THREE ESSAYS ON ECONOMICS OF REHYPOTHECATION

BY

HYE JIN PARK

DISSERTATION

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Doctoral Committee:

Professor Charles M. Kahn, Chair
Professor Daniel M. Bernhardt
Assistant Professor Jorge Lemus
Assistant Professor Guillermo Marshall

ABSTRACT

In the first chapter of my thesis, I develop a theoretical model of rehypothecation, a practice in which financial institutions re-use or re-pledge collateral pledged by their clients for their own purposes. I show that rehypothecation has trade-off effects; it enhances provision of funding liquidity to the economy so that additional productive investments can be undertaken, but incurs deadweight cost by misallocating the asset among the agents when it fails. Next, I show that the intermediary's choices of rehypothecation may not achieve a socially optimal outcome. The direction of the conflict between the objectives of the intermediary and social efficiency depends on haircuts of the contract between the intermediary and the borrower; if the contract involves over-collateralization, there tends to be an excessive use of rehypothecation by the intermediary, and if the contract involves under-collateralization, there tends to be an insufficient use of rehypothecation.

In the second chapter, I extend the previous model into a dynamic economy with aggregate uncertainty that financial intermediaries might default having repledged their clients' collateral. I discuss how individual reuse decisions, allocation of collateral, and aggregate output vary across (i) different temporary shocks and (ii) different persistent shocks. I show that when negative temporary shocks reduce the borrower's willingness to allow rehypothecation, the economy drops further, but it recovers faster. In addition, I show that a more protracted period of good shocks can lead to a greater fall in output in the future.

In the third chapter, I consider the competition between direct financing and rehypothecation. A borrower faces two alternative ways of financing: one option is to borrow funding directly from a cash holder (direct financing), another option is to borrow funding through an intermediary (indirect financing). With direct financing, the borrower delivers collateral directly to the cash holder with some transaction costs. With indirect financing, the

borrower delivers collateral to the intermediary who then lends (rehypothecates) it to the cash holder, and this exposes the borrower with the risk of losing collateral in case that the intermediary defaults. I investigate how the severity of the borrower's moral hazard problem affects the borrower's choice between these two alternative ways of financing. If the intermediary's default risk is exogenous, as the moral hazard problem gets more severe, the borrower is less concerned about the default risk of the intermediary, resulting in indirect financing being chosen more frequently. However, if the intermediary's default risk is endogenous, as the moral hazard problem gets more severe, the cost of indirect financing also increases, resulting in indirect financing not being chosen in both cases that the severity of the moral hazard problem is too small or too large.

To Juhan and my family.

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CHAPTER 1

COLLATERAL, REHYPOTHECATION AND EFFICIENCY

1.1 Introduction

Most financial contracts are in the form of promises to pay a certain amount of money or exchange assets on a later date at pre-arranged terms. But often these promises cannot be warranted themselves, and they need to be backed by an eligible asset or property, called collateral, such as Treasury bills in repo transactions and residential houses in mortgage contracts. Generally, collateral in finanancial contracts plays two crucial roles as emphasized in Mills and Reed (2012): (i) first, collateral provides a borrower with incentives to repay to avoid forfeiting it; (ii) second, collateral provides a lender with some insurance allowing him to collect some revenue by liquidating it in the event that the borrower defaults. In order that a certain asset can be used as collateral, however, it has to be sufficiently valuable especially to the borrower so that the lender can be assured that the borrower will repay the loan to get back the collateral.

Nonetheless, such assets that can be used as collateral are scarce in the economy and the cost of generating these assets also non-negligible. In particular, as the volume of financial transactions has sharply increased over the last few decades, the demand for collateral has also significantly increased, and economizing on the existing limited amount of collateral has become an important issue for market participants.[1]

Probably the easiest way to save on collateral is by re-using it. In most cases, collateral sits idle in the lender's account until the borrower repays the

[1] Krishnamurthy and Vissing-Jorgensen (2012) estimated the liquidity and safety premium on Treasuries paid by investors on average from 1926 to 2008 was 72 basis points per year, which supports the idea that there has been a large and persistent demand for safe and liquid assets in the economy. Similarly, Greenwood, Hanson, and Stein (2012) emphasize the monetary premium embedded in short-term Treasury bills, which has a lower yield than would be in a conventional asset-pricing literature.

loan to get it back. Clearly, during the time that the collateral is deposited in the lender's account, it ties up capital that the lender might have other profitable uses for. One way that the lender can access that capital is to make a loan by re-pledging the collateral (initially pledged by his borrower) to another party. From the view of liquidity provision, this re-using collateral is socially beneficial because it reduces the cost of holding collateral for the lender, and ultimately it would benefit the borrower since the lender would be willing to provide more funding against the same unit of the collateral posted by the borrower. From the view of the economy as a whole, the same collateral is used to support more than one transaction, and it creates a 'collateral chain' in the system which increases interdependence among the agents.

Inarguably, rehypothecation has been one of the most popular devices for many broker-dealder banks to serve their own funding liquidity needs before the crisis. However, as reported by Singh (2010, 2011), after the failure of Lehman Brothers in 2008, rehypothecation significantly dropped as hedge funds (the clients of those investment banks) became wary of losing access to their collateral, and limited the amount of the assets that are permitted to be re-pledged. At the same time, regulation on rehypothecation has also been advocated by legislators and policy-makers.[2] Nevertheless, understaning of the economics underlying this practice is still incomplete, and there are still considerable debates on how to regulate rehypothecation, as evidenced by the wide variation in the rules on rehypothecation across different nations.[3]

In this chapter, I address some basic, but open questions about this practice of re-using collateral: under what circumstances 'rehypothecation' – the practice in which the receiver of collateral re-uses, re-pledges, or sometimes even sells the collateral to another party for its own trading or borrowing – arises, how it creates a collateral chain in the system; what benefits and costs it produces, and whether decentralized decisions made by each individual to participate in rehypothecaion achieves a socially efficient outcome.[4]

[2]On the regulatory side, the Dodd-Frank Act requires the collateral in most swap contracts to be held in a segragated account of a central counterparty.

[3]Under SEC rule 15c3-3, a prime broker may rehypothecate assets to the value of 140% of the client's liability to the prime broker. In the U.K., there is no limit on the amount that can be rehypothecated. See Monnet (2011) for more detailed explanation on the difference in regulatory regimes on rehypothecation across countries.

[4]The material in this chapter is extended in Kahn and Park (2016a). Applications to policy are considered in Kahn and Park (2016b).

To answer these questions, I adopt the framework of Bolton and Oehmke (2014) that is in turn based on Biais, Heider, and Hoerova (2012), in which a borrower is subject to a moral hazard problem and must post collateral to prevent him from engaging in risk-taking actions. In this framework, there can be positive NPV investments which a borrower with limited liability cannot undertake without posting collateral.[5] Previous models, however, have not considered the risk on the other side that the lender might fail to return the collateral as well as any incentives to use it for their own purposes. In contrast, our model incorporates the possibility of re-using collateral by the counterparty and the risk associated with it, thereby offering the first formal welfare analysis on rehypothecation.

Another important feature of my model is that the borrower transfers collateral to the lender at the time of the beginning of the contract. In other words, collateral is like a repurchase agreement in Mills and Reed (2012): the borrower transfers his asset to the lender at the time a contract is initiated and buys it back at a later point. This contrasts to most of the previous works on collateral in which collateral is transferred to the lender after final pay-offs are realized, or at the time when the default of the borrower actually occurs.

This early transfer of collateral, however, introduces the risk that the lender may not be able to return collateral at the time when the borrower wants to repurchase it.[6] Indeed, as observed from the failure of Lehman Brothers in 2008 and MF Global in 2011, this is not simply a theoretical possibility. In consideration of this, I introduce counterparty risk – the lender might be unable to return the collateral – into the baseline framework, and I show that if the risk is too high, it makes it too costly for the borrower to post its asset as collateral. As a result, the positive NPV project of the borrower cannot be undertaken in this case since non-collateralized borrowing is not feasible when the borrower is subject to moral hazard.

Building on this basic intuition in the two-player model, I extend it into

[5] Holmström and Tirole (1998, 2011) shows that the moral hazard problem of the borrower makes the firm's pledgeable income less than its total value, which leads to a shortage of liquidity for its investment in some states. Also, Shleifer and Vishny (1992), Bernanke, Gertler, and Gilchrist (1994), and Kiyotaki and Moore (1997) concern a firm's financing problem constrained by its net wealth.

[6] Mills and Reed (2012) discuss the effect of this counterparty risk on the form of the optimal contract in a different context.

the three-player model to more explicitly describe how rehypothecation introduces the risk of counterparty failure and specifies the condition under which rehypothecation is socially efficient. Our results show that the efficiency of rehypothecation is determined by the relative size of the two fundamental effects. Rehypothecation lowers the cost of holding collateral and makes the illiquid collateral more liquid, thereby providing more funding liquidity into the market. On the other hand, rehypothecation failure – the counterparty failure to return the collateral to the borrower who posted it – may incur deadweight costs in the economy.

One difficulty in this general argument is that it is not obvious through which channel the rehypothecation failure incurs deadweight costs, and this has not been clearly addressed in most of the previous works on rehypothecation[7] While there could be several channels through which the rehypothecation failure incurs deadweight costs in the economy, this paper focuses on the possibility that rehypothecation failure leads to misallocation of the assets posted as collateral.

This misallocation of assets crucially depends on the following two types of market frictions: (i) I assume that the asset is 'illiquid' in the sense that the asset is likely to be more valuable to the initial owner than to other agents – for example, a stock, bond, or security included in A's portfolio is likely to fit better for A's portfolio but not for the other's; (ii) I also consider a possibility that some traders may not have access to some parts of the markets, and they can trade indirectly each other only through an intermediary (who has an access to all the markets). In the model, the asset provider and the investor make separate contracts with the intermediary who transfers the collateral between them. Taken together, if the intermediary fails, the asset ends up in the wrong hands: the asset cannot be returned to the initial owner (the asset provider) who values it the most, but instead it is seized by the third party (the investor) for whom the collateral may not be as useful.

Finally, I examine whether an individual agent's decision to participate in rehypothecation achieves a socially optimal outcome. To answer this question, I endogenize each individual's participation decision in the rehypothecation, and investigate whether their objectives are aligned with social efficiency. I show that in general, the ex-post objective of the intermediary (the

[7] See the literature review in Kahn and Park (2016a) for further discussion.

lender of the initial borrower in the model) may conflict with what would be ex-ante efficient.

The direction of this conflict between the intermediary's objective and the socially efficient choice depends on the terms of the contract between the intermediary and his borrower. If the contract involves over-collateralization, in the sense that the value of collateral to the borrower exceeds the payment for recovering it, there tends to be an excessive use of rehypothecation by a holder of collateral. Intuitively, this is because when the contract involves over-collateralization, there is a negative externality of not returning the collateral to the borrower, which is reflected in the spread between the borrower's private value on his collateralized asset and the payment for recovering it. From the perspective of the borrower, he is supposed to repurchase the collateral at a price lower than his valuation on it. However, the intermediary does not internalize this private cost to the borrower from failure of returning collateral, and thus he sometimes wants to participate in rehypothecation even when rehypothecation is inefficient – that is, when the social cost (which includes this private cost to the borrower from rehypothecation failure) exceeds the benefit. Similarly, if the contract involves under-collateralization, there tends to be an insufficient use of rehypothecation.

1.2 A Baseline Model

There are two periods, date 0 and date 1, and two types of agents in the economy, a firm A and an outside investor B. All agents are risk-neutral and consume at the end of date 1. For simplicity, the price of date 1 consumption good is normalized to be 1.

At date 0, firm A has an opportunity of an investment which requires an immediate input at date 0 to produce an outcome at date 1. We assume that the outcome of the investment is uncertain and can take two values; if the investment succeeds, the investment produces $R > 1$ units of the date 1 good (measured per unit of inputs) and if it fails, it produces zero units,

$$\text{outcome of investment} = \begin{cases} R & \text{if the investment succeeds} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

These outcomes are costlessly observable to the outside investors.

However, at date 0, A is endowed with no capital that can be spent as an input for his investment, but only one unit of indivisible asset which is illiquid in the following two senses. First, it yields the consumption goods only at the end of date 1. Second, it produces more output when it is in the hands of the initial owner, firm A, than in the hands of the outside investor, B – think of the asset as an intermediate good that the initial owner uses it for its own production and he has a better skill to manage it than do the other agents in the economy. Specifically, I assume that the asset yields Z units of the good if it is held by A at the end of date 1, while it yields $Z_0 (< Z)$ units of the good if it is held by B at that time.

On the other hand, the outside investor B is endowed with a large amount of capital that can be spent as an input for A's investment. Thus, A cannot undertake the project alone, and has to borrow capital from B. For simplicity, we assume that A tries to borrow funds for his investment from B by issuing simple debt; A receives cash from B at date 0 by promising to pay a certain amount of his investment outcome to B at date 1.

1.2.1 Moral Hazard and Limited Liability

Following the approach of Bolton and Oehmke (2014), we assume that the probability of the success of the borrower's investment depends on his hidden action. We assume that A can choose either a safe or a risky action, denoted by $a \in \{s, r\}$ where $a = s$ represents the safe action and $a = r$ represents the risky action. The safe action leads to a high probability of success of the investment, which we take for simplicity to be 1 and the risky action lower probability of success, $p < 1$,

$$\text{probability of success} = \begin{cases} 1 & \text{if A takes safe action } (a = s) \\ p & \text{if A takes risky action } (a = r). \end{cases} \tag{1.2}$$

On the other hand, the risky action gives firm A a private benefit $b > 0$ (measured per unit of inputs).

Taken together, the (expected) average return of the investment is given by

$$\text{return of the investment} = \begin{cases} R & \text{if A takes safe action } (a = s) \\ pR + b & \text{if A takes risky action } (a = r). \end{cases} \tag{1.3}$$

In addition we assume that the parameters satisfy the following two assumptions.

Assumption 1. $\min\{R, pR + b\} > 1 > pR$.

The first inequality implies that it is efficient for A to undertake the project regardless of his action. The second inequality implies that, from the persepective of B, it is profitable only if A takes the safe action. To understand the second inequality, suppose B invests capital I into A's project. Then, if A takes the safe action, the maximum level of the expected payment by A is RI, which is greater than the investment cost I by the first inequality, and if A takes the risky action, it reduces to pRI (note that the private benefit bI cannot be pledgeable), which is smaller than the investment cost I by the second inequality.

Assumption 2. $R - 1 < p(R - 1) + b$.

This assumption implies that when A invests with the borrowed money from B, A will always find it profitable to take the risky action rather than the safe action for any given contract (I, X) such that $X \geq I$ (or, equivalently, for any contract with a positive interest rate). To see this, notice that the left side represents A's expected net surplus (per unit of the inputs) after paying out the investment cost to B if he takes the safe action and the right side represents A's expectied net surplus if he takes the risky action.

Combining assumptions 1 and 2, one can conclude that A cannot borrow funds for his investment from B, because B expects that A will take the risky action after borrowing, and he will end up with negative profit. Formally, Assumption 2 implies that A will always take the risky action after borrowing, and the expected loan payment by A will be at most pR (per unit of inputs), but this is not enough to cover the cost that B spent for A's project due to Assumption 1.

1.2.2 Benchmark: Uncollateralized Borrowing

Let us start by considering the contracting problem between A and B where A issues debt which is solely backed by the future return of the investment. For simplicity, we assume that A has all the bargaining power and makes a take-it-or-leave-it offer to B. We assume that B's outside option pays utility of $J \geq 0$, and B will accept the offer as long as he can receive utility greater or at least equal to J.

Timing is as follows. At date 0, A borrows the investment cost, denoted by I_a, from B, and then takes either the safe or risky action, denoted by $a \in \{s, r\}$. At date 1, the investment outcome is realized and A pays a part of the return of the investment, denoted by X_a, to B (hereafter, the subscript a stands for which type of action is taken by A).

Note that depending on A's action, the optimal contract takes either of the two forms: in one case, A takes the safe action and in the other case, A takes the risky action. Let us first consider the case in which A takes the safe action, $a = s$. In this case, the contracting problem is to choose (I_s, X_s) which solves the following maximization problem.

$$\max_{I_s, X_s} RI_s - X_s \tag{1.4}$$

subject to

$$RI_s - X_s \geq p(RI_s - X_s) + bI_s \tag{IC_s}$$

$$X_s - I_s \geq J \tag{P_s}$$

$$RI_s \geq X_s \tag{R_s}$$

The objective function is A's expected utility when A takes the safe action. With probability 1, the investment yields the return RI_s and A has the remaining amount after paying off the loan X_s out of this to B. The incentive constraint (IC_s) implies that A's expected profit when A takes the safe action on the left side is greater than that when A takes the risky action on the right hand side. The participation constraint (P_s) ensures that B's expected profit from lending cannot be less than the reservation value J. Lastly, the resource constraint (R_s) says that A cannot pay more than what he has, that is, the payment is bounded above by the return from the investment when it succeeds, RI_s (note that if the investment fails, it yields zero ouput, and A does not make any payments to B).

Next, consider the case in which A takes the risky action, $a = r$. In this case, the contracting problem is to choose (I_r, X_r) which solves the following problem,

$$\max_{I_r, X_r} p(RI_r - X_r) + bI_r \tag{1.5}$$

subject to

$$RI_r - X_r \leq p(RI_r - X_r) + bI_r \tag{IC_r}$$

$$pX_r - I_r \geq J \tag{P_r}$$

$$RI_r \geq X_r \tag{R_r}$$

The objective function is A's expected utility when A takes the risky action. A obtains the return RI_r from the investment and pays off the loan X_r with probability p and also receives the private benefit bI_r from misbehavior. The incentive constraint (IC_r) implies that A's expected profit when A takes the risky action which is on the right hand side is greater than that when A takes the safe action which is on the left hand side. The participation constraint (P_r) and the resource constraint (R_r) are the same as in the previous case.

Taking the two subcases together, the optimal contract is to choose a profile of (I_a, X_a, a) where $a \in \{s, r\}$ which solves the following problem.

$$\max_{a \in \{s,r\}} \mathbf{1}_S(a)(RI_s - X_s) + (1 - \mathbf{1}_S(a))[p(RI_r - X_r) + bI_r] \tag{1.6}$$

where $\mathbf{1}_S(\cdot)$ is the indicator function where $S = \{s\}$ and (I_s, X_s) solves subproblem (1.4) and (I_r, X_r) solves subproblem (1.5).

However, the solution to the maximization problem above may not exist under some parameter values. In other words, it may not be feasible to finance A's project by issuing simple debt, which is solely backed by the future return of the project.

Lemma 1. *Suppose Assumption 1 and 2 hold. Then, uncollateralized debt financing for A's project is not feasible.*

1.2.3 Collateralized Borrowing

In the previous section, we showed that A's project cannot be funded with uncollateralized debt if Assumption 1 and 2 hold. Suppose now that A is required to post his endowed asset, which is worth Z to A himself and $Z_0 < Z$ to B, as collateral. In this section, we show that in such case, posting collateral helps A's investment to be funded in the following two ways: (i) posting collateral incentivizes A to take the safe action by introducing the risk of forfeiting it if he defaults; (ii) collateral provides B with some compensation in case that A defaults by allowing B to seize it.

To show this formally, let us consider the contracting problem between A and B when A posts his asset as collateral. As before, A is assumed to have all the bargaining power and makes a take-it-or-leave-it offer to B. At date 0, A borrows the investment cost I_a from B and deposits his endowed asset in B's account (or, pledges it as collateral), and then takes either the safe or risky action, $a \in \{s, r\}$. At date 1, if the investment succeeds, A makes the promised payment X_a to B, or if the investment fails, A defaults and B seizes the asset posted by A.

As in the previous section, in order to solve for the optimal contract, we consider the two possible cases separately. First, we begin with the case in which A takes the safe action, $a = s$. The contracting problem between A and B in this case is to choose (I_s, X_s) to solve the following maximization problem.

$$\max_{I_s, X_s} \; RI_s - X_s \tag{1.7}$$

subject to

$$RI_s - X_s \geq p(RI_s - X_s) + bI_s - (1-p)Z \tag{IC'_s}$$

$$X_s - I_s \geq J \tag{P'_s}$$

$$RI_s \geq X_s \tag{R'_s}$$

The objective function is as before. The incentive constraint (IC'_s) now has the additional term $-(1-p)Z$ on the right hand side. This captures the fact that there is now an additional loss from taking the risky action, which is calculated by the probability of default, $1-p$ times the private value of collateral to A, Z. In constrast, if A takes the safe action, A will always get back his collateral. Hence, when posting collateral, the return when taking the safe action increases relative to that when taking the risky action, thereby incentivizing A to take the safe action. The participation constraint (P_s') and the resource constraint (R_s') are the same as before.

Next, consider the case in which A takes the risky action. In this case, the contracting problem is to choose (I_r, X_r) which solves the following problem,

$$\max_{I_r, X_r} p(RI_r - X_r) + bI_r - (1-p)Z \tag{1.8}$$

subject to

$$RI_r - X_r \leq p(RI_r - X_r) + bI_r - (1-p)Z \tag{IC_r'}$$

$$pX_r - I_r + (1-p)Z_0 \geq J \tag{P_r'}$$

$$RI_r \geq X_r \tag{R_r'}$$

The objective function is A's expected utility when A takes the risky action which is the same as before except that there is additional term $-(1-p)Z$, which captures the cost of losing collateral in case of default with probability $1-p$ when A takes the risky action. The incentive constraint (IC_r') shows that compared to the case without collateral, the return from taking the risky action on the right hand side decreases by $(1-p)Z$ due to the loss of value from forfeiting it in case of default. The participation constraint (P_r') has the additional term $(1-p)Z_0$, which means the compensation value that B earns from liquidating collateral, Z_0, if A defaults with probability $1-p$. Again the resource constraint (R_r') is the same as in the previous case.

Taking these together, the optimal solution is a profile of (I_a, X_a, a) which solves the following problem.

$$\max_{a \in \{s,r\}} \mathbf{1}_S(a)(RI_s - X_s) + (1 - \mathbf{1}_S(a))[p(RI_r - X_r) + bI_r - (1-p)Z] \tag{1.9}$$

where $\mathbf{1}_S(\cdot)$ is the indicator function where $S = \{s\}$ and (I_s, X_s) solves subproblem (1.7) and (I_r, X_r) solves subproblem (1.8).

Optimal Contract

Our next result shows that if B's outside utility J is sufficiently small, there exists a solution to the problem described above, and to characterize it.

Proposition 1 (Optimal Contract under Collateralized Borrowing)**.** *Suppose Assumption 1 and 2 hold. If* $J \leq \min\left\{\frac{1-p}{b}(R-1)Z, \frac{1-p}{b}\left(pR + b(Z_0/Z) - 1\right)Z\right\}$, *there exists an optimal solution to problem (1.9). In this solution*

$$(I_a, X_a; a) = \begin{cases} \left(\frac{Z-J}{1-\mathcal{B}}, \frac{Z-\mathcal{B}J}{1-\mathcal{B}}; s\right) & \text{if} \quad U_s \geq U_r \\ \left(\frac{(1-p)Z_0-J}{1-pR}, R\frac{(1-p)Z_0-J}{1-pR}; r\right) & \text{if} \quad U_s < U_r \end{cases} \tag{1.10}$$

where $\mathcal{B} \equiv R - \frac{b}{1-p}$, $U_s \equiv RI_s - X_s$, *and* $U_r \equiv (pR+b)I_r - pX_r - (1-p)Z$.

Depending on parameter values, either the safe or the risky action can arise in the optimal contract. In either case, the participation constraint is binding at the optimum; player B receives exactly J in value.

In the subcase where the risky action is optimal ($a = r$) the resource constraint is also binding; thus I_r and X_r are defined by the two equalities:

$$pX_r + (1-p)Z_0 = J + I_r \tag{1.11}$$

$$RI_r = X_r \tag{1.12}$$

In other words, when the investment is successful the entirety of the payout is given to B. Since this is not enough alone to compensate for the initial investment by B, the remnant of the compensation comes from the value of the collateral to B; the more valuable the collateral, the larger the initial investment.

Roughly speaking, the parameter J measures the profitability of B's lending activity. If lending is sufficiently competitive (J close to 0), the investment in the risky subcase tends to be undercollateralized; at least relative to B's valuation, collateral is less than the required repayment, $Z_0 < X_r$.

In the subcase where the safe action is optimal repayment cannot be pushed to the limit of the resource constraint, for if A were forced to pay out the full amount of the proceeds of the investment, he would not be willing to take the safe action. Instead the incentive constraint binds first, and so I_s and X_s are defined by the two equalities:

$$RI_s - X_s = p(RI_s - X_s) + bI_s - (1-p)Z \tag{1.13}$$

$$X_s - I_s = J \tag{1.14}$$

As before, increases in the value of the collateral relax the constraints on the problem and increase the amount of investment. Here, however, the relevant value is the value to the borrower, not the lender, because the collateral is being used as an incentive, not a repayment. Again roughly speaking, the need to maintain the incentives for safe behavior increases the collateral needed to back the borrowing. As J approaches 0, whether this extra consideration is sufficient to lead to overcollateralization depends on the sign of the quantity $\mathcal{B}$; if it is negative, then $X_s < Z$.

The effect of parameter values on the choice between the safe and risky subcases can be analyzed by using the results of the proposition. For example in the case where $J = 0$ and $p = 0$, the formulas for A's utility under the two subcases reduce to

$$U_s = \frac{(R-1)Z}{1+b-R}, \quad U_r = bZ_0 - Z.$$

The risky action becomes relatively more attractive as the private benefit b increases and B's evaluation of collateral, Z_0, increases. The safe action becomes relatively more attractive as its return increases and as the value to A of retaining the collateral increases.

1.3 Rehypothecation model

1.3.1 Players and Endowments

There are three periods, date 0, 1 and 2, and three players, A, B, and C. I assume that all the agents are risk-neutral and consume only at the end of date 2. For simplicity, I take the price of date 2 consumption good to be 1.

At date 0, A has an opportuntiy of an investment which has the same feature as in the previous baseline model, except that it produces the outcome after two periods, at date 2, not date 1. Also, I assume that at date 0, A is endowed with a single unit of indivisible illiquid asset, which is again the same as in the previous model, except that it yields the goods at the end of date 2. On the other hand, B is endowed with a large amount of capital which can be used as an input for A's investment at date 0.

At date 1, B has an investment opportunity which requires an immediate input at that time to produce an outcome at date 2, but B does not have capital that can be spent as an input for his project nor any other pledgeable assets.[8] On the other hand, C does not have access to B's investment, but has a large amount of capital that can be spent as an input for B's investment. Thus, in order that B's investment is undertaken, capital must be transferred from C to B. As in the previous section, I assume that B issues simple debt to borrow funds for his investment; B borrows funds from C at date 1 by promising to pay a part of his investment outcome to C at date 2.

B's project produces a positive outcome, Y (measured per unit of the investment cost) if it succeeds or zero if it fails. Let θ be the probability of success.

$$\text{outcome of B's investment} = \begin{cases} Y & \text{with prob. } \theta \\ 0 & \text{with prob. } 1-\theta \end{cases} \tag{1.15}$$

In addition, B's investment is productive in the sense that the expected return of B's investment is greater than the investment cost (both are measured per unit of inputs).

Assumption 3. $\theta Y > 1$.

However, the outcome of B's investment is *not* verifiable to its creditor, C. For example, even when the project succeeds, B can falsely report that his investment fails, and can avoid paying the loan to C. This implies that debt financing solely backed by the future return of the investment is not feasible for B.[9]

[8] Equivalently, I can assume that B's endowment at date 0 cannot be storable until the next period when he wants to use it as an input for his investment, for example, B faces a liquidity mismatch problem.

[9] In general, I may assume that some of the future return of B's investment can be pledgeable, but as long as it is not fully pledgeable, B cannot raise enough funding for the investment soley backed by its future returns.

1.3.2 Sequential Contracts: Collateral Chain

Suppose B is now allowed to repledge A's collateral deposited in his account to borrow funds from C, in other words, B can rehypothecate A's collateral. This can help to raise funds for B's investment, which otherwise cannot be funded on its own. By posting A's collateral, B can assure C that he will make the payment to recover the collateral, so that he can receive the payment by returning it to A – note that in effect the debt between A and B is also transferred to C when the collateral is transferred from B to C.[10] In addition, collateral provides some compensation to C even by allowing C to seize the collateral in case that B defaults.

Formally, under rehypothecation, the same single unit of collateral is used to support more than one transaction, the contract between A and B at date 0 and that between B and C at date 1, thereby creating collateral chain in the economy. In this three-period model, these two contracts arise sequentially, and the timing of the model is as follows.

- At date 0, A borrows funds for his investment, denoted by $I_a^\dagger$, from B by pledging his asset, which is worth Z to himself and $Z_0 (< Z)$ to the others, and promises to pay a part of the investment outcome, denoted by $X_a^\dagger$, to B, conditional B's returning collateral to A. After entering the contract, A then takes an action, either safe or risky, $a \in \{s, r\}$.

- At date 1, B has an investment opportuntiy, and decides whether to repledge A's collateral and undertake the investment, or keep A's collateral to return it safely to A in the next period. If B decides to rehypothecate, B borrows funds for his investment, denoted by $I_a^\ddagger$, from C by repledging A's collateral, and promises to pay a part of the investment outcome, denoted by $X_a^\ddagger$, to C for recovering A's collateral from C.

- At date 2, both A's and B's investment outcomes are realized and the contracts are executed according to the prearranged terms. If B pays

[10]For the transferability of debt, see Kahn and Roberts (2007) and Donaldson and Micheler (2015). As the conditions for debt to be transferable, Kahn and Roberts (2007) assume that the debtor and an entity who holds the debt claim can meet at some point in the future and the enforceability of debts does not diminish when it is transferred between agents. In our model, neither of these two conditions hold, and debt can be transferrable only when the collateral supporting it is transferred simultaneously.

$X_a^{\ddagger}$ he repurchases A's collateral from C, and then returns it to A to receive $X_a^{\dagger}$. However, if B defaults, C seizes A's collateral repledged by B, and it cannot be returned to A, while at the same time, A is also exempt from paying the loan $X_a^{\dagger}$ to B.

1.4 Model Solution

Next, I solve for the optimal contract in this model. Throughout this section, I focus on the case in which B's decision whether to rehypothecate A's collateral at date 1 is exogenously given – I will endogenize this into the model later on. First, consider the case in which B does not rehypothecate A's collateral at date 1. In that case, the model involves only one contract made between A and B, and effectively, it boils down to the previous two-player model.

Next, consider the case in which B rehypothecates at date 1. In that case, the model has a sequence of two contracts, the date 0 contract between A and B and the date 1 contract between B and C. Formally, the date 0 contract between A and B is defined as a profile $(I_a^{\dagger}, X_a^{\dagger}; a)$ where $I_a^{\dagger}$ represents the investment cost that A borrows from B by pledging his asset as collateral at date 0, $X_a^{\dagger}$ represents the payment promised by A to pay for getting back his asset from B at date 2, and $a \in \{s, r\}$ denotes the type of action taken by A. Similarly, the date 1 contract between B and C is defined by a profile $(I_a^{\ddagger}, X_a^{\ddagger})$ where $I_a^{\ddagger}$ represents the investment cost that B borrows from C by repledging A's collateral at date 1 and $X_a^{\ddagger}$ represents the payment promised by B to repurchase A's collateral from C at date 2, and similarly as above, these also depend on the type of action taken by A, $a \in \{s, r\}$.

To solve for the optimal solution to this problem, I use backward induction; first, I solve for the contracting problem between B and C at date 1 by taking the date 0 contract between A and B as given, and then solve for the contracting problem between A and B at date 0.

1.4.1 Contracting Problem between B and C at Date 1

Let us consider the contracting problem between B and C at date 1. I assume that B has all the bargaining power and makes a take-it-or-leave-it offer to

C, and if C rejects the offer, C receives reservation value 0. In the case in which A takes the safe action, $a = s$, the contracting problem between B and C is to choose $(I_s^\ddagger, X_s^\ddagger)$ which solves the following maximization problem.

$$\max_{I_s^\ddagger, X_s^\ddagger} \quad \theta(YI_s^\ddagger - X_s^\ddagger + X_s^\dagger) \tag{1.16}$$

subject to

$$I_s^\ddagger \leq \theta X_s^\ddagger + (1-\theta)Z_0 \tag{P_c}$$

$$X_s^\ddagger \leq X_s^\dagger \tag{R}$$

The objective function is B's expected utility when he makes the investment with cash borrowed by repledging A's collateral. With probability θ, the investment returns $YI_s^\ddagger$, and B pay a part of the return $X_s^\ddagger$ to repurchase A's collateral from C, and then B delivers this collateral to receive $X_s^\dagger$ from A. The participation constraint (P_C) states that C's utility must be at least the reservation value, 0. The right side of (P_C) is the expected revenue from lending; C receives $X_s^\ddagger$ from B with probability θ and Z_0 by seizing the collateral when B defaults with probability $1-\theta$. The left side of (P_C) is the cost $I_s^\ddagger$ that C provides to B's investment. The last constraint (R) is the resource constraint which implies that B's promise, $X_s^\ddagger$, cannot be greater than what B is going to earn by recovering the collateral from C, the expected payment that B would receive by returning the collateral to A, $X_s^\dagger$. If this does not hold, B will find it more profitable not to recover the collateral from C.

In the case in which A takes the risky action, the contracting problem is to choose $(I_r'^\ddagger, X_r^\ddagger)$ which solves the following problem.

$$\max_{I_r^\ddagger, X_r^\ddagger} \quad \theta(YI_r^\ddagger - X_r^\ddagger + pX_r^\dagger + (1-p)Z_0) \tag{1.17}$$

subject to

$$I_r^\ddagger \leq \theta X_r^\ddagger + (1-\theta)Z_0 \tag{P_c'}$$

$$X_r^\ddagger \leq pX_r^\dagger + (1-p)Z_0 \tag{R'}$$

The objective function is B's expected profit. With probability θ, B's investment returns $\theta YI_r^\ddagger$, and B repurchases A's collateral from C at a prearranged price $X_r^\ddagger$, and then with probability p, B receives $X_r^\dagger$ from A in exchange for the collateral when A's investment succeeds and with probability $1-p$, seizes the collateral (which is worth Z_0 to B) when A defaults. The participation constraint (P_c') is the same as (P_C), which implies that C's utility must be at least the reservation value, 0. Lastly, the constraint (R') implies that B's promise, $X_r^\ddagger$, cannot be greater what he is going to earn by recovering the collateral from C; B receives $X_s^\dagger$ from A in exchange for the collateral with probability p and Z_0 by seizing it if A defaults with probability $1-p$.

Then, Assumption 3 and the linearity of the problem ensure that in both cases, all the constraints are binding at the optimum, and by solving these equations simultaneously, I can write the optimal contract at date 1 as a function of the date 0 contract.

Lemma 2. *Suppose Assumption 3 holds and the date 0 contract between A and B is given. Then, the optimal contract between B and C at date 1 can be written as a function of the date 0 contract between A and B, $(I_s^\dagger, X_s^\dagger)$ if A takes the safe action and $(I_r^\dagger, X_r^\dagger)$ if A takes the risky action.*

$$(I_a^\ddagger, X_a^\ddagger) = \begin{cases} (\theta X_s^\dagger + (1-\theta)Z_0,\ X_s^\dagger) & \text{if} \quad a = s \\ (p\theta X_r^\dagger + (1-p\theta)Z_0,\ pX_r^\dagger + (1-p)Z_0) & \text{if} \quad a = r \end{cases} \tag{1.18}$$

In other words, B passes the collateral along to C. If neither A nor B fails, A's payment $X^\dagger$ is passed along to C. Otherwise C retains the collateral valued at Z_0. The weighted average of these two quantities is the amount that C lends to B.

1.4.2 Contracting Problem between A and B at date 0

Moving backward, I consider the contracting problem between A and B at date 0. First, I consider the case in which A takes the safe action. To facilitate analysis, I plug the results in Lemma 2 into the objective function of the date 1 problem, so that B's expected utility can be written as a function of $(I_s^\dagger, X_s^\dagger)$ as follows.

$$\theta(YI_s^\ddagger - X_s^\ddagger + X_s^\dagger) - I_s^\dagger = \theta Y(\theta X_s^\dagger + (1-\theta)Z_0) - I_s^\dagger. \tag{1.19}$$

Again, I assume that A has all the bargaining power and makes a take-it-or-leave-it offer to B, and if B rejects the offer, he receives reservation value, $J > 0$. In this case, the contracting problem is to choose $(I_s^\dagger, X_s^\dagger)$ which solves the following problem.

$$\max_{I_s^\dagger, X_s^\dagger} \quad RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z \tag{1.20}$$

subject to

$$RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z \geq p(RI_s^\dagger - \theta X_s^\dagger) + bI_s^\dagger - (1-p\theta)Z \tag{IC_s}$$

$$\theta Y[\theta X_s^\dagger + (1-\theta)Z_0] - I_s^\dagger \geq J \tag{P_b}$$

$$RI_s^\dagger \geq X_s^\dagger \tag{R_s}$$

The objective function is A's expected utility when A takes the safe action where $1-\theta$ is the probability of default of B. The incentive constraint (IC_s) implies that A prefers to take the safe action rather than the risky action, which is the same as in the previous two-player model when the counterparty risk is $1-\theta$. The participation constraint (P_b) implies that B's utility (in case that he is supposed to rehypothecate A's collateral at date 1) must be at least his outside utility, J. Lastly, the resource constraint (R_s) implies that A cannot promise to pay more than what he has, i.e., the return from the investment, $RI_s^\dagger$, since he has no other source of income.

Next, consider the case in which A takes the risky action. Again, I write B's utility as a function of the date 0 contract, $(I_r^\dagger, X_r^\dagger)$ by plugging the results in Lemma 2 into the objective function of the date 1 problem described above.

$$\theta(YI_r^\ddagger - X_r^\ddagger + pX_r^\dagger + (1-p)Z_0) - I_r^\dagger = \theta Y[p\theta X_r^\dagger + (1-p\theta)Z_0] - I_r^\dagger. \tag{1.21}$$

Then, the contracting problem between A and B at date 0 is to choose $(I_r^\dagger, X_r^\dagger)$ which solves the following problem.

$$\max_{I_r^\dagger, X_r^\dagger} \quad (pR+b)I_r^\dagger - p\theta X_r^\dagger - (1-p\theta)Z \tag{1.22}$$

subject to

$$RI_r^\dagger - \theta X_r^\dagger - (1-\theta)Z \leq p(RI_r^\dagger - \theta X_r^\dagger) + bI_r^\dagger - (1-p\theta)Z \qquad (IC_r)$$

$$\theta Y[p\theta X_r^\dagger + (1-p\theta)Z_0] - I_r^\dagger \geq J \qquad (P_B')$$

$$RI_r^\dagger \geq X_r^\dagger \qquad (R_r)$$

The objective function is A's expected utility when A takes the risky action where p is the probability of the success of A's investment and $1-\theta$ is the probability that B loses A's collateral. The incentive constraint (IC_r) is the reverse of (IC_s), which implies that A prefers to take the risky action than the safe action. The participation constraint (P_B') and the resource constraint (R_r) are the same as before.

Taken together, the optimal date 0 contract between A and B is a profile $(I_a^\dagger, X_a^\dagger; a)$ which solves the following problem.

$$\max_{a\in\{s,r\}} \mathbf{1}_S(a)[RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z] + (1-\mathbf{1}_S(a))[(pR+b)I_r^\dagger - p\theta X_r^\dagger - (1-p\theta)Z] \qquad (1.23)$$

where $S = \{s\}$ and $(I_s^\dagger, X_s^\dagger)$ solves subproblem (1.20) and $(I_r^\dagger, X_r^\dagger)$ solves subproblem (1.22).

To solve for this problem, I make the following two parametric assumptions on θ and Y to maintain consistency between this extended model with rehypothecation and the previous baseline model without rehypothecation.

Assumption 4. $\max\{\theta^2 YR, \theta Y(pR+b)\} > 1 > p\theta^2 YR.$

Assumption 4 is analogous to Assumption 1 in the baseline model, which implies that it is efficient for A to undertake the project regardless of his action, but from the perspective of B, providing funds for A's project is profitable only if A takes the safe action. Suppose A's collateral is worthless to B and C, that is, $Z_0 = 0$. Then, the expected repayment from A can be up to θR (per unit of inputs) if A takes the safe action, and $p\theta R$ (per unit of inputs) when A takes the risky action – note that B cannot receive the payment from A in case that B defaults with probability $1-\theta$. However, the total profit to B is multiplied by $\theta Y > 1$ since B can repledge the payment from A and invest that amount into his project with expected unit return of θY. To summarize, this assumption implies that even in the case that B rehypothecates, it is efficient for A to undertake the project, but from the persepective of B, A's project is efficient only if A takes the safe action.

Assumption 5. $R - \theta R < p(R - \theta R) + b$.

Similarly, Assumption 5 is analogous to Assumption 2 when A's payment can be up to θR (which might be either greater or smaller than 1. The left side is A's expected net surplus after paying the loan to B if A takes the safe action and the right side is that if A takes the risky action. Again, Assumption 4 and 5 together imply that, when B is supposed to rehypothecate, A must post collateral in order to borrow funding for his project.

Then, I show that as long as B's outside utility, J is sufficiently small, there exists a solution to the problem.

Lemma 3. *Suppose Assumption 3, 4, and 5 hold. If* $J \leq \min\Big\{(1-\theta)\theta Y Z_0 + \frac{\theta(\theta^2 YR-1)}{\theta R-\mathcal{B}} Z, (1-p\theta)Z_0 - \frac{\theta(1-p\theta^2 YR)}{\theta R - \mathcal{B}} Z\Big\}$*, there exists an optimal solution to problem (1.23).*

And this optimal solution, denoted by $(I_a^\dagger, X_a^\dagger; a)$, takes the following form.

$$(I_a^\dagger, X_a^\dagger; a) = \begin{cases} \left(\frac{\theta Y[\theta Z+(1-\theta)Z_0]-J}{1-\mathcal{B}\theta Y}, \frac{Z+(1-\theta)\mathcal{B}YZ_0-(\theta/\mathcal{B})J}{1-\mathcal{B}\theta Y}; s\right) & \text{if} \quad U_s^\dagger \geq U_r^\dagger \\ \left(\frac{(1-p\theta)\theta Y Z_0 - J}{1-p\theta^2 YR}, \frac{(1-p\theta)\theta YRZ_0-RJ}{1-p\theta^2 YR}; r\right) & \text{if} \quad U_s^\dagger < U_r^\dagger \end{cases} \tag{1.24}$$

where $\mathcal{B} \equiv R - \frac{b}{1-p}$, $U_s^\dagger \equiv RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z$, and $U_r^\dagger \equiv (pR+b)I_r^\dagger - p\theta X_r^\dagger - (1-p\theta)Z$.

Finally, taking all the results obtained so far together, the optimal contract in this rehypothecation model can be characterized as follows.

Proposition 2. *Suppose Assumption 3, 4, 5 hold, and* $J \leq \min\Big\{(1-\theta)\theta Y Z_0 + \frac{\theta(\theta^2 YR-1)}{\theta R-\mathcal{B}} Z, (1-p\theta)Z_0 - \frac{\theta(1-p\theta^2 YR)}{\theta R-\mathcal{B}} Z\Big\}$*. The optimal contract under rehypothecation is a profile* $(I_a^\dagger, X_a^\dagger, I_a^\ddagger, X_a^\ddagger; a)$ *where* $a \in \{s, r\}$ *such that*

$$(I_a^\dagger, X_a^\dagger, I_a^\ddagger, X_a^\ddagger; a) = \begin{cases} \left(I_s^\dagger, X_s^\dagger, I_s^\ddagger, X_s^\ddagger; s\right) & \textit{if} \quad U_s^\dagger \geq U_r^\dagger \\ \left(I_r^\dagger, X_r^\dagger, I_r^\ddagger, X_r^\ddagger; r\right) & \textit{if} \quad U_s^\dagger < U_r^\dagger \end{cases} \tag{1.25}$$

where

$$
\begin{aligned}
(I_s^\dagger, X_s^\dagger) &= \Big(\frac{\theta Y[\theta Z + (1-\theta)Z_0] - J}{1-\mathcal{B}\theta Y}, \frac{Z + (1-\theta)\mathcal{B}YZ_0 - (\theta/\mathcal{B})J}{1-\mathcal{B}\theta Y}\Big),\\
(I_r^\dagger, X_r^\dagger) &= \Big(\frac{(1-p\theta)\theta YZ_0 - J}{1-p\theta^2 YR}, \frac{(1-p\theta)\theta YRZ_0 - RJ}{1-p\theta^2 YR}\Big),\\
(I_s^\ddagger, X_s^\ddagger) &= (\theta X_s^\dagger + (1-\theta)Z_0, X_s^\dagger),\\
(I_r^\ddagger, X_r^\ddagger) &= (p\theta X_r^\dagger + (1-p\theta)Z_0, pX_r^\dagger + (1-p)Z_0),
\end{aligned}
$$

$\mathcal{B} \equiv R - \frac{b}{1-p}$, $U_s^\dagger \equiv RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z$, *and* $U_r^\dagger \equiv (pR+b)I_r^\dagger - p\theta X_r^\dagger - (1-p\theta)Z$.

To summarize, the optimal solution to the rehypothecation model consists of the inital contract between A and B and the subsequent constract between B and C and the solution uniquely exists under some reasonable assumptions on parameters.

1.5 Welfare Analysis: Trade-off Effects of Rehypothecation

In this section, I evaluate the social welfare with and without rehypothecation, and compare them to examine the benefits and costs of rehypothecation in the economy. I illustrate the components of the trade-off: On one side, rehypothecation helps provide more funding liquidity to the economy so that additional productive investment can be undertaken, by enabling the agent to use the limited amount of collateral to support multiple transactions. On the other side, it introduces the additional risk that the intermediary might default having repledged his borrower's collateral to the third party, which is often called 'rehypothecation failure.'

One potential cost associated with this failure of the intermediary is the deadweight cost of misallocating the asset, which arises in the presence of illiquidity of the asset and trading frictions; for example, the initial owner of the asset is likely to put a higher value on it than the other agents in the market, and if the intemediary repledges the initial owner's collateral to the third party, it is likely to be the case that this third party and the initial owner are indirectly connected through the intermediary, and they cannot trade on their own. This implies that, if the intermediary fails, or equivalently, rehypothecation fails, the asset remains with the third party to whom it is less valuable than to its initial owner.

Formally, if B rehypothecates A's collateral, the social welfare, denoted by W_R, can be represented by the sum of A's, B's and C's utility, respectively. First, in the case that A takes the safe action, the social welfare is given by

$$\begin{aligned} W_R(a=s) = &[RI_s^\dagger - \theta X_s^\dagger + \theta Z - Z] + [\theta \bar{Y} I_s^\ddagger - \theta X_s^\ddagger + \theta X_s^\dagger - I_s^\dagger] \\ &+ [\theta X_s^\ddagger + (1-\theta)Z_0 - I_s^\ddagger]. \end{aligned} \tag{1.26}$$

where $(I_a^\dagger, X_a^\dagger, I_a^\ddagger, X_a^\ddagger)_{a\in\{s,r\}}$ are from Proposition 2.

Similarly, in the case that A takes the risky action, the social welfare is given by

$$\begin{aligned} W_R(a=r) = &[(pR+b)I_r^\dagger - p\theta X_r^\dagger + p\theta Z - Z] \\ &+ [\theta Y I_r^\ddagger - \theta X_r^\ddagger + p\theta X_r^\dagger + (1-p)\theta Z_0 - I_r^\dagger] + [\theta X_r^\ddagger + (1-\theta)Z_0 - I_r^\ddagger]. \end{aligned} \tag{1.27}$$

Simplifying, the social welfare under rehypothecation takes the following form.

$$W_R \equiv \begin{cases} (R-1)I_s^\dagger + (\theta Y - 1)I_s^\ddagger - (1-\theta)(Z - Z_0) & \text{if} \quad a = s \\ (pR+b-1)I_r^\dagger + (\theta Y - 1)I_r^\ddagger - (1-p\theta)(Z - Z_0) & \text{if} \quad a = r \end{cases} \tag{1.28}$$

In other words, the social welfare under rehypothecation consists of three components: (i) the surplus generated from A's investment, which is captured by the terms $(R-1)I_s^\dagger$ and $(pR+b-1)I_r^\dagger$; (ii) the surplus generated from B's investment, which is captured by the terms $(\theta Y - 1)I_s^\ddagger$ and $(\theta Y - 1)I_r^\ddagger$; and (iii) the cost generated from the misallocation of the asset in case of rehypothecation failure, which is captured by the terms $(1-\theta)(Z-Z_0)$ and $(1-p\theta)(Z-Z_0)$.

On the other hand, if rehypothecation is not allowed, the social welfare, denoted by W_0, consists of A's utility and B's utility only, since further transactions between B and C cannot happen. Using the same approach as above, I can show that the social welfare without rehypothecation takes the following form.

$$W_0 \equiv \begin{cases} (R-1)I_s & \text{if} \quad a = s \\ (pR+b-1)I_r - (1-p)(Z-Z_0) & \text{if} \quad a = r \end{cases} \tag{1.29}$$

where $(I_a, X_a)_{a\in\{s,r\}}$ are from Proposition 1. Note that the surplus is now generated only from A's investment, since B's investment cannot be undertaken. Also, provided that A's choice of action is not changed whether B rehypothecates or not,[11] rehypothecation tends to increase the cost of misallocating the asset from 0 to $(1-\theta)(Z-Z_0)$ if $a = s$, and from $(1-p)(Z-Z_0)$ to $(1-p\theta)(Z-Z_0)$ if $s = r$. This is because rehypothecation introduces the model with the risk that the counterparty B loses A's collateral, which occurs with the probability $1-\theta$.

As a result, the efficiency of rehypothecation is determined by the relative size of these trade-off effects discussed so far. It enhances the provision of funding liquidity to the system so that additional productive investment can be undertaken, B's investment at date 1 in our model, but it introduces the counterparty risk to lose collateral, which may incur the deadweight loss by allocating the asset inefficiently.

1.6 Conflict between Intermediary's Ex-Post Decision to Rehypothecate and Ex-Ante Efficiency

I have so far assumed that B's (the intermediary's) decision to rehypothecate at date 1 is exogenously given. In this section, I relax this assumption and assume that it is endogenously determined within the model. B can decide whether to rehypothecate A's collateral or not when the investment opportunity arrives at date 1, by comparing the expected return if he rehypothecates A's collateral for the investment versus the expected return if he just keeps it safe until A makes the payment for recovering it from B at a later date. The question is then whether this decision made by B achieves a socially optimal outcome or not, in other words, whether B's ex post objective is in alignment or conflict with what would be an ex ante efficient decision.

[11] In general, A's choice of action can vary with rehypothecation, and I will address this issue in the next section.

In this section, I build numerical examples where B's and the society's objectives are in conflict. In the first example B prefers to rehypothecate even when rehypothecation is inefficient, and the (ex ante) social welfare will be even greater if B can commit not to rehypothecate. In the second example B does not rehypothecate even when rehypothecation is efficient , and the social welfare will be even greater if B can commit to rehypothecate.

The direction of this conflict between B's ex post objectives and the ex ante social efficiency depends on terms of the contract between A and B; to be specific, whether the contract involves over-collateralization or under-collateralization from the persepective of A, or equivalently, whether A's private value of collateral, Z, is greater or smaller than the payment for recovering it from B, $X^{\dagger}$.

First, in the case that the contract involves over-collateralization, B tends to be excessively eager to rehypothecate than the socially efficient level. Intuitively, this is because B does not internalize a negative externality that A suffers from not getting back his collateral in case of rehypothecation failure – this negative externality of rehypothecation failure is reflected in the spreads between A's private value on his collateral and the payment for recovering it. As a result, without considering this external cost to A when rehypothecation fails, B sometimes chooses to rehypothecate, even when rehypothecation is not socially efficient.

Similarly, in the case that the contract involves under-collateralization, B tends to be excessively cautious to rehypothecate, as B does not internalize a positive externality that A enjoys from not paying the loan when B loses A's collateral in case of rehypothecation failure. Hence, B sometimes chooses not to rehypothecate, even when rehypothecation is socially efficient.

1.6.1 Over-Collateralization and Excessive Rehypothecation

In this subsection, I build an example in which the contract involves over-collateralization and there arises excessive rehypothecation. I will build this example in parallel with the example from the previous section, picking parameter values that also induce safe behavior in the optimal contract. The setup of the example is analogous to that of the models described in the previous sections. For simplicity, I assume that B's outside utility is $J = 0$.

In addition, I set the parameter values to satisfy the following additional conditions as well as Assumption 1 $\sim$ 5.

(i) $Z_0 = 0$

(ii) $\mathcal{B} \equiv R - \frac{b}{1-p} < 0$

(iii) $\theta \in \left(\frac{1}{\theta Y}, \frac{1}{\theta Y}\left(\frac{1-\mathcal{B}\theta Y}{1-\mathcal{B}}\right)\right)$ where θY is a fixed positive number.

These conditions are intuitive. First, condition (i) implies that it can never be optimal for A to choose the risky action in any cases, either nonrehypothecation or rehypothecation, so that I can focus on the case in which A takes the safe action. Next, Condition (ii) implies, combined with condition (i), the optimal contract takes the form of over-collateralization. Lastly, condition (iii) implies that B prefers to rehypothecate if $\theta > \frac{1}{\theta Y}$, but rehypothecation becomes efficient only if $\theta > \frac{1}{\theta Y}\left(\frac{1-\mathcal{B}\theta Y}{1-\mathcal{B}}\right)$, which is above B's cutoff to participate in rehypothecation by condition (ii).[12] Then, under these assumptions, I can show the following.

Proposition 3. *Under assumptions 1 $\sim$ 5 and (i)-(iii), B prefers to rehypothecate despite the fact that rehypothecation is inefficient,*

Proof. First, note that B prefers to rehypothecate at date 1 if and only if

$$\theta Y(\theta X_s + (1-\theta)Z_0) > X_s \tag{1.30}$$

where X_s is the payment promised by A for recovering his collateral from B. I can show that under condition (i) and (iii), $Z_0 = 0$ and $\theta > \frac{1}{\theta Y}$, the left hand side of Equation (1.30) is greater than the right hand side, and thus B wants to rehypothecate.

On the other hand, rehypothecation is not socially efficient if,

$$RI_s - X_s > RI_s^{\dagger} - \theta X_s^{\dagger} - (1-\theta)Z \tag{1.31}$$

where I_s, X_s are from Proposition 1 and $I_s^{\dagger}$, $X_s^{\dagger}$ are from Proposition 2. The left hand side is A's utility under nonrehypothecation and the right hand side

[12]Note that since the contract involves over-collateralization, i.e., $\mathcal{B} < 0$, B's cutoff where she participates in rehypothecation is smaller than the cutoff where rehypothecation becomes efficient.

is that under rehypothecation – this equals to the social welfare as I assumed that A has all the bargaining power.

This inequality holds under the parametric restrictions (i), (ii), and (iii). First, note that by substituting the incentive constraints to A's utility function, I can represent it as a function of I_s or $I_s^\dagger$ as follows.

$$\begin{aligned} RI_s - X_s &= (R - \mathcal{B})I_s - Z, \\ RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z &= (R - \mathcal{B})I_s^\dagger - Z. \end{aligned} \tag{1.32}$$

Also, by condition (iii), I can show that

$$I_s = \frac{Z}{1-\mathcal{B}} > \frac{\theta^2 Y Z}{1-\mathcal{B}\theta Y} = I_s^\dagger. \tag{1.33}$$

Finally, plugging these into I_s and $I_s^\dagger$ in Equation (1.32), one can derive Inequality (1.31).

□

1.6.2 Under-Collateralization and Insufficient Rehypothecation

Next, I build an example where the contract between A and B involves under-collateralization and there arises insufficient rehypothecation.

Again, the setup of the example is analogous to that in the previous models, and I assume that B's outside utility is $J = 0$. In addition, I make the following restrictions on the parameters beyond Assumption 1 $\sim$ 5.

(i') $Z_0 = 0$

(ii') $\mathcal{B} \equiv R - \frac{b}{1-p} > 0$.

(iii') $\theta \in \left(\frac{1}{\theta Y}\left(\frac{1-\mathcal{B}\theta Y}{1-\mathcal{B}}\right), \frac{1}{\theta Y}\right)$ where θY is a fixed positive number.

As in the previous example, condition (i') implies that choosing the risky action is never optimal for A, and thus I can focus on the case in which A takes the safe action. Condition (ii'), combined with condition (i'), implies that the optimal contract between A and B involves under-collateralization – recall that by the incentive constraint (IC_s), $X_s < Z$ and $X_s^\dagger < Z$, if and only if $\mathcal{B} < 0$. Lastly, condition (iii') implies rehypothecation is efficient if

$\theta > \frac{1}{\theta Y}\left(\frac{1-\mathcal{B}\theta Y}{1-\mathcal{B}}\right)$, but B wants to rehypothecates only if $\theta > \frac{1}{\theta Y}$, which is above the cutoff where rehypothecation is efficient. Then, under these assumptions, I can show the following.

Proposition 4. *Under assumptions 1 ∼ 5 and (i')-(iii'), B does not want to rehypothecate despite the fact that rehypothecation is efficient.*

Proof. First, note that if conditions (i') and (iii') hold, i.e., $Z_0 = 0$ and $\theta > \frac{1}{\theta Y}$, for all X_s, the following inequality holds,

$$\theta Y(\theta X_s + (1-\theta)Z_0) < X_s \tag{1.34}$$

which implies that the expected revenue from rehypothecation, which is on the left hand side, is smaller than the expected revenue from just keeping collateral, which is on the right hand side, and thus B does not rehypothecate .

Next, I want to show that condition (i'), (ii'), and (iii') imply that rehypothecation is efficient,

$$RI_s - X_s < RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z \tag{1.35}$$

where I_s, X_s are from Proposition 1, and $I_s^\dagger$, $X_s^\dagger$ are from Proposition 2. And, the left hand side is A's expected utility under nonrehypothecation and the right hand side is that under rehypothecation.

To show this inequality, I first plug the incentive constraints (IC_s') and (IC_s) into A's utility function, so that represent it as a function of only I_s and $I_s^\dagger$, respectively.

$$\begin{aligned} RI_s - X_s &= (R-\mathcal{B})I_s - Z, \\ RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z &= (R-\mathcal{B})I_s^\dagger - Z. \end{aligned} \tag{1.32}$$

Then, I can compare the welfare in each case by simply comparing I_s with $I_s^\dagger$. Also, by condition (iii'), I can show that

$$I_s = \frac{Z}{1-\mathcal{B}} < \frac{\theta^2 YZ}{1-\mathcal{B}\theta Y} = I_s^\dagger. \tag{1.36}$$

Finally, plugging these into I_s and $I_s^\dagger$ in Equation (1.32), Inequality (1.35) is derived.

□

1.7 Conclusion

Rehypothecation is the re-use of the same collateral to support multiple transactions. Rehypothecation helps to provide more funding liquidity to the system by allowing the lender to use her borrower's collateral sitting idle in her account for another productive investment. However, it can incur deadweight cost of misallocating the asset when the lender fails having repledged the borrower's collateral to the third party to whom the collateral is less likely to be as valuable to the initial owner.

I show that the individuals' incentives to participate in rehypothecation may not be aligned with economic efficiency. In other words, cases arise in which agents choose socially inefficient levels of rehypothecation. Importantly, the direction of the inefficiency depends on terms of the agents' contracts. If the contract involves over-collateralization, the intermediary tends to be overly eager to rehypothecate; if the contract involves under-collateralization, the intermediary tends to be overly cautious to rehypothecate.

Several natural extensions are worth consideration. Thus far I have not incorporated insurance motives, random fluctuation in the value of the collateral, and the effects of aggregate and idiosyncratic shocks. These important practical considerations are left for future research.

CHAPTER 2

REHYPOTHECATION, COLLATERAL MISMATCH AND FINANCIAL CRISES

2.1 Introduction

At the center of the financial crisis, there were failures of large dealer banks that intermediated in the market for over-the-counter (OTC) derivatives and repurchase agreements (repos) such as Lehman Brothers and Bear Sterns. The failures of these banks were prompted and accelerated by runs of their clients. For example, as Lehman Brothers headed towards bankruptcy, hedge funds who were concerned about the solvency of their dealer banks – these also included major dealer banks such as Morgan Stanley and Goldman Sachs – raced to withdraw their assets from these dealer banks, shifting them into other commercial banks deemed safer, thereby weakening the banks' capital position.

The concerns of these hedge funds over the solvency of the dealer banks were rooted in the deepest part of the business of these banks: *rehypothecation*, a practice in which banks reuse or repledge parts of their clients' collateral to support their own loans. Before the crisis, rehypothecation was commonly used by dealer banks to raise extra liquidity. After the failure of Lehman Brothers, however, many clients of these dealer banks, especially, hedge funds, were concerned about losing access to their collateral,[1] and did not allow the banks to reuse their collateral.[2] Since then, collateral reuse

[1]This is because in most case, if a bank defaults having repledged their clients' collateral, that collateral will be seized by third parties to whom the bank repledged it. For example, hedge funds that used Lehman Brothers such as GLG Partners, Amber Capital and Ramius were not able to access the assets of $40 billion held by the European branch of Lehman Brothers (Farrell 2008), and Olivant, the investment company that used Lehman Brothers could not participate in voting at UBS's shareholder meeting since its 700 million pound shares in UBS were tied up at Lehman (Mackintosh 2008).

[2]Singh (2010) reports that the value of rehypothecatable assets that Morgan Stanley received from its clients decreased from $953 million in May 2008 (the peak) to $294 million in Nov 2008. Over the same period, the decrease in the amount of rehypothecatable

has received extensive attention as an important factor that contributed the severity of the crisis. In particular, the need to study the economic effects of this practice has been emphasized.[3]

This paper studies the dynamics of an economy relying on collateral reuse subject to aggregate uncertainty that collateral chains might fail. Building on the model of the first chapter, I consider how a shock to collateral chains affects individual reuse decision, collateral allocation and aggregate output. In particular, I contrast outcomes across (i) different temporary shocks and (ii) different persistent shocks, discussing how the size of crisis and the speed of recovery vary with different temporary shocks, and whether a more protracted period of good states will lead to a greater output, or smaller.

As shown in the first chapter, the equilibrium haircut solves incentive problems, and this leads to a wedge between shadow values of the collateral to parties in the collateral chain – for example, if the haircut is positive, the private value of the collateral to the borrower exceeds the payment, that is, the shadow value of the collateral for the lender. If the loan is over-collateralized, parties down the chain might be tempted to overuse the collateral provided them, and this in turn, causes parties up the chain to be unwilling to extend permission for reuse. In this chapter, I extend this work to a dynamic setting with aggregate uncertainty that collateral chains might fail, deriving the consequences of collateral chain shocks for the re-use decisions and the evolution of the allocation of collateral and aggregate output.

The model highlights the trade-offs of collateral re-use, which are the main driving forces behind the reuse decision making: (1) a positive effect that it helps finance additional productive investments which cannot be financed otherwise by allowing the scarce collateral to support multiple loans at once; (2) a negative effect that it might lead to misallocation of collateral in the event that the intermediary in the middle of the collateral chain defaults. The misallocation of collateral arises from the assumption that collateral is not perfectly liquid in the sense that: (a) the shadow value of collateral is

collateral by Goldman Sachs and Merrill Lynch are about 66% and 38%, respectively.

[3]However, there are still differences in the regulation of collateral reuse across countries. For example, problems stemming from rehypothecation failure are more severe in Europe than in the US, since in Europe, especially in the UK, there was no limit on the amount that can be rehypothecated by broker-dealers, but in the US, SEC rule 15c3-3 restricts the amount of client assets that can be rehypothecated to no more than 140% of the value of the client's liability.

higher for borrowers than for lenders;[4] and (b) the market for collateral is not frictionless, so that buying or selling collateral is costly. Combining these two assumptions yields the result that collateral is misallocated in the event that the intermediary fails, since collateral cannot be returned to borrowers who value them highest.

A key variable that determines the relative size of these two trade-off effects is the risk of failure of the intermediary. If the risk of default of the intermediary is lower, the positive effect to finance additional investment is more likely to exceed the negative effect to cause collateral misallocation, so that the initial owner is more likely to extend permission to reuse his collateral; and the opposite happens if the risk of default of the intermediary is higher. This implies that given that the risk of failure of the intermediary tends to be counter-cyclical, collateral reuse occurs more frequently in booms than in recessions.

Using this model, I study the effect of shocks that increase the risk of failures of the collateral chain on dynamics of the economy. In particular, I compare outcomes across (i) different temporary shocks and (ii) different persistent shocks. Endogenizing the reuse decisions complicates the characterization of the evolution of collateral allocation. As the shock is worse, more collateral cannot be returned to the borrowers, increasing the misallocation of collateral, while at the same time, the borrowers are less willing to allow their counterparty to reuse their collateral with concerns of losing the collateral, decreasing the misallocation of collateral. It follows that aggregate output can be even greater after the adverse shock depending on the relative size of these two opposing effects; if the effect of the shock that causes borrowers not to allow collateral reuse exceeds the effect that increases failures of returning collateral, the mismatch of collateral decreases, setting the stage for a greater output once the shock disappears.

First, I introduce an unexpected temporary shock that increases the risk of failure of the intermediary in the collateral chain, and compare the size of crisis and the speed of recovery across shocks with different magnitude. As the shock increases, the expected return of the intermediary's project decreases, which in turn causes the intermediary to reduce funding provided for the

[4] In this chapter, the shadow values of collateral are drived endogenously from the persistence of investment oportunities that agents with investment opportunity today are more likely to have investment opportunity again tomorrow than others without it.

borrowers' projects, both leading to a further decrease in output. Moreover, the size of the crisis can be even greater if borrowers stop allowing collateral reuse in response to the shock. The reason is that forbidding collateral reuse not only causes the intermediary reduce funding for the borrowers' project, but also make the intermediary's project not be undertaken at all, decreasing aggregate output even further. The speed of recovery, however, can be faster as a shock is worse if borrowers preclude collateral reuse at the time of the shock. This is because precluding collateral reuse has a positive effect to mitigate the misallocation of collateral in the future, leading to a faster increase in aggregate output as the effect of shock disappears.

Next, I consider random aggregate shocks that follow a standard two-state Markov process, and compare outcomes across two different histories of shocks, one is uniformly better than another. In particular, I describe the effect of continued good or bad shocks on reuse decisions and aggregate variables. I show that as bad shocks continue, borrowers are more cautious to allow collateral reuse, and the misallocation of collateral decreases gradually, setting the stage for a greater increase in output once the recession ends. Conversely, as good shocks continue, borrowers are more eager to allow collateral reuse, and the misallocation of collateral accrues, sowing the seeds for a greater decrease in output once the boom ends.

The paper starts with a simple discrete time, infinite horizon model with a continuum of ex-ante identical agents. At the beginning of each period each agent receives either an investment opportunity or consumption goods, but cannot have both at the same time. I assume that investment opportunities are persistent; if an agent has an investment opportunity today, she is more likely to have an investment opportunity again tomorrow. I assume that consumption goods depreciate completely if not consumed immediately. An investment opportunity requires consumption goods as input to produce more consumption goods at the end of the period.

An investment project thus can be undertaken only if it is financed from the outside. However, if the borrower wants to divert the funds received for his private benefit, borrowing only backed by the future return of the project might not be feasible. One way to restore credit in such circumstance is to require the borrower to post other valuable assets (which is separate from the project) as collateral; the borrower has now an incentive to repay the loan in order to get back his collateral – thus, collateral works as *hostage*. I assume that there is one type of collateralizable asset, a long-term indivisible asset that delivers a constant dividend for an infinite period of time (e.g. a perpetuity). For simplicity, I assume that it is costly to hold more than one unit of collateral, and thus an agent can hold only up to one unit of collateral.

Therefore, at any given point of time, there are four types of agents in the economy: those who have both a project and collateral (denoted by type AZ), those who have only a project but no collateral (type A0), those who have both consumption goods and collateral (type CZ), and those who have only consumption goods but no collateral (type C0). Since agents can raise funding only with collateral, only type AZ can become borrowers. On the other hand, since holding more than one unit of collateral is costly, only type C0 will be lenders. I assume that borrowers and lenders are randomly matched in the beginning of each period.

The contracting problem in a single period is similar to the model of collateral chain in Kahn and Park (2016), in which there is an intermediary who provides funding to the borrower (type AZ, here), and then subsequently repledges the borrower's collateral to borrow funding for his own project from a third party (type C0, here). The optimal contract within a single period is then obtained by backward induction. Based on that, I derive the value of collateral to the borrower and for the lender endogenously.

An equilibrium is then a set of (i) the optimal reuse decision made by borrowers in each period, (ii) the optimal contract within a single period, and (iii) the evolution of the distribution of agent types that is consistent with the individual optimal decisions. I show that the optimal reuse decision made by the borrower is negatively related with the risk of failure by the intermediary: as the probability of failure by the intermediary increases, allowing collateral reuse is less attractive for the borrower. Also I show that provided that the reuse decision held constant, as the probability of failure by the intermediary increases, there are more transitions from type AZ to A0 and from type C0 to type CZ, implying that there are more mismatch between collateral (Z) and investment opportunities (A).

Next, I introduce a temporary unexpected shock that increases the risk of failure by the intermediary. As mentioned earlier, the size of crisis increases as the shock is worse, but the speed of recovery can be faster as the shock is worse if the borrowers preclude collateral reuse at the time of the shock. Then, I investigate the case in which aggregate shock on the collateral chain follows a standard two-state Markov process, comparing outcomes in two different histories of shocks, one uniformly better than another. In particular, I compare the reuse decision, mismatch of collateral, and aggregate output along those two different paths, showing that continued good states can lead to a greater decrease in output once the boom ends by accumulation of misallocated collateral.

2.2 Model

Time is descrete with an infinite horizon and each period is indexed by t. There are three types of agents: an entrepreneur, an intermediary, and an investor all risk-neutral. An intermediary lives only one period and entrepreneurs and investors live forever and share a common discount factor $\beta \in (0, 1)$. An agent can become either entrepreneur or investor in each period, depending on his endowment in that period: he might have an investment opportunity but has no input, or he might be endowed with numeraire good that can be used as input for the investment but has no investment opportunity.

Entrepreneurs and investors can trade only through the intemediary.[5] One can interpret enterpreneurs and investors as clients of the intermediary on opposite sides – for example, hedge funds and money market mutual funds of large dealer banks.

The investment opportunity requires numeraire good as an input to produce more numeraire good, $R > 1$ per unit of input, at the end of the period. This implies that it is potentially useful to transfer funding from an investor to an entrepreneur's project.

However, an entrepreneur has the option to divert the funds that he borrowed for his private benefit. The private benefit from diverting is b per unit of funds diverted, which is assumed to be not transferable. If this benefit from diverting is sufficiently large, without any external means of forcing repayment of the loan, self-financing will not be feasible.[6] Therefore, an entrepreneur must post *collateral* to borrowing funding for his project.

In this setting, there are a collateral asset, numeraire good, and non-

[5] This assumption is relaxed in the third chapter.
[6] See the first chapter for details.

transferable project. I assume that numeraire good is not storable, depreciating completely if not consumed immediately. In contrast, the asset is storable, producing fixed amount of numeraire good at the end of each period, such as a perpetuity with constant dividends. I assume that each agent can store only one unit of the collateral asset, that is, the cost of holding more than one unit of the asset is sufficiently large.[7] For simplicity, I also assume that the asset is endowed to agents only in the initial period, $t = 0$, and so the total amount of the asset is fixed in all periods.

I assume that the probability of receiving an investment opportunity in the next period depends on the state of an agent in the current period; if an agent has an investment opportunity today, then he will have an investment opportunity again tomorrow with probability p, and with symmetry, if that person does not have an investment opportunity today, he will not receive an investment opportunity with probability p. When $p > \frac{1}{2}$, the investment opportunity is persistent.

Assumption 6. $p > \frac{1}{2}$

As a result, there are two state variables: (1) an agent has an investment opportunity or not (have numeraire good instead of the investment opportunity); and (2) an agent has the collateral asset or not. This implies that at any given point of time there are four types of agents: entrepreneurs with collateral, denoted by type "AZ," entrepreneurs without collateral, type "$A0$," investors with collateral, type "CZ," and lastly, investors without collateral, type "$C0$."

Note that only entrepreneurs with collateral, type AZ, can undertake the project and only investors without collateral, type $C0$, and be lenders – by assumption, investors can hold only up to one unit of collateral at one time and so, type CZ agents remain outside of the market and consume their numeraire good.

I assume that type AZ and $C0$ are randomly matched through the intermediary in each period. For the time being, I assume that in any period t, the mass of type AZ (borrowers), denoted by M_{AZt}, is less than the mass of type $C0$ (lenders), denoted by M_{C0t}.[8]

[7] This assumption is made in order to simplify tracking the evolution of the distribution of asset holdings over time.

[8] Later, I will show that this condition holds under reasonable assumptions on parameters.

Assumption 7. *The initial distribution of agent types* $(M_{AZ_0}, M_{A0_0}, M_{CZ_0}, M_{C0_0})$ *satisfies*

– $M_{A0_0} + M_{C0_0} \geq M_{AZ_0} + M_{CZ_0}$

– $M_{C0_0} + M_{CZ_0} \geq M_{AZ_0} + M_{A0_0}$

– $M_{C0_0} \geq M_{AZ_0}$

First, $M_{A0_0} + M_{C0_0} \geq M_{AZ_0} + M_{CZ_0}$ states that the mass of cash holders, type $A0$ and $C0$, exceeds the mass of collateral holders, type AZ and CZ, implying that collateral is scarce. Second, $M_{C0_0} + M_{CZ_0} \geq M_{AZ_0} + M_{A0_0}$ states that the mass of agents without investment opportunities, type C, exceeds the mass of agents with investment opportunities, type A, implying that investment opportunities are scarce. Lastly, $M_{C0_0} \geq M_{AZ_0}$ states that the number of lenders exceeds the number of borrowers.

2.2.1 Single period problem

I assume that the same single-period problem repeats every period, and each period is divided into three subperiods as follows:

- At the beginning of each period, an agent becomes an entrepreneur or an investor depending on his endowment in that period, as described in the previous section. The entrepreneur's project requires an immediate input and he borrows funding from the intermediary by posting collateral.

- Next the intermediary has an investment opportunity as well, but he has no input for his project. If the borrower has allowed the intemediary to reuse his collateral in advance, the intermediary borrows funding for his project by transferring the borrower's collateral to the investor.

- At the end of the period, both the entrepreneur's and intermediary's projects mature. If the intermediary's project succeeds, he recovers the borrower's collateral from the investor, and returns it to the borrower in exchange for receiving the payment. However, if the intermediary's project fails, the investor seizes the collateral, and it cannot be returned to the borrower.

2.2.2 Optimal Contract within a Single Period

Note that two contracts arise sequentially: one is the contract between the entrepreneur of type AZ (denoted by A, hereafter) and the intermediary (denoted by B), and another is the contract between the intermediary and the investor of type $C0$ (denoted by C). The optimal contract can be obtained via backward induction. I solve the contracting problem between B and C in the second stage taking the contract between A and B as given and then solve the contracting problem between A and B in the first stage.

Contract between Intermediary and Investor in the Second Stage

I assume that B's project yields $Y > 1$ per unit of inputs if successful and 0 if unsuccessful. The probability of success is denoted by $\theta \in (0, 1)$.

Assumption 8. $\theta Y > 1$.

However, the return from B's project is not pledgeable and B has to rehypothecate A's collateral in order to borrow funding from C. Throughout this section, I assume that collateral reuse has already been allowed by A, but it will be relaxed later on.

I assume that the contract between A and B in the first subperiod is given by (I_A, X_A) where I_A is the amount that A borrowed from B and X_A is the amount that A promised to repay in return for his collateral. Then, taking the contract terms (I_A, X_A) between A and B in the first subperiod as given, the optimal collateralized debt contract between B and C, (I_B, X_B), solves the following problem.[9]

$$\max_{I_B, X_B} \theta(YI_B - X_B + X_A) \tag{2.1}$$

subject to

$$\theta X_B + (1-\theta)V_C \geq I_B \tag{IR_C}$$

$$V_C \geq X_B \tag{P_B}$$

The objective function is B's expected revenue from undertaking the project by reusing A's collateral; θ is the probability of success of B's project; YI_B

[9]For simplicity, I assume that all the bargaing power goes to B.

is the gross return from the project; X_B is the payment in exchange of retrieving A's collateral; and X_A is the payment received from A in exchange for returning the collateral to A.

The first constraint (IR_C) is C's participation constraint which implies that the expected revenue from the collateralized lending (on the right side) must not be less than the amount of funding he provided (on the left side). The second constraint (P_B) is the resource constraint which states that any promise by B, X_B, that exceeds the value of collateral to the lender, V_C is not credible.[10] I also assume that the value function, V_C, is given throught this section, but will endogenize it later on.

Assumption 8 and the linearity of the problem then imply that both contraints bind at the optimum. Solving these two equations simultaneously yields the optimal contract between B and C in the second stage as follows:

Lemma 4. *Suppose Assumption 8 holds and the value functions of A and C are exogenously given by V_A and V_C, respectively. Taking the contract between A and B in the first stage, (I_A, X_A), as given, the optimal contract between B and C in the second stage, (I_B, X_B), is given by*

$$I_B = X_B = V_C \tag{2.2}$$

Hence, in this simple setting, the size of loan for B is solely determined by the value of the collateral to C. This is mainly because of the assumptions that the contract between B and C concerns only the insurance aspect of collateral and C cannot condition the terms of contract between A and B.

Contract between Entrepreneur and Intermediary in the First Stage

Next, I solve for the optimal contract between the entrepreneur (A) and the intermediary (B) in the first stage. As in the first chapter, the optimal contract between A and B, (I_A, X_A), solves

$$\max_{I_A, X_A} \; RI_A + \theta\{1 + \beta[pV_A + (1-p)V_C] - X_A\} \tag{2.3}$$

[10]For simplicity, I assume that C cannot make a deal conditional on the terms of contract between A and B (even if C has a knowledge of the previous contracts). For example, B cannot pledge the promise by A, X_A, even in the case that it might be more profitable for B to promise to pay X_A rather than V_C. For discussion of the latter case, see Kahn and Park (2016a).

subject to

$$RI_A + \theta\{1 + \beta[pV_A + (1-p)V_C] - X_A\} \geq bI_A \qquad (IC_A)$$
$$\theta(YI_B - X_B + X_A) \geq I_A \qquad (IR_B)$$

The objective function is the entrepreneur's payoff from undertaking the project with collateralized borrowing; RI_A is the gross return from the project as before; θ is the probability that A receives the collateral from B; X_A is the amount paid to B in exchange for receiving the collateral; and number 1 is the dividend from collateral and $\beta[pV_{AZ} + (1-p)V_{CZ}]$ is the expected value of the collateral to A as before. Note that if B defaults, A cannot retrieve the collateral, but at the same time, he does not need to pay the loan.

The first constraint (IC_A) is the incentive constraint of A which states that A finds it less profitable diverting funding for his own personal benefits than exerting efferts. The second constraint is the participation constraint of B, which can be rewritten,

$$\theta(YI_B - X_B + X_A) = \theta(Y-1)V_C + \theta X_A \geq I_A \qquad (IR_B)$$

where the equality holds by Lemma 4.

Then, under Assumption 8, the linearity of the problem ensures that both contraints (IC_A) and (IR_B) bind at the optimum. Solving these two equations simultaneously yields the optimal contract between A and B.

Lemma 5. *Suppose Assumption 8 holds and the value functions of A and C are given by V_A and V_C, respectively. The optimal contract between A and B in the first stage, (I_A, X_A), is given by*

$$I_A = \theta \frac{1 + \beta[pV_A + (1-p)V_C] + (Y-1)V_C}{1 + b - R} \qquad (2.4)$$
$$X_A = \frac{1 + \beta[pV_A + (1-p)V_C] + (b-R)(Y-1)V_C}{1 + b - R} \qquad (2.5)$$

2.2.3 Stationary Equilibrium

Next, I solve for a stationary equilibrium and provide comparative statics at the steady state.

Definition 1. *A stationary equilibrium is a profile* $(I_A, X_A, I_B, X_B, M_{AZ}, M_{A0}, M_{CZ}, M_{C0})$ *that satisfies:*

- (I_A, X_A, I_B, X_B) *solves the single-period problem in Lemma 4 and 5.*
- $(M_{AZ}, M_{A0}, M_{CZ}, M_{C0})$ *is the stationary distribution of agent types consistent with the agents' optimization problem.*

Stationary Distribution of Types of the Agents

First, note that the transition of types between any period t and $t+1$ is given by,

$$\begin{bmatrix} M_{AZ_{t+1}} \\ M_{A0_{t+1}} \\ M_{CZ_{t+1}} \\ M_{C0_{t+1}} \end{bmatrix} = \begin{bmatrix} p\theta + (1-p)(1-\theta) & 0 & 1-p & 0 \\ (2p-1)(1-\theta) & p & 0 & 1-p \\ (1-p)\theta + p(1-\theta) & 0 & p & 0 \\ -(2p-1)(1-\theta) & 1-p & 0 & p \end{bmatrix} \begin{bmatrix} M_{AZ_t} \\ M_{A0_t} \\ M_{CZ_t} \\ M_{C0_t} \end{bmatrix} \tag{2.6}$$

where $p \in \left(\frac{1}{2}, 1\right]$ by Assumption 6.

Then, combining these result with Assumption 7 yields the equation for the mass of each type as follows.

Lemma 6. *Under Assumption7, the mass of each type of agent in period t is given by*

$$M_{AZ_t} = \left\{(1-p)\left[\frac{1-(2p-1)^t\theta^t}{1-(2p-1)\theta}\right] + (2p-1)^t\theta^t\right\} Z \tag{2.7}$$

$$M_{A0_t} = M_{A_t} - M_{AZ_t} \tag{2.8}$$

$$M_{CZ_t} = Z - M_{AZ_t} = Z - \left\{(1-p)\left[\frac{1-(2p-1)^t\theta^t}{1-(2p-1)\theta}\right] + (2p-1)^t\theta^t\right\} Z \tag{2.9}$$

$$M_{C0_t} = M_{C_t} - M_{CZ_t} \tag{2.10}$$

where

$$M_{A_t} \equiv M_{AZ_t} + M_{A0_t} = \frac{1}{2}\left\{[1+(2p-1)^t]Z + [1-(2p-1)^t]\right\} \tag{2.11}$$

$$M_{C_t} \equiv C_{AZ_t} + C_{A0_t} = \frac{1}{2}\left\{[1-(2p-1)^t]Z + [1+(2p-1)^t]\right\} \tag{2.12}$$

Note that this result is consistent with Assumption 7:

$$M_{AZ_t} = \frac{1}{2}[1 + (2p-1)^t]Z < \frac{1}{2}[1 + (2p-1)^t] = M_{C0_t} \quad \forall t \tag{2.13}$$

Finally, the limiting distribution for $(M_{AZ_t}, M_{A0_t}, M_{CZ_t}, M_{C0_t})$ in Lemma 6 becomes the stationary distribution of agent types.

Corollary 1. *Suppose Assumption 7 holds. There exists a unique stationary distribution of types of the agent* $(M_{AZ}, M_{A0}, M_{CZ}, M_{C0})$ *such that*

$$M_{AZ} = \frac{1-p}{1-(2p-1)\theta}Z \tag{2.14}$$

$$M_{A0} = \frac{1}{2}(Z+1) - M_{AZ} \tag{2.15}$$

$$M_{CZ} = \frac{p-(2p-1)\theta}{1-(2p-1)\theta}Z \tag{2.16}$$

$$M_{C0} = \frac{1}{2}(Z+1) - M_{CZ}. \tag{2.17}$$

Note that

$$\frac{\partial M_{AZ}}{\partial p} < 0, \ \frac{\partial M_{A0}}{\partial p} > 0, \ \frac{\partial M_{CZ}}{\partial p} > 0, \text{ and } \frac{\partial M_{C0}}{\partial p} < 0 \tag{2.18}$$

Value Functions

First, the value function of investor with collateral is given by,

$$V_C = 1 + \beta[pV_C + (1-p)V_A] \tag{2.19}$$

where number 1 is the earning from the asset today and $\beta[pV_C + (1-p)V_A]$ is the expected value of holding the asset until tomorrow. With probability p, he remains as investor and with probability $1-p$, he becomes an entrepreneur.

The value function of entrepreneur is given by,

$$V_A = \max_{I_A, X_A} \ RI_A + \theta[-X_A + \{1 + \beta[pV_A + (1-p)V_C]\}] \tag{2.20}$$

which is as described in Lemma 5 .

Solving these equations (2.19) and (2.20) simultaneously reveals that the value functions of entrepreneurs (A) and cash holders (C) take the following forms.

Lemma 7. *The value function of enterpreneurs (A) and cash holders (C) are given by,*

$$V_A = \frac{\gamma\theta[Y - \beta(2p-1)]}{1 - \beta\{p + \gamma\theta[p + (1-p)(Y-1) - \beta(2p-1)]\}} \tag{2.21}$$

$$V_C = \frac{1 - \gamma\theta\beta(2p-1)}{1 - \beta\{p + \gamma\theta[p + (1-p)(Y-1) - \beta(2p-1)]\}} \tag{2.22}$$

where $\gamma \equiv \frac{b}{1+b-R} > 1$.

This yields the following comparative statics immediately.

Corollary 2 (Comparative Statics on the Value Functions)**.**

$$V_A > V_C \tag{2.23}$$

$$\frac{\partial V_A}{\partial p} = \begin{cases} > 0 & if \quad Y < 2 - \beta \\ < 0 & otherwise \end{cases} \quad \frac{\partial V_C}{\partial p} < 0 \tag{2.24}$$

It is noteworthy that the value function of A, V_A, is now a hump-shaped function of the persistence of investment opportunities, p; the function is increasing in p if $Y < 2 - \beta$ and decreasing in p if $Y < 2 - \beta$. The reason is as follows. First, an increase in p has a direct positive effect on V_A; it increases V_A by increasing the chance for A to receive a future investment opportunity. However, an increase in p has another indirect effect on V_A by changing the value function of C. In particular, an increase in p decreases V_C (as it makes even more difficult for C to receive the investment opportunity in the future). This, in turn, decreases funding that A can credibly borrow by posting that collateral, thereby decreasing V_A (recall that the value of the collateral to A is a sum of the dividend from the collateral, and V_A and V_C, that is, $1 + \beta[pV_A + (1-p)V_C]$).

As a result, an increase in p may increase or decrease V_A depending on the relative size of these two opposing effects; if the return from reusing the collateral is sufficiently low, that is, $Y < 2 - \beta$, the former effect outweighs the latter effect, and so V_C increases in p, but if $Y > 2 - \beta$, the latter effect outweighs the former effect, and so V_C decreases in p.

Finally, I find the unique stationary equilibrium in this economy.

Proposition 5. *If Assumption 7 and 6 hold, there exists a unique stationary equilibrium.*

And the equilibrium can be represented by a profile $(I_A, X_A, I_B, X_B, M_{AZ}, M_{A0}, M_{CZ}, M_{C0})$ where

- (I_A, X_A, I_B, X_B) is a solution to the single-period problem in Lemma 4 and 5.
- $(M_{AZ}, M_{A0}, M_{CZ}, M_{C0})$ is as described in Corollary 1.

2.2.4 Borrower's choices of allowing rehypothecation

Up to this point, I have assumed that B can always reuse A's collateral whenever he has an investment opportunity. In practice, however, A has the right to restrict B's ability to reuse his collateral, and if the risk of losing the collateral and the cost of recovering it are too high, he may want to preclue B's ability to reuse collateral. I now investigate how the probability of losing collateral by intermediary B, $1-\theta$, affects A's decisions on whether to allow collateral re-use.

Proposition 6. *An enterpreneur's (A's) decision on whether to allow collateral re-use or not depends on the risk of intermediaries' failure $1-\theta$ as follows:*

- *he allows collateral reuse if*

$$1-\theta < \frac{1-\beta(2p-1)}{1-\beta(2p-1)+(Y-1)[1-\gamma\beta(2p-1)]} \tag{2.25}$$

  and does not allow it if the inequality is reversed.

Allowing the intermediary to reuse collateral or not imposes both the benefits and costs on the entrepreneur. As mentioned earlier, allowing the intermediary to reuse collateral reduces the opportunity costs of keeping the collateral. This, in turn, benefits the enterpreneur since the intermediary now would be willing to provide more funding against the same collateral. For example, if the return from the intermediary's project Y increases, the right side of Equation (2.25) increases, and the entrepreneur is more likely to allow collateral reuse. However, allowing the collateral reuse exposes the entrepreneur to the risk of losing collateral if the intermediary fails. For example, if the probability of defaults of the intermediary, $1 - \theta$, increases, the left side of Equation (2.25) increases, and the entrepreneur is less willing to allow collateral reuse.

2.3 Dynamics

2.3.1 The Effect of Temporary Shocks on Output Dynamics

In this section I introduce a shock that increases the risk of default by the intermediary and investigate its effect on output dynamics. I find that even when magnitudes of shocks are almost the same, the output dynamics triggered by them can be significantly different dependingon its effect on the entrepreneur's choice over collateral re-use.

Specifically, this section addresses the following questions: when does a shock induce the entrepreneurs to continue to allow collateral re-use or preclude it?; how does the allocation of collateral evolve after the shock?; and how does the size of the crisis and the speed of recovery differ between different shocks?

First, I show that an adverse shock increases the misallocation of collateral, this leads to a decrease in aggregate output. However, there is another indirect effect of the shock on the output dynamics by altering the entrepreneurs' preferences over collateral re-use. If they continue to allow collateral reuse when the shock arrives, the size of the crisis will be relatively small, but it leads to a long economic downturn by increasing the mismatch of collateral and investment opportunities in the long run. In contrast, if they discontinue to allow collateral re-use, the output drops further, but it helps the economy to recover fast by reducing the potential misallocation of collateral due to rehypothecation failure.

Formally, I begin with a steady-state economy in which collateral reuse is initially allowed as described in Proposition 5 and introduce a temporary negative shock in period t.[11] For a given set of parameters $(\beta, \gamma, \theta, p, Y)$, I characterize the range of shocks in which the borrower allows collateral

[11]Dynamics without rehypothecation is trivial since the aggregate output varies only with the shocks on the borrowers' side.

re-use.

Lemma 8. *Suppose there is a temporary negative shock $\Delta \in (0,1)$ in period t that decreases the probability of success of B's project from θ to $(1-\Delta)\theta$. The borrower allows the collateral reuse in that period only if Δ satisfies,*

$$(1-\Delta)\theta < \frac{1+\beta[pV_A+(1-p)V_C]}{1+\beta[pV_A+(1-p)V_C]+(Y-1)V_C} \tag{2.26}$$

where V_A and V_C are as described in Proposition 5.

The borrower will be more cautious to allow collateral reuse as the shock Δ increases.

Next, I investigate how the distribution of agents' types evolve after the shock.

Proposition 7 (The Distribution of Types of Agents after the Shock)**.** *Suppose the distribution of types of the agents at $t = T$ is given by the stationary distribution, $(M_{AZ}, M_{A0}, M_{CZ}, M_{C0})$ as described in Proposition 5.*
(i) If Δ satisfies the condition in Lemma 8, the mass of each type evolves as follows:

$$M_{AZ_T} = [1-\Delta(2p-1)^{T-t}\theta^{T-t}]M_{AZ} \tag{2.27}$$

$$M_{A0_T} = \frac{1}{2}(Z+1) - M_{AZ_T} \tag{2.28}$$

$$M_{CZ_T} = M_{CZ} + \Delta(2p-1)^{T-t}\theta^{T-t}M_{AZ} \tag{2.29}$$

$$M_{C0_T} = \frac{1}{2}(Z+1) - M_{CZ_T} \tag{2.30}$$

(ii) If Δ does not satisfy the condition in Lemma 8, the mass of each type of the agents at $T \geq t$ evolves as follows:

$$M_{AZ_T} = \begin{cases} [1+\Delta(2p-1)(1-\theta)]M_{AZ} & \text{if} \quad T = t+1 \\ [1+\Delta(2p-1)^{T-t}\theta^{T-t-1}(1-\theta)]M_{AZ} & \text{if} \quad T \geq t+2 \end{cases} \tag{2.31}$$

$$M_{A0_T} = \frac{1}{2}(Z+1) - M_{AZ_T} \tag{2.32}$$

$$M_{CZ_T} = \begin{cases} M_{CZ} - \Delta(2p-1)(1-\theta)M_{AZ} & \text{if} \quad T = t+1 \\ M_{CZ} - \Delta(2p-1)^{T-t}\theta^{T-t-1}(1-\theta)M_{AZ} & \text{if} \quad T \geq t+2 \end{cases} \tag{2.33}$$

$$M_{C0_T} = \frac{1}{2}(Z+1) - M_{CZ_T} \tag{2.34}$$

If collateral re-use is allowed at the time of the shock, the mass of type AZ (entrepreneur with collateral) in period $t+1$ falls immediately from M_{AZ}, to $[1-\Delta(2p-1)\theta]M_{AZ}$, but then it starts to increase and converges to the steady state level, M_{AZ}, as the effect of the shock disappears. At that same time, the mass of type $A0$ and CZ move in the opposite direction of the mass of type AZ.

If collateral re-use is prohibited at the time of the shock, however, the mass of type AZ in period $t+1$ rises from M_{AZ} to $[1+\Delta(2p-1)(1-\theta)]M_{AZ}$ at $t+1$, but then it decreases again and converges to the steady state, M_{AZ}, as the effect of the shock disappears and the collateral reuse is allowed again. Similarly, the mass of type $A0$ and CZ move in the opposite direction of that of type AZ during this period.

To summarize, continuing collateral re-use at the time of the shock alleviates the crisis at the time of the shock and increases the misallocation of collateral in the future, while stopping the collateral re-use at the time of the shock increases the crisis and prevents the misallocation of collateral in the future.

This natually leads to the following questions: what are the implication of the collateral allocation (or misallocation) for output dynamics? And, how does the borrower's choice over collateral reuse affect the speed of recovery after the shock?

Proposition 8 (The Output Dynamics after the Shock)**.** *Suppose I_A and I_B are as described in Lemma 4 and 5, and $I_A(1)$ indicates I_A when $\theta = Y = 1$. Denote M_{AZ_T} and M'_{AZ_T} the mass of type AZ in (i) and (ii) of Proposition 7, respectively.*

- *If Δ satisfies the condition in Lemma 8, the aggregate output in period $T \geq t$,Y_T, evolves as follows:*

$$Y_T = \begin{cases} [RI_A([1-\Delta]\theta) + (1-\Delta)\theta Y I_B([1-\Delta]\theta)]M_{AZ_T} & \text{if} \quad T = t \\ [RI_A(\theta) + \theta Y I_B(\theta)]M_{AZ_T} & \text{if} \quad T \geq t+1 \end{cases} \tag{2.35}$$

- *If Δ does not satisfy the condition in Lemma 8, the borrower in period t does not allow the collateral reuse, and the aggregate output in period*

$T \geq t$ *evolves as follows:*

$$Y_T = \begin{cases} RI_A(1)M_{AZ_T} & \text{if} \quad T = t \\ [RI_A(\theta) + \theta YI_B(\theta)]M'_{AZ_T} & \text{if} \quad T \geq t+1 \end{cases} \tag{2.36}$$

If the collateral re-use is allowed in period t, the immediate effect of the crisis on output is relatively small: although the funding for A's project decreases from $I_A(\theta)$ to $I_A([1-\Delta]\theta)$, B's project is funded as well. However, this has a negative effect to increase the collateral misallocation in the subsequent periods, and thus the recovery is slower relative to the case in which collateral re-use is prohibited – note that M_{AZ_T} is greater than M'_{AZ_T} for all $T \geq t+1$.

Figure 2.1 illustates how output evolves after different temporary shocks. I assume that the economy is initially at the steady state. The parameters are $Z = 0.9, R = 1, b = 2, Y = 2, \beta = 0.5, p = 0.9, \theta = 0.9$.

In period 3 I introduce temporary negative shocks Δ that decrease the probability of success of the intermediary from θ to $(1-\Delta)\theta$. When the size of the shock is smaller than $\Delta = 35\%$, the borrower allows collateral reuse even after the shock. When the size of the shock is $\Delta = 36\%$, however, the borrower switches not to allow collateral reuse after the shock. Note that the downfall of output at the time of the shock increases as the shock is larger. Taking the borrower's decision unchanged in case that $\Delta = 20\%, 33\%$, the crisis is more severe and it takes more time to recover as the size of the shock increases. On the other hand, if the borrower's decision changed after the shock as in $\Delta = 36\%$, the size of the crisis is even larger, but the economy recovers faster than the other cases, implying that if the shock affects the borrower's decision to allow rehypothecation, and the dynamics after the shock can be significantly different.

2.3.2 Stochastic Aggregate Shocks

I now consider the possibility that aggregate shocks randomly arrive. I assume that the counterparty shock θ follows a two-state Markov chain where $\theta \in \{\theta_L, \theta_H\}$ with $\theta_L \leq \theta_H$ and let

$$\pi_{ij} = \Pr(\theta_{t+1} = \theta_j | \theta_t = \theta_i) \quad i, j \in \{H, L\}. \tag{2.37}$$

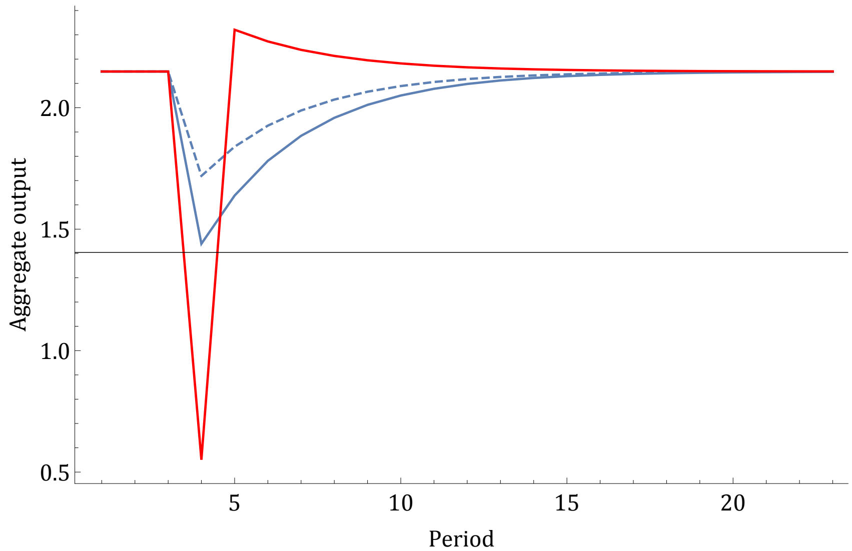

Figure 2.1: Output dynamics after different temporary shocks (the size of shock is $\Delta = 20\%$ (blue dashed), $\Delta = 33\%$ (blue solid), and $\Delta = 36\%$ (red))

That is, π_{LL} and π_{HH} capture the persistence of bad and good states: higher π_{LL} and π_{HH}, greater persistence in bad and good states, respectively. If π_{LL} or π_{HH} equals 1, the state remains to be either θ_L or θ_H forever, once reached (and, it boils down to the previous case with no persistent aggregate shocks).

Equilibrium

As in the previous analysis, in equilibrium, the entrepreneurs' collateral reuse decision in any period t maximizes expected payoff given a state $\theta_t \in \{\theta_L, \theta_H\}$, the optimal single-period contract $(I_A(\theta_t), X_t(\theta_t), I_B(\theta_t), X_B(\theta_t))$ as in Lemma 4 and 5, and a present value of holding on collateral to the next period, $\sum_{j\in\{H,L\}} \pi_{ij}[pV_A(\theta_j) + (1-p)V_C(\theta_j)]$. This determines the evolution of the distribution of types of the agents over time along any path of aggregate shocks $(\theta_1, \theta_2, \ldots)$. As a result, the equilibrium consists of the optimal decision rule on collateral reuse, the optimal single-period contract, the value functions of each type of the agents, and the distribution of types of the agents over time that is consistent with the optimization by agents.

First, the value function of entrepreneurs (A) is given by,

$$V_A(\theta_i) \equiv \max\Bigg\{RI_A(\theta_i)+\theta_i\Big[1+\beta\sum_{j\in\{H,L\}}\pi_{ij}[pV_A(\theta_j)+(1-p)V_C(\theta_j)]-X_A(\theta_i)\Big],$$
$$R\hat{I}_A + 1 + \beta\sum_{j\in\{H,L\}}\pi_{ij}[pV_A(\theta_j)+(1-p)V_C(\theta_j)] - \hat{X}_A\Bigg\} \quad (2.38)$$

where $(I_A(\theta_i), X_A(\theta_i))$ are in Proposition 5 and $(\hat{I}_A, \hat{X}_A) \equiv (I_A(\theta = 1, Y = 1), X_A(\theta = 1, Y = 1))$. The first term means the value of allowing collateral reuse and the second term means the value of not allowing collateral reuse at state θ_i. This shows that enterpreneurs choose whether to allow collateral reuse or not to allow it optimally. And the value function of cash holders (C) is given by,

$$V_C(\theta_i) \equiv 1 + \beta\sum_{j\in\{H,L\}}\pi_{ij}[(1-p)V_A(\theta_j) + pV_C(\theta_j)] \quad (2.39)$$

The evolution of the distribution of types that are consistent with the collateral reuse decisions is given by,

$$\begin{bmatrix} M_{AZ_{t+1}} \\ M_{A0_{t+1}} \\ M_{CZ_{t+1}} \\ M_{C0_{t+1}} \end{bmatrix} =$$
$$\begin{bmatrix} \lambda_t[p\theta_t + (1-p)(1-\theta_t)] + (1-\lambda_t)(1-p) & 0 & 1-p & 0 \\ \lambda_t[(2p-1)(1-\theta_t)] & p & 0 & 1-p \\ \lambda_t[(1-p)\theta_t + p(1-\theta_t)] + (1-\lambda_t)(1-p) & 0 & p & 0 \\ -\lambda_t[(2p-1)(1-\theta_t)] & 1-p & 0 & p \end{bmatrix} \begin{bmatrix} M_{AZ_t} \\ M_{A0_t} \\ M_{CZ_t} \\ M_{C0_t} \end{bmatrix} \quad (2.40)$$

where $\theta_t \in \{\theta_L, \theta_H\}$, $p \in \left(\frac{1}{2}, 1\right]$ as in Assumption 6, and $\lambda_t \in \{0, 1\}$ where $\lambda_t = 1$ means that collateral reuse is allowed and $\lambda_t = 0$ means that collateral reuse is not allowed.

If π_{LL} or π_{HH} equals 1, the state θ_L or θ_H is absorbing in the sense that it is never left once reached.[12] Then, with the state constant over time, the mass of entrepreneurs with collateral, M_{AZ}, converges to either $\frac{(1-p)Z}{1-\theta_H(2p-1)}$ or

[12]The analysis here closely follows [2].

$\frac{Z}{2}$ (where Z is the initial endowment of collateral), the mass of enterpreneurs with collateral induced by an arbitrarily long sequence of $\theta = \theta_H$ and $\theta = \theta_L$, respectively – by assumption above, if $\theta = \theta_L$, collateral reuse does not occur, so that M_{AZ} is independent of θ.

Let $M_{AZ_t}^{\{\theta_\tau\}_{\tau=0}^t}(M_{AZ_0})$ denote the mass of entrepreneurs with collateral t periods after following sequence of states $\{\theta_\tau\}_{\tau=0}^t$, starting from the mass M_{AZ_0}. Then it leads to the following lemma immediately.

Lemma 9. *Suppose Assumption 7 holds. For $\theta \in \{\theta_L, \theta_H\}$, if collateral reuse is allowed only when $\theta = \theta_H$, then,*

(i) The interval $\left[\frac{(1-p)Z}{1-\theta_H(2p-1)}, \frac{Z}{2}\right]$ is absorbing in the sense that given any initial $M_{AZ_0} \in \left[\frac{(1-p)Z}{1-\theta_H(2p-1)}, \frac{Z}{2}\right]$, for any sequence of states $\{\theta_\tau\}_{\tau=0}^t$, $M_{AZ_t}^{\{\theta_\tau\}_{\tau=0}^t}(M_{AZ_0}) \in \left[\frac{(1-p)Z}{1-\theta_H(2p-1)}, \frac{Z}{2}\right]$

(ii) For any $M_{AZ_0} \notin \left[\frac{(1-p)Z}{1-\theta_H(2p-1)}, \frac{Z}{2}\right]$,

$$\lim_{t\to\infty} Prob\left(\left\{\{\theta_\tau\}_{\tau=0}^t \;\middle|\; M_{AZ_t}^{\{\theta_\tau\}_{\tau=0}^t}(M_{AZ_0}) \in \left[\frac{(1-p)Z}{1-\theta_H(2p-1)}, \frac{Z}{2}\right]\right\}\right) = 1$$

Effects of Prolonged Shocks on Output Dynamics

Here, I focus on the case in which enterpreneurs allows collateral reuse if $\theta = \theta_H$ and do not allow it if $\theta = \theta_L$, that is, the decision on collateral reuse depends on the state of the economy. I study the impact of persistence of aggregate shocks – that is, either good (θ_H) or bad (θ_L) states continues for a long time – on the output dynamics.

Specifically, I contrast the output dynamics in two economies that start from the same state, but one is followed by a prolonged period of good states $(\theta_H, \theta_H, \ldots, \theta_H)$ and another is followed by a prolonged period of bad states, $(\theta_L, \theta_L, \ldots, \theta_L)$.

A key factor that determines aggregate output in this economy is the amount of collateral held by entrepreneurs, M_{AZ}: collateral can be used for more efficient investments only if it is held by entrepreneurs, otherwise it simply yields a fixed dividend which is less than the outcome of the project. Other things equal, as more collateral is assigned to entrepreneurs, the output tends to be larger.

Counterintuitively, the next result shows that the long period of continued good shocks might lead to a smaller output in the future. The intuition is as follows. In a boom, the risk of losing collateral is relatively small and so borrowers tend to allow rehypothecation too frequently, thereby accumulating misallocated collateral in the economy. In contrast, in recession, the risk of losing collateral is relatively large and so borrowers are reluctant to allow rehypothecation, thereby preventing potential misallocation of collateral due to rehypothecation failure. To summarize, when the borrower's decisions are affected by shocks, the number of entrepreneurs with collateral, M_{AZ}, diminishes over time as long as good shocks continue, while it rises over time as long as bad shocks continue. Taking these results with the fact that aggregate output is an increasing function of M_{AZ}, this implies that output can be even smaller after continued good shocks than after continued bad shocks.

Figure 2.2 illustates how continued good shocks can lead to smaller output than continued bad shocks. I assume that the economy is initially in a long run equilibrium in the sense that M_{AZ} initially lies within the interval, $\left[\frac{(1-p)Z}{1-\theta_H(2p-1)}, \frac{Z}{2}\right]$. I set parameters as follows: $\beta = .5$, $Z = .9$, $R = 1.45, b = 1.5, Y = 1.4, p = .95, \theta = .92$, and $\pi_{ij} = .5$ for all $i, j \in \{H, L\}$. And under these parameter values, it can be also verified that: (i) the borrower wants to allow rehypothecation only in the good state H; (ii) the intermediary wants to reuse collateral ex post if rehypothecation has been allowed in advance; (iii) the borrower prefers to invest by delivering collateral rather than holding on to it in both good and bad states.

Note that if good shocks continue, M_{AZ} decreases over time, while if bad shocks continue, M_{AZ} increases over time, showing that misallocation of collateral increases during the boom (see Figure 2.3). As explained above, this plays a major role in determining aggregate output, and Figure 2.2 shows that aggregate output following bad shocks exceeds that following good shocks in period 22.

2.4 Conclusion

Collateral is important for financing investments; without it, firms may find it difficult to obtain loans to fund their projects. However, collateral is scarce, and one way to economize on the limited amount of collateral is to"re-use"

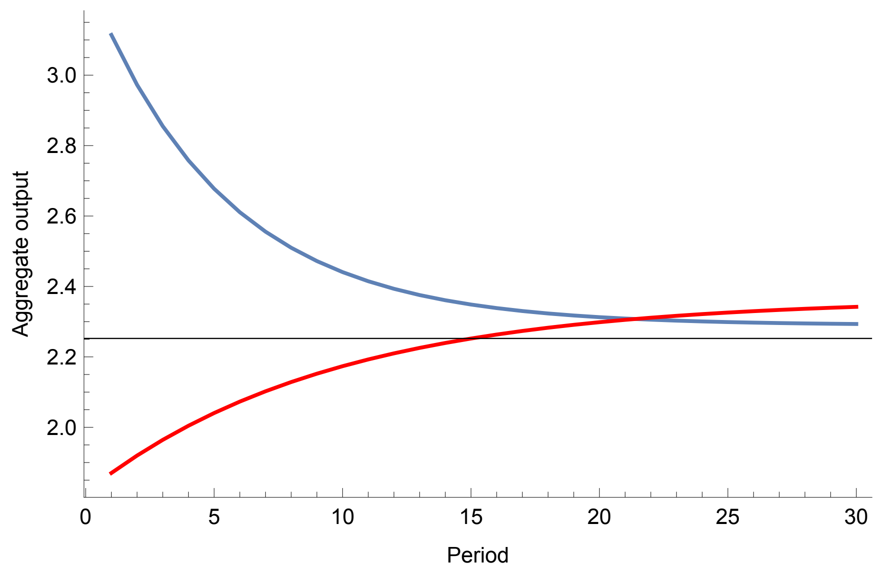

Figure 2.2: Output dynamics with continued good shocks (blue) and bad shocks (red)

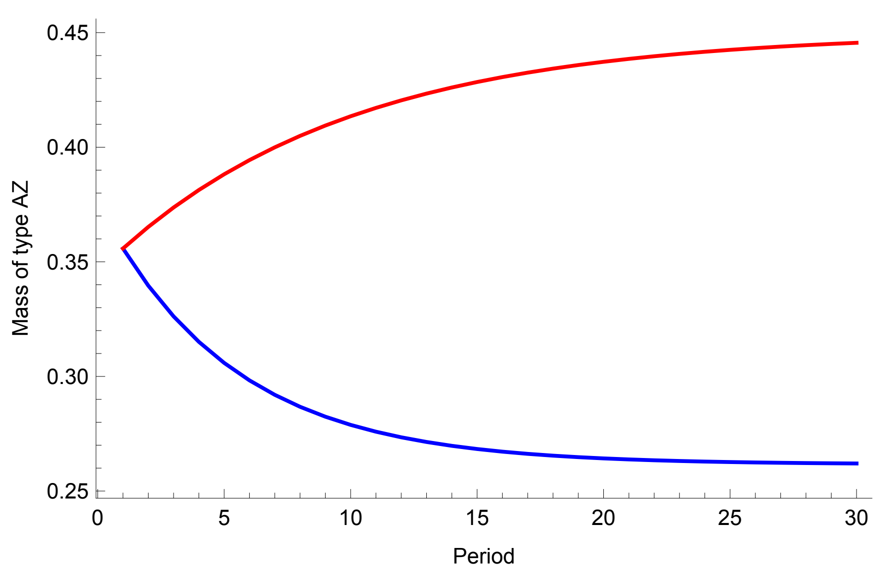

Figure 2.3: Evolution of M_{AZ} with continued good shocks (blue) and bad shocks (red)

or rehypothecate it, making it support multiple transactions at once.

I have constructed a dynamic model in which collateral reuse plays an important role in output fluctuations. In the model, the allocation of collateral is a key determinant of aggregate output and the risk of default by the intermediary initiates output fluctuations; a shock that increases the risk of failure of the collateral chain leads to the mismatch of collateral and investment opportunities, decreasing output. In addition, the output dynamics after the shock depend on the borrowers' decision on collateral reuse; with concerns of losing collateral, they might preclude the intermediary from reusing their collateral. If they prohibit collateral reuse, then the crisis is more severe while it helps the economy to recover faster by preventing potential mismatch of collateral in the future.

A future research is to extend the model into a richer setting such as: (i) the borrowers have heterogeneous valuations on collateral, and so the choices of allowing collateral reuse might differ across them; (ii) the asset market is available, and the borrower who lose collateral can repurchase it from the market with some costs.

CHAPTER 3

EMERGENCE OF REHYPOTHECATION

3.1 Introduction

In this chapter, I develop a simple model in which rehypothecation occurs as an optimal choice when a borrower faces two alternative ways of financing. One option is to borrow funding directly from a cash holder (direct financing), another option is to borrow funding through an intermediary (indirect financing). With direct financing, the borrower delivers collateral directly to the cash holder, and this incurs some transaction costs. With indirect financing, the borrower delivers collateral to the intermediary who then lends (rehypothecates) it to the cash holder, and this exposes the borrower with the risk of losing collateral in case that the intermediary defaults. Thus, the borrower's choice of financing depends on the relative cost associated with each method of finance.

In particular, I investigate the effect of moral hazard on the part of the borrower on the cost of each financing and the borrower's choice of method of finance. If the intermediary's default risk is exogenous, as the moral hazard problem gets more severe, the borrower is less concerned by the default risk of the intermediary, inducing the borrower to choose indirect financing more frequently. However, if the intermediary's default risk is endogenous, as the moral hazard problem gets more severe, the cost of effort made by the intermediary to prevent default also increases. As a result, indirect financing is inferior to direct financing both if the severity of the moral hazard problem is too small or too large.

3.2 Model

3.2.1 Direct financing

All agents are risk neutral; there is no discounting. Agent A has a project that yields a payoff of $R > 1$ tomorrow per unit of input invested today. However, A has no inputs for the project and has to borrow funding. There are a large number of competitive agents C, each has (a sufficiently large amount of) inputs for A's project. However A can avoid repaying the loan: he can divert what he borrows to gain a private benefit of b per unit of funds diverted.

Assumption 9. *$R > b > R - 1$.*

The first inequality implies that investing in the project is more efficient than diverting as the return from the project R is greater than the private benefit to the borrower from diverting b (both measured per unit invested). However, the second inequality means that the benefit from diverting b is greater than the project's net return $R - 1$, and thus self-financing (or uncollateralized borrowing) is not feasible.

Suppose A delivers an asset that he possesses as collateral to C in exchange for funding for his project. The collateral asset is indivisible and has the value of holding it tomorrow is $Z > 0$ to A but zero to C. Thus A wants to retrieve it from C (and C does not want to retain it) at the end.[1] I assume that the surplus that C gains from temporarily holding the collateral asset is given by $V_c > 0$.[2] In addition, there is a transaction cost of $T > 0$ which is borne by A.

In such scenario a contract between A and C is a profile (I, X) where I is the amount of funds that A initially borrows from C in exchange for pledging the asset as collateral and X is the amount that A pays C in exchange for receiving the collateral back. The optimal contract maximizes A's payoff subject to (1) a incentive constraint that prevents A from diverting the funds

[1]Heterogeneity in the value of the asset can arise for various reasons; the asset might be an important component of the borrower's portfolio, the borrower needs the asset to gain voting rights, or the borrower might have technologies and opportunities to use his asset efficiently, which are not available to others.

[2]For example, C borrows the asset from A to sell it short, or to hedge other investments, or to own the record of dividends for some accounting or tax reasons, as documented by Duffie, Gârleanu, and Pedersen (2002).

borrowed and (2) a participation constraint that ensures C's payoff to be greater than a certain reservation value. In what follows I assume that A has all the bargaining power and set the reservation value of B to 0. Thus, the optimal contracting problem is as follows:

$$\max_{I,X} RI - X + Z - T \tag{3.1}$$

subject to

$$RI - X + Z \geq bI \tag{IC_A}$$

$$V_c + X - I \geq 0 \tag{P_C}$$

Lemma 10. *There exists a solution to the contracting problem* (3.1)*. The solution is unique and takes a form of* (I, X) *where*

$$I = \frac{V_c + Z}{1 + b - R}, \quad X = \frac{(R-b)V_c + Z}{1 + b - R}. \tag{3.2}$$

Proof. This is a special case of Lemma 11, proved below. □

To interpret this result, I write the incentive constraint of the borrower as follows:

$$(R - b)I + Z \geq X \tag{3.3}$$

implying that the most that lenders can be promised is the sum of the pledgeable income $(R - b)I$ – the remaining outcome of the project after paying a rent of bI to the borrower – and the value of collateral to the borrower Z. Similarly the participation constraint of the lender can be rewritten as

$$V_c + X \geq I \tag{3.4}$$

implying that the most that lenders are willing to provide is the sum of the value of collateral to lenders (or the value of holding collateral temporarily) and the amount that lenders are promised to be paid. Combining these, a maximum investment scale is

$$I = \frac{1}{1 - (R - b)}(V_c + Z) \tag{3.5}$$

where $\frac{1}{1-(R-b)} > 1$ is the multiplier, the inverse of the minimum equity ratio:

for each unit of investment, the borrower can raise $R - b$ from outside, but must cover the remaining part $1 - (R - b)$ by his own wealth. For example, a borrower with a minumum equity ratio of 25% can invest a maximum of $4(= 1/.25)$ per unit of own asset value.

Finally, at the maximum investment scale the borrower's payoff is

$$bI = \frac{b}{1 - (R - b)}(V_c + Z)$$

Since $R > 1$ by assumption, the rate of return on the borrower's collateral $\frac{b}{1-(R-b)}$ is greater than 1. As a result, the internal value of collateral to the borrower, the right side of the above equation, exceeds the market value of collateral to lenders, $V_c + Z$.

3.2.2 Indirect financing

Consider the following scenario: A borrows funding from an intermediary B in exchange for delivering his asset and specifies that B may reuse the collateral. Having provided funding for A, B then lends (rehypothecates) the collateral to C who needs it temporarily. The transaction through intermediary B saves A the transaction costs – in other words, T becomes zero – but exposes A to the risk of not getting back the collateral if B defaults, which occurs with probability $1 - p \in (0, 1)$.[3]

The contracting problem in this scenario is solved by backward induction: the arrangement between B and C is solved by taking the terms agreed on between A and B as given, and with foreknowledge of the contract between B and C in the second stage, A and B make the initial contract.

The contract between B and C is straightforward: because C is competitive, the revenue to B from lending (rehypothecating) the collateral is the surplus to C from temporarily using collateral, V_c. Then, the conditions of the contract between A and B can be described as before: A receives funding I for his project from B in exchange for delivering the asset; if B does not default, A makes the prearranged payment X to B in return for the collateral and if B defaults, A loses the collateral and does not pay X.

Assuming that A has all the bargaining power and B's reservation utility

[3]In this section, I assume that this default probability is exogenous and not correlated with rehypothecation. In the subsequent section, I consider the case where p is endogenous.

is zero, the optimal contract between A and B solves

$$\max_{I,X} RI + p(Z - X) \tag{3.6}$$

subject to

$$RI + p(Z - X) \geq bI \tag{IC_A}$$

$$V_c + pX - I \geq 0 \tag{P'_B}$$

Lemma 11. *Suppose $p > 0$. There exists a solution to the contracting problem* (3.6). *The solution is unique and takes a form of (I, X) where*

$$I = \frac{V_c + pZ}{1 + b - R}, \quad X = \frac{p^{-1}(R - b)V_c + Z}{1 + b - R} \tag{3.7}$$

Proof. If $p > 0$, a necessary and sufficient condition for a solution (I, X) is the existence of a quartet (I, X, λ, μ) satisfying the following conditions:

$$R + \lambda(R - b) - \mu = 0$$
$$-p - p\lambda + p\mu = 0$$
$$RI + p(Z - X) \geq bI;\ \lambda \geq 0$$
$$V_c + pX - I \geq 0\ \mu \geq 0$$

where λ and μ are the multipliers on (IC_A) and (P'_B), respectively, and the pairs of inequalities hold with complementary slackness. The equalities simplify to

$$\lambda = \frac{R - 1}{b - R + 1}$$
$$\mu = \frac{b}{b - R + 1}$$

both of which are positive by Assumption 9. Therefore, both conditions (IC_A) and (P'_B) bind, and (3.7) follows. Note that by Assumption 9, both I and X in (3.7) are positive. Also note that when $p = 1$ this problem reduces to the previous section. □

Note that A's payoff in this problem becomes

$$\frac{b}{1-(R-b)}(V_c + pZ) \tag{3.8}$$

which can be interpreted in a similar way as in the previous case.

The payoff in this problem has to be compared with the payoff from direct financing in Equation (3.2.1) to derive the condition under which A chooses intermediation over direct financing.

Proposition 9. *The borrower chooses indirect financing if and only if*

$$(1-p)\frac{b}{1-(R-b)}Z < T. \tag{3.9}$$

The left side is the expected loss of value from losing collateral when the intermediary defaults: the probability of default $1-p$ multiplied by the internal value of collateral to the borrower $\frac{b}{1-(R-b)}Z$ (where $\frac{b}{1-(R-b)}$ is the rate of return on collateral if the borrower invests and Z is the value of collateral to the borrower). The right side is the transaction cost of the direct financing. As the value of collateral to the borrower decreases or the transaction cost increases, the borrower is more likely to choose the indirect financing.

Equation (3.9) also shows that b has two opposing effects on the borrower's choice: as b increases, the rent to the borrower increases while the equity multiplier, the inverse of the minumum equity ratio, $\frac{1}{1-(R-b)}$ decreases. However, if $R > 1$, the latter effect dominates the former effect,

$$\frac{d}{db}\left(\frac{b}{1-(R-b)}\right) = \frac{1-R}{[1-(R-b)]^2} < 0,$$

which yields the following result.

Corollary 3. *As b increases the borrower is more likely to choose the indirect financing.*

In other words, if the default risk is exogenous, as the moral hazard problem gets more severe, the borrower is more likely to choose the indirect financing rather than the direct financing.

3.3 Endogenous default risk

In the previous analysis I assumed that the probability that the intermediary fails $1 - p$ is exogenous. Suppose instead that this probability varies with the efforts made by the intermediary: increasing effort by the intermediary lowers the probability of default and its cost to the intermediary.

Let $p(x)$ be the repayment probability where x represents the intermediary's effort costing x. I assume the function p is increasing and strictly concave, with $p(0) = 0$, $\lim_{x\to\infty} p(x) = 1$, $p'(0) = \infty$, and $\lim_{x\to\infty} p'(x) = 0$.

Given X, the intermediary chooses x to maximize

$$p(x)X - x.$$

So that the choice is defined by

$$p'(x)X = 1 \tag{IC_B}$$

The contracting problem chooses x, X, and I to maximize

$$RI + p(x)(Z - X) \tag{3.10}$$

subject to

$$\begin{aligned} RI + p(x)(Z - X) &\geq bI && (IC_A) \\ V_c + p(x)X - x &\geq I && (P_B) \\ p'(x)X &= 1 && (IC_B) \\ x \geq 0,\ I \geq 0,\ X &\geq 0 \end{aligned}$$

To solve this problem, I need to simplify the problem as follows.

Lemma 12. *In any optimum, $V_c + p(x)X - x = I$.*

Proof. Suppose not. Then increase I until this is the case; it does not violate any constraint, and improves the objective. □

Thus I can rewrite the problem as follows.

$$\max_{I,X,x} R[V_c + p(x)X - x] + p(x)(Z - X) \tag{3.11}$$

subject to

$$(R-b)[V_c+p(x)X-x]+p(x)(Z-X)\geq 0 \qquad (IC_A)$$

$$p'(x)X=1 \qquad (IC_B)$$

$$x\geq 0,\ I\geq 0,\ X\geq 0$$

Lemma 13. *There exists a solution to this problem.*

Proof. The set of feasible solutions is non empty (let $X=Z$ and x satisfy $p'(x)Z=1$, for example). Now the right side of (IC_A)

$$(R-b)[V_c+p(x(X))X-x(X)]+p(x(X))(Z-X)$$

where $x(X)=p'^{-1}\left(\frac{1}{X}\right)$. This is decreasing in X for $X>Z$, since $(R-b-1)p(x)+[p'(x)Z-1]x'(X)<0$ for $X\geq Z$. There exists $\bar{X}\in(Z,\infty)$ such that (IC_A) does not hold for any $X>\bar{X}$. As a result, the feasible set X satisfying (IC_A) is bounded above by $\bar{X}$ and so the set of feasible pairs (X,x) is closed and bounded. Since the objective function is continuous, a maximum exists for the problem. □

Now rewrite the problem further as

$$\max_{X}\ R[V_c+p(x(X))X-x(X)]+p(x(X))(Z-X) \qquad (3.12)$$

subject to

$$(R-b)[V_c+p(x(X))X-x(X)]+p(x(X))(Z-X)\geq 0 \qquad (IC_A)$$

$$X\geq 0$$

where $x(X)=p'^{-1}\left(\frac{1}{X}\right)$ and p'^{-1} is the inverse function of p'.

Lemma 14. *In any optimum, $X>Z$.*

Proof. Suppose $X\leq Z$. Then we know that

$$(R-b)[V_c+p(x(X))X-x(X)]+p(x(X))(Z-X)>0$$

because the conditions on the function p guarantee that $p(x(X))X-x>0$ for all X. So that we consider the unconstrained maximum: first order conditions are

$$(R-1)p(x(X)) + p'(x(X))x'(X)(Z-X) = 0$$

At any such unconstrained maximum we have

$$X - Z = \frac{(R-1)p(x(X))}{p'(x(X))x'(X)} > 0.$$

However, this contradicts $X \leq Z$. Therefore, $X > Z$. □

Lemma 15. *There is a unique value of X, call it X_1, such that the constraint (IC_A) holds with equality.*

Proof. From above, we know the constraint does not bind for any $X \leq Z$. Also the derivative of the right side of (IC_A) is negative for $X > Z$. Therefore, there is unique value of $X_1 (> Z)$ where the constraint is binding. □

Lemma 16. *If the function*

$$(R-1)p(x(X)) - x'(X)[1 - p'(x(X))Z]$$

is positive in the range $X \in (Z, X_1)$ where $x(X) = p'^{-1}\left(\frac{1}{X}\right)$, the solution to the maximization problem is $X = X_1$, $I = V_c + p(x(X_1))X_1 - x(X_1)$, and $x(X) = p'^{-1}\left(\frac{1}{X_1}\right)$.

Proof. Clear, since the objective function is monotonic over the interval. □

One simple sufficient conditions for this is

$$(R-1)p(x(Z)) - x'(Z) > 0 \quad \text{and} \quad x'' < 0. \tag{3.13}$$

which is equivalent to

$$(R-1)p(p'^{-1}(Z^{-1})) + \frac{1}{p''(p'^{-1}(Z^{-1}))Z^2} > 0$$

and

$$2Z[p''(p'^{-1}(Z^{-1}))]^2 > p'''(p'^{-1}(Z^{-1})).$$

To summarize, we reach to the following proposition.

Proposition 10. *If* $(R-1)p(p'^{-1}(Z^{-1}))+\frac{1}{p''(p'^{-1}(Z^{-1}))Z^2} > 0$ *and* $2Z[p''(p'^{-1}(Z^{-1}))]^2 > p'''(p'^{-1}(Z^{-1}))$, *there exists a unique solution to the problem (3.12) such that* $X = X_1$, $I = V_c + p(x(X_1))X_1 - x(X_1)$, *and* $x(X_1) = p'^{-1}\left(\frac{1}{X_1}\right)$.

Comparing A's payoff in each case, the condition under which intermediation is preferred to direct trade is

$$\frac{b}{1-(R-b)}[\{1-p(x(X))\}Z + x(X)] < T. \tag{3.14}$$

For the purpose of analysis, let $p(x) = 1-\frac{1}{1+ax}$ where $a \in \mathbb{R}^+$. At optimum,

$$\begin{aligned} x(X) &= \sqrt{\frac{X}{a}} - \frac{1}{a} \\ p(x(X)) &= 1 - \frac{1}{\sqrt{aX}} \\ I(X) &= \frac{1}{1-(R-b)}\left[V_c + \left(1 - \frac{1}{\sqrt{aX}}\right)Z - \left(\sqrt{\frac{X}{a}} - \frac{1}{a}\right)\right] \end{aligned}$$

and X is a solution to the following equation

$$\frac{1}{1-(R-b)}\left\{(R-b)\left(1-\frac{1}{\sqrt{aX}}\right)^{-1}\left[V_c - \left(\sqrt{\frac{X}{a}} - \frac{1}{a}\right)\right] + Z\right\} - X = 0. \tag{3.15}$$

Using the result in Equation (3.14), the indirect financing is preferred to the direct financing if and only if

$$\frac{b}{1-(R-b)}\left[\frac{1}{\sqrt{aX}}Z + \left(\sqrt{\frac{X}{a}} - \frac{1}{a}\right)\right] < T$$

Note that the rate of return of investment, $\frac{b}{1-(R-b)}$, increases in b. Provided that X decreases in b, the probability of default increases in b and the cost of effort decreases in b. Therefore, the effect of b on the choice of financing is non-monotonic: if b is small, the probability of default is large, and if b is large, the cost of efforts is large, in both cases, the indirect financing tends to be less profitable than the direct financing.

Corollary 4. *If* $T > \underline{T}$, *there exists* $\bar{b}$ *and* $\underline{b}$ *such that the borrower chooses the indirect financing if and only if* $b \in (\underline{b}, \bar{b})$ *and chooses the direct financing otherwise.*

The following figures illustate the effects of b on the choice of financing if $R = 2, V_C = 4, Z = 11$ and $a = 1$.[4]

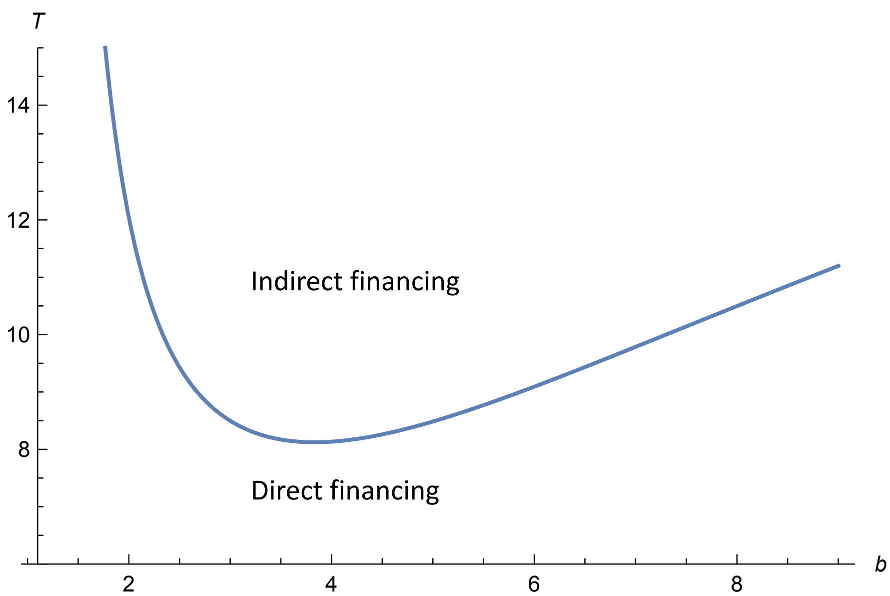

Figure 3.1: Effect of b on the cost of intermediation

3.4 Conclusion

A fundamental tradeoff between indirect financing and direct financing is that indirect financing saves the cost of finding lenders (transaction cost), but it exposes the borrower to the risk of losing collateral in case that the intermediary defaults. The borrower may then prefer direct financing which guarantees the return of collateral but incurs the transaction cost.

In particular, I investigate how the choice of financing is affected by moral hazard of the borrower. Taking the intermediary's default risk as given constant, as the borrower's moral hazard is more severe, the borrower is less concerned of losing collateral, and the indirect financing is chosen more frequently. However, if the intermediary's default risk is endogenous, the result is non monotonic. If the borrower's moral hazard is small, the intermediary does not make sufficient effort to avoid default. On the other hand, if the moral hazard is severe, the intermediary has to make too much costly effort

[4]This set of parameters satisfy the condition in Proposition 10.

to alleviate it. As a result, in both cases, the cost of the indirect financing tends to be large.

Important future works would be to consider how rehypothecation differs from other common ways of intermediation such as collecting deposits and issuing loans.

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