

DYNAMICS OF FREE GROUP AUTOMORPHISMS AND A SUBGROUP  
ALTERNATIVE FOR  $\text{OUT}(F_N)$

BY

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DISSERTATION

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# Abstract

This thesis is motivated by a foundational result of Thurston which states that pseudo-Anosov mapping classes act on the compactified Teichmüller space with north-south dynamics. We prove that several analogues of pseudo-Anosov mapping classes in the  $\text{Out}(F_N)$  setting act on the space of projective geodesic currents with generalized north-south dynamics. As an application of our results, we prove several structural theorems for subgroups of  $\text{Out}(F_N)$ .

*To Meltem and Yumuk*

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# Chapter 1

## Introduction

### 1.1 Motivation from surface theory

For a compact, oriented surface  $S$ , the mapping class group  $\text{Mod}(S)$  of  $S$  is the group of isotopy classes of orientation preserving homeomorphisms from  $S$  to itself. The group  $\text{Mod}(S)$  is one of the most prevalent objects in mathematics; it plays an important role in the study of geometry and topology of 3- and 4-manifolds, and has deep connections to dynamics, group theory, algebraic geometry, and complex analysis.

In his foundational work, Thurston [47] provides a  $\text{Mod}(S)$  equivariant compactification  $\overline{\mathcal{T}}$  of the Teichmüller space  $\mathcal{T}$  of marked hyperbolic structures on  $S$  by the space of *projective measured laminations*  $\mathbb{P}\mathcal{ML}$ , and gives a classification of individual elements in  $\text{Mod}(S)$  using the action of  $\text{Mod}(S)$  on  $\overline{\mathcal{T}}$ . In particular, he shows that if  $S$  is a hyperbolic surface and  $f \in \text{Mod}(S)$  is a *pseudo-Anosov homeomorphism*, then  $f$  acts on  $\mathbb{P}\mathcal{ML}(S)$  with north-south dynamics. In other words, there are two fixed points of this action,  $[\mu_+]$  and  $[\mu_-]$  called stable and unstable laminations, and any point  $[\mu] \in \mathbb{P}\mathcal{ML}(S)$  other than  $[\mu_-]$  and  $[\mu_+]$  converges to  $[\mu_+]$  under positive iterates of  $f$ , and converges to  $[\mu_-]$  under negative iterates of  $f$ . Moreover, there is a constant  $\lambda > 1$ , called the *dilatation*, such that  $f\mu_+ = \lambda\mu_+$  and  $f^{-1}\mu_- = \frac{1}{\lambda}\mu_-$ . In fact, this convergence is uniform on compact sets by work of Ivanov [26].

### 1.2 Free groups

Let  $F_N$  be a free group of rank  $N \geq 2$ . The outer automorphism group of  $F_N$ ,  $\text{Out}(F_N)$ , is the quotient group  $\text{Aut}(F_N)/\text{Inn}(F_N)$ . An important analogy between the outer automorphism group  $\text{Out}(F_N)$  of a free group  $F_N$  and the mapping class group  $\text{Mod}(S)$  illuminates much of the current research in  $\text{Out}(F_N)$ . This analogy is fueled by the following two observations: First, the Dehn–Nielsen–Baer theorem states that, for a closed surface  $S$ ,  $\text{Mod}(S)$  can be identified with an index two subgroup of  $\text{Out}(\pi_1(S))$ . Second, every outer automorphism



of the free group  $F_N$  can be represented by a homotopy equivalence from a finite connected graph to itself, which allows one to regard them as 1-dimensional mapping class groups.

A successful approach for studying the group  $\text{Out}(F_N)$  has been to investigate to what extent the analogy between  $\text{Mod}(S)$  and  $\text{Out}(F_N)$  can be formalized. Two rather different spaces on which  $\text{Out}(F_N)$  acts serve as analogues to the Teichmüller space and its compactification: One is Culler-Vogtmann’s *Outer space*  $cv_N$  [21], which is the space of marked metric graphs or equivalently the space of minimal, free, discrete isometric actions of  $F_N$  on  $\mathbb{R}$ -trees. The space  $cv_N$  and its projectivization  $CV_N$ , obtained as the quotient by the action of  $\mathbb{R}_+$ , acting by scaling the metrics, both have natural bordifications  $\overline{cv_N}$  and  $\overline{CV_N}$  with respect to *Gromov-Hausdorff* topology, [4].

Another space on which  $\text{Out}(F_N)$  acts naturally is Bonahon’s space of *geodesic currents*  $\text{Curr}(F_N)$ , which is the space of locally finite Borel measures on  $\partial^2 F_N = \{\partial F_N \times \partial F_N - \Delta\}$  (where  $\Delta$  is the diagonal) that are  $F_N$  invariant and *flip* invariant [10]. The space of *projective geodesic currents*, denoted by  $\mathbb{P}\text{Curr}(F_N)$  is the quotient of  $\text{Curr}(F_N)$ , where two currents are equivalent if they are positive scalar multiples of each other.

An *intersection form* introduced by Kapovich and Lustig [30], analogous to Thurston’s geometric intersection number for measured laminations [47], intimately intertwines the space of geodesic currents and the Outer space, see section 2.4.

The first  $\text{Out}(F_N)$  analogue of pseudo-Anosov mapping classes are *fully irreducible* outer automorphisms, also known as *iwip* (irreducible with irreducible powers). They are characterized by the property that no power fixes the conjugacy class of a nontrivial proper free factor of  $F_N$ . Here,  $A < F_N$  is a *free factor* of  $F_N$  if there exists another subgroup  $B < F_N$  such that  $F_N = A * B$ . The second analogue of pseudo-Anosov mapping classes in this setting are *atoroidal (or hyperbolic)* outer automorphisms. An element  $\varphi \in \text{Out}(F_N)$  is called *atoroidal* if no power of  $\varphi$  fixes the conjugacy class of a non-trivial element in  $F_N$ .

We remark that these two notions do not coincide. Namely, there are automorphisms that are atoroidal but not fully irreducible, and conversely there are automorphisms that are fully irreducible but not atoroidal. On the other hand, both of these notions are “generic” in a certain probabilistic sense, which roughly says that a “randomly” chosen element in  $\text{Out}(F_N)$  will be both atoroidal and fully irreducible; see [43, 45, 46] for precise statements.

### 1.3 Statement of results

Thurston’s north south dynamics result on  $\mathbb{P}\mathcal{ML}(S)$  has several different generalizations in the  $\text{Out}(\mathbb{F}_N)$  context. The first such generalization is due to Levitt and Lustig. In [35] they show that if  $\varphi \in \text{Out}(\mathbb{F}_N)$  is fully irreducible, then it acts on the compactified outer space  $\overline{\text{CV}}_N$  with uniform north-south dynamics.

Reiner Martin, in his unpublished 1995 thesis [38], proves that if  $\varphi \in \text{Out}(\mathbb{F}_N)$  is fully irreducible and atoroidal, then  $\varphi$  acts on  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  with north-south dynamics. We also give an alternative proof of this theorem using the Kapovich–Lustig intersection form, and the Levitt–Lustig’s north-south dynamics result on the closure of the outer space, see section 3.1. This proof appears in [48].

In joint work with Martin Lustig [36], we generalize R. Martin’s result to atoroidal (but not necessarily fully irreducible) outer automorphisms  $\varphi \in \text{Out}(\mathbb{F}_N)$  such that  $\varphi$  and  $\varphi^{-1}$  admit train-track representatives. We define “generalized north and south poles” in  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$ , i.e. disjoint, finite, convex cells  $\Delta_+(\varphi)$  and  $\Delta_-(\varphi)$ , and show:

**Theorem 1.3.1.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be an atoroidal outer automorphism with the property that both  $\varphi$  and  $\varphi^{-1}$  admit absolute train-track representatives. Then  $\varphi$  acts on  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  with “generalized uniform north-south dynamics from  $\Delta_-(\varphi)$  to  $\Delta_+(\varphi)$ ” in the following sense:*

*Given a neighborhood  $U$  of  $\Delta_+(\varphi)$  and a compact set  $K \subset \mathbb{P}\text{Curr}(\mathbb{F}_N) \setminus \Delta_-(\varphi)$ , there exists an integer  $M \geq 1$  such that  $\varphi^n(K) \subset U$  for all  $n \geq M$ .*

*Similarly, given a neighborhood  $V$  of  $\Delta_-(\varphi)$  and a compact set  $K' \subset \mathbb{P}\text{Curr}(\mathbb{F}_N) \setminus \Delta_+(\varphi)$ , there exists an integer  $M' \geq 1$  such that  $\varphi^{-n}(K') \subset V$  for all  $n \geq M'$ .*

The proof of Theorem 1.3.1 uses train-track techniques and is built on our earlier results (joint with Martin Lustig) about dynamics of reducible substitutions [37] which generalizes the classical Perron–Frobenius theorem.

On the other end of the spectrum, we describe the dynamics of fully irreducible and non atoroidal  $\varphi \in \text{Out}(\mathbb{F}_N)$  as a special case of a more general theorem about dynamics of pseudo-Anosov homeomorphisms on surfaces with  $b \geq 1$  boundary components.

Let  $S$  be a compact hyperbolic surface with  $b \geq 1$  boundary components  $\alpha_1, \alpha_2, \dots, \alpha_b$ . We think of  $S$  as a subset of a complete, hyperbolic surface  $S'$ , obtained from  $S$  by attaching  $b$  flaring ends. A geodesic current on  $S$  is a locally finite Borel measure on the space of unoriented bi-infinite geodesics on the universal cover  $\tilde{S}'$  of  $S'$ , which is  $\pi_1(S)$  invariant, and

whose support projects into  $S$ . Let  $\mathbb{P}\text{Curr}(S)$  be the space of projective geodesic currents on  $S$ .

Let  $\mu_{\alpha_i}$  denote the current corresponding to the boundary curve  $\alpha_i$ . Let us define  $\Delta, H_-(f), H_+(f) \subset \mathbb{P}\text{Curr}(S)$  as follows:

$$\Delta := \{[a_1\mu_{\alpha_1} + a_2\mu_{\alpha_2} + \dots + a_b\mu_{\alpha_b}] \mid a_i \geq 0, \sum_{i=1}^b a_i > 0\}.$$

$$H_-(f) := \{[t_1\mu_- + t_2\nu] \mid [\nu] \in \Delta, t_1, t_2 \geq 0\}$$

and

$$H_+(f) := \{[t'_1\mu_+ + t'_2\nu] \mid [\nu] \in \Delta, t'_1, t'_2 \geq 0\}.$$

**Theorem 1.3.2.** *Let  $f$  be a pseudo-Anosov homeomorphism on  $S$ . Let  $K$  be a compact set in  $\mathbb{P}\text{Curr}(S) \setminus H_-(f)$ . Then, for any open neighborhood  $U$  of  $[\mu_+]$ , there exist  $m \in \mathbb{N}$  such that  $f^n(K) \subset U$  for all  $n \geq m$ . Similarly for a compact set  $K' \subset \mathbb{P}\text{Curr}(S) \setminus H_+(f)$  and an open neighborhood  $V$  of  $[\mu_-]$ , there exist  $m' \in \mathbb{N}$  such that  $f^{-n}(K') \subset V$  for all  $n \geq m'$ .*

Theorem 1.3.2 implies that if  $[\nu] \in \mathbb{P}\text{Curr}(S) \setminus (H_-(f) \cup H_+(f))$ , then  $\lim_{n \rightarrow \infty} f^n[\nu] = [\mu_+]$  and  $\lim_{n \rightarrow \infty} f^{-n}[\nu] = [\mu_-]$ . Moreover, it is not hard to see that  $f$  has simple dynamics on  $H_-(f) \cup H_+(f)$ :

If  $[\mu] = [t_1\mu_+ + t_2\nu]$  where  $t_1 > 0, [\nu] \in \Delta$ , then

$$\lim_{n \rightarrow \infty} f^n([\mu]) = [\mu_+]$$

and

$$\lim_{n \rightarrow \infty} f^{-n}([\mu]) = [t_2\nu].$$

If  $[\mu] = [t'_1\mu_- + t'_2\nu]$  where  $t'_1 > 0, [\nu] \in \Delta$ , then

$$\lim_{n \rightarrow \infty} f^{-n}([\mu]) = [\mu_-]$$

and

$$\lim_{n \rightarrow \infty} f^n([\mu]) = [t_2\nu].$$

Using the natural identification between  $\mathbb{P}\text{Curr}(S)$  and  $\mathbb{P}\text{Curr}(\pi_1(S)) = \mathbb{P}\text{Curr}(F_N)$  and a Theorem of Bestvina–Handel (see Theorem 4.4.1), as a particular case of Theorem 1.3.2 for surfaces with one boundary component, we obtain the following result about dynamics of non-atoroidal and fully irreducible elements on  $\mathbb{P}\text{Curr}(F_N)$ .

**Theorem 1.3.3.** *Let  $\varphi \in \text{Out}(F_N)$  be a non-atoroidal and fully irreducible element. Then the action of  $\varphi$  on the space of projective geodesic currents,  $\mathbb{P}\text{Curr}(F_N)$ , has generalized uniform north-south dynamics in the following sense: Given an open neighborhood  $U$  of the stable current  $[\mu_+]$  and a compact set  $K_0 \subset \mathbb{P}\text{Curr}(F_N) \setminus H_-(\varphi)$ , there is an integer  $M_0 > 0$  such that for all  $n \geq M_0$ ,  $\varphi^n(K_0) \subset U$ . Similarly, given an open neighborhood  $V$  of the unstable current  $[\mu_-]$  and a compact set  $K_1 \subset \mathbb{P}\text{Curr}(F_N) \setminus H_+(\varphi)$ , there is an integer  $M_1 > 0$  such that for all  $m \geq M_1$ ,  $\varphi^{-m}(K_1) \subset V$ .*

In analogy with Ivanov’s classification of subgroups of the mapping class group, Handel–Mosher [23], and Horbez [25] show that any subgroup of  $H < \text{Out}(F_N)$  either contains a fully irreducible element, or there exist a finite index subgroup  $H_0 < H$  and a non-trivial proper free factor  $F_k < F_N$  such that  $H_0[F_k] = [F_k]$ . We complement their result by characterizing precisely when an irreducible subgroup contains an atoroidal and fully irreducible element.

**Theorem 1.3.4.** *Let  $H \leq \text{Out}(F_N)$  and suppose that  $H$  contains a fully irreducible element  $\varphi$ . Then one of the following holds:*

1.  *$H$  contains an atoroidal and fully irreducible element.*
2.  *$H$  is geometric, i.e.  $H \leq \text{Mod}^\pm(S) \leq \text{Out}(F_N)$  where  $S$  is a compact surface with one boundary component with  $\pi_1(S) = F_N$  such that  $\varphi \in H$  is induced by a pseudo-Anosov homeomorphism of  $S$ . In particular, the current corresponding to the boundary curve is fixed by all elements of  $H$ , and hence  $H$  contains no atoroidal elements.*

Moreover, if the original fully irreducible element  $\varphi \in H$  is non-atoroidal and (1) happens, then  $H$  contains a free subgroup  $L$  of rank two such that every nontrivial element of  $L$  is atoroidal and fully irreducible. See Remark 5.2.5 below.

In [11, 12], using the Handel–Mosher [23] subgroup classification, Carette–Francaviglia–Kapovich–Martino showed that every nontrivial normal subgroup of  $\text{Out}(F_N)$  contains a fully irreducible element for  $N \geq 3$ . And they asked whether every such subgroup contains an atoroidal and fully irreducible element. As a corollary of Theorem 1.3.4 we answer this question in the affirmative direction:

**Corollary 1.3.5.** *Let  $N \geq 3$ . Then, every nontrivial normal subgroup  $H < \text{Out}(\mathbb{F}_N)$  contains an atoroidal fully irreducible element.*

As a final application of Theorem 1.3.3, we show that when restricted to a smaller subset  $\mathcal{M}_N$  of  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$ , non-atoroidal and fully irreducible elements act with uniform north-south dynamics, hence recovering a previous claim of R. Martin [38].

The *minimal set*  $\mathcal{M}_N \subset \mathbb{P}\text{Curr}(\mathbb{F}_N)$ , introduced by R. Martin [38], is the closure of the set

$$\{[\eta_g] \mid g \in \mathbb{F}_N \text{ is primitive element}\}$$

in  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$ . By a result of Kapovich-Lustig [29]  $\mathcal{M}_N$  is the unique smallest non-empty closed  $\text{Out}(\mathbb{F}_N)$ -invariant subset of  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$ . Concretely,  $\mathcal{M}_N$  is equal to the closure of the  $\text{Out}(\mathbb{F}_N)$  orbit of  $[\eta_g]$  for a primitive element  $g \in \mathbb{F}_N$ . Note that for every non-atoroidal fully irreducible  $\varphi \in \text{Out}(\mathbb{F}_N)$ , its stable current  $[\mu_+]$  belongs to  $\mathcal{M}_N$ . Indeed, for every primitive element  $g \in \mathbb{F}_N$  the positive iterates  $\varphi^n([\eta_g])$  converge to  $[\mu_+]$  by Theorem 1.3.2, and therefore  $[\mu_+] \in \mathcal{M}_N$ . For similar reasons  $[\mu_-] \in \mathcal{M}_N$ . As a direct consequence of Theorem 1.3.3 we obtain:

**Corollary 1.3.6.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be a non-atoroidal fully irreducible element, the action of  $\varphi$  on  $\mathcal{M}_N$  has uniform north-south dynamics. In other words, given a compact set  $K_0 \subset \mathcal{M}_N \setminus \{[\mu_-]\}$  and an open neighborhood  $U$  of  $[\mu_+]$  in  $\mathcal{M}_N$ , there is an integer  $M_0 > 0$  such that  $\varphi^n(K_0) \subset U$  for all  $n \geq M_0$ . Similarly, given a compact set  $K_1 \subset \mathcal{M}_N \setminus \{[\mu_+]\}$  and an open neighborhood  $V$  of  $[\mu_-]$  in  $\mathcal{M}_N$ , there is an integer  $M_1 > 0$  such that  $\varphi^{-m}(K_1) \subset V$  for all  $m \geq M_1$ .*

## 1.4 Outline

In Chapter 2, we give some preliminaries about free group automorphisms and tools for studying them including geodesic currents, laminations, train-track maps, and Culler–Vogtmann’s outer space. Further, we describe several variations on north-south dynamics without any particular reference to free group automorphisms.

In Chapter 3, we describe the dynamics of atoroidal outer automorphisms. We first give<sup>1</sup> a short proof of the north-south dynamics result for the irreducible case using Levitt–Lustig’s north-south dynamics result on the closure of the Outer space and Kapovich–Lustig

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<sup>1</sup>This proof appears in *Dynamics of hyperbolic iwips*. Conform. Geom. Dyn. 18 (2014), 192-216.

intersection form. We then give the proof for the general case (based on joint work<sup>2</sup> with Martin Lustig) which spans several sections where the convergence estimates are carefully studied.

In Chapter 4, we describe<sup>3</sup> geodesic currents on surfaces and dynamics of pseudo-Anosov mapping classes on the space of geodesic currents on surfaces.

In Chapter 5, we apply the main result of Chapter 4 to obtain several structural results about subgroups of  $\text{Out}(F_N)$ .

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<sup>2</sup>*North-South dynamics of hyperbolic free group automorphisms on the space of currents.* arXiv:1509.05443

<sup>3</sup>The contents of Chapter 4 and Chapter 5 appeared as *Generalized north-south dynamics on the space of geodesic currents.* *Geom. Dedicata*, 177 (2015), 129-148. It is reproduced here with kind permission from Springer.

# Chapter 2

## Preliminaries

### 2.1 North-south dynamics

In this section we describe some general considerations for maps with north-south dynamics. We will keep the notation simple and general; at no point we will refer to the specifics of geodesic currents on free groups. However, in this section we will prove the main criteria used to establish the north-south dynamics of atoroidal outer automorphisms.

**Convention 2.1.1.** Throughout this section we will denote by  $X$  a compact space, and by  $f : X \rightarrow X$  a homeomorphism of  $X$ .

**Definition 2.1.2.** (a) A map  $f : X \rightarrow X$  as in Convention 2.1.1 is said to have (*pointwise*) *north-south dynamics* if  $f$  has two distinct fixed points  $P_+$  and  $P_-$ , called *attractor* and *repeller*, such that for every  $x \in X \setminus \{P_+, P_-\}$  one has:

$$\lim_{t \rightarrow \infty} f^t(x) = P_+ \quad \text{and} \quad \lim_{t \rightarrow -\infty} f^t(x) = P_-$$

(b) The map  $f : X \rightarrow X$  is said to have *uniform north-south dynamics* if the following hold: There exist two distinct fixed points  $P_-$  and  $P_+$  of  $f$ , such that for every compact set  $K \subset X \setminus \{P_-\}$  and every neighborhood  $U_+$  of  $P_+$  there exists an integer  $t_+ \geq 0$  such that for every  $t \geq t_+$  one has:

$$f^t(K) \subset U_+.$$

Similarly, for every compact set  $K \subset X \setminus \{P_+\}$  and every neighborhood  $U_-$  of  $P_-$  there exists an integer  $t_- \leq 0$  such that for every  $t \leq t_-$  one has:

$$f^t(K) \subset U_-.$$

**Remark 2.1.3.** It is easy to see that uniform north-south dynamics implies pointwise north-south dynamics. Conversely, the main result of [24] implies that for a compact metric space

$X$ , pointwise north-south dynamics implies uniform north-south dynamics.

**Definition 2.1.4.** A homeomorphism  $f : X \rightarrow X$  is said to have *generalized uniform north-south dynamics* if there exist two disjoint compact  $f$ -invariant sets  $\Delta_+$  and  $\Delta_-$  in  $X$ , such that the following hold:

- (i) For every compact set  $K \subset X \setminus \Delta_-$  and every neighborhood  $U_+$  of  $\Delta_+$  there exists an integer  $t_+ \geq 0$  such that for every  $t \geq t_+$  one has:

$$f^t(K) \subset U_+$$

- (ii) For every compact set  $K \subset X \setminus \Delta_+$  and every neighborhood  $U_-$  of  $\Delta_-$  there exists an integer  $t_- \leq 0$  such that for every  $t \leq t_-$  one has:

$$f^t(K) \subset U_-$$

To be more specific, we say that a map  $f$  as in Definition 2.1.4 *has generalized uniform north-south dynamics from  $\Delta_-$  to  $\Delta_+$* . Notice that, here we interpret “ $f$ -invariant” in its strong meaning, i.e.  $f(\Delta_+) = \Delta_+$  and  $f(\Delta_-) = \Delta_-$ . It is easy to see that, for example, the next proposition holds also under the weaker assumption  $f(\Delta_+) \subset \Delta_+$  and  $f(\Delta_-) \subset \Delta_-$ , but the advantage of our strong interpretation is that then any map with uniform generalized north-south dynamics determines uniquely the “generalized north and south poles”  $\Delta_+$  and  $\Delta_-$ .

**Proposition 2.1.5.** *Let  $f : X \rightarrow X$  be as in Convention 2.1.1, and assume that  $X$  is sufficiently separable, for example metrizable. Let  $Y \subset X$  be dense subset of  $X$ , and let  $\Delta_+$  and  $\Delta_-$  be two  $f$ -invariant sets in  $X$  that are disjoint. Assume that the following criterion holds:*

*For every neighborhood  $U$  of  $\Delta_+$  and every neighborhood  $V$  of  $\Delta_-$  there exists an integer  $m_0 \geq 1$  such that for any  $m \geq m_0$  and any  $y \in Y$  one has either  $f^m(y) \in U$  or  $f^{-m}(y) \in V$ .*

*Then  $f^2$  has generalized uniform north-south dynamics from  $\Delta_-$  to  $\Delta_+$ .*

*Proof.* Let  $K \subset X \setminus \Delta_-$  be compact, and let  $U$  and  $V$  be neighborhoods of  $\Delta_+$  and  $\Delta_-$  respectively.

Since by Convention 2.1.1  $X$  is compact, for any open neighborhood  $W$  of  $K$  the closure  $\bar{W}$  is compact. Then  $V_1 := V \setminus \bar{W}$  is again an open neighborhood of  $\Delta_-$ , moreover it is disjoint from  $\bar{W}$ . Let  $U_1$  be a neighborhood of  $\Delta_+$  which has the property that its closure



is contained in the interior of  $U$ . Such a neighborhood exists because we assumed that  $X$  is “sufficiently separable”.

Let  $m_0$  be as postulated in the criterion, applied to the neighborhoods  $U_1$  and  $V_1$ , and pick any  $m \geq m_0$ . Consider any  $y \in Y \cap f^m(\bar{W})$ . Notice that  $f^{-m}(y)$  is contained in  $\bar{W}$ , which is disjoint from  $V_1$ . Thus, by the assumed criterion,  $f^m(y)$  must be contained in  $U_1$ .

Since  $W$  is open and  $f$  a homeomorphism, any dense subset of  $X$  must intersect  $f^m(\bar{W})$  in a subset that is dense in  $f^m(\bar{W})$ . This implies that  $f^m(f^m(\bar{W})) \subseteq \bar{U}_1 \subset U$ . Since  $K \subset \bar{W}$ , this shows that  $f^{2m}(K) \subset U$ .

Using the analogous argument for the inverse iteration we see that  $f^2$  has generalized uniform north-south dynamics from  $\Delta_-$  to  $\Delta_+$ .  $\square$

**Proposition 2.1.6.** *Let  $f : X \rightarrow X$  be as in Convention 2.1.1, with disjoint  $f$ -invariant sets  $\Delta_+$  and  $\Delta_-$ , and assume that some power  $f^s$  with  $s \geq 1$  has generalized uniform north-south dynamics from  $\Delta_-$  to  $\Delta_+$ .*

*Then  $f$  too has generalized uniform north-south dynamics from  $\Delta_-$  to  $\Delta_+$ .*

*Proof.* Let  $K \subset X \setminus \Delta_-$  be compact, and let  $U$  be an open neighborhood of  $\Delta_+$ .

Set  $K' := K \cup f(K) \cup \dots \cup f^{s-1}(K)$ , which is again compact. Note that the fact that  $K \subset X \setminus \Delta_-$  and  $f^{-1}(\Delta_-) = \Delta_-$  implies that  $K' \subset X \setminus \Delta_-$ . Indeed,  $x \in K'$  implies that  $x = f^t(y)$  for some  $y \in K$  and for some  $0 \leq t \leq s-1$ .

Thus  $x \in \Delta_-$  would imply that  $y = f^{-t}(x) \in f^{-1}(\Delta_-) = \Delta_-$ , contradicting the assumption  $K \cap \Delta_- = \emptyset$ .

From the hypothesis that  $f^s$  has generalized uniform north-south dynamics from  $\Delta_-$  to  $\Delta_+$  it follows that there is a bound  $t_0$  such that for all  $t' \geq t_0$  one has  $f^{t's}(K') \subset U$ .

Hence, for any point  $x \in K$  and any integer  $t \geq st_0$ , we can write  $t = k + st'$  with  $t' \geq t_0$  and  $0 \leq k \leq s-1$  to obtain the desired fact

$$f^t(x) = f^{k+st'}(x) = f^{st'} f^k(x) \in f^{st'}(K') \subset U.$$

The analogous argument for  $f^{-1}$  finishes the proof of the Proposition.  $\square$

## 2.2 Graphs and graph maps

A graph  $\Gamma$  is a one dimensional cell complex where 0-cells of  $\Gamma$  are called vertices and 1-cells of  $\Gamma$  are called topological edges. The set of vertices is denoted by  $V\Gamma$  and the set of topological edges is denoted by  $E\Gamma$ . We choose an orientation for each edge, and denote

the set of positively oriented edges with  $E^+\Gamma$ . Given an edge  $e \in E^+\Gamma$ , the initial vertex of  $e$  is denoted by  $o(e)$  and the terminal vertex of  $e$  is denoted by  $t(e)$ . The edge  $e$  with the opposite orientation is denoted by  $\bar{e}$ .

An *edge path*  $\gamma$  in  $\Gamma$  is a concatenation  $e_1e_2\dots e_n$  of edges of  $\Gamma$  where  $t(e_i) = o(e_{i+1})$ . An edge path  $\gamma$  is called *reduced* if  $e_i \neq \bar{e}_{i+1}$  for all  $i = 1, \dots, n-1$ . A reduced edge path is called *cyclically reduced* if  $t(e_n) = o(e_1)$  and  $e_n \neq \bar{e}_1$ . An edge path  $\gamma$  is *trivial* if it consists of a vertex.

The graph  $\Gamma$  is equipped with a natural metric called the *simplicial metric* which is obtained by identifying each edge  $e$  of  $\Gamma$  with the interval  $[0, 1]$ . The *simplicial length* of an edge path  $\gamma$  in  $\Gamma$  is denoted by  $|\gamma|_\Gamma$ , and if it is clear from the context, we suppress  $\Gamma$  and write  $|\gamma|$ .

A *graph map*  $f : \Gamma \rightarrow \Gamma$  is an assignment that sends vertices to vertices, and edges to edge paths. We say that  $f$  *has no contracted edges* if the path  $f(e)$  is non-trivial for all  $e \in E\Gamma$ . A graph map is called *tight* if  $f(e)$  is reduced for each edge  $e \in E\Gamma$ .

A *turn* in  $\Gamma$  is a pair  $(e_1, e_2)$  where  $o(e_1) = o(e_2)$ . A turn is called *non-degenerate* if  $\bar{e}_1 \neq e_2$ , otherwise it is called *degenerate*. A graph map  $f : \Gamma \rightarrow \Gamma$  with no contracted edges induces a *derivative map*  $Df : E\Gamma \rightarrow E\Gamma$  where  $Df(e)$  is the first edge of the edge path  $f(e)$ . The derivative map induces a map  $Tf$  on the set of turns defined as

$$Tf((e_1, e_2)) := (Df(e_1), Df(e_2)).$$

A turn  $(e_1, e_2)$  is called *legal* if  $Tf^n((e_1, e_2))$  is non-degenerate for all  $n \geq 0$ . An edge path  $\gamma = e_1e_2\dots e_n$  is called *legal* if every turn  $(e_i, \bar{e}_{i+1})$  in  $\gamma$  is legal. A graph map  $f : \Gamma \rightarrow \Gamma$  is called a *train track map* if for every edge  $e$  the edge paths  $f^n(e)$  are legal for all  $n \geq 1$ .

## 2.3 Markings and topological representatives

The *rose*  $R_N$  with  $N$  petals is a finite graph with one vertex  $q$ , and  $N$  edges attached to the vertex  $q$ . We identify the fundamental group  $\pi_1(R_N, q)$  with  $F_N$  via the isomorphism obtained by orienting and ordering the petals and sending the homotopy class of the  $j^{\text{th}}$  oriented petal to  $j^{\text{th}}$  generator of  $F_N$ . A *marking* on  $F_N$  is pair  $(\Gamma, \alpha)$  where  $\Gamma$  is a finite, connected graph with no valence-one vertices such that  $\pi_1(\Gamma) \cong F_N$  and  $\alpha : (R_N, q) \rightarrow (\Gamma, \alpha(q))$  is a homotopy equivalence.

Let  $\alpha : R_N \rightarrow \Gamma$  be a marking and  $\sigma : \Gamma \rightarrow R_N$  a homotopy inverse. Every homotopy equivalence  $f : \Gamma \rightarrow \Gamma$  *determines* an outer automorphism  $(\sigma \circ f \circ \alpha)_*$  of  $F_N = \pi_1(R_N, p)$ . Let

$\varphi \in \text{Out}(\mathbb{F}_N)$ , the map  $f : \Gamma \rightarrow \Gamma$  is called a *topological representative* of  $\varphi$  if  $f$  determines  $\varphi$  as above,  $f$  is tight, and  $f$  has no contracted edges. A graph map  $f : \Gamma \rightarrow \Gamma$  is called a *train track representative* for  $\varphi$  if  $f$  is a topological representative for  $\varphi$  and  $f$  is a train-track map.

**Definition 2.3.1.** A self-map  $f : \Gamma \rightarrow \Gamma$  is called *expanding* if for every edge  $e \in E\Gamma$  there is an exponent  $t \geq 1$  such that the edge path  $f^t(e)$  has simplicial length  $|f^t(e)| \geq 2$ .

**Remark 2.3.2.** If a self-map  $f : \Gamma \rightarrow \Gamma$  represents an atoroidal outer automorphism  $\varphi$  of  $\mathbb{F}_N$ , then the hypothesis that  $f$  be expanding is always easy to satisfy: It suffices to contract all edges which are not expanded by any iterate  $f^t$  to an edge path of length  $\geq 2$ : The contracted subgraph must be a forest, as otherwise some  $f^t$  would fix a non-contractible loop and hence  $\varphi^t$  would fix a non-trivial conjugacy class of  $\pi_1(\Gamma) \cong \mathbb{F}_N$ , contradicting the assumption that  $\varphi$  is atoroidal.

Given an (not necessarily reduced) edge path  $\gamma \in \Gamma$ , let  $[\gamma]$  denote the reduced edge path which is homotopic to  $\gamma$  relative to endpoints. The following is a classical fact for free group automorphisms:

**Lemma 2.3.3** (Bounded Cancellation Lemma [15]). *Let  $f : \Gamma \rightarrow \Gamma$  be a homotopy equivalence. There exist a constant  $C_f$ , depending only on  $f$ , such that for any reduced path  $\rho = \rho_1\rho_2$  in  $\Gamma$  one has*

$$|[f(\rho)]| \geq |[f(\rho_1)]| + |[f(\rho_2)]| - 2C_f.$$

*That is, at most  $C_f$  terminal edges of  $[f(\rho_1)]$  are cancelled against  $C_f$  initial edges of  $[f(\rho_2)]$  when we concatenate them to obtain  $[f(\rho)]$ .*

**Definition 2.3.4.** A path  $\eta$  in  $\Gamma$  which crosses over precisely one illegal turn is called a *periodic indivisible Nielsen path* (or *INP*, for short), if for some exponent  $t \geq 1$  one has  $[f^t(\eta)] = \eta$ . The smallest such  $t$  is called the period of  $\eta$ . A path  $\gamma$  is called a *pre-INP* if its image under  $f^{t_0}$  is an INP for some  $t_0 \geq 1$ . The illegal turn on  $\eta = \gamma' \circ \bar{\gamma}$  is called the *tip* of  $\eta$ , while the two maximal initial legal subpaths  $\gamma'$  and  $\gamma$ , of  $\eta$  and  $\bar{\eta}$  respectively, are called the *branches* of  $\eta$ . A *multi-INP* or a *Nielsen path* is a legal concatenation of finitely many INP's.

## 2.4 Geodesic currents on free groups and Outer space

Let  $F_N$  be a finitely generated free group of rank  $N \geq 2$ . Let us denote the Gromov boundary of  $F_N$  by  $\partial F_N$  and set

$$\partial^2 F_N := \{(\xi, \zeta) \mid \xi, \zeta \in \partial F_N, \text{ and } \xi \neq \zeta\}.$$

A geodesic current on  $F_N$  is a positive locally finite Borel measure on  $\partial^2 F_N$ , which is  $F_N$ -invariant and  $\sigma_f$ -invariant, where  $\sigma_f : \partial^2 F_N \rightarrow \partial^2 F_N$  is the *flip* map defined by

$$\sigma_f(\xi, \zeta) = (\zeta, \xi)$$

for  $(\xi, \zeta) \in \partial^2 F_N$ . We will denote the space of geodesic currents on  $F_N$  by  $\text{Curr}(F_N)$ . The space  $\text{Curr}(F_N)$  is endowed with the weak\* topology so that, given  $\nu_n, \nu \in \text{Curr}(F_N)$ ,  $\lim_{n \rightarrow \infty} \nu_n = \nu$  if and only if  $\lim_{n \rightarrow \infty} \nu_n(S_1 \times S_2) = \nu(S_1 \times S_2)$  for all disjoint closed-open subsets  $S_1, S_2 \subseteq \partial F_N$ .

For a Borel subset  $S$  of  $\partial^2 F_N$  and  $\varphi \in \text{Aut}(F_N)$ ,

$$\varphi\nu(S) := \nu(\varphi^{-1}(S))$$

defines a continuous, linear left action of  $\text{Aut}(F_N)$  on  $\text{Curr}(F_N)$ . Moreover,  $\text{Inn}(F_N)$  acts trivially, so that the action induces an action by the quotient group  $\text{Out}(F_N)$ .

Let  $\nu_1, \nu_2$  be two non-zero currents, we say  $\nu_1$  is equivalent to  $\nu_2$ , and write  $\nu_1 \sim \nu_2$ , if there is a positive real number  $r$  such that  $\nu_1 = r\nu_2$ . Then, the space of *projective geodesic currents* on  $F_N$  is defined by

$$\mathbb{P}\text{Curr}(F_N) := \{\nu \in \text{Curr}(F_N) : \nu \neq 0\} / \sim.$$

We will denote the projective class of the current  $\nu$  by  $[\nu]$ . The space  $\mathbb{P}\text{Curr}(F_N)$  inherits the quotient topology and the above  $\text{Aut}(F_N)$  and  $\text{Out}(F_N)$  actions on  $\text{Curr}(F_N)$  descend to well defined actions on  $\mathbb{P}\text{Curr}(F_N)$  as follows: For  $\varphi \in \text{Aut}(F_N)$  and  $[\nu] \in \mathbb{P}\text{Curr}(F_N)$ ,

$$\varphi[\nu] := [\varphi\nu].$$

Given a marking  $\alpha : R_N \rightarrow \Gamma$ , the map  $\alpha$  induces an isomorphism  $\alpha_* : \pi_1(R_N, q) \rightarrow \pi_1(\Gamma, p)$  on the level of fundamental groups. The induced map  $\alpha_*$  gives rise to natural

$F_N$ -equivariant homeomorphisms  $\tilde{\alpha} : \partial F_N \rightarrow \partial \tilde{\Gamma}$  and  $\partial^2 \alpha : \partial^2 F_N \rightarrow \partial^2 \tilde{\Gamma}$ .

The *cylinder set* associated to a reduced edge-path  $\gamma$  in  $\tilde{\Gamma}$  (with respect to the marking  $\alpha$ ) is defined as follows:

$$Cyl_\alpha(\gamma) := \{(\xi, \zeta) \in \partial^2 F_N \mid \gamma \subset [\tilde{\alpha}(\xi), \tilde{\alpha}(\zeta)]\},$$

where  $[\tilde{\alpha}(\xi), \tilde{\alpha}(\zeta)]$  is the geodesic from  $\tilde{\alpha}(\xi)$  to  $\tilde{\alpha}(\zeta)$  in  $\tilde{\Gamma}$ .

Let  $v$  be a reduced edge-path in  $\Gamma$ , and  $\gamma$  be a lift of  $v$  to  $\tilde{\Gamma}$ . Then, we set

$$\langle v, \mu \rangle_\alpha := \mu(Cyl_\alpha(\gamma)).$$

In what follows we will suppress the letter  $\alpha$  and write  $\langle v, \mu \rangle$ . It is easy to see that the quantity  $\mu(Cyl_\alpha(\gamma))$  is invariant under the action of  $F_N$ , so the right-hand side of the above formula does not depend on the choice of the lift  $\gamma$  of  $v$ . Hence,  $\langle v, \mu \rangle$  is well defined. In [28], it was shown that, if we let  $\mathcal{P}\Gamma$  denote the set of all finite reduced edge-paths in  $\Gamma$ , then a geodesic current is uniquely determined by the set of values  $(\langle v, \mu \rangle)_{v \in \mathcal{P}\Gamma}$ . In particular, given  $\mu_n, \mu \in \text{Curr}(F_N)$ ,  $\lim_{n \rightarrow \infty} \mu_n = \mu$  if and only if  $\lim_{n \rightarrow \infty} \langle v, \mu_n \rangle = \langle v, \mu \rangle$  for every  $v \in \mathcal{P}\Gamma$ .

Given a marking  $(\Gamma, \alpha)$ , the *weight* of a geodesic current  $\mu \in \text{Curr}(F_N)$  with respect to  $(\Gamma, \alpha)$  is denoted by  $w_\Gamma(\mu)$  and defined as

$$w_\Gamma(\mu) := \sum_{e \in E\Gamma} \langle e, \mu \rangle,$$

where  $E\Gamma$  is the set of oriented edges of  $\Gamma$ . In [28] Kapovich gives a useful criterion for convergence in  $\mathbb{P}\text{Curr}(F_N)$ .

**Lemma 2.4.1.** *Let  $[\mu_n], [\mu] \in \mathbb{P}\text{Curr}(F_N)$ , and  $(\Gamma, \alpha)$  be a marking. Then,*

$$\lim_{n \rightarrow \infty} [\mu_n] = [\mu]$$

*if and only if for every  $v \in \mathcal{P}\Gamma$ ,*

$$\lim_{n \rightarrow \infty} \frac{\langle v, \mu_n \rangle}{w_\Gamma(\mu_n)} = \frac{\langle v, \mu \rangle}{w_\Gamma(\mu)}.$$

**Definition 2.4.2** (Rational Currents). Let  $g \in F_N$  be a nontrivial element such that  $g \neq h^k$  for any  $h \in F_N$  and  $k > 1$ . Define the *counting current*  $\eta_g$  as follows: For a closed-open subset  $S$  of  $\partial^2 F_N$ ,  $\eta_g$  is the number of  $F_N$ -translates of  $(g^{-\infty}, g^\infty)$  and  $(g^\infty, g^{-\infty})$  that are

contained in  $S$ . For an arbitrary non-trivial element  $g \in F_N$  write  $g = h^k$ , where  $h$  is not a proper power, and define  $\eta_g := k\eta_h$ . Any nonnegative scalar multiple of a counting current is called a *rational current*.

An important fact about rational currents is that, the set of rational currents is dense in  $\text{Curr}(F_N)$ , see [27, 28]. Note that for any  $h \in F_N$  we have  $(hgh^{-1})^{-\infty} = hg^{-\infty}$  and  $(hgh^{-1})^\infty = hg^\infty$ . From here, it is easy to see that  $\eta_g$  depends only on the conjugacy class of the element  $g$ . So, from now on, we will use  $\eta_g$  and  $\eta_{[g]}$  interchangeably.

The action of  $\text{Out}(F_N)$  on rational currents is given explicitly by the formula

$$\varphi\eta_g = \eta_{\varphi(g)}.$$

Let  $c$  be a circuit in  $\Gamma$ . For any edge path  $v$  define *number of occurrences of  $v$  in  $c$* , denoted by  $\langle v, c \rangle$ , to be the number of vertices in  $c$  such that starting from that vertex, moving in the positive direction on  $c$  one can read off  $v$  or  $\bar{v}$  as an edge path. Then for an edge path  $v$ , and a conjugacy class  $[g]$  in  $F_N$  one has

$$\langle v, \eta_{[g]} \rangle = \langle v, c(g) \rangle,$$

where  $c(g) = \alpha(g)$  is the unique reduced circuit in  $\Gamma$  representing  $[g]$ , see [28].

**Definition 2.4.3** (Outer Space). The space of minimal, free and discrete isometric actions of  $F_N$  on  $\mathbb{R}$ -trees up to  $F_N$ -equivariant isometry is denoted by  $\text{cv}_N$  and called the *unprojectivized Outer Space*. The closure of the Outer Space,  $\overline{\text{cv}_N}$ , consists precisely of *very small, minimal, isometric* actions of  $F_N$  on  $\mathbb{R}$ -trees, see [3]. It is known that [20] every point in the closure of the outer space is uniquely determined by its *translation length function*  $\|\cdot\|_T : F_N \rightarrow \mathbb{R}$  where  $\|g\|_T = \min_{x \in T} d_T(x, gx)$ . There is a natural continuous right action of  $\text{Aut}(F_N)$  on  $\overline{\text{cv}_N}$ , which in the level of translation length functions is defined by

$$\|g\|_{T\varphi} = \|\varphi(g)\|_T$$

for any  $T \in \overline{\text{cv}_N}$  and  $\varphi \in \text{Aut}(F_N)$ . It is easy to see that for any  $h \in F_N$ ,  $\|hgh^{-1}\|_T = \|g\|_T$ . So  $\text{Inn}(F_N)$  is in the kernel of this action, hence the above action factors through  $\text{Out}(F_N)$ .

The closure  $\overline{\text{CV}_N}$  of the projectivized Outer space is precisely the projectivized space of very small, minimal, isometric actions of  $F_N$  on  $\mathbb{R}$ -trees. The above  $\text{Out}(F_N)$  action on  $\overline{\text{cv}_N}$  induces a well defined action on  $\overline{\text{CV}_N}$  that leaves  $\text{CV}_N$  invariant.

Levitt and Lustig [35] showed that a fully irreducible element acts on  $\overline{\text{CV}}_N$  with north-south dynamics.

**Theorem 2.4.4** (Theorem 1.1 of [35]). *Every fully irreducible element  $\varphi \in \text{Out}(\mathbb{F}_N)$  acts on  $\overline{\text{CV}}_N$  with exactly two fixed points  $[T_+]$  and  $[T_-]$ . Further, for any other  $[T] \in \overline{\text{CV}}_N$  such that  $[T] \neq [T_-]$  it holds that*

$$\lim_{n \rightarrow \infty} [T\varphi^n] = [T_+].$$

*The trees  $[T_+]$  and  $[T_-]$  are called attracting and repelling trees of  $\varphi$ . The attracting and repelling trees of  $\varphi^{-1}$  are  $[T_-]$  and  $[T_+]$  respectively.*

A useful tool relating geodesic currents to Outer space is the intersection form introduced by Kapovich–Lustig.

**Proposition-Definition 2.4.5.** [30] There exists a unique continuous map  $\langle, \rangle : \overline{\text{cv}}_N \times \text{Curr}(\mathbb{F}_N) \rightarrow \mathbb{R}_{\geq 0}$  with the following properties:

1.  $\langle T, c_1\nu_1 + c_2\nu_2 \rangle = c_1 \langle T, \nu_1 \rangle + c_2 \langle T, \nu_2 \rangle$  for any  $T \in \overline{\text{cv}}_N$ ,  $\nu_1, \nu_2 \in \text{Curr}(\mathbb{F}_N)$  and non-negative scalars  $c_1, c_2$ .
2.  $\langle cT, \nu \rangle = c \langle T, \nu \rangle$  for any  $T \in \overline{\text{cv}}_N$  and  $\nu \in \text{Curr}(\mathbb{F}_N)$  and  $c \geq 0$ .
3.  $\langle T\varphi, \nu \rangle = \langle T, \varphi\nu \rangle$  for any  $T \in \overline{\text{cv}}_N$ ,  $\nu \in \text{Curr}(\mathbb{F}_N)$  and  $\varphi \in \text{Out}(\mathbb{F}_N)$ .
4.  $\langle T, \eta_g \rangle = \|g\|_T$  for any  $T \in \overline{\text{cv}}_N$ , any nontrivial  $g \in \mathbb{F}_N$ .

A detailed discussion of geodesic currents on free groups can be found in [27, 29, 30, 31].

## 2.5 Laminations on free groups

An *algebraic lamination* on  $\mathbb{F}_N$  is a closed subset of  $\partial^2 \mathbb{F}_N$  which is flip-invariant and  $\mathbb{F}_N$ -invariant. In analogy with the geodesic laminations on surfaces (see section 4.1), the elements  $(X, Y)$  of an algebraic lamination are called *leaves* of the lamination. The set of all algebraic laminations on  $\mathbb{F}_N$  is denoted by  $\Lambda^2 \mathbb{F}_N$ .

Let  $(\Gamma, \alpha)$  be a marking. For  $(X, Y) \in \partial^2 \mathbb{F}_N$ , let us denote the bi-infinite geodesic in  $\tilde{\Gamma}$  joining  $\tilde{\alpha}(X)$  to  $\tilde{\alpha}(Y)$  by  $\tilde{\gamma}$ . The reduced bi-infinite path  $\gamma$ , which is the image of  $\tilde{\gamma}$  under the covering map, is called the *geodesic realization* of the pair  $(X, Y)$  and is denoted by  $\gamma_\Gamma(X, Y)$ .

We say that a set  $\mathcal{A}$  of reduced edge paths in  $\Gamma$  *generates* a lamination  $L$  if the following condition holds: For any  $(X, Y) \in \partial^2 \mathbb{F}_N$ ,  $(X, Y)$  is a leaf of  $L$  if and only if every reduced subpath of the geodesic realization of  $(X, Y)$  belongs to  $\mathcal{A}$ .

Here we describe several important examples of algebraic laminations, all of which will be used in Section 3.1.

**Example 2.5.1** (Diagonal closure of a lamination). The following construction is due to Kapovich-Lustig, see [33] for details. For a subset  $S$  of  $\partial^2 \mathbb{F}_N$  the diagonal extension of  $S$ ,  $diag(S)$ , is defined to be the set of all pairs  $(X, Y) \in \partial^2 \mathbb{F}_N$  such that there exists an integer  $n \geq 1$  and elements  $X_1 = X, X_2, \dots, X_n = Y \in \partial \mathbb{F}_N$  such that  $(X_{i-1}, X_i) \in S$  for  $i = 1, \dots, n-1$ . It is easy to see that for a lamination  $L \in \Lambda^2 \mathbb{F}_N$ , the diagonal extension of  $L$ ,  $diag(L)$  is still  $\mathbb{F}_N$  invariant and flip-invariant but it is not necessarily closed. Denote the closure of  $diag(L)$  in  $\partial^2 \mathbb{F}_N$  by  $\overline{diag}(L)$ . For an algebraic lamination  $L \in \Lambda^2 \mathbb{F}_N$ , the diagonal closure of  $L$ ,  $\overline{diag}(L)$  is again an algebraic lamination.

**Example 2.5.2** (Support of a current). Let  $\mu \in \text{Curr}(\mathbb{F}_N)$  be a geodesic current. The *support* of  $\mu$  is defined to be  $supp(\mu) := \partial^2 \mathbb{F}_N \setminus \mathcal{U}$  where  $\mathcal{U}$  is the union of all open subsets  $U \subset \partial^2 \mathbb{F}_N$  such that  $\mu(U) = 0$ . For any  $\mu \in \text{Curr}(\mathbb{F}_N)$ ,  $supp(\mu)$  is an algebraic lamination. Moreover, it is not hard to see that  $(X, Y) \in supp(\mu)$  if and only if for every reduced subword  $v$  of the geodesic realization  $\gamma_\Gamma(X, Y)$  of  $(X, Y)$ , we have  $\langle v, \mu \rangle_\alpha > 0$ , see [31].

**Example 2.5.3.** If  $(\Gamma, \alpha)$  is a marking, and  $\mathcal{P}$  is a family of finite reduced paths in  $\Gamma$ , the lamination  $L(\mathcal{P})$  “*generated by  $\mathcal{P}$* ” consists of all  $(X, Y) \in \partial^2 \mathbb{F}_N$  such that for every finite subpath  $v$  of the geodesic realization of  $(X, Y)$  in  $\Gamma$ ,  $\gamma_\Gamma(X, Y)$ , there exists a path  $v'$  in  $\mathcal{P}$  such that  $v$  is a subpath of  $v'$  or of  $\bar{v}'$ .

**Example 2.5.4** (Laminations dual to an  $\mathbb{R}$ -tree). Let  $T \in \overline{\text{cv}_N}$ . For every  $\epsilon > 0$  consider the set

$$\Omega_\epsilon(T) = \{1 \neq [w] \in \mathbb{F}_N : \|w\|_T \leq \epsilon\}.$$

Given a marking  $\Gamma$ , define  $\Omega_{\epsilon, \Gamma}(T)$  as the set of all closed cyclically reduced paths in  $\Gamma$  representing conjugacy classes of elements of  $\Omega_\epsilon(T)$ . Define  $L_{\epsilon, \Gamma}(T)$  to be the algebraic lamination generated by the family of paths  $\Omega_{\epsilon, \Gamma}(T)$ . Then, the *dual algebraic lamination*  $L(T)$  associated to  $T$  is defined as:

$$L(T) := \bigcap_{\epsilon > 0} L_{\epsilon, \Gamma}(T).$$

It is known that this definition of  $L(T)$  does not depend on the choice of a marking  $\Gamma$ .



A detailed discussion about laminations on free groups can be found in a sequence of papers by Coulbois–Hilion–Lustig, [16, 17, 18].

## 2.6 Non-negative matrices, substitutions and symbolic dynamics

The standard sources for this section are [42] and [44].

A non-negative integer  $(n \times n)$ -matrix  $M$  is called *irreducible* if for any  $1 \leq i, j \leq n$  there exists an exponent  $k = k(i, j)$  such that the  $(i, j)$ -th entry of  $M^k$  is positive. The matrix  $M$  is called *primitive* if the exponent  $k$  can be chosen independent of  $i$  and  $j$ . The matrix  $M$  is called *reducible* if  $M$  is not irreducible.

A *substitution*  $\zeta$  on a finite set  $A = \{a_1, a_2, \dots, a_n\}$  (called the *alphabet*) of *letters*  $a_i$  is given by associating to every  $a_i \in A$  a finite word  $\zeta(a_i)$  in the alphabet  $A$ :

$$a_i \mapsto \zeta(a_i) = x_1 \dots x_n \quad (\text{with } x_i \in A)$$

This defines a map from  $A$  to  $A^*$ , by which we denote the free monoid over the alphabet  $A$ . The map  $\zeta$  extends to a well defined monoid endomorphism  $\zeta : A^* \rightarrow A^*$  which is usually denoted by the same symbol as the substitution.

The combinatorial length of  $\zeta(a_i)$ , denoted by  $|\zeta(a_i)|$ , is the number of letters in the word  $\zeta(a_i)$ . We call a substitution  $\zeta$  *expanding* if there exists  $k \geq 1$  such that for every  $a_i \in A$  one has

$$|\zeta^k(a_i)| \geq 2.$$

It follows directly that this is equivalent to stating that  $\zeta$  is *non-erasing*, i.e. none of the  $\zeta(a_i)$  is equal to the empty word, and that  $\zeta$  doesn't act periodically on any subset of the generators.

A substitution  $\zeta$  on  $A$  is called *irreducible* if for all  $1 \leq i, j \leq n$ , there exist  $k = k(i, j) \geq 1$  such that  $\zeta^k(a_j)$  contains the letter  $a_i$ . It is called *primitive* if  $k$  can be chosen independent of  $i, j$ . A substitution is called *reducible* if it is not irreducible. Note that any irreducible substitution  $\zeta$  (and hence any primitive  $\zeta$ ) is expanding, except if  $A = \{a_1\}$  and  $\zeta(a_1) = a_1$ .

Given a substitution  $\zeta : A \rightarrow A^*$ , there is an associated *incidence matrix*  $M_\zeta$  defined as follows: The  $(i, j)^{th}$  entry of  $M_\zeta$  is the number of occurrences of the letter  $a_i$  in the word  $\zeta(a_j)$ . Note that the matrix  $M_\zeta$  is a non-negative integer square matrix. It is easy to verify that an expanding substitution  $\zeta$  is irreducible (primitive) if and only if the matrix  $M_\zeta$  is

irreducible (primitive). It also follows directly that  $M_{\zeta^t} = (M_{\zeta})^t$  for any exponent  $t \in \mathbb{N}$ .

For any letter  $a_i \in A$  and any word  $w \in A^*$  we denote the number of occurrences of the letter  $a_i$  in the word  $w$  by  $|w|_{a_i}$ .

We observe directly from the definitions that the resulting *occurrence vector*  $\vec{v}(w) := (|w|_{a_i})_{a_i \in A}$  satisfies:

$$M_{\zeta} \cdot \vec{v}(w) = \vec{v}(\zeta(w)) \quad (2.6.1)$$

The following result, which is proved in [37], generalizes a classical theorem for primitive substitutions.

**Proposition 2.6.1.** *Let  $\zeta : A \rightarrow A^*$  be an expanding substitution. Then, up to replacing  $\zeta$  by a power, the frequencies of factors converge: For any word  $w \in A^*$  of length  $|w| \geq 1$  and any letter  $a \in A$  the limit frequency*

$$f_w(a) := \lim_{t \rightarrow \infty} \frac{|\zeta^t(a)|_w}{|\zeta^t(a)|}$$

*exists. If,  $\zeta$  is primitive, then the above limit is independent of the letter  $a$ .*

The proof of the above proposition also implies the following.

**Lemma 2.6.2** (Remark 3.3 of [37]). *Let  $\zeta : A^* \rightarrow A^*$  be an expanding substitution. Then (up to replacing  $\zeta$  by a positive power) there exists a constant  $\lambda_{a_i} > 1$  for each  $a_i \in A$  such that:*

$$\lim_{t \rightarrow \infty} \frac{|\zeta^{t+1}(a_i)|}{|\zeta^t(a_i)|} = \lambda_{a_i}.$$

*If,  $\zeta$  is primitive, then  $\lambda$  is independent of the letter  $a$  and is equal to the Perron-Frobenius eigenvalue for  $M(\zeta)$ .*

Let  $X_{\zeta}$  be the set of semi-infinite words such that for every  $a_n \in X_{\zeta}$ , every subword of  $a_n$  appears as a subword of  $\zeta^k(x)$  for some  $k \geq 0$  and for some  $x \in A$ . Let  $T : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$  be the shift map, which erases the first letter of each word. The following unique ergodicity result is an important ingredient of the Proof of Lemma 3.1.8. It is due to Michel [39], and a proof can be found in [42, Proposition 5.6].

**Theorem 2.6.3.** *For a primitive substitution  $\zeta$ , the system  $(X_{\zeta}, T)$  is uniquely ergodic. In other words, there is a unique  $T$ -invariant, Borel probability measure on  $X_{\zeta}$ .*

Let  $(\Gamma, \alpha)$  be a marking. Let  $\Omega(\Gamma)$  denote the set of semi-infinite reduced edge paths in  $\Gamma$ . Let  $T_{\Gamma} : \Omega(\Gamma) \rightarrow \Omega(\Gamma)$  be the shift map. Define the one sided cylinder  $Cyl_{\Omega}(v)$  for an

edge-path  $v$  in  $\Gamma$  to be the set of all  $\gamma \in \Omega(\Gamma)$  such that  $\gamma$  starts with  $v$ . It is known that the set  $\{Cyl_\Omega(v)\}_{v \in \mathcal{P}\Gamma}$  generates the Borel  $\sigma$ -algebra for  $\Omega(\Gamma)$ , [28].

Let  $\mathcal{M}(\Omega(\Gamma))$  denote the space of finite, positive Borel measures on  $\Omega(\Gamma)$  that are  $T_\Gamma$ -invariant. Define  $\mathcal{M}'(\Omega(\Gamma)) \subset \mathcal{M}(\Omega(\Gamma))$  to be the set of all  $\nu \in \mathcal{M}(\Omega(\Gamma))$  that are symmetric, i.e. for any reduced edge path  $v$  in  $\Gamma$ ,

$$\nu(Cyl_\Omega(v)) = \nu(Cyl_\Omega(\bar{v})).$$

**Proposition 2.6.4** (Proposition 4.5 of [28]). *The map  $\tau : \text{Curr}(F_N) \rightarrow \mathcal{M}'(\Omega(\Gamma))$  defined as*

$$\mu \mapsto \tau\mu,$$

where  $\tau\mu(Cyl_\Omega(v)) = \langle v, \mu \rangle$  is an affine homeomorphism.

## 2.7 Train-track maps reinterpreted as substitutions

Let  $f : \Gamma \rightarrow \Gamma$  be an expanding train-track map that represents an atoroidal outer automorphism  $\varphi$ . We interpret  $E\Gamma$  as a finite alphabet and consider the occurrences of a path  $\gamma$  as subpath in a path  $\gamma'$ . As before, the number of such occurrences is denoted by  $|\gamma'|_\gamma$ . We denote the number of occurrences of  $\gamma$  or of  $\bar{\gamma}$  as subpath in a path  $\gamma'$  by  $\langle \gamma, \gamma' \rangle$  and obtain:

$$\langle \gamma, \gamma' \rangle = |\gamma'|_\gamma + |\gamma'|_{\bar{\gamma}} \tag{2.7.1}$$

The map  $f$  induces a substitution

$$\zeta_f : E\Gamma^* \rightarrow E\Gamma^*$$

but in general  $\zeta_f$ -iterates of reduced paths in  $\Gamma$  will be mapped to non-reduced paths. An exception is when the path  $\gamma'$  is legal: In this case all  $f^t(\gamma')$  will be reduced as well:

$$[f^t(\gamma')] = f^t(\gamma')$$

where  $[\rho]$  denotes as in section 2.2 the path obtained from an edge path  $\rho$  via reduction relative to its endpoints. Hence the occurrences of any path  $\gamma$  or of  $\bar{\gamma}$  in  $[f^t(\gamma')]$  are given by

$$\langle \gamma, [f^t(\gamma')] \rangle = |f^t(\gamma')|_\gamma + |f^t(\gamma')|_{\bar{\gamma}} \tag{2.7.2}$$

for any integer  $t \geq 0$ .

We are now ready to prove:

**Proposition 2.7.1.** *Let  $f : \Gamma \rightarrow \Gamma$  be an expanding train-track map that represents an atoroidal outer automorphism, and let  $e \in E\Gamma$ . Then, after possibly replacing  $f$  by a positive power, for any reduced edge path  $\gamma$  in  $\Gamma$  the limit*

$$\lim_{n \rightarrow \infty} \frac{\langle \gamma, f^n(e) \rangle}{|f^n(e)|} = a_\gamma$$

*exists and the set of these limit values defines a unique geodesic current  $\mu_+(e)$  on  $F_N$  through setting  $\langle \gamma, \mu_+(e) \rangle = a_\gamma$  for any  $\gamma \in \mathcal{P}(\Gamma)$ .*

*Moreover, when  $\varphi$  is irreducible and hence  $f$  is primitive, the current  $\mu_+(e)$  is independent of the edge  $e$ .*

*Proof.* We will give the proof when  $\varphi$  is irreducible, the reducible case is a straightforward generalization. Let  $\rho = \lim_{n \rightarrow \infty} f^n(e_0)$ , where  $e_0$  is a periodic edge. For an edge  $e \in E\Gamma$  we have two possibilities:

*Type 1 :* Either only  $e$  occurs or only  $\bar{e}$  occurs in  $\rho$ .

*Type 2 :* Both  $e$  and  $\bar{e}$  occur in  $\rho$ .

**Claim.** There are two disjoint cases:

1. Every edge  $e \in E\Gamma$  is of Type 1.
2. Every edge  $e \in E\Gamma$  is of Type 2.

Let us assume that for an edge  $e$  both  $e$  and  $\bar{e}$  occur in  $\rho$ . Now look at  $f(e)$ . Since  $M(f) > 0$ , for an arbitrary edge  $e_i$ , it means that either  $e_i$  occurs in  $f(e)$  or  $\bar{e}_i$  or possibly both of them occur in  $f(e)$ . If both of them occur in  $f(e)$  they occur in  $\rho$  as well and we are done, otherwise assume that only one of them occurs in  $f(e)$ , say  $e_i$ . In that case  $\bar{e}_i$  occurs in  $f(\bar{e})$  so that both  $e_i$  and  $\bar{e}_i$  occur in  $\rho$ . For the second case, assume that for an edge  $e$  either only  $e$  occurs or only  $\bar{e}$  occurs on  $\rho$ . We claim that this is the case for every other edge. Assume otherwise, and say that for some edge  $e_j$  both  $e_j$  and  $\bar{e}_j$  occur in  $\rho$ , but from first part that would imply that both  $e$  and  $\bar{e}$  occur in  $\rho$  which is a contradiction. We now continue with the proof of the Lemma.

**Case 1** (Every edge  $e \in E\Gamma$  is of Type 1). Split  $E\Gamma = E_+ \cup E_-$ , where

$$E_+ = \{e \mid e \text{ occurs in } \rho \text{ only with positive sign}\}$$

and

$$E_- = \{e \mid e \text{ occurs in } \rho \text{ only with negative sign}\}.$$

So  $f$  splits into two primitive substitutions:  $f_+ : A_0 \rightarrow A_0^*$  where  $A_0 = E_+$  and  $f_- : A_1 \rightarrow A_1^*$  where  $A_1 = E_-$ . Proposition 2.6.1, together with the observation that  $(v, f^n(e)) = (\bar{v}, f^n(\bar{e}))$  gives the required convergence.

**Case 2** (Every edge  $e \in E\Gamma$  is of Type 2). In this case we can think of  $\bar{e}$  as a distinct edge, then  $f$  becomes a primitive substitution on the set  $A = E\Gamma$  and the result follows from Proposition 2.6.1.

This completes the first half of the proof of the Lemma. For the second assertion, let us define

$$q_+(v) = \{e \in E\Gamma \mid ve \in \mathcal{P}\Gamma\}, \quad q_-(v) = \{e \in E\Gamma \mid ev \in \mathcal{P}\Gamma\}.$$

We will show that above set of numbers satisfies the switch conditions as in [28].

- (1) It is clear that for any  $v \in \mathcal{P}\Gamma$  we have  $0 \leq a_v < 1 < \infty$ .
- (2) It is also clear from the definition that  $\{a_v\} = \{a_{\bar{v}}\}$ .
- (3) We need to show that

$$\sum_{e \in q_+(v)} \lim_{n \rightarrow \infty} \frac{\langle ve, f^n(e_0) \rangle}{\ell_\Gamma(f^n(e_0))} = \lim_{n \rightarrow \infty} \frac{\langle v, f^n(e_0) \rangle}{\ell_\Gamma(f^n(e_0))} = \sum_{e \in q_-(v)} \lim_{n \rightarrow \infty} \frac{\langle ev, f^n(e_0) \rangle}{\ell_\Gamma(f^n(e_0))}$$

For the first equality, under a finite iterate of  $f$ , the only undercount of occurrences of  $ve$  in  $f^n(e_0)$  can happen if  $v$  is the last subsegment of  $f^n(e_0)$  or  $\bar{v}$  is the first subsegment of  $f^n(e_0)$ . Hence

$$\left| \frac{\langle v, f^n(e_0) \rangle}{\ell_\Gamma(f^n(e_0))} - \sum_{e \in q_+(v)} \frac{\langle ve, f^n(e_0) \rangle}{\ell_\Gamma(f^n(e_0))} \right| \leq \frac{2|q_+(v)|}{\ell_\Gamma(f^n(e_0))} \rightarrow 0$$

as  $n \rightarrow \infty$ . Second equality can be shown similarly.  $\square$

We now want to show that the currents  $\mu_+(e)$  are projectively  $\varphi$ -invariant. For this purpose we start by stating two lemmas; the first one is elementary:

**Lemma 2.7.2.** *For any graph  $\Gamma$  without valence 1 vertices there exists a constant  $K \geq 0$  such that for any finite reduced edge path  $\gamma$  in  $\Gamma$  there exists an edge path  $\gamma'$  of length  $|\gamma'| \leq K$  such that the concatenation  $\gamma \circ \gamma'$  exists and is a reduced loop.*  $\square$

**Lemma 2.7.3.** *Let  $f : \Gamma \rightarrow \Gamma$  as in Proposition 2.7.1, and let  $K_1 \geq 0$  be any constant. For all integers  $t \geq 0$  let  $\gamma'_t \in E\Gamma^*$  be any element with  $|\gamma'_t| \leq K_1$ . Set  $\gamma_t := f^t(e) * \gamma'_t$ , where*

$f^t(e)^*$  is obtained from  $f^t(e)$  by erasing an initial and a terminal subpath of length at most  $K_1$ . Then for any reduced path  $\gamma$  in  $\Gamma$  one has

$$\lim_{t \rightarrow \infty} \frac{\langle \gamma, \gamma_t \rangle}{|\gamma_t|} = \langle \gamma, \mu_+(e) \rangle$$

*Proof.* From the hypotheses  $|\gamma'_t| \leq K_1$  and  $|f^t(e)^*| \geq |f^t(e)| - 2K_1$ , and from the fact that  $f$  is expanding and hence  $|f^t(e)| \rightarrow \infty$  for  $t \rightarrow \infty$ , we obtain directly

$$\lim_{t \rightarrow \infty} \frac{\langle \gamma, \gamma_t \rangle}{\langle \gamma, f^t(e) \rangle} = 1$$

and

$$\lim_{t \rightarrow \infty} \frac{|\gamma_t|}{|f^t(e)|} = 1.$$

Hence the claim follows directly from Proposition 2.7.1.  $\square$

**Proposition 2.7.4.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be an atoroidal outer automorphism which is represented by an expanding train-track map  $f : \Gamma \rightarrow \Gamma$ . We assume that  $\varphi$  and  $f$  have been replaced by positive powers according to Proposition 2.7.1.*

*Then there exist a constant  $\lambda_e > 1$  such that  $\varphi(\mu_+(e)) = \lambda_e \mu_+(e)$ .*

*Proof.* For the given graph  $\Gamma$  let  $K \geq 0$  be the constant given by Lemma 2.7.2, and for any integer  $t \geq 0$  let  $\gamma'_t \in \mathcal{P}(\Gamma)$  with  $|\gamma'_t| \leq K$  be the path given by Lemma 2.7.2 so that  $\gamma_t =: f^t(e)\gamma'_t \in \mathcal{P}(\Gamma)$  is a reduced loop. Let  $[w_t] \subset \mathbb{F}_N \cong \pi_1\Gamma$  be the conjugacy class represented by  $\gamma_t$ , and note that the rational current  $\eta_{[w_t]}$  satisfies  $\|\eta_{[w_t]}\| = |\gamma_t|$ .

Similarly, consider  $f(\gamma_n) = f^{t+1}(e)f(\gamma'_t)$ , and notice that  $|f(\gamma'_t)|$  is bounded above by the constant  $K_0 = K \max\{|f(e)| \mid e \in E\Gamma\}$ . Since  $f$  is a train track map, the path  $f^{t+1}(e)$  is reduced. Hence the reduced loop  $\gamma''_t := [f(\gamma_n)] = [f^{t+1}(e)f(\gamma'_t)]$  can be written as product  $f^{t+1}(e)^*\gamma''_t$  with  $|\gamma''_t| \leq K_1$  and  $|f^{t+1}(e)^*| \geq |f^{t+1}(e)| - 2K_1$ , where  $f^{t+1}(e)^*$  is a subpath of  $f^{t+1}(e)$  and  $K_1$  is the maximum of  $K_0$  and the cancellation bound  $C_f$  of  $f$  (see Lemma 2.3.3). Thus we can apply Lemma 2.7.3 twice to obtain for any reduced path  $\gamma$  in  $\Gamma$  that

$$\lim_{t \rightarrow \infty} \frac{\langle \gamma, \gamma_t \rangle}{|\gamma_t|} = \langle \gamma, \mu_+(e) \rangle$$

and

$$\lim_{t \rightarrow \infty} \frac{\langle \gamma, \gamma''_t \rangle}{|\gamma''_t|} = \langle \gamma, \mu_+(e) \rangle.$$

The first equality implies that the rational currents  $\eta_{[w_t]}$  satisfy

$$\lim_{t \rightarrow \infty} \frac{\eta_{[w_t]}}{\|\eta_{[w_t]}\|} = \mu_+(e).$$

From the continuity of the  $\text{Out}(\mathbb{F}_N)$ -action on current space and from  $\varphi\eta_{[w_t]} = \eta_{\varphi[w_t]}$  (see equality (2.4) from section 2.4) we thus deduce

$$\lim_{t \rightarrow \infty} \frac{\eta_{\varphi[w_t]}}{\|\eta_{\varphi[w_t]}\|} = \varphi\mu_+(e).$$

However, since the reduced loops  $\gamma_t''$  represent the conjugacy classes  $\varphi[w_t]$ , the second of the above equalities implies that

$$\lim_{t \rightarrow \infty} \frac{\eta_{\varphi[w_t]}}{\|\eta_{\varphi[w_t]}\|} = \mu_+(e).$$

Since  $\lim_{t \rightarrow \infty} \frac{|\gamma_t|}{|f^t(e)|} = 1$  and  $\lim_{t \rightarrow \infty} \frac{|\gamma_t''|}{|f^{t+1}(e)|} = 1$ , with  $|\gamma_t| = \|\eta_{[w_t]}\|$  and  $|\gamma_t''| = \|\eta_{\varphi[w_t]}\|$ , the conclusion follows from Lemma 2.7.5 below.  $\square$

**Lemma 2.7.5.** *For every edge  $e$  of  $\Gamma$  there exists a real number  $\lambda_e > 1$  which satisfies:*

$$\lim_{t \rightarrow \infty} \frac{|f^{t+1}(e)|}{|f^t(e)|} = \lambda_e$$

*Proof.* This is a direct consequence of Lemma 2.6.2 and of the definition of  $\zeta_f$ .  $\square$

We now define  $\Delta_-(\varphi)$  and  $\Delta_+(\varphi)$  that are used in the next section:

**Definition 2.7.6.** Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be an atoroidal outer automorphism. Assume that  $\varphi$  is replaced by a positive power such that both,  $\varphi$  and  $\varphi^{-1}$  are represented by expanding train-track maps as in Proposition 2.7.1. Let  $f : \Gamma \rightarrow \Gamma$  be the representative of  $\varphi$ . Then the *simplex of attraction* is defined as follows:

$$\Delta_+(\varphi) = \left\{ \left[ \sum_{e_i \in E^+\Gamma} a_i \mu_+(e_i) \right] \mid a_i \geq 0, \sum a_i > 0 \right\}.$$

Analogously, we define the *simplex of repulsion* as  $\Delta_-(\varphi) = \Delta_+(\varphi^{-1})$ .

**Remark 2.7.7.** Note that when  $\varphi$  is fully irreducible, the simplex of attraction consists of a single point  $[\mu_+]$  called the *stable current* and the simplex of repulsion consists of a single point  $[\mu_-]$  called the *unstable current*.

# Chapter 3

## Dynamics of atoroidal automorphisms

### 3.1 The irreducible case

The main theorem of this section is the following:

**Theorem 3.1.1.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be an atoroidal fully irreducible element. Then the action of  $\varphi$  on  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  has two fixed points  $[\mu_-]$  and  $[\mu_+]$ , called the unstable and stable currents respectively. Moreover, for any  $[\mu] \in \mathbb{P}\text{Curr}(\mathbb{F}_N) \setminus \{[\mu_+], [\mu_-]\}$  we have*

$$\lim_{n \rightarrow \infty} \varphi^n([\mu]) = [\mu_+] \text{ and } \lim_{n \rightarrow \infty} \varphi^{-n}([\mu]) = [\mu_-].$$

By a result of Kapovich-Lustig [32, Lemma 4.7], also by Remark 2.1.3 this implies uniform north-south dynamics for the action of atoroidal fully irreducible elements on the space of projective geodesic currents.

**Lemma 3.1.2** ([6]). *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be fully irreducible. Then, for some  $k \geq 1$ , the automorphism  $\varphi^k$  admits a train track representative  $f : \Gamma \rightarrow \Gamma$  with the following properties:*

1. *Every periodic Nielsen path has period 1.*
2. *There is at most one indivisible Nielsen path (INP) in  $\Gamma$  for  $f$ . Moreover, if there is an INP, the illegal turn in the INP is the only illegal turn in the graph  $\Gamma$ .*

**Convention 3.1.3.** In what follows we will pass to a power of  $\varphi$  so that it satisfies Lemma 3.1.2, and the corresponding substitution has a positive transition matrix.

**Notation 3.1.4.** Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be an atoroidal fully irreducible element. Denote the stable and the unstable currents corresponding to the action of  $\varphi$  on  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  by  $[\mu_+]$  and  $[\mu_-]$  respectively, as defined in Remark 2.7.7. Let  $T_-$  and  $T_+$  denote representatives in  $\overline{\text{cv}}_N$  of repelling and attracting trees for the right action of  $\varphi$  on  $\overline{\text{CV}}_N$ , where  $T_+\varphi = \lambda_+T_+$  and  $T_-\varphi^{-1} = \lambda_-T_-$  for some  $\lambda_-, \lambda_+ > 1$ , [35].



**Remark 3.1.5.** Note that by Proposition 2.1.6, if  $\varphi$  is an atoroidal fully irreducible element,  $k \geq 1$  is an integer and if the conclusion of Theorem 3.1.1 holds for  $\varphi^k$ , then Theorem 3.1.1 holds for  $\varphi$  as well. Therefore, for the remainder of this section, we pass to appropriate powers and make the same assumptions as in Convention 3.1.3.

Let  $f : \Gamma \rightarrow \Gamma$  be a train-track map representing an atoroidal fully irreducible element  $\varphi \in \text{Out}(\mathbb{F}_N)$ . Then, the *Bestvina-Feighn-Handel lamination*  $L_{BFH}(\varphi)$  is the lamination generated by the family of paths  $f^k(e)$ , where  $e \in E\Gamma$ , and  $k \geq 0$ , [2].

**Proposition 3.1.6.** *Let  $f$  be a train-track map representing an atoroidal fully irreducible element  $\varphi \in \text{Out}(\mathbb{F}_N)$ . Then, the Bestvina-Feighn-Handel lamination  $L_{BFH}(\varphi)$  is uniquely ergodic. In other words, there exists a unique geodesic current  $[\mu] \in \mathbb{P}\text{Curr}(\mathbb{F}_N)$  such that  $\text{supp}(\mu) \subset L_{BFH}(\varphi)$ , namely  $[\mu] = [\mu_+]$ .*

*Proof.* Note that we are still working with a power of the outer automorphism  $\varphi$  which satisfies 3.1.3. There are two cases to consider in terms of the type of the train track map  $f$  as in Proposition 2.7.1. First assume that  $f$  is of *Type 2*. Define  $\mathcal{L}_f$  to be the set of all finite edge-paths  $v$  in  $\Gamma$  such that there exists an edge  $e \in \Gamma$  and an integer  $n \geq 0$  such that  $v$  is a subword of  $f^n(e)$ . Let  $X_f$  be the set of all semi-infinite reduced edge paths  $\gamma$  in  $\Gamma$  such that every finite subword of  $\gamma$  is in  $\mathcal{L}_f$ . Note that the map  $\tau : \text{Curr}(\mathbb{F}_N) \rightarrow \mathcal{M}'(\Omega(\Gamma))$  as defined in Section 2.6, gives an affine homeomorphism from the set

$$A = \{\mu \in \text{Curr}(\mathbb{F}_N) \mid \text{supp}(\mu) \subset L_{BFH}(\varphi)\}$$

to the set

$$B = \{\nu \in \mathcal{M}'(\Omega(\Gamma)) \mid \text{supp}(\nu) \subset X_f\}.$$

Since  $X_f$  is uniquely ergodic by Theorem 2.6.3, this implies that  $L_{BFH}(\varphi)$  is uniquely ergodic. Now, let the map  $f$  be of *Type 1*. Partition the edges of  $\Gamma$  as in Proposition 2.7.1,  $E\Gamma = E_+ \cup E_-$ , and let  $f_+ : E_+ \rightarrow E_+$  and  $f_- : E_- \rightarrow E_-$  be the corresponding primitive substitutions. Define  $L_{f_+}$  and  $X_{f_+}$  similarly. Let  $\Omega_+(\Gamma)$  be the set of all semi-infinite reduced edge-paths in  $\Gamma$  where each edge is labeled by an edge in  $E_+$ . Let  $\mathcal{M}(\Omega_+(\Gamma))$  be the set of positive Borel measures on  $\Omega_+(\Gamma)$  that are shift invariant. Then, the map

$$\sigma : \{\nu \in \mathcal{M}(\Omega_+(\Gamma)) \mid \text{supp}(\nu) \subset X_{f_+}\} \rightarrow \{\mu \in \text{Curr}(\mathbb{F}_N) \mid \text{supp}(\mu) \subset L_{BFH}(\varphi)\},$$

which is defined by  $\langle v, \mu \rangle_\Gamma = \nu(\text{Cyl}(v))$  for a positive edge path  $v$ ,  $\langle v, \mu \rangle = \nu(\text{Cyl}(v^{-1}))$  for a negative edge path  $v$ , and  $\langle v, \mu \rangle = 0$  otherwise, is an affine homeomorphism. Since  $X_{f_+}$  is

uniquely ergodic, so is  $L_{BFH}(\varphi)$ . Note that because of the way  $\mu_+$  is defined, see Proposition 2.7.1,  $\text{supp}(\mu_+) \subset L_{BFH}(\varphi)$ . Hence,  $[\mu_+]$  is the only current whose support is contained in  $L_{BFH}(\varphi)$ .  $\square$

**Proposition 3.1.7.** *Let  $\mu \in \text{Curr}(\mathbb{F}_N)$  be a geodesic current, and  $\alpha : R_N \rightarrow \Gamma$  be a marking.*

1. *If  $\langle v, \mu \rangle_\alpha > 0$ , then there exist  $\epsilon, \delta \in \{-1, 1\}$  and a finite path  $z$  such that  $\langle v^\epsilon z v^\delta, \mu \rangle > 0$ .*
2. *If  $\langle v, \mu \rangle_\alpha > 0$ , then for every  $r \geq 2$  there exists a path  $v_r = v^{\epsilon_1} z_1 v^{\epsilon_2} \dots z_{r-1} v^{\epsilon_r}$ , where  $\epsilon_i \in \{-1, 1\}$  such that  $\langle v_r, \mu \rangle_\alpha > 0$ .*

*Proof.* The above proposition seems to be well known to experts in the field, but for completeness we will provide a sketch of the proof here. Let  $T = \tilde{\Gamma}$ , and normalize  $\mu$  such that  $\langle T, \mu \rangle = 1$ . There exists a sequence  $\{w_n\}$  of conjugacy classes such that

$$\mu = \lim_{n \rightarrow \infty} \frac{\eta_{w_n}}{\|w_n\|_\Gamma}.$$

This means that there exists an integer  $M > 0$  such that for all  $n \geq M$ ,

$$\frac{\langle v, \eta_{w_n} \rangle_\alpha}{\|w_n\|_\Gamma} \geq \frac{\epsilon}{2}.$$

Note that without loss of generality we can assume  $\|w_n\|_\Gamma \rightarrow \infty$ . Otherwise,  $\mu$  would be a rational current for which the conclusion of the Proposition clearly holds. From here, it follows that for some  $\epsilon_1 > 0$ , we have

$$\frac{m(n)\ell_\Gamma(v)}{\|w_n\|_\Gamma} \geq \epsilon_1,$$

where  $m(n)$  is the maximal number of disjoint occurrences of  $v^{\pm 1}$  in  $w_n$ . Let  $u_{n_i}$  be the complementary subwords in  $w_n$  as in Figure 3.1.

Let us set  $K = \frac{\ell_\Gamma(v)}{\epsilon_1}$ . Observe that for all  $n \geq M$  we have  $\min \ell_\Gamma(u_{n_i}) \leq K$ , otherwise we would have

$$\|w_n\|_\Gamma \geq m(n)K + m(n)\ell_\Gamma(v)$$

and hence,

$$\frac{m(n)\ell_\Gamma(v)}{\|w_n\|_\Gamma} \leq \epsilon_1,$$

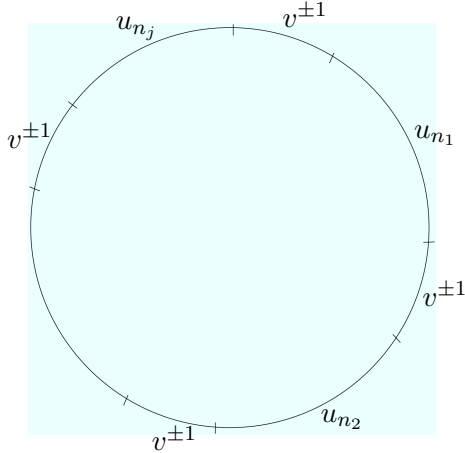


Figure 3.1:  $w_n$

which is a contradiction. Let us call complementary subwords  $u_{n_i}$  of length  $\ell_\Gamma(u_{n_i}) \leq K$  “short”. By using a similar reasoning it is easy to see that *short*  $u_{n_i}$  cover a definite proportion of  $w_n$  for all  $n \geq M$ .

Since there are only finitely many edge-paths  $\rho$  of length  $\ell_\Gamma(\rho) \leq K$  in  $\Gamma$ , for each  $w_n$  we can look at the short  $u_{n_i}$  which occurs most in  $w_n$ . This particular  $u_{n_i}$  covers a definite amount of  $w_n$ . Now, take a subsequence  $n_k$  so that it is the same short  $u$  for every  $n_k$ . This means that,  $v^\pm u v^\pm$  covers a definite proportion of  $w_{n_k}$ . Since  $\mu$  is the limit of  $\eta_{w_n}$ ’s, this shows that

$$\langle v^{\pm 1} u v^{\pm 1}, \mu \rangle_\alpha > 0.$$

This completes the proof of part (1) of Prop 3.1.7. Part (2) now follows from part (1) by induction.  $\square$

The standard proof of the following lemma uses the result that an atoroidal fully irreducible  $\varphi \in \text{Out}(\mathbb{F}_N)$  acts on  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  with north-south dynamics; but since we are proving that result in this paper we need a different argument.

**Lemma 3.1.8.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be an atoroidal fully irreducible element. Let  $[\mu] \neq [\mu_+]$  be a geodesic current and  $T_-$  be as in 3.1.4. Then,  $\langle T_-, \mu \rangle \neq 0$ . Similarly, for a geodesic current  $[\mu] \neq [\mu_-]$  and  $T_+$  as in 3.1.4, we have  $\langle T_+, \mu \rangle \neq 0$ .*

*Proof.* We will prove the first statement. The proof of the second statement is similar. Let  $(\Gamma, \alpha)$  be a marking and  $f : \Gamma \rightarrow \Gamma$  be a train-track representative for  $\varphi \in \text{Out}(\mathbb{F}_N)$ . Assume

that for a geodesic current  $\mu \in \text{Curr}(\mathbb{F}_N)$  we have  $\langle T_-, \mu \rangle = 0$ . By a result of Kapovich-Lustig [31], this implies that

$$\text{supp}(\mu) \subset L(T_-),$$

where  $L(T_-)$  is the dual algebraic lamination associated to  $T_-$  as explained in Example 2.5.4. It is shown in [33] that,  $L(T_-) = \overline{\text{diag}(L_{BFH}(\varphi))}$  and moreover,  $L(T_-) \setminus (L_{BFH}(\varphi))$  is a finite union of  $\mathbb{F}_N$  orbits of leaves  $(X, Y) \in \partial^2 \mathbb{F}_N$ , where geodesic realization  $\gamma$  in  $\Gamma$  of  $(X, Y)$  is a concatenation of eigenrays at either an INP or an unused legal turn.

**Claim.**  $\text{supp}(\mu) \subset L_{BFH}(\varphi)$ .

Assume that this is not the case, this means that there is a leaf  $(X, Y)$  in the support of  $\mu$  such that  $(X, Y) \in L(T_-) \setminus (L_{BFH}(\varphi))$ . By a result of Kapovich–Lustig, [33] a geodesic representative of  $(X, Y) \in L(T_-) \setminus (L_{BFH}(\varphi))$ ,  $\gamma_\Gamma(X, Y)$  can be one of the following two types of singular leaves. See Figure 3.2.

1.  $\gamma_\Gamma(X, Y) = \rho^{-1}\eta\rho'$ , where  $\rho$  and  $\rho'$  are again *combinatorial eigenrays* of  $f$ , and  $\eta$  is the unique *INP* in  $\Gamma$ . In this case turns between  $\eta$  and  $\rho$ , and between  $\eta$  and  $\rho'$  are legal (and may or may not be used), and  $\gamma_\Gamma(X, Y)$  contains exactly one occurrence of an illegal turn, namely the tip of the INP  $\eta$ .
2.  $\gamma_\Gamma(X, Y) = \rho^{-1}\rho'$ , where  $\rho$  and  $\rho'$  are *combinatorial eigenrays* of  $f$  satisfying  $f(\rho) = \rho$  and  $f(\rho') = \rho'$ , and where the turn between  $\rho$  and  $\rho'$  is legal but not used. In this case all the turns contained in  $\rho$  and  $\rho'$  are used.

First, recall that a bi-infinite geodesic  $\gamma$  is in the support of  $\mu$  if and only if for every subword  $v$  of  $\gamma$ ,

$$\langle v, \mu \rangle_\alpha > 0.$$

Now, let  $e_2^{-1}e_1$  be either the unused subword at the concatenation point as in the second case or the *tip* of the INP as in the first case. Since  $\langle e_2^{-1}e_1, \mu \rangle > 0$ , Proposition 3.1.7 implies that there exists a subword  $v = (e_2^{-1}e_1)^{\pm 1} \dots (e_2^{-1}e_1)^{\pm 1} \dots (e_2^{-1}e_1)^{\pm 1} \dots (e_2^{-1}e_1)^{\pm 1}$  which is in the support of  $\mu$ . This is a contradiction to the fact that support of  $\mu$  consists precisely of

1. bi-infinite used legal paths, and
2. bi-infinite paths with one singularity as in Figure 3.2.

Therefore,  $\text{supp}(\mu) \subset L_{BFH}(\varphi)$ . Now, Proposition 3.1.6 implies that  $[\mu] = [\mu_+]$ .  $\square$

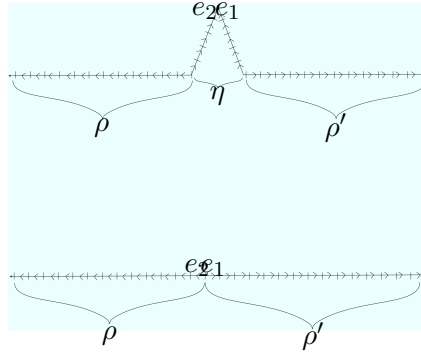


Figure 3.2: Singular Leaves

*Proof of Theorem 3.1.1.* We will prove the first assertion, the proof of the second assertion is similar. Suppose that this is not the case. Then, there exists a subsequence  $\{n_k\}$  such that

$$\lim_{n_k \rightarrow \infty} \varphi^{n_k}([\mu]) = [\mu'] \neq [\mu_+].$$

This means that there exists a sequence of positive real numbers  $\{c_{n_k}\}$  such that

$$\lim_{n_k \rightarrow \infty} c_{n_k} \varphi^{n_k}(\mu) = \mu'.$$

We first note that, by invoking Proposition 2.4.5, we have

$$\langle T_+, \mu' \rangle = \left\langle T_+, \lim_{n_k \rightarrow \infty} c_{n_k} \varphi^{n_k}(\mu) \right\rangle = \lim_{n_k \rightarrow \infty} c_{n_k} \lambda_+^{n_k} \langle T_+, \mu \rangle,$$

which implies that  $\lim_{n_k \rightarrow \infty} c_{n_k} = 0$ .

Similarly, using Proposition 2.4.5, we get

$$\langle T_-, \mu' \rangle = \left\langle T_-, \lim_{n_k \rightarrow \infty} c_{n_k} \varphi^{n_k}(\mu) \right\rangle = \lim_{n_k \rightarrow \infty} c_{n_k} \langle T_- \varphi^{n_k}, \mu \rangle = \lim_{n_k \rightarrow \infty} \frac{c_{n_k}}{\lambda_-^{n_k}} \langle T_-, \mu \rangle = 0,$$

which is a contradiction to the Lemma 3.1.8. This finishes the proof of the Theorem 3.1.1.  $\square$

## 3.2 The general case

**Convention 3.2.1.** Let  $\Gamma$  be a finite connected graph, and that  $f : \Gamma \rightarrow \Gamma$  be an expanding train-track map that represents an atoroidal outer automorphism, and  $C_f$  be the bounded cancellation constant given by Lemma 2.3.3. We also assume that  $f$  has been replaced by a positive power so that for some integers  $\lambda'' \geq \lambda' > 1$  we have, for any edge  $e$  of  $\Gamma$ :

$$\lambda'' \geq |f(e)| \geq \lambda' \quad (3.2.1)$$

and  $\lambda', \lambda''$  is attained for some edges.

### 3.2.1 Goodness

The following terminology was introduced by R. Martin in his thesis [38].

**Definition 3.2.2.** Let  $f : \Gamma \rightarrow \Gamma$  and  $C_f$  be as in Convention 3.2.1. Define the *critical constant*  $C$  for  $f$  as  $C := \frac{C_f}{\lambda' - 1}$ . Any edge  $e$  in  $\gamma$  that is at least  $C$  edges away from an illegal turn on  $\gamma$  is called *good*, where the distance (= number of edges traversed) is measured on  $\gamma$ . An edge is called *bad* if it is not good. Edge paths or loops, in particular subpaths of a given edge path, which consist entirely of good (or entirely of bad) edges are themselves called *good* (or *bad*).

For any edge path or a loop  $\gamma$  in  $\Gamma$  we define the *goodness* of  $\gamma$  as the following quotient:

$$\mathbf{g}(\gamma) := \frac{\#\{\text{good edges of } \gamma\}}{|\gamma|} \in [0, 1]$$

We will now discuss some basic properties of the goodness of paths and loops. We first consider any legal path  $\gamma$  in  $\Gamma$  of length  $|\gamma| = C$  and compute:

$$|f(\gamma)| \geq \lambda' |\gamma| = \lambda' C = C_f + C \quad (3.2.2)$$

**Lemma 3.2.3.** *For any reduced loop  $\gamma$  in  $\Gamma$  we have:*

$$\#\{\text{good edges in } [f(\gamma)]\} \geq \lambda' \cdot \#\{\text{good edges in } \gamma\}$$

*Proof.* If  $\gamma$  is legal, then every edge is good, and the claim follows directly from the definition

of  $\lambda'$  in Convention 3.2.1. Now assume that the path  $\gamma$  has at least one illegal turn. Let

$$\gamma = \gamma_1 B_1 \gamma_2 B_2 \dots \gamma_n B_n$$

be a decomposition of  $\gamma$  into maximal good edge paths  $\gamma_i$  and maximal bad edge paths  $B_i$ . Note that each  $B_i$  can be written as an illegal concatenation  $B_i = a_i b_i c_i$  where  $a_i, c_i$  are legal segments of length  $\geq C$  and  $b_i$  is a bad edge path.

Note that since  $|f(a_i)| \geq \lambda'|a_i| \geq \lambda'C = C_f + C$ , Lemma 2.3.3 implies that  $[f(B_i)]$  is an edge path of the form  $[f(B_i)] = a'_i b'_i c'_i$  where  $a'_i, c'_i$  are legal edge paths such that  $|a'_i|, |c'_i| \geq C$ . Moreover, the turn at  $f(\gamma_i)a'_i$  is legal. Since by Convention 3.2.1 every edge grows at least by a factor of  $\lambda'$ , this implies the required result.  $\square$

**Remark 3.2.4.** It is easy to see that statement and proof of Lemma 3.2.3 apply as well if  $\gamma$  is an edge path rather than a loop. Furthermore, we observe from the proof that any good subpath  $\gamma'$  of  $\gamma$  has the property that no edge of  $f(\gamma') = [f(\gamma')]$  is cancelled when  $f(\gamma)$  is reduced, and that it consists entirely of edges that are good in  $[f(\gamma)]$ .

On the other hand, for any reduced loop  $\gamma$  the number of bad edges is related to the number of illegal turns on  $\gamma$ , which we denote by  $ILT(\gamma)$ , via:

$$ILT(\gamma) \leq \#\{\text{bad edges in } \gamma\} \leq 2C \cdot ILT(\gamma) \quad (3.2.3)$$

Since the number of illegal turns on  $\gamma$  can only stay constant or decrease under iteration of the train track map, we obtain directly

$$\#\{\text{bad edges in } [f^t(\gamma)]\} \leq 2C \cdot ILT(\gamma) \leq 2C \cdot \#\{\text{bad edges in } \gamma\} \quad (3.2.4)$$

for all positive iterates  $f^t$  of  $f$ .

Notice however that the number of bad edges may actually grow (slightly) faster than the number of good edges under iteration of  $f$ , so that the goodness of  $\gamma$  does not necessarily grow monotonically under iteration of  $f$ . Nevertheless we have:

**Proposition 3.2.5.** (a) *There exists an integer  $s \geq 1$  such that for every reduced loop  $\gamma$  in  $\Gamma$  one has:*

$$\mathfrak{g}([f^s(\gamma)]) \geq \mathfrak{g}(\gamma)$$

In particular, for any integer  $t \geq 0$  one has

$$\mathfrak{g}([(f^s)^t(\gamma)]) \geq \mathfrak{g}(\gamma)$$

(b) If  $0 < \mathfrak{g}(\gamma) < 1$ , then for any integer  $s' > s$  we have

$$\mathfrak{g}([f^{s'}(\gamma)]) > \mathfrak{g}(\gamma)$$

and thus, for any  $t \geq 0$ :

$$\mathfrak{g}([(f^{s'})^t(\gamma)]) > \mathfrak{g}(\gamma)$$

*Proof.* (a) We set  $s \geq 1$  so that  $\lambda^s \geq 2C$  and obtain from Lemma 3.2.3 for the number  $g'$  of good edges in  $[f^s(\gamma)]$  and the number  $g$  of good edges in  $\gamma$  that:

$$g' \geq \lambda^s g$$

For the number  $b'$  of bad edges in  $[f^s(\gamma)]$  and the number  $b$  of bad edges in  $\gamma$  we have from equation (3.2.4) that:

$$b' \leq 2Cb$$

Thus we get

$$\frac{g'}{b'} \geq \frac{\lambda^s g}{2C b} \geq \frac{g}{b} \text{ and hence } \frac{g'}{g' + b'} \geq \frac{g}{g + b}$$

which proves  $\mathfrak{g}([f^s(\gamma)]) \geq \mathfrak{g}(\gamma)$ . The second inequality in the statement of the lemma follows directly from an iterative application of the first.

(b) The proof of part (b) follows from the above given proof of part (a), since  $\lambda^s > 2C$  unless there is no good edge at all in  $\gamma$ , which is excluded by our hypothesis  $0 < \mathfrak{g}(\gamma)$ . Since the hypothesis  $\mathfrak{g}(\gamma) < 1$  implies that there is at least one illegal turn in  $\gamma$ , in the above proof we get  $b \geq 1$ , which suffices to show

$$\frac{g'}{b'} > \frac{g}{b},$$

and thus  $\mathfrak{g}([f^s(\gamma)]) > \mathfrak{g}(\gamma)$ . □

From the inequalities at the end of part (a) of the above proof one derives directly, for  $g > 0$ , the inequality

$$\mathfrak{g}([f^s(\gamma)]) \geq \frac{1}{1 + \frac{2C}{\lambda^s} \left( \frac{1}{\mathfrak{g}(\gamma)} - 1 \right)}.$$

Hence we obtain:



**Corollary 3.2.6.** *Let  $\delta > 0$  and  $\epsilon > 0$  be given. Then there exist an integer  $M' = M'(\delta, \epsilon) \geq 0$  such that for any loop  $\gamma$  in  $\Gamma$  with  $\mathbf{g}(\gamma) \geq \delta$  we have  $\mathbf{g}([f^m(\gamma)]) \geq 1 - \epsilon$  for all  $m \geq M'$ .  $\square$*

### 3.2.2 Illegal turns and iteration of the train track map

The following lemma (and also other statements of this subsection) are already known in differing train track dialects (compare for example Lemma 4.2.5 in [6]); for convenience of the reader we include here a short proof. Recall the definition of an INP and a pre-INP from Definition 2.3.4.

**Lemma 3.2.7.** *Let  $f : \Gamma \rightarrow \Gamma$  be as in Convention 3.2.1. Then, there are only finitely many INP's and pre-INP's in  $\Gamma$  for  $f$ . Furthermore, there is an efficient method to determine them.*

*Proof.* We consider the set  $\mathcal{V}$  of all pairs  $(\gamma_1, \gamma_2)$  of legal edge paths  $\gamma_1, \gamma_2$  with common initial vertex but distinct first edges, which have combinatorial length  $|\gamma_1| = |\gamma_2| = C$ . Note that  $\mathcal{V}$  is finite.

From the definition of the cancellation bound and the inequalities (3.2.1) it follows directly that every INP or every pre-INP  $\eta$  must be a subpath of some path  $\bar{\gamma}_1 \circ \gamma_2$  with  $(\gamma_1, \gamma_2) \in \mathcal{V}$ . Furthermore, we define for  $i = 1$  and  $i = 2$  the initial subpath  $\gamma_i^*$  of  $\gamma_i$  to consist of all points  $x$  of  $\gamma_i$  that are mapped by some positive iterate  $f^t$  into the backtracking subpath at the tip of the unreduced path  $f^t(\bar{\gamma}_1 \circ \gamma_2)$ . We observe that the interior of any INP-subpath or pre-INP-subpath  $\eta$  of  $\bar{\gamma}_1 \circ \gamma_2$  must agree with the subpath  $\bar{\gamma}_1^* \circ \gamma_2^*$ . Thus any pair  $(\gamma_1, \gamma_2) \in \mathcal{V}$  can define at most one INP-subpath of  $\bar{\gamma}_1 \circ \gamma_2$ . Since  $\mathcal{V}$  is finite and easily computable, we obtain directly the claim of the lemma.  $\square$

**Convention 3.2.8.** From now on we assume in addition to Convention 3.2.1 that the edges of  $\Gamma$  have been subdivided in such a way that every endpoint of an INP or pre-INP is a vertex. By the finiteness result proved in Lemma 3.2.7 this can be done through introducing finitely many new vertices, while keeping the property that  $f$  maps vertices to vertices.

**Lemma 3.2.9.** *There exists an exponent  $M_1 \geq 1$  with the following property: Let  $\gamma$  be a path in  $\Gamma$ , and assume that it contains precisely two illegal turns, which are the tips of INP-subpaths or pre-INP-subpaths  $\eta_1$  and  $\eta_2$  of  $\gamma$ . If  $\eta_1$  and  $\eta_2$  overlap in a non-trivial subpath  $\gamma'$ , then  $f^{M_1}(\gamma)$  reduces to a path  $[f^{M_1}(\gamma)]$  which has at most one illegal turn.*

*Proof.* For any point  $x_1$  in the interior of one of the two legal branches of an INP or a pre-INP  $\eta$  there exists a point  $x_2$  on the other legal branch such that for a suitable positive power of  $f$  one has  $f^t(x_1) = f^t(x_2)$ . Hence, if we pick a point  $x$  in the interior of  $\gamma'$ , there are points  $x'$  on the other legal branch of  $\eta_1$  and  $x''$  on the other legal branch of  $\eta_2$  such that for some positive iterate of  $f$  one has  $f^t(x') = f^t(x) = f^t(x'')$ . It follows that the decomposition  $\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3$ , which uses  $x'$  and  $x''$  as concatenation points, defines legal subpaths  $\gamma_1$  and  $\gamma_3$  which yield  $[f^t(\gamma)] = [f^t(\gamma_1) \circ f^t(\gamma_3)]$ , which has at most one illegal turn.

Since by Convention 3.2.8 the overlap  $\gamma'$  is an edge path, it follows from the finiteness result proved in Lemma 3.2.7 that there are only finitely many constellations for  $\eta_1$  and  $\eta_2$ . This shows that there must be a bound  $M_1$  as claimed.  $\square$

**Lemma 3.2.10.** *For every train track map  $f : \Gamma \rightarrow \Gamma$  there exists a constant  $M_2 = M_2(\Gamma) \geq 0$  such that every path  $\gamma$  with precisely 1 illegal turn satisfies the following:*

*Either  $\gamma$  contains an INP or pre-INP as a subpath, or else  $[f^{M_2}(\gamma)]$  is legal.*

*Proof.* Similar to the set  $\mathcal{V}$  in the proof of Lemma 3.2.7 we define the set  $\mathcal{V}_+$  be the set of all pairs  $(\gamma_1, \gamma_2)$  of legal edge paths  $\gamma_1, \gamma_2$  in  $\Gamma$  which have combinatorial length  $0 \leq |\gamma_1| \leq C$  and  $0 \leq |\gamma_2| \leq C_f$ , and which satisfy:

The paths  $\gamma_1$  and  $\gamma_2$  have common initial point, and, unless one of them (or both) are trivial, they have distinct first edges. Note that  $\mathcal{V}_+$  is finite and contains  $\mathcal{V}$  as subset.

Following ideas from [19] we define a map

$$f_{\#} : \mathcal{V}_+ \rightarrow \mathcal{V}_+$$

induced by  $f$  through declaring the  $f_{\#}$ -image of a pair  $(\gamma_1, \gamma_2) \in \mathcal{V}_+$  to be the pair  $f_{\#}(\gamma_1, \gamma_2) := (\gamma_1'', \gamma_2'') \in \mathcal{V}_+$  which is defined by setting for  $i \in \{1, 2\}$

$$f(\gamma_i) =: \gamma_i' \circ \gamma_i'' \circ \gamma_i''' ,$$

where  $\gamma_i'$  is the maximal common initial subpath of  $f(\gamma_1)$  and  $f(\gamma_2)$ , where  $|\gamma_i''| \leq C$ , and where  $\gamma_i'''$  is non-trivial only if  $|\gamma_i''| = C$ .

Then for any  $(\gamma_1, \gamma_2) \in \mathcal{V}_+$  we see as in the proof of Lemma 3.2.7 that there exists an exponent  $t \geq 0$  such that the iterate  $f_{\#}^t(\gamma_1, \gamma_2) =: (\gamma_3, \gamma_4) \in \mathcal{V}_+$  satisfies one of the following:

1. One of  $\gamma_3$  or  $\gamma_4$  (or both) are trivial.
2. The turn defined by the two initial edges of  $\gamma_3$  and  $\gamma_4$  is legal.

3. The path  $\bar{\gamma}_3 \circ \gamma_4$  contains an INP as subpath.

From the finiteness of  $\mathcal{V}_+$  it follows directly that there is an upper bound  $M_2 \geq 0$  such that for  $t \geq M_2$  one of the above three alternatives must be true for  $f^t_\#(\gamma_1, \gamma_2) =: (\gamma_3, \gamma_4)$ .

Consider now the given path  $\gamma$ , and write it as illegal concatenation of two legal paths  $\gamma = \gamma'_1 \circ \gamma'_2$ . Then the maximal initial subpaths  $\gamma_1$  of  $\bar{\gamma}'_1$  and  $\gamma_2$  of  $\gamma'_2$  of length  $|\gamma_i| \leq C$  form a pair  $(\gamma_1, \gamma_2)$  in  $\mathcal{V}_+$ . In the above cases (1) or (2) it follows directly that  $f^t(\gamma)$  is legal. In alternative (3) the path  $f^t(\gamma)$  contains an INP.  $\square$

**Proposition 3.2.11.** *There exists an exponent  $r = r(f) \geq 0$  such that every finite path  $\gamma$  in  $\Gamma$  with  $ILT(\gamma) \geq 1$  satisfies*

$$ILT([f^r(\gamma)]) < ILT(\gamma),$$

*unless every illegal turn on  $\gamma$  is the tip of an INP or pre-INP subpath  $\eta_i$  of  $\gamma$ , where any two  $\eta_i$  are either disjoint subpaths on  $\gamma$ , or they overlap precisely in a common endpoint.*

*Proof.* Through considering maximal subpaths with precisely one illegal turn we obtain the claim as direct consequence of Lemma 3.2.10 and Lemma 3.2.9.  $\square$

**Remark 3.2.12.** From the same arguments we also deduce that for every path  $\gamma$  in  $\Gamma$  there is a positive iterate  $f^t(\gamma)$  which reduces to a path  $\gamma' := [f^t(\gamma)]$  which is *pseudo-legal*, meaning that it is a legal concatenation of legal paths and INP's. The analogous statement holds for loops instead of paths. The exponent  $t$  needed in either case depends only on the number of illegal turns in  $\gamma$  (or  $\gamma$ ).

Indeed, since the number of illegal turns in  $f^t(\gamma)$  non-strictly decreases for increasing  $t$ , we can assume that for sufficiently large  $t$  it stays constant. It follows from Lemma 3.2.10 (after possibly passing to a further power of  $f$ ) that every illegal turn of  $f^t(\gamma)$  is the tip of some INP-subpath  $\eta_i$  of  $f^t(\gamma)$ . From Lemma 3.2.9 we obtain furthermore that any two such  $\eta_i$  and  $\eta_j$  that are adjacent on  $f^t(\gamma)$  can not overlap non-trivially.

In the next subsection we also need the following lemma, where “illegal (cyclic) concatenation” means that the path (loop) is a concatenation of subpaths where all concatenation points must be illegal turns.

**Lemma 3.2.13.** *Let  $\gamma$  be a reduced loop in  $\Gamma$  and let  $\gamma' = [f(\gamma)]$  be its reduced image loop. Assume that for some  $t \geq 1$  a decomposition  $\gamma' = \gamma'_1 \circ \gamma'_2 \circ \dots \circ \gamma'_t$  as illegal cyclic concatenation is given. Then there exists a decomposition as illegal cyclic concatenation  $\gamma = \gamma_1 \circ \gamma_2 \circ \dots \circ \gamma_t$*

with the property that the reduced image paths  $[f(\gamma_i)]$  contain the paths  $\gamma'_i$  as subpaths, for  $i = 1, 2, \dots, t$ .

*Proof.* We cut the loop  $\gamma$  at some illegal turn to get a (closed) path  $\gamma = e_1 e_2 \dots e_q$ . We consider the initial subpaths  $\gamma(k) = e_1 e_2 \dots e_k$  of  $\gamma$  for  $k = 1, 2, \dots$ . Let  $k'$  be the smallest positive integer such that the reduced image path  $[f(\gamma(k'))]$  contains the path  $\gamma'_1$  as a subpath. Since  $k'$  is the smallest such integer, it follows that the path  $f(e_{k'})$  passes through the last edge of  $\gamma'_1$ . Note that  $f(e_{k'})$  is a legal path and that  $\gamma'_i$  terminates at an illegal turn, so that the endpoint of  $f(e_{k'})$  must lie somewhere in the backtracking subpath of the possibly unreduced path  $[f(e_1 e_2 \dots e_{k'})] \circ [f(e_{k'+1} e_{k'+2} \dots e_q)]$ . It follows that the reduced image path  $[f(e_{k'+1} e_{k'+2} \dots e_q)]$  contains the path  $\gamma'_2 \circ \dots \circ \gamma'_t$  as subpath.

Thus we define  $\gamma_1 := \gamma(k')$ , and proceed iteratively in precisely the same fashion, thus finding iteratively  $\gamma_2, \gamma_3, \dots, \gamma_{t-1}$ . As above, it follows that the “left-over” terminal subpath of  $\gamma$  has the property that its reduced image contains  $\gamma'_t$  as subpath, so that we can define this left-over subpath to be the final factor path  $\gamma_t$ .  $\square$

### 3.2.3 Goodness versus illegal turns

We recall from Definition 2.3.4 that a multi-INP is a legal concatenation of finitely many INPs along their endpoints.

**Convention 3.2.14.** For the rest of this section we assume that  $f : \Gamma \rightarrow \Gamma$  the train-track map as in Convention 3.2.8, and that there is an upper bound  $A(f)$  to the number of INP factor paths in any multi-INP path  $\gamma \in \Gamma$ , which is equal to  $ILT(\gamma)$  in this case.

Note that this condition implies in particular that in  $\Gamma$  there can not be a non-trivial loop which is a cyclic legal concatenation of INPs.

**Lemma 3.2.15.** *Any expanding train track map  $f : \Gamma \rightarrow \Gamma$ , that represents an atoroidal outer automorphism satisfies the hypotheses of Convention 3.2.14.*

*Proof.* There is an upper bound  $A \geq 0$  for number of INPs in  $\Gamma$ , see Lemma 3.2.7. Hence, any multi-INP with more than  $A(f) := 2A$  factors would have to run over the same INP twice in the same direction. Thus we obtain as subpath a non-trivial loop which is fixed by some positive power of  $f$ , violating the given assumption.  $\square$

**Lemma 3.2.16.** *There exists an exponent  $s \geq 1$  such that for any reduced path  $\gamma \in \Gamma$  with reduced image path  $\gamma' = [f^s(\gamma)]$  the following holds: If  $\mathfrak{g}(\gamma') = 0$  and satisfies  $ILT(\gamma') \geq A(f) + 1$ , then*

$$ILT(\gamma) > ILT(\gamma').$$

*Proof.* Since the map  $f$  is expanding, for the critical constant  $C \geq 0$  given in Definition 3.2.2 there exists an integer  $s \geq 1$  such that  $|f^s(e)| \geq 2C + 1$  for every edge  $e \in \Gamma$ . We can furthermore assume  $s \geq r$ , where  $r = r(f)$  is given by Lemma 3.2.11. Thus, by Lemma 3.2.11 we have

$$ILT(\gamma) > ILT(\gamma'),$$

unless every illegal turn on  $\gamma$  is the tip of an INP or pre-INP  $\eta_i$ , and any two such subpaths  $\eta_i$  are disjoint or they overlap precisely at a common endpoint.

The case where two such paths are disjoint can be excluded as follows: If there is an edge  $e$  in  $\gamma$  which doesn't belong to any of the  $\eta_i$ , then  $f^s(e)$  is a legal subpath of  $[f^s(\gamma)] = \gamma'$  of length greater than  $2C + 1$ , contradicting the assumption that  $\mathfrak{g}(\gamma') = 0$ .

Thus we can assume that any two of the subpaths  $\eta_i$  overlap precisely in a common endpoint, and that there is no non-trivial initial or final subpath of  $\gamma$  outside of the concatenation of all the  $\eta_i$ . Therefore, for some  $s' \geq 0$ , the iterate  $f^{s'}(\gamma')$  is a multi-INP with  $ILT(\gamma') \geq A(f) + 1$  factors, which contradicts Convention 3.2.14. Hence the conclusion of the Lemma follows.  $\square$

**Lemma 3.2.17.** *Let  $A(f)$  be the upper bound for the number of INP factors in any multi-INP, as given by Convention 3.2.14. Then there exists a constant  $\delta$  with  $0 < \delta \leq 1$  so that the following holds: Every reduced loop  $\hat{\gamma}$  in  $\Gamma$  with  $ILT(\hat{\gamma}) \geq A(f) + 1$  can be written as cyclic illegal concatenation*

$$\hat{\gamma} = \gamma_1 \circ \gamma_2 \dots \circ \gamma_{2m}$$

*such that for every odd index  $j$ , the subpath  $\gamma_j$  is either trivial or satisfies:*

$$\mathfrak{g}(\gamma_j) \geq \delta$$

*For every even index  $k$  the subpath  $\gamma_k$  is non-trivial and satisfies  $\mathfrak{g}(\gamma_k) = 0$ ; moreover, we have:*

$$A(f) + 1 \leq ILT(\gamma_k) \leq 2A(f) + 1$$

*Proof.* Let  $L_\gamma$  be the collection of maximal legal subpaths  $\gamma'_i$  of  $\gamma$  of length  $|\gamma'_i| \geq 2C + 1$ , for  $C \geq 0$  as given in Definition 3.2.2. Note that any two distinct elements  $\gamma'_i, \gamma'_j$  in this

collection are either disjoint or overlap at at single point. In the latter case the turn at the overlap point is illegal, as otherwise they would merge into a longer legal subpath, which violates the maximality of the  $\gamma'_i$ . For any two (on  $\gamma$ ) consecutive paths  $\gamma'_i$  and  $\gamma'_{i+1}$  in  $L_\gamma$ , if the path  $\beta_i$  between them is trivial or satisfies  $ILT(\beta_i) \leq A(f)$ , then we erase  $\gamma'_i$  and  $\gamma'_{i+1}$  from the collection  $L_\gamma$  and add in the new path  $\gamma'_i\beta_i\gamma'_{i+1}$ . We continue this process iteratively until all the complementary subpaths  $\beta_i$  between any two consecutive elements in our collection satisfies  $ILT(\beta_i) \geq A(f) + 1$ . We call the obtained collection of subpaths  $C_\gamma$ .

We now pick a path  $\gamma_j$  in the collection  $C_\gamma$  and consider its “history” as being obtained iteratively through joining what amounts to  $\ell$  paths  $\gamma'_i$  from  $L_\gamma$ . Thus  $\gamma_j$  can be written as illegal concatenation

$$\gamma_j = \gamma'_1 \circ \beta_1 \circ \gamma'_2 \circ \dots \circ \gamma'_{\ell-1} \circ \beta_{\ell-1} \circ \gamma'_\ell,$$

where each  $\gamma'_i$  is legal and of length  $|\gamma'_i| \geq 2C + 1$ . Each  $\beta_i$  has at most  $A(f)$  illegal turns, and the legal subpaths of  $\beta_i$  between these illegal turns have length  $\leq 2C$ , so that we get  $|\beta_i| \leq (A(f) + 1)2C$ . Thus the set of good edges on  $\gamma_j$  is given precisely as disjoint union over all the  $\gamma'_i$  of the sets of the  $|\gamma'_i| - 2C$  edges of  $\gamma'_i$  that are not on the two boundary subpaths of length  $C$  of  $\gamma'_i$ . Hence we compute for the goodness:

$$\begin{aligned} \mathfrak{g}(\gamma_j) &= \frac{\sum_{i=1}^{\ell} (|\gamma'_i| - 2C)}{|\gamma_j|} = \frac{\sum_{i=1}^{\ell} (|\gamma'_i| - 2C)}{\sum_{i=1}^{\ell} |\gamma'_i| + \sum_{i=1}^{\ell-1} |\beta_i|} \\ &\geq \frac{\ell}{(2C + 1) \cdot \ell + (A(f) + 1)2C \cdot \ell} = \frac{1}{2C(A(f) + 2) + 1} \end{aligned}$$

We finally add to the collection  $C_\gamma$  a suitable set of trivial subpaths at illegal turns to get a collection  $C'_\gamma$ , where these trivial paths are chosen as to get as complementary subpaths of  $C'_\gamma$  only subpaths  $\gamma_k$  which satisfy:

$$A(f) + 1 \leq ILT(\gamma_k) \leq 2A(f) + 1$$

Thus setting

$$\delta = \frac{1}{2C(A(f) + 2) + 1}$$

finishes the proof. □

**Proposition 3.2.18.** *Let  $s \geq 1$  be the integer given by Lemma 3.2.16 and  $\delta > 0$  be the constant given by Lemma 3.2.17. Then there exists a constant  $R > 1$  such that for any*

reduced loop  $\gamma$  in  $\Gamma$  either

$$(1) \quad \mathfrak{g}([f^s(\gamma)]) \geq \frac{\delta}{2}$$

or

$$(2) \quad ILT(\gamma) \geq R \cdot ILT([f^s(\gamma)]).$$

*Proof.* By Remark 3.2.12 there is an exponent  $t \geq 0$  such that for any loop  $\gamma$  with less than  $A(f) + 1$  illegal turns the loop  $\gamma' = [f^t(\gamma)]$  is pseudo-legal. From the assumption that  $f$  satisfies Convention 3.2.14 it follows that  $\gamma'$  is not a legal concatenation of INPs, so that it must have at least one good edge. Since iteration of  $f$  only decreases the number of illegal turns, we obtain from equality (3.2.3) that

$$\mathfrak{g}(\gamma) \geq \frac{1}{2C(A(f) + 1) + 1}.$$

Thus the first inequality from our assertion follows for a suitable choice of  $s$  from Proposition 3.2.5.

Thus we can now assume that  $ILT(\gamma) \geq A(f) + 1$ , and thus that  $[f^s(\gamma)] = \gamma'_1 \circ \dots \circ \gamma'_{2m}$  is a decomposition as given by Lemma 3.2.17. There are two cases to consider: Assume that

$$\sum_{j \text{ odd}} |\gamma'_j| \geq \sum_{k \text{ even}} |\gamma'_k|.$$

For any odd index  $j$  the non-trivial path  $\gamma'_j$  has  $\mathfrak{g}(\gamma'_j) \geq \delta$ , which together with the last inequality implies:

$$\mathfrak{g}([f^s(\gamma)]) \geq \frac{\delta}{2}$$

Now, assume that:

$$\sum_{j \text{ odd}} |\gamma'_j| \leq \sum_{k \text{ even}} |\gamma'_k|$$

Let  $\gamma = \gamma_1 \circ \dots \circ \gamma_{2m}$  be an illegal cyclic concatenation given by Lemma 3.2.13 so that, for each  $i = 1, \dots, 2m$ , the reduced image  $[f^s(\gamma_i)]$  contains  $\gamma'_i$  as a subpath. Hence, for every even index, we apply Lemma 3.2.16 to see that

$$ILT(\gamma_k) > ILT([f^s(\gamma_k)]) \geq ILT(\gamma'_k).$$

Since the number of illegal turns never increases when applying a train-track map, for each

odd index  $j$ , we have

$$ILT(\gamma_j) \geq ILT([f^s(\gamma_j)]) \geq ILT(\gamma'_j).$$

Combining last two inequalities we obtain:

$$ILT(\gamma) > ILT([f^s(\gamma)]) + m. \quad (3.2.5)$$

We also observe that the number of illegal turns in  $[f^s(\gamma)]$  is equal to the sum of the number of illegal turns in the odd indexed subpaths, the number of illegal turns in the even indexed subpaths, and the number of illegal turns at concatenation points:

$$ILT([f^s(\gamma)]) \leq \sum_{j \text{ odd}} ILT(\gamma'_j) + \sum_{k \text{ even}} ILT(\gamma'_k) + 2m \quad (3.2.6)$$

Now, note that

$$\sum_{j \text{ odd}} ILT(\gamma'_j) \leq \sum_{j \text{ odd}} |\gamma'_j| \leq \sum_{k \text{ even}} |\gamma'_k|, \quad (3.2.7)$$

by assumption.

For each  $\gamma'_k$  (with even index  $k$ ), since Lemma 3.2.17 assures  $\mathfrak{g}(\gamma'_k) = 0$  and  $ILT(\gamma'_k) \leq 2(A(f) + 1)$ , we have

$$|\gamma'_k| \leq 2C(2(A(f) + 1) + 1),$$

which together with (3.2.7) implies that:

$$\sum_{j \text{ odd}} ILT(\gamma'_j) \leq 2mC(2(A(f) + 1) + 1) \quad (3.2.8)$$

Furthermore, from  $ILT(\gamma'_k) \leq 2(A(f) + 1)$  for even index  $k$  we deduce

$$\sum_{k \text{ even}} ILT(\gamma'_k) \leq 2m(A(f) + 1) \quad (3.2.9)$$

Using (3.2.6), (3.2.8) and (3.2.9) we obtain:

$$ILT([f^s(\gamma)]) \leq 2mC(2(A(f) + 1) + 1) + 2m(A(f) + 1) + 2m \quad (3.2.10)$$



Using (3.2.5) and (3.2.10), we have:

$$\begin{aligned}
\frac{ILLT(\gamma)}{ILLT([f^s(\gamma)])} &\geq 1 + \frac{m}{ILLT([f^s(\gamma)])} \\
&\geq 1 + \frac{m}{2mC(2(A(f)+1)+1) + 2m(A(f)+1) + 2m} \\
&\geq 1 + \frac{1}{2C(2(A(f)+1)+1) + 2(A(f)+1) + 2}
\end{aligned}$$

Therefore, the conclusion of the lemma holds for:

$$R = 1 + \frac{1}{2C(2(A(f)+1)+1) + 2(A(f)+1) + 2}$$

□

### 3.2.4 Uniform goodness in the future or the past

As before, we consider in this subsection train track maps that satisfy Convention 3.2.14.

For the convenience of the reader we first prove a mild generalization of Proposition 3.2.18, which will be a crucial ingredient in the proof of the next proposition.

**Proposition 3.2.19.** *Given any constants  $0 < \delta_1 < 1$  and  $R_1 > 1$ , there exist an integer  $s_1 > 0$  so that for any reduced loops  $\gamma$  and  $\gamma'$  in  $\Gamma$  with  $[f^{s_1}(\gamma')] = \gamma$  one has either*

$$(i) \mathfrak{g}([f^{s_1}(\gamma)]) \geq \delta_1$$

or

$$(ii) ILLT(\gamma') \geq R_1 \cdot |\gamma|_\Gamma.$$

*Proof.* We first replace  $f$  by a positive power (say  $f^r$ , cited at the end of the proof) as given by Proposition 3.2.5 so that for the rest of the proof we can assume that goodness is monotone. Let  $s, \delta$  and  $R$  be as in Lemma 3.2.18 and let  $\gamma$  be any reduced loop in  $\Gamma$ . Assume that for  $\gamma$  that alternative (1) of Lemma 3.2.18 holds, i.e.  $\mathfrak{g}([f^s(\gamma)]) \geq \frac{\delta}{2}$ . Then by Corollary 3.2.6 there is an exponent  $M \geq 1$  such that

$$\mathfrak{g}([(f^s)^m(\gamma)]) \geq \delta_1$$

for all  $m \geq M$ . On the other hand, if  $\mathfrak{g}([f^s(\gamma)]) < \frac{\delta}{2}$ , then Lemma 3.2.18 assures that

$$ILT(\gamma) \geq R \cdot ILT([f^s(\gamma)]).$$

We now claim that

$$ILT(\gamma_1) \geq R \cdot ILT(\gamma)$$

for any reduced loop  $\gamma_1$  with  $[f^s(\gamma_1)] = \gamma$ . To see this, apply Lemma 3.2.18 to the loop  $\gamma_1$ . If one had

$$\mathfrak{g}([f^s(\gamma_1)]) = \mathfrak{g}(\gamma) \geq \frac{\delta}{2},$$

then by monotonicity of goodness this would also imply  $\mathfrak{g}([f^s(\gamma)]) \geq \frac{\delta}{2}$ , which contradicts with our assumption. Hence for  $\gamma_1$  alternative (2) of Lemma 3.2.18 holds, giving indeed

$$ILT(\gamma_1) \geq R \cdot ILT(\gamma).$$

Repeating the same argument iteratively shows that for any sequence of reduced loops  $\gamma_{M'}$ , defined iteratively through  $[f^s(\gamma_{M'})] = \gamma_{M'-1}$  for any  $M' \geq 2$ , the inequality

$$\gamma_{M'} \geq R^{M'} \cdot ILT(\gamma).$$

holds. Also, notice that since  $\mathfrak{g}(\gamma) < \frac{\delta}{2}$  we have

$$\frac{\#\{\text{bad edges in } \gamma\}}{|\gamma|_{\Gamma}} \geq 1 - \frac{\delta}{2},$$

and by the inequalities (3.2.3) we have furthermore

$$\#\{\text{bad edges in } \gamma\} \leq 2C \cdot ILT(\gamma).$$

Hence we obtain

$$\begin{aligned} ILT(\gamma_{M'}) &\geq R^{M'} \cdot ILT(\gamma) \\ &\geq R^{M'} \cdot \frac{\#\{\text{bad edges in } \gamma\}}{2C} \\ &\geq R^{M'} \cdot \left(1 - \frac{\delta}{2}\right) \frac{1}{2C} |\gamma|_{\Gamma} \end{aligned}$$

Let  $M' \geq 1$  be such that  $R^{M'} \cdot \left(1 - \frac{\delta}{2}\right) \frac{1}{2C} \geq R_1$ . Then  $s_1 = \max\{M, M'\}$  satisfies the

requirements of the statement of Proposition 3.2.19 up to replacing  $f$  by the power  $f^r$  as done at the beginning of the proof.  $\square$

In addition to the train track map  $f$  we now consider a similar train track map  $g : \Gamma' \rightarrow \Gamma'$ , i.e.  $g$  satisfies the requirements of Conventions 3.2.1, 3.2.8 and 3.2.14, which is related to  $f$  via maps  $h : \Gamma \rightarrow \Gamma'$  and  $h' : \Gamma' \rightarrow \Gamma$  such that  $f$  and  $h'g'h$  are homotopy inverses. We also assume that (with respect to the simplicial metrics) the lift to the universal coverings of  $h$  and  $h'$  are quasi-isometries, so that there exists a bi-Lipschitz constant  $B > 0$  which satisfies for any two “corresponding” reduced loops  $\gamma$  in  $\Gamma$  and  $\gamma' := [h(\gamma)]$  the inequalities

$$\frac{1}{B} |\gamma'|_{\Gamma'} \leq |\gamma|_{\Gamma} \leq B |\gamma'|_{\Gamma'} \quad (3.2.11)$$

We denote the goodness for the map  $g$  by  $\mathfrak{g}'$ , and the critical constant for  $g$  from Definition 3.2.2 by  $C'$ .

**Proposition 3.2.20.** *Given a real number  $\delta$  so that  $0 < \delta < 1$ , there exist a bound  $M > 0$  such that (up to replacing  $f$  and  $g$  by a common power) for any pair of corresponding reduced loops  $\gamma$  in  $\Gamma$  and  $\gamma'$  in  $\Gamma'$ , either*

$$(a) \quad \mathfrak{g}([f^n(\gamma)]) \geq \delta$$

or

$$(b) \quad \mathfrak{g}'([g^n(\gamma')]) \geq \delta$$

holds for all  $n \geq M$ .

*Proof.* Let  $B$  be a bi-Lipschitz constant for the transition from  $\Gamma$  to  $\Gamma'$  as in (3.2.11) above. Set  $R_1 = 4C'B^2$ , and apply Proposition 3.2.19 to the loop  $\gamma$ . Assume first that alternative (i) of this proposition holds. Then Corollary 3.2.6, applied to  $f^{s_1}$ , gives a bound  $L \geq 0$  so that inequality (a) holds for all  $n \geq L$  (after having replaced  $f$  by  $f^{s_1}$ ).

Now assume that for  $\gamma$  alternative (ii) of Proposition 3.2.19 holds. Then we have the following inequalities, where  $\gamma''$  denotes the reduced loop in  $\Gamma$  corresponding to  $g^{s_1}(\gamma')$  (which implies  $[f^{s_1}(\gamma'')] = \gamma$ ):

$$\begin{aligned} |g^{s_1}(\gamma')|_{\Gamma'} &\geq \frac{1}{B} |\gamma''|_{\Gamma} \geq \frac{1}{B} ILT(\gamma'') \\ &\geq \frac{1}{B} R_1 \cdot |\gamma|_{\Gamma} = 4C'B \cdot |\gamma|_{\Gamma} \\ &= 4C' \cdot |\gamma'|_{\Gamma'} \end{aligned}$$

This however implies that  $\mathfrak{g}'(g^{s_1}(\gamma')) \geq 1/2$ , since for any  $t \geq 0$  the number of bad edges in  $g^t(\gamma')$  is bounded above by

$$2C' \cdot ILT(g^t(\gamma')) \leq 2C' \cdot ILT(\gamma') \leq 2C' \cdot |\gamma'|_{\Gamma'}.$$

Thus, by invoking Corollary 3.2.6 again, there exist  $L'$  such that

$$\mathfrak{g}'(g^{s_1 L'}(\gamma')) \geq \delta.$$

Hence for  $M' = \max\{L, L'\}$  the conclusion of the Lemma follows.  $\square$

### 3.2.5 Convergence estimates and the dynamics

Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be an atoroidal outer automorphism which is represented by a train track representative  $f : \Gamma \rightarrow \Gamma$  as in Convention 3.2.14. Let  $[w]$  be a conjugacy class in  $\mathbb{F}_N$ . Represent  $[w]$  by a reduced loop  $\gamma$  in  $\Gamma$ . Then the goodness of  $w$ , denoted by  $\mathfrak{g}([w])$ , is defined by:

$$\mathfrak{g}([w]) := \mathfrak{g}(\gamma)$$

**Lemma 3.2.21.** *Given a neighborhood  $U$  of the simplex of attraction  $\Delta_+(\varphi) \in \mathbb{P}\text{Curr}(\mathbb{F}_N)$ , there exist a bound  $\delta > 0$  and an integer  $M = M(U) \geq 1$  such that, for any  $[w] \in \mathbb{F}_N$  with  $\mathfrak{g}([w]) \geq \delta$ , we have*

$$(\varphi^M)^n[\eta_w] \in U$$

for all  $n \geq 1$ .

*Proof.* We first replace  $\varphi$  by a positive power as in Proposition 3.2.5 so that the goodness function for the train-track map  $f : \Gamma \rightarrow \Gamma$  becomes monotone.

Recall from Section 2.4 that  $[\nu] \in \mathbb{P}\text{Curr}(\mathbb{F}_N)$  is close to  $[\nu'] \in \mathbb{P}\text{Curr}(\mathbb{F}_N)$  if there exists  $\epsilon > 0$  and  $R \gg 0$  such that for all reduced edge paths  $\gamma$  with  $|\gamma| \leq R$  we have

$$\left| \frac{\langle \gamma, \nu \rangle}{\|\nu\|_{\Gamma}} - \frac{\langle \gamma, \nu' \rangle}{\|\nu'\|_{\Gamma}} \right| < \epsilon.$$

Thus, since  $\Delta_+(\varphi)$  is compact, there exist  $\epsilon > 0$  and  $R \in \mathbb{R}$  such that the above inequalities imply for  $\nu' \in \Delta_+(\varphi)$  that  $\nu \in U$ .

We proved the pointwise convergence for edges in Proposition 2.7.1. Since there are only finitely many edges and finitely many edge paths  $\gamma$  in  $\Gamma$  with  $|\gamma| \leq R$ , we can pick an integer

$M_0 \geq 0$  such that

$$\left| \frac{\langle \gamma, f^n(e) \rangle}{|f^n(e)|} - \langle \gamma, \mu_+(e) \rangle \right| < \epsilon/4 \quad (3.2.1)$$

for all  $n \geq M_0$ , for all  $\gamma$  with  $|\gamma| \leq R$  and for all edges  $e$  of  $\Gamma$ .

Let  $\lambda', \lambda'' > 1$  be the minimal and the maximal expansion factors respectively as given in Convention 3.2.1. For any reduced loop  $c$  in  $\Gamma$  with  $\mathbf{g}(c) \geq \frac{1}{1+\epsilon/4}$  iterative application of the fact, that  $f$  maps any good edge in  $c$  to a path in  $[f(c)]$  which consists entirely of good edges and has length at least  $\lambda'$ , implies:

$$|f^n(c)| \geq \frac{1}{1+\epsilon/4} |c| (\lambda')^n.$$

Thus for any integer  $M_1 > \log_{\lambda'} R(1 + \frac{4}{\epsilon})$  and all  $n \geq M_1$  we get the following inequalities:

$$\frac{R|c|}{|f^n(c)|} \leq \frac{R|c|_{\Gamma}(1 + \epsilon/4)}{|c|(\lambda')^n} \leq \frac{R(1 + \epsilon/4)}{(\lambda')^{M_1}} \leq \frac{R(1 + \epsilon/4)}{R(1 + 4/\epsilon)} < \frac{\epsilon}{4} \quad (3.2.2)$$

We note that for any integer  $m \geq 1$  and for each edge the minimum expansion factor for  $f^m$  is at least  $(\lambda')^m$  and the maximum expansion factor for  $f^m$  is at most  $(\lambda'')^m$ .

For the rest of the proof set

$$M = \max\{M_0, M_1\}$$

and

$$\delta := \max\left\{ \frac{1}{1 + \epsilon/4}, \frac{1}{1 + (\frac{\lambda'}{\lambda''})^M \epsilon/4} \right\},$$

and let  $c$  be a reduced loop in  $\Gamma$  which represents a conjugacy class  $w$  with  $\mathbf{g}(w) \geq \delta$ .

The assertion of Lemma 3.2.21 now follows if we show that for all integers  $n \geq 1$  the current  $(\varphi^M)^n([\eta_w])$  is  $(\epsilon, R)$ -close (in the above sense) to some point in  $\Delta_+(\varphi)$ . Indeed, since by the first paragraph of the proof the goodness function is monotone, it suffices to assume  $n = 1$  and apply the resulting statement iteratively.

For simplicity we denote from now on  $f^M$  by  $f$ . Another auxiliary computation gives:

$$\frac{(\lambda'')^M \cdot \#\{\text{bad edges in } c\}}{(\lambda')^M \cdot \#\{\text{good edges in } c\}} = \frac{(\lambda'')^M (1 - \mathbf{g}(w)) |c|}{(\lambda')^M \mathbf{g}(w) |c|} = \left(\frac{\lambda''}{\lambda'}\right)^M \left(\frac{1}{\mathbf{g}(w)} - 1\right) \leq \left(\frac{\lambda''}{\lambda'}\right)^M \left(\frac{1}{\delta} - 1\right) \leq \frac{\epsilon}{4}. \quad (3.2.3)$$

We now write  $c = c_1 c_2 \dots b_1 \dots c_r c_{r+1} \dots b_2 \dots b_k c_s$ , where the  $c_i$  denote good edges and the  $b_j$  denote maximal bad subpaths of  $c$ . Note for the second of the inequalities below that the definition of ‘‘good’’ implies that there can be no cancellation in  $f(c)$  between adjacent

$f(c_i)$  and  $f(c_{i+1})$  nor between adjacent  $f(c_i)$  and  $f(b_j)$ , see Remark 3.2.4.

Then we calculate:

$$\begin{aligned}
& \left| \frac{\langle \gamma, f(c) \rangle}{|f(c)|} - \frac{\langle \gamma, |f(c_1)|\mu_+(c_1) + \dots + |f(c_s)|\mu_+(c_s) \rangle}{\sum_{i=1}^s |f(c_i)|} \right| \\
& \leq \left| \frac{\langle \gamma, f(c) \rangle}{|f(c)|} - \sum_{i=1}^s \frac{\langle \gamma, f(c_i) \rangle}{|f(c)|} \right| \\
& + \left| \sum_{i=1}^s \frac{\langle \gamma, f(c_i) \rangle}{|f(c)|} - \frac{\sum_{i=1}^s \langle \gamma, f(c_i) \rangle}{\sum_{i=1}^s |f(c_i)|} \right| \\
& + \left| \frac{\sum_{i=1}^s \langle \gamma, f(c_i) \rangle}{\sum_{i=1}^s |f(c_i)|} - \frac{\langle \gamma, |f(c_1)|\mu_+(c_1) + \dots + |f(c_s)|\mu_+(c_s) \rangle}{\sum_{i=1}^s |f(c_i)|} \right| \\
& \leq \frac{2R|c|}{|f(c)|} + \sum_{j=1}^k \frac{\langle \gamma, [f(b_j)] \rangle}{|f(c)|} \\
& + \left| \frac{\sum_{i=1}^s \langle \gamma, f(c_i) \rangle}{\sum_{i=1}^s |f(c_i)| + \sum_{j=1}^k |[f(b_j)]|} - \frac{\sum_{i=1}^s \langle \gamma, f(c_i) \rangle}{\sum_{i=1}^s |f(c_i)|} \right| \\
& + \left| \frac{\sum_{i=1}^s \langle \gamma, f(c_i) \rangle}{\sum_{i=1}^s |f(c_i)|} - \frac{\langle \gamma, |f(c_1)|\mu_+(c_1) + \dots + |f(c_s)|\mu_+(c_s) \rangle}{\sum_{i=1}^s |f(c_i)|} \right| \\
& < \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon.
\end{aligned}$$

Here the first inequality is just a triangle inequality. In the second inequality the last two terms are unchanged, and first two terms come from the first term in the previous quantity and follows from counting frequencies as follows: An occurrence of an edge path  $\gamma$  or its inverse can occur either inside the image of a good edge  $f(c_i)$  or inside of the image of a bad segment  $[f(b_j)]$ , or it might cross over the concatenation points. This observation gives the claimed inequality. In the final inequality, the first  $\epsilon/4$  follows from the equation (3.2.2). The second one follows from the equation (3.2.3) as follows:

$$\sum_{j=1}^k \frac{\langle \gamma, [f(b_j)] \rangle}{|f(c)|} \leq \sum_{j=1}^k \frac{|[f(b_j)]|}{|f(c)|} \leq \sum_{j=1}^k \frac{|b_j|(\lambda'')^M}{(\lambda')^M \mathfrak{B} \cdot \#\{\text{good edges in } c\}} \leq \frac{\epsilon}{4}.$$

The third  $\epsilon/4$  comes from the following observation:

$$\begin{aligned}
& \left| \frac{\sum_{i=1}^s \langle \gamma, f(c_i) \rangle}{\sum_{i=1}^s |f(c_i)| + \sum_{j=1}^k |[f(b_j)]|} - \frac{\sum_{i=1}^s \langle \gamma, f(c_i) \rangle}{\sum_{i=1}^s |f(c_i)|} \right| \\
&= \left| \frac{(\sum_{i=1}^s \langle \gamma, f(c_i) \rangle)(\sum_{j=1}^k |[f(b_j)]|)}{(\sum_{i=1}^s |f(c_i)|)(\sum_{i=1}^s |f(c_i)| + \sum_{j=1}^k |[f(b_j)]|)} \right| \\
&\leq \left| \frac{\sum_{i=1}^s \langle \gamma, f(c_i) \rangle}{\sum_{i=1}^s |f(c_i)|} \cdot \frac{\sum_{j=1}^k |[f(b_j)]|}{\sum_{i=1}^s |f(c_i)|} \right| \\
&\leq \frac{(\lambda'')^M \sum_{j=1}^k |b_j|}{(\lambda')^M \sum_{i=1}^s |c_i|} \leq \epsilon/4
\end{aligned}$$

by (3.2.3).

Finally, the last  $\epsilon/4$  can be verified using (3.2.1) as follows:

$$\begin{aligned}
& \left| \frac{\sum_{i=1}^s \langle \gamma, f(c_i) \rangle}{\sum_{i=1}^s |f(c_i)|} - \frac{\langle \gamma, |f(c_1)|\mu_+(c_1) + \dots + |f(c_s)|\mu_+(c_s) \rangle}{\sum_{i=1}^s |f(c_i)|} \right| \\
&= \left| \frac{\sum_{i=1}^s |f(c_i)| \left( \frac{\langle \gamma, f(c_i) \rangle}{|f(c_i)|} - \langle \gamma, \mu_+(c_i) \rangle \right)}{\sum_{i=1}^s |f(c_i)|} \right| \\
&\leq \frac{\sum_{i=1}^s |f(c_i)| \epsilon/4}{\sum_{i=1}^s |f(c_i)|} = \epsilon/4
\end{aligned}$$

Since after applying  $\varphi^M$  to a conjugacy class with  $\mathbf{g}(w) \geq \delta$  we still have  $\mathbf{g}(\varphi^M(w)) \geq \delta$  we get  $(\varphi^M)^n([\eta_w]) \in U$  for all  $n \geq 1$ .  $\square$

**Lemma 3.2.22.** *For any  $\delta > 0$  and any neighborhood  $U$  of  $\Delta_+(\varphi)$  there exists an integer  $M(\delta, U) > 0$  such that (up to replacing  $\varphi$  by a positive power) for any conjugacy class  $w$  with goodness  $\mathbf{g}(w) > \delta$  we have*

$$\varphi^n[\eta_w] \in U$$

for all  $n \geq M$ .

*Proof.* This is a direct consequence of Corollary 3.2.6 and Lemma 3.2.21.  $\square$

**Proposition 3.2.23.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be a hyperbolic outer automorphism such that  $\varphi$  and  $\varphi^{-1}$  both admit properly expanding train-track representatives.*

Given neighborhoods  $U$  of the simplex of attraction  $\Delta_+(\varphi)$  and  $V$  of the simplex of repulsion  $\Delta_-(\varphi)$  in  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$ , then (up to replacing  $\varphi$  by a positive power) there exist an integer  $M \geq 0$  such that for any conjugacy class  $w \in \mathbb{F}_N$  we have

$$\varphi^n[\eta_w] \in U \quad \text{or} \quad \varphi^{-n}[\eta_w] \in V$$

for all  $n \geq M$ .

*Proof.* This is a direct consequence of Proposition 3.2.20 and Lemma 3.2.22.  $\square$

The following result also proves Theorem 1.3.1 from the Introduction:

**Theorem 3.2.24.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be a hyperbolic outer automorphism such that  $\varphi$  and  $\varphi^{-1}$  admit absolute train-track representatives. Then,  $\varphi$  acts on  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  with uniform north-south dynamics from  $\Delta_-(\varphi)$  to  $\Delta_+(\varphi)$ :*

*Given an open neighborhood  $U$  of the simplex of attraction  $\Delta_+(\varphi)$  and a compact set  $K \subset \mathbb{P}\text{Curr}(\mathbb{F}_N) \setminus \Delta_-(\varphi)$  there exists an integer  $M > 0$  such that  $\varphi^n(K) \subset U$  for all  $n \geq M$ .*

*Proof.* We first pass to common positive powers of  $\varphi$  and  $\varphi^{-1}$  that have expanding train track representatives. We can then combine Propositions 2.1.5, 2.1.6 and 3.2.23 to obtain directly the required result. For the application of Proposition 2.1.5 we need that  $\Delta_-(\varphi)$  and  $\Delta_+(\varphi)$  are disjoint, which is shown in Remark 3.2.25 below.  $\square$

Recall that the simplex of attraction  $\Delta_+ := \Delta_+(\varphi)$  is in general only a “degenerate simplex”, i.e. the convex linear hull of finitely many (possibly linearly dependent) points, called the *vertices* of  $\Delta_+$ . A subset  $\Delta' \subset \Delta_+$  is called a *face* of  $\Delta_+$  if it is the convex linear span of a subset of the vertices of  $\Delta_+$ . If there is no further vertex of  $\Delta_+$  which is linearly dependent on the vertices in  $\Delta'$ , then the face  $\Delta'$  is called *full*. Clearly every face  $\Delta''$  of  $\Delta_+$  is contained in a well defined *minimal full* face, which is the intersection of all full faces that contain  $\Delta''$ .

**Remark 3.2.25 (Dynamics within  $\Delta_+(\varphi)$ ).** (1) It is proved in Proposition 2.7.1 that every vertex of the convex cell  $\Delta_+ = \Delta_+(\varphi)$  is an *expanding  $\varphi$ -invariant* current, i.e. a projective current  $[\mu]$  for which there exist  $\lambda > 1$  and  $t \geq 1$  such that  $\varphi^t(\mu) = \lambda\mu$ .

For the rest of this remark we replace  $\varphi$  by a suitable positive power so that every vertex current of  $\Delta_+$  is projectively fixed by  $\varphi$ . Of course, this implies that  $\varphi$  fixes also every face  $\Delta'$  of  $\Delta_+$  (but not necessarily pointwise).



(2) A *uniform face*  $\Delta'$  is a face of  $\Delta_+$  which is spanned by vertices  $[\mu_+^1], \dots, [\mu_+^k]$  that all have the same stretch factor  $\lambda > 1$ , i.e.  $\varphi(\mu_+^j) = \lambda\mu_+^j$  for all  $1 \leq j \leq k$ . In this case it follows directly that the minimal full face which contains  $\Delta'$  is also uniform.

(3) Since the action of  $\varphi$  on  $\text{Curr}(\mathbb{F}_N)$  is linear, all the uniform faces of  $\Delta_+$  are pointwise fixed. Any non-vertex current  $[\mu]$  is always contained in the interior of some full face  $\Delta' \subset \Delta_+$ . Then the sequence of  $\varphi^n([\mu])$  converges towards a point in the uniform face  $\Delta'_+$  of  $\Delta'$  which is spanned by all vertices that have maximal stretch factor among all vertices of  $\Delta'$ .

(4) Similarly, under backwards iteration  $n \rightarrow -\infty$  the sequence of  $\varphi^n([\mu])$  converges towards a point in the uniform face  $\Delta'_-$  of  $\Delta' \subset \Delta_+$  which is spanned by all vertices that have minimal stretch factor among all vertices of  $\Delta'$ .

(5) As a consequence, it follows that a current  $[\mu]$  can not belong to both,  $\Delta_+(\varphi)$  and  $\Delta_-(\varphi)$ : By symmetry in the previous paragraph, any  $[\mu] \in \Delta_-(\varphi)$  converges under backwards iteration of  $\varphi^{-1}$  to an expanding  $\varphi^{-1}$ -invariant current. But backwards iteration of  $\varphi^{-1}$  is the same as forward iteration of  $\varphi$ , and the limit current can not be simultaneously expanding  $\varphi$ -invariant and expanding  $\varphi^{-1}$ -invariant.

Moreover, pointwise dynamics for a dense subset follows from our machinery:

**Theorem 3.2.26.** *Let  $\eta_g$  be a rational current in  $\text{Curr}(\mathbb{F}_N)$ . Then, there exist  $[\mu_\infty] \in \Delta_+$  such that*

$$\lim_{t \rightarrow \infty} \varphi^t[\eta_g] = [\mu_\infty].$$

*Proof.* Let  $\gamma$  be a reduced loop representing the conjugacy class  $g$ . Apply a power  $\varphi^k$  to  $g$ , hence  $f^k(\gamma)$ , the edge path representing  $\varphi^k(g)$  is pseudo-legal, see Remark 3.2.12. Then the arguments in Lemma 3.2.21 give the pointwise convergence by looking at the set of non-INP edges in  $f^k(\gamma)$ .  $\square$

# Chapter 4

## Dynamics of surface homeomorphisms

### 4.1 Classification of surface homeomorphisms

The classical sources for this section are Thurston's original manuscript [47] and the book by Casson and Bleiler [13].

Let  $S$  be a surface of finite type. The mapping class group  $\text{Mod}(S)$  of  $S$  is the group of isotopy classes of orientation-preserving homeomorphisms of  $S$ ; in other words,

$$\text{Mod}(S) = \text{Homeo}^+(S) / \text{Homeo}_0(S)$$

where  $\text{Homeo}_0(S)$  is the connected component of the identity in the orientation preserving homeomorphism group  $\text{Homeo}^+(S)$ .

We will give the Nielsen-Thurston classification of elements in  $\text{Mod}(S)$ , and recall the main tools that go into the Thurston's proof [47].

**Definition 4.1.1.** A marked hyperbolic surface is a pair  $(X, f)$  where

1.  $X = \mathbb{H}^2 / \Gamma$  is a hyperbolic surface, and
2.  $f : S \rightarrow X$  is an orientation-preserving homeomorphism.

Given a marked hyperbolic surface  $(X, f)$ , we can pull back the hyperbolic structure on  $X$  by  $f$  to one on  $S$ . Conversely, given a hyperbolic structure on  $S$ , the identity map  $id : S \rightarrow S$  makes  $(S, id)$  into a marked hyperbolic surface.

**Definition 4.1.2.** The *Teichmüller space* of  $S$  is the set  $\mathcal{T}(S) = \{(X, f)\} / \sim$  of equivalence classes of marked hyperbolic surfaces, where two hyperbolic surfaces  $(X, f)$  and  $(Y, g)$  are *equivalent* if  $g \circ f^{-1}$  is homotopic to an isometry from  $X$  to  $Y$ .

**Definition 4.1.3.** A *geodesic lamination* on  $S$  is a closed subset  $\mathcal{L}$  of  $S$  that is a union of finitely many disjoint, simple, complete geodesics on  $S$ . The geodesics in  $\mathcal{L}$  are called *leaves*

of the lamination. A *transverse measure*  $\mu$  on  $\mathcal{L}$  is an assignment of a locally finite Borel measure  $\mu|_k$  on each arc  $k$  transverse to  $\mathcal{L}$  so that

1. If  $k'$  is a subarc of an arc  $k$ , then  $\mu|_{k'}$  is the restriction to  $k'$  of  $\mu|_k$ ;
2. Transverse arcs which are transverse isotopic have the same measure.

A measured lamination is a pair  $(\mathcal{L}, \mu)$  of a geodesic lamination together with a transverse measure. The set of measured laminations on  $S$  is denoted by  $\mathcal{ML}(S)$ .

The space of *projective measured laminations* is defined as the quotient

$$\mathbb{P}\mathcal{ML}(S) = \mathcal{ML}(S)/\mathbb{R}_+.$$

Thurston discovered a  $\text{Mod}(S)$ -equivariant compactification of the Teichmüller Space  $\mathcal{T}(S)$  by the space of projective measured laminations  $\mathbb{P}\mathcal{ML}(S)$ , and using the action of  $\text{Mod}(S)$  on  $\overline{\mathcal{T}} = \mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$  showed:

**Theorem 4.1.4** (Nielsen-Thurston classification). *Each  $f \in \text{Mod}(S)$  is either periodic, reducible or pseudo-Anosov. Furthermore, pseudo-Anosov mapping classes are neither periodic nor reducible.*

Here  $f \in \text{Mod}(S)$  is called *periodic* if there exist a  $k \geq 0$  such that  $f^k$  is isotopic to the identity,  $f$  is called *reducible* if there is a collection  $\mathcal{C}$  of disjoint essential, simple, closed curves on  $S$  such that  $f(\mathcal{C})$  is isotopic to  $\mathcal{C}$ . Finally,  $f \in \text{Mod}(S)$  is called a *pseudo-Anosov* if there exist a pair of transverse measured laminations  $(\mathcal{L}^+, \mu_+)$  and  $(\mathcal{L}^-, \mu_-)$ , a number  $\lambda$  called *dilatation*, and a representative homeomorphism  $f'$  of  $f$  such that

$$f'(\mathcal{L}^+, \mu_+) = (\mathcal{L}^+, \lambda\mu_+)$$

and

$$f'(\mathcal{L}^-, \mu_-) = (\mathcal{L}^-, \frac{1}{\lambda}\mu_-).$$

The measured laminations  $(\mathcal{L}^+, \mu_+)$  and  $(\mathcal{L}^-, \mu_-)$  are called *stable lamination* and *unstable lamination* respectively. We will suppress the  $\mathcal{L}$  and write  $\mu_+$  and  $\mu_-$  respectively.

Denote the projective class of a non-zero measured lamination  $\mu$  by  $[\mu]$ . Thurston [47] further showed that a pseudo-anosov element  $f$  acts on  $\mathbb{P}\mathcal{ML}(S)$  with north-south dynamics. More precisely,

**Theorem 4.1.5.** *The action of a pseudo-Anosov mapping class  $f$  on  $\mathbb{P}\mathcal{ML}(S)$  has exactly two fixed points, the projective classes of the stable lamination  $[\mu_+]$  and the unstable lamination  $[\mu_-]$ . For any point  $[\mu] \neq [\mu_-]$  in  $\mathbb{P}\mathcal{ML}(S)$   $\lim_{k \rightarrow \infty} f^k([\mu]) = [\mu_+]$ , and for any point  $[\mu] \neq [\mu_+]$  in  $\mathbb{P}\mathcal{ML}(S)$   $\lim_{k \rightarrow \infty} f^{-k}([\mu]) = [\mu_-]$ .*

In fact, this convergence is uniform on compact sets by work of Ivanov [26].

## 4.2 Geodesic currents on surfaces

Let  $S$  be a closed surface of genus  $g \geq 2$ . Let us fix a hyperbolic metric on  $S$  and consider the universal cover  $\tilde{S}$  with the pull-back metric. There is a natural  $\pi_1(S)$  action on  $\tilde{S}$  by isometries. We will denote the space of bi-infinite, unoriented, unparameterized geodesics in  $\tilde{S}$  by  $G(\tilde{S})$ , given the quotient topology from the compact open topology on parameterized bi-infinite geodesics. Since such a geodesic is uniquely determined by the (unordered) pair of its distinct end points on the boundary at infinity of  $\tilde{S}$ , a more concrete description can be given by

$$G(\tilde{S}) = ((\tilde{S}_\infty \times \tilde{S}_\infty) - \Delta) / ((\xi, \zeta) = (\zeta, \xi)).$$

A *geodesic current* on  $S$  is a locally finite Borel measure on  $G(\tilde{S})$  which is  $\pi_1(S)$  invariant. The set of all geodesic currents on  $S$ , denoted by  $\text{Curr}(S)$ , is a metrizable topological space with the weak\* topology, see [8, 9].

As a simple example, consider the preimage in  $\tilde{S}$  of any closed curve  $\gamma \subset S$ , which is a collection of complete geodesics in  $\tilde{S}$ . This is a discrete subset of  $G(\tilde{S})$  which is invariant under the action of  $\pi_1(S)$ . Dirac (counting) measure associated to this set on  $G(\tilde{S})$  gives a geodesic current on  $S$ , which is denoted by  $\mu_\gamma$ . Note that this construction gives an injection from the set of closed curves on  $S$  to  $\text{Curr}(S)$ .

As another example, one can consider a measured geodesic lamination as a geodesic current, see [8, 34]. Therefore, the set  $\mathcal{ML}(S)$  of measured geodesic laminations on  $S$ , is a subset of  $\text{Curr}(S)$ .

Recall that *geometric intersection number* of  $\alpha, \beta$  for any two homotopy classes of closed curves is the minimum number of intersections of  $\alpha'$  and  $\beta'$  for  $\alpha' \simeq \alpha$  and  $\beta' \simeq \beta$ . One can show that this minimum is realized when  $\alpha', \beta'$  are geodesic representatives.

In [8], Bonahon constructed a continuous, symmetric, bilinear function  $i(\cdot, \cdot) : \text{Curr}(S) \times \text{Curr}(S) \rightarrow \mathbb{R}_{\geq 0}$  such that for any two homotopy classes of closed curves  $\alpha, \beta$ ,  $i(\mu_\alpha, \mu_\beta)$  is the geometric intersection number between  $\alpha$  and  $\beta$ . We will call  $i(\cdot, \cdot)$  the *intersection number function*.

We say that a geodesic current  $\mu \in \text{Curr}(S)$  is *binding* if  $i(\nu, \mu) > 0$  for all  $\nu \in \text{Curr}(S)$ . For example, let  $\beta = \{\beta_1, \dots, \beta_m\}$  be a filling set of simple closed curves, i.e. union of geodesic representatives of  $\beta_i$  cuts  $S$  up into topological disks. Then  $\mu_\beta = \sum \mu_{\beta_i}$  is a binding current. One of the facts proved in [9] that we will use repeatedly in our arguments is as follows:

**Proposition 4.2.1** (Bonahon). *Let  $\beta$  be a binding current. Then the set of  $\nu \in \text{Curr}(S)$  with  $i(\nu, \beta) \leq M$  for  $M > 0$  is compact in  $\text{Curr}(S)$ .*

Proposition 4.2.1 also implies that the *space of projective geodesic currents*,  $\mathbb{P}\text{Curr}(S)$ , is compact where

$$\mathbb{P}\text{Curr}(S) = (\text{Curr}(S) \setminus 0) / \sim$$

and where  $\nu_1 \sim \nu_2$  if and only if  $\nu_1 = c\nu_2$  for some  $c > 0$ . Similar to the elements of  $\mathbb{P}\mathcal{ML}$  we will denote the projective class of a non-zero geodesic current  $\nu$  by  $[\nu]$ .

In [41], Otal used the intersection number function to distinguish points in  $\text{Curr}(S)$ .

**Theorem 4.2.2.** [41] *Given  $\nu_1, \nu_2 \in \text{Curr}(S)$ ,  $\nu_1 = \nu_2$  if and only if  $i(\nu_1, \alpha) = i(\nu_2, \alpha)$  for every closed curve  $\alpha$  in  $S$ .*

Recently, in [22], Duchin–Leininger–Rafi used Theorem 4.2.2 to construct a metric on  $\text{Curr}(S)$ , which will be crucial in our argument.

**Theorem 4.2.3.** [22] *Let  $\beta$  be a filling set of simple closed curves. Enumerate all the closed curves on  $S$  by  $\{\alpha_1, \alpha_2, \alpha_3 \dots\}$ . Let  $x_k = \frac{\alpha_k}{i(\alpha_k, \beta)}$ , then*

$$d(\nu_1, \nu_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} |i(\nu_1, x_k) - i(\nu_2, x_k)|$$

*defines a proper metric which is compatible with the weak\* topology.*

In general one can define a geodesic current on an oriented surface  $S$  of finite type, but here we restrict our attention to compact surfaces with  $b$  boundary components and give the definition for this particular case. Given a compact surface  $S$  with  $b$  boundary components, think of it as a subset of a complete, hyperbolic surface  $S'$ , obtained from  $S$  by attaching  $b$  *flaring ends*, see Figure 4.1.

A *geodesic current* on  $S$  is a geodesic current on  $S'$ , a locally finite Borel measure on  $G(\tilde{S}')$  which is  $\pi_1(S')$  invariant, with the property that support of this measure projects into  $S$ .

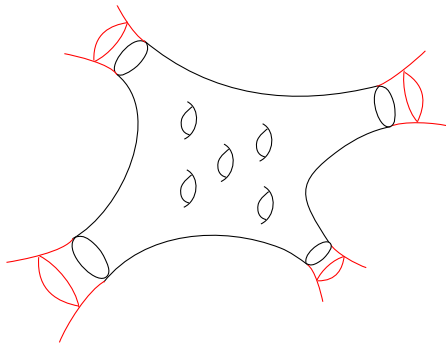


Figure 4.1: Attaching flaring ends

An alternative definition can be given as follows: Let  $S$  be an oriented surface of genus  $g \geq 1$  with  $b \geq 1$  boundary components. Consider the double of  $S$ ,  $DS$ , which is a closed, oriented surface obtained by gluing two copies of  $S$  along their boundaries by the identity map. We equip  $DS$  with a hyperbolic metric such that  $S$  has geodesic boundary. A geodesic current on  $S$  can be thought as a geodesic current on  $DS$  whose support projects entirely into  $S$ . Building upon work of Bonahon [8], Duchin–Leininger–Rafi [22] showed that intersection number on  $\text{Curr}(S)$  is in fact the restriction of the intersection number defined on  $\text{Curr}(DS)$ . For a detailed discussion of geodesic currents on surfaces we refer reader to [1, 8, 9, 22].

### 4.3 Dynamics on surfaces

Let  $S$  be a compact surface of genus  $g \geq 1$  with  $b \geq 1$  boundary components and let  $f : S \rightarrow S$  be a *pseudo-Anosov* map which fixes  $\partial S$  pointwise. Let  $\varphi \in \text{Out}(\pi_1(S)) = \text{Out}(F_N)$  be the induced outer automorphism. Let  $\mu_+$  and  $\mu_-$  be the stable and the unstable lamination for  $f$ . Up to scaling, we will always assume that  $i(\mu_-, \mu_+) = 1$ . We fix a hyperbolic metric on  $S$  and denote the geodesic boundary components of  $S$  by  $\alpha_1, \alpha_2, \dots, \alpha_b$ . We extend the map  $f$  on  $S$  to double surface  $DS$  by taking the identity map on the other side and will continue to denote our extension to  $DS$  by  $f$ . We will also adhere to the point of view that a geodesic current on  $S$  is a geodesic current on  $DS$  whose support projects entirely into  $S$ . Let  $\mu_{\alpha_i}$  be the current corresponding to the boundary curve  $\alpha_i$ .

Let us define  $\Delta, H_-(f), H_+(f) \subset \mathbb{P}\text{Curr}(S)$  as follows:

$$\Delta := \{[a_1\mu_{\alpha_1} + a_2\mu_{\alpha_2} + \dots + a_b\mu_{\alpha_b}] \mid a_i \geq 0, \sum_{i=1}^b a_i > 0\}.$$

$$H_-(f) := \{[t_1\mu_- + t_2\nu] \mid [\nu] \in \Delta\} \text{ and } H_+(f) := \{[t'_1\mu_+ + t'_2\nu] \mid [\nu] \in \Delta\}.$$

First, we will prove a general result about geodesic currents which will be useful throughout this article.

**Proposition 4.3.1.** *Let  $f$  be a pseudo-Anosov homeomorphism on a surface  $S$  with  $b$  boundary components  $\{\alpha_1, \alpha_2, \dots, \alpha_b\}$ . Let  $[\mu_+]$  and  $[\mu_-]$  be the corresponding stable and unstable laminations for  $f$ . Assume that  $\nu \in \text{Curr}(S)$  is a geodesic current on  $S$  such that  $i(\nu, \mu_+) = 0$ . Then,*

$$\nu = a_1\mu_{\alpha_1} + a_2\mu_{\alpha_2} + \dots + a_b\mu_{\alpha_b} + c\mu_+$$

for some  $a_i, c \geq 0$ . Similarly, if  $i(\nu, \mu_-) = 0$ , then

$$\nu = a'_1\mu_{\alpha_1} + a'_2\mu_{\alpha_2} + \dots + a'_b\mu_{\alpha_b} + c'\mu_-$$

for some  $a'_i, c' \geq 0$ .

*Proof.* We will prove the first part of the proposition. The proof of the second part is similar. Here we use the structure of geodesic laminations and in particular the structure of the stable and unstable laminations. For a detailed discussion see [13, 47, 34].

Let  $S$  be a surface with  $b \geq 1$  boundary components and  $DS$  the double of  $S$ . Let  $\mu_+$  be the stable lamination corresponding to the pseudo-Anosov map  $f$ . Assume that for a geodesic current  $\nu$ , the intersection number  $i(\nu, \mu_+) = 0$ . We'll now investigate possible geodesics in the support of  $\nu$ . Since  $i(\nu, \mu_+) = 0$ , it follows from the definition of Bonahon's intersection form that projection of any geodesic in the support of  $\nu$  onto  $S$  cannot intersect the leaves of the lamination transversely. Indeed, such an intersection would contribute to intersection number positively. Therefore, for each geodesic  $\ell$  in the support of  $\nu$  one of the following happens:

1.  $\ell$  projects onto a leaf of the stable lamination or a boundary curve, or
2.  $\ell$  projects onto a geodesic that is disjoint from the leaves of the stable lamination or the boundary curves.

If (1) happens for every geodesic  $\ell$  in the support of  $\nu$ , then there is nothing to prove. If not, then  $\ell$  projects to a geodesic, which is disjoint from the stable lamination. So if we cut  $S$  along the stable lamination to get ideal polygons or crowns [13] (see Figure 4.2), then one of these contains the image of the geodesic  $\ell$ .

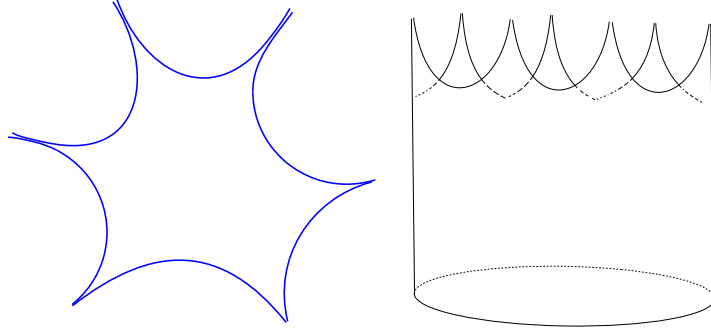


Figure 4.2: Complementary Regions

From here, it follows that there is a geodesic  $\ell'$  (possibly different than  $\ell$ ) in the support of  $\nu$  which is *isolated*, i.e. there is some open set  $U$  in  $G(\tilde{S})$ , which intersects the support of  $\nu$  in the single point  $\{\ell'\}$ . The geodesic  $\ell'$  must have positive mass assigned by  $\nu$ . This geodesic will project onto a biinfinite geodesic in  $S$ , which must therefore accumulate since  $S$  is compact. Take a short geodesic segment  $L$  in  $S$ , which the projection of  $\ell'$  intersects infinitely often. Consider a lift  $\tilde{L}$  of  $L$  to  $\tilde{S}$ . There are infinitely many translates of  $\ell'$  that intersect  $\tilde{L}$  which also form a compact set. By the  $\pi_1(S)$ -invariance of the measures, this set must have infinite measure, which contradicts the fact that  $\nu$  is a locally finite Borel measure. Therefore (2) does not happen. So as a result,  $\nu$  will be a combination of some multiple of the stable lamination and a linear combination of currents corresponding to the boundary curves.  $\square$

The following lemma is due to N. Ivanov.

**Lemma 4.3.2.** [26] *Let  $\mu$  be a measured geodesic lamination on  $DS$ , if  $[\mu] \notin H_-(f)$  then*

$$\lim_{n \rightarrow \infty} \lambda^{-n} f^n(\mu) = i(\mu, \mu_-)\mu_+,$$



where  $\lambda$  is the dilatation. Similarly if  $[\mu] \notin H_+(f)$ , then

$$\lim_{n \rightarrow \infty} \lambda^{-n} f^{-n}(\mu) = i(\mu, \mu_+) \mu_-.$$

The first step toward a generalization of the Lemma 4.3.2 for actions of pseudo-Anosov homeomorphisms on the space of projective geodesic currents can be stated as follows:

**Lemma 4.3.3.** *Let  $[\nu] \notin H_-(f)$  be an arbitrary geodesic current, then*

$$\lim_{n \rightarrow \infty} f^n[\nu] = [\mu_+].$$

*Proof.* Considering  $\mathbb{P}Curr(S) \subset \mathbb{P}Curr(DS)$  as a closed subset, since  $\mathbb{P}Curr(DS)$  is compact, there exist a subsequence  $\{n_k\}$  such that  $\lim_{n_k \rightarrow \infty} f^{n_k}[\nu]$  exists. This means that there is a sequence  $\{w_{n_k}\}$  of positive real numbers such that

$$\lim_{n_k \rightarrow \infty} w_{n_k} f^{n_k}(\nu) = \nu_* \neq 0$$

where  $\nu_* \in \mathbb{P}Curr(S)$ . Let  $\beta$  be a filling set of simple closed curves in  $DS$ . We have

$$\begin{aligned} 0 \neq C &= i(\nu_*, \beta) \\ &= i\left(\lim_{n_k \rightarrow \infty} w_{n_k} f^{n_k}(\nu), \beta\right) \\ &= \lim_{n_k \rightarrow \infty} i(w_{n_k} \nu, f^{-n_k}(\beta)) \\ &= \lim_{n_k \rightarrow \infty} w_{n_k} \lambda^{n_k} i(\nu, \lambda^{-n_k} f^{-n_k}(\beta)) \\ &= \lim_{n_k \rightarrow \infty} w_{n_k} \lambda^{n_k} \lim_{n_k \rightarrow \infty} i(\nu, \lambda^{-n_k} f^{-n_k}(\beta)) \\ &= \lim_{n_k \rightarrow \infty} w_{n_k} \lambda^{n_k} i(\nu, \mu_-) i(\mu_+, \beta). \end{aligned}$$

From here we can deduce that

$$\lim_{n_k \rightarrow \infty} w_{n_k} \lambda^{n_k} = C_1 \neq 0,$$

since  $i(\nu, \mu_-) \neq 0$  and  $i(\mu_+, \beta) \neq 0$ . This means that without loss of generality we can choose

$w_{n_k} = \lambda^{-n_k}$ . Now, look at

$$\begin{aligned}
i(\nu_*, \mu_+) &= i\left(\lim_{n_k \rightarrow \infty} \lambda^{-n_k} f^{n_k}(\nu), \mu_+\right) \\
&= \lim_{n_k \rightarrow \infty} i(\nu, \lambda^{-n_k} f^{-n_k}(\mu_+)) \\
&= \lim_{n_k \rightarrow \infty} \lambda^{-2n_k} i(\nu, \mu_+) \\
&= 0.
\end{aligned}$$

Therefore, by Proposition 4.3.1,

$$\nu_* = t\mu_0 + s\mu_+$$

where  $t, s \geq 0$ ,  $t + s > 0$  and  $\mu_0$  is a non-negative, non-trivial linear combination of currents corresponding to boundary curves.

**Claim 1.**  $s \neq 0$ , i.e  $\nu_* \neq t\mu_0$ .

We will show that  $\nu_*$  has non-zero intersection number with  $\mu_-$  which will imply that  $\nu_* \neq \mu_0$ .

$$\begin{aligned}
i(\nu_*, \mu_-) &= i\left(\lim_{n_k \rightarrow \infty} \lambda^{-n_k} f^{n_k}(\nu), \mu_-\right) \\
&= \lim_{n_k \rightarrow \infty} i(\nu, \lambda^{-n_k} f^{-n_k}(\mu_-)) \\
&= \lim_{n_k \rightarrow \infty} i(\nu, \mu_-) \\
&= i(\nu, \mu_-) \neq 0,
\end{aligned}$$

since  $\nu \notin H_-$ .

**Claim 2.**  $\lim_{n \rightarrow \infty} f^n[\nu] = [\mu_+]$ .

Assume not, then there exists a subsequence  $\{n_k\}$  such that

$$\lim_{n_k \rightarrow \infty} f^{n_k}[\nu] = [\nu_*] \neq [\mu_+].$$

First observe that

$$\lim_{m \rightarrow \infty} f^{-m}[\nu_*] = [\mu_0],$$

since

$$f^{-m}[t\mu_0 + s\mu_+] = [t\mu_0 + \lambda^{-m}s\mu_+].$$

We also have,  $\forall i > 0$

$$\lim_{n_k \rightarrow \infty} f^{n_k - i}[\nu] = f^{-i}[\nu_*].$$

Let  $d'$  be a metric on  $\mathbb{P}Curr(DS)$  which induces the quotient topology from the weak-\* topology. Then,  $\forall m \geq 1$  there exist an integer  $i_m \geq 1$  such that

$$d'([\mu_0], f^{-i_m}[\nu_*]) < \frac{1}{2m}.$$

Since  $\lim_{n_k \rightarrow \infty} f^{n_k - i_m}[\nu] = f^{-i_m}[\nu_*]$ , pick  $n_k = n_k(m)$  such that

$$d'(f^{-i_m}[\nu_*], f^{n_k - i_m}[\nu]) < \frac{1}{2m} \text{ and } n_k - i_m > m.$$

Now, set  $j_m = n_k - i_m$ , then by triangle inequality we have

$$d'(f^{j_m}[\nu], [\mu_0]) < \frac{1}{m},$$

which implies  $\lim_{j_m \rightarrow \infty} f^{j_m}[\nu] = [\mu_0]$ , which contradicts with the previous claim. This completes the proof of the lemma.  $\square$

**Theorem 4.3.4.** *Let  $f$  be a pseudo-Anosov homeomorphism on a compact surface  $S$ . Let  $[\nu] \notin H_-(f)$  be a geodesic current. Then,*

$$\lim_{n \rightarrow \infty} \lambda^{-n} f^n(\nu) = i(\nu, \mu_-)\mu_+,$$

where  $\mu_+$  and  $\mu_-$  are the stable and the unstable laminations for  $f$ . Similarly, for a geodesic current  $[\nu] \notin H_+(f)$ , we have

$$\lim_{n \rightarrow \infty} \lambda^{-n} f^{-n}(\nu) = i(\nu, \mu_+)\mu_-.$$

*Proof.* Lemma 4.3.3 implies that there exists a sequence  $\{w_n\}$  of positive real numbers such that  $\lim_{n \rightarrow \infty} w_n f^n(\nu) = \mu_+$ . Recall that we assumed  $i(\mu_+, \mu_-) = 1$ . Hence, we have

$$\begin{aligned} 1 &= i\left(\lim_{n \rightarrow \infty} w_n f^n(\nu), \mu_-\right) \\ &= \lim_{n \rightarrow \infty} i(w_n \nu, f^{-n}(\mu_-)) \\ &= \lim_{n \rightarrow \infty} w_n \lambda^n i(\nu, \mu_-), \end{aligned}$$

which implies that  $\lim_{n \rightarrow \infty} w_n \lambda^n = \frac{1}{i(\nu, \mu_-)}$ . Therefore, we have

$$\lim_{n \rightarrow \infty} \lambda^{-n} f^n(\nu) = \lim_{n \rightarrow \infty} \frac{w_n}{w_n \lambda^n} f^n(\nu) = i(\nu, \mu_-) \mu_+.$$

The proof of the second assertion is similar. □

Now we are ready to prove the main theorem of this section.

**Theorem 4.3.5.** *Let  $f$  be a pseudo-Anosov homeomorphism on a compact surface with boundary. Assume that  $K$  is a compact set in  $\mathbb{P}\text{Curr}(S) \setminus H_-(f)$ . Then for any open neighborhood  $U$  of the stable current  $[\mu_+]$ , there exist  $m \in \mathbb{N}$  such that  $f^n(K) \subset U$  for all  $n \geq m$ . Similarly, for a compact set  $K' \subset \mathbb{P}\text{Curr}(S) \setminus H_+(f)$  and an open neighborhood  $V$  of the unstable current  $[\mu_-]$ , there exist  $m' \in \mathbb{N}$  such that  $f^{-n}(K') \subset V$  for all  $n \geq m'$ .*

*Proof.* We will prove the first statement. The proof of the second one is similar. To prove this theorem we utilize the metric on the space of geodesic currents introduced by Duchin–Leininger–Rafi in [22]. Let  $d$  be the metric on  $\text{Curr}(DS)$  as discussed in Section 4.2. Let  $\{\alpha_0, \alpha_1, \alpha_2, \dots\}$  be an enumeration of all the closed curves on  $DS$ . Let us set  $x_k = \frac{\alpha_k}{i(\alpha_k, \beta)}$  where  $\beta$  is a filling set of simple closed curves on  $DS$ .

Let us take a cross section  $\bar{K}$  of  $K$  in  $\text{Curr}(S)$  by picking the representative  $\nu \in \text{Curr}(S)$  with  $i(\nu, \mu_-) = 1$  for any  $[\nu] \in K$ . From Theorem 4.3.4, we know that

$$\lim_{n \rightarrow \infty} \lambda^{-n} f^n(\nu) = i(\nu, \mu_-) \mu_+ = \mu_+$$

So it suffices to show that for any  $\epsilon > 0$ , there exist  $m > 0$  such that

$$d(\lambda^{-n} f^n(\nu), \mu_+) < \epsilon$$

for all  $n \geq m$  and for all  $\nu \in \bar{K}$ .

**Claim.**

$$|i(\mu_+, x_k) - i(\lambda^{-n} f^n(\nu), x_k)|$$

is uniformly bounded  $\forall \nu \in \bar{K}, \forall k \geq 1, \forall n \geq 1$ .

By triangle inequality  $|i(\mu_+, x_k) - i(\lambda^{-n} f^n(\nu), x_k)| \leq |i(\mu_+, x_k)| + |i(\lambda^{-n} f^n(\nu), x_k)|$ , so it suffices to bound the two quantities on the right. Since  $\{x_k\}$  is precompact we have

$$i(\mu_+, x_k) \leq R_0$$

for some  $R_0 > 0$ .

For the second quantity we have

$$i(\lambda^{-n} f^n(\nu), x_k) = i(\nu, \lambda^{-n} f^{-n}(x_k))$$

Since  $\nu$  comes from a compact set, it suffices to show that  $\{\lambda^{-n} f^{-n}(x_k)\}$  is precompact. Now we have,

$$i(\lambda^{-n} f^{-n}(x_k), \beta) = i(x_k, \lambda^{-n} f^n(\beta)) \leq R_1$$

for some  $R_1 > 0$  since  $\{x_k\}$  is precompact and  $\lim_{n \rightarrow \infty} \lambda^{-n} f^n(\beta) = i(\beta, \mu_-)\mu_+$ . Therefore by proposition 4.2.1,  $\{\lambda^{-n} f^{-n}(x_k)\}$  is precompact, and hence the second quantity is uniformly bounded, and the claim follows.

By the claim there is some  $R > 0$  so that

$$|i(\mu_+, x_k) - i(\lambda^{-n} f^n(\nu), x_k)| \leq R.$$

Now we have;

$$d(\mu_+, \lambda^{-n} f^n(\nu)) \leq \sum_{j=1}^M \frac{1}{2^j} |i(\mu_+, x_j) - i(\lambda^{-n} f^n(\nu), x_j)| + \sum_{j=M+1}^{\infty} \frac{1}{2^j} R. \quad (4.3.1)$$

The second sum can be made as small as we want by choosing  $M$  big enough, so we make it less than  $\epsilon/2$ . We need to show that first sum goes to 0 uniformly over all  $\nu \in \bar{K}$ . Since there are only finitely many terms we will show that for a fixed  $x_j$

$$|i(\mu_+, x_j) - i(\lambda^{-n} f^n(\nu), x_j)| \rightarrow 0$$

uniformly  $\forall \nu \in \bar{K}$ . Now, because  $i(\nu, \mu_-) = 1$ , we have

$$\begin{aligned} |i(\mu_+, x_j) - i(\lambda^{-n} f^n(\nu), x_j)| &= |i(\nu, \mu_-)i(\mu_+, x_j) - i(\nu, \lambda^{-n} f^{-n}(x_j))| \\ &= |i(\nu, i(\mu_+, x_j)\mu_-) - i(\nu, \lambda^{-n} f^{-n}(x_j))|. \end{aligned}$$

Since  $\nu$  comes from a compact set, we can choose a small neighborhood  $V$  of  $i(\mu_+, x_j)\mu_-$  such that for any  $\nu' \in V$

$$|i(\nu, i(\mu_+, x_j)\mu_-) - i(\nu, \nu')| < \epsilon/2M$$

for all  $\nu \in \bar{K}$ . We already know by Theorem 4.3.4 that

$$\lim_{n \rightarrow \infty} \lambda^{-n} f^{-n}(x_j) = i(\mu_+, x_j) \mu_-.$$

So there exist  $m \in \mathbb{N}$  such that for all  $n \geq m$  one has  $\lambda^{-n} f^{-n}(x_j) \in V$  and therefore

$$|i(\nu, i(\mu_+, x_j) \mu_-) - i(\nu, \lambda^{-n} f^{-n}(x_j))| < \epsilon/2M.$$

Hence, by (4.3.1) we have

$$d(\mu_+, \lambda^{-n} f^n(\nu)) \leq \epsilon$$

for all  $n \geq m$  and for all  $\nu \in \bar{K}$ . □

## 4.4 Dynamics of non-atoroidal, fully irreducible automorphisms

For a non-atoroidal and fully irreducible element  $\varphi \in \text{Out}(\mathbb{F}_N)$ , using a theorem of Bestvina and Handel we will be able to transfer the question about the dynamics of the action of  $\varphi$  on  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  to a problem in surface theory. Using the result we established in the previous section, we will prove a variant of uniform north-south dynamics on the space of geodesic currents for non-atoroidal fully irreducible elements.

The result of Bestvina–Handel we need is the following.

**Theorem 4.4.1.** *[7] Let  $\varphi \in \text{Out}(\mathbb{F}_N)$ . Then  $\varphi$  is non-atoroidal and fully irreducible if and only if  $\varphi$  is induced by a pseudo-Anosov homeomorphism  $f$  of a compact surface  $S$  with one boundary component and  $\pi_1(S) \cong \mathbb{F}_N$ .*

**Remark 4.4.2.** Note that with the definition we gave at the end of Section 2.4, a geodesic current on  $\mathbb{F}_N$  is precisely a geodesic current on a surface  $S$  with  $\pi_1(S) = \mathbb{F}_N$ . Therefore,  $\text{Curr}(\mathbb{F}_N) = \text{Curr}(S)$ .

For  $\varphi$  and  $f$  as in Theorem 4.4.1, define  $H_-(\varphi) := H_-(f)$ ,  $H_+(\varphi) = H_+(f) \subset \mathbb{P}\text{Curr}(S) = \mathbb{P}\text{Curr}(\mathbb{F}_N)$ . Combining the above remark and Theorem 4.4.1, a special case of Theorem 4.3.5 for one boundary component gives the following:

**Theorem 4.4.3.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be non-atoroidal and fully irreducible. Then the action of  $\varphi$  on the space of projective geodesic currents  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  has uniform north-south dynamics*

in the following sense: Given an open neighborhood  $U$  of the stable current  $[\mu_+]$  and a compact set  $K_0 \subset \mathbb{P}\text{Curr}(\mathbb{F}_N) \setminus H_-(\varphi)$  there exist a power  $M_0 > 0$  such that for all  $n \geq M_0$ ,  $\varphi^n(K_0) \subset U$ . Similarly, given an open neighborhood  $V$  of the unstable current  $[\mu_-]$  and a compact set  $K_1 \subset \mathbb{P}\text{Curr}(\mathbb{F}_N) \setminus H_+(\varphi)$ , there exist a power  $M_1 > 0$  such that for all  $m \geq M_1$ ,  $\varphi^{-m}(K_1) \subset V$ .

We complement the above theorem by giving a complete picture in terms of fixed points of the action of a non-atoroidal fully irreducible element  $\varphi$  on the space of projective geodesic currents.

**Proposition 4.4.4.** *Let  $\varphi$  be non-atoroidal and fully irreducible. Then the action of  $\varphi$  on  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  has exactly three fixed points: the stable lamination (current)  $[\mu_+]$ , the unstable lamination (current)  $[\mu_-]$  and the current corresponding to the boundary curve  $[\mu_\alpha]$ .*

*Proof.* A straightforward computation shows that points in  $(H_-(\varphi) \cup H_+(\varphi))$  except  $[\mu_+]$ ,  $[\mu_-]$  and  $[\mu_\alpha]$  are not fixed. For any other point  $[\nu] \in \mathbb{P}\text{Curr}(\mathbb{F}_N) \setminus (H_-(\varphi) \cup H_+(\varphi))$ , Theorem 4.4.3 and the fact that  $[\mu_+] \neq [\mu_-]$  implies that  $\varphi([\nu]) \neq [\nu]$ .  $\square$

# Chapter 5

## Applications to subgroup structure of $\text{Out}(\mathbb{F}_N)$

### 5.1 Dynamical results

Recall that the minimal set  $\mathcal{M}_N$  in  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  is the closure of the set

$$\{[\eta_g] \mid g \in \mathbb{F}_N \text{ is primitive element}\}$$

in  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$ . Equivalently,  $\mathcal{M}_N$  is equal to the closure of the  $\text{Out}(\mathbb{F}_N)$  orbit of  $[\eta_g]$  for a primitive element  $g \in \mathbb{F}_N$ . As a consequence of Theorem 4.3.5, we obtain the following result, which was claimed without proof by R. Martin [38]. A sketch of the proof following a different approach was given by Bestvina-Feign in [5].

**Corollary 5.1.1.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be a non-atoroidal fully irreducible element with stable and unstable currents  $[\mu_+]$  and  $[\mu_-]$ . Then the action of  $\varphi$  on  $\mathcal{M}_N$  has uniform north-south dynamics. Namely, given a compact set  $K_0 \subset \mathcal{M}_N \setminus \{[\mu_-]\}$  and an open neighborhood  $U$  of  $[\mu_+]$  in  $\mathcal{M}_N$ , there is an integer  $M_0 > 0$  such that  $\varphi^n(K_0) \subset U$  for all  $n \geq M_0$ . Similarly, given a compact set  $K_1 \subset \mathcal{M}_N \setminus \{[\mu_+]\}$  and an open neighborhood  $V$  of  $[\mu_-]$  in  $\mathcal{M}_N$ , there is an integer  $M_1 > 0$  such that  $\varphi^{-m}(K_1) \subset V$  for all  $m \geq M_1$ .*

*Proof.* Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be non-atoroidal and fully irreducible. Then  $\varphi$  is induced by a pseudo-Anosov homeomorphism  $f$  of a compact surface  $S$  with one boundary component  $\alpha$  and  $\pi_1(S) \cong \mathbb{F}_N$ . Note that the current  $\mu_\alpha$  corresponding to the boundary curve  $\alpha$  does not belong to the minimal set  $\mathcal{M}_N$ . Indeed, it is well known that if a current  $[\nu] \in \mathcal{M}_N$  and  $A$  is free basis for  $\mathbb{F}_N$ , then Whitehead graph of support of  $[\nu]$  with respect to  $A$  is either disconnected or connected but has a cut vertex, [50, 5]. Pick a basis  $A$  such that  $\mu_\alpha$  corresponds to product of commutators. It is straightforward to check that  $\mu_\alpha \notin \mathcal{M}_N$  by using this criteria. From here it also follows that any element in  $(H_- \cup H_+)$  other than  $[\mu_+]$  and  $[\mu_-]$  is not in  $\mathcal{M}_N$  since the closure of the  $\varphi$  orbit, and hence the  $\text{Out}(\mathbb{F}_N)$  orbit, of any such element will contain  $[\mu_\alpha]$ . Therefore, Theorem 4.3.5 implies that  $\varphi$  has uniform



north-south dynamics on the minimal set  $\mathcal{M}_N$ . □

As another corollary of our theorem we prove a unique ergodicity type result for non-atoroidal fully irreducible elements that is analogous to a theorem of Kapovich–Lustig [31] about atoroidal and fully irreducible elements. Recall that, by [35], the action of a fully irreducible  $\varphi \in \text{Out}(\mathbb{F}_N)$  on the projectivized outer space  $\overline{\text{CV}}_N$  has exactly two fixed points,  $[T_+]$  and  $[T_-]$ , called attracting and repelling trees for  $\varphi$ . For any  $[T] \neq [T_-]$ ,  $\lim_{n \rightarrow \infty} [T]\varphi^n = [T_+]$  and for any  $[T] \neq [T_+]$ ,  $\lim_{n \rightarrow \infty} [T]\varphi^{-n} = [T_-]$ . Moreover, there are constants  $\lambda_+, \lambda_- > 1$  such that  $T_+\varphi = \lambda_+T_+$  and  $T_-\varphi^{-1} = \lambda_-T_-$ . In fact, for a non-atoroidal, fully irreducible  $\varphi \in \text{Out}(\mathbb{F}_N)$ , one has  $\lambda_- = \lambda_+ = \lambda$ .

**Theorem 5.1.2.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be non-atoroidal and fully irreducible. Let  $T_+$  and  $T_-$  be representatives of attracting and repelling trees, respectively, in  $\overline{\text{cv}}_N$  corresponding to the right action of  $\varphi$  on  $\overline{\text{cv}}_N$ . Then,*

$$\langle T_+, \mu \rangle = 0 \iff [\mu] = [a_0\mu_- + b_0\mu_\alpha]$$

for some  $a_0 \geq 0, b_0 \geq 0$ . Similarly,

$$\langle T_-, \mu \rangle = 0 \iff [\mu] = [a_1\mu_+ + b_1\mu_\alpha]$$

for some  $a_1 \geq 0, b_1 \geq 0$ .

*Proof.* We will prove the first assertion, the second one is symmetric. The “If” direction follows from the properties of the intersection form (see Proposition 2.4.5). Specifically, we have

$$\langle T_+, \mu_- \rangle = \langle T_+\varphi, \varphi^{-1}\mu_- \rangle = \lambda\lambda_+ \langle T_+, \mu_- \rangle,$$

which implies  $\langle T_+, \mu_- \rangle = 0$ . Similarly,

$$\langle T_+, \mu_\alpha \rangle = \langle T_+\varphi, \varphi^{-1}\mu_\alpha \rangle = \lambda_+ \langle T_+, \mu_\alpha \rangle,$$

which implies that  $\langle T_+, \mu_\alpha \rangle = 0$  as well. Therefore,

$$\langle T_+, a_0\mu_- + b_0\mu_\alpha \rangle = a \langle T_+, \mu_- \rangle + b \langle T_+, \mu_\alpha \rangle = 0.$$

Conversely, let  $\langle T_+, \mu \rangle = 0$ . Assume that  $\mu$  is not a linear combination of  $\mu_-$  and  $\mu_\alpha$ . Then,

there exist a sequence of positive real numbers  $\{a_n\}$  such that

$$\lim_{n \rightarrow \infty} a_n \varphi^n(\mu) = \mu_+.$$

Therefore by continuity of the intersection number we have

$$0 \neq \langle T_+, \mu_+ \rangle = \left\langle T_+, \lim_{n \rightarrow \infty} a_n \varphi^n(\mu) \right\rangle = \lim_{n \rightarrow \infty} a_n \lambda_+^n \langle T_+, \mu \rangle = 0,$$

which is a contradiction.  $\square$

The other direction of this unique ergodicity type result is the same as for atoroidal case.

**Theorem 5.1.3.** *Let  $\varphi$  be non-atoroidal and fully irreducible. Let  $T_-, T_+$  be as in Theorem 5.1.2. Let  $\mu_+$  and  $\mu_-$  be representatives of stable and unstable currents corresponding to action of  $\varphi$  on  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$  accordingly. Then*

$$\langle T, \mu_{\pm} \rangle = 0 \iff [T] = [T_{\mp}].$$

*Proof.* We will prove that  $\langle T, \mu_- \rangle = 0 \iff [T] = [T_+]$ . The proof of the other assertion is similar. We have already proved in the previous theorem that  $\langle T_+, \mu_- \rangle = 0$ . Let us assume that  $\langle T, \mu_- \rangle = 0$  but  $[T] \neq [T_+]$ . Then, by [35], there exist a sequence of positive real numbers  $\{b_n\}$  such that

$$\lim_{n \rightarrow \infty} b_n T \varphi^{-n} = T_-.$$

Therefore, by continuity of the intersection number we get

$$0 \neq \langle T_-, \mu_- \rangle = \left\langle \lim_{n \rightarrow \infty} b_n T \varphi^{-n}, \mu_- \right\rangle = \left\langle \lim_{n \rightarrow \infty} b_n T, \varphi^{-n} \mu_- \right\rangle \lim_{n \rightarrow \infty} \lambda_-^n b_n \langle T, \mu_- \rangle = 0,$$

which is a contradiction.  $\square$

## 5.2 A subgroup alternative for $\text{Out}(\mathbb{F}_N)$

The following lemma is an adaptation of Lemma 3.1 of [14].

**Lemma 5.2.1.** *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be non-atoroidal and fully irreducible. Let  $[\mu_-], [\mu_+], [\mu_\alpha]$  be the unstable current, stable current and current corresponding to boundary curve respectively. Denote the convex hull of  $[\mu_-]$  and  $[\mu_\alpha]$  by  $H_-$  and the convex hull of  $[\mu_+]$  and  $[\mu_\alpha]$  by  $H_+$ .*

Assume that  $\psi \in \text{Out}(\mathbb{F}_N)$  is such that  $\psi H_+ \cap H_- = \emptyset$ . Then there exist an integer  $M \geq 1$  such that for all  $m \geq M$ , the element  $\varphi^m \psi$  is atoroidal.

*Proof.* Recall that, since  $\varphi$  is non-atoroidal and fully irreducible,  $\varphi$  is induced by a pseudo-Anosov  $g \in \text{Mod}^\pm(S)$ , where  $S$  is a compact surface with single boundary component and  $\pi_1(S) \cong \mathbb{F}_N$ . Therefore,  $\lambda_-(\varphi) = \lambda_+(\varphi) = \lambda$ , where  $\lambda$  is the dilatation for  $g$ . Let  $T_+$  and  $T_-$  be representatives of the attracting and repelling trees for  $\varphi$  in  $\overline{\text{cv}}_N$  so that  $T_+ \varphi = \lambda T_+$  and  $T_- \varphi^{-1} = \lambda T_-$ . Then for all  $m \geq 0$  and  $\nu \in \text{Curr}(\mathbb{F}_N)$

$$\langle T_+, \varphi^m \psi \nu \rangle = \langle T_+ \varphi^m, \psi \nu \rangle = \lambda^m \langle T_+, \psi \nu \rangle,$$

and

$$\langle T_- \psi, \psi^{-1} \varphi^{-m} \nu \rangle = \langle T_- \varphi^{-m}, \nu \rangle = \lambda^m \langle T_-, \nu \rangle.$$

Now define

$$\alpha_1(\nu) = \max\{\langle T_+, \nu \rangle, \langle T_- \psi, \nu \rangle\}$$

and

$$\alpha_2(\nu) = \max\{\langle T_+, \psi \nu \rangle, \langle T_-, \nu \rangle\}.$$

Then,

$$\alpha_1(\varphi^m \psi \nu) \geq \langle T_+, \varphi^m \psi \nu \rangle = \lambda^m \langle T_+, \psi \nu \rangle$$

and

$$\alpha_1(\psi^{-1} \varphi^{-m} \nu) \geq \langle T_- \psi, \psi^{-1} \varphi^{-m} \nu \rangle = \lambda^m \langle T_-, \nu \rangle$$

Hence

$$\max\{\alpha_1(\varphi^m \psi \nu), \alpha_1(\psi^{-1} \varphi^{-m} \nu)\} \geq \lambda^m \alpha_2(\nu)$$

Now  $\alpha_2(\nu) = 0$  if and only if  $\langle T_+, \psi \nu \rangle = 0$  and  $\langle T_-, \nu \rangle = 0$ . By Theorem 5.1.2  $\langle T_-, \nu \rangle = 0 \iff [\nu] \in H_+$ . Since by assumption  $\psi H_+ \cap H_- = \emptyset$ , this implies that  $\langle T_+, \psi \nu \rangle \neq 0$  again by Theorem 5.1.2. Therefore  $\alpha_2(\nu) > 0$ . So the ratio  $\alpha_1(\nu)/\alpha_2(\nu)$  defines a continuous function on the compact space  $\mathbb{P}\text{Curr}(\mathbb{F}_N)$ . Thus there exist a constant  $K$  such that  $\alpha_1(\nu)/\alpha_2(\nu) < K$  for all  $\nu \in \text{Curr}(\mathbb{F}_N) - \{0\}$ . Pick  $M \geq 1$  such that  $\lambda^M \geq K$ . Then we have

$$\max\{\alpha_1(\varphi^m \psi \nu), \alpha_1(\psi^{-1} \varphi^{-m} \nu)\} > \alpha_1(\nu)$$

for all  $m \geq M$  and for all  $\nu \in \text{Curr}(\mathbb{F}_N) - \{0\}$ .

**Claim.** The action of  $\varphi^m \psi$  on  $\text{Curr}(\mathbb{F}_N) - \{0\}$  does not have periodic orbits.

Let us set  $\theta = \varphi^m \psi$ . Assume that there exist a  $\nu \in \text{Curr}(\mathbb{F}_N) - \{0\}$  such that  $\theta^k(\nu) = \nu$  for some  $k \geq 1$ . Since  $\max\{\alpha_1(\theta\nu), \alpha_1(\theta^{-1}\nu)\} > \alpha_1(\nu)$  there are two cases to consider. If  $\alpha_1(\theta\nu) > \alpha_1(\nu)$  then by induction it is straightforward to show that  $\alpha_1(\theta^n\nu) > \alpha_1(\theta^{n-1}\nu) > \dots > \alpha_1(\nu)$  for all  $n \geq 1$ . Similarly,  $\alpha_1(\theta^{-1}\nu) > \alpha_1(\nu)$  implies that  $\alpha_1(\theta^{-n}\nu) > \alpha_1(\theta^{-(n-1)}\nu) > \dots > \alpha_1(\nu)$  for all  $n \geq 1$ . In any case, it is clear that  $\theta^k(\nu) \neq \nu$  for all  $k \geq 1$ , which is a contradiction.

Now, observe that if  $\theta = \varphi^m \psi$  had a periodic conjugacy class that would mean that  $\theta$  acts on  $\text{Curr}(\mathbb{F}_N) - \{0\}$  with a periodic orbit. So  $\varphi^m \psi$  does not have a periodic conjugacy class and hence it is atoroidal. □

**Proposition 5.2.2.** [14] *Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be a fully irreducible outer automorphism. Let  $[T_+]$  and  $[T_-]$  be the corresponding attracting and repelling trees in the closure of the projectivized Outer Space  $\overline{\text{CV}}_N$ . Assume  $\psi \in \text{Out}(\mathbb{F}_N)$  is such that  $[T_+\psi] \neq [T_-]$ . Then there is an  $M \geq 0$  such that for  $m \geq M$  the element  $\varphi^m \psi \in \text{Out}(\mathbb{F}_N)$  is fully irreducible.*

**Remark 5.2.3.** Note that  $\psi H_+ \cap H_- = \emptyset$  in fact implies that  $[T_+\psi] \neq [T_-]$ . Assume otherwise and look at the intersection number

$$0 = \langle T_+\psi, a\mu_+ + b\mu_\alpha \rangle = \langle T_+, \psi(a\mu_+ + b\mu_\alpha) \rangle \neq 0$$

by Theorem 5.1.2, which is a contradiction. Let  $\varphi \in \text{Out}(\mathbb{F}_N)$  be non-atoroidal and fully irreducible, and  $\psi \in \text{Out}(\mathbb{F}_N)$  be an element such that  $\psi H_+ \cap H_- = \emptyset$ . Now let  $M$  be the largest of the two in previous the two lemmas, then for all  $m \geq M$  the element  $\varphi^m \psi$  is a atoroidal and fully irreducible.

**Theorem 5.2.4.** *Let  $H \leq \text{Out}(\mathbb{F}_N)$  such that  $H$  contains a fully irreducible element  $\varphi$ . Then one of the following holds:*

1.  $H$  contains an atoroidal and fully irreducible element.
2.  $H$  is geometric, i.e.  $H$  contains no atoroidal elements and  $H \leq \text{Mod}^\pm(S) \leq \text{Out}(\mathbb{F}_N)$  where  $S$  is a compact surface with one boundary component with  $\pi_1(S) = \mathbb{F}_N$  such that  $\varphi \in H$  is induced by a pseudo-Anosov homeomorphism of  $S$ .

*Proof.* If the fully irreducible element  $\varphi$  is atoroidal, then (1) holds and there is nothing to prove. Suppose now that the fully irreducible element  $\varphi$  is non-atoroidal. Then  $\varphi$  is induced by a pseudo-Anosov homeomorphism on a surface  $S$  with one boundary component  $\alpha$  and

$\pi_1(S) \cong F_N$ . Note that if  $[g]$  is the conjugacy class in  $F_N$  corresponding to the boundary curve  $\alpha$  of  $S$ , then  $\varphi$  fixes  $[g]$  up to a possible inversion. In this situation, the extended mapping class group  $\text{Mod}^\pm(S)$  is naturally included as a subgroup of  $\text{Out}(F_N)$ . Moreover, by the Dehn-Nielsen-Baer theorem [40], the subgroup of  $\text{Out}(F_N)$ , consisting of all elements of  $\text{Out}(F_N)$  which fix  $[g]$  up to inversion is exactly  $\text{Mod}^\pm(S)$ . If  $H \leq \text{Mod}^\pm(S)$ , then part (2) of Theorem holds and there is nothing to prove. Assume now that  $H$  is not contained in  $\text{Mod}^\pm(S)$ . Then there exist an element  $\psi \in H$  such that  $\psi([g]) \neq [g^{\pm 1}]$ .

**Claim.** Let  $\psi \in \text{Out}(F_N)$  be an element such that  $\psi([g]) \neq [g^{\pm 1}]$ . Then,

$$\psi(H_+) \cap H_- = \emptyset.$$

First,  $\psi[t_1\mu_\alpha + t_2\mu_+] = [t'_1\mu_\alpha + t'_2\mu_-]$  implies  $\psi[\mu_+] = [\mu_-]$ . Indeed, only periodic leaves in the support of the right hand side are leaves labeled by powers of  $g$ . Therefore,  $t_1 = 0$ . Similarly, it is easy to see that  $t'_1 = 0$ . From here we note that  $[T_+\psi] = [T_-]$ . To see this, look at the intersection number

$$0 \neq \langle T_+, \mu_- \rangle = \langle T_+, \psi\mu_+ \rangle = \langle T_+\psi, \mu_+ \rangle,$$

which implies that  $T_+\psi = cT_-$  for some  $c > 0$  by Theorem 5.1.3. Therefore we get

$$0 = c\|g\|_{T_-} = \|g\|_{T_+\psi} = \|\psi(g)\|_{T_+},$$

which implies  $\psi([g]) = [g^{\pm 1}]$  since only conjugacy classes that have translation length 0 are peripheral curves. This contradicts the choice of  $\psi$ . Hence  $\psi(H_+) \cap H_- = \emptyset$ , and the Claim is verified.

Since  $\psi(H_+) \cap H_- = \emptyset$ , Remark 5.2.3 implies that  $[T_+\psi] \neq [T_-]$ . Therefore, by Lemma 5.2.1 and Proposition 5.2.2, there exists  $m \geq 1$  such that the element  $\varphi^m\psi \in H$  is atoroidal and fully irreducible.  $\square$

**Remark 5.2.5.** Now assume that the original  $\varphi \in \text{Out}(F_N)$  in the statement of Theorem 5.2.4 is non-atoroidal and (1) holds, i.e. there is an element  $\theta \in H$  which is atoroidal and fully irreducible. Then  $H$  is not virtually cyclic, since otherwise some positive power of a non-atoroidal fully irreducible  $\varphi$  would be equal to some positive power of an atoroidal and fully irreducible  $\theta$ . Therefore,  $H$  is a subgroup of  $\text{Out}(F_N)$  which is not virtually cyclic and contains an atoroidal fully irreducible element  $\theta$ . Therefore by Corollary 6.3 of [32],  $H$  contains a free subgroup  $L$  of rank 2 such that all nontrivial elements of  $L$  are atoroidal and

fully irreducible.

**Corollary 5.2.6.** *Let  $N \geq 3$  and  $H \leq \text{Out}(\mathbb{F}_N)$  be a nontrivial normal subgroup. Then  $H$  contains an atoroidal and fully irreducible element.*

*Proof.* By Lemma 5.1 of [11], the subgroup  $H$  contains a fully irreducible element  $\varphi$ . If  $\varphi$  is atoroidal, we are done. Assume that  $\varphi$  is non-atoroidal and hence induced by a pseudo-Anosov map on a compact surface  $S$  with one boundary component  $\alpha$ .

**Claim.** If  $H_0 \leq \text{Mod}^\pm(S)$  contains a fully irreducible element  $\varphi$ , then  $H_0$  is not normal in  $\text{Out}(\mathbb{F}_N)$ .

Suppose, on the contrary, that  $H_0$  is normal in  $\text{Out}(\mathbb{F}_N)$ . Choose an atoroidal element  $\eta \in \text{Out}(\mathbb{F}_N)$  (such  $\eta$  exists since  $N \geq 3$ ). Put  $[g_2] = \eta[g_1]$ , where  $[g_1]$  represents the boundary curve of  $S$ . Put  $\varphi_1 = \eta\varphi\eta^{-1}$ . Since  $H_0$  is normal, then  $\varphi_1 \in H_0$ . Since  $\eta$  has no periodic conjugacy classes, we have  $[g_2] \neq [g_1^{\pm 1}]$ . Then  $\varphi_1[g_2] = \eta\varphi\eta^{-1}[g_2] = [g_2]$ . Since  $\varphi$  is non-atoroidal and fully irreducible, the element  $\varphi_1 = \eta\varphi\eta^{-1}$  is also non-atoroidal and fully irreducible, and  $[g_2]$  is a periodic conjugacy class for  $\varphi_1$ . A non-atoroidal fully irreducible element has a unique, up to inversion, nontrivial periodic conjugacy class  $[g]$  such that  $g \in \mathbb{F}_N$  is not a proper power. Since  $\varphi_1[g_2] = [g_2]$ ,  $[g_2] \neq [g_1^{\pm 1}]$  and  $g_2$  is not a proper power, it follows that  $\varphi_1[g_1] \neq [g_1^{\pm 1}]$ . Hence  $\varphi_1 \notin \text{Mod}^\pm(S)$ , contrary to the assumption that  $H_0 \leq \text{Mod}^\pm(S)$ . This verifies the Claim.

Since  $H$  is normal in  $\text{Out}(\mathbb{F}_N)$  and contains a non-atoroidal and fully irreducible element  $\varphi$  coming from a pseudo-Anosov element of  $\text{Mod}^\pm(S)$ , the Claim implies that  $H$  is not contained in  $\text{Mod}^\pm(S)$ . Therefore by Theorem 5.2.4,  $H$  contains an atoroidal fully irreducible element, as required.  $\square$

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