STABILITY THRESHOLDS FOR SIGNED LAPLACIANS ON LOCALLY–CONNECTED NETWORKS

BY

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DISSERTATION

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In this work we are interested in the stability bifurcations of the dynamical systems defined on graphs, and we use signed graph Laplacians as our tool. In chapter 1, we give the formal definition of the Laplacian matrix for a graph, and point out several references on it. In chapter 2, we give the main result from one of the references, along with other preliminaries we need for our results. In chapter 3, we give our first main result – finding the stable point for the Laplacians of one family of graphs, namely $C_n^2$. In chapter 4, we extend the previous definition and question to $C_n^{(k)}$, and we give the exact result for $k = 3$, along with a procedure to find the answer for general $k > 3$. We then give several conjectures about this topic, which are supported by numerical experiments.
To my parents, for their love and support.
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It is well known that many problems concerning networks can be studied as problems on graph theory. In particular, one object called Laplacian matrix for a graph is a useful tool to analysis the dynamics of the graph. We are very interested in this method and partial results have been achieved on this area.

In formal definition, we consider a simple, loop–free, connected, undirected, weighted labeled-graph $\Gamma = (V(\Gamma), E(\Gamma))$, where $V(\Gamma) = \{1, 2, \cdots, |V|\}$ is the vertices of $\Gamma$ (this is the meaning of labeled-graph), and $E(\Gamma)$ is the edges of $\Gamma$. Edge $(i, j)$ connects vertices $i$ and $j$ and will have weight $\gamma_{ij}$. Of course $\gamma_{ij} = 0$ means $i$ and $j$ are not adjacent, otherwise the weights can be both positive and negative. Then the signed Laplacian matrix is defined by:

$$L(\Gamma)_{ij} = \begin{cases} 
\gamma_{ij}, & i \neq j, \\
-\sum_{k \neq i} \gamma_{ik}, & i = j.
\end{cases}$$

It is a symmetric matrix, so all the eigenvalues must be real. We are interested in the number of positive, zero, negative eigenvalues, denoting these by $n_+, n_0$ and $n_-$. The signed Laplacian matrix has been studied in [7], but has a long history, see [15], [9], [3], [1], [4], [8] and [2], where in the last reference the authors used terminology Kirchhoff matrix and $-1$ for edge weights.
2.1 Previous Results

In [7], possible bounds on $n_+, n_0, n_-$ above have been given only by the topological information of the graph, i.e., connectivity and the sign of edge weights.

For the graph $\Gamma$ satisfying conditions above, define two subgraphs of $\Gamma$, namely $\Gamma_+$ and $\Gamma_-$, to be the subgraphs with same vertices set as $\Gamma$, but with only positive-weighted edges and negative-weighted edges in $\Gamma$, respectively. Let $\mathcal{L}_+, \mathcal{L}_-$ stand for the Laplacian matrix of $\Gamma_+$ and $\Gamma_-$, when graph $\Gamma$ is unambiguous.

Clearly the subgraphs $\Gamma_+, \Gamma_-$ are not necessarily connected. Denote $c(\Gamma_+)$ and $c(\Gamma_-)$ the number of connected components of these subgraphs.

We define the flexibility of a weighted graph as the number $\tau(\Gamma) = n + 1 - c(\Gamma_+) - c(\Gamma_-)$, where $n$ stands for the order of the graph $\Gamma$. It turns out the flexibility is always non-negative and when $\tau(\Gamma) = 0$, the graph is called rigid. In particular, the flexibility $\tau(\Gamma)$ is of great importance.

The main result is as following [[7], Theorem 2.10]:

**Theorem 2.1.** Let $\Gamma$ be a connected signed graph, and let $n_+(\Gamma)$, $n_0(\Gamma)$, $n_-(\Gamma)$ be the number of negative, zero, and positive eigenvalues, respectively. Then for any choice of weights one has the following inequalities:

\[
\begin{align*}
  c(\Gamma_+) - 1 & \leq n_+(\Gamma) \leq n - c(\Gamma_-), \\
  c(\Gamma_-) - 1 & \leq n_-(\Gamma) \leq n - c(\Gamma_+), \\
  1 & \leq n_0(\Gamma) \leq n + 2 - c(\Gamma_-) - c(\Gamma_+). 
\end{align*}
\]  

(2.1)

Furthermore, these bounds are tight: for any given graph there exist open
sets of weights giving the maximal number of negative eigenvalues

\[ n_+(\Gamma) = c(\Gamma_+) - 1, \quad n_-(\Gamma) = n - c(\Gamma_+), \quad n_0(\Gamma) = 1, \]

as well as open sets of weights giving the maximal number of positive eigenvalues

\[ n_+(\Gamma) = n - c(\Gamma_-), \quad n_-(\Gamma) = c(\Gamma_-) - 1, \quad n_0(\Gamma) = 1. \]

**Remark 2.2 ([7], Remark 2.11).** In each inequality in (2.1), the difference between the upper and lower bounds is the flexibility of the graph \( \tau(\Gamma) \). This shows that \( \tau(\Gamma) \) counts the number of eigenvalue crossings: there are \( c(\Gamma_+) - 1 \) eigenvalues which are always negative, \( c(\Gamma_-) - 1 \) eigenvalues which are always positive, and \( \tau(\Gamma) = n + 1 - c(\Gamma_-) - c(\Gamma_+) \) eigenvalues whose signs depend on the choice of weights. For rigid graphs there are no eigenvalue crossings and the index is independent of the choice of weights.

The proof for the theorem relies on a family of one-parameter graphs \( \Gamma(t) \overset{\text{def}}{=} \Gamma_+ + t\Gamma_- \). Clearly \( \Gamma = \Gamma(1) \) and \( \mathcal{L}(\Gamma(t)) = \mathcal{L}_+ + t\mathcal{L}_- \). Not only does the proof require the family of weighted graphs \( \Gamma(t) \), but nice properties come with them as well.

As in dynamical systems, we call the graph Laplacian **stable** if \( n_+(\mathcal{L}(\Gamma)) = 0 \), i.e., \( \mathcal{L}(\Gamma) \) has no positive eigenvalues. Back to our graphs \( \Gamma(t) \), by the Lemma 2.18 of [7], we know that \( n_+(\mathcal{L}(\Gamma(t))) \) is nondecreasing on \( t \). So if we define

\[ t^*(\Gamma) \overset{\text{def}}{=} \sup_{t \geq 0} \{ n_+(\mathcal{L}(\Gamma(t))) = 0 \}, \]

then we have stable Laplacian when \( t \leq t^*(\Gamma) \). Since there is a bifurcation at the special point \( t^* \), we really want to analyze it and try to compute this value. The computation of this number for various families of graphs is the topic of this thesis.

Notice 0 is always an eigenvalue of the Laplacian matrix \( \mathcal{L} \). For graph with \( n \) vertices, order the \( n \) eigenvalues in \( \mathcal{L} \) such that \( \lambda_1 = 0 \), then define

\[ \mathcal{M}(\Gamma) = \frac{(-1)^{n-1}}{n} \prod_{i=2}^{n} \lambda_i. \quad (2.2) \]

Then stable is equivalent to the case where \( \mathcal{M}(\Gamma(t)) \geq 0 \).
2.2 \( M(\Gamma) \) and the Characteristic Polynomial

In previous section we mentioned how to get the desired \( t^* \) through \( M(\Gamma(t)) \). Here we claim that this value is connected with the characteristic polynomial of Laplacian matrix \( L(\Gamma(t)) \).

**Definition 2.1.** In our work, \( p_A(\lambda) \overset{\text{def}}{=} \det(A - \lambda I) \) is called the characteristic polynomial of matrix \( A \), where \( I \) is the identical matrix.

**Theorem 2.3.** Let \( A \) be a matrix with a single zero eigenvalue. The product of the non-zero eigenvalues is the negative linear coefficient in the characteristic polynomial.

**Proof.** Let \( \lambda_i \)'s be eigenvalues of \( A \) s.t. \( \lambda_1 = 0 \).

For the characteristic polynomial of \( A \), we have

\[
P_A(\lambda) = \det(A - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda). \tag{2.3}
\]

Consider the constant item of \( P_A(\lambda) \), it is \( \det(A) = \prod_{i=1}^{n}(\lambda_i) = 0 \), since \( \lambda_1 = 0 \).

Consider the linear term, it is

\[
\left( \sum_{i=1}^{n} \prod_{j \neq i} \lambda_j \right) (-\lambda) = -\lambda \prod_{j \neq 1} \lambda_j - \lambda_1 \lambda \sum_{i=2}^{n} \prod_{j \neq 1, i} \lambda_j \tag{2.4}
\]

\[
= - \left( \prod_{j=2}^{n} \lambda_j \right) \lambda.
\]

So the coefficient of the linear term is \(-1\) times the product of nonzero eigenvalues.

**Remark 2.4.** Recall from (2.2) that

\[
M(\Gamma) = \frac{(-1)^{n-1}}{n} \prod_{i=2}^{n} \lambda_i. \tag{2.5}
\]

So \( M(\Gamma) \) is \( \frac{(-1)^n}{n} \) times (coefficient of) the linear term in the Characteristic Polynomial of the Laplace matrix.
2.3 Preliminaries

We now introduce some preliminaries for the main theorems of this work.

2.3.1 Deletion–contraction theorem

Following lemma is the re-writing of [7], Theorem 3.2.

**Lemma 2.5 (Deletion–Contraction Theorem).** If \( e = (i, j) \) be an edge of graph \( \Gamma \) and it is not a loop, then

\[
\mathcal{M}(\Gamma) = \mathcal{M}(\Gamma \setminus e) + \gamma_e \mathcal{M}(\Gamma \cdot e),
\]

(2.6)

where \( \Gamma \cdot e \) stands for the image of \( \Gamma \) by projecting the two vertices of \( e \) into one vertex. Precisely, \( \Gamma \cdot e \) is obtained from \( \Gamma \) by identifying node \( i \) and node \( j \) as a single node \( i^* \), along with edge weight \( \gamma_{i^*,h} = \gamma_{i,h} + \gamma_{j,h} \) for any \( h \notin \{i, j\} \).

This proof can be found in Chapter 13.2 of [12].

2.3.2 Kirchhoff’s Matrix Tree Theorems

One way of getting \( \mathcal{M}(\Gamma) \) is through the Kirchhoff’s matrix tree theorem.

**Definition 2.2.** Let \( \mathcal{L} = \mathcal{L}(\Gamma) \) stand for the Laplacian matrix of some simple, loop-free, connected, undirected, weighted labeled-graph \( \Gamma = (V, E) \). Let minor \( M_{i,j} \) of \( \mathcal{L} \) to be the determinant of the sub-matrix by deleting \( i \)th row and \( j \)th column of \( \mathcal{L} \). Let \( C_{i,j} = (-1)^{i+j} M_{i,j} \) to be the \((i, j)\)-cofactor. Let \( Q_{i,j} \) stand for the complement of \((i, j) - (i, j)\) minor of \( \mathcal{L} \), i.e.,

\[
Q_{i,j} = \det(\text{submatrix of } \mathcal{L} \text{ by removing } i \text{th and } j \text{th rows and columns of } \mathcal{L}).
\]

**Lemma 2.6 (Kirchhoff’s Matrix Tree Theorem).** For an un-weighted connected graph \( \Gamma \) (i.e. weight = 1 on every edge) with at least two vertices, all the cofactors of Laplacian \( \mathcal{L}(\Gamma) \) are equal, and the value of each cofactor is \( \pm \#\mathcal{T}(\Gamma) \). Precisely, denote \( n = |V| \) to be the order of \( \Gamma \), then for any \( 1 \leq i, j \leq n \),

\[
\#\mathcal{T}(\Gamma) = (-1)^{n+1} C_{i,j}.
\]

(2.7)
This is the re-writing of the theorem in [2], where the authors used notation Kirchhoff matrix and $-1$ as edge weight, rather than our Laplacian matrix and $1$ weight setting. And that is why we have the term $(-1)^{n+1}$ which is different from the original reference.

**Lemma 2.7 (Weighted matrix tree theorem, [15] Theorem VI.29).** Let $\Gamma$ be a connected, weighted graph, then

$$M(\Gamma) = \sum_{T \in \mathcal{ST}(\Gamma)} \pi(T),$$

(2.8)

where $\pi(T)$ is the product over the edge weights in tree $T$, $\mathcal{ST}(\Gamma)$ is the set of all the spanning trees in $\Gamma$.

**Remark 2.8.** Notice both Lemma 2.7 and Lemma 2.6 are variants of Kirchhoff’s matrix tree theorem.

### 2.3.3 Edge lemma

As Kirchhoff’s matrix tree theorems are about the relation between cofactor $C_{i,j}$ and spanning trees, we have discovered similar results for $Q_{i,j}$, that is, relation between second order cofactor and spanning trees under certain conditions.

**Lemma 2.9.** Let $n = |V|$ be the order of $\Gamma$.

1. If $(i, j) \in E$, then $(-1)^n Q_{i,j}$ equals the number of spanning trees containing this edge $(i, j)$.

2. If $(i, j) \notin E$, then $(-1)^n Q_{i,j}$ equals the number of spanning trees in the projection graph $\Gamma^*$, where $\Gamma^*$ is obtained from $\Gamma$ by identifying node $i$ and node $j$ as a single node $i^*$, along with edge weight $\gamma_{i^* h} = \gamma_{i h} + \gamma_{j h}$ for any $h \notin \{i, j\}$.

**Proof.** 1. Without loss of generality, we can assume this edge is $(1, 2) \in E$, otherwise we can always re-label the nodes of $\Gamma$ to keep the desired edge as $(1, 2)$, in the meantime the number of spanning trees along with the corresponding minors of the Laplacian matrix won’t change.
Let $\Gamma' = \Gamma \setminus (i, j)$, and denote $M_{i,j}$ and $M'_{i,j}$ as the minor of $\Gamma$ and $\Gamma'$, respectively. By Kirchhoff’s Matrix Tree Theorem 2.6, we know

$$\#ST(\Gamma) = (-1)^{n+1} C_{i,j} = (-1) M_{1,1},$$
$$\#ST(\Gamma') = (-1)^{n+1} C_{i,j} = (-1) M'_{1,1}.$$

On the other hand,

$$\mathcal{L}(\Gamma') = \mathcal{L} - \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}_{n \times n}.$$

So

$$M'_{1,1} = M_{1,1} - \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0\end{bmatrix}_{(n-1) \times 1} \text{ submatrix of } \mathcal{L} \text{ by removing first two rows and columns }_{(n-1) \times (n-1)}.$$

So we have $\#ST(\Gamma) - \#ST(\Gamma') = (-1)^{n+1}(M_{1,1} - M'_{1,1}) = (-1)^{n} Q_{1,2}$, where $\#ST(\Gamma) - \#ST(\Gamma')$ is the number of spanning trees containing edge $(1, 2)$.

2. Define $\Gamma^+$ to be the augmented graph of $\Gamma$ by adding edge $(i, j)$. By Deletion-Contraction Theorem 2.5, we know

$$\#ST(\Gamma^+) = \#ST((\Gamma^+ \setminus (i, j)) + \#ST((\Gamma^+).(i, j))$$
$$= \#ST(\Gamma) + \#ST(\Gamma^*).$$

On the other hand, corresponding $Q_{i,j}$ for $\Gamma^+$ is the same as that for $\Gamma$,
since \(i\)th and \(j\)th rows and columns are removed. By part (1) we know

\[
(-1)^n Q_{i,j} = \#(\text{spanning trees containing edge } (i, j) \text{ in } \Gamma^+) \\
= \#ST(\Gamma^+) - \#ST(\Gamma) \\
= \#ST(\Gamma^*).
\]

2.4 Extend domain of the Laplacian

**Question:** If we take \(\Omega\) to be some domain in \(\mathbb{R}^n\), and define the Laplacian operator of a function \(f(x)\) on \(\Omega\) as following:

\[
\mathcal{B} f(x) \overset{\text{def}}{=} \int_\Omega \gamma(x, y)(f(y) - f(x)) \, dy. \tag{2.9}
\]

\[
Q(f) \overset{\text{def}}{=} \langle f, -\mathcal{B} f \rangle = -\int_\Omega f(x) \mathcal{B} f(x) \, dx. \tag{2.10}
\]

We can ask: what conditions on \(\gamma(x, y)\) suffice to make \(Q(f) \geq 0\)?

Remember if we consider the discrete Laplacian Matrix,

\[
\mathcal{L}(\Gamma)_{ij} = \begin{cases} 
\gamma_{i,j}, & i \neq j \\
- \sum_{k \neq i} \gamma_{ik}, & i = j,
\end{cases} \tag{2.11}
\]

under the assumptions:

\[
\gamma_{ij} \geq 0(i \neq j), \quad \gamma_{ij} = \gamma_{ji}, \quad \sum_j \gamma_{ij} = 0,
\]
we will have \( x^T(-\mathcal{L})x \geq 0 \) for any vector \( x = (x_1, x_2, \ldots, x_n)^T \).

\[
x^T(-\mathcal{L}(\Gamma))x = -\sum_i \gamma_{ii} x_i^2 - \sum_{i \neq j} \gamma_{ij} x_i x_j \\
= \sum_i \left( \sum_{k \neq i} \gamma_{ik} x_i^2 - \sum_{i \neq j} \gamma_{ij} x_i x_j \right) \\
= \sum_i \sum_{j \neq i} \gamma_{ij} (x_i^2 + x_i x_j) \\
= \sum_{i<j} \gamma_{ij} (x_i^2 + 2x_i x_j + x_j^2) \\
= \sum_{i<j} (x_i - x_j)^2 \geq 0.
\]

**Proposition 2.10.** \( Q(f) \) is positive semi-definite if

\[
\gamma(x, y) \geq 0 (x \neq y), \quad \gamma(x, y) = \gamma(y, x).
\]

**Proof.**

\[
Q(f) = -\int_\Omega f(x) B f(x) dx \\
= -\int_\Omega f(x) \int_\Omega \gamma(x, y) (f(y) - f(x)) dy dx \\
= -\int_{\Omega^2} f(x) \gamma(x, y) (f(y) - f(x)) dy dx \\
= \int_{\Omega^2} \gamma(x, y) [f(x)^2 - f(x) f(y)] dy dx \\
= \int_{\Omega^2} \gamma(y, x) [f(x)^2 - f(x) f(y)] dy dx \\
= \int_{\Omega^2} \gamma(y, x) [f(x)^2 - f(x) f(y)] dx dy.
\]

The last step is due to Fubini’s Theorem. Swap \( x \) and \( y \), we also have

\[
Q(f) = \int_{\Omega^2} \gamma(x, y) [f(y)^2 - f(y) f(x)] dy dx.
\]
So
\[Q(f) = \frac{1}{2} \iint_{\Omega^2} \gamma(x, y)[f(x)^2 + f(y)^2 - 2f(y)f(x)]dy \, dx.
\]
\[= \frac{1}{2} \iint_{\Omega^2} \gamma(x, y)[f(x) - f(y)]^2dy \, dx \geq 0.
\]
This chapter is about one concrete example for finding the stable point \( t^* \) for the Laplacians of a family of graphs. This is our first main theorem.

**Definition 3.1.** For \( n \geq 5 \), the Square of a Cycle \( C_n^2 \) is a graph on vertices set \( \{1, 2, \cdots, n\} \), where the weighted-edge \( e_{ij} = (i, j) \) has weight:

\[
e_{ij} = \begin{cases} 
1 & |j - i| \leq 2 \pmod n, \\
0 & \text{otherwise.} 
\end{cases} \tag{3.1}
\]

Below is the graph for \( C_8^2 \).

![Graph of \( C_8^2 \)](image)

Again we denote \( ST(G) \) as the set of all spanning trees in graph \( G \). By the results of Baron[2] (or Kleitman[13], independent works), we have:

**Theorem 3.1.** For \( n \geq 5 \), the number of spanning trees of the square of a cycle is given by \( nF_n^2 \), where \( F_n \) is the Fibonacci Numbers (see Appendix Example A.4) defined by

\[
F_n = \begin{cases} 
F_{n-1} + F_{n-2}, & n \geq 3, \\
1, & n \in \{1, 2\}. 
\end{cases}
\]

In other words, \( \#ST(C_n^2) = nF_n^2, n \geq 5 \).
Both proofs in [13] and [2] are concise where the former one is a pure combinational proof and the latter one is achieved by Kirchhoff’s Matrix Tree Theorem. However here we prefer the latter proof since we can modify the arguments for our following results.

**Definition 3.2.** Consider a weighted graph $\Gamma_n(t)$ as following: for $n \geq 5$, the vertex set is $\{1, 2, \cdots, n\}$, and we define the weights

$$
\gamma_{ij} = \begin{cases} 
-t & i = 1, j = 2 \\
1 & |j - i| \leq 2 \mod n, \quad \{i, j\} \neq \{1, 2\} \\
0 & \text{otherwise}
\end{cases} \tag{3.2}
$$

So we can consider $\Gamma_n(t)$ as graph $C^2_n$ with weight 1 on each edge except weight $-t$ on the edge $(1, 2)$.

Inheriting notations in previous chapter, we have $\mathcal{L}(\Gamma_n(t))$, $\mathcal{M}(\Gamma_n(t))$, respectively.

Also define

$$t^*(\Gamma_n) \overset{def}{=} \sup_{t \geq 0} \{n_+ (\mathcal{L}(\Gamma_n(t))) = 0\},$$

then we have stable Laplacian when $t \leq t^*(\Gamma_n)$. As mentioned before, $\mathcal{L}(\Gamma_n)$ is stable when $\mathcal{M}(\Gamma_n) \geq 0$. In this sense, $t^* = \sup_{t \geq 0} \{\mathcal{M}(\Gamma_n(t)) \geq 0\}$.

The main theorem of this chapter is that we can get the exact limit of $t^*(\Gamma_n)$ as $n \to \infty$.

**Theorem 3.2.** If we let $n \to \infty$, then we get a corresponding weighted graph $\Gamma_\infty$ and graph Laplacian $\mathcal{L}(\Gamma_\infty(t))$. For the stable bifurcation point, we have

$$t^*(\Gamma_\infty) \overset{def}{=} \lim_{n \to \infty} t^*(\Gamma_n) = \sqrt{5} - 1. \tag{3.3}$$

Before the proof we shall introduce some new variables here. By Lemma 2.5, if we fix edge $(1, 2)$ in graph $\Gamma_n$ to be edge $e$, and define $B_n = \mathcal{M}(\Gamma_n \setminus e)$ and $R_n = \mathcal{M}(\Gamma_n.e)$, we have

$$\mathcal{M}(\Gamma_n) = \mathcal{M}(\Gamma_n \setminus e) - \mathcal{M}(\Gamma_n.e) t = B_n - t R_n, \tag{3.4}$$

$$t^*(\Gamma_n) = \sup_{t \geq 0} \{\mathcal{M}(\Gamma_n) \geq 0\} = \sup_{t \geq 0} \{B_n - t R_n \geq 0\} = \frac{B_n}{R_n}. \tag{3.5}$$
Now let think about the explanations of $B_n$ and $R_n$, by Lemma 2.7:

$$B_n = \mathcal{M}(\Gamma_n \setminus e) = \sum_{T \in \mathcal{ST}(\Gamma_n \setminus e)} \pi(T) = \#\mathcal{ST}(\Gamma_n \setminus e), \quad (3.6)$$

the last equality is because each edge weight in $\Gamma_n \setminus e$ is 1, so is each $\pi(T)$.

We use black lines to represent positive edges while red lines for negative, and in this example the only red edge is $e = (1, 2)$. By computation above, $B_n$ stands for the number of spanning trees in $\Gamma_n \setminus e$, which we can consider as the trees in the original graph $\Gamma_n$ with only the positive (black) edges. Conversely, $R_n$ stands for the number of spanning trees in $\Gamma_n$ containing the negative edge $e$. In this sense we can call $B_n$ the number of Black spanning trees and $R_n$ for the number of Red spanning trees, respectively, in $\Gamma_n$.

Notice $B_n + R_n = \#\mathcal{ST}(C_2^n) = nF_2^n$, and this is because the spanning trees of $\Gamma_n$ is the disjoint union of black and red spanning trees. So our question can be simplified to calculate $R_n$, the number of Red trees.

We use following notations for convenience.

**Definition 3.3.** For a sequence $\{x_n\}_{n \geq 0}$, we call $T$ the forward shift operator if $Tx_n = x_{n+1}$ holds for any $n \geq 0$.

If the sequence $\{x_n\}_{n \geq 0}$ has a recursive relation, i.e.,

$$\lambda_k x_{n+k} + \lambda_{k-1} x_{n+k-1} + \cdots + \lambda_0 x_n = 0, \quad (3.7)$$

holds for any $n \geq 0$ and some constants $\lambda_0, \lambda_2, \cdots, \lambda_k$, we define

$$p(T) \overset{\text{def}}{=} \lambda_k T^k + \lambda_{k-1} T^{k-1} + \cdots + \lambda_0, \quad (3.8)$$

then we say

$$p(T) (x_n) = 0, \quad \forall n \geq 0.$$ 

We also say the sequence $\{x_n\}_{n \geq 0}$ is in the kernel of $p(T)$.

**Lemma 3.3 ([2], Lemma 4).** The sequence $\{nF_2^n\}_{n \geq 1}$ is in the kernel of

$$p_2(T) \overset{\text{def}}{=} T^6 - 4T^5 + 10T^3 - 4T + 1. \quad (3.9)$$
Proof. From Appendix Example A.4, we know that

\[ F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n, \quad n \geq 1, \quad (3.10) \]

then

\[ nF_n^2 = \frac{n}{5} \left[ \left( \frac{3 + \sqrt{5}}{2} \right)^n + \left( \frac{3 - \sqrt{5}}{2} \right)^n - 2(-1)^n \right]. \quad (3.11) \]

By Appendix Theorem A.3, the recursive relation should have double-roots \( \frac{3 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2} \), so \( nF_n^2 \) is in the kernel of

\[
\left( T - \frac{3 + \sqrt{5}}{2} \right)^2 \cdot \left( T - \frac{3 - \sqrt{5}}{2} \right)^2 \cdot (T + 1)^2 = (T^2 - 3T + 1)^2 \cdot (T + 1)^2 = T^6 - 4T^5 + 10T^3 - 4T + 1.
\]

Lemma 3.4. Sequences \( \{ (n+1)F_{n-1}F_n \}_{n \geq 2} \) and \( \{ (n - 1)F_nF_{n+1} \}_{n \geq 2} \) are also in the kernel of \( p_2(T) \) as in (3.9).

Proof. Similar as in the proof of Lemma 3.3.

By (3.10), we have

\[
(n + 1)F_{n-1}F_n = \frac{n + 1}{5} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \cdot \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n-1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n-1} \right] \\
= \frac{n + 1}{5} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{2n-1} + \left( \frac{1 - \sqrt{5}}{2} \right)^{2n-1} + (-1)^n \right] \\
= \frac{\sqrt{5} - 1}{10} (n + 1) \left( \frac{3 + \sqrt{5}}{2} \right)^n + \frac{1 + \sqrt{5}}{10} (n + 1) \left( \frac{3 - \sqrt{5}}{2} \right)^n \\
+ \frac{n + 1}{5} (-1)^n.
\]

By Appendix Theorem A.3, we have same double-roots as in Lemma 3.3, so we shall also have the same recursive relation as in (3.9). Same argument for
the item \((n - 1)F_n F_{n+1}\).

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** For \(n \geq 5\), following matrix stands for graph Laplacian \(L(C_n^2)\).

\[
\begin{array}{cccccccccccc}
-4 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\
1 & -4 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\
1 & 1 & -4 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & -4 & 1 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & -4 & 1 & 1 \\
1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & -4 & 1 \\
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & -4 \\
\end{array}
\]

Let \(V_{n-1}\) be its cofactor at position \((n, n)\), and \(v_k = \det V_k\), then we have \(X_n \stackrel{\text{def}}{=} \#ST(C_n^2) = (-1)^{n+1} \cdot v_{n-1}\) by Kirchhoff’s Matrix Tree Theorem.
Straightforward calculation shows following:

\[
v_k = \det V_k = \det \begin{pmatrix} -4 & 1 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -4 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & -4 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 1 & -4 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 & -4 \\ 1 & 0 & \cdots & 0 & 1 & 1 & -4 \end{pmatrix}
\]

\[
= (-1)^{k+1} \det \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 1 \\ -4 & 1 & 1 & 0 & \cdots & 0 \\ 1 & -4 & 1 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 1 & -4 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 & -4 \\ 1 & 1 & -4 & 1 & 1 \end{pmatrix} + \det \begin{pmatrix} -4 & 1 & 1 & 0 & \cdots & 0 \\ 1 & -4 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & -4 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 & -4 \\ 1 & 1 & -4 & 1 & 1 \end{pmatrix}
\]

\[
= (-1)^{k+1+k} \det \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{pmatrix} + (-1)^{k+1} \det \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ -4 & 1 & 1 & 0 & \cdots & 0 \\ 1 & -4 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 1 & -4 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 & -4 \\ 1 & 1 & -4 & 1 & 1 \end{pmatrix}
\]

\[
+ (-1)^{k+1} \det \begin{pmatrix} 1 & -4 & 1 \\ 1 & 1 & -4 \\ 0 & \cdots & 0 & 1 & 1 \\ 1 & 1 & -4 \\ 0 \end{pmatrix} + \det \begin{pmatrix} -4 & 1 & 1 & 0 & \cdots & 0 \\ 1 & -4 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & -4 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & 1 & -4 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 & -4 \\ 1 & 1 & -4 \end{pmatrix}
\]

\[
= -\det A_{k-2} + (-1)^{k+1} \det C_{k-1} + (-1)^{k+1} \det C_{k-1}^T + \det A_k
\]

\[
= a_k - a_{k-2} + 2(-1)^{k+1} c_{k-1}.
\]

whereas \(a_k, c_k\) stand for the determinants of the matrices below, respectively:

\[
A_k \overset{\text{def}}{=} \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ \vdots & \ddots & \ddots \\ 1 & 1 & -4 \\ 1 & 1 & -4 \end{pmatrix}_{(k \times k)}, \quad C_k \overset{\text{def}}{=} \begin{pmatrix} -1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & A_{k-1} \\ -1 & -1 \end{pmatrix}_{(k \times k)}
\]

By [2] Lemma 3, we know sequence \(\{a_n\}\) is in the kernel of

\[
p_a(T) \overset{\text{def}}{=} T^5 + 5T^4 + 5T^3 - 5T^2 - 5T - 1 = (T - 1)(T^2 + 3T + 1)^2;
\]
while by [2] Lemma 2, \( \{c_n\} \) is in the kernel of
\[
p_c(T) \overset{\text{def}}{=} T^4 - T^3 - 4T^2 - T + 1 = (T + 1)^2(T^2 - 3T + 1).
\]

Notice \( X_{n+1} = (-1)^n v_n \), this means sequence \( \{X_n\} \) is in the kernel of the common multiple of \( p_a(-T) \) and \( p_c(T) \). And \( p_2(T) = (T + 1)^2(T^2 - 3T + 1) \) is the least common multiple, so \( p_2(T)(x_n) = 0 \). On the other hand, by Lemma 3.3, we know that \( nF_n^2 \) is also in the kernel of \( p_2(T) \). By comparing the initial cases \( n = 5, 6, \cdots, 10 \), we can show the two sequences are equal, see Table 3.1.

**Table 3.1:** compare initial values of \( X_n \) and \( nF_n^2 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_n )</td>
<td>125</td>
<td>384</td>
<td>1183</td>
<td>3528</td>
<td>10404</td>
<td>30250</td>
</tr>
<tr>
<td>( nF_n^2 )</td>
<td>125</td>
<td>384</td>
<td>1183</td>
<td>3528</td>
<td>10404</td>
<td>30250</td>
</tr>
</tbody>
</table>

**Remark 3.5.** The proof above is the re-writing of the main Theorem in [2]. However, since we shared different definitions for the Laplacian matrix (off by \(-1\) at any position), the reader would expect every recurrence should be applied with \( T \rightarrow -T \). To avoid the confusion driven by different notations, we present their results accordingly to make our work self-consistent.

Similar to the proof of Theorem 3.1, we can use Kirchhoff’s Matrix Tree Theorem (Lemma 2.6) to show following results.

**Lemma 3.6.** The sequence \( \{R_n\}_{n \geq 5} \) is in the kernel of
\[
p_a(-T) = T^5 - 5T^4 + 5T^3 + 5T^2 - 5T + 1.
\]

*Proof.* Keeping the notations as in previous proof, then matrix \( A_{n-2} \) can be considered as removing the first two rows and columns of the Laplacian matrix \( \mathcal{L}(C_n^2) \). Then by Lemma 2.9, we know
\[
(-1)^n \cdot R_{n+2} = \det(A_n) = a_n.
\]

Thus sequence \( \{R_n\} \) is in the kernel of \( p_a(-T) \).
Theorem 3.7.

\[ R_n = \frac{1}{5}[(n + 1)F_{n-1}F_n + (n - 1)F_nF_{n+1}], \quad n \geq 5. \]  

(3.14)

Proof. Notice

\[(T^5 - 5T^4 + 5T^3 + 5T^2 - 5T + 1) \cdot (T + 1) = T^6 - 4T^5 + 10T^3 - 4T + 1.\]

So \(R_n\) is also in the kernel of (3.9). By Lemma 3.4, we know right hand side of (3.14) also satisfies (3.9). Notice it is a 6-degree recurrence, so it is sufficient to compare the first 6 elements of both sequences. Remember we assume \(n \geq 5\) for graph \(\Gamma_n\), so we prove the result by checking the initial cases \(n = 5, 6, \cdots, 10\), see Table 3.2.

Table 3.2: compare initial values of \(R_n\) and the other sequence

<table>
<thead>
<tr>
<th>(n)</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_n)</td>
<td>50</td>
<td>160</td>
<td>494</td>
<td>1491</td>
<td>4420</td>
<td>12925</td>
</tr>
<tr>
<td>(\frac{1}{5}[(n + 1)F_{n-1}F_n + (n - 1)F_nF_{n+1}])</td>
<td>50</td>
<td>160</td>
<td>494</td>
<td>1491</td>
<td>4420</td>
<td>12925</td>
</tr>
</tbody>
</table>

Now we are in the position to prove the main theorem of this chapter.

Proof of Theorem 3.2. From Appendix A.4, we know that

\[ \frac{F_{n+1}}{F_n} \to \frac{\sqrt{5} + 1}{2} = \phi, \]

where \(\phi\) is the Golden Ratio, so we have

\[ R_n B_n + R_n = \frac{1}{5}[(n + 1)F_{n-1} + (n - 1)F_{n+1}] \xrightarrow{n\to\infty} \frac{1}{5}(\phi^{-1} + \phi) = \frac{1}{\sqrt{5}}. \]  

(3.15)

Then by equation (3.5), we have

\[ t^*(\Gamma_n) = \frac{B_n}{R_n} = \frac{B_n + R_n}{R_n} - 1 \to \sqrt{5} - 1, \]  

(3.16)

\[ t^*(\Gamma_\infty) = \lim_{n\to\infty} t^*(\Gamma_n) = \sqrt{5} - 1. \]  

(3.17)
Proposition 3.8 ([7], Proposition 3.6). Let $\Gamma$ be a $N$-vertex graph with one single negative edge, and $\gamma_{i,j} = \pm 1$. Assume $\Gamma_+$ is connected and set $\Gamma(t) = \Gamma_+ + t\Gamma_-$, then

$$t^*(\Gamma(t)) \in \left[\frac{1}{N-1}, \frac{N-2}{2}\right].$$

(3.18)

The extremes are attained by the ring graph and the complete graph, respectively.

Remark 3.9. Clearly our example fits Proposition 3.8. And it gives a non-trivial case: there exists a family of graphs whose $t^*_n$ is bounded above and below as $n$ goes to infinity. Also notice that $t^*(\Gamma_{\infty}) = \sqrt{5} - 1 = 2\Phi$, where $\Phi = \frac{\sqrt{5}-1}{2}$ is the Golden Ratio Conjugate.
CHAPTER 4

MAIN THEOREM 2, EXTENSION ON $C_N^{(K)}$ FOR $K \geq 3$

Definition 4.1. For $n \geq 2k + 1$, the circulant graph $C_n^{(k)}$ is a graph on vertices set \{1, 2, \cdots, n\}, where the weighted-edge $e_{ij} = (i, j)$ has weight:

\[
e_{ij} = \begin{cases} 
1 & |j - i| \leq k \mod n, \\
0 & \text{otherwise}. 
\end{cases} \quad (4.1)
\]

Remark 4.1. Notice $C_n^{(k)}$ is a generalization of $C_n^{2}$ in equation (3.1). Accordingly, for $n \geq 2k + 1$, we can redefine (3.2) as following.

Definition 4.2. For $n \geq 2k + 1$, graph $\Gamma_n^{(k)}$ is defined to be the graph with vertices set \{1, 2, \cdots, n\} and following weights:

\[
\gamma_{ij}^{(k)} = \begin{cases} 
-t & i = 1, j = 2 \\
1 & |j - i| \leq k \mod n, \quad \{i, j\} \neq \{1, 2\} \\
0 & \text{otherwise}. 
\end{cases} \quad (4.2)
\]

In another word, we consider graph $\Gamma_n^{(k)}$ as the circulant graph $C_n^{(k)}$ with negative weight $-t$ on $(1, 2)$-edge.

Keep notations $L, M, t^*$ as in previous chapters. A natural question arises: for general $k > 2$, can we have similar results as in Theorem 3.2 and get a close form for the bifurcation point $t^*(\Gamma_n^{(k)})$ as $n \to \infty$?

We have several positive results in this direction, although the question is more complex than it is for $k = 2$. An apparent difficulty for extending the result from $C_n^{2}$ to $C_n^{(k)}$ is that the recurrence relation may not be as simple as a six degree recurrence in (3.1).
4.1 Analytic Proofs

We are going to give an result for $k = 3$, and introduce a theoretical method of finding such recurrences for general $k$. To do that, we need to find the recurrence of $\#ST(C_n^{(k)})$ first.

4.1.1 Recurrence of $\#ST(C_n^{(k)})$ for $k \geq 3$

**Definition 4.3.** For $n \geq 2$, matrix $S_n = (s_{ij})_{n \times n}$ is defined to be

$$s_{i,j} = \begin{cases} 
1, & j = i + 1 \mod n, \\
0, & \text{otherwise}.
\end{cases}$$

i.e.,

$$S_n \overset{\text{def}}{=} \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 & \cdots \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}_{n \times n}.$$

Naturally $(S_n)^n = I_n$, the identity matrix.

For real $\lambda$, define $D_n(\lambda) = \det(S_n + (S_n)^{-1} - \lambda I_n)$.

Further, define $D_n^{(2)}(\lambda) = \det(S_n + (S_n)^{2} + (S_n)^{-1} + (S_n)^{-2} - \lambda I_n)$.

For general $k \geq 3$, define

$$D_n^{(k)}(\lambda) = \det \left( \sum_{i=1}^{k} (S_n)^i + \sum_{i=1}^{k} (S_n)^{-i} - \lambda I_n \right).$$

**Remark 4.2.** Using notations above, then the Laplacian matrix for $C_n^{(k)}$ is

$$\mathcal{L}(C_n^{(k)}) = \sum_{i=1}^{k} (S_n)^i + \sum_{i=1}^{k} (S_n)^{-i} - 2k I_n.$$ 

Set $\lambda = 2k$, then $D_n^{(k)}(2k) = D_n^{(k)}(\lambda) \big|_{\lambda=2k} = \det \left( \mathcal{L}(C_n^{(k)}) \right) = 0$.

**Theorem 4.3.** We can get the number of spanning trees in $C_n^{(k)}$ by computing
the first derivative of $D_n^{(k)}(\lambda)$. Precisely,

$$
\left. \frac{dD_n^{(k)}(\lambda)}{d\lambda} \right|_{\lambda = 2k} = (-1)^n \cdot n \cdot \#ST(C_n^{(k)}).
$$

(4.3)

Proof. Fix $n$ and $k$, within this proof we use $L = L(C_n^{(k)})$ for short.

Let *minor* $M_{i,j}$ of $L$ to be the determinant of the sub-matrix by deleting $i$th row and $j$th column of $L$.

Let $C_{i,j} = (-1)^{i+j} M_{i,j}$ to be the $(i,j)$-cofactor.

Notice $L$ is the Laplacian matrix for an unweighted graph $C_n^{(k)}$, so by Kirchhoff’s Matrix Tree Theorem 2.6, we know for all $1 \leq i, j \leq n$, cofactors $C_{i,j}$’s are equal, and $\#ST(C_n^{(k)}) = (-1)^{n+1} C_{i,j}$. In particular,

$$
\#ST(C_n^{(k)}) = (-1)^{n+1} C_{i,j} = (-1)^{n+1} C_{i,i} = (-1)^{n+1} \cdot (-1)^{i+j} M_{i,i} = (-1)^{n+1} M_{1,1}.
$$

(4.4)

Let $L = (\overrightarrow{l}_1, \overrightarrow{l}_2, \cdots, \overrightarrow{l}_n)$, where $\overrightarrow{l}_i \in \mathbb{R}^n$. Let $\{\overrightarrow{e}_i\}_{i=1}^n$ be the standard orthogonal basis of $\mathbb{R}^n$. Then

$$
\det(L - \varepsilon I_n) = \left| \overrightarrow{l}_1 - \varepsilon \overrightarrow{e}_1, \overrightarrow{l}_2 - \varepsilon \overrightarrow{e}_2, \cdots, \overrightarrow{l}_n - \varepsilon \overrightarrow{e}_n \right|
$$

$$
= \left| \overrightarrow{l}_1, \overrightarrow{l}_2 - \varepsilon \overrightarrow{e}_2, \cdots, \overrightarrow{l}_n - \varepsilon \overrightarrow{e}_n \right| - \varepsilon \left| \overrightarrow{e}_1, \overrightarrow{l}_2 - \varepsilon \overrightarrow{e}_2, \cdots, \overrightarrow{l}_n - \varepsilon \overrightarrow{e}_n \right|
$$

$$
= \left| \overrightarrow{l}_1, \overrightarrow{l}_2, \cdots, \overrightarrow{l}_n - \varepsilon \overrightarrow{e}_n \right| - \varepsilon \left| \overrightarrow{e}_1, \overrightarrow{l}_2, \cdots, \overrightarrow{l}_n - \varepsilon \overrightarrow{e}_n \right|
$$

$$
- \varepsilon \left| \overrightarrow{e}_1, \overrightarrow{l}_2, \cdots, \overrightarrow{l}_n - \varepsilon \overrightarrow{e}_n \right| + \varepsilon^2 \left| \overrightarrow{e}_1, \overrightarrow{e}_2, \cdots, \overrightarrow{l}_n - \varepsilon \overrightarrow{e}_n \right|
$$

$$
= \det(L) - \varepsilon \left| \overrightarrow{e}_1, \overrightarrow{l}_2, \cdots, \overrightarrow{l}_n \right| - \cdots - \varepsilon \left| \overrightarrow{l}_1, \cdots, \overrightarrow{l}_{n-1}, \overrightarrow{e}_n \right| + O(\varepsilon^2)
$$

$$
= -\varepsilon \sum_{i=1}^n M_{i,i} + O(\varepsilon^2),
$$

where the last step is due to the fact $\det(L) = 0$. Set $\varepsilon = \lambda - 2k$, then

$$
D_n^{(k)}(\lambda) = \det(L - (\lambda - 2k) I_n) = \det(L - \varepsilon I_n) = -\varepsilon \sum_{i=1}^n M_{i,i} + O(\varepsilon^2)
$$

$$
= -(\lambda - 2k) \sum_{i=1}^n M_{i,i} + O((\lambda - 2k)^2).
$$

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By the definition of derivative, we have

\[
\frac{d}{d\lambda} D^{(k)}_n(\lambda) = \lim_{h \to 0} \frac{D^{(k)}_n(\lambda + h) - D^{(k)}_n(\lambda)}{h}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left[ - (\lambda + h - 2k) \sum_{i=1}^{n} M_{i,i} + O((\lambda + h - 2k)^2) 
\right.
\]

\[
+ (\lambda - 2k) \sum_{i=1}^{n} M_{i,i} - O((\lambda - 2k)^2) \left. \right]
\]

\[
= \lim_{h \to 0} - \sum_{i=1}^{n} M_{i,i} + \frac{1}{h} O((\lambda + h - 2k)^2) - \frac{1}{h} O((\lambda - 2k)^2).
\]

Then at the point \( \lambda = 2k \), we have

\[
\left. \frac{dD^{(k)}_n}{d\lambda} \right|_{\lambda=2k} = \lim_{h \to 0} - \sum_{i=1}^{n} M_{i,i} + \frac{1}{h} O(h^2) = - \sum_{i=1}^{n} M_{i,i} = -n M_{1,1}.
\]

Combined with equation (4.4), we then have the desired result.

**Remark 4.4.** There is similar result for the second derivative of \( D^{(k)}_n(\lambda) \) on \( \lambda \), and it also connected with the number of spanning trees under certain conditions. We shall discuss it later, see Theorem 4.23.

**Theorem 4.5.** If sequence \( D^{(k)}_n(\lambda) \) fulfills some \( l \)-th order recurrence with coefficients \( \{a_j(\lambda)\}_{0 \leq j \leq l} \), then sequence \( \left. \frac{dD^{(k)}_n}{d\lambda} \right|_{\lambda=2k} \) fulfills the \( l \)-th order recurrence with coefficients \( \{a_j(2k)\} \).

**Proof.** Let’s assume for \( n \) sufficiently large, \( D^{(k)}_n(\lambda) \) fulfills

\[
\sum_{j=n}^{n+l} a_j(\lambda) D^{(k)}_j(\lambda) = 0.
\]

Taking the derivative, we have

\[
\sum_{j=n}^{n+l} \frac{d}{d\lambda} a_j(\lambda) D^{(k)}_j(\lambda) + a_j(\lambda) \frac{d}{d\lambda} D^{(k)}_j(\lambda) = 0.
\]

Set \( \lambda = 2k \), then \( D^{(k)}_n(2k) \) stands for the determinant of the Laplacian matrix
of \( C_n^{k} \), which is zero. Then we have
\[
\sum_{j=n}^{n+l} a_j (2k) \frac{d D^{(k)}_j}{d \lambda} (\lambda) \bigg|_{\lambda=2k} = 0.
\]

By above two theorems, the number of spanning trees of \( C_n^{(k)} \) can be calculated via sequence \( D_n^{(k)}(\lambda) \). So we are interested in getting (recurrence of) \( D_n^{(k)}(\lambda) \) from \( D_n(\lambda) \).

To do that we first introduce the tensor product of (monic) polynomials. Here we inherit the notation \( \otimes \) from [11] and [14], but with slightly modifications for our own purpose. More discussions and the origin on this topic can be found in [6] and [5], where the authors used notations \( \circ \) and \( \diamond \), for tensor product and general binary operation, respectively.

**Definition 4.4.** A monic polynomial is a univariate polynomial in which the nonzero coefficient of highest degree is equal to 1.

Let \( P(x) = \prod_{i=1}^{n} (x-r_i) \) and \( Q(x) = \prod_{j=1}^{m} (x-s_j) \) be two monic polynomials, then define their tensor product \( P(x) \otimes Q(x) \) to be the polynomial of degree \( mn \) with roots \( \{r_i s_j : 1 \leq i \leq n, 1 \leq j \leq m\} \) and initial coefficient 1, i.e.,
\[
P \otimes Q(x) = P(x) \otimes Q(x) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x - r_i s_j). \quad (4.5)
\]

Of course \( P \otimes Q \) is also a monic polynomial, with degree \( mn \).

**Remark 4.6.** Theorem 2.1 in [11] states that, if we denote
\[
P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,
\]
\[
Q(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,
\]
\[
P \otimes Q(x) = x^{mn} + c_{mn-1}x^{mn-1} + \cdots + c_1 x + c_0,
\]
then each \( c_k \) can be viewed as a polynomial in \( a_0, a_1, \ldots, a_{n-1} \) and \( b_0, b_1, \ldots, b_{m-1} \).
For example,

\[
(x^3 + a_2 x^2 + a_1 x + a_0) \otimes (x^2 + b_1 x + b_0) = \\
x^6 - a_2 b_1 x^5 + (a_2^2 b_0 + a_1 b_1^2 - 2a_1 b_0)x^4 + (3a_0 b_0 b_1 - a_1 a_2 b_0 b_1 - a_0 b_1^3)x^3 \\
+ (a_0 a_2 b_0 b_1^2 + a_1^2 b_0^2 - 2a_0 a_2 b_0^2)x^2 - a_0 a_1 b_0^2 b_1 x + a_0^2 b_0^3.
\]

\textbf{Proposition 4.7.} Let \( P, Q \) and \( R \) be monic polynomials in \( x \), and \( \deg(Q) = m \). Notation \( \cdot \) stands for point-wise product, then we have following statements.

1. \( P \otimes Q = Q \otimes P \).
2. \( P \otimes (Q \otimes R) = (P \otimes Q) \otimes R = P \otimes Q \otimes R \).
3. \( (P \cdot Q) \otimes R = (P \otimes R) \cdot (Q \otimes R) \).
4. \( (x - 1) \otimes Q = Q \).
5. \( (x + 1) \otimes Q(x) = (-1)^m \cdot Q(-x) \).

\textit{Proof.} (1)(2) By symmetry.

(3) Assuming \( P = \prod_{i=1}^{n}(x - r_i) \), \( Q = \prod_{j=1}^{m}(x - s_j) \), and \( R = \prod_{k=1}^{l}(x - t_k) \), then

\[
(P \otimes R) \cdot (Q \otimes R) = \left( \prod_{i,k}(x - r_i t_k) \right) \cdot \left( \prod_{j,k}(x - s_j t_k) \right) \\
= \prod_{k=1}^{l} \left( \prod_{i=1}^{n}(x - r_i t_k) \prod_{j=1}^{m}(x - s_j t_k) \right) = (P \cdot Q) \otimes R.
\]

The last step is due to the fact \( P \cdot Q = \prod_{i=1}^{n}(x - r_i) \prod_{j=1}^{m}(x - s_j) \).

(4) Set \( P(x) = x - 1 \). Then the only zero of \( P \) is 1, by definition, \( P \otimes Q \) shall have the same zero set as \( Q \), so \( P \otimes Q = Q \).

(5) Set \( P(x) = x + 1 \). Assuming \( Q = \prod_{j=1}^{m}(x - s_j) \), then

\[
P \otimes Q = \prod_{j=1}^{m}(x + s_j) = (-1)^m \prod_{j=1}^{m}(-x - s_j) = (-1)^m \cdot Q(-x).
\]
Remark 4.8. If \( P_1 \) and \( P_2 \) are two monic polynomials with same degree, then by propositions above we have

\[
(x + 1)P_1(x) \otimes (x + 1)P_2(x) \\
= ((x + 1) \otimes (x + 1)) \cdot ((x + 1) \otimes P_1) \cdot ((x + 1) \otimes P_2) \cdot (P_1 \otimes P_2) \\
= (x - 1) \cdot P_1(-x) \cdot P_2(-x) \cdot (P_1 \otimes P_2).
\]

**Theorem 4.9.** Given two monic polynomials \( P \) and \( Q \). If sequence \( \{x_n\} \) is in the kernel of \( P(T) \), \( \{y_n\} \) is in the kernel of \( Q(T) \), then the product sequence \( \{x_n \cdot y_n\} \) is in the kernel of \( P \otimes Q(T) \).

**Proof.** Assume \( P \) and \( Q \) has simple roots \( \{r_i\}_{i=1}^{\deg(P)} \) and \( \{s_j\}_{j=1}^{\deg(Q)} \) respectively, then by Appendix Theorem A.1,

\[
x_n = \sum_i a_i r_i^n, \quad y_n = \sum_j b_j s_j^n,
\]

for some constants \( a_i \)'s and \( b_j \)'s. Then \( x_n \cdot y_n = \sum_{i,j} a_i b_j (r_i s_j)^n \). Then

\[
P \otimes Q(T) (x_n y_n) = 0.
\]

Now assume \( P \) has a duplicate root \( r \) with multiplicity \( m_r \), and \( Q \) has a duplicate root \( s \) with multiplicity \( m_s \), then by Appendix Theorem A.2, the expressions for \( x_n, y_n \) contain terms \((a_1 + a_2 n + \cdots + a_{m_r-1} n^{m_r-1})r^n\) and \((b_1 + b_2 n + \cdots + b_{m_s-1} n^{m_s-1})s^n\) respectively. So item \((r s)^n\) is in the general term of \( x_n y_n \), with coefficient being a polynomial in \( n \) with degree \((m_r + m_s - 2)\). This means term \((T - rs)^{m_r + m_s - 1}\) must be a factor of the recurrence polynomial of \( \{x_n \cdot y_n\} \). However, by definition, we know term \((T - rs)^{m_r + m_s}\) is in \( P \otimes Q \). So we will still have \( P \otimes Q(T) (x_n y_n) = 0 \).

**Remark 4.10.** Notice in previous proof, \( r, s \)'s are not necessarily distinct and \( m_r + m_s - 1 < m_r + m_s \) when both of \( m_r \) and \( m_s \) is greater than 1. This means in general, the lowest–degree recurrence of the product sequence may have lower degree than the product of the degrees of the original sequences.

**Theorem 4.11.** Sequence \( \{D_n(\lambda)\}_{n \geq 3} \) is in the kernel of the 3rd-degree recurrence

\[
h_{\lambda}(T) \overset{\text{def}}{=} (1 + T)(1 + \lambda T + T^2) = T^3 + (1 + \lambda)T^2 + (1 + \lambda)T + 1.
\]
In other word, $D_{n+3} + (1 + \lambda)D_{n+2} + (1 + \lambda)D_{n+1} + D_n = 0$.

Proof.

$$D_k(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 0 & \cdots & 0 & 1 \\ 1 & -\lambda & 1 & 0 & \cdots & 0 \\ 0 & 1 & -\lambda & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -\lambda & 1 \\ 1 & 0 & \cdots & 0 & 1 & -\lambda \end{bmatrix},$$

$$= (-1)^{k+1} \det \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ -\lambda & 1 & 0 & \cdots & 0 \\ 1 & -\lambda & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 1 & -\lambda & 1 \end{bmatrix} + \det \begin{bmatrix} -\lambda & 1 & 0 & \cdots & 0 & 1 \\ 1 & -\lambda & 1 & 0 & \cdots & 0 \\ 0 & 1 & -\lambda & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -\lambda \end{bmatrix},$$

$$= (-1)^{k+1+k} \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda & 1 \\ \vdots & \vdots & \ddots \\ 1 & -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} + (-1)^{k+1} \det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\lambda & 1 & 0 & \cdots & 0 \\ 1 & -\lambda & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 1 & -\lambda \end{bmatrix} + (-1)^{k+1} \det \begin{bmatrix} 1 & -\lambda & 1 \\ 0 & 1 & -\lambda & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -\lambda \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$= - \det(R_{k-2}) + 2(-1)^{k+1} + \det(R_k).$$

whereas

$$\det(R_k) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda & 1 \\ \vdots & \vdots & \ddots \\ 1 & -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = -\lambda \det(R_{k-1}) - \det(R_{k-2}).$$

This means $\{D_n(\lambda)\}$ fulfills the recurrence of $\{\det(R_n)\}$ and $\{(-1)^{n+1}\}$. The former is in the kernel of $T^2 + \lambda T + 1$, the latter is in the kernel of $T + 1$. Two recurrence polynomials are prime, so $\{D_n(\lambda)\}$ is in the kernel of their product.

**Theorem 4.12.** Sequence $\{D_n^{(2)}(\lambda)\}_{n \geq 5}$ fulfills a $9$th-degree recurrence with coefficients in $\lambda$. 

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Proof. Let \( W_n = S_n + (S_n)^{-1} \), then \( D_n(\lambda) = \det(W_n - \lambda I_n) \) and

\[
D_n^{(2)}(\lambda) = \det \left( S_n + (S_n)^2 + (S_n)^{-1} + (S_n)^{-2} - \lambda I_n \right)
= \det \left( (W_n)^2 + W_n - (\lambda + 2)I_n \right)
= \det(W_n - \mu_+ I_n) \cdot \det(W_n - \mu_- I_n)
= D_n(\mu_+) \cdot D_n(\mu_-),
\]

where \( \mu_+ \) and \( \mu_- \) are the two roots of \( w^2 + w - (\lambda + 2) = 0 \).

By Theorem 4.11, we know that \( D_n(\mu_+) \) is in the kernel of

\[
h_+(T) = (T + 1)(1 + \mu_+ T + T^2) = (T + 1) \cdot g_+(T);
\]

\( D_n(\mu_-) \) is in the kernel of

\[
h_-(T) = (T + 1)(1 + \mu_- T + T^2) = (T + 1) \cdot g_-(T),
\]

whereas \( g_\pm(T) = 1 + \mu_\pm T + T^2 \).

Then by Theorem 4.9, \( D_n^{(2)} = D_n(\mu_+) \cdot D_n(\mu_-) \) is in the kernel of \( h_+(T) \otimes h_-(T) \).

By Remark 4.8,

\[
h_+(T) \otimes h_-(T) = (T - 1) \cdot g_+(T) \cdot g_-(T) \cdot (g_+(T) \otimes g_-(T)).
\]

By Vieta’s formulas, we know \( \mu_+ \mu_- = -\lambda - 2 \) and \( \mu_+ + \mu_- = -1 \), so we have

\[
g_+ \otimes g_- = (1 + \mu_+ T + T^2) \otimes (1 + \mu_- T + T^2)
= T^4 - (\mu_+ \mu_-)T^3 + (\mu_+^2 + \mu_-^2 - 2)T^2 - (\mu_+ \mu_-)T + 1
= T^4 + (\lambda + 2)T^3 + (2\lambda + 3)T^2 + (\lambda + 2)T + 1,
\]

\[
g_+(T) \cdot g_-(T) = (1 - \mu_+ T + T^2) \cdot (1 - \mu_- T + T^2)
= T^4 - (\mu_+ + \mu_-)T^3 + (2 + \mu_+ \mu_-)T^2 - (\mu_+ + \mu_-)T + 1
= T^4 + T^3 - \lambda T^2 + T + 1.
\]

Then \( h_+(T) \otimes h_-(T) \) has degree 9 and coefficients in \( \lambda \).

**Corollary 4.13.** Sequence \( \{\#ST(C_n^2)\}_{n \geq 5} \) is in the kernel of the 6th-degree recurrence \( p_2(T) \) as in (3.9).

(We got this result before in the proof of Theorem 3.1.)
Proof. Apply $\lambda = 4$ on Theorem 4.12 and Theorem 4.5, we know $\frac{dD_n^{(2)}(\lambda)}{d\lambda} \bigg|_{\lambda=4}$ is in the kernel of

\[(T - 1)(T^4 + T^3 - 4T^2 + T + 1)(T^4 + 6T^3 + 11T^2 + 6T + 1)
= (T - 1)^3(T^2 + 3T + 1)^3
= (T - 1)^3 \left( T - \frac{3 - \sqrt{5}}{2} \right)^3 \left( T - \frac{3 + \sqrt{5}}{2} \right)^3.
\]

By Appendix Theorem A.2, $\frac{dD_n^{(2)}(\lambda)}{d\lambda} \bigg|_{\lambda=4}$ has triple roots $1$, $-\frac{3+\sqrt{5}}{2}$ and $-\frac{3-\sqrt{5}}{2}$ in the general terms.

By Theorem 4.3, we know $H_n \overset{\text{def}}{=} n \cdot \#ST(C_n^2) = (-1)^n \cdot \frac{dD_n^{(2)}(\lambda)}{d\lambda} \bigg|_{\lambda=4}$. So $H_n$ have roots $-1$, $\delta_1 = \frac{3+\sqrt{5}}{2}$ and $\delta_2 = \frac{3-\sqrt{5}}{2}$ with general term

\[H_n = A_1 (-1)^n + A_2 n(-1)^n + A_3 n^2(-1)^n + B_1 n\delta_1^n + B_2 n\delta_1^n + B_3 n^2\delta_1^n + C_1 \delta_2^n + C_2 n\delta_2^n + C_3 n^2\delta_2^n,
\]

for some coefficients $A_i, B_i, C_i$’s, $1 \leq i \leq 3$.

Numerically we solved that (See Appendix Example B.1):

\[A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = 0, A_3 = -\frac{2}{5}, B_3 = C_3 = \frac{1}{5}.
\]

So we know $H_n = -\frac{2}{5} n^2(-1)^n + \frac{1}{5} n^2\delta_1^n + \frac{1}{5} n^2\delta_2^n$, and

\[\#ST(C_n^2) = \frac{H_n}{n} = -\frac{2}{5} n(-1)^n + \frac{1}{5} n\delta_1^n + \frac{1}{5} n\delta_2^n.
\]

Then by Appendix Theorem A.3, $\#ST(C_n^2)$ is in the kernel of

\[(T + 1)^2(T - \delta_1)^2(T - \delta_2)^2 = T^6 - 4T^5 + 10T^2 - 4T + 1. \quad (4.6)
\]

**Theorem 4.14.** Sequence $\{D_n^{(3)}(\lambda)\}_{n \geq 7}$ fulfills a 27th-degree recurrence with coefficients in $\lambda$.

**Proof.** Apply same idea as in the proof of Theorem 4.12.
Let \( W_n = S_n + (S_n)^{-1} \), then \( D_n(\lambda) = \det(\lambda I_n) \) and

\[
D_n^{(3)}(\lambda) = \det(S_n + (S_n)^{-1} + (S_n)^{-2} + (S_n)^{-3} - \lambda I_n) \\
= \det((W_n)^3 + (W_n)^2 - 2W_n - (\lambda + 2)I_n) \\
= \det(W_n - \nu_1 I_n) \cdot \det(W_n - \nu_2 I_n) \cdot \det(W_n - \nu_3 I_n) \\
= D_n(\nu_1) \cdot D_n(\nu_2) \cdot D_n(\nu_3),
\]

where \( \nu_1, \nu_2 \) and \( \nu_3 \) are the roots of \( w^3 + w^2 - 2w - (\lambda + 2) = 0 \).

By Theorem 4.11, we know for \( i = 1, 2, 3 \), \( D_n(\nu_i) \) is in the kernel of

\[
h_i(T) = (T + 1)(1 + \nu_i T + T^2) = (T + 1) \cdot g_i(T),
\]

whereas \( g_i(T) = 1 + \nu_i T + T^2 \).

Then by Theorem 4.9, \( D_n^{(3)} = D_n(\nu_1) \cdot D_n(\nu_2) \cdot D_n(\nu_3) \) is in the kernel of

\[
h_1(T) \otimes h_2(T) \otimes h_3(T).
\]

\[
h_1(T) \otimes h_2(T) \otimes h_3(T) \\
= [(T + 1) \cdot g_1(T)] \otimes [(T + 1) \cdot g_2(T)] \otimes [(T + 1) \cdot g_3(T)] \\
=(T + 1) \cdot \left[ \prod_{i=1}^{3} g_i \right] \cdot \left[ \prod_{(a,b) \in \{1,2\}, \{2,3\}, \{3,1\}} g_a \otimes g_b \otimes (1 + T) \right] \cdot [g_1 \otimes g_2 \otimes g_3].
\]

\[(4.7)\]

We know \( \deg(g_i) = 2 \), \( \deg(g_a \otimes g_b \otimes (T + 1)) = 4 \), \( \deg(g_1 \otimes g_2 \otimes g_3) = 8 \), so

\[\deg(h_1 \otimes h_2 \otimes h_3) = 1 + 2 \times 3 + 4 \times 3 + 8 = 27.\]

Apply Vieta’s formulas, we know

\[
\nu_1 \nu_2 \nu_3 = \lambda + 2,
\]
\[
\nu_1 \nu_2 + \nu_2 \nu_3 + \nu_3 \nu_1 = -2,
\]
\[
\nu_1 + \nu_2 + \nu_3 = -1.
\]
So for items in the right hand side of (4.7), we have

\[ g_a \otimes g_b \otimes g_3 = T^8 + \nu_1 \nu_2 \nu_3 T^7 \]

\[ + (4 - 2 \nu_1^2 - 2 \nu_2^2 - 2 \nu_3^2 + \nu_1^2 \nu_2^2 + \nu_2^2 \nu_3^2 + \nu_3^2 \nu_1^2) T^6 \]

\[ + \nu_1 \nu_2 \nu_3 (\nu_1^2 + \nu_2^2 + \nu_3^2 - 5) T^5 \]

\[ + (\nu_1^2 \nu_2^2 \nu_3^2 + \nu_1^4 + \nu_2^4 + \nu_3^4 - 4 \nu_1^2 - 4 \nu_2^2 - 4 \nu_3^2 + 6) T^4 \]

\[ + \nu_1 \nu_2 \nu_3 (\nu_1^2 + \nu_2^2 + \nu_3^2 - 5) T^3 \]

\[ + (4 - 2 \nu_1^2 - 2 \nu_2^2 - 2 \nu_3^2 + \nu_1^2 \nu_2^2 \nu_3^2) T^2 \]

\[ + \nu_2^2 \nu_3^2 + \nu_3^2 \nu_1^2) T + \nu_1 \nu_2 \nu_3 T + 1 \]

\[ = T^8 + (\lambda + 2) T^7 + (2 \lambda + 2) T^6 + (\lambda^2 - 1) T^4 \]

\[ + (2 \lambda + 2) T^2 + (\lambda + 2) T + 1. \]

\[ g_a \otimes g_b \otimes (1 + T) = T^4 + \nu_1 \nu_2 \nu_3 T^3 + (\nu_1^2 + \nu_2^2 - 2) T^2 + \nu_1 \nu_2 \nu_3 T + 1, \]

\[ \prod_{(a,b) \in (1,2),(2,3),(3,1)} g_a \otimes g_b \otimes (1 + T) = T^{12} - 2 T^{11} + (2 - \lambda) T^{10} \]

\[ + (\lambda^2 + 3 \lambda - 2) T^9 - (3 \lambda^2 + 8 \lambda) T^8 \]

\[ + (6 \lambda^2 + 14 \lambda + 2) T^7 - (7 \lambda^2 + 16 \lambda + 2) T^6 \]

\[ + (6 \lambda^2 + 14 \lambda + 2) T^5 - (3 \lambda^2 + 8 \lambda) T^4 \]

\[ + (\lambda^2 + 3 \lambda - 2) T^3 + (2 - \lambda) T^2 - 2 T + 1. \]

Then \( h_1 \otimes h_2 \otimes h_3 \) has degree 27 and coefficients in \( \lambda \).

**Theorem 4.15.** Sequence \( \{\#ST(C_n^3)\}_{n \geq 7} \) is in the kernel of

\[ p_3(T) \overset{\text{def}}{=} (T - 1)^2(T^4 - 4 T^3 - T^2 - 4 T + 1)^2(T^4 + 3 T^3 + 6 T^2 + 3 T + 1)^2. \]  

\[(4.8)\]

**Proof.** Keep the notations as in previous Theorem, set \( \lambda = 6 \), we have

\[ \prod_{i=1}^{3} g_i \bigg|_{\lambda=6} = (T + 1)^2(T^4 - 3 T^3 + 6 T^2 - 3 T + 1), \]

\[ \prod_{(a,b) \in (1,2),(2,3),(3,1)} g_a \otimes g_b \otimes (1 + T) \bigg|_{\lambda=6} \]

\[ = (T^4 + 4 T^3 - T^2 + 4 T + 1)(T^4 - 3 T^3 + 6 T^2 - 3 T + 1)^2, \]

\[ g_1 \otimes g_2 \otimes g_3 \bigg|_{\lambda=6} = (T^4 + 4 T^3 - T^2 + 4 T + 1)^2. \]
Then by Theorem 4.5 we know \( \frac{dD_n^{(3)}(\lambda)}{d\lambda} \bigg|_{\lambda=6} \) is in the kernel of

\[
p_h(T) \overset{\text{def}}{=} h_1 \otimes h_2 \otimes h_3 \bigg|_{\lambda=6} = (T+1)^3(T^4+4T^3-T^2+4T+1)^3(T^4-3T^3+6T^2-3T+1)^3 = (T^9+2T^8-6T^7+21T^6+21T^3-6T^2+2T+1)^3.
\]

By Theorem 4.3, we know

\[
n \cdot \#ST(C_n^{(3)}) = (-1)^n \cdot \frac{dD_n^{(3)}(\lambda)}{d\lambda} \bigg|_{\lambda=6},
\]

where sequence \( \{(-1)^n\} \) is in the kernel of \((T+1)\), so \( \{n \cdot \#ST(C_n^{(3)})\} \) is in the kernel of

\[
(T+1) \otimes p_h(T) = (-1)^9 \cdot p_h(-T)
\]

\[
= (T-1)^3(T^4-4T^3-T^2-4T+1)^3(T^4+3T^3+6T^2+3T+1)^3
\]

\[
= (T^9-2T^8-6T^7+21T^6+21T^3+6T^2+2T-1)^3.
\]

By Appendix Theorem A.2, we know that the general expression of sequence \( \{n \cdot \#ST(C_n^{(3)})\} \) shall have 9 triple-roots. Denote them as \( \{\delta_i : 1 \leq i \leq 9\} \), then

\[
n \cdot \#ST(C_n^{(3)}) = \sum_{i=1}^{9} A_i \cdot \delta_i^n + B_i \cdot n \cdot \delta_i^n + C_i \cdot n^2 \cdot \delta_i^n,
\]

for some coefficients \( A_i, B_i, C_i \)'s, \( 1 \leq i \leq 9 \).

Further we can calculate the roots as

\[
\begin{align*}
\delta_1 &= 1, \\
\delta_{2,3} &= \frac{\sqrt{7}}{4} i + \frac{1}{2} \sqrt{-\frac{7}{2} - \frac{3\sqrt{7}}{2} i - \frac{3}{4}}, \\
\delta_{4,5} &= -\frac{\sqrt{7}}{4} i + \frac{1}{2} \sqrt{-\frac{7}{2} + \frac{3\sqrt{7}}{2} i - \frac{3}{4}}, \\
\delta_{6,7} &= -\frac{1}{2} \left( \sqrt{7} + i \sqrt{4\sqrt{7} - 7} \right), \\
\delta_{8,9} &= \frac{1}{2} \left( \sqrt{7} \pm i \sqrt{4\sqrt{7} + 7} \right).
\end{align*}
\]

Numerically we solve the coefficients as (see Appendix Example B.2 for more
detail):

\[
\begin{cases}
A_i = B_i = 0, & 1 \leq i \leq 9, \\
C_1 = \frac{2}{7}, \\
C_2 = C_3 = C_4 = C_5 = -\frac{1}{7}, \\
C_6 = C_7 = C_8 = C_9 = \frac{1}{14}.
\end{cases}
\]

So we know \( n \cdot \#ST(C_n^{(3)}) = \sum_{i=1}^{9} n^2 \cdot C_i \cdot \delta_i^n \), then \( \#ST(C_n^{(3)}) = \sum_{i=1}^{9} n \cdot C_i \cdot \delta_i^n \).

By Appendix Theorem A.3 we know that \( \{ \#ST(C_n^{(3)}) \} \) is in the kernel of

\[
\prod_{i=1}^{9} (T - \delta_i)^2 = (T - 1)^2 (T^4 - 4T^3 - T^2 - 4T + 1)^2 (T^4 + 3T^3 + 6T^2 + 3T + 1)^2.
\]

**Remark 4.16.** From \( \#ST(C_n^{(3)}) = \sum_{i=1}^{9} n \cdot C_i \cdot \delta_i^n \) we know it’s also true that

\[
\frac{\#ST(C_n^{(3)})}{n} = \sum_{i=1}^{9} C_i \cdot \delta_i^n.
\]

This means sequence \( \{ \frac{\#ST(C_n^{(3)})}{n} \} \) is in the kernel of \( (p_3(T))^\frac{1}{2} \).

These results fit our numerical data, see equation (4.36).

Further, for each \( n \geq 7 \), the ratio of \( \#ST(C_n^{(3)}) \) over \( n \) is an integer. This is because the total number of spanning trees in graph \( C_n^{(3)} \) is always a multiple of \( n \), as the graph is symmetric with respect to any vertex.

In general, \( \frac{\#ST(C_n^{(k)})}{n} \) is an integer for any \( k \geq 3 \) and \( n \geq 2k + 1 \) with the same reason.

**Theorem 4.17.** Sequence \( \{ D_n^{(4)}(\lambda) \}_{n \geq 9} \) fulfills a 81th-degree recurrence with coefficients in \( \lambda \).

**Proof.** Apply same idea as in the proof of Theorem 4.12.
Let \( W_n = S_n + (S_n)^{-1} \), then \( D_n(\lambda) = \det(W_n - \lambda I_n) \) and

\[
D_n^{(4)}(\lambda) = \det \left( S_n + S^2 + (S_n)^3 + (S_n)^4 
+ (S_n)^{-1} + (S_n)^{-2} + (S_n)^{-3} + (S_n)^{-4} - \lambda I_n \right)
\]

\[
= \det \left( (W_n)^4 + (W_n)^3 - 3W_n^2 - 2W_n - \lambda I_n \right)
\]

\[
= \prod_{i=1}^{4} \det(W - \nu_i I_n)
\]

\[
= \prod_{i=1}^{4} D_n(\nu_i),
\]

where \( \nu_i \)'s are the roots of \( w^4 + w^3 - 3w^2 - 2w - \lambda = 0 \). By Theorem 4.11, we know for \( 1 \leq i \leq 4 \), \( D_n(\nu_i) \) is in the kernel of

\[
h_i(T) = (T + 1)(1 + \nu_i T + T^2) = (T + 1) \cdot g_i(T),
\]

whereas \( g_i(T) = 1 + \nu_i T + T^2 \).

Then by Theorem 4.9, \( D_n^{(4)} = D_n(\nu_1) D_n(\nu_2) D_n(\nu_3) D_n(\nu_4) \) is in the kernel of

\[
h_1(T) \otimes h_2(T) \otimes h_3(T) \otimes h_4(T) = [(T + 1) \cdot g_1] \otimes \cdots \otimes [(T + 1) \cdot g_4]
\]

\[
=(T - 1) \cdot \left[ \prod_{i=1}^{4} g_i \otimes (T + 1) \right] \cdot \left[ \prod_{a<b} g_a \otimes g_b \otimes (T - 1) \right]
\]

\[
\cdot \left[ \prod_{a<b<c} g_a \otimes g_b \otimes g_c \otimes (T + 1) \right] \cdot [g_1 \otimes g_2 \otimes g_3 \otimes g_4]. \tag{4.9}
\]

We know \( \deg(g_i \otimes (T + 1)) = 2 \), \( \deg(g_a \otimes g_b \otimes (T - 1)) = \deg(g_a \otimes g_b) = 4 \),
\( \deg(g_a \otimes g_b \otimes g_c \otimes (T + 1)) = \deg(g_a \otimes g_b \otimes g_c) = 8 \), \( \deg(g_1 \otimes g_2 \otimes g_3 \otimes g_4) = 16 \),
and all these items have coefficients symmetric on \( \nu_i \)'s.

\[
\deg(h_1 \otimes h_2 \otimes h_3 \otimes h_4) = 1 + 4 \times 2 + 6 \times 4 + 4 \times 8 + 16 = 81,
\]

So, \( D_n^{(4)}(\lambda) \) fulfills a 81-th recurrence with coefficients in \( \lambda \), the latter part is by Vieta’s formulas and the symmetry on the roots.

**Theorem 4.18. Generalization.** Sequence \( \left\{ D_n^{(k)}(\lambda) \right\}_{n \geq 2k+1} \) is in the ker-
nel of the following recurrence

\[ P(k, \lambda; T) \overset{\text{def}}{=} (T - 1) \left( g_1 \otimes \cdots \otimes g_k \right) \cdot \prod_{m=1}^{k-1} \prod_{1 \leq s_1 < \cdots < s_m \leq k} \left( T + (-1)^{k-m+1} \right) \otimes g_{s_1} \otimes \cdots \otimes g_{s_m}, \]

(4.10)

whereas \( g_i(T) = (1 + \nu_i(\lambda)T + T^2) \), where \( \{\nu_i(T) \mid 1 \leq i \leq k\} \) are functions on \( \lambda \).

Also, the degree of \( P(k, \lambda; T) \) is 3^k.

**Proof.** Let \( p_\lambda(s) = \sum_{i=1}^{k} (s^i + s^{-i}) - \lambda \) be a rational function on variable \( s \). Then we can rewrite \( p_\lambda(s) \) as a polynomial on \( s + s^{-1} \), with degree \( k \) and coefficients on \( \lambda \), and we call this new polynomial \( q_\lambda(s + s^{-1}) \). Let \( \{\nu_i \mid 1 \leq i \leq k\} \) stand for all the roots of \( q_\lambda = 0 \), then by Vieta’s formulas, each \( \nu_i \) is a function on \( \lambda \), so \( \nu_i = \nu_i(T) \).

Let \( W_n = S_n + (S_n)^{-1} \), then

\[ D_n^{(k)}(\lambda) = \det \left( \sum_{i=1}^{k} ((S_n)^i + (S_n)^{-i}) - \lambda I_n \right) = \det (p_\lambda(S_n)) \]

\[ = \det (q_\lambda(S_n + (S_n)^{-1})) = \det (q_\lambda(W_n)) \]

\[ = \det \left( \prod_{i=1}^{k} (W - \nu_i I_n) \right) = \prod_{i=1}^{k} \det (W - \nu_i I_n) = \prod_{i=1}^{k} D_n(\nu_i). \]

By Theorem 4.11, \( D_n(\nu_i) \) is in the kernel of

\[ h_i(T) = (T + 1)(1 + \nu_i T + T^2) = (T + 1) \cdot g_i(T). \]

Then by Theorem 4.9, \( \prod_{i=1}^{k} D_n(\nu_i) \) is in the kernel of

\[ h_1(T) \otimes \cdots \otimes h_k(T) = [(T + 1) \cdot g_1] \otimes \cdots \otimes [(T + 1) \cdot g_k]. \]

Apply Proposition 4.8 multiple times, we then get recurrence as in (4.10).

Further, \( \deg (h_1(T) \otimes \cdots \otimes h_k(T)) = \prod_{i=1}^{k} \deg(h_i) = 3^k \).

**Remark 4.19.** Keep the notations in (4.10), set \( \lambda = 2k \), then by Theorem 4.5, we know \( \{ \frac{dD_n^{(k)}(\lambda)}{d\lambda} \}_{\lambda=2k} \) is in the kernel of \( P(k, 2k; T) \); while by Theorem 4.3, we know \( \{ n \cdot \#ST(U_n^{(k)}) \} \) is in the kernel of \( P(k, 2k; -T) \).
Conjecture 4.20. The sequence of the total number of spanning trees in graph $C_{n}^{(k)}$ fulfills some $(2 \cdot 3^{k-1})$-degree recurrence relation.

Remark 4.21. The conjecture above is valid for $k = 2, 3$, as been proved in Corollary 4.13 and theorem 4.15. And it remains valid for $k = 4$ under numerical computation.

Notice the degree of $P(k, \lambda; T)$ is $3k$, so the recurrence of $\{n \cdot \#ST(C_{n}^{(k)})\}$ is of degree $3k$. Then it’s reasonable to expect the recurrence of $\{\#ST(C_{n}^{(k)})\}$ has lower degree $2 \cdot 3^{k-1}$.

4.1.2 Recurrence of $R_{n}^{(3)}$

In previous section, we have found the recurrence of $\#ST(C_{n}^{2})$ and $\#ST(C_{n}^{(3)})$, and discussed a general method for finding $\#ST(C_{n}^{(k)})$. In this section we are going to study the sequence of the number of Black and Red Spanning trees, starting on the definitions:

Definition 4.5. Fix the negative edge $(1, 2)$ in graph $\Gamma_{n}^{(k)}$ to be edge $e$, and define $B_{n}^{(k)} = M(\Gamma_{n}^{(k)} \setminus e)$ and $R_{n}^{(k)} = M(\Gamma_{n}^{(k)}.e)$. Then by Lemma 2.5, we have

$$M(\Gamma_{n}^{(k)}) = M(\Gamma_{n}^{(k)} \setminus e) - M(\Gamma_{n}^{(k)}.e) t = B_{n}^{(k)} - t R_{n}^{(k)},$$

(4.11)

$$t^{*}(\Gamma_{n}^{(k)}) = \sup_{t \geq 0} \{M(\Gamma_{n}^{(k)}) \geq 0\} = \sup_{t \geq 0} \{B_{n}^{(k)} - t R_{n}^{(k)} \geq 0\} = \frac{B_{n}^{(k)}}{R_{n}^{(k)}}.$$  

(4.12)

Remark 4.22. By Lemma 2.7, we know that

$$B_{n}^{(k)} = M(\Gamma_{n}^{(k)} \setminus e) = \sum_{T \in ST(\Gamma_{n}^{(k)} \setminus e)} \pi(T) = \#ST(\Gamma_{n}^{(k)} \setminus e),$$

(4.13)

the last equality is because each edge weight in $\Gamma_{n}^{(k)} \setminus e$ is 1, so is each $\pi(T)$.

Keep our convention that black stands for positive and red stands for negative edges. Then we can consider $B_{n}^{(k)}$ as the number of black spanning trees in $\Gamma_{n}^{(k)}$. On the other hand, $R_{n}^{(k)}$ stands for the number of spanning trees with the red edge $e$ (we will call them red spanning trees for abbreviation).

Notice when set $t = -1$, $B_{n}^{(k)} + R_{n}^{(k)} = \#ST(C_{n}^{(k)})$. So we can get $B_{n}^{(k)}$ or $R_{n}^{(k)}$ if knowing either one of them.
**Theorem 4.23.** Keep \( D_n^{(k)}(\lambda) = \det \left( \sum_{i=1}^{k} (S_n)^i + \sum_{i=1}^{k} (S_n)^{-i} - \lambda I_n \right) \) as in Definition 4.3, then we have

\[
\left. \frac{d^2 D_n^{(k)}(\lambda)}{d\lambda^2} \right|_{\lambda = 2k} = 2 \sum_{i<j} Q_{i,j} = n \sum_{j=2}^{n} Q_{1,j}, \quad (4.14)
\]

where \( Q_{i,j} \) stands for determinant of the sub-matrix by removing \( i \)th and \( j \)th rows and columns of the Laplacian matrix \( \mathcal{L}(C_n^{(k)}) \).

In particular, \( Q_{1,2} = (-1)^n \cdot R_n^{(k)} \).

**Proof.** For the latter part, by Lemma 2.9, \((-1)^n Q_{1,2}\) equals the number of spanning trees containing the negative (red) edge, which is \( R_n^{(k)} \).

For the main statement, let \( \mathcal{L} = \left( \vec{l}_1, \vec{l}_2, \cdots, \vec{l}_n \right) \), where \( \vec{l}_i \in \mathbb{R}^n \). Let \( \{e_i\}_{i=1}^n \) be the standard orthogonal basis of \( \mathbb{R}^n \). Then

\[
\det(\mathcal{L} - \varepsilon I_n) = \left| \vec{l}_1 - \varepsilon e_1, \vec{l}_2 - \varepsilon e_2, \cdots, \vec{l}_n - \varepsilon e_n \right|
\]

\[
= \det(\mathcal{L}) - \varepsilon \sum_{i=1}^{n} M_{i,i} + \varepsilon^2 \sum_{i<j} Q_{i,j} + O(\varepsilon^3).
\]

Due to the fact \( \det(\mathcal{L}) = 0 \), set \( \varepsilon = \lambda - 2k \), then

\[
D_n^{(k)}(\lambda) = \det(\mathcal{L} - (\lambda - 2k) I_n) = -\varepsilon \sum_{i=1}^{n} M_{i,i} + \varepsilon^2 \sum_{i<j} Q_{i,j} + O(\varepsilon^3)
\]

\[
= -(\lambda - 2k) \sum_{i=1}^{n} M_{i,i} + (\lambda - 2k)^2 \sum_{i<j} Q_{i,j} + O((\lambda - 2k)^3).
\]

Compare with the Taylor expression of \( D_n^{(k)}(\lambda) \) at point \( \lambda = 2k \), we have

\[
\left. \frac{d^2 D_n^{(k)}(\lambda)}{d\lambda^2} \right|_{\lambda = 2k} = 2 \sum_{i<j} Q_{i,j} = n \sum_{j=2}^{n} Q_{1,j},
\]

where the last step is because the symmetry of the nodes in \( C_n^{(k)} \).

**Theorem 4.24.** Sequence \( \{R_n^{(3)}\}_{n \geq 7} \) fulfills the same 18-degree recurrence as \( \{\#ST(C_n^{(3)})\}_{n \geq 7} \).

i.e., \( p_3(T) (R_n^{(3)}) = 0 \) where

\[
p_3(T) = (T - 1)^2(T^4 - 4T^3 - T^2 - 4T + 1)^2(T^4 + 3T^3 + 6T^2 + 3T + 1)^2
\]
as in (4.8).

We shall prove some preliminaries first.

**Definition 4.6.** For $n \geq 2$, matrix $T_n = (t_{ij})_{n \times n}$ is defined to be

$$t_{i,j} = \begin{cases} 
1, & j = i + 1, \\
0, & \text{otherwise.}
\end{cases}$$

i.e.,

$$T_n \overset{\text{def}}{=} \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{n \times n}.$$

Naturally, $(T_n)^n = 0$, zero matrix.

Let $T_n^\top$ be the transpose of $T_n$ and define the followings for $n \geq 7$:

$$L_n = L(C_n^{(3)}) = S_n + (S_n)^2 + (S_n)^3 + (S_n)^{-1} + (S_n)^{-2} + (S_n)^{-3} - 6I_n,$$

$$U_n = S_n + (T_n)^2 + (T_n)^3 + (S_n)^{-1} + (S_n)^{-2} + (T_n^\top)^3 - 6I_n,$$

$$V_n = S_n + (T_n)^2 + (T_n)^3 + (S_n)^{-1} + (T_n^\top)^2 + (T_n^\top)^3 - 6I_n,$$

$$W_n = T_n + (T_n)^2 + (T_n)^3 + (T_n^\top) + (T_n^\top)^2 + (T_n^\top)^3 - 6I_n,$$

and let $l_n$, $u_n$, $v_n$ and $w_n$ be the corresponding determinants.

We can get $U_{n-1}$ by deleting the first row and column of $L_n$, and get $V_{n-2}$ by deleting the first two rows and columns. Then by Lemma 2.9 and Theorem 4.23, we know

$$(-1)^n \cdot \#TS(C_n^{(3)}) = \det(U_n) = u_n,$$

$$(-1)^n \cdot R_{n+2}^{(3)} = \det(V_n) = v_n.$$  \hspace{1cm} (4.15) \hspace{1cm} (4.16)

Our goal is to find the recurrence of sequence $\{R_n^{(3)}\}$, which is connected with the recurrence of $\{v_n\}$. We shall introduce two lemmas first to find $\{v_n\}$ recurrence.
Lemma 4.25. Define a $k \times k$ matrix $C_k$ to be the sub-matrix of $V_{k+1}$ by removing its first column and last row. Let $c_k$ stand for the determinant of $C_k$. Then the sequence $\{c_n\}_{n=7}^\infty$ is in the kernel of the 13-degree recurrence below:

$$p_c(T) \overset{\text{def}}{=} (T - 1)(T^4 - 4T^3 - T^2 - 4T + 1)(T^4 + 3T^3 + 6T^2 + 3T + 1)^2.$$ \hfill (4.17)

Proof. Define matrix $F_k$, $G_k$, $H_k$ as following and $f_k$, $g_k$, $h_k$ as their determinants:

$$F_k \overset{\text{def}}{=} \begin{bmatrix} -6 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots & \ddots & \ddots \\ \end{bmatrix}, \quad G_k \overset{\text{def}}{=} \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots & \ddots & \ddots \\ \end{bmatrix}, \quad H_k \overset{\text{def}}{=} \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots & \ddots & \ddots \\ \end{bmatrix}.$$
By expanding the first row of determinant of $C_k$, we have:

$$c_k = \det C_k = \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ -6 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-1} & \end{bmatrix}$$

$$= c_{k-1} - \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ -6 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-2} & \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ -6 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-3} & \end{bmatrix}$$

$$= c_{k-1} - f_{k-2}$$

$$- 6 \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ -6 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-3} & \end{bmatrix} - \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-3} & \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-4} & \end{bmatrix}$$

$$= c_{k-1} - f_{k-2} - 6f_{k-3} - g_{k-3}$$

$$+ \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-4} & \end{bmatrix} + 6 \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-4} & \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-5} & \end{bmatrix}$$

$$= c_{k-1} - f_{k-2} - 6f_{k-3} - g_{k-3} + g_{k-4} + 6h_{k-4}$$

$$+ \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-5} & \end{bmatrix} - \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-4} & \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & \vdots \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ C_{k-6} & \end{bmatrix}$$

$$= c_{k-1} - f_{k-2} - 6f_{k-3} - g_{k-3} + g_{k-4} + 6h_{k-4} + h_{k-5} - c_{k-5} + c_{k-6}.$$  

(4.18)
Continue expanding $f_{k-2}$, $g_{k-3}$, $h_{k-4}$, we have

$$f_{k-2} = \det \begin{bmatrix} -6 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & C_{k-2} \\ 1 & 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \vdots & \end{bmatrix} = -6c_{k-2}$$

$$= -6c_{k-2} - \det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & C_{k-3} \\ 1 & 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \vdots & \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & C_{k-4} \\ 1 & 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \vdots & \end{bmatrix}$$

$$= -6c_{k-2} - g_{k-3} + f_{k-4} - \det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & C_{k-4} \\ 1 & 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \vdots & \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & C_{k-5} \\ 1 & 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \vdots & \end{bmatrix}$$

$$= -6c_{k-2} - g_{k-3} + f_{k-4} - h_{k-4} + g_{k-5} + 6c_{k-5} + h_{k-6}.$$  \hspace{1cm} (4.19)

$$g_{k-3} = \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & C_{k-3} \\ 1 & 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \vdots & \end{bmatrix}$$

$$= c_{k-3} - \det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & C_{k-4} \\ 1 & 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \vdots & \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & C_{k-5} \\ 1 & 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \vdots & \end{bmatrix}$$

$$= c_{k-3} + f_{k-5} - c_{k-5} + h_{k-6}.$$ \hspace{1cm} (4.20)

$$h_{k-4} = \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & C_{k-4} \\ 1 & 0 & \vdots & & \vdots & \\ 0 & \vdots & & & \vdots & \end{bmatrix} = c_{k-4} - c_{k-5} + f_{k-6}.$$ \hspace{1cm} (4.21)

Set $n = k - 6$ and $T : X_n \rightarrow X_{n+1}$ as the shifting operator, we can rewrite
equations above as

\[
\begin{align*}
(T^6 - T^5 + T - 1)c_n &= (-T^4 - 6T^3)f_n + (T^2 - T^3)g_n + (6T^2 + T)h_n \\
(T^4 - T^2)f_n &= (-6T^4 + 6T)C_n + (T - T^3)g_n + (1 - T^2)h_n \\
(T^3 - 1)g_n &= (T^3 - T)c_n + T f_n - T^2 h_n \\
T^2 h_n &= (T^2 - T)c_n + f_n \\
\end{align*}
\]

(4.22)

We can do substitutions and cancel the items \(f_n, g_n, h_n\), and we will get a polynomial \(p_c(T)\) of \(T\) with \(p_c(T)(c_n) = 0\). However, we can also use linear algebra to solve \(p_c(T)\) here.

Let

\[
M = \begin{pmatrix}
T^6 - T^5 + T - 1 & T^4 + 6T^3 & T^3 - T^2 & -6T^2 - T \\
6T^4 - 6T & T^4 - T^2 & T^3 - T & T^2 - 1 \\
T^3 - T & T & -T^3 + 1 & -T^2 \\
T^2 - T & 1 & 0 & -T^2 \\
\end{pmatrix},
\]

then we have

\[
M \begin{pmatrix} c_n \\ f_n \\ g_n \\ h_n \end{pmatrix} = \vec{0}.
\]

Since \(\det(M) = (T - 1)^3(T^4 - 4T^3 - T^2 - 4T + 1)(T^4 + 3T^3 + 6T^2 + 3T + 1)^2\), we know that \(C_M^\top \cdot M = \det(M) I_4\), where \(C_M\) is the cofactor matrix of \(M\). Thus we have \(\det(M) \cdot c_n = 0\). What’s more, notice term \(T - 1\) is a common divider for both the first and the third column of \(M\), which means that \((T - 1)^2\) is a common divider for each cofactor of \(M\), and thus a common divider of \(C_M\). So we have

\[
\frac{1}{(T - 1)^2} C_M^\top \cdot M = \frac{\det(M)}{(T - 1)^2} I_4.
\]

Let \(p_c(T) \overset{\text{def}}{=} \frac{\det(M)}{(T - 1)^2} = (T - 1)(T^4 - 4T^3 - T^2 - 4T + 1)(T^4 + 3T^3 + 6T^2 + 3T + 1)^2\) and we will have \(p_c(T)(c_n) = 0\).

**Lemma 4.26.** The sequence \(\{w_n\}_{n \geq 7}\) is in the kernel of the 15-degree re-
\[ p_{w}(T) \overset{\text{def}}{=} (T + 1)^2(T^4 + 4T^3 - T^2 + 4T + 1)^2(T^4 - 3T^3 + 6T^2 - 3T + 1). \]  
(4.23)

**Proof.** Define matrix \( A_k, B_k, E_k \) as following and \( a_k, b_k, e_k \) as their determinants:

\[
A_k \overset{\text{def}}{=} \begin{bmatrix}
1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & W_k & & & \\
1 & 0 & & & & \\
& \vdots & & & & \\
0 & & & & & \\
\end{bmatrix}, \quad E_k \overset{\text{def}}{=} \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 0 & & & & & \\
& \vdots & & & & & \\
0 & & & & & & \\
\end{bmatrix}, \quad B_k \overset{\text{def}}{=} \begin{bmatrix}
-6 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 0 & & & & \\
1 & 0 & W_k & & & \\
& \vdots & & & & & \\
0 & & & & & & \\
\end{bmatrix}.
\]
By expanding the first column of determinant of $W_k$, we have:

$$w_k = \det W_k = \det \begin{bmatrix}
-6 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & -6 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & -6 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & -6 & 1 & 1 & 1 & \cdots & 0 \\
0 & 1 & 1 & 1 & -6 & 1 & 1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 1 & 1 & -6 & 1 & 1 \\
0 & \cdots & 0 & 1 & 1 & 1 & -6 & 1 & \cdots \\
0 & \cdots & 0 & 1 & 1 & 1 & -6 & 1 & \cdots \\
\end{bmatrix}_{k \times k}$$

$$= \det \begin{bmatrix}
-6 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 1 & 1 & -6 \\
0 & \cdots & 0 & 1 & 1 & 1 & -6 \\
0 & \cdots & 0 & 1 & 1 & 1 & -6 \\
\end{bmatrix}$$

$$= -6w_{k-1} - \det W_{k-2}$$

$$+ \det W_{k-3} - \det W_{k-4}.$$

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where

\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\
-6 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & 0 & 1 & 1 & 0 & \cdots & 0 \\
0 & \vdots & 0 & 1 & 0 & \cdots & 0 \\
\end{vmatrix} = W_{k-3}
\]

\[
\begin{vmatrix}
-6 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & 0 & 1 & 1 & 0 & \cdots & 0 \\
0 & \vdots & 0 & 1 & 0 & \cdots & 0 \\
\end{vmatrix} + \begin{vmatrix}
1 & 1 & 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \vdots & 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\end{vmatrix} = a_{k-3} - 6a_{k-4} - e_{k-5} + \det \begin{vmatrix}
1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
0 & \vdots & 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
-6 & 1 & 1 & 0 & \cdots & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & 0 & 1 & 1 & 0 & \cdots & 0 \\
0 & \vdots & 0 & 1 & 0 & \cdots & 0 \\
\end{vmatrix} - \begin{vmatrix}
1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\
-6 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & 0 & 1 & 1 & 0 & \cdots & 0 \\
0 & \vdots & 0 & 1 & 0 & \cdots & 0 \\
\end{vmatrix} = a_{k-3} - b_{k-3} + (-6a_{k-4} - e_{k-4} + e_{k-5} - w_{k-5} + e_{k-6}),
\]
\[ \begin{vmatrix} -6 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 - 6 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots \\ 0 & \vdots & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ W_{k-4} & & & & & & & \\ \end{vmatrix} = \begin{vmatrix} -6 & b_k - 4 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots \\ 0 & \vdots & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ W_{k-5} & & & & & & & \\ \end{vmatrix} \]

So we know

\[ w_k = -6w_{k-1} - a_{k-2} + (a_{k-3} - b_{k-3} + \cdots) \]

\[ - [(a_{k-4} - b_{k-4} - \cdots) + \cdots] \]

\[ - [(a_{k-3} - b_{k-3} - \cdots) + \cdots] \]

\[ + (-6b_{k-4} - \cdots) ] \]

\[ = -6b_{k-4} + (-w_{k-4} + e_{k-5}) + (e_{k-5} + 6w_{k-5} + w_{k-6}). \]

Set \( n = k - 7 \) and \( T : X_n \to X_{n+1} \) as the shifting operator, we can rewrite equation above as

\[ (T^5 - T^4 + 13T^3 - 6T^2)a_n + (T^4 - 7T^3)b_n \]

\[ + (2T^3 - T^2 - T + 1)e_n + (T^7 + 6T^6 - T^3 + 8T^2)w_n = 0. \]
Continue expanding $a_k$, $b_k$, $e_k$, we have

\[
a_k = \det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ \\ 1 & 1 & 1 & 0 & \cdots \\ \\ 1 & 1 & 1 & 1 & 0 \\ \\ 0 & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & \vdots & \vdots & \vdots & \vdots \end{bmatrix} W_k = \det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ \\ 1 & 1 & 1 & 0 & \cdots \\ \\ 1 & 1 & 1 & 1 & 0 \\ \\ 0 & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & \vdots & \vdots & \vdots & \vdots \end{bmatrix} + \det \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \\ 1 & 1 & 0 & \cdots \\ \\ 1 & 1 & 1 & 0 & \cdots \\ \\ 0 & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & \vdots & \vdots & \vdots & \vdots \end{bmatrix} W_k \]

\[
= e_k + \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & \vdots \end{bmatrix} W_{k-2} = e_k + (a_{k-2} - b_{k-2} - 6a_{k-3} - e_{k-3} + e_{k-4} - w_{k-4} + e_{k-5}). \tag{4.26}
\]

\[
b_k = \det \begin{bmatrix} -6 & 1 & 1 & 0 & \cdots & 0 \\ \\ 1 & 1 & 1 & 0 & \cdots \\ \\ 1 & 1 & 1 & 1 & 0 \\ \\ 0 & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & \vdots & \vdots & \vdots & \vdots \end{bmatrix} W_k = -6w_k - \det \begin{bmatrix} 0 & 1 \\ \vdots & \vdots \\ 0 & \vdots \end{bmatrix} W_{k-1} + \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \\ 1 & 1 & 1 & 1 & 0 & \cdots \\ \\ 1 & 1 & 1 & 1 & 1 & 0 \\ \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} W_{k-2}
\]

\[
= -6w_k - e_{k-1} + (a_{k-2} - b_{k-2}). \tag{4.27}
\]

\[
e_k = \det \begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ \\ 1 & 1 & 1 & 1 & 0 & \cdots \\ \\ 1 & 1 & 1 & 1 & 1 & 0 \\ \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} W_k = w_k - a_{k-1}. \tag{4.28}
\]

Still using $T : X_n \to X_{n+1}$ as the shifting operator, we can rewrite equa-
tions above as

\[
\begin{cases}
(T^5 - T^3 + 6T^2)a_n + T^3 b_n + (-T^5 + T^2 - T - 1)e_n + T w_n = 0, \\
a_n - (T^2 + 1)b_n - T e_n - 6T^2 w_n = 0, \\
a_n + T e_n - T w_n = 0.
\end{cases}
\]  

(4.29)

Let

\[
M_1 = \begin{pmatrix}
T^5 - T^3 + 13T^3 - 6T^2 & T^4 - 7T^3 & 2T^3 - T^2 - T + 1 & T^7 + 6T^6 - T^3 + 8T^2 \\
T^5 - T^3 + 6T^2 & T^3 - T^5 + T^2 - T - 1 & T & T
\end{pmatrix}
\]

then we have

\[
M_1 \begin{pmatrix}
a_n \\
b_n \\
e_n \\
w_n
\end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.
\]

Using the same argument as in the previous proof for sequence \( \{c_n\} \), we know the term \( T \) is a common divider for the last column of \( M_1 \), then it is also a common divider for each cofactor of \( M_1 \). Let \( C_1 \) be the cofactor matrix of \( M_1 \), then we have

\[
\frac{1}{T} C_1^T \cdot M_1 = \frac{\det(M)}{T} I_4,
\]

whereas \( \det(M_1) = T(T-1)^2(T^4-4T^3-T^2-4T+1)^2(T^4+3T^3+6T^2+3T+1) \).

Let \( p_w(T) \) be defined \( \frac{\det(M)}{T} = (T - 1)^2(T^4 - 4T^3 - T^2 - 4T + 1)^2(T^4 + 3T^3 + 6T^2 + 3T + 1) \) and we will have \( p_w(T)(w_n) = 0. \)

Now we are ready to prove Theorem 4.24.
**Proof of Theorem 4.24.** Notice that

\[
\begin{bmatrix}
-6 & 1 & 1 & 1 & 0 & \cdots & 0 & 1 \\
1 & -6 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & -6 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & -6 & 1 & 1 & \cdots & 0 \\
0 & 1 & 1 & 1 & -6 & 1 & \cdots & 0 \\
\ddots \ & \ddots \ & \ddots \ & \ddots \\
0 & \cdots & 0 & 1 & 1 & 1 & -6 & 0 \\
1 & 0 & \cdots & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\]

\[v_n = \text{det} \begin{bmatrix}
\end{bmatrix} \]

\[= (-1)^n \text{det} \begin{bmatrix}
1 & -6 & 1 & 1 & 1 & 0 & \cdots \\
1 & 1 & -6 & 1 & 1 & 0 & \cdots \\
1 & 1 & 1 & -6 & 1 & 1 & \cdots \\
0 & 1 & 1 & 1 & -6 & 1 & \cdots \\
0 & \cdots & 0 & 1 & 1 & 1 & -6 \\
1 & 0 & \cdots & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\[= (n-1) \times (n-1)\]

\[+ \text{det} \begin{bmatrix}
-6 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\
1 & -6 & 1 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & -6 & 1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & -6 & 1 & 1 & \cdots & 0 \\
0 & 1 & 1 & 1 & -6 & 1 & \cdots & 0 \\
\ddots \ & \ddots \ & \ddots \ & \ddots \\
0 & \cdots & 0 & 1 & 1 & 1 & -6 & 0 \\
1 & 0 & \cdots & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\]

\[= 2(-1)^n \cdot c_n + w_n - w_{n-2}. \]  \hspace{1cm} (4.30)

By equation (4.16), \((-1)^n \cdot R_{n+2}^{(3)} = v_n\), which means sequence \(\{R_{n}^{(3)}\}\) is in the kernel of the least common multiple of \(p_c(T)\) and \(p_w(-T)\), by (4.17) and (4.23),

\[p_c(T) = (T - 1)(T^4 - 4T^3 - T^2 - 4T + 1)(T^4 + 3T^3 + 6T^2 + 3T + 1)^2,\]

\[p_w(-T) = (T - 1)^2(T^4 - 4T^3 - T^2 - 4T + 1)^2(T^4 + 3T^3 + 6T^2 + 3T + 1),\]

then \(R_{n}^{(3)}\) is in the kernel of

\[(T - 1)^2(T^4 - 4T^3 - T^2 - 4T + 1)^2(T^4 + 3T^3 + 6T^2 + 3T + 1)^2.\]

Now we present the main theorem for this chapter.

**Theorem 4.27.** If we let \(n \to \infty\), then we get corresponding weighted graph

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\( \Gamma^{(3)} \) and graph Laplacian \( \mathcal{L}(\Gamma^{(3)}(t)) \). For the stable bifurcation point, we have

\[
t^*(\Gamma^{(3)}) \overset{\text{def}}{=} \lim_{n \to \infty} t^*(\Gamma_n^{(3)}) = \frac{2\sqrt{2} \sqrt[4]{7}/4 - \sqrt{7} - 1}{\sqrt{7} + 1}. \quad (4.31)
\]

**Proof.** By equation (4.12), we have

\[
t^*(\Gamma^{(3)}) = \lim_{n \to \infty} t^*(\Gamma_n^{(3)}) = \lim_{t \to \infty} \frac{B_n^{(3)}}{R_n^{(3)}} - 1
\]

\[
= \lim_{t \to \infty} \frac{B_n^{(3)} + R_n^{(3)}}{R_n^{(3)}} - 1
\]

By Theorem 4.24, we know sequence \( \{\#ST(C_n^{(3)})\} \) and \( \{R_n^{(3)}\} \) are in the kernel of the same 18-degree recurrence

\[
p_3(T) = (T - 1)^2(T^4 - 4T^3 - T^2 - 4T + 1)^2(T^4 + 3T^3 + 6T^2 + 3T + 1)^2.
\]

In Theorem 4.15, we have found its 9 triple roots \( \{\delta_i: 1 \leq i \leq 9\} \), where \( \delta_9 \) has the greatest norm, which means the general expression of the two sequences can be dominated by \( O(n \delta_9^2) \). Also we have showed the leading coefficient of item \( n \delta_9^2 \) in \( \#ST(C_n^{(3)}) \) is \( \frac{1}{14} \), now we only need to compute the corresponding leading coefficient for \( \{R_n^{(3)}\} \). Precisely, by A.2 we know its general expression is

\[
R_n^{(3)} = \sum_{i=1}^{9} a_i \cdot \delta_i^n + b_i \cdot n \cdot \delta_i^n.
\]

Comparing with the initial terms (see Appendix Example B.3), we solved
that
\[
\begin{align*}
a_1 &= -\frac{1}{14}, \\
a_2 &= a_3 = a_4 = a_5 = \frac{1}{98}, \\
a_6 &= a_7 = a_8 = a_9 = -\frac{1}{196}, \\
b_1 &= 0, \\
b_2 &= -b_3 = \frac{b_4}{b_5} = \frac{i(\sqrt{7} - 1)}{28\sqrt{2}7^{3/4}}, \\
b_7 &= \frac{i(\sqrt{7} - 1)}{28\sqrt{2}7^{3/4}}, \\
b_9 &= -b_8 = \frac{\sqrt{7} + 1}{28\sqrt{2}7^{3/4}}.
\end{align*}
\]

Then we have
\[
\lim_{t \to \infty} \frac{\#ST(C_n^{(3)})}{R_n^{(3)}} - 1 = \frac{14}{b_9} - 1 = \frac{2\sqrt{2}7^{3/4} - \sqrt{7} - 1}{\sqrt{7} + 1} \approx 2.33873. \tag{4.32}
\]

**Remark 4.28.** Again, the results are consistent with our numerical observation:

1. Sequences \{\#ST(C_n^{(3)})\}, \{R_n^{(3)}\} and \{B_n^{(3)}\} fulfill the same 18-degree recurrence as in equation (4.36).

2. \(t^{*}(\Gamma_{\infty}^{(3)}) \approx 2.33873\), where our numerical result shows that the ratio of sequence \(B_n^{(3)}\) and \(R_n^{(3)}\) is close to \(\frac{7}{3}\) when \(n\) is big. See comment after equation (4.36).

3. Numerically we found that \(t^{*}(\Gamma_{n}^{(3)})\) is decreasing as \(n\) goes bigger. Our best numerical approximation is that it should be smaller than 2.33953, which is the simulation result of \(t^{*}(\Gamma_{n}^{(3)})\) \(\big|_{n=1000}\).

### 4.1.3 Expected Conclusions on General \(k\)

Based on various numerical results and our proofs in this chapter, we make following claims on general extension.

1. The sequence of the total number of spanning trees in graph \(C_n^{(k)}\) fulfills some \((2 \cdot 3^{k-1})\)-degree recurrence relation.

2. The sequence of the number of Red spanning trees \(R_n^{(k)}\) fulfills the same recurrence relation as above.
3. The leading coefficients of the dominated terms for both sequences above are positive constants.

4. The value of \( t^*(\Gamma_{\infty}^{(k)}) \) is only determined by these two leading coefficients. It is an exact positive constant close to \( k \).

**Remark 4.29.** Claim 1 is the Conjecture 4.20 in previous section.

**Remark 4.30.** Keep the notations for matrices \( S_n \) and \( T_n \) as in previous sections. Define matrices

\[
E_n^{(k,p)} \overset{\text{def}}{=} \sum_{i=1}^{k-p} ((S_n)_i + (S_n)_i^{-1}) + \sum_{i=k+1-p}^{k} ((T_n)_i + (T_n^\top)_i) - 2kI_n. \tag{4.33}
\]

Let \( A(k, p; n) = \det (E_n^{(k,p)}) \) be the determinants. Then by Lemma 2.9 and Theorem 4.23, we know

\[
A(k, 1; n) = \det (E_n^{(k,1)}) = (-1)^n \cdot \#ST(E_{n+1}^{(k)}), \tag{4.34}
\]

\[
A(k, 2; n) = \det (E_n^{(k,2)}) = (-1)^n \cdot R_{n+2}^{(k)}. \tag{4.35}
\]

Numerically we found that sequences \( \{A(k_0, p; n)\}_n \) fulfill the same recurrence for all \( 1 \leq p \leq k_0 \) when fix \( k_0 \) to be 2, 3 and 4. We would expect this result holds for general \( k > 4 \). And it is equivalent to Claim 2.

**Remark 4.31.** Claim 3 and Claim 4 are inferred from our numerical observations on \( t^* \) in the next chapter.

### 4.2 Numerical Approaches

#### 4.2.1 Simulation on \( t^* \)

As Remark 2.4 pointed out, there is a natural connection between \( \mathcal{M} \) with the characteristic polynomial of \( \mathcal{L} \), then numerically, we can get \( \mathcal{M}(\Gamma_{\infty}^{(k)}) \) and corresponding \( t^*(\Gamma_{\infty}^{(k)}) \) by computing the characteristic polynomial of \( \mathcal{L}(\Gamma_{\infty}^{(k)}) \). Then at least we can compute the first finite items and see the pattern. After done that, we have some numerical observations:
1. For fixed \( k_0 \geq 3 \), \( t^*(\Gamma_n^{(k_0)}) \) is decreasing as \( n \) goes bigger, and has exponential decay.

Figure 4.1: \( t^*(\Gamma_n^{(k_0)}) \) for \( k_0 = 3 \) and \( k_0 = 4 \)

2. Given large \( N \), \( t^*(\Gamma_N^{(k)}) \) is increasing as \( k \) goes bigger, approximately linear growth on \( k \).

Figure 4.2: \( t^*(\Gamma_{100}^{(k)}) \) for \( 2 \leq k \leq 40 \)

4.2.2 Simulation on Sequence \( B_n^{(k)} \) and \( R_n^{(k)} \)

Another approach is to consider the number of black \( (B_n^{(k)}) \) and red spanning trees \( (R_n^{(k)}) \) in graphs \( \Gamma_n^{(k)} \). Then with the help of Kirchhoff’s Matrix Tree Theorem, we can numerically generating the sequence \( \{B_n^{(k)}\} \) and \( \{R_n^{(k)}\} \) and its ratio \( t^*(\Gamma_n) = \frac{B_n^{(k)}}{R_n^{(k)}} \). We have the following numerical results:
1. For $k = 3$, both $B_n^{(3)}$ and $R_n^{(3)}$ fulfill the same 18-degree recursive relation, and they are in the kernel of

\[
(T - 1)^2(T^4 - 4T^3 - T^2 - 4T + 1)^2(T^4 + 3T^3 + 6T^2 + 3T + 1)^2
= 1 - 4T - 8T^2 - 18T^3 + 120T^4 + 252T^5 + 483T^6 - 72T^7
- 272T^8 - 964T^9 - 272T^{10} - 72T^{11} + 483T^{12} + 252T^{13}
+ 120T^{14} - 18T^{15} - 8T^{16} - 4T^{17} + T^{18}. \tag{4.36}
\]

The ratio $\frac{B_n^{(k)}}{R_n^{(k)}}$ is about to converge to $\frac{7}{3}$.

2. Left right side of (4.36) is the square of

\[
p_z(T) \overset{\text{def}}{=} 1 - 2T - 6T^2 - 21T^3 + 21T^6 + 6T^7 + 2T^8 - T^9.
\]

And $Z_n \overset{\text{def}}{=} \#\mathcal{ST}(C_n^3)/n$ fulfills $p_z(T)(Z_n) = 0$.

3. For $k = 4$, let

\[
p_f(T) \overset{\text{def}}{=} -1 + 3T + 12T^2 + 36T^3 + 195T^4 - 345T^5 - 720T^6
+ 120T^7 - 2697T^8 + 2211T^9 + 9699T^{10} - 3513T^{11}
+ 7335T^{12} - 11085T^{13} - 11085T^{14} + 7335T^{15}
- 3513T^{16} + 9699T^{17} + 2211T^{18} - 2697T^{19} + 120T^{20}
- 720T^{21} - 345T^{22} + 195T^{23}
+ 36T^{24} + 12T^{25} + 3T^{26} - T^{27}.
\]

Then $\{\#\mathcal{ST}(C_n^4)/n\}$ is in the kernel of $p_f(T)$, and corresponding $B_n^{(4)}$, $R_n^{(4)}$ are in the kernel of the same 54-degree recurrence $(p_f(T))^2$.

4.2.3 Change the Red Edge

In (3.2), we have set edge $(1, 2)$ to be red. Consider the symmetric of $C_n^2$, it is reasonable to set $(1, 3)$ to be the red edge.

Numerically, we have conclusion that the corresponding $t^* = \frac{1}{\sqrt{\phi - 1}}$, the reciprocal of the original answer.

Numerically, we have also showed followings are true:
Conjecture 4.32. For fixed $k$ and $1 \leq p \leq k$, take edge $e = (1, 1+p)$ on $C_n^{(k)}$ with negative weight $-t$. Name this graph to be $G_n$. Define $b_n = \mathcal{M}(G_n \setminus e)$ and define $r_n = \mathcal{M}(G_n \cdot e)$. Then sequences $\{b_n\}$ and $\{r_n\}$ will fulfill the same recurrence as $\{\#ST(C_n^{(k)})\}_n$. Further the ratio $b_n/r_n$ is equal to the stable point $t^*(G_n)$ for the graph Laplacian. And $t^*(G_n)$ converges to an exact positive constant.
APPENDIX A

LINEAR HOMOGENEOUS EQUATIONS
WITH CONSTANT COEFFICIENTS

This part is from chapter 3.2 of [10], and here we re-write several theorems as we need for our results with our conventions. All the proofs can be found in [10] with slight changes in the notation, so we omit them here.

The target is the \( k \)th order difference equation below

\[
x_{n+k} + p_1 x_{n+k-1} + p_2 x_{n+k-2} + \cdots + p_k x_n = 0, \quad (A.1)
\]

where the \( p_i \)'s are constants and \( p_k \neq 0 \).

**Definition A.1.**

\[
\lambda^k + p_1 \lambda^{k-1} + \cdots + p_k = 0
\]

is called the characteristic equation of equation (A.1), and its roots \( \lambda \) are called characteristic roots. None of the characteristic roots is equal to zero as \( p_k \neq 0 \).

**Theorem A.1.** Suppose \( \lambda_1, \lambda_2, \cdots, \lambda_k \) are distinct, then the set 

\( \{\lambda_1^n, \lambda_2^n, \cdots, \lambda_k^n\} \) is a fundamental set of solutions.

As a consequence, the general solution of equation (A.1) is

\[
x_n = \sum_{i=1}^{k} a_i \lambda_i^n, \quad a_i \in \mathbb{C}.
\]

**Theorem A.2.** Suppose the distinct characteristic roots are \( \lambda_1, \lambda_2, \cdots, \lambda_r \) with multiplicity’s \( m_1, m_2, \cdots, m_r \), respectively, then the general solution of equation (A.1) is given by

\[
x_n = \sum_{i=1}^{r} \left( a_{i,0} + a_{i,1} n + a_{i,2} n^2 + \cdots + a_{i,m_i-1} n^{m_i-1} \right) \lambda_i^n, \quad a_{i,j} \in \mathbb{C}. \quad (A.2)
\]
Corollary A.3. If the general expression of sequence \( \{x_n\} \) is given by (A.2), then conversely we also know

\[
(T - \lambda_1)^{m_1}(T - \lambda_2)^{m_2} \cdots (T - \lambda_r)^{m_r}(x_n) = 0,
\]

where \( T : x_n \to x_{n+1} \) is our shift operator.

Example A.4 (The Fibonacci Sequence). Define the Fibonacci sequence \( \{F_n : n \geq 1\} \) as

\[
F_n = \begin{cases} 
F_{n-1} + F_{n-2}, & n \geq 3, \\
1, & n \in \{1, 2\}.
\end{cases}
\]

Then the characteristic equation is

\[
\lambda^2 - \lambda - 1 = 0,
\]

with two characteristic roots \( \phi = \frac{\sqrt{5} + 1}{2} \) and \( \overline{\phi} = \frac{1 - \sqrt{5}}{2} \). So by Theorem A.1, the general solution is given by

\[
F_n = a_1 \phi^n + a_2 (\overline{\phi})^n.
\]

By plug in the initial conditions \( F_1 = F_2 = 1 \), we solve that

\[
a_1 = \frac{1}{\sqrt{5}}, a_2 = -\frac{1}{\sqrt{5}}.
\]

As a consequence,

\[
F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{\sqrt{5} + 1}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]. \tag{A.3}
\]

\[
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi \approx 1.618. \tag{A.4}
\]

We call \( \phi \) as Golden Ration and \( \Phi = \phi^{-1} \) as Golden Ration Conjugate.
APPENDIX B

NUMERICAL METHOD FOR SOLVING COEFFICIENTS FOR BIG LINEAR SYSTEM

In our work we often need to solve the coefficients in the general solution of linear equations as in (A.1), by evaluating the initial values. Here we present several numerical ways used in our work.

If we know a sequence \( \{x_n : n \geq n_0\} \) is in the kernel of \( p(T) \), and numerically we can generate first finite terms of \( x_n \). Let \( r \) be the rank of \( p \). If the characteristic roots of \( p \) are \( \{\lambda_i : 1 \leq i \leq r\} \), then by Appendix Theorem A.2, the general solution of \( x_n \) will be a summation of item \( a_{i,j} n^j \lambda_i^n \)'s. In theory by plugging in the first \( r \) terms, we can solve the coefficients \( a_{i,j} \)'s from a linear system \( \textbf{M} \cdot \vec{a} = \vec{0} \). In practice, however, we can not always solve this linear system especially when the rank \( r \) is high.

B.1 Simplifying Assumptions

We can assume some \( a_{i,j} \)'s are zero and reduce our linear system to some lower rank. And if the reduced system \( \textbf{M}_1 \vec{b} = \vec{0} \) can be solved, then we can consider the solutions \( \vec{b} \) along with the zero-assumption \( a_{i,j} \)'s as a candidate solution \( \hat{\vec{a}} \) for the original system. And if the first \( r \) terms of the general solution derived by candidate solution and the original data are a match, then by linear algebra we can claim that the candidate solution \( \hat{\vec{a}} \) is the real solution for the original system \( \textbf{M} \cdot \vec{a} = \vec{0} \).

Example B.1. Now we are going to solve the 9-degree linear system in the proof of Corollary 4.13:

\[
H_n = A_1 (-1)^n + A_2 n(-1)^n + A_3 n^2(-1)^n + B_1 \delta_1^n + B_2 n\delta_1^n + B_3 n^2\delta_1^n + C_1 \delta_2^n + C_2 n\delta_2^n + C_3 n^2\delta_2^n,
\]

(B.1)

where \( \delta_1 = \frac{3+\sqrt{5}}{2} \) and \( \delta_2 = \frac{3-\sqrt{5}}{2} \). We need to find the coefficients \( A_i, B_i, C_i \),
$C_i$'s, $1 \leq i \leq 3$.

We know $H_n = n \cdot \#ST(C_n^2) = (-1)^n \frac{d D_n^{(2)}(\lambda)}{d \lambda} \bigg|_{\lambda=4}$. So we can generate sequence $\{H_n\}$ using Mathematica.

```mathematica
S[N_] := SparseArray[{Band[{1, 2}] -> 1, {N, 1} -> 1}, {N, N}]

In[19]:= H2 = Table[D[D2[n, \[Lambda]], \[Lambda]]/(-1)^n, {n, 5, 35}]
/. \[Lambda] -> 4
Out[19]= {625, 2304, 8281, 28224, 93636, 302500, 958441, 2985984, 9174841}

Then we try to solve the linear system $M \cdot \vec{a} = \vec{0}$:

```mathematica
\[ Delta \]1 = 1/2 (3 - Sqrt[5]); \[ Delta \]2 = 1/2 (3 + Sqrt[5]);
m = Table[((-1)^n, n*(-1)^n, n^2*(-1)^n, \[Delta\]1^n,
n*(\[Delta\]1)^n, n^2*(\[Delta\]1)^n, \[Delta\]2^n,
n*(\[Delta\]2)^n, n^2*(\[Delta\]2)^n), {n, 5, 13}];

In[25]:= LinearSolve[m, H2]
Out[25]= \{CenterEllipsis\} \{CenterEllipsis\}
large output

The system can not be solved. We then assume $A_i = B_i = C_i = 0$ for $i = 1, 2$, and reduce the system to rank 3:

```mathematica
l = Table[n^2*(-1)^n, n^2*(\[Delta\]1)^n, n^2*(\[Delta\]2)^n], {n, 5, 7}];

In[28]:= LinearSolve[l, Take[H2, 3]] // Simplify
Out[28]= \{-2/5, 1/5, 1/5\}

Generate $\hat{H}_n$ for $5 \leq n \leq 13$ and compare with the original sequence:

```mathematica
In[29]:= Table[-(2/5) n^2 (-1)^n + 1/5 n^2 (\[Delta\]1)^n +
1/5 n^2 (\[Delta\]2)^n, {n, 5, 13}] // Simplify
Out[29]= \{625, 2304, 8281, 28224, 93636, 302500, 958441, 2985984, 9174841\}

In[30]:= \% == H2
Out[30]= True

Then we know

\[
\begin{align*}
A_i &= B_i = C_i = 0, & i & \in \{1, 2\} \\
A_3 &= -\frac{2}{5} \\
B_3 &= C_3 = \frac{1}{5}
\end{align*}
\]

is a genuine solution for equation (B.1).
B.2 Find Smaller Recurrence for Component Sequence

Sometimes by observation, we may find us a simple function \( f \) and sequence \( \{y_n\} \) such that \( x_n = f(y_n) \), and sequence \( \{y_n\} \) has simpler recurrence than \( \{x_n\} \). Then we can solve the general solution for \( \{y_n\} \) first and then use \( f(y_n) \) to get original \( x_n \). Sometimes we can also find multi-variant function \( F \) and sequences \( \{y^n_i\} \) with \( x_n = F(y^n_1, y^n_2, \ldots, y^n_r) \). And we can always start with the component sequences \( \{y^n_i\} \).

Example B.2. Now we are going to solve the 27-degree linear system in the proof of Theorem 4.15:

\[
 n \cdot \# ST(C_{n}^{(3)}) = \sum_{i=1}^{9} A_i \cdot \delta_i^n + B_i \cdot n \cdot \delta_i^n + C_i \cdot n^2 \cdot \delta_i^n, \tag{B.2}
\]

where \( \delta_i \)'s are roots of \( p_3(T) = 0 \). We need to find the coefficients \( A_i, B_i, C_i \)'s, \( 1 \leq i \leq 9 \).

We know \( H_3^{\text{def}} \equiv n \cdot \# ST(C_{n}^{(3)}) = (-1)^n \cdot \frac{dD_{n}^{(3)}(\lambda)}{d\lambda} \bigg| \lambda=6 \). So we can generate sequence \( \{H_3^{(3)}\} \) using Mathematica.

\[
 S[N_] := \text{SparseArray}\left\{\text{Band}\{1,2\} \rightarrow 1, \{N,1\} \rightarrow 1\right\},\{N,N\} \right.
\]

In[32]:= \( H3 = \text{Table}[D3[n, \Lambda], \Lambda] /\Lambda \rightarrow 6 \)
\]
Out[32]= \( \{117649, 663552, 3709476, 20352200, 108513889, 570949632, \ldots \}
\]

The (distinct) roots of \( p_3(T) \) are:

\[
 \text{In}[33]:= u = \text{Solve}[-1 + 2 r + 6 r^-2 + 21 r^-3 - 21 r^-6 - 6 r^-7 - 2 r^-8 + r^-9 == 0, r]
\]
\]
Out[33]= \( \{3/4 + (3 Sqrt[7])/4 + 1/2 Sqrt[-(7/2) - (3 I Sqrt[7])/2], 3/4 + (3 Sqrt[7])/4 + 1/2 Sqrt[-(7/2) - (3 I Sqrt[7])/2], 3/4 - (3 Sqrt[7])/4 + 1/2 Sqrt[-(7/2) + (3 I Sqrt[7])/2], 3/4 - (3 Sqrt[7])/4 + 1/2 Sqrt[-(7/2) + (3 I Sqrt[7])/2], 1/2 (2 Sqrt[7] - I Sqrt[-7 + 4 Sqrt[7]]), 1/2 (2 Sqrt[7] + I Sqrt[-7 + 4 Sqrt[7]]), 1/2 (2 Sqrt[7] - Sqrt[7 + 4 Sqrt[7]]), 1/2 (2 Sqrt[7] + Sqrt[7 + 4 Sqrt[7]]) \}
\]

60
We can set the linear system as before. Again the system can not be solved. As before we observe that \( H_n \) is always a multiple of \( n^2 \), so we assume \( A_i = B_i = 0 \). But the system is still hard to solve.

However, we have further observation that \( H_n = (n A_n)^2 \) where \( \{ A_n \} \) is a sequence valued on field \( \mathbb{Z}(\sqrt{2}) \). And numerically we find \( \{ A_n \} \) is in the kernel of

\[
p_n(T) = T^4 - \sqrt{2}T^3 - T^2 - \sqrt{2}T + 1.
\]

The roots of \( p_n(T) = 0 \) are

\[
\begin{align*}
d_{1,4} &= \frac{1}{2} \left( \sqrt{\frac{7}{2}} \pm \sqrt{\frac{7}{2}} + \frac{1}{\sqrt{2}} \right), \\
d_{2,3} &= \frac{1}{2} \left( \sqrt{\frac{7}{2}} - \sqrt{\frac{7}{2}} + i \sqrt{\frac{7}{2}} \right).
\end{align*}
\]

By A.1, we know \( A_n \) has general solution:

\[
A_n = \sum_{i=1}^{4} \varepsilon_i d_i^n. \tag{B.3}
\]

Now the problem has been reduced to a rank 4 linear system \( M_2 \cdot \varepsilon = 0 \).

We still fail to solve this rank 4 system. However, we also observe that \( \varepsilon_i \)'s are close in norm and roots \( d_i \)'s share symmetries. We then assume \( \varepsilon_{1,4} = -\varepsilon_{2,3} \), and reduce above into a one variable system:

\[
\begin{align*}
in[80] &= m3 = Table[rts[[k]] - n, \{k, 1, 4\}] . \{1, -1, -1, 1\} / \ n -> 7; \\
\quad Solve[m3 \backslash \{\text{Epsilon}\} == a3[[1]], \{\text{Epsilon}\}] \ // \ Simplify
\end{align*}
\]

So we have candidate solutions \( \widehat{\varepsilon}_{1,4} = -\widehat{\varepsilon}_{2,3} = \frac{1}{\sqrt{14}} \).

First we can check \( \widehat{\varepsilon} \) is a solution for equation (B.3):
So we have \( A_n = \frac{1}{\sqrt{14}} (d_1^n - d_2^n - d_3^n + d_4^n) \). Then

\[
\hat{H}_n^3 = n^2 A_n^2 = \frac{n^2}{14} (d_1^n - d_2^n - d_3^n + d_4^n)^2 ,
\]

this means the general solution for \( \hat{H}_n^3 \) has roots \( d_i d_j \)'s. Compare \( d_i d_j \) with \( \delta_k \), we have

\[
\begin{align*}
\delta_1 &= 1 = d_2 d_3, \\
\delta_2 &= d_3 d_4, \quad \delta_3 = d_1 d_2, \quad \delta_4 = d_2 d_3, \quad \delta_5 = d_1 d_3, \\
\delta_6 &= d_3^2, \quad \delta_7 = d_2^2, \quad \delta_8 = d_1^2, \quad \delta_9 = d_4^2.
\end{align*}
\]

And we can get candidate solution for the original equation (B.2):

\[
\begin{cases}
\hat{A}_i = \hat{B}_i = 0, & 1 \leq i \leq 9, \\
\hat{C}_1 = \frac{2}{7}, \\
\hat{C}_2 = \hat{C}_3 = \hat{C}_4 = \hat{C}_5 = -\frac{1}{7}, \\
\hat{C}_6 = \hat{C}_7 = \hat{C}_8 = \hat{C}_9 = \frac{1}{14}.
\end{cases}
\]

Generate \( \hat{H}_n^3 \) for \( 7 \leq n \leq 33 \) using \( \{\hat{A}_i, \hat{B}_i, \hat{C}_i\} \), and compare with \( H_n^3 \):

\[
\text{In[97]:=} B = \{2/7, -1/7, -1/7, -1/7, -1/7, 1/14, 1/14, 1/14, 1/14\};
\]
\[
\Delta[1] = \{1, \text{rts[[3]] \ rts[[4]] \ rts[[1]] \ rts[[2]]}, \]
\[
\text{rts[[2]] \ rts[[4]], \ rts[[1]] \ rts[[3]], \ rts[[3]]^2,} \]
\[
\text{rts[[2]]^2, \ rts[[1]]^2, \ rts[[4]]^2} \} / \text{Simplify;}
\]
\[
\text{Table[n^2 B.\[Delta]1^n, \{n, 7, 33\}] / \text{Simplify}}
\]
\[
\%
\]
\[
\text{Out[99]=} \{117649, 663552, 3709476, 20352200, 108513889, 570949632, 296273761, 15180485768, 77022900900, 387312648192, 1932302945329, ...
\]
\[
\text{Out[100]=} \text{True}
\]

So far we proved \( \{\hat{A}_i, \hat{B}_i, \hat{C}_i\} \) is a genuine solution for equation (B.2).

### B.3 Find Backward Items

Notice results in Appendix A are general true for all \( n \in \mathbb{N} \). This means from \( \{x_n: n \geq n_0\} \), we can sometimes go backward to get \( x_n \) with \( n < n_0 \), and have the same general solution, in particular, same coefficients. In practice, we prefer items \( x_n \) with \( n \) around the origin (both signs), as the general
expressions at these positions are sometimes concise, which makes our linear system easier to solve.

**Example B.3.** Now we are going to solve the 18-degree linear system in the proof of Theorem 4.27:

\[
R_n^{(3)} = \sum_{i=1}^{9} a_i \cdot \delta^n_i + b_i \cdot n \cdot \delta^n_i
\]  
(B.4)

where \( \delta_i \)’s are roots of \( p_3(T) = 0 \). We need to find the coefficients \( a_i \) and \( b_i \) for \( 1 \leq i \leq 9 \).

Apply same procedure as in B.2 first.

Keep sequence \( \{A_n\} \) as in previous proof, we observe that \( R_n^{(3)} = A_n Y_n \) where \( \{Y_n\} \) is also a sequence valued on field \( \mathbb{Z}(\sqrt{2}) \). And numerically we find \( \{Y_n\} \) is in the kernel of

\[
p_g(T) = (p_a(T))^2.
\]

So \( p_g(T) \) has double roots \( d_i \)'s. By A.2, we know \( Y_n \) has general solution:

\[
Y_n = \sum_{i=1}^{4} (\alpha_i d^n_i + \beta_i n d^n_i).
\]  
(B.5)

Now the problem has been reduced to a rank 8 linear system.

We still fail to solve this rank 8 system. However, by the recursive relation, we can go backward and find smaller \( Y_n \)'s, see Table B.1

We use \( \{Y_n: -3 \leq n \leq 4\} \) as initial values, and try to solve the linear system.
Table B.1: values of $Y_n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_n$</td>
<td>2</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>2</td>
<td>$4\sqrt{2}$</td>
<td>16</td>
<td>$28\sqrt{2}$</td>
<td>98</td>
</tr>
</tbody>
</table>

We observe that $\alpha_i$’s are close in norm and pair $\beta_{1,4}$ and pair $\beta_{2,3}$ have zero sum. We then assume $\alpha_{2,3} = -\alpha_{1,4} = \nu_1$, $\beta_4 = -\beta_1 = \nu_2$, $\beta_3 = -\beta_2 = \nu_3$, and reduce above into a three-variables system:

```
In[151]:= m5 = Table[Table[rts[[k]]^n, {k, 1, 4}]~Join~
(n*Table[rts[[k]]^n, {k, 1, 4}]), {n, -3, 4}];
LinearSolve[m5, {2, 1/Sqrt[2], 0, 0, 0, 1/Sqrt[2], 2,
4 Sqrt[2]}] // Simplify;
% // N
Out[151]= {-0.0190901 + 8.67362*10^-19 I, 0.0190901 + 1.73472*10^-18 I,
0.0190901 + 6.93889*10^-18 I, -0.0190901 + 0. I,
-0.0800487 - 6.93889*10^-18 I, -4.33681*10^-18 - 0.0361353 I,
-1.73472*10^-18 + 0.0361353 I, 0.0800487}
```

So we have candidate solutions $\hat{\alpha}_{2,3} = -\hat{\alpha}_{1,4} = \frac{1}{14\sqrt{14}}$, $\hat{\beta}_4 = -\hat{\beta}_1 = \frac{\sqrt{7+1}}{28\frac{\sqrt{7}}{7}}$ and $\hat{\beta}_3 = -\hat{\beta}_2 = \frac{\sqrt{7-1}}{28\frac{\sqrt{7}}{7}}$.

First we can check $\hat{\alpha}_i$’s and $\hat{\beta}_i$’s form a solution for equation (B.5):

```
In[156]:= Simplify[Table[Table[rts[[k]]^n, {k, 1, 4}]~Join~
(n*Table[rts[[k]]^n, {k, 1, 4}]), {n, 7, 14}].xx
/. {\[Nu]1 -> 1/(14 Sqrt[14]), \[Nu]2 -> (1 + Sqrt[7])/(28 7^(-1/4)),
\[Nu]3 -> (-1 + Sqrt[7])/(28 7^(-1/4))}]
Out[156]= {98, 168 Sqrt[2], 562, 1863/Sqrt[2], 3054, 4960 Sqrt[2],
16002, 51303 Sqrt[2]}
```

```
Out[157]= True
```

Since $\hat{R}^{(3)}_n = A_n Y_n$, so the general solution for $\hat{R}^{(3)}_n$ has roots $d_i d_j$’s. We
can get candidate solution for the original equation (B.4):

\[
\begin{align*}
    a_1 &= -\frac{1}{14}, \\
    a_2 &= a_3 = a_4 = a_5 = \frac{1}{98}, \\
    a_6 &= a_7 = a_8 = a_9 = -\frac{1}{196}, \\
    b_1 &= 0, \\
    b_2 &= -b_3 = \overline{b_4} = \overline{b_5} = \frac{i(\sqrt{7} - 1) - \sqrt{7} - 1}{28\sqrt{2}\sqrt{7}^{3/4}}, \\
    b_7 &= -b_6 = \frac{i(\sqrt{7} - 1)}{28\sqrt{2}\sqrt{7}^{3/4}}, \\
    b_9 &= -b_8 = \frac{\sqrt{7} + 1}{28\sqrt{2}\sqrt{7}^{3/4}}.
\end{align*}
\]

Generate \( \hat{R}_n^{(3)} \) for \( 7 \leq n \leq 24 \) using the candidate solution, and compare with \( R_n^{(3)} \):

\[
\text{In[160]:= Table[a.\[Delta]1^n + n b.\[Delta]1^n, \{n, 7, 24\}] // Simplify}
\]

\[
\text{Out[160]= } \{4802, 24192, 120268, 594297, 2892138, 13967360, 67000374, \\
319258569, 1513500604, 714238720, 33566992190, 157199798144, \ldots
\]

\[
\text{Out[161]= True}
\]

So far we proved \( \{a_i, b_i, 1 \leq i \leq 9\} \) is a genuine solution for equation (B.4).
REFERENCES


