every $v \in V_i$.

Claim 3.4.11. *For $i \in \{1,2\}$ and $u \in V_i$, there is a vertex $v \in V_{3-i} - N[u]$ of shared degree at most 4.*

Proof. Let $S = V_{3-i} - N(u)$ and $s = |S|$. Suppose that $\mathrm{sd}(v) > 4$ for every $v \in S$. Then by the property (d) of shared degrees, $\sum_{w\in S} \mathrm{sd}(w) \geq \frac{13}{3}s$. By Lemma 3.4.4 and properties (b) and (c) of shared degrees, $\sum_{x\in V_{3-i}-S} \mathrm{sd}_{3-i}(x) \geq n - s$ and, since each vertex in $V_{3-i} - S$ is also a yellow neighbor of u, we have that $\sum_{x\in V_{3-i}-S} \mathrm{sd}(x) \geq 2(n - s)$. Combining these two sums, we see that $2e_{3-i} + e_3 = \sum_{x\in V_{3-i}} \mathrm{sd}(x) \geq \frac{13}{3}s + 2(n - s)$.

If $i = 1$, then by Lemma 3.4.9, $e_i = e_1 \geq n(1 - \frac{3}{C})$. If $i = 2$, then $\sum_{x\in V_i - u} d(x) \geq 2n - 2$. In both cases the yellow neighbors of u were not included in the sum, so we have that

$$\sum_{x\in V_i} d(x) \geq 2n\left(1 - \frac{3}{C}\right) + (n - s).$$

By definition, $\mathcal{D} \geq (d_3(u) - 4) + \Delta_{3-i} \geq n - s - 3$. These inequalities and property (a) of shared degrees yield,

$$\begin{aligned}2(e_1 + e_2 + e_3 + \mathcal{D}) &\geq 2n\left(1 - \frac{3}{C}\right) + (n - s) + 2(n - s) + \frac{13}{3}s + 2(n - s - 3)\\ &= \left(7 - \frac{6}{C}\right)n - \frac{2}{3}s - 6 > 6n - 6.\end{aligned}$$

By (3.16), this is at most $6n - 2C$, a contradiction. □

Lemma 3.4.12. *Let* $F := \sqrt{\dfrac{C}{11}} = 195$*. Then* $D_1, D_2 \geq F$*.*

Proof. Suppose that $D_1 \leq D_2$ and $D_1 < F = \sqrt{C/11}$; the proof for D_2 is similar. By Theorem 3.4.1, $D_2F + D_2 \geq D_2D_1 + \max\{D_1, D_2\} \geq n/2$, so $D_2 \geq n/(2F + 2)$. Consider a vertex $w \in V_2$ of maximum degree. By the choice, $d(w) \geq D_2$. By (3.17), $d_2(w) < n-C+2$. By Claim 3.4.11, V_1 contains a non-neighbor v of w with $\mathrm{sd}(v) \leq 4$. In particular, by the definition of shared degree, $d(v) \leq 4$. Let $N_1(v) := \{v_1, \ldots, v_s\}$. We wish to find an independent set $\{w_1, \ldots, w_s\} \subset V_2 - N_2[w]$ such that each w_i has degree at most 3 and is not adjacent to v_i.

By Claim 3.4.10, at least $\frac{2C-16}{3}$ vertices in $V_2 - N_2[w]$ have degree at most 3. At most $F - 1$ of them are adjacent to v_1. So, we can choose $w_1 \in V_2 - N_2[w] - N(v_1)$ with $d(w_1) \leq 3$. Continuing in this way for $j = 2, \ldots, s$, at least $\frac{2C-16}{3} - 4(j - 1)$ vertices in $V_2 - N_2[w] - \bigcup_{i=1}^{j-1} N[w_i]$ have degree at most 3. Again, at most $F - 1$ of them are adjacent to v_j. Since $s \leq 4$ and $\frac{2C}{3} - 5 - 4(s - 1) - F \geq \frac{2C-16}{3} - 17 - F > 0$, we can choose $w_j \in V_2 - N_2[w] - \bigcup_{i=1}^{j-1} N[w_i] - N(v_j)$ with $d(w_j) \leq 3$.

We now create a new graph triple $\mathbf{G'} = (G'_1, G'_2, G'_3)$ by removing $\{w, v, w_1, \ldots, w_s, v_1, \ldots, v_s\}$ and adding new yellow edges between $N_1(v_i)$ and $N_2(w_i)$ for each $1 \leq i \leq s$ and then deleting the matched pairs. Through this process, since the set $\{w_1, \ldots, w_s, w\}$ is independent, we have removed at least $d(v) + d(w) +$

$\sum_{i=1}^{s}(d_1(v_i)-1+d_2(w_i))-|E(G_1[N_1(v)])|$ edges, and added at most $3\sum_{i=1}^{s}(d_1(v_i)-1)-2|E(G_1[N_1(v)])|$ edges. We have increased $\mathcal{D}$ by at most $\max\{\max_i(d_1(v_i)-1),\max_j d_2(w_j)\}\leq F-1$. Thus, we have

$$\partial(\mathbf{G},\mathbf{G}')\geq d(v)+d(w)+\sum_{i=1}^{s}d_2(w_i)-2\sum_{i=1}^{s}(d_1(v_i)-1)-F+|E(G_1[N_1(v)])|+1,$$

and therefore

$$\partial(\mathbf{G},\mathbf{G}')\geq d(w)-2\sum_{i=1}^{s}(d_1(v_i)-1)-F. \tag{3.28}$$

If $s\leq 2$, then $\sum_{i=1}^{s}(d_1(v_i)-1)\leq 2F-2$. If $s=3$, then since $\mathrm{sd}(v)\leq 4$, at least two neighbors of v have degree less than 15, so in this case $\sum_{i=1}^{s}(d_1(v_i)-1)\leq 2\cdot 13+F-1=25+F\leq 2F-2$. If $s=4$, then since $\mathrm{sd}(v)\leq 4$, all 4 neighbors of v have degree less than 15. So in this case $\sum_{i=1}^{s}(d_1(v_i)-1)\leq 4\cdot 13\leq 2F-2$. So since $d(w)\geq D_2\geq \frac{n}{2(F+1)}\geq\frac{C}{2F+2}$, by (3.28) and the definitions of C and F,

$$\partial(\mathbf{G},\mathbf{G}')\geq\frac{C}{2F+2}-2(2F-2)-F=\frac{C}{2F+2}-5F+4\geq 15\geq 3(s+1).$$

It follows that (3.16) holds for $\mathbf{G}'$. Also by above, $\mathcal{D}'-\mathcal{D}\leq F-1$. Thus by (3.17),

$$\mathcal{D}'\leq\mathcal{D}+F-1\leq n+2-C+F-1=(n'+s+1)+1-C+F<n'-5,$$

and (3.15) holds for $\mathbf{G}'$. So $\mathbf{G}'$ packs by the minimality of $\mathbf{G}$, and then $\mathbf{G}$ also packs, a contradiction. □

Lemma 3.4.13. *Let* $K:=\dfrac{F}{13}=15$. *Let* $i\in\{1,2\}$ *and* $v\in V_i$ *with* $d(v)=t\leq 4$ *be not adjacent to some vertex* $w\in V_{3-i}$ *of degree at least* F.

(a) Then v *has a neighbor in* V_i *of degree at least* $\frac{13K}{3t+1}$.

(b) Moreover, if $2\leq t\leq 3$ *and* v *has* $t-1$ *neighbors of degree at most* 2*, then* v *has a neighbor in* V_i *of degree at least* $\frac{13K}{5}$.

Proof. Suppose Statement (a) of the lemma fails for $i=1$ (the proof for $i=2$ is the same). This means that for a vertex $v\in V_1$ of degree t in $\mathbf{G}$, all of its neighbors in V_1 have degree less than $\frac{13K}{3t+1}$ and some non-neighbor $w\in V_2$ of v has $d(w)\geq F$. Let $N_1(v):=\{v_1,\ldots,v_s\}$. By definition, $s\leq t\leq 4$. We wish to find an independent set $\{w_1,\ldots,w_s\}\subset V_2-N_2[w]$ such that each w_i has degree at most 3 and is not adjacent to v_i.

By Claim 3.4.10, at least $\frac{2C-16}{3}$ vertices in $V_2-N_2[w]$ have degree at most 3. Less than $\frac{13K}{3t+1}-1$ of them are adjacent to v_1. So, we can choose $w_1\in V_2-N_2[w]-N(v_1)$ with $d(w_1)\leq 3$. Continuing in this way for $j=2,\ldots,s$, at least $\frac{2C-16}{3}-4(j-1)$ vertices in $V_2-N_2[w]-\bigcup_{i=1}^{j-1}N[w_i]$ have degree at most 3. Again less than $\frac{13K}{3t+1} - 1$ of them are adjacent to v_j. Since $\frac{2C-16}{3} - 4s - \frac{13K}{3t+1} \geq \frac{2C-16}{3} - 16 - \frac{13K}{3t+1} > 0$, we can choose $w_j \in V_2 - N_2[w] - \bigcup_{i=1}^{j-1} N[w_i] - N(v_j)$ with $d(w_j) \leq 3$.

Finally, we can map v to w, vertices $v_1, \dots, v_s$ to $w_1, \dots, w_s$, respectively, delete the matched pairs, and for each pair $\{v_i, w_i\}$, introduce yellow edges between the remaining vertices of $N_1(v_i)$ and $N_2(w_i)$. This creates a new graph triple $\mathbf{G}' = (G'_1, G'_2, G'_3)$. During this process, we have deleted at least $d(w) + d(v)$ edges, added in strictly less than $3s(\frac{13K}{3t+1} - 1)$ new yellow edges, and increased $\mathcal{D}$ by at most $\max\{3, \max_i\{d_1(v_i) - 1\}\} \leq \frac{13K}{3t+1} - 1$. Therefore since $F = 13K$,

$$\begin{aligned} \partial(\mathbf{G}, \mathbf{G}') &> d(v) + d(w) - (3s+1)\left(\frac{13K}{3t+1} - 1\right) \\ &\geq s + d(w) - 13K + (3s+1) \\ &\geq F - 13K + (4s+1) \geq 3s + 2. \end{aligned} \tag{3.29}$$

Now, we need $\partial(\mathbf{G}, \mathbf{G}') \geq 3s + 3$ but since we added *strictly* less than $3s(\frac{13K}{3t+1} - 1)$ yellow edges, we have a strict inequality which, in combination with the fact that both $\partial(\mathbf{G}, \mathbf{G}')$ and $3s+2$ are integers, in fact gives $\partial(\mathbf{G}, \mathbf{G}') \geq 3s + 3$. Since $\partial(\mathbf{G}, \mathbf{G}')$ is sufficiently large and $\mathbf{G}$ is a minimal counterexample, $\mathbf{G}'$ packs unless (3.15) is violated. However, by (3.17), this violation would have to occur at some vertex in some $N_1(v_i)$ or $N_2(w_i)$ but the degrees of these vertices only increase by at most 3 or $(\frac{13K}{3t+1} - 1) < 4K$, neither of which could get us to have a vertex of degree $(n - s - 1) - 2 \geq n - 7$. Hence, $\mathbf{G}'$ packs and this packing extends to a packing of $\mathbf{G}$, as we constructed above. This proves (a).

To prove (b), we repeat the argument of (a) with $\frac{13K}{5}$ in place of $\frac{13K}{3t+1}$ until we count the number of added yellow edges. We have added less than $3\left((s-1) + \frac{13K}{5}\right)$ edges and increased $\mathcal{D}$ by at most $\frac{13K}{5} - 1$. So, instead of (3.29), we will have

$$\begin{aligned} \partial(\mathbf{G}, \mathbf{G}') &> d(v) + d(w) - 3(s-1) - 4\left(\frac{13K}{5} - 1\right) \\ &\geq s + 13K - 3(s-1) - \frac{4 \cdot 13K}{5} + 4 \\ &= \frac{13K}{5} - 2s + 7 > 3s + 3. \end{aligned}$$

Then again we simply repeat the last paragraph of the proof of (a). □

3.4.3 At Most One Vertex in V_1 is a donor

Recall that by Lemma 3.4.5 we assume (see (3.24)) that V_2 has no vertices of degree 1. A *donor* is a vertex in V_1 adjacent to at least two vertices of degree 1. The goal of this section is to prove that V_1 contains at most one donor.

Lemma 3.4.14. *Suppose V_1 contains donors v and v'. If $w \in V_2$ with $d(w) = 2$, then $N(w) \subset V_2$ and $d(w') \geq 2K$ for each $w' \in N(w)$.*

Proof. Suppose the lemma fails for some $w \in V_2$ with $d(w) = 2$. Let $x, y \in V_1$ be degree one neighbors of v and let $x', y' \in V_1$ be degree one neighbors of v'. By Lemma 3.4.13, $d(v), d(v') \geq 3K$.

Case 1: $N(w) = \{w_1, w_2\} \subset V_2$. By symmetry, assume $d(w_2) < 2K$. Begin by mapping x and y to w_1 and w_2 , respectively, and adding new yellow edges from $N_2(w_1) \cup N_2(w_2) - \{w\}$ to v. Since v is the only neighbor of x and y, this assignment is permitted and adding the yellow edges ensures that any permissible extension of the mapping will not violate the packing property. After mapping x and y, w is adjacent only to v and so v' may be mapped to w. This in turn causes x' and y' to be newly isolated vertices. After removing these 3 pairs of vertices and adding the yellow edges, let $z \in V_2 - \{w, w_1, w_2\}$ be the vertex of V_2 of highest degree and map x' to z.

We now have a new graph triple $\mathbf{G}' := (G_1', G_2', G_3')$. Note that $\Delta_1', \Delta_2' \leq n' - 2$ since (3.17) holds for $\mathbf{G}$ so that (3.15) is only violated if $d_3'(v) = n - 4$. However,

$$d_3'(v) \leq (d_3(v) + d_2(w_1)) + d_2(w_2) \leq (\mathcal{D} + 4) + 2K \leq n - C + 6 + 2K < n - 4,$$

so (3.15) is satisfied for $\mathbf{G}'$ as well. Now, we will consider $\partial(\mathbf{G}, \mathbf{G}')$. In particular, we have deleted at least $d(w_1) + d(w_2) - \|w_1, w_2\|$ edges adjacent to w_1 and w_2 and exactly 2 edges adjacent to x and y. We then added at most $(d_2(w_1) - 1) + (d_2(w_2) - 1) - |N_2(w_1) \cap N_2(w_2) - \{w\}| - 2\|w_1, w_2\|$ yellow edges. Finally, we deleted at least $d(v') - 1 - \|v', \{w_1, w_2\}\|$ edges adjacent to v and at least $d(z) - \max\{0, \|z, \{w_1, w_2\}\| - 1\}$ edges adjacent to z. To see this, note that if $\|z, \{w_1, w_2\}\| \neq 0$, then we save one additional edge, since vz must now be a yellow edge in the modified graph (either $vz \in E_3$ and we didn't need to add it to begin with, or it was added and the degree of z grew by one before we deleted it). In any event, $|N_2(w_1) \cap N_2(w_2) - \{w\}| - \max\{0, \|z, \{w_1, w_2\}\| - 1\} \geq 0$. Thus,

$$d(w_1) + d(w_2) + \|w_1, w_2\| \geq d_2(w_1) + d_2(w_2) + \|v', \{w_1, w_2\}\|.$$

Therefore, the total change in the number of edges is:

$$e(\mathbf{G}) - e(\mathbf{G}') \geq d(v') + d(z) + 1. \tag{3.30}$$

Next, consider the difference $\mathcal{D} - \mathcal{D}'$. If $\mathcal{D} - \mathcal{D}' \geq -1$, then $\partial(\mathbf{G}, \mathbf{G}') \geq d(v') + d(z) \geq 12$ and $\mathbf{G}'$ packs by the inductive assumption. If $\mathcal{D} - \mathcal{D}' \leq -2$, then we must have that $\mathcal{D}' = d_3'(v) + \Delta_2' - 4$. In particular, since $d(z) \geq \Delta_2'$, $\Delta_2 \geq d_2(w_1)$, and $d_3(v) - d_3'(v) \geq 2 - d_2(w_1) - d_2(w_2)$,

$$\mathcal{D} - \mathcal{D}' \geq 2 - d_2(w_1) - d_2(w_2) + d_2(w_1) - d(z) = 2 - d_2(w_2) - d(z).$$

Combining this with (3.30) , we see that

$$\partial(\mathbf{G}, \mathbf{G}') \geq (d(v') + d(z) + 1) + (2 - d_2(w_2) - d(z)) = d(v') - d_2(w_2) + 3.$$

Since $d(v') \geq 3K$ and $d(w_2) \leq 2K$, we have $\partial(\mathbf{G}, \mathbf{G}') \geq 12$. By the minimality of $\mathbf{G}$, we conclude that $\mathbf{G}'$ packs. And we can extend any packing of $\mathbf{G}'$ to a packing of $\mathbf{G}$.

Case 2: $N_2(w) = \{w'\}$. This case follows in a similar fashion to Case 1. Since $d_3(w) = 1$, we may assume that $v' \notin N(w)$. We begin by mapping x to w' and adding new yellow edges from v to $N_2(w') - w$. We then map v' to w and choose a remaining vertex $z \in V_2$ of maximum degree to have x' map to z. Then we delete the matched pairs. This process creates a new graph triple $\mathbf{G}'' := (G_1'', G_2'', G_3'')$. Again, the only way (3.15) is violated is if $d_3'(v) = n - 3$, but this is not the case, since

$$d_3('(v) \leq d_3(v) + d_2(w') \leq \mathcal{D} + 4 \leq n + 6 - C < n - 3.$$

During this process, we removed $d(w')$ edges adjacent to w', one edge adjacent to x, one yellow edge adjacent to w, at most $d(v') - 1 - \|v', w'\|$ edges adjacent to v', and $d(z) - \|w', z\|$ edges adjacent to z. We have added in $d_2(w') - 1 - \|w', z\|$ new yellow edges. Since $d(w') \geq d_2(w') + \|v', w\|$, we see that:

$$e(\mathbf{G}) - e(\mathbf{G}'') \geq d(v') + d(z) + 2.$$

As in Case 1, if $\mathcal{D} - \mathcal{D}' \geq -1$, then $\partial(\mathbf{G}, \mathbf{G}') \geq d(v') + d(z) \geq 12$ and $\mathbf{G}''$ packs by the inductive assumption. If $\mathcal{D} - \mathcal{D}' \leq -2$, then we must have that $\mathcal{D}' = d_3'(v) + \Delta_2' - 4$. Since $d(z) \geq \Delta_2'$, $\Delta_2 \geq d_2(w')$, and $d_3(v) - d_3'(v) \geq 1 - d_2(w')$, we must have that

$$\mathcal{D} - \mathcal{D}' \geq 1 - d_2(w') + d_2(w') - d(z) = 1 - d(z).$$

Thus,

$$\partial(\mathbf{G}, \mathbf{G}') \geq (d(v') + d(z) + 1) + (1 - d(z)) \geq d(v') + 2 \geq 9.$$

By the minimality of $\mathbf{G}$, triple $\mathbf{G}'$ has a packing, which we can extend to a packing of $\mathbf{G}$. □

Corollary 3.4.15. *Suppose V_1 contains donors v and v'. Then* $2e_2 + e_3 = \sum_{v \in V_2} d(v) \geq 3n$.

Proof. Consider the following discharging. For each vertex $v \in V_2$, assign v charge $d(v)$. The total charge allocated is $\sum_{v \in V_2} d(v) = 2e_2 + e_3$. Now, each vertex of degree at least 6 will give charge $\frac{1}{2}$ to each neighbor and save $d(v)/2 \geq 3$ for itself. By Lemma 3.4.14, each vertex of degree 2 is adjacent to two vertices in V_2 with degree at least $2K \geq 30$. Thus, after discharging each vertex has charge at least 3. So the total charge is at least $3n$ and $2e_2 + e_3 \geq 3n$, as needed. □

Remark 3.4.16. Suppose V_1 contains donors v and v'. If $w \in V_2$ with $d(w) = 3$ and $v'w \notin E(\mathbf{G})$, then w has a neighbor in V_2 of degree at least $K + 1$.

Proof. If w has no yellow neighbors, this follows from Lemma 3.4.13. Otherwise, suppose the remark fails for some $w \in V_2$ with $d(w) = 3$. Then each of the neighbor(s) w_1 and w_2 (if it exists) of w in V_2 has degree at most K. Map w to v' and map two degree one neighbors of v to w_1 and w_2. Next, form a new graph triple $\mathbf{G}'$ by adding new yellow edges from v to $W := N_2(w_1) \cup N_2(w_2) - \{w, w_1, w_2\}$ and deleting the previously matched pairs. We have deleted at least $d(v') + 2 + d_2(w_1) + d_2(w_2) - \|w_1, w_2\|$ edges and added $|W|$ new yellow edges. We have increased $\mathcal{D}$ by at most $|W|$. Since $d(w_1) + d(w_2) - \|w_1, w_2\| - 1 \geq |W|$ (in fact, it is at least $|W| + 1$ if w_2 exists), $\partial(\mathbf{G}, \mathbf{G}') \geq d(v') + 3 - |W|$. Now $|W| \leq 2K - 2$ and $d(v') \geq 3K$, so that $\partial(\mathbf{G}, \mathbf{G}') \geq 12$. In particular, by the minimality of $\mathbf{G}$, $\mathbf{G}'$ has a packing, and it extends to a packing of $\mathbf{G}$, a contradiction. □

Lemma 3.4.17. *Suppose V_1 contains donors v and v'. Then* $\mathcal{D} \leq \frac{9n}{4K}$.

Proof. Suppose $\mathcal{D} > \frac{9n}{4K}$. By Lemma 3.4.9, $e_1 \geq n(1 - 3/C)$.

Consider the following discharging on $V_2 \cup E_3$. The initial charge, $ch(v)$, of every $v \in V_2$ is $d(v)$ and of every edge in E_3 is 1. The total sum of charges, $ch(w)$, over $w \in V_2 \cup E_3$ is $2(e_2 + e_3)$. We use two rules.

(R1) Each vertex $w \in V_2$ of degree at least 5 gives to every neighbor in V_2 charge $\frac{d(w)-4}{d(w)}$.

(R2) Each edge in E_3 gives charge 1 to its end in V_2.

Let $ch^*(w)$ denote the new charge of $w \in V_2 \cup E_3$. By (R2), $ch^*(w) = 0$ for every $w \in E_3$. By (R1), if $w \in V_2$ and $d(w) \geq 4$, then $ch^*(w) \geq 4$. If $d(w) = 3$ then by (R1), (R2) and Lemma 3.4.13, $ch^*(w) \geq 3 + (1 - \frac{4}{K})$. If $d(w) = 2$ then by Lemmas 3.4.13 and 3.4.14,

$$ch^*(w) \geq 2 + 2(1 - \frac{2}{K}) = 4 - \frac{4}{K}.$$

Since the total sum of charges did not change, we conclude that

$$2(e_2+e_3)=\sum_{w\in V_2} ch^*(w)\geq 4n\left(1-\frac{1}{K}\right).$$

It follows that

$$\begin{aligned} e_1+e_2+e_3+\mathcal{D} &\geq n\left(1-\frac{3}{C}\right)+n\left(2-\frac{2}{K}\right)+n\left(\frac{9}{4K}\right) \\ &\geq 3n+n\left(-\frac{3}{C}+\frac{1}{4K}\right). \end{aligned}$$

Since $4K\leq \frac{C}{3}$, this contradicts (3.16). ☐

For $v\in V_1$, let $L(v)$ be the set of neighbors of v of degree 1.

Lemma 3.4.18. *Suppose V_1 contains donors v and v'. Then $|L(x)|\leq d(x)/2$ for every $x\in V_1$.*

Proof. Suppose $x\in V_1$, $\ell=|L(x)|>d(x)/2$ and $L(x)=\{x_1,\ldots,x_\ell\}$. By Lemma 3.4.13, $d(x)\geq K$. Thus, x is a donor, so we may assume $x=v$.

Case 1: *There is a vertex $w\in V_2-N_3(v)$ with $d_2(w)\leq 2$.* Let w_1 be a white neighbor of w and, if it exists, let w_2 be the other white neighbor of w. We wish to find a vertex in $V_2-\{w,w_1,w_2\}$ with low degree that is adjacent to none of w_1, w_2, or v'. By Lemma 3.4.17 and since $K=15$, we have $\mathcal{D}\leq\frac{9n}{4K}=\frac{3n}{20}$. By definition, $d_2(w_1)+(d_3(v')-4)\leq\mathcal{D}$. Therefore,

$$|V_2-N[\{w_1,w_2,v'\}]|\geq(n-3)-\mathcal{D}-(\mathcal{D}+4)\geq\frac{14n}{20}-7\geq\frac{n}{2}.$$

Since $\sum_{w\in V_2} d(v)<4n$ by Lemma 3.4.8 and (3.16), the average degree of the vertices in $V_2-N[\{w_1,w_2,v'\}]$ is less than 8. So, there exists a vertex $w'\in V_2-N[\{w_1,w_2,v'\}]$ with $d(w')\leq 7$.

Construct a packing in the following way. Since $\ell\geq\frac{13}{8}K>7$, we may send $x_1,\ldots,x_{d_2(w')}$ to the white neighbors of w'. Send two degree 1 neighbors of v' to w_1 and w_2. Finally, send v to w and v' to w'. Let $\mathbf{G}'$ be obtained by deleting the matched pairs. Then $n-n'\leq 11$. By Lemma 3.4.13, we have deleted at least $d(v)+d(v')-\|v,v'\|\geq\frac{13}{2}K-1\geq 36$ edges and (3.15) still holds, so $\mathbf{G}'$ packs. This packing extends to a packing of $\mathbf{G}$, a contradiction.

Case 2: *Every vertex $w\in V_2-N_3(v)$ has $d_2(w)\geq 3$.* If there is a vertex $w\in V_2$ with $d(w)=2$, then $N(w)\subset V_2$ by Lemma 3.4.14 and we have Case 1. So, $d(w)\geq 3$ for all $w\in V_2$. If every vertex in

$X := V_1 - N_1[v] - N_1[v']$ has degree at least 3, then

$$\begin{aligned}\sum_{x\in V_1} d(x) + 2\mathcal{D} &= \sum_{x\in N_1(v)\cup N_1(v')} d(x) + \sum_{y\in X} d(y) + d(v) + d(v') + 2\mathcal{D} \\ &\geq d_1(v) + d_1(v') + 3(n-2-d_1(v)-d_1(v')) + d(v) + d(v') + 2\mathcal{D} \qquad (3.31) \\ &\geq 3n-6.\end{aligned}$$

Since every vertex in V_2 has degree at least 3, we get

$$\sum_{x\in V} d(x) + 2\mathcal{D} \geq (3n-6) + 3n \geq 6n-6,$$

a contradiction to (3.16). So there is a vertex $v_0 \in V_1 - N_1[v] - N_1[v']$ with $d(v_0) \leq 2$.

By Lemma 3.4.8 and (3.16), $\sum_{v\in V_2} d(v) + \mathcal{D} \leq 4n - 2C + 12$ and so there are at least $2C + \mathcal{D} - 12$ vertices of degree 3 in V_2. Moreover, since $d_3(v) \leq \mathcal{D} + 4$, there is a vertex $w \in V_2 - N_3(v)$ with $d(w) = 3$. By Case 1, all neighbors of w are white so let $\{w_1, w_2, w_3\} = N_2(w)$ with

$$d_2(w_1) \geq d_2(w_2) \geq d_2(w_3) \geq 3. \qquad (3.32)$$

Similarly to Case 1, we wish to find a vertex in V_2 with low degree that is adjacent to none of w_1, w_2, w_3, v'. As in Case 1, we use $d_2(w_1) + (d_3(v') - 4) \leq \mathcal{D}$. This yields that

$$|V_2 - N[\{w_1, w_2, w_3, v'\}]| \geq (n-4) - 2\mathcal{D} - (\mathcal{D}+4) \geq \frac{11n}{20} - 8 \geq \frac{n}{2}.$$

Since $\sum_{w\in V_2} d(v) < 4n$ by Lemma 3.4.8 and (3.16), the average degree of $V_2 - N[\{w_1, w_2, w_3, v'\}]$ is less than 8 and there exists a vertex w' in this set with degree at most 7.

Let j be the largest index such that $v_0 w_j \notin E_3$ and $j \leq 3$. Since $d(v_0) \leq 2$ and v_0 has a neighbor in V_1, $\|v_0, \{w_1, w_2, w_3\}\| \leq 1$. So, $j \geq 2$.

Since $\ell \geq \frac{13}{8}K > 7$, we may send $x_1, \ldots, x_{d_2(w')}$ to the white neighbors of w'. Send two degree 1 neighbors of v' to the vertices in $\{w_1, w_2, w_3\} - w_j$ and v_0 to w_3. Send v to w and v' to w'. Finally, add yellow edges between the white neighbors of v_0 and the white neighbors of w_j. Delete the matched pairs. The resulting triple $\mathbf{G}'$ has order $n - 5 - d_2(w')$. We added at most $d_1(v_0)(d_2(w_j)-1) \leq 2(d_2(w_j)-1)$ yellow edges, and

$$\mathcal{D}' \leq \mathcal{D} + \max\{2, d_2(w_j)-1\} \leq 2\mathcal{D} - 1. \qquad (3.33)$$

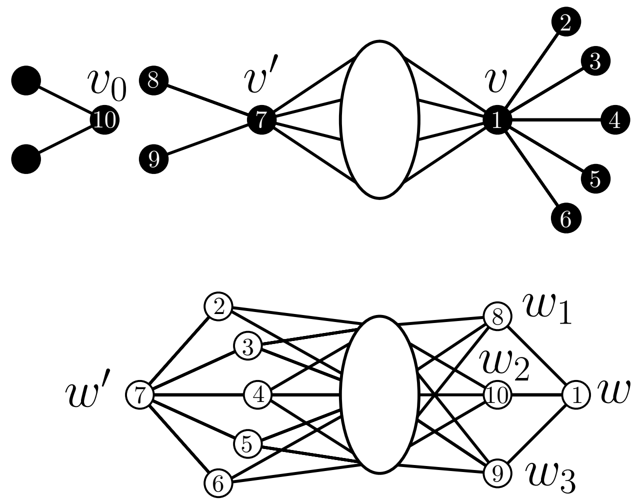

Figure 3.5: Sketch of the packing used in Lemma 3.4.18

By Lemma 3.4.17 and (3.33), (3.15) holds. The number of deleted edges is at least

$$d_2(w') + d_2(w_1) + d_2(w_2) + d_2(w_3) - |E(G_2[\{w_1, w_2, w_3\}])| + d(v) + d(v') - \|v, v'\| + d(v_0).$$

$$\geq d_2(w') + d_2(w_1) + d_2(w_2) + d_2(w_3) - 4 + d(v) + d(v') + d(v_0). \tag{3.34}$$

Case 2.1: $j = 3$. Then by (3.33), the number of added yellow edges plus $\mathcal{D}' - \mathcal{D}$ is at most $3(d_2(w_3) - 1) + \max\{3 - d_2(w_3), 0\}$. Since $d_2(w_3) \geq 1$, by (3.32), this is at most $d_2(w_1) + d_2(w_2) + d_2(w_3) - 1$. So by (3.34) and because $d(w') \leq 7$,

$$\partial(\mathbf{G}, \mathbf{G}') \geq d_2(w') + d(v) + d(v') - 2 \geq d_2(w') + \frac{13}{2}K - 2 \geq 3(d_2(w') + 5). \tag{3.35}$$

Therefore, $\mathbf{G}'$ packs by the minimality of $\mathbf{G}$, and this packing extends to a packing of $\mathbf{G}$, a contradiction.

Case 2.2: $j = 2$. By the choice of j, this means $v_0 w_3 \in E_3$. Since $d(v_0) \leq 2$ and v_0 has a white neighbor, $d(v_0) = 2$ and $d_1(v_0) = 1$. It follows that we have added at most $d_2(w_2) - 1$ yellow edges, and so by (3.34), similarly to (3.35), we get

$$\partial(\mathbf{G}, \mathbf{G}') \geq d_2(w') + d_2(w_3) + d(v) + d(v') - 2 \geq d_2(w') + \frac{13}{2}K - 2 \geq 3(d_2(w') + 5),$$

which similarly yields a contradiction. □

Lemma 3.4.19. *V_1 contains at most one donor.*

Proof. Suppose v and v' are donors in V_1. Consider the following discharging.

At start, we let $ch(v) = d(v)+\mathcal{D}+4$, $ch(v') = d(v')+\mathcal{D}+4$, and $ch(u) = d(u)$ for each $u \in V(\mathbf{G})-v-v'$. By definition, the total sum of charges is $\sum_{v\in V(\mathbf{G})} d(v) + 2\mathcal{D} + 8 = 2F(\mathbf{G}) + 8$. We redistribute charges according to the following rules.

(R1) Each vertex u not adjacent to 1-vertices with $d(u) \geq 4$ gives to each neighbor charge $\frac{d(u)-4}{d(u)}$ (and keeps 4 for itself).

(R2) Each vertex x adjacent to 1-vertices (it must be in V_1 and have degree at least $3K$) gives to each $z \in L(x)$ charge $\frac{4}{3}$ and to each $z' \in N(x) - L(x)$ charge $\frac{|N(x)-L(x)|-\frac{1}{3}|L(x)|-3}{|N(x)-L(x)|}$.

(R3) Each of v, v', in addition, gives 1 to each yellow neighbor.

We will show that the resulting charge, ch^*, satisfies

$$ch^*(x) \geq \frac{7}{3} \quad \text{for each } x \in V_1 \qquad \text{and} \qquad ch^*(y) \geq \frac{11}{3} \quad \text{for each } y \in V_2. \tag{3.36}$$

This would mean that $\sum_{v\in V(\mathbf{G})} d(v) + 2\mathcal{D} + 8 \geq \frac{7}{3}n + \frac{11}{3}n = 6n$, a contradiction to (3.16).

If $d(u) = 1$, then $u \in V_1$ and by (R2), $ch^*(u) = d(u) + \frac{4}{3} = \frac{7}{3}$, as claimed. If $d(u) = 2$ and $u \in V_1$, then by Lemma 3.4.13, u has a neighbor x with $d(x) \geq \lceil \frac{13K}{7} \rceil = 28$. If x has no neighbors of degree 1, then by (R1) it gives to u charge $\frac{d(x)-4}{d(x)} \geq 1 - \frac{4}{28} > \frac{1}{3}$. Otherwise, by (R2), it gives to u charge $\frac{|N(x)-L(x)|-\frac{1}{3}|L(x)|-3}{|N(x)-L(x)|}$. By Lemmas 3.4.18 and 3.4.13, this is at least $1 - \frac{1}{3} - \frac{3}{|N(x)-L(x)|} \geq \frac{2}{3} - \frac{3}{28/2} > \frac{1}{3}$. If $d(u) = 2$ and $u \in V_2$, then by Lemma 3.4.14, both neighbors of u are in V_2, and each of them has degree at least $2K$. So by (R1), $ch^*(u) \geq 2 + 2\frac{2K-4}{2K} = 4 - \frac{4}{K} = 4 - \frac{4}{15} > \frac{11}{3}$.

If $d(u) \geq 3$, $u \in V_1$ and u has no neighbors of degree 1, then either u keeps all its original charge (when $d(u) \leq 4$) or keeps for itself charge 4 by (R1). In both cases, $ch^*(u) \geq 3$. If $d(u) \geq 3$, $u \in V_1 - v - v'$ and u has a neighbor of degree 1, then by Lemma 3.4.13, $d(u) \geq 3K$. By Lemma 3.4.18, $|N(u) - L(u)| - \frac{1}{3}|L(u)| \geq \frac{1}{3}d(u) \geq K = 15$. So, after giving away charges by (R2), u keeps for itself charge at least 3. If $u \in \{v, v'\}$, then it originally had extra $\mathcal{D} + 4$ of charge and it gives out by (R3) at most $\mathcal{D} + 4$.

If $u \in V_2$ and $d(u) \geq 4$, then by (R1), it keeps 4 for itself. Suppose finally that $u \in V_2$ and $d(u) = 3$. If it is adjacent to v or v', then by (R3), $ch^*(u) \geq 3 + 1 = 4$. Otherwise, by Remark 3.4.16, u has a neighbor $y \in V_2$ with degree at least $K + 1$ and by (R1) receives from y charge $1 - \frac{4}{K+1} > \frac{2}{3}$. □

3.4.4 Weak Vertices and Sponsors

A *weak* vertex is either a 1-vertex or a 2-vertex with a neighbor of degree 2. The *sponsor*, $s(u)$, of a weak vertex u is the unique neighbor of u of degree at least 3. By Lemma 3.4.13, $d(s(u)) \geq \frac{13}{5}K$ for each weak u.

A *supersponsor* is a vertex with at least two neighbors that are weak. Notice that, for example, every donor is also a supersponsor. By definition, each supersponsor is the sponsor for each of its weak neighbors.

Lemma 3.4.20. *Either V_1 or V_2 contains more than one supersponsor.*

Proof. Suppose not. Choose $v_0 \in V_1$ and $w_0 \in V_2$ so that no $x \in V(\mathbf{G}) - v_0 - w_0$ is a supersponsor. For $x \in V(\mathbf{G})$, let $W(x)$ denote the set of weak neighbors of x. By our assumption, $|W(x)| \leq 1$ for each $x \in V(\mathbf{G}) - v_0 - w_0$. Consider the following discharging.

To start we let $ch(v_0) = d(v_0) + 2\mathcal{D} + 7$, $ch(w_0) = d(w_0) + 3$, $ch(u) = d(u)$ for each $u \in V(\mathbf{G}) - v_0 - w_0$.

$$\text{The total charge is } 2(e_1 + e_2 + e_3 + \mathcal{D} + 5). \tag{3.37}$$

We redistribute charges according to the following rules.

(R1) Each vertex u of degree at least 4 not adjacent to weak vertices gives to each neighbor charge $\frac{d(u)-3}{d(u)}$ (and keeps 3 for itself).

(R2) Each vertex $u \in V(\mathbf{G}) - v_0 - w_0$ with $d(u) = 3$ gives to each neighbor of degree 2 charge 1/4.

(R3) Each sponsor $u \in V(\mathbf{G}) - v_0 - w_0$ (then its degree is at least $\frac{13}{5}K$ by Lemma 3.4.13(b)) gives to each $x \in W(u)$ charge 2 and to each other neighbor charge $\frac{d(u)-5}{d(u)}$, and leaves charge at least $5 - 2 \cdot |W(u)| \geq 3$ for itself.

(R4) Vertex v_0 gives 2 to each neighbor and leaves $(2\mathcal{D} + d(v_0) + 7) - 2d(v_0) \geq 3$ for itself.

(R5) Vertex w_0 gives 1 to each neighbor and leaves 3 for itself.

We will show that the resulting charge, $ch^*(x)$, is at least 3 for each $x \in V(\mathbf{G})$. Together with (3.37), this will contradict (3.16).

Indeed, if x is weak and has degree 1, then it must be in V_1 and so it will get 2 by (R3) or by (R4). If it is weak and degree 2, then it gets at least 1 by (R3), (R4), or (R5). If $d(x) = 2$, and x is not weak, then x gets at least $1 - \frac{5 \cdot 7}{13K} = 1 - \frac{7}{39}$ from its neighbor of degree at least $\frac{13K}{7}$ and at least $\frac{1}{4}$ from another neighbor; in total, more than 1. If $d(x) = 3$, then x gets at least $\frac{K-5}{K} = \frac{2}{3}$ from its neighbor of degree at least K, and gives away at most $\frac{2}{4}$ by (R2). Similarly, if $d(x) \geq 4$, then by (R1),(R3),(R4) or (R5), it reserves charge 3 for itself. □

Lemma 3.4.21. *If V_i contains at least two supersponsors, then for each weak $w \in V_{3-i}$, the unique sponsor of w is also contained in V_{3-i}.*

Proof. Suppose a weak $w \in V_{3-i}$ is adjacent to a vertex $x_1 \in V_i$ of degree at least $\frac{13}{5}K$. By Lemma 3.4.4, $d(w) = 2$ and w has a neighbor $w' \in V_{3-i}$ with $d(w') = 2$. Let w'' be the other neighbor of w' (possibly,

$w'' \in V_i$). By the conditions of the lemma, there is a supersponsor $x_2 \in V_i - x_1$. By Claim 3.4.10, there is a vertex $x_3 \in V_i - N[x_2] - w''$ of degree at most 3. Send x_2 to w, x_3 to w', and, if $w'' \in V_{3-i}$, join w'' with the white neighbors of x_3 (there are at most 3 of them) by yellow edges. This way we eliminate all $d(x_2) + d(w) + 1$ edges incident with x_2 or w or w', add at most 3 yellow edges and increase $\mathcal{D}$ by at most 3. Moreover, the remaining graph triple $\mathbf{G}'$ satisfies (3.15) since for $i = 1, 2, 3$,

$$\Delta_i \leq \Delta_i + 3 \leq (\mathcal{D} + 4) + 3 \leq n + 9 - C < (n - 2) - 2.$$

Since $d(x_2) + d(w) + 1 \geq \frac{13}{5}K + 3 \geq 18$, we see that $\partial(\mathbf{G}, \mathbf{G}') \geq 18 - 3 - 3 = 12$. Hence, we are able to pack the remaining graph triple since $\mathbf{G}$ was a minimal counterexample. □

Lemma 3.4.22. *Each of V_1 and V_2 contains at least two supersponsors.*

Proof. Suppose V_i contains at most one supersponsor and, if this supersponsor exists, it is w_0. Then by Lemma 3.4.20, V_{3-i} contains two supersponsors x_1 and x_2. By Lemma 3.4.21, the sponsor of each weak vertex in V_i is also in V_i. By Lemma 3.4.19, $\mathbf{G}$ has at most one donor. Let v_0 denote such a vertex, if it exists. By (3.24), $v_0 \in V_1$, and by definition it is a supersponsor.

Case 1: $i = 2$. We use the following discharging. Let $ch(u) = d(u)$ for each $u \in V - v_0 - w_0$. If w_0 and/or v_0 exist, then let $ch(v_0) = d(v_0) + \Delta_1 + \Delta_{3|1} + 4$, and $ch(w_0) = d(w_0) + \Delta_2 + \Delta_{3|2} + 4$. By the definition of $\mathcal{D}$,

$$\Delta_1 + \Delta_{3|1} + \Delta_2 + \Delta_{3|2} \leq 2\mathcal{D} + 8,$$

so the total charge is at most $2(e_1 + e_2 + e_3 + \mathcal{D} + 8)$.

Then we redistribute the charges using the following set of rules.

(R1) Each vertex u of degree at least 5 not adjacent to weak vertices gives to each neighbor charge $\frac{d(u)-19/6}{d(u)} \geq \frac{1}{3}$ (and keeps $\frac{19}{6}$ for itself).

(R2) Each vertex $u \in V(\mathbf{G})$ with $d(u) = 3$ or $d(u) = 4$ gives to each neighbor of degree 2 charge $\frac{1}{3}$.

(R3) Each sponsor $u \in V(\mathbf{G})$ (then by Lemma 3.4.13(b) its degree is at least $\frac{13K}{5} = 39$) but not a supersponsor gives charge $\frac{13}{6}$ to its weak neighbor, and charges $\frac{d(u)-4.5}{d(u)}$ to each other neighbor.

(R4) Each supersponsor $u \notin \{v_0, w_0\}$ gives $\frac{13}{6}$ to each adjacent 1-vertex (by Lemma 3.4.19 and the definition of v_0, there is at most 1 such neighbor) and $\frac{d(u)-4.5}{d(u)}$ to each other neighbor.

(R5) Each of w_0 and v_0 gives $\frac{11}{6}$ to each neighbor.

We will show that the resulting charge, $ch^*(y)$, is at least $\frac{17}{6}$ for each $y \in V_1$ and at least $\frac{19}{6}$ for each $y \in V_2$. This would mean the total charge is at least $6n$, a contradiction to (3.16).

Indeed, if y is a 1-vertex, then it is in V_1 and will get $\frac{11}{6}$ by (R3), (R4), or (R5). If y is a weak 2-vertex and not adjacent to a supersponsor, then it will get $\frac{13}{6}$ from its sponsor by (R3). If y is a weak 2-vertex adjacent to a supersponsor and $y \in V_1$, then by (R4) or (R5), it will get at least $1 - \frac{4.5}{39} > \frac{5}{6}$ from its sponsor, and its resulting charge will be at least $\frac{17}{6}$. If y is a weak 2-vertex in V_2 adjacent to a supersponsor, then by Lemma 3.4.21, this supersponsor is w_0, and y gets $\frac{11}{6}$ from w_0.

If $d(y) = 2$, and y is not weak, then by Lemma 3.4.13(a), y has a neighbor of degree at least $\lceil \frac{13K}{7} \rceil = 28$. So y gets from it at least $1 - \frac{4.5}{28}$ (by (R1), (R3), (R4) or (R5)) and at least $\frac{1}{3}$ from another neighbor (by one of (R1)–(R5)). Then $ch^*(y) \geq 3 - \frac{4.5}{28} + \frac{1}{3} > \frac{19}{6}$. If $d(y) = 3$ and y has two neighbors of degree 2, then by Lemma 3.4.13(b), y has a neighbor x of degree at least $\frac{13K}{5} = 39$, so it gets from x at least $\frac{39-4.5}{39} \geq \frac{5}{6}$, and gives away at most $\frac{2}{3}$ by (R2). If $d(y) = 3$ and y has at most one neighbor of degree 2, then it gets from its neighbor of degree at least $\lceil \frac{13K}{10} \rceil = 20$ charge at least $\frac{15.5}{20}$ and gives away at most $\frac{1}{3}$. If $d(y) = 4$, then y gets at least $\frac{K-5}{K} = \frac{2}{3}$ from it neighbor of degree at least K and gives away at most $3 \cdot \frac{1}{3} = 1$ by (R2). If $d(y) \geq 5$ and y has no weak neighbors, then it leaves $\frac{19}{6}$ for itself by (R1).

If y has a weak neighbor and $y \notin \{v_0, w_0\}$, then $d(y) \geq 39$ and by (R3) or (R4), it reserves for itself charge

$$d(y) - \frac{13}{6} - (d(y) - 1)\frac{d(y) - 4.5}{d(y)} = -\frac{13}{6} + \frac{5.5d(y) - 4.5}{d(y)} = \frac{10}{3} - \frac{4.5}{d(y)} \geq \frac{10}{3} - \frac{4.5}{39} > \frac{19}{6}.$$

The vertex w_0 gives away charge $\frac{11}{6}d_2(w_0) + \frac{11}{6}d_3(w_0) \leq d(w_0) + \Delta_2 + \Delta_{3|2}$ and saves more than 4 for itself. Similarly, v_0 saves more than 4 for itself. This proves the case.

Case 2: $i = 1$. In this case either v_0 does not exist, or $v_0 = w_0$. The discharging is very similar to that in Case 1, but a bit simpler. Let $ch(u) = d(u)$ for each $u \in V - w_0$. If w_0 exists, then let $ch(w_0) = d(w_0) + 2\mathcal{D} + 4$. So, the total charge is at most $2(e_1 + e_2 + e_3 + \mathcal{D} + 4)$. The first 3 rules of discharging are again (R1)–(R3), but instead of (R4) and (R5), we have

(Q4) Each supersponsor $u \neq w_0$ gives $\frac{d(u)-4.5}{d(u)}$ to each neighbor.

(Q5) Vertex w_0 gives $\frac{13}{6}$ to each neighbor.

Symmetrically to Case 1, we will show that the resulting charge, $ch^*(y)$, is at least $\frac{19}{6}$ for each $y \in V_1$ and at least $\frac{17}{6}$ for each $y \in V_2$, again yielding a contradiction to (3.16).

If y is a 1-vertex, then it is in V_1 and its neighbor also is in V_1. Since all supersponsors apart from w_0 are in V_2, Rule (Q4) does not apply to y, so y will get $\frac{13}{6}$ by (R3) or (Q5). If y is a weak 2-vertex and not adjacent to a supersponsor, then it will get $\frac{13}{6}$ from its sponsor by (R3). If y is a weak 2-vertex adjacent to a supersponsor and $y \in V_2$, then by (Q4) or (Q5), it will get at least $1 - \frac{4.5}{13K/5} = 1 - \frac{3}{26}$ from its sponsor, so that its resulting charge will be more than $\frac{17}{6}$. If y is a weak 2-vertex in V_1 adjacent to a supersponsor, then by Lemma 3.4.21, this supersponsor is w_0, and y gets $\frac{13}{6}$ from w_0.

Counting of charges for other vertices apart from w_0 simply repeats that in Case 1 (using (Q4) and (Q5) in place of (R4) and (R5)). Since the starting charge of w_0 was at least $3d(w_0)$, by (Q5), its new charge is at least $\frac{5}{6}d(w_0) + 4 > 4$. □

3.4.5 List packing of G_1 and G_2.

By Lemma 3.4.22, V_1 contains supersponsors x_1 and x_2 and V_2 contains supersponsors y_1 and y_2. Let v_1 (respectively w_1) be a weak neighbor of x_1 (of y_1), let v_1' (w_1') be the other neighbor of it which is of degree 2 if it exists, and let v_1'' (w_1'') be the other neighbor of v_1' (of w_1'). Let v_2 (w_2) be a weak neighbor of x_2 (of y_2) that is *not adjacent* to v_1 (to w_1); this is possible since x_2 (y_2) is adjacent to multiple weak vertices. Let v_2' (w_2') be the other neighbor of it which is again of degree 2 if it exists, and let v_2'' (w_2'') be the other neighbor of v_2' (of w_2').

We are now ready to construct our packing. For $j = 1, 2$, begin by placing x_j on w_j, and v_j on y_{3-j}. Notice that by Lemma 3.4.21, $v_j \in V_1$ and $w_j \in V_2$ so this assignment is well defined. Since the weak vertices have only one sponsor, v_j is not adjacent to x_{3-j}, y_1, nor y_2, and w_j is not adjacent to y_{3-j}, x_1, nor x_2. Together with the fact that v_1 (w_1) was chosen to be not adjacent to v_2 (w_2), we see that these mappings do not violate the packing property.

As we extend this packing, we only need to ensure that v_j' is not mapped to a vertex in $N_2(y_{3-j})$ and no vertex in $N_1(x_j)$ is mapped to w_j'. This can only be an issue if $v_j' \in V_1$ ($w_j' \in V_2$) and in this case, we will find an appropriate assignment for v_j'. If $v_j' \in V_2$ ($w_j' \in V_1$), we will simply ignore this part of the construction.

By Claim 3.4.10, there is a vertex $x_1' \in V_1 - N(x_1) - \bigcup_{i=1,2}\{v_i, v_i', v_i'', w_i, w_i', w_i''\}$ ($y_1' \in V_2 - N(y_1) - \bigcup_{i=1,2}\{v_i, v_i', v_i'', w_i, w_i', w_i''\}$) with degree at most 3. Similarly, there are vertices $x_2' \in V_1 - N(x_2) - x_1' - \bigcup_{i=1,2}\{v_i, v_i', v_i'', w_i, w_i', w_i''\}$ and $y_2' \in V_2 - N(y_2) - y_1' - \bigcup_{i=1,2}\{v_i, v_i', v_i'', w_i, w_i', w_i''\}$ of degree at most 3.

For the following mappings, refer to Figure 3.6. If $w_j' \in V_2$, then send x_j' to w_j' and, if $w_j'' \in V_2$, add the yellow edges connecting w_j'' with the at most 3 white neighbors of x_j'. Similarly, if $v_j' \in V_1$, then send v_j' to y_{3-j}' (if $v_j' \in V_1$) and, if $v_j'' \in V_1$, add the yellow edges connecting v_j'' with the at most three white neighbors of y_{3-j}'.

Let $\mathbf{G}'$ be the triple obtained by deleting the assigned vertices. By construction, if $\mathbf{G}'$ packs, then together with our placement, we get a packing of $\mathbf{G}$. We decreased n by at most 8 and decreased the number of edges by at least $d(x_1) + d(x_2) + d(y_1) + d(y_2) - 16 \geq 12K - 16$. We have increased $\mathcal{D}$ by at most

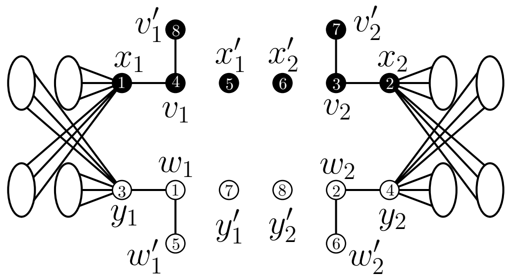

Figure 3.6: Sketch of Packing

6 (with the new yellow edges). So, $\partial(\mathbf{G}, \mathbf{G}') \geq 12K - 22 \geq 24 = 3(n - n')$. Since $d_i(v) \leq \mathcal{D} + 4 \leq n - C + 6$ for every $v \in V$ (and $C \geq 8$), (3.15) holds for $\mathbf{G}'$. Thus $\mathbf{G}'$ (and hence $\mathbf{G}$) packs, a contradiction to the choice of $\mathbf{G}$. □

Chapter 4

Cyclic Stable Matchings

The results from this chapter are from joint work with Sarah Behrens and Nicholas Kosar. The work was completed during the 2013 REGS program at the University of Illinois at Urbana-Champaign [1].

4.1 Introduction

The Stable Marriage problem was first introduced by Gale and Shapley in 1962 [21]. In the original, 2-dimensional case, an instance of the stable matching problem consists of a group of n men and n women, together with a set of rankings. Each woman has a ranking of the n men and each man has a ranking of the n women. The goal is to pair off the men and women in a manner that is consistent with their preferences.

A *matching* is a set of n pairs such that each each pair contains exactly one man and one woman, and each person is in exactly one pair. We say that a matching M is *stable* if there is no man m and woman w such that m and w are not paired in M, but prefer each other to their assigned partners. If such an m and w exist, we say that (m, w) is a *blocking pair*. In other words, a matching is stable if no two people have the incentive to leave their respective partners for one another. The stable marriage problem asks the following question.

Question 4.1.1. Given n men, n women, and their preferences, is it always possible to find a stable matching?

In their paper, Gale and Shapley show that the answer to this question is yes.

Theorem 1.4.1. *[21] Every instance of the stable marriage problem admits a stable matching.*

Moreover, Gale and Shapley provided an algorithm, now known as the Gale-Shapley Algorithm, to find such a matching in polynomial time. This celebrated result eventually led to Shapley receiving the 2012 Nobel Prize in Economics. Theorem 1.4.1 and the Gale-Shapley algorithm have led to other results regarding stable marriages, including algorithms to determine all possible stable matchings for a given instance, an ordering of the set of stable matchings, etc. We refer the reader to [23, 41] for additional information.

[1]The author acknowledges support from National Science Foundation grant DMS 08-38434 “EMSW21-MCTP: Research Experience for Graduate Students”.

Given the positive result of Gale and Shapley, it is natural to consider the situation when there are more than 2 genders. Indeed, in 1976, Knuth first proposed extending the stable marriage problem to 3 dimensions [36]. For the three dimensional case, a third group, say dogs, is introduced. There are now $3n$ *agents*, consisting of n men, n women, and n dogs, and the goal is to form a matching of n triples. Each triple, also called a *family*, contains exactly one man, woman, and dog and each agent is contained in exactly one triple.

However, there are multiple ways to generalize the notion of a preference list. The extension proposed by Knuth has each agent rank the n^2 pairs formed by taking one member from each of the other two genders. That is, each man ranks the n^2 woman-dog pairs, each woman ranks the n^2 man-dog pairs, and each dog ranks the n^2 man-woman pairs. A matching M is stable if there is no triple (m, w, d), not matched in M, such that m, w, and d each prefer (m, w, d) to their assigned triple.

It was proved by Alkan in 1988 that, unlike the 2-dimensional case, not all instances of the 3-dimensional stable matching problem admit a stable matching [1]. In fact, it was later proved by Ng and Hirschberg that determining whether or not a given instance contains a stable matching is NP-complete [44]. This led to an alternate approach to defining preference lists.

The notion of the *cyclic 3-dimensional stable matching problem* (cyclic 3-DSM) was introduced by Ng and Hirschberg in [44], though they credit the problem to Knuth. This version of the problem asks each man to rank the n women, each woman to rank the n dogs, and each dog to rank the n men. A matching M is stable if there is no triple (m, w, d), not matched in M, such that m, w, and d each *strictly* prefers (m, w, d) to their assigned triple. As a person may now be indifferent to changing families, cycle 3-DSM specifies that strict preference is required. For example, given a man m, woman w and dogs d and d', since the men do not rank the dogs, m is indifferent between the triples (m, w, d) and (m, w, d').

For any positive integer s, this problem can be extended to s dimensions by having s genders of n agents each, ordered cyclically, and having each member of a gender rank the n members of the successive gender. We call the s genders and their preference lists an *instance* of cyclic s-DSM of size n. In 2004, Boros, Gurvich, Jaslar, and Krasner proved the first result for this problem:

Theorem 1.4.3. *[7] Let $n, s \in \mathbb{N}$ with $2 \leq n \leq s$, then each instance of cyclic s-DSM of size n admits a stable matching.*

In 2006, Eriksson, Sjöstrand, and Strimling [20] were able to increase n by one in the special case when $s = 3$.

Theorem 1.4.4. *[20] Each instance of cyclic 3-DSM of size 4 admits a stable matching.*

The main result of Chapter 4 is the following generalization of Theorem 1.4.4 to cyclic s-DSM.

Theorem 1.4.5. *For $s \geq 3$ and $n \leq s+1$, any instance of cyclic s-DSM of size n has a stable matching.*

This theorem was proved independently by Hofbauer in 2016 [28]. The result in [28] uses a similar algorithm to find a stable matching, though our analysis is somewhat different.

We will also consider one final variation of the stable marriage problem, the *cyclic s-dimensional strongly stable matching problem* (cyclic s-DSSM). The setup of this problem is the same as in cyclic s-DSM, however we now require a stronger notion of stability. A matching M is *strongly stable* if there is no triple (m, w, d), not matched in M, such that m, w, and d each either prefers (m, w, d) to their assigned triple or is indifferent.

Biró and McDermid [3] showed that in cyclic 3-DSSM not all instances contain a strongly stable matching by providing an example with $n = 6$. Recently, Irving provided an example of 3-DSSM with $n = 3$ that admits no strongly stable matching [41, p. 280]. We improve on this result by providing a (different) instance of cyclic 3-DSSM with no strongly stable matching and extending it to an example of s-DSSM for $s \geq 3$.

Theorem 1.4.6. *Let $s \geq 3$.*

(i) *If $n \leq 2$, every instance of cyclic s-DSSM of size n admits a strongly stable matching, and*

(ii) *If $n \geq 3$, there exists an instance of cyclic s-DSSM of size n that admits no strongly stable matching.*

The remainder of Chapter 4 is outlined as follows. The next section outlines the notation and tools that we will use throughout the paper. Section 4.3 contains a proof of Theorem 1.4.5. In Section 4.4, we consider instances of cyclic s-DSSM and prove Theorem 1.4.6.

4.2 Setup and Notation

Let $s, n \in \mathbb{N}$. Let A be a set of (sn) agents, partitioned evenly into s *genders* of size n, $\{A^1, \ldots, A^s\}$. For each $i \in [s]$, $A^i = \{a^i_j : 1 \leq j \leq n\}$. For each $i \in s$ and each agent $a \in A^i$, there exists a linear order, or *preference list*, $>_a$ on the elements of A^{i+1}. For $x, y \in A^{i+1}$, we say agent a *prefers* x to y, denoted $x >_a y$, if and only if x is ranked above y on a's preference list. For brevity, when discussing a specific agent $a^i_j \in A^i$, we may use the vector $(j_1, \ldots, j_n)$ to denote the preference list $a^{i+1}_{j_1} >_{a^i_j} \cdots >_{a^i_j} a^{i+1}_{j_n}$.

The order $>_a$ can be extended to a partial order of the set $A^1 \times \cdots \times A^s$ in the following way. Let $X, Y \in A^1 \times \cdots \times A^s$ and $x, y \in A^{i+1}$ be the $(i+1)^{\text{st}}$ entry in X and Y, respectively. Agent a *prefers* the tuple X to Y, denoted $X \succ_a Y$ if $x >_a y$ and is *indifferent* if $x = y$. We write $X \succeq_a Y$ if a prefers X to Y or is indifferent.

A *matching* M is a set of families $\{F_i : 1 \leq i \leq n\}$ where each family is an element in $A^1 \times \ldots \times A^s$ and each element of A is represented in exactly one family. A *blocking family* for a matching M is an element $T \in A^1 \times \ldots \times A^s$ such that $T \notin M$ and, if F_x is the family in M that contains x, $T \succ_x F_x$ for each agent $x \in T$. A matching is called *stable* if no blocking family exists. Similarly, a *weakly blocking family* for M is an element $T \in A^1 \times \ldots \times A^s$ such that $T \notin M$ and $T \succeq_x F_x$ for each agent $x \in T$. A matching is *strongly stable* if there is no weakly blocking family.

Finally, we define the *favorite function* $f : A \to A$. For an agent $a \in A$, we define $f(a)$ to be the first agent of a's preference list. Note that if $a \in A^i$, then $f(a) \in A^{i+1}$ and $f^\ell(a) \in A^{i+\ell \pmod{s}}$ for any $\ell \in \mathbb{N}$. For an agent a, define the s-tuple $(a, f(a), \ldots, f^{s-1}(a))$ to be the family *generated* by a. Such a family will be called *preferred* if $f^s(a) = a$.

Notice that if there is no preferred family, then the image $f^{s-1}(A^1)$ contains at least 2 elements. This is particularly important because in any instance of s-DSM, either there is a preferred family or there exist two agents $a, a' \in A^1$ such that $\{a, f(a), \ldots, f^{s-1}(a)\}$ and $\{a', f(a'), \ldots, f^{s-1}(a')\}$ are disjoint.

The main idea of our proof will use of a greedy approach first introduced by Boros et al. [7]. The goal is to use the favorite function f to recursively generate families in a matching. Since we will repeatedly make use of this method, we outline it here.

Let π be an arbitrary permutation of the agents in A^1. The *matching generated by* π, M_π, is obtained using the following algorithm:

Step 1: Add family $F_1 = (\pi(1), f(\pi(1)), \ldots, f^{s-1}(\pi(1))$ to M_π.

Step i (for $2 \leq i \leq n$*)*: Remove agents from $F_1, \ldots, F_{i-1}$. Let $\widetilde{f}$ be the function that, to an agent a, assigns its favorite remaining agent. Add the family $F_i = (\pi(i), \widetilde{f}(\pi(i)), \ldots, \widetilde{f}^{s-1}(\pi(i))$ to M_π.

4.3 Proof of Theorem 1.4.5

If $s = 3$, then the result follows from Theorem 1.4.4, so we only consider fixed $s \geq 4$. Similarly, if $n \leq s$, the result follows from Theorem 1.4.3, so we assume $n = s + 1$.

For each agent $a \in A$, consider the ranking of a on $f^{s-1}(a)$'s preference list. Choose a^* to be the agent such that this ranking is as high as possible. We assume $a^* \in A^1$ and that a^* is ranked r^{th} on $f^{s-1}(a^*)$'s preference list. Notice that if $r = 1$, then the family $(a^*, f(a^*), \ldots, f^{s-1}(a^*))$ is a preferred family. By Theorem 1.4.3, there is a stable matching M in the instance of s-DSM (of size $n - 1$) obtained by removing the agents $\{a^*, f(a^*), \ldots, f^{s-1}(a^*)\}$ from A. Since no agent in a preferred family can be in a blocking triple and there is no blocking family in the remaining agents, we see that $M \cup (a^*, f(a^*), \ldots, f^{s-1}(a^*))$ is a stable matching.

Alternatively, if $r = n$, then a^* is the least preferred agent on $f^{s-1}(a^*)$'s list. Agent a^* was chosen to make this rating as high as possible so, for each $a \in A^1$, agent a is the least preferred agent on $f^{s-1}(a)$'s preference list. Hence, for distinct $a, a' \in A^1$, $f^{s-1}(a) \neq f^{s-1}(a')$ and the families generated by $a_1^1, \ldots, a_n^1$ are disjoint. In particular, these families form a matching. Since each agent from A^1 is matched with its favorite, no agent from A^1 is in a blocking family and the matching is stable. Thus, if a^* is at the top or the bottom of $f^{s-1}(a^*)$'s preference list, then we are able to find a stable matching. The following lemma shows that if a^* is near the bottom of the list, we are still able to find a matching.

Lemma 4.3.1. *If a^* appears as the $n-1^{st}$ agent on $f^{s-1}(a^*)$'s list (i.e. $r = n-1$), then the given instance of s-DSM admits a stable matching*

Proof. For $i \in [n]$, let $C_i = \{x \in A^1 : f^{s-1}(x) = a_i^{s-1}\}$ be the $(s-1)^{\text{st}}$ preimage of agent a_i^{s-1} under f. Observe that $C_1, \ldots, C_n$ partitions the agents of A^1 into n possibly empty sets. For each i, the class C_i contains at most 2 agents, as otherwise some $a \in A^1$ is ranked in position $n-2$ on $f^{s-1}(a)$'s preference list, contradicting the optimality of a^*.

Since $n \geq 5$, the pigeonhole principle implies $|f^{s-1}(A^1)| \geq 3$. In particular, there are three agents, $x, y, z \in A^1$, each in different parts of the partition $\{C_i : i \in [s]\}$. Hence, the families generated by x, y and z are pairwise disjoint. Let π be a permutation of A_1 with $\pi(1) = x$, $\pi(2) = y$, and $\pi(3) = z$. We claim that M_π is stable. Assume that the agents are labeled such that $(a_1^1, \ldots, a_1^s), \ldots, (a_n^1, \ldots, a_n^s)$ are the families in M_π, where $(a_i^1, \ldots, a_i^s)$ was the i^{th} family added to M_π.

If M_π is not stable, then there is a blocking family $(a_{\ell_1}^1, \ldots, a_{\ell_s}^s)$. For each $i \in [s]$, agent $a_{\ell_i}^i$ must prefer $a_{\ell_{i+1}}^{i+1}$ to $a_{\ell_i}^{i+1}$, and so agent $a_{\ell_{i+1}}^{i+1}$ must have been matched before the i^{th} iteration of the algorithm. In particular, $\ell_{i+1} \leq \ell_i - 1$. Since the index ℓ_i is at most n for any choice of i, we have

$$n \geq \ell_1 \geq \ell_2 + 1 \geq \cdots \geq (s-2) + \ell_{s-1} = (n-3) + \ell_{s-1}.$$

This implies that $\ell_{s-1} \leq 3$. However, by the choice of π, $(a_1^1, \ldots, a_1^s)$, $(a_2^1, \ldots, a_2^s)$, and $(a_3^1, \ldots, a_3^s)$ are the family generated by agents x, y, and z, respectively. So $a_{\ell_{s-1}}^{s-1}$ is already matched in M_π to it's favorite, contradicting that $(a_{\ell_1}^1, \ldots, a_{\ell_s}^s)$ is a blocking family. We conclude that M_π is stable. □

Using this lemma, we may assume that a^* is ranked between 2 and $n-2$ on $f^{s-1}(a^*)$'s preference list. Consider two agents of A^1, say x and y, such that each is ranked below a^* on $f^{s-1}(a^*)$'s preference list. We will choose a permutation π with $\pi(1) = a^*$ and show that the matching generated by π is stable.

The choice of π will depend on the structure of the set of agents $\widetilde{A} = A - \{a^*, \ldots, f^{s-1}(a^*)\}$. Let $(a^*, \ldots, f^{s-1}(a^*)) = (a_1^1, \ldots, a_1^s)$. Let $\widetilde{f}$ be the function that, to an agent a, assigns its favorite agent in $\widetilde{A}$. Recall that either there exists a preferred family in $\widetilde{A}$ or there exist two agents $a_i^1, a_j^1 \in \widetilde{A} - \{a^*\}$ such that the families generated by a_i^1 and a_j^1 agents are disjoint.

Case 1: *There is a preferred family in* $\widetilde{A}$*.* Since x and y are both in A^1, both agents cannot be in the family. Assume that x is not in the preferred family. Choose a permutation π such that the member of the preferred family in $\widetilde{A} \cap A^1$ is $\pi(2)$ and $\pi(n) := x$.

Case 2: *There is no preferred family in* $\widetilde{A}$*.* As in Lemma 4.3.1, the gender $A^1 - \{a^*\}$ can be partitioned into $n-1$ (possibly empty) classes $C_2, \ldots, C_n$ by assigning $a \in C_j$ if $\widetilde{f}^{s-1}(a) = a_j^s$. Since there is no preferred family in $\widetilde{A}$, at least 2 of these classes are non-empty. We can choose two agents, $a, b \in A^1 - \{a^*\}$, from separate classes and the families in $\widetilde{A}$ generated by these two agents are disjoint. Moreover, since the partition $C_2, \ldots, C_n$ contains $n-1 \geq 4$ agents, one of a or b can be chosen to be neither x nor y. We may assume that x is neither a nor b and define the permutation π so that $\pi(2) = a$, $\pi(3) = b$, and $\pi(n) = x$.

Given this permutation π, we now show that the matching M_π is stable. By relabeling, we will assume that the families in M_π are $(a_1^1, \ldots, a_1^s), \ldots, (a_n^1, \ldots, a_n^s)$, where the family $(a_i^1, \ldots, a_i^s)$ was the i^{th} family added to the matching. In particular, $a_1^1 = a^*$ and $a_n^1 = x$. If a_j^i and $a_{j'}^{i+1}$ are both in a blocking family, then $j' < j$. Thus, if a blocking family containing only agents in $\widetilde{A}$, the family is is $(a_n^1, a_{n-1}^2 \ldots, a_3^{s-1}, a_2^s)$. However, either a_2^s already receives its first choice in $\widetilde{A}$ (Case 1) or a_3^{s-1} already receives its first choice in $\widetilde{A}$ (Case 2), so there is no such family.

Thus, a blocking family must contain an agent from outside of $\widetilde{A}$. As $(a_1^1, \ldots, a_1^s)$ is the family generated by a_1^1, the only option is for agent a_1^s to be in the blocking family. Further, $a_n^1 = x$ is below $a_1^1 = a^*$ on a_1^s's preference list, so agent a_n^1 is not in a blocking family. Therefore, the only possible blocking family is $(a_{n-1}^1, \ldots, a_{n-i}^i, \ldots, a_1^s)$. Suppose this is the case.

Agent $a_1^s = f^{s-1}(a_1^1)$ strictly prefers a_{n-1}^1 to a_1^1. We claim that $f^{s-1}(a_{n-1}^1) = a_1^s$. Consider $i \in [s-2]$. If $f^{i-1}(a_{n-1}^1) = a_{n-i}^i$, then since a_{n-i}^i is in the blocking family, $f^i(a_{n-1}^1) = a_j^{i+1}$ for some $j \leq n-(i+1)$. On the other hand, if $f^{i-1}(a_{n-1}^1) = a_j^i$ for some $j \leq n-(i+1)$, then since the families were chosen using the greedy algorithm, $f^i(a_{n-1}^1) = a_j^{i+1}$ for some $j \leq n-(i+1)$.

In particular, this implies that $f^{s-1}(a_{n-1}^1) = a_1^s$. However, this is a contradiction since a_{n-1}^1 is above a_1^1 on a_1^s's preference list, contradicting the choice of $a^* = a_1^1$. We conclude that M_π is stable.

$a_1^1 : (1,2,5,3,4)$	$a_1^2 : (1,2,4,5,3)$	$a_1^3 : (1,2,4,3,5)$	$\mathbf{a_1^4} : (\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{2},\mathbf{5})$
$a_2^1 : (1,2,3,5,4)$	$a_2^2 : (2,3,5,1,4)$	$\mathbf{a_2^3} : (\mathbf{2},\mathbf{1},\mathbf{3},\mathbf{4},\mathbf{5})$	$a_2^4 : (3,5,2,4,1)$
$a_3^1 : (3,1,2,5,4)$	$\mathbf{a_3^2} : (\mathbf{3},\mathbf{5},\mathbf{1},\mathbf{2},\mathbf{4})$	$a_3^3 : (5,4,3,2,1)$	$a_3^4 : (1,2,3,4,5)$
$\mathbf{a_4^1} : (\mathbf{3},\mathbf{4},\mathbf{1},\mathbf{5},\mathbf{2})$	$a_4^2 : (3,4,1,2,5)$	$a_4^3 : (2,4,1,5,3)$	$a_4^4 : (1,2,3,4,4)$
$a_5^1 : (5,3,4,1,2)$	$a_5^2 : (1,5,3,2,4)$	$a_5^3 : (3,5,1,2,4)$	$a_5^4 : (1,4,5,2,3)$

Table 4.1: The above table is an example of an instance of cyclic 4-DSM. In the above preference lists, choosing $a = a_1^1$ optimizes the position of a on $f^{s-1}(a)$'s preference list. After removing the family generated by a_1^1, the permutation $a_2^1 a_3^1 a_4^1 a_5^1$ generates a matching that is stable when restricted to the remaining agents. Since a_1^4 prefers a_1^1 to a_5^1, the only possible blocking family is $(a_4^1, a_3^2, a_2^3, a_1^4)$, shown above in bold. However, a_3^2 is already matched with its favorite, so this is not a blocking family and so the matching is stable.

4.4 A result on strongly stable matchings

In this section we consider, for $s \geq 3$, instances of cyclic s-dimensional strongly stable matchings of size n. We prove that if $n \leq 2$, then every instance of cyclic s-DSSM admits a strongly stable matching but, if $n \geq 3$, then there exists an instance that does not admit a strongly stable matching. We will prove Theorem 1.4.6 using several lemmas.

Lemma 4.4.1. *Let $s \geq 3$. Every instance of cyclic s-DSSM of size one or two admits a strongly stable matching.*

Proof. When $n = 1$, assigning all the agents to the same family is a stable matching, so assume $n = 2$. If there exists a preferred family, then let M be matching containing the preferred family as one family and the remaining s agents as the other family. Suppose there is a blocking family. Since there are only two families in M, a blocking family must contain agents from both families. In particular, there is an agent a in both the blocking family and the preferred family. Since a either prefers his partner in the blocking family to his partner in the preferred family or is indifferent, $f(a)$ is also in both the blocking family and the preferred family. This implies that the blocking family is the family generated by a, a contradiction since the family generated by a is already in M.

So we may assume that there is no preferred family. Let a and b be the two agents in A^1. Since there does not exist a preferred family, $f^{s-1}(a) \neq f^{s-1}(b)$ and the families generated by a and b are disjoint. Let M be the matching consisting of these two families. Suppose there is a blocking family and let x^i be the representative from A^i in the blocking family. For $i \neq s$, then x^i is matched in M to its favorite, so $f(x^i)$ must also be part of the blocking family. In particular, if a (or, respectively, b) is in the blocking family, then the blocking family is the family generated by a (respectively, the family generated by b). This is a contradiction, since this family is already in the matching and cannot be a blocking family. We conclude that M is stable □

In order to prove that there is an instance of cyclic s-DSSM of size at least 3 that admits no strongly stable matching, we first construct an example for $s = 3$ and $n = 3$ and then extend the example to larger values of s.

Lemma 4.4.2. *There exists an instance of cyclic* 3*-DSSM of size* 3 *that admits no strongly stable matching.*

$a_1^1 : (2, 1, 3)$	$a_1^2 : (2, 1, 3)$	$a_1^3 : (2, 1, 3)$
$a_2^1 : (1, 3, 2)$	$a_2^2 : (1, 3, 2)$	$a_2^3 : (1, 3, 2)$
$a_3^1 : (2, 3, 1)$	$a_3^2 : (2, 3, 1)$	$a_3^3 : (2, 3, 1)$

Table 4.2: Table representing the preference lists for an instance of cyclic 3-DSSM that admits no strongly stable matching

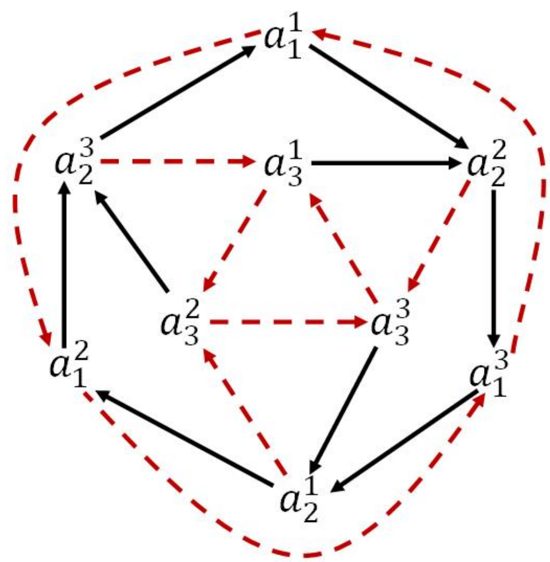

Figure 4.1: Preference diagram for the instance cyclic 3-DSSM given in Table 4.2. A solid line from x to y represents that y is the first choice of x. A dashed line from x to z represents that y is the second choice of z. If there is no line from $x \in A^i$ to $w \in A^{i+1}$, then w is the third choice of x.

Proof. Consider the preference lists shown in Table 4.2. (The preference list are also displayed in a preference diagram in Figure 4.1.) For each $i \in \{1, 2, 3\}$, define the set $S_i := \{a_i^1, a_i^2, a_i^3\}$. By construction, each element of S_i has the same preference list.

First, we will show that in any strongly stable matching, a family may contain at most one agent from S_1. Without loss of generality, consider a matching M such that a_1^1 and a_1^2 are in the same family. If either a_1^3 or a_3^3 is the third member, then $a_1^1 a_1^2 a_2^3$ is a blocking family, so we may assume that $a_1^1 a_1^2 a_2^3$ is a family in M. Now, a_1^3 must be in the same family as a_2^1 or else $a_1^1 a_2^2 a_1^3$ will be a blocking family. However, this implies $a_2^1 a_3^2 a_3^3$ is a blocking family. Thus, by symmetry, in any strongly stable matching, each family contains exactly one element from S_1.

Next, suppose that a_2^3 is in the family containing a_1^1. If a_3^2 were the third member, then $a_1^1 a_1^2 a_2^3$ would be a blocking family. On the other hand, if $a_1^1 a_2^2 a_2^3 \in M$, then $a_2^1 a_1^2 a_1^3$ is a blocking family. We conclude that that a_1^1 must be matched with a_3^3 in any stable matching. Further, by symmetry we see that a_1^2 and a_3^1 are in the same family and also that a_1^3 and a_3^2 are in the same family. So our matching M must be $\{(a_1^1 a_2^2 a_3^3), (a_2^1 a_3^2 a_1^3), (a_3^1 a_1^2 a_2^3)\}$. Now, $a_3^1 a_3^2 a_3^3$ is a blocking family M. We conclude that there is no strongly stable matching for this set of preference lists. □

The preferences in Table 4.2 can be modified to allow the introduction of additional genders. The following result shows that, even after the additions, the example still admits no strongly stable matching.

Lemma 4.4.3. *There exists an instance of cyclic* 4*-DSSM of size* 3 *that admits no strongly stable matching.*

$a_1^1 : (2,1,3)$	$a_1^2 : (2,1,3)$	$a_1^3 : (2,1,3)$	$a_1^4 : (1,2,3)$
$a_2^1 : (1,3,2)$	$a_2^2 : (1,3,2)$	$a_2^3 : (1,3,2)$	$a_2^4 : (2,3,1)$
$a_3^1 : (2,3,1)$	$a_3^2 : (2,3,1)$	$a_3^3 : (2,3,1)$	$a_3^4 : (3,2,1)$

Table 4.3: Table representing the preference lists for an instance of cyclic 3-DSSM that admits no strongly stable matching

Proof. Consider the preference lists as shown in Table 4.3. Notice that the agents in A^1, A^2, and A^3 have the same preference lists as in Table 4.2. Suppose M is a matching such that a_i^4 is in the family as a_i^1 for all $i \in \{1,2,3\}$. Since each agent in A^4 is given its first choice, if a_i^4 is in a blocking family, a_i^1 must also be in the blocking family. Hence, we may view the pair $a_i^1 a_i^4$ as a single agent and the proof follows exactly as in Lemma 4.4.2. Thus, there must exist an i such that a_i^4 and a_i^1 are not in the same family.

Suppose first that a_1^1 is in the same family as a_1^4, so each of the three families in a matching must contain one of the following pairs: $a_1^1 a_1^4$, $a_3^1 a_2^4$, or $a_2^1 a_3^4$. If a_1^2 is not matched with a_2^3, then $a_2^1 a_1^2 a_1^3 a_2^4$ is a blocking family. So $a_1^2 a_2^3$ must be paired together and this pair must share a family with $a_3^4 a_2^1$, as otherwise $a_2^1 a_3^2 a_3^3 a_2^4$ is blocking family. So $a_2^1 a_1^2 a_2^3 a_3^4$ is a family in any stable matching. Finally, if a_2^2 is in a family with $a_1^4 a_1^1$, then $a_3^1 a_2^2 a_1^3 a_2^4$ is a blocking family. However, if a_2^2 is not in a family with $a_1^4 a_1^1$, then $a_1^1 a_1^2 a_2^3 a_1^4$ is a blocking family. Therefore, we see that there can be no stable matching such that a_1^1 and a_1^4 share a family.

On the other hand, if a_1^1 and a_1^4 are in separate families, then a_1^3 must be matched with a_2^4 as otherwise $a_1^1 a_2^2 a_1^3 a_1^4$ is a blocking family. Similarly, if a_1^1 is not matched with a_2^2, then $a_1^1 a_1^2 a_2^3 a_1^4$ is a blocking family. If, together, these pairs form a family $a_1^1 a_2^2 a_1^3 a_2^4 \in M$, then $a_3^1 a_2^2 a_1^3 a_2^4$ is a blocking family. So the pairs $a_1^3 a_2^4$ and $a_1^1 a_2^2$ must be in separate families.

Now, a_3^4 is the only remaining agent in A^4 that may be matched with $a_1^1 a_2^2$. Further, since a_1^3 is already paired with a_2^4, we see that a_3^3 must complete the family with a_1^1, a_2^2 and a_3^4. Finally, since $a_1^1 a_2^2 a_3^3 a_3^4$ is a family and $a_3^1 >_{a_3^4} a_1^1$, we see that $a_3^1 a_2^2 a_3^3 a_3^4$ is a blocking family and the matching is not stable. We conclude that this set of preference lists has no strongly stable matching. $\square$

Finally, we show that the preference lists used in the proof of Lemma 4.4.3 can be modified to include an arbitrary number of genders an still admit no strongly stable matching.

Lemma 4.4.4. *Let $s \geq 5$. There exists an instance of cyclic s-DSSM of size 3 that admits no strongly stable matching.*

$a_1^1 : (2,1,3)$	$a_1^2 : (2,1,3)$	$a_1^3 : (2,1,3)$	$a_1^4 : (1,2,3)$	$\cdots$	$a_1^s : (1,2,3)$
$a_2^1 : (1,3,2)$	$a_2^2 : (1,3,2)$	$a_2^3 : (1,3,2)$	$a_2^4 : (2,3,1)$	$\cdots$	$a_2^s : (2,3,1)$
$a_3^1 : (2,3,1)$	$a_3^2 : (2,3,1)$	$a_3^3 : (2,3,1)$	$a_3^4 : (3,2,1)$	$\cdots$	$a_3^s : (3,2,1)$

Table 4.4: Table representing the preference lists for an instance of cyclic 3-DSSM that admits no strongly stable matching

Proof. Construct an instance of cyclic s-DSSM such that that preference lists for agents in $A^1, \ldots, A^4$ are as shown in Table 4.3. Add additional genders A^i, for $5 \leq i \leq s$ and, for each $i \geq 5$, let each a_j^i have the same preference list as a_j^4 (See Table 4.4). We claim this set of preference lists has no strongly stable matching. As in the proof of Lemma 4.4.2, for each $i \in \{1,2,3\}$, define the set $S_i := \{a_i^1, \ldots, a_i^s\}$.

Let M be any matching. Consider the instance of cyclic 4-DSSM and associated matching M' formed by removing the agents in $A^5, \ldots, A^s$. By Lemma 4.4.3, the matching M' is not strongly stable so there exists a blocking family (a,b,c,d). We will use these 4 agents to find a blocking family for M.

If $a \in A^1$ and $d \in A^4$ are from the same family $F \in M$, then we obtain a blocking family for M by removing the representatives of A^2 and A^3 from F and replacing them with b and c, respectively. Alternatively, if $a, d \in S_i$ for some $i \in \{1,2,3\}$, then $(a,b,c,d,a_i^5,\ldots,a_i^s)$ is a blocking family.

We now assume a and d are from different families in M and, for each $i \in \{1,2,3\}$, either $a \notin S_i$ or $d \notin S_i$. Fix $i,j \in \{1,2,3\}$ such that $a \in S_i$ and $d \in S_j$. Let $(a_{\ell_1}^1, a_{\ell_2}^2, a_{\ell_3}^3, d, a_{\ell_5}^5, \ldots, a_{\ell_s}^s) \in M$ be the family containing d. Consider the largest integer $k \in \{4,\ldots,s\}$, such that $a_{\ell_{k+1}}^{k+1} \in S_i$ (with addition taken modulo s). The family $(a,b,c,d,a_{\ell_5}^5,\ldots,a_{\ell_j}^k, a_i^{k+1},\ldots,a_i^s)$ is a blocking family in M. To see this, notice that other than a, b and c, each agent in the blocking family receives the same partner as in M or is matched with its favorite. Since (a,b,c,d) was a blocking family in M', we see that adding these additional agents creates a blocking family in M. We conclude that this is an instance of s-DSSM admits no strongly stable matching. $\square$

Proof of Theorem 1.4.6. Fix $s \geq 3$ and $n \geq 3$. By Lemmas 4.4.2, 4.4.3, and 4.4.4, there is an instance of cyclic s-DSSM of size 3 that admits no strongly stable matching. If $n = 3$, the proof is complete. If $n \geq 4$, we create an instance of cyclic s-DSM by adding $n-3$ agents of each gender and adjusting the preference lists as follows. The original $3s$ agents keep the first three choices in their preference lists unchanged and rank the remaining $s-3$ agents in any order. The preference lists of the new agents are chosen so that the $s(n-3)$ agents form $(n-3)$ preferred families.

Consider a matching M. Let a be newly added agent. Since a is in a preferred family, either the family generated by a is a blocking family, in which case M is not stable, or the family generated by a is in M. Therefore, each of the $s-3$ preferred families are in M. Since there is no stable matching among the original $3s$ agents, we conclude that M is not stable. $\square$

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