ON RESOURCE DISTRIBUTION IN NETWORK COORDINATION GAME

BY

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THESIS

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Abstract

Product adoption is an important topic from the marketing perspective. People want to understand how a new product penetrates a market. Researchers propose many models to capture product adoption growth as well as the decision-making process for individuals. In this work, we identify the lack of consideration in current research of how much time people spend on their friends and families. We develop a new game-theoretical framework to model how people spend their time. Also, we analyze the extreme cases in this model. In addition, we do extensive simulations to understand the average case performance of the steady state. We find that our resource allocation model is a potential game and the efficiency of the steady state of the game is very good on average.

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Chapter 1

INTRODUCTION

In this work, we study product adoption. Product adoption is an important topic. It helps new companies to develop strategies to penetrate markets. It also helps existing companies to understand how to hold their market shares. The topic is very challenging by its nature. It is intrinsically hard to capture how people make their decisions and how individual decisions affect aggregate outcomes.

Marketing researchers, economists, and computer science people have done an extensive amount of research in this regard. Marketing researchers develop different population models to analyze the curve adoption growth and product life-cycle [2, 17, 21]. Economists and computer scientists take a bottom-up approach. They designed network models and use a game theoretical framework to capture strategic behavior at the individual level. They analyze aggregate outcomes and the network effects on these outcomes [1, 4, 19, 12].

The main focus of this work is the resource allocation in the context of product adoption. We focus on those products with strong network effects. Examples are online social networks like Facebook and Google+. The traditional approach in analyzing network effects do not consider the fact that people have limited time to spend on these products, and the time constraints do affect the benefit of using different products. We identify the lack of research on this topic and propose a game theoretical framework to understand resource allocation.

By understand how people allocate their resources, we know how efficient are people in terms of using their time to work with others. From the theoretical perspective, it is hard in general to coordinate the whole network and the result can be bad, players in games may use time very inefficiently. But surprisingly, in our simulation, people use their time very efficiently. One can also use our model to analyze product adoption as well as general coordination game on networks.

Our contributions are:

- Propose a new framework for technology adoption with limited resources.
- Develop a new model for resource allocation.
- Prove that our resource allocation game is a potential game and always admits at least one Nash equilibrium.
- Study the best and the worst Nash equilibrium in resource allocation game.
- Do extensive simulations to analyze the average quality of Nash equilibrium in resource allocation game.

The rest of the work is organized as follows: we introduce related work in the second chapter. And we discuss models in the Chapter 3. We discuss theoretical results and simulation results in Chapter 4. And we conclude our work in Chapter 5.

Chapter 2

RELATED WORK

In this chapter, we discuss related work regarding product adoption. We mainly discuss two approaches: population model and network model. In population models, people study product adoption from a global perspective. They try to characterize the adoption process by looking at the effect of the total number of adoptions on the adoption rate. In network models, researchers take a different approach. They first model the decision process of individuals and then look at the effect of individual adoption on aggregate outcomes.

In addition to the product adoptions, we also look at behavioral models. Behavioral models generalize product adoptions. One can think of product adoptions as a special case in behavioral models with binary action space. We introduce some works regarding behavioral models with binary action space. These works focus on the steady state. For example, if people can choose between two products, what will they choose in the long run. These works will offer us more insights into how to capture the steady state of product adoption.

And we also discuss behavioral models with continuous action space. Throughout our research, we found that product adoption is very relevant to how people allocate their resources. Therefore, we discuss some works in this vein. These works show us how to characterize the steady state when the action space is continuous.

We introduce some works in influence maximization in the end. These works illustrate how to use network effects to maximize the number of product adoptions. A classic problem in this regard is: given a network and limited budgets, how do we maximize product adoptions by sending free samples to some users in the network. Computer science people have done extensive research in this vein and we will introduce both the theoretical model as well as popular algorithms.

The rest of the chapter is organized as follows. We introduce population models in the first section. And then we discuss network models in the second section. In the third and fourth section, we focus on behavioral models with binary and continuous action space. And we end the chapter by introducing works in influence maximization.

2.1 PRODUCT ADOPTION: POPULATION MODELS

Population models study behavior from a global point of view. They try to capture collective behaviors and do not model strategic behaviors at the individual level.

Peres, Muller, and Mahajan grouped the research in population models into six categories: (1) the drivers for the growth of product adoption (2) the shape of adoption curves (3) the effect of individual adoptions to aggregate behaviors (4) marketing multiple products (5) cross-country influence, (6) the effect of competition on growth [21]. Due to the large volume of works in this area, we focus only on Bass model. See [17, 21] for detail surveys in population models.

Bass model [2] is the best-known model in this vein and many following works are based on this model. Bass developed a theoretical model to capture the growth of adoption of a new product. He also did an extensive analysis to compare the model with the real world data and showed that the model is able to capture the trend of real world data.

Bass assumed that the adoption rate at a time is bilinear to the number of previous adopters as well as the number of non-adopters. More specifically, they consider a monopoly market and a new product A. They want to analyze the fraction of adopters in the market at time T. Let $f(T)$ be the fraction of adopters of A at time T. Bass claimed that

$$\frac{\mathrm{d}f(t)}{\mathrm{d}dt} = p \underbrace{(1 - f(t))}_{\text{non-adopters}} + q \underbrace{f(t)}_{\text{adopters}} \underbrace{(1 - f(t))}_{\text{non-adopters}} \tag{2.1}$$

Here, p represents external influences and q represents internal influences. An example for external influence is the advertisement of the product. And an example for internal influence is the influence or pressure of adopters on non-adopters. The close form solution for the differential equation is

$$f(t) = \frac{1 - \exp(-(p+q)t)}{1 + (\frac{q}{p}) \exp(-(p+q)t)} \tag{2.2}$$

The contribution of this work is that it is the first theoretical model the accurately captures growth curves of products adoption and explains the underlying behavioral rationale. Many works use Bass model to analyze the diffusion patterns of different products. For example, Kobrin studied the pattern of oil production [**kobrin:95**] and Gatignon, Eliashberg, and Robertson extended Bass models to estimate cross-country diffusion process and analyze six different products like dishwashers and car radios [9].

2.2 PRODUCT ADOPTION: NETWORK MODELS

Network models study product adoption from the individual perspective. Unlike population models, network models capture strategic behaviors at the individual level.

In this section, we first introduce how researchers use game theory to capture individual decision-making process, and then discuss two papers. Morris designed a game-theoretic framework to analyze product adoption. And Immorlica et al. discussed the role of compatibility in technology adoption.

2.2.1 GAME THEORY BACKGROUNDS

Economists use game theory to understand how people make decisions and how these decisions affect others. And coordination game is the main tool researchers use to capture the relationship between network effects and individual decision.

In a coordination game, people have two choices: A and B. One can think of A as adopting a product, and choice B as not adopting a product. A person receives payoff through an interaction if the person is able to coordinate with the other side of the interaction.

Given an edge represents the interaction between two people, we can use a matrix Table 2.1 to represent how much a person will receive in different configurations. We call two people 'row player' and 'column player' respectively. In Table 2.1, the first row represents the options for column player, and the first column represents the options for row player. The cells containing tuples represents how much each player receives. For example, if both players take action A, they will both receive q_A.

One important topic here is how people choose strategies. We use the term solution profile to describe the decisions of all the players. An important tool to characterize how people make their choices is Nash equilibrium. Nash et al. defined the notion of Nash equilibrium. A Nash equilibrium is a solution profile that, if any player deviates from their strategy in that solution profile, her payoff will decrease. Because if people act rationally, they will not make such deviation, Nash equilibrium is a stable state.

To apply coordination game in the analysis of product adoption, we have to first discuss coordination games on networks. Coordination games on networks are a set of coordination games on each edge of a network. People try to coordinate with all their neighbors to maximize their payoff. The best outcome from the global perspective is that all people coordinate on the same action, either A or B. But such goal is hard in general and it may happen that two actions coexist in a game.

	A	B
A	(q_A, q_A)	$(0, 0)$
B	$(0, 0)$	(q_B, q_B)

Table 2.1: Example of a payoff matrix

Through social links, people affect each other through their own actions. Researchers call such phenomenon 'contagion', 'diffusion of behaviors', or 'word-of-mouth effect'.

2.2.2 CONTAGION

Morris characterized the conditions for *contagion* to occur [19]. Contagion is an outcome of network effects. Through social links, product adopters will influence surrounding families and friends. If the influence signals are strong enough, more people will adopt the product and cause the diffusion of adoption. Contagion refers to the state in which the whole population adopts a new product.

Morris developed a model to capture how people affect each other on their choice of product adoption in networks. In his game, there are two options: to adopt a new product (A) or use an existing product (B). He assumes that A is superior to B so $q_A > q_B$. The whole game starts with all people in the network adopting the inferior option B. And the question Morris tries to ask is: if I pick a set of players and force them to use the other option A, is it possible for all the population to adopt A because these players switch their actions? He calls such phenomenon 'contagion'. And if contagion is possible, can we characterize contagion by the network structure? Contagion effect is so interesting because Morris points out that even A is superior to B, it can still happen that people are not switching to A.

Instead of looking at simple graph properties like degree and number of players, Morris proposed another feature 'cohesion'. Given a set of players X, cohesion measure how connected are the people in the set X. More specifically, if all the players in X have at least proportion p of their connections within X, we say the group is p-cohesive.

Intuitively, a group that is p-cohesive with high p is tightly connected. The concept is important because a p-cohesive group can stop the contagion. More specifically, in a game with a group X that is p-cohesive, if we pick a set of players Y with no overlap with X and make players in Y switch to A. Then to influence people in X, q_A needs to be high to offer strong incentives for people in X because people in X have more connections within the group so they receive high payoff by staying at B.

Morris characterized the game with this cohesion property. He pointed out that contagion occurs if q_A is high enough so there is no group that is $(1 - q_A)$-cohesive or higher. The strength of his work is that he was able to determine the conditions for contagion to occur in a robust yet elegant way. The major weakness of his work is that it is hard in general to check if a graph has a $(1 - q_A)$-cohesive subgraph. And also, the model is rather simple and fail to provide explanations for the case when we have more complicated action space. We also do not know what happens when the payoff matrix is not diagonal or not symmetric.

2.2.3 THE ROLE OF COMPATIBILITY

Morris motivated Immorlica et al. to understand the role of compatibility in the diffusion of adoptions.

Immorlica et al. extended Morris' work by adding compatibility option and compatibility cost. More specifically, the strategy space in this game consists of three options: a new product A, an existing product B, and both product A and B. People can choose to either adopt one product or adopt both. But to adopt both products, they have to pay an extra cost compatibility cost c. Without loss of generality, we say A is the superior product, which gives $q_A > q_B$. The benefit for people using both products is that they are able to interact with anyone. They can use product B to interact with people adopting product B and use A to interact with who adopts A. The more interesting case is when they interact with people using both products. Because players are rational in game theory, they will both choose to use A and receive q_A. We can use a similar payoff matrix to Table 2.1 to represent this scenario. We present the payoff matrix here in Table 2.2.

	A	B	AB
A	(q_A, q_A)	$(0, 0)$	$(q_A, q_A - c)$
B	$(0, 0)$	(q_B, q_B)	$(q_B, q_B - c)$
AB	$(q_A - c, q_A)$	$(q_B - c, q_B)$	$(q_A - c, q_A - c)$

Table 2.2: Payoff matrix for coordination game with compatibility cost.

In their game, all players adopt product B in the beginning, which is the same as the setting in Morris' game. Immorlica et al. discovered that the compatibility cost affects contagion in a non-convex way [12]. They draw the contagion region, the two dimensions are the quality of product A and the compatibility cost. The contagion region is not convex. More specifically, when the compatibility cost is higher than a threshold, if $q_A > q_B$, contagion will occur. But they find that there is a range of cost such that q_A has to be even bigger to make contagion happen. This implies that inferior product may be able to hold market share if they manage to provide compatibility to their customers with low cost.

In addition to characterize the contagion region, Immorlica et al. characterized a blocking structure. That is, if such structure exists in the network, the contagion cannot happen and both products will exist in the game. The p-cohesive group in [19] is an example of such block structure.

The strength of their work is their discovery of the role of the compatibility cost, as well as their extension of the concept of cohesion and characterize blocking structures. The main weakness is that their analysis heavily relies on d-regular graph. It is not clear if one can extend the same argument to a game with more general network structure.

Morris and Immorlica et al. both analyzed coordination game with binary action space on networks. Their work offer insights into why an inferior product may survive, and how the network structure affects this. But their analyses are still restrictive and it is not easy to extend their arguments to arbitrary graph or other strategy space.

2.3 OTHER BEHAVIORAL MODELS

In this section, we introduce some other behavioral models. Behavioral models capture how people choose their behavior. And one can think of product adoption as a special case in the behavioral model with binary action space. It is important to understand behavioral models because it offers insights into how to model different decision process and capture different elements.

The rest of the section is organized as follows. We first discuss the behavioral models using binary action space. And we introduce models with continuous action space.

2.3.1 BINARY ACTION SPACE

Behavioral models with binary action space capture many aspects of network effects on the decision-making process. For example, how people choose between two products.

We have discussed two works in this vein [19, 12]. Morris and Immorlica et al. analyzed games in a deterministic approach. Players in their games do best-response deterministically. But people in the real world does not always act rationally. Economists use the phrase 'bounded rationality' to describe this fact.

So here, we introduce two works that try to capture 'bounded rationality'. They add randomness when people are making decisions. They found that randomness makes it easier to analyze game dynamics and also illustrates how the whole population select between equilibria.

Kandori, Mailath, and Rob studied coordination games with *mutation* setting [13]. The basic game setting is the same as Morris' games. The extra feature is the mutation setting. With mutation setting, any player can change her strategy with a probability p_ϵ. A direct result is that, in this game, all possible outcomes can happen. The problem becomes which outcome is more likely to happen than the others, or which outcome happens more often than the others in the long run.

Let q be the threshold fraction such that if there are more than proportion q of a player i's neighbors choosing strategy A, i will choose A. Kandori, Mailath, and Rob proved that if p_ϵ is small and q is not too close to $\frac{1}{2}$, people choose the risk-dominant equilibrium with high probability.

To be clear, in coordination game, if there are more than one equilibria $N_1, N_2, \cdots N_N$, the risk-dominant equilibrium is the Nash equilibrium that if any player deviates from her choice in the equilibrium, many players will suffer a huge decrease in payoff. Harsanyi, Selten, et al. proposed this solution concept to solve the problem that Nash equilibrium is not unique and we do not know how people choose between Nash equilibrium.

Kandori, Mailath, and Rob analyzed the outcome of considering bounded rationality. And also, he justified that people will choose risk-dominant equilibrium in games on networks. But one weakness of their work is that the interaction model in their game is global. One can think of it as a game played on a complete network. Such interaction model is not realistic because in real-world, no person interacts with all people in a social network. Another problem with their argument of people choosing risk-dominant equilibrium more often than the others is the mutation probability. Ellison showed that the time span of the 'long-run' depends on p_ϵ. If p_ϵ is small, it actually takes a long time before the population shift from one equilibrium to the other. And such long time is not practical in practice.

Ellison refined the model and propose a *local interaction model* [8]. One can think of it as a game played on an incomplete network. Ellison discovered that in local interaction model, the shift from one equilibrium to the other takes much less time, which in turns justify the observation that people choose one equilibrium more often than the others. The major contribution of their work is that they refine the interaction model and also justify that risk-dominant equilibrium is indeed preferable. Moreover, Ellison showed that network structure does not affect the timespan for shifting from one equilibrium to another.

The major weakness of the work is, still, the underlying graph structure. Although they introduce local interaction notion and study an incomplete network. They still assume that every player interacts with k other players. So we can think of it as a game played on k-regular graph.

2.3.2 CONTINUOUS ACTION SPACE

We discussed different models in games with binary action space. Another line of works is the models with continuous action space. Continuous action space capture aspects lack in binary action space. For example, if we want to analyze how people allocate their time among their friends, the action space will be continuous. The challenge here is how to capture the game dynamics when the action space is continuous. And also, how to identify the role of network structure in such game settings.

Ballester, Calvó-Armengol, and Zenou studied a game with continuous action space and quadratic utility function [1]. They focus on actions that are strategic substitutes. In their game, players spend the effort to earn utility. And their payoff depends both on their utility as well as the effort spends on others. They propose the following utility function.

$$U_i(x_1, \cdots, x_n) = \alpha x_i + \frac{1}{2}\sigma_{ii}x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j \tag{2.3}$$

That is, the utility of player i is quadratic to its own effort x_i, and linear to his friends' efforts $x_j, j \neq i$. Notice that the σ_{ij} terms represent the neighborhood relationship in the sense that if $\sigma_{ij} = 0$, we can say that there is no connection between i and j. We can use one σ matrix that contains all σ_{ij} to characterize such game and network structure. Ballester, Calvó-Armengol, and Zenou proved that [1], with some conditions, there exists only one Nash equilibrium. And they use Bonacich centrality [3] to characterize the unique Nash equilibrium. They also study the key-player problem: given a game, how can one find the most valuable player in the network such that by removing this key player, one can reduce the aggregated effort of the whole population.

The strength of their work is the novel construction of the game, they use one matrix to capture both of the utility function and the graph structure. And also, they identify the role of network structure in this game. They use results from spectral graph theory to characterize the unique Nash equilibrium. They also provide extensive examples on how to apply their theorems in the real world. Examples are crime networks and collaboration networks.

The main weakness of their work is that they study specific quadratic utility functions. The function nicely captures linear externalities in players actions but its restricted. Bramoullé, Kranton, and D'amours generalized Ballester, Calvó-Armengol, and Zenou's games as well as others and provide a unified analysis [4].

Bramoullé, Kranton, and D'amours generalized this model and studied all possible strategic interaction. Close to the setting of the previous work, their game consists of a set of players as well as an underlying graph $\mathbf{G}$ representing the relationship between players. They examine all utility functions of the form:

$$U_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G})$$

with best response:

$$f_i(\mathbf{x}_{-i}), \delta, \mathbf{G}) = \max\left(0, c_i - \delta \sum_{i=1}^{n} g_{ij} x_j\right)$$

Where δ is the parameter that represents the degree of substitutability: how does an action of one player substitutes for the other players actions. Bramoullé, Kranton, and D'amours discovered that the lowest eigenvalue characterize the existence of stable Nash equilibrium. More precisely, by examining the potential function, they realize that the smallest eigenvalue $\lambda_{min}(\mathbf{G})$ of the graph $\mathbf{G}$ plays an important role in determining uniqueness Nash equilibrium. And also, they discover that λ_{min} measures how the network $\mathbf{G}$ amplifies the substitutability. Their conclusion is that if

$$\lambda_{min}(G) < \frac{1}{\delta} \tag{2.4}$$

, then there is only one Nash equilibrium. In other cases, there may be more than one

Nash equilibrium. The use perturbation analysis of quadratic programming to analyze the stability of Nash equilibrium. They find that the unique Nash equilibrium maximizing potential function value is stable when Eq. (2.4) is satisfied. More generally, given a Nash equilibrium S, let A be the set of active agents and let G_A be the subgraph consisting of active agents, they realize that $\lambda_{min}(G_A) < \frac{1}{\delta}$ implies the stability of S. And they also point out that many stable Nash equilibria have inactive agents(free riders).

The strength of their work is that they are able to characterize Nash equilibrium for a wide range of games. They develop methods to connect the network structure to Nash equilibrium through the help of spectral graph theory. Moreover, they identify that the direction of previous works, which mainly focused on symmetric equilibria, is wrong. They also provide extensive analysis on many real-world cases like crime networks and social networks.

The weaknesses of their works are as follows. First of all, in these models, action space is $\mathbb{R}$. This means that they can only model the case where people have only one action to interact with all other players. But in reality, we also need to model the case that people have different strategies for different friends. Secondly, many existing works assume that given a player i, all its neighbors affect i in the same way. However, people do have preferences on their neighbors. Thirdly, they assume linear externality, and the connection to graph theory heavily rely on the linearity. It will be interesting to examine more complicated externality functions.

2.3.3 MORE ON NETWORK EFFECTS

One interesting topic in product adoption is how to use network effects to maximize product adoption rate? For example, if a company wants to promote their products, one approach is to persuade some people in the social network to use their products by giving them free samples. Through the social influence, they may influence their neighbors and cause contagion. The major challenge here is that dynamics on networks are hard to capture. And it is hard to choose the set of players to give free samples with limiting budget.

Domingos and Richardson first studied this topic [7, 22]. They study the network value of a user. In target marketing, firms analyze the behaviors of users to decide who may be a potential customer and then promote their products to these users. However, many traditional works focus more on analyzing and mining historical data of a user but ignore the network effect of that user. In reality, friends behavior may strongly affect a person's decision. Viral marketing, which makes use of this word-of-mouth effect, can be very effective. Domingos and Richardson identified the lack of pricing such network effect in existing approaches and use a graphical model to capture the social influence of players. They prove the effectiveness by running experiments on a movie database and prove that their method can find profitable (binary)marketing strategy. Richardson and Domingos extends their work to continuous marketing strategy and apply such strategies to data from some knowledge-sharing sites, such as the platform for customers to exchange information, writing product reviews, etc.

Domingos and Richardson motivated Kempe, Kleinberg, and Tardos to develop the theoretical framework for influence maximization [14]. Kempe, Kleinberg, and Tardos attacked the problem from theoretical approach and propose a game-theoretical model to capture the diffusion process. Their main focus is: given a weighted network and a budget, how to maximize the influence of a product. Kempe, Kleinberg, and Tardos studied two popular diffusion models: Linear Threshold (LT) model and Independent Cascading (IC) model. Kempe, Kleinberg, and Tardos proved the influence maximization problem is NP-hard. But they examine the influence spread function $\sigma(S)$, which is the function that returns the coverage of diffusion given a set of nodes S. And they find that the influence spread functions $\sigma(S)$ is monotone and submodular. This property helps them develop an approximation algorithm with the approximation ratio of $1 - \frac{1}{e}$.

The weakness of the algorithm is the scalability. Chen, Wang, and Yang showed that influence spread of a set of nodes is $\#p$-hard [6], so Kempe, Kleinberg, and Tardos had to find a way around the computation issue. They propose to do Monte Carlo simulations 10,000 times to approximate the value. Such simulation is inefficient and cannot scale to a massive network. To solve this problem, many people propose different approaches, examples are: CELF [16], CELF++ proposed by [10], and MIP [6, 5].

Leskovec et al. proposed CELF to optimize the greedy algorithm by exploiting the submodularity of influence spread [16]. Let marginal gain of influence spread by adding a node

u be $\Delta_u(S) = \sigma(S \cup \{u\}) - \sigma(S)$. Leskovec et al. sort all the nodes based on the marginal gain. Because the marginal gain always goes down, they can update those nodes with higher marginal gain and save significant time by reducing the total number of simulations in each round. Goyal, Lu, and Lakshmanan proposed CELF+ to refine this idea [10]. They observe that in one round if the node with highest marginal gain is the same as the last round, we do not have to recompute the marginal gain. This observation leads to $17 - 61\%$ improvement.

Chen, Wang, and Yang proposed another approach to maximize influence [6, 5]. They first discover that computing influence spared in independent cascade(IC) model is $\#p$-hard in general. They propose another model: Maximum Influence Arborescence(MIA) to approximate IC model. The idea of MIA is to constraint influence propagates through the path with the highest probability. For example, given two nodes u and v, they restrict the only way u can influence v is through a path $p_{u \sim v}$ such that the probability of that path is the highest among all possible paths $\{p'_{u \sim v}\}$. With this more structured model, they develop a much more efficient approximation algorithms to maximize influence in this MIA model.

There are an extensive amount of works in this vein. We know how to model influence maximization problem, and are able to use approximation algorithm to select a set of nodes to achieve our goals. See [23] for more details of influence maximization algorithms.

Chapter 3

MODELS AND DEFINITIONS

We discussed many games on networks. These works emphasize the importance of local interactions or social ties. But they do not look deeper into the local interaction model. There are two major weaknesses that motivate our work. First, they do not capture preferences of players. Second, they do not consider the number of times people interact with each other. Third, the do not consider the resource constraint. People do not have infinite time to interact with all their friends and families.

The preferences of people are important because this is how they value each social tie. For example, people weight family higher than their friends. The number of times people interact with each other is also important because social influence comes from the fact that people interact. And the amount of time they interact with each other change the value of that social tie. The time constraint is important because it limits how people spend the time among their friends. So we suggest capturing these three features.

In this section, we introduce the coordination games with limited resources. And because of the complexity of the model, we focus on the resource distribution game, which is the basis for coordination games with limited resources. And the coordination games with limited resources is our future work.

3.1 COORDINATION GAME WITH LIMITED RESOURCE

The works of Morris [19] and Immorlica [12] motivate our work. Our game setting is close to their games. Our game consists of N players $\mathcal{P}$ and an underlying graph $\mathcal{G} = (V, E)$. Each node of the graph represents a player, and each edge represents a social link between players. In the game, players interact with each other through technologies. And there are two technologies: A and B. Players can choose between A and B, or choose to use A and B at the same time. For simplicity, we denote the strategy of using A and B at the same time as AB.

By using technology to interact with their friends, a player receives certain payoff. The number times a player interacts with her friend decide the payoff she will receive from the friend. And we also assume that they have limited time, which implies there is an upper bound on the total number of interactions per user.

We define some notations. For simplicity, we use players and users interchangeably. For a given graph $\mathcal{G} = (V, E)$, we use $N(i) = \{j|(i, j)\}$ to deonte the neighborhood of player i. We use $N_+(i) = N(i) \cup \{i\}$ to denote the neighborhood of a player i and the player itself. We denote $deg(i)$ as the degree of a node i.

The graph is weighted and undirected. We associate each edge (i, j) with two rational numbers w_{ij} and w_{ji}. The weight w_{ij} captures the preference of a player i on a player j and the weight w_{ji} is the preference of j on i. We use these weights to compute utility, which we will define in the next section. For simplicity, weights are all normalized, that is

$$\sum_{j \in N(i)} w_{ij} = 1 \tag{3.1}$$

We also define $w_{ii} = 0$. So we also have

$$\sum_{j \in N_+(i)} w_{ij} = 1 \tag{3.2}$$

Every player has some (limited) resource to interact with their neighbors. They receive payoff through the interaction with others. Players can also earn payoff from participating the game. For example, a player can still earn payoff by browsing contents in Instagram even she does not interact with any friend at all. For simplicity, we do not consider this type of payoff. We denote the resource of player i as β_i.

Recall that players use technologies to interact with their friends, we use T to denote the technology space $\{A, B, AB\}$. Consider both the time and the technology, the strategy space consists of two parts: resource(time) allocation and technology choices.

The strategy space for time allocation is a N-dimensional vector, $\vec{f_i} = (f_1, f_2, \cdots, f_n) \in \mathbb{R}^N$, which represents the resource(time) a player i wants to spend on each player. Each element in the vector takes value in $[0, \beta]$. Due to the resource constraint setting, we have the following equality

$$\sum_j f_{ij} = \beta_i \tag{Resource Constraint}$$

Given the space for time allocation and technology choices, the strategy space of every player is $T \times [0, \beta]^N$. An example for a strategy is $(\vec{f_i}, t_i)$ where $t_i \in T$.

Given the strategy space of each player, we denote the action of every player i as $x_i = (\vec{f_i}, t_i)$.A solution profile is a set of actions used by every players $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$. For simplicity, we let $\mathbf{x}_{-i} = \{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}$. That is, $\mathbf{x}_{-i}$ is a solution profile without the action x_i of player i.

We denote $u_i : (T \times [0, \beta]^N)^N \mapsto \mathbb{R}$ as the utility function for player i. The input is a solution profile and the output is a real number. In the next section, we will define the utility function.

We discussed the settings for our games and now we discuss some notions in game theory and how we represent them in our game setting. In game theory, given a solution profile, the best response of a player i is the strategy that maximizes her payoff. We denote $b_i(\mathbf{x}_{-i})$ as the best response function and

$$b_i(\mathbf{x}_{-i}) = \underset{x_i \in T \times [0,\beta]^N}{\arg\max}\ u_i(x_i, \mathbf{x}_{-i}) \tag{3.3}$$

A player may deviate from any given solution profile. We are interested in the stable solution profile, which we call Nash equilibrium.

Definition 1 (Nash equilibrium) *Given a game, a solution profile* $\mathbf{x}$ *is a Nash equilibrium if and only if for every player* i*,*

$$u_i(x_i, \mathbf{x}_{-i}) \geq u_i(x_i', \mathbf{x}_{-i})$$

The Nash equilibrium $\mathbf{x}$ *is also a solution for the following set of equations*

$$x_i = b_i(\mathbf{x}_{-i}) \ \forall i$$

In our game, there can be more than one Nash equilibria. We do not know which equilibrium the population will choose in general, so we are interested in the worst case and the best case. That is, how bad and how good this stable state can be. To evaluate the quality of a Nash equilibrium, we define the social welfare of a game as the aggregation of the players' utilities:

Definition 2 (Social Welfare) *Given a solution profile* $\mathbf{x}$*, the social welfare of a game* $(\mathcal{P}, \mathcal{G}, \beta, T)$ *is the aggregation of all players utilities in that solution profile* $\mathbf{x}$*. That is:*

$$SW(\mathbf{x}) = \sum_{i \in \mathcal{P}} u_i(\mathbf{x}) \tag{3.4}$$

Let the solution profile maximizing the social welfare be OPT, we can then define two popular efficiency metrics to evaluate a Nash equilibrium:

Definition 3 (Efficiency Metrics) *Given a game* $(\mathcal{P}, \mathcal{G}, \beta, T)$*, there is a solution profile* OPT *maximizing social welfare value, a Nash equilibrium* NE_{max} *maximizing the social welfare value, and a Nash equilibrium* NE_{min} *minimizing the social welfare value. The Price of Anarchy(PoA) of the game is*

$$PoA((\mathcal{P}, \mathcal{G}, \beta, T)) = \frac{SW(OPT)}{SW(NE_{min})} \tag{3.5}$$

And similarly, the Price of Stability(PoS) of the game is

$$PoS((\mathcal{P}, \mathcal{G}, \beta, T)) = \frac{SW(OPT)}{SW(NE_{min})} \tag{3.6}$$

The PoA measures how bad the stable state can be in a game and the PoS measures how good the stable state can be in a game.

The game setting for the coordination game with limited resources is complicated. The strategy space of every player consists of a continuous action space and a discrete action space. Also, we do not have tools to analyze the coordination game with continuous action space in our setting. So we focus on a special case of our games: the game with one technology only. Our goal is to understand how people distribute their time among their neighbors.

In the next section, we will first define resource distribution game. And then we will introduce the utility function as well as the weighting system that is used in our work. We then introduce a broad class of resource distribution game which we call *uniform-weighting resource distribution game* that admits a potential function. We also explain the best response dynamic as well as the convex programs based on best response dynamic. And at the end of this section, we explain the importance of starting state in our game and introduce *proportional starting state*.

3.2 RESOURCE DISTRIBUTION GAME

In this section, we define resource distribution game. And look deeper into the setting.

A resource distribution game is a specialized coordination game with limited resources. It as a special case where there is only one technology available to players. The difference between this game and general coordination game we discussed in the last section is the set of technologies T. Because T has only one element in the current setting, we simplify the notation and use $(\mathcal{P}, \mathcal{G}, \beta)$ to represent the game. And also, the utility function u_i now maps from $[0, \beta]^N$ to a real number.

For simplicity, we assume β-regularity for resources: every player has the same limited resources β. We use min function to capture the coordination. That is, for every two (connected) players i and j, they will interact with each other with interaction frequency $\min(f_{ij}, f_{ji})$. We denote this frequency as f^*_{ij} so

$$f^*_{ij} = \min(f_{ij}, f_{ji})$$

For clarity, we give the formal definition of resource distribution game as follows.

Definition 4 (Resource Distribution Game) *A resource distribution game is a tuple $(\mathcal{P}, \mathcal{G}, \beta)$ consists of a set of player $\mathcal{P}$, an underlying graph $\mathcal{G} = (V, E, W)$, and a real number β.*

The graph $\mathcal{G}$ represents a social network. The graph is undirected and weighted. Unless otherwise specified, the graph should admit no multi-edge and no self-loop. Every node in the graph represents a player, and every edge (i, j) in the graph represent a social link between two players i and j. We associate two weights with each edge: w_{ij} and w_{ji}, each capture i and js preference over the other respectively.

The action space for each player i is a vector $\vec{f_i} \in [0, \beta]^N$. Each component of $\vec{f_i}$ is f_{ij}, which represents the proposal of player i to player j. Because the resource is limited, we have

$$\sum_{j=1}^{N} f_{ij} = \sum_{j \in N(i)} f_{ij} \leq \beta \quad \forall i \in \mathcal{P} \tag{3.7}$$

For simplicity, we define $f_{ii} = \beta - \sum_{j \in N(i)}$. And define $N_+(i) = N(i) \cup \{i\}$ In this way, we can write

$$\sum_{j \in N_+(i)} f_{ij} = \beta \quad \forall i \in \mathcal{P} \tag{3.8}$$

For every two connected players i and j, we define $f^*_{ij} = \min(f_{ij}, f_{ji})$ as the agreed value of two player. We also use the phrse 'effective frequency' to describe f^*_{ij}. As we will see in the later section, effective frequency decide the utility i receives from interaction with j.

3.3 UTILITY FUNCTION AND WEIGHTING SYSTEM

In our game, people receive utility from interacting with others. There are two steps to compute utility. Every player first computes the utility per interaction with a neighbor, and then scale that utility by the interaction frequency she spends on the neighbor. The total utility of each player is the summation of all the utilitie she receives from all her neighbors.

To compute the utility per interaction with a neighbor, we use weighting system. Recall that weightings capture the preferences of players. Our approach is to scale the row utility per interaction by the weight. For example, if a player i interacts one time with player j, let the raw utility per interaction be u_{raw}, the utility i receives for this interaction is $w_{ij}u_{raw}$. We can normalize this u_{raw} for the whole graph and the utility of i interacts with a neighbor j once is w_{ij}.

Determining the weight on graphs is interesting. Existing works in the research of contagion and influence maximization does not tell how they decide the weights. We define a global ranking to capture the social status in the real world. And based on that status, we define the corresponding weighting.

Example 1 (Global-Ranking Weighting System) *A global ranking is a function* $GR : \mathcal{P} \mapsto \mathbb{Z}^+$. *By imposing this function on a game* $(\mathcal{P}, \mathcal{G}, \beta)$, *we can associate each player i with a number* $W(i)$. *Then in the global ranking weighting system, the weight* w_{ij} *of an edge* (i, j) *in the graph* $\mathcal{G}$ *is defined as*

$$w_{ij} = \frac{W(j)}{\sum_{j \in N(i)} W(i)} \tag{3.9}$$

We can define the weighting system in other ways. But this global ranking weighting system captures the notion of social status and it also guarantees a potential game.

Another part of utility is the total utility a player received from its neighbor. In general, the utility function for a player i is $u_i(\cdot)$. The utility player i gets from player j is $u_{ij} : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$. The function takes the proposals of both palyer i and player j as inputs. The first theoretical result we have is, if the form of utility function is the symmetric for all players, that is $u_{ij} = u_{ji}\ \forall i \neq j$, then the game is a potential game.

Proposition 1 (Potential Game) *Given a game $(\mathcal{P}, \mathcal{G}, \beta)$, if the game adopts global ranking weighting system, and the utility functions are symmetric for all the players, then $(\mathcal{P}, \mathcal{G}, \beta)$ is a potential game.*

We will present the proof for Proposition 1 and the potential function in Section 4.1. To end this section, we introduce one specific utility function that we will study later in this work:

Example 2 (Quadrdatic Utility Function) *Given a player i, let f^*_{ij} be the effective interaction frequency for i's interaction with j, the utility i receives is:*

$$u_i(j) = w_{ij} f^*_{ij}(\beta - f^*_{ij}) \tag{3.10}$$

As you can see, this is a quadratic function that maximizes when $f^*_{ij} = \frac{\beta}{2}$. This function captures two important features of real-world interactions: First of all, the utility increases sin the beginning when the interaction increases. Secondly, utility decreases after it reaches some threshold. The threshold we have here is $\frac{\beta}{2}$. From now on, unless otherwise specified, we assume that the game is adopting Global ranking weighting system and this quadratic utility function.

3.4 LOCAL AND GLOBAL INTERACTION

We can examine the game from two perspectives. From the local perspective, players are trying to maximize their own utility, which leads to a convex program for local interaction. From the global perspective, if we are a coordinator coordinating everything of the game, one interesting question to ask is: can I make every player happy? One way to capture this notion is to compute the **social welfare** of a game. To maximize this value, we have another convex program for this global interaction.

As we will see in a later section, if a game takes global ranking weighting, then its PoS is one, which means the quality of its Nash equilibrium can be good.

From the local perspective, the one-degree ego network centered on each player i in the game forms a set of local interactions. In our game, we are using best-response dynamic, so the player i will want to maximize his payoff. There are two constraints: first of all, at the time point player i is making his move, i knows all his neighbors' proposals. Because i will act rationally, it will not propose more than f_{ji} to its neighbor j. This leads to the first constraint:

$$f_{ij} \leq f_{ji} \tag{3.11}$$

The second constraint is Eq. (3.8).

We call the quadratic program *local convex program*:

Definition 5 (Local Convex Program) *For every player i, let $N(i)$ be her neighborhood. Use f_{ij} to denote her proposal to j, and f_{ji} is j's proposal to i. i will want to solve a convex program to maximize her own payoff, the convex program is the local convex program $LC(i)$ of player i, which is*

$$\begin{aligned} \min_{\vec{f}} F_L(\vec{f}) = \sum_{j \in N_+(i)} w_{ij} f_{ij}(f_{ij} - \beta) \\ s.t. f_{ij} \leq f_{ji} \quad \forall i \in \mathcal{P} \quad \forall j \in N(i) \\ \sum_{j \in N_+(i)} f_{ij} = \beta \quad \forall i \in \mathcal{P} \\ f_{ij} \geq 0 \quad \forall i \in \mathcal{P} \quad \forall j \in N_+(i) \end{aligned} \tag{3.12}$$

Note that we do minimization here, but the objective function has opposite sign to the utility function, so we are maximizing the utility.

After computing the local convex program, notice that f_{ii} represents the *excess* of interaction frequency. That is, no mater how i spends f_{ii}, she cannot increase her payoff. It is not clear what i will do. For clarity, we define two types of dynamics: *aggressive* and *conservative.*

Definition 6 (Aggressive and Conservative Best-response) *In a player i takes aggressive best-response, she will first compute her local convex program. And then i will spend f_{ii} on each edge proportionally. That is, for every neighbor j, let the frequency in the solution of local convex program be f^L_{ij}, we have*

$$f_{ij} = f^L_{ij} + f_{ii} \times w_{ij}$$

If a player takes the conservative approach, then we say the player will store the frequency f_{ii} *and do not spend it on any edge.*

We will show that no matter how players choose between these two dynamics, the game is a potential game, and also, the price of stability can all be one.

From the global perspective, we want to maximize the social welfare (SW), which is the total utility. To maximize the social welfare, we have the following convex program

Definition 7 (Global Convex Program) *For a game* $(\mathcal{P}, \sim, \beta)$*. The global convex program* $GC((\mathcal{P}, \mathcal{G}, \beta))$ *of the game is a convex program trying to maximize the social welfare function. There are two types of variables:* $\vec{f}$ *and* $\vec{f^*}$*.* $\vec{f}$ *is the proposal that players make. For example* f_{ij} *is the proposal of player* i *to player* j*. And* $\vec{f^*}$ *is the agreed value between two people. For example,* $f^*_{ij} = \min(f_{ij}, f_{ji})$*. Same constraints in Definition 5 are imposed here for every player* i*.*

$$
\begin{aligned}
\min_{\vec{f}, \vec{f^*}} & F_G(\vec{f}) = \sum_{i \in \mathcal{P}} \sum_{j \in N(i)} w_{ij} f^*_{ij}(f^*_{ij} - \beta) \\
s.t. & f^*_{ij} \leq f_{ij} \quad \forall i \in \mathcal{P} \quad \forall j \in N(i) \\
& f^*_{ij} = f^*_{ji} \quad \forall i \in \mathcal{P} \quad \forall j \in N(i) \\
& \sum_{j \in N(i)} f_{ij} = \beta \quad \forall i \in \mathcal{P} \\
& f^*_{ij} \geq 0 \quad \forall i \in \mathcal{P} \quad \forall j \in N(i)
\end{aligned}
\tag{3.13}
$$

Note that the objective function is the opposite of total utility. So the minimization is equivalent to maximizing the total utility.

These are the two critical convex program in our game. One can observe from the local convex program that the initial state is important. This is because we have the constraint Eq. (3.11), which implies that if in the beginning, if all the neighbors of i propose 0 to i, then i will not be able to propose anything and will not receive any utility. But in the real world, this seldom helps. In the next section, we introduce proportional starting state to solve this problem.

3.5 PROPORTIONAL STARTING STATE

One observation of the real world is that people distribute their time according to their preference in the beginning. For example, when you are working with your friends and some acquaintances, very likely you will spend more time with your friends and then adjust the distribution based on the result. To capture this, we propose proportional starting state: the first proposals people make is proportional to their preferences.

Definition 8 (Proportional Starting State) *In the beginning of the game* $(\mathcal{P}, \mathcal{G}, \beta)$*, every player* i *will propose initial frequency to their friends with no history data. We call the set of initial values of all players starting state. Proportional starting state strategy defines that player* i *will propose* $w_{ij}\beta$ *to player* j*.*

Chapter 4

RESULTS

4.1 THEORETICAL RESULTS

We present three major theoretical results in this section. First, we present the potential function for our game and prove that our game is a potential game. Based on this result, we know that Nash equilibrium always exists in our game. The second and the third results are the quality of Nash equilibria. We show that with bad initial frequency proposal, we can have low-quality Nash equilibria. On the other hand, we also show that optimal solution for the global convex problem is also a Nash equilibria. There is a significant difference between PoS and PoA. In the next section, we present our simulations and show that, on average, a game will converge to a Nash equilibria with good quality.

The following results are proved in the Appendix.

4.1.1 CHARACTERIZING POTENTIAL GAME

Given a game $(\mathcal{P}, \mathcal{G}, \beta)$ adopting global ranking weighting and symmetric utility function, let $NW(i) = \sum_{j \in N(i)} W(j)$. We define potential function $\mathbb{U} : \mathbf{x} \mapsto \mathbb{R}$ as follows:

$$\Phi(\mathbf{x}) = \sum_{i \in \mathcal{P}} W(i) NW(i) \sum_{j \in N(i)} w_{ij} u_{ij} \tag{4.1}$$

The intuition of this potential function is two observations: First of all, because players compute utility based on effective frequency. If a player change its proposed frequency and the effective frequency changed accordingly, the difference is the same for both the player and the corresponding neighbor. Secondly, recall that $w_{ij} = \frac{W(i)}{NW(i)}$. By scaling the utility by $W(i)NW(i)$, we will have $W(i)W(j)$ for every $j \in N(i)$, and this is symmetric. Whenever player i is making a rational move, the total value increases. See the detailed proof for the potential function in Appendix A.

4.1.2 HOW GOOD AND HOW BAD CAN A NASH EQUILIBRIUM BE?

Because the potential value is given by this function always increases when any player makes a rational move, we can prove Proposition 1:

Proposition 1 (Potential Game) *Given a game $(\mathcal{P}, \mathcal{G}, \beta)$, if the game adopts global ranking weighting system, and the utility functions are symmetric for all the players, then $(\mathcal{P}, \mathcal{G}, \beta)$ is a potential game.*

Notice that this result does not rely on the form of the utility function, nor does it rely on the best-response dynamic used by players. If utility function is symmetric for all the player, the game is a potential game.

Because a game with global ranking weighting system and symmetric utility function is a potential game, it always admits at least one Nash equilibrium. The next question we address is how bad and how god can its Nash equilibria be. It turns out that the Nash equilibrium can be arbitrarily bad.

Theorem 1 *The price of anarchy of a game $(\mathcal{P}, \mathcal{G}, \beta)$ adopting global-ranking weighting scheme can be arbitrarily high.*

Proof: Consider the game with players on a two-dimensional grid presented in Fig. 4.1. In the game, every players in row k has rank α^k. So given a player i in row k, she has one neighbor in row $k+1$, two neighbors in row k, and one neighbor in row $k-1$. We present two confiugrations in Fig. 4.1 that correspond to a Nash equilibrium with good social welfare(Fig. 4.1a), and a Nash equilibrium with bad social welfare(Fig. 4.1b).

Let the solution profile in Fig. 4.1a be NE_{good} and the other one be NE_{bad}. First, these two profiles are both Nash equilibria. The reason is because every player has reach agreements. No player has extra frequency so based on the local convex program, nobody will deviate from the solution profiles.

In Fig. 4.1a, every player will receive payoff

$$\frac{\beta^2}{4}(\frac{\alpha^2+1}{(\alpha+1)^2})$$

And in Fig. 4.1b, every player will receive payoff

$$\frac{\beta^2}{4}(\frac{\alpha}{(\alpha+1)^2})$$

So if the game have N players, we have

$$SW(NE_{good}) = N \times \frac{\beta^2}{4}(\frac{\alpha^2+1}{(\alpha+1)^2})$$

and

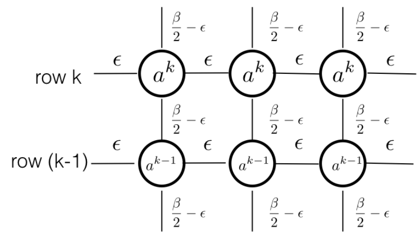
$$SW(NE_{bad}) = N \times \frac{\beta^2}{4}(\frac{\alpha}{(\alpha+1)^2})$$

Let OPT be the solution profile that maximizes the social welfare, and let NE_{worst} be the Nash equilibrium that minimizes the social welfare, we have

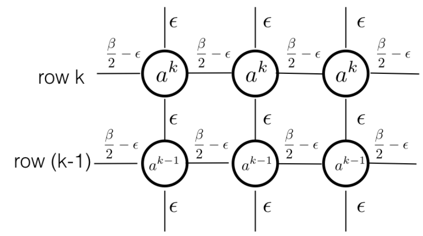
$$PoA = \frac{SW(OPT)}{SW(NE_{worst})} \geq \frac{SW(NE_{good})}{SW(NE_{bad})} = \frac{\alpha^2+1}{2\alpha}$$

We can make this PoA arbitrarily big by tuning the parameter α. Notice that this result does not depend on the size of the grid, so the game can have bad Nash equilibrium even with a small number of players. ■

We now want to know how good a Nash equilibrium can be. The next theorem shows that, if a game adopts quadratic utility function(2), we can guarantee that optimal solution is also a Nash equilibrium.



(a) N.E. with high social welfare.



(b) N.E. with low social welfare.

Figure 4.1: Example of game with bad PoA.

Theorem 2 (Stability of Optimal Solution) *Given a game $(\mathcal{P}, \mathcal{G}, \beta)$ adopting global-ranking weighting scheme and uniform utility function. If the utility function used by players are all quadratic utility function (Example 2), then the optimal solution is also a Nash equilibrium.*

The detailed proof is left to Appendix B. And this leads to the next corollary.

Corollary 1 *Given a game $(\mathcal{P}, \mathcal{G}, \beta)$ adopting global-ranking weighting scheme and quadratic utility function, the price of stability is 1.*

4.2 EXPERIMENTAL RESULTS

As we have seen in the last section, the gap between the price of anarchy and price of stability is very large. One interesting topic is the average case performance in the real world In this section, we look at average case performance by conducting experiments.

Our simulation consists of a regular graph with a thousand nodes. And they all have limited resource $\beta = 200$. Unless otherwise specified, we adopt proportional starting state and use global ranking weighting. We use quadratic utility function for all players. In the following sections, we run simulation 80 times and compute the average and the standard deviation of total utility. In order to make the stats more meaningful, we scale the total utility by the optimal utility of each game. We define average performance as

$$\frac{SW(Nash)}{SW(OPT)} \tag{4.2}$$

for each Nash equilibrium. This is exactly the inverse of the price of anarchy and it measures how efficient a game is.

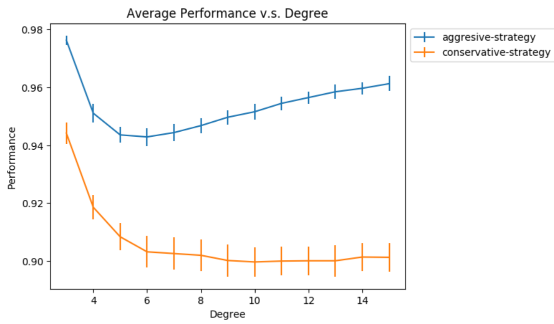
The results are organized in the following manner. We first present the average performance for regular graphs with different degrees. And then compare games with the different initial states. There are many initialization strategies that we can choose from: proportional, quadratic proportional, etc. The third comparison is between different role distributions. Recall that we use global ranking weighting scheme. The global ranking captures the social status in the real world. We are interested in how the role distributions in the population can affect the efficiency. In the end of the section, we look at global ranking from another perspective. In global-ranking weighting system, we compute weighting based on ranking and normalization. There are many other ways to compute weighting: proportional to the square of rank, exponential to the square of rank, etc.

4.2.1 REGULAR GRAPH OF DIFFERENT DEGREE

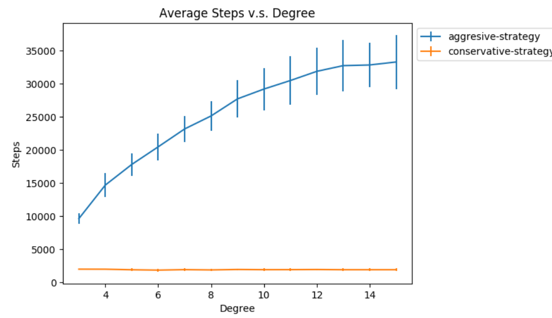
We do simulations on regular graphs with different degrees. The results are presented in Fig. 4.2. The first observation is that for conservative strategy, the average performance decreases as the degree of regular graph increases. But for the aggressive strategy, the average performance decreases in the beginning, but for the graphs with the degree more than 6, the average performance increase. The second observation is that aggressive strategy takes much more time than conservative strategy.

4.2.2 DIFFERENT STARTING STATES

There are many different ways to configure initial proposal for players in games. In the previous sections, we have seen proportional starting states. There are two other interesting starting states that we would like to discuss. Instead of distribution frequency based on the weighting, we can also distribute frequency based on the square of weighting, which we called



(a) Average performance.



(b) Average steps.

Figure 4.2: Simulations for regular graphs with different degrees.

quadratic-proportional starting state. And also, we can distribute frequency based on the exponential to the weighting, which we called exponential weighting. The formal definition is as follows.

Example 3 (Different starting states.) *Given a game adopting the global-ranking weighting system, we have quadratic-proportional weighting system, which defines the initial proposal for i to j as:*

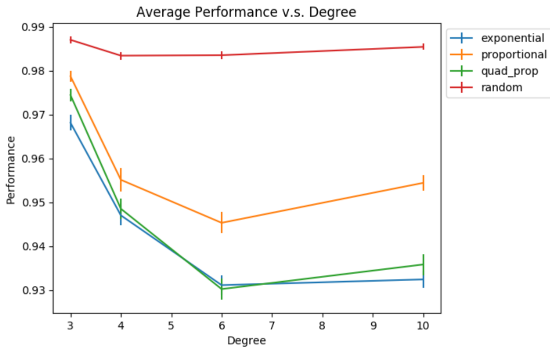
$$f_{ij}^{quad-prop} = \beta \times \frac{w_{ij}^2}{\sum_{k \in N(i)} w_{ik}^2} \tag{4.3}$$

And we can also define exponential weighting in the similar way:

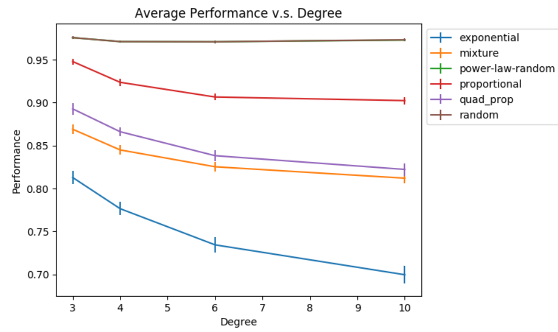
$$f_{ij}^{exp} = \beta \times \frac{exp(w_{ik})}{\sum_{j \in N(i)} exp(w_{ik})} \tag{4.4}$$

The simulation results are presented in Fig. 4.3. We also include the results when some players adopt proportional starting state and some adopt quadratic-proportional starting state.

The first thing we observe is that the average performance degrades as the degree of regular graph increases. As we can see in the figure, exponential starting state performs like proportional starting state and perform really well. The average utility is around 90% of the optimal total utility. On the other hand, the games adopting quadratic-proportional starting state are not very efficient compared to other starting state. The mixture of proportional starting state and quadratic-proportional starting locates in the between. The conclusion is proportional starting state is a good strategy overall.



(a) Aggressive best-response.



(b) Conservative best-response.

Figure 4.3: Average performance of different initial proposals.

4.2.3 DIFFERENT ROLE DISTRIBUTIONS

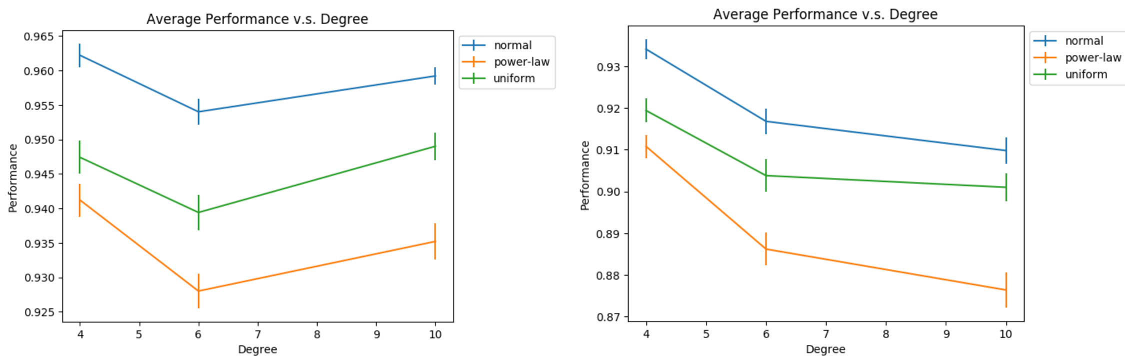
There are many different ways to decide the global ranking and the roles. We analyze this problem from a statistical perspective and interested in the following question. What kinds of role distributions perform the best? We adopt uniform distribution as a baseline case, as well as Gaussian distribution and power-law distribution. The later two distribution better capture the fact that in the social network, there are lots of people who have similar social status, but only a few people have very high or low social status. We show the results in Fig. 4.4.

Our observation is that by using the normal distribution to distribute the roles, the efficiency of a game can be as high as 96% in average for the aggressive case. And 93% for the conservative case. On the other hand, games with power-law role distribution do not perform well in average.

4.2.4 DIFFERENT WEIGHTING SCHEMES

We also compare different weighting schemes. Previously we discussed global-ranking weighting, which is basically a proportional weighting scheme. There are two other interesting weighting schemes: quadratic-proportional weighting and exponential weighting.

Example 4 (Different weighting systems) *Given a game* $(\mathcal{P}, \mathcal{G}, \beta)$ *and a global ranking* GR*, we can associate each player* i *with a rank* $W(i) = GR(i)$*. Quadratic-proportional*



(a) Aggressive best-response. (b) Conservative best-response.

Figure 4.4: Average performance of different role distributions.

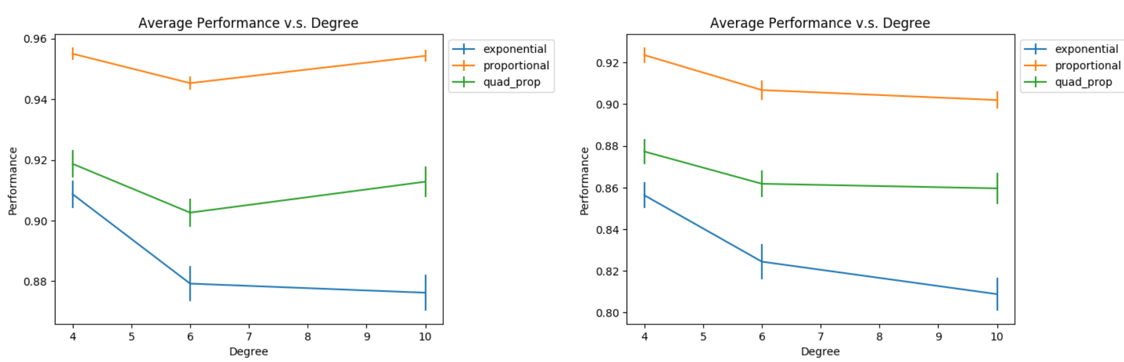
weighting defines w_{ij} *as:*

$$w_{ij} = \frac{W^2(i)}{\sum_{j \in N(i)} W^2(j)} \tag{4.5}$$

And the exponential weighting defines w_{ij} *as:*

$$w_{ij} = \frac{exp(W(i))}{\sum_{j \in N(i)} exp(W(j))} \tag{4.6}$$

The results are presented in Fig. 4.5. We can see that in general, average performance is a lot better than other weighting schemes, both in the aggressive case and conservative case.



(a) Aggressive best-response. (b) Conservative best-response.

Figure 4.5: Average performance of different weighting.

Throughout these simulations, we found that average efficiency for our game is high, both in the case of using aggressive best-response and conservative best-response. And also, we found that games with aggressive best-response almost always have better efficiency than the games with conservative best-response.

Chapter 5

CONCLUSION

In this work, we study product adoption with limited resources. We started from defining the general framework for coordination games with limited resources. And because of the complexity of the framework. We restricted our attention to a special case: the resource distribution game on networks. We developed game theory model to capture how people spend their resources on their friends.

Our contributions are:

- Propose a new framework for technology adoption with limited resources.
- Develop a new model for resource allocation.
- Prove that our resource allocation game is a potential game and always admits at least one Nash equilibrium.
- Study the best and the worst Nash equilibrium in resource allocation game.
- Do extensive simulations to analyze the average quality of Nash equilibrium in resource allocation game.

Our future work is to study coordination games on networks considering resource constraints. We expect to extend our results to coordination games and examine the properties of contagion studied in [19] and [12]. We also want to analyze influence maximization problem in our framework and study existing algorithms for influence maximization.

Another direction is community formation. If we consider a resource distribution game on the complete network, then after the game reaches steady state, we can remove those connections with low interaction frequency. An interesting topic is whether the resulting network can reflect community formation in the real world.

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Appendix A

PROOF FOR POTENTIAL GAME

In this chapter, we discuss the proof for Proposition 1. Because the game adopts global-ranking weighting scheme, every weight w_{ij} can be written as

$$w_{ij} = \frac{W(j)}{\sum_{j \in N(i)} W(i)} \tag{3.9}$$

Recall the potential function we proposed in Eq. (4.1). If a player k makes a deviation from $\mathbf{x} = (x_k, \mathbf{x}_{-k})$ to $\mathbf{x}' = (x'_k, \mathbf{x}_{-k})$, we have:

$$\begin{aligned}
&\Phi(x_k, \mathbf{x}_{-k}) - \Phi(x'_k, \mathbf{x}_{-k}) \\
=&\sum_{i \in \mathcal{P}} W(i) NW(i) \sum_{j \in N(i)} w_{ij}(u_i(x_k, \mathbf{x}_{-k}) - u_i(x'_k, \mathbf{x}_{-k})) \\
=&\sum_{i \in \mathcal{P}} W(i) \sum_{j \in N(i)} W(j)(u_i(x'_k, \mathbf{x}_{-k}) - u_i(x'_k, \mathbf{x}_{-k}))
\end{aligned}$$

Notice that only k's neighbors are affected, so

$$\begin{aligned}
&\Phi(x_k, \mathbf{x}_{-k}) - \Phi(x'_k, \mathbf{x}_{-k}) \\
=&\sum_{i \in \mathcal{P}} W(i) \sum_{j \in N(i)} W(j)(u_i(x'_k, \mathbf{x}_{-k}) - u_i(x'_k, \mathbf{x}_{-k})) \\
=&W(k) \sum_{j \in N(k)} W(j)(u_k(x_k, \mathbf{x}_{-k}) - u_k(x'_k, \mathbf{x}_{-k})) + \sum_{j \in N(k)} W(j)W(k)(u_j(x'_k, \mathbf{x}_{-k}) - u_j(x'_k, \mathbf{x}_{-k}))
\end{aligned}$$

Let u_{jk} be the payoff j receives from k in $\mathbf{x}$, and u'_{jk} is the payoff j receives from k in $\mathbf{x}'$. Recall that utility function is symmetric, we have:

$$
\begin{aligned}
&\Phi(x_k, \mathbf{x}_{-k}) - \Phi(x'_k, \mathbf{x}_{-k}) \\
=&W(k) \sum_{j \in N(k)} W(j)(u_{jk} - u'_{jk}) + \sum_{j \in N(k)} W(j)W(k)(u_{kj} - u'_{kj}) \\
=&W(k) \sum_{j \in N(k)} W(j)(u_{jk} - u'_{jk}) + \sum_{j \in N(k)} W(j)W(k)(u_{jk} - u'_{jk}) \\
=&2W(k) \sum_{j \in N(k)} W(j)(u_{jk} - u'_{jk}) \\
=&2W(k)NW(k) \sum_{j \in N(k)} w_{kj}(u_{jk} - u'_{jk}) \\
=&2W(k)NW(k)(u_k(x_k, \mathbf{x}_{-k}) - u_k(x'_k, \mathbf{x}_{-k}))
\end{aligned}
$$

The difference of potential value after the move of a player k is the difference of k's utility-scale by a constant. This proves that the game is a weighted potential game and Nash equilibrium always exists [18].

Appendix B

PROOF FOR PRICE OF STABILITY

In this section, we assume that the game adopts global ranking weighting system and all players are using quadratic utility function. Because the utility functions are symmetric for all players, the game is a potential game.

We first analyze the solution for the global convex program and local convex program using lagrangian multiplier and KKT conditions. We then analyze games with players using aggressive best-response and with players using conservative best-response. Our goal is to prove that given the solution for the global convex program, we can construct the solution profile that maximizes the social welfare value and also satisfy the constraints for the local convex program.

B.1 Solution for Global Convex Program

We first compute the lagrangian of the global convex program:

$$\begin{aligned}\mathcal{L}^G(\vec{f}, \vec{f^*}, \vec{\lambda^G}, \vec{\mu^G}, \vec{\eta^G}) =& F_G(\vec{f}) + \sum_{i \in \mathcal{P}} \sum_{j \in N_+(i)} \lambda^G_{ij}(f^*_{ij} - f_{ij}) + \sum_{i \in \mathcal{P}} \sum_{j \in N_+(i)} \mu^G_{ij}(f^*_{ij} - f^*_{ji}) \\ &+ \sum_{i \in \mathcal{P}} \eta^G_i (\sum_{j \in N_+(i)} f_{ij} - \beta)\end{aligned}$$

By setting $\nabla F_G(\vec{f}) = 0$, we have:

$$\nabla F_G(\vec{f}) = \left(2w_{ij}f^*_{ij} - w_{ij}\beta\right) \hat{f^*}_{ij}$$

where $\hat{f^*}_{ij}$ is the unit vector. The first-order condition for the global convex program is:

$$\frac{\partial \mathcal{L}}{\partial f^*_{ij}} = 2w_{ij}f^*_{ij} - w_{ij}\beta + \lambda^G_{ij} + \mu^G_{ij} - \mu^G_{ji} = 0 \tag{B.1}$$

$$\frac{\partial \mathcal{L}}{\partial f_{ij}} = -\lambda^G_{ij} + \eta^G_i = 0 \tag{B.2}$$

$$\lambda^G_{ij}(f^*_{ij} - f_{ij}) = 0 \tag{B.3}$$

So the solution $(\vec{f}^{OPT}, \vec{f}^{*,OPT})$ for the global convex program must satisfies:

$$f^{*,OPT}_{ij} = \frac{\beta}{2} - \frac{\eta^G_i + \mu^G_{ij} - \mu^G_{ji}}{2w_{ij}} \tag{B.4}$$

$$\eta^G_i(f^{*,OPT}_{ij} - f^{OPT}_{ij}) = 0 \tag{B.5}$$

We can now examine one feasibility constraints that will help use simplify the expression for $\vec{f}^{*,OPT}$. Because $f^{*,OPT}_{ij} = f^{*,OPT}_{ji}$, we have:

$$f^{*,OPT}_{ij} = f^{*,OPT}_{ji} \tag{B.6}$$

$$\Rightarrow \frac{\eta^G_i + \mu^G_{ij} - \mu^G_{ji}}{w_{ij}} = \frac{\eta^G_j + \mu^G_{ji} - \mu^G_{ij}}{w_{ji}} \tag{B.7}$$

$$\Rightarrow w_{ji}(\eta^G_i + \mu^G_{ij} - \mu^G_{ji}) = w_{ij}(\eta^G_j + \mu^G_{ji} - \mu^G_{ij}) \tag{B.8}$$

$$\Rightarrow (w_{ij} + w_{ji})(\mu_{ij} - \mu_{ji}) = w_{ij}\eta^G_j - w_{ji}\eta^G_i$$

$$\Rightarrow \mu_{ij} - \mu_{ji} = \frac{w_{ij}}{w_{ij} + w_{ji}}\eta^G_j - \frac{w_{ji}}{w_{ij} + w_{ji}}\eta^G_i \tag{B.9}$$

Put this back to $f^{*,OPT}_{ij}$, we have

$$f^{*,OPT}_{ij} = \frac{\beta}{2} - \frac{\eta^G_i + \eta^G_j}{2(w_{ij} + w_{ji})} \tag{B.10}$$

The effective frequency is characterized by the sum of two parameters η_i^G and η_j^G.

Because the only constraint for $\vec{f}^{OPT}$ is

$$\sum_{j \in N(i)} f_{ij}^{OPT} = \beta$$

the game can have many optimal solutions. This is due to the coordination we choose. Because we take minimization of proposals, if one player has slack frequency after coordination, she cannot improve her utility no matter where she spends her slack frequency.

In order to construct the solution profile that is a Nash equilibrium, we have to specify f_{ij}^{OPT} for all edges (i, j). For the conservative one, we have

$$f_{ij}^{OPT} = f_{ij}^{*,OPT} \qquad \text{(conservative case)}$$

Then the solution profile is guaranteed to be a Nash equilibrium. This is because there is no extra frequency on any edge, and no player can deviate from this solution. So we have the following claim

Claim 1 (Optimal Solution in Conservative Best-Response) *Given the optimal solution, construct a solution profile based on Eq.* (conservative case). *The solution profile is a Nash equilibrium*

The case for aggressive best-response is more complicated, we redistribute f_{ii} to every edge (i, j).

Let

$$\delta_{ij} = w_{ij} \left(\beta - \sum_{j \in N(i)} f_{ij}^{*,OPT} \right)$$

We define $\vec{f}^{OPT}$ as follows:

$$f_{ij}^{OPT} = f_{ij}^{*,OPT} + \delta_{ij} \qquad \text{(B.11)}$$

We have the following lemma:

Lemma 1 *Given the optimal solution for global convex program, and given the solution profile constructed by redistribute f_{ii} in the way we just discussed. If a player i has $f_{ij}^{OPT} >$*

$f_{ij}^{*,OPT}$ *in this solution profile for some* j*, then for all* $k \in N(i)$*,* $f_{ik}^{OPT} > f_{ik}^{*,OPT}$*. If* $f_{ij}^{OPT} >$ $f_{ij}^{*,OPT}$

Proof: By construction, if $\delta_{ij} > 0$ for some j, then $\delta_{ij} > 0$ for all $j \in N(i)$. ■

We construct a solution profile $\vec{f}^{OPT}$ that maximizes the social welfare value in this section. We have the following claim:

Claim 2 (Optimal Solution in Aggressive Best-response) $\vec{f}^{OPT}$ *is a Nash equilibrium for the game using aggressive best-response.*

To prove the claim, we have to examine the local convex program. We analyze the behavior of $\vec{f}^{OPT}$ in the local convex program and show that it is a Nash equilibrium.

B.2 Solution for Local Convex Program

We now examine the local convex program for a player i, with the help of lagrangian, a solution of local convex program f_{ij}^{L} must satisfy KKT conditions:

$$2w_{ij}f_{ij} - w_{ij}\beta + \lambda_j^L + \eta^L = 0 \tag{B.12}$$

$$\lambda_j^L(f_{ij} - f_{ji}) = 0 \tag{B.13}$$

We want to put in our solution for global convex program, which are:

$$f_{ij}^{*,OPT} = \frac{\beta}{2} - \frac{\eta_i^G + \eta_j^G}{2(w_{ij} + w_{ji})} \tag{B.14}$$

$$f_{ij}^{OPT} = f_{ij}^{*,OPT} + \delta_i \tag{B.15}$$

Let $f_{ij}^{L} = f_{ij}^{OPT}$, with some work, we can get:

$$\lambda_j^L + \eta^L = \frac{w_{ij}}{w_{ij} + w_{ji}}(\eta_i^G + \eta_j^G) \tag{B.16}$$

I now claim the solution for the local convex program is:

$$f_{ij}^L = \frac{\beta}{2} - \frac{\lambda_j^L + \eta^L}{2w_{ij}} \tag{B.17}$$

where

$$\begin{cases} \lambda_j^L = \frac{w_{ij}}{w_{ij}+w_{ji}}\eta_j^G \\ \eta^L = \frac{w_{ij}}{w_{ij}+w_{ji}}\eta_i^G \end{cases} \tag{B.18}$$

By construction, this solution already satisfies Eq. (B.12). We want to show that it satisfies Eq. (B.13)

Recall that for a player i, and the solution $\vec{f}^{*,OPT}$, if i has an excess on one edge (i,j), it will have excess on every edge (i,k) $\forall k \in N(i)$. We have to examine two cases:

- i has an extra frequency on every edge.
- i has no extra frequency on any edge.

For the first case, because i is already at the maximum on every edge, if i moves some frequency from a set of players $K = \{k_1, k_2, \cdots, k_l\}$ and send to a set of other players $J = \{j_1, j-2, \cdots, j_m\}$. Then i cannot increase its utility because of i already at the maximum for the player in J. On the other hand, if i decrease interaction with those players in K, it may even decrease its own utility because effective frequency will reduce. Therefore, the solution profile also satisfies i's local convex program. So i will not move.

For the second case, there are two types of neighbors: those neighbors who have excess, and those neighbors who have no extra frequency. For those neighbors who have no extra frequency, we have $f_{ij} = f_{ji}$, so Eq. (B.13) is satisfied.

For those neighbors who have excess, let j be one of these neighbors, its clear that $f_{ji}^{OPT} > f_{ij}^{OPT}$. Based on Eq. (B.5), we know that$\eta_j^G = 0$. Based on Eq. (B.18), we have

$$\lambda_j^L = \frac{w_{ij}}{w_{ij}+w_{ji}}\eta_j^G = 0$$

So we also satisfy Eq. (B.13) here. So we prove the claim Claim 2.

Based on Claim 1 and Claim 2, we prove Theorem 2.