INFORMATION, INSIDER TRADING AND TAKEOVER ANNOUNCEMENTS

BY

WEI QIN

DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics in the Graduate College of the University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

Doctoral Committee:

Professor Renming Song, Chair
Professor Richard Sowers
Associate Professor Runhuan Feng
Assistant Professor Alexandra Chronopoulou
ABSTRACT

This thesis focuses on the effect of takeover announcements in financial markets. We want to use a math model to analyze the inside traders’ behavior when there is a potential takeover in the market. The thesis starts with a math model to capture the stock price dynamics, and then it states the term structure behaviors under the model. The thesis also contains numerical methods in the model calibration and validation.
Based on Mike Lipkin’s ideas and discussions with Tim Johnson
TABLE OF CONTENTS

CHAPTER 1   INTRODUCTION ................................................. 1

CHAPTER 2   BEHAVIOR WITH INSIDER INFORMATION ................... 4
   2.1   Buy the stock ................................................. 4
   2.2   Sell long-term premium via Calender spread ................. 4
   2.3   Do the near term 1-by-many for a credit .................... 5
   2.4   Consequence .................................................. 5

CHAPTER 3   THE MODEL .................................................. 6
   3.1   Our Model ..................................................... 6
   3.2   Background Information ...................................... 6
   3.3   The Option Price ............................................. 8
   3.4   Properties of the Option Price .............................. 11

CHAPTER 4   TERM STRUCTURE ........................................ 14
   4.1   Term Structure when $T \to \infty$ .............................. 14
   4.2   Term Structure when $\lambda \to \infty$ ......................... 16

CHAPTER 5   NUMERICAL CALIBRATION ................................. 29
   5.1   One By Many .................................................. 29
   5.2   Calender Spread .............................................. 32

CHAPTER 6   FULL MODEL .............................................. 39
   6.1   Risk neutral model .......................................... 39
   6.2   Calibration of the Jump rate ................................ 39
   6.3   Dynamics of option price ................................... 40

CHAPTER 7   FURTHER DISCUSSION .................................. 43
   7.1   CIR PROCESS .................................................. 43

APPENDIX A   HOW DO THE STRATEGIES WORK ....................... 48
   A.1   One by many .................................................. 48
   A.2   Calender spread ............................................. 50

REFERENCES .......................................................... 51
Takeovers involve conversion of a company’s stock to another type of asset (either cash or the stock of another company). This occurs at a specific time and at a specific conversion factor, so it reflects a discontinuity in the valuation of the company’s assets. This thesis focuses on the effect of advance information about takeovers. Although the public perception of takeovers is one of immediacy, this is far from the case. For example the stock GENZ (Genzyme corporation) was the subject of multiple takeover rumors over many years. And finally it acquiesced to a reworked offer with a process that took over 9 months. The stock TLAB (Tellabs corporation) has been the subject of takeover rumors for at least 10 years, but has never been acquired.

However, there are people in the market who know some inside information, and they benefit themselves by doing something illegal: insider trading. Insider trading happens every year, and people convicted insider trading for many cases. One of the most famous cases, Martha Stewart, is a good example. In 2004, Stewart was convicted of charges related to the ImClone insider trading affair and sentenced to prison. Although insider trading is illegal, people can’t detect it when it happens. In this paper, we are trying to build up a model, so that we can detect the insider trading when it happens.

What makes the extended nature of takeover processes non-trivial from a financial perspective is that there are price consequences for the equity and its derivative securities as the process moves along. Stocks being acquired trade, often wildly, with large volumes for many months and option prices exhibit an even larger variety of behaviors.

There are several different types of takeovers: cash, stock, a mixture of cash and stock. Furthermore, these deals may have many flavors involving Dutch auctions, two-tier deals; deals requiring spin-offs; hostile, friendly, take-unders, etc. etc. Although the analysis we discuss in this paper is applicable to all, for simplicity and clarity of exposition we will narrow the focus to simple cash deals only. In these, one party, the acquirer, makes a cash tender offer of a specific amount to purchase all the outstanding shares of stock in a second company, the acquiree. As a rule of thumb: After the announcement happens, the
implied volatility of the stock will decrease.

Although simple cash deal sounds straightforward, it is not. Rumors, may precede an actual tender offer. Once an offer is made the deal is subject to a host of intermediating events involving accounting discovery, governmental review, and the possibility of counter-offers. Deals sometimes fall through. Only after a period of time (which is rarely less than 3 months) will a deal consummate. At the termination point, all stock in the acquirer is converted to cash at the ultimate agreed amount, an amount which need not be the original tender price.

To effectively model the price evolution of both the acquiree and its options we need to focus on the timeline of these events. We shall see that this timeline need not have a starting point nor a terminating point. TLAB which has been the subject of continuous takeover rumors has never been acquired, yet its options reflect the degree to which the market perceives a takeover as likely. We shall refer in the paper to the pre-announcement time period, the period of time when option and stock prices may reflect a heightened possibility of takeover. One striking consequence of very large takeover rumors is the pleating of the volatility surface in the near-term option series.

time line: announcement, takeover, and the announcement price is almost the takeover price, the plot of takeover of underlying stock. possibilities: cash, large companies, or fail.

If a tender offer is made the consequences are immediate and dramatic. The stock price of the acquiree will make a jump (up except in the case of a take-under) and the option volatility surface will make an extreme twist. The announcement of a tender offer marks a singularity in the timeline of a stock. Although it is a fundamental necessity in all stocks
which are eventually acquired it also marks the transition from a pre-announcement process to a post-announcement process. In the post-announcement phase, stocks are also subject to rumors of varying likelihood, in this case involving the possibilities of delay, counter-offers, restructuring and break-up. A striking consequence of increasing likelihood of delay and break-up is the exceptional volatility of the just-out-of-the-money near-term options.

As a deal approaches completion, the premia of long dated options must fall to zero. On the other hand, as the possibility of break-up or delay increases, these same options may regain some of the lost premia. A consequence of this analysis will convince the reader that small- and medium-cap stocks will in general exhibit decreasing term-structures of volatility.

To model these effects, we let stock, $S$, the acquiree, have pricing dependent on an auxiliary parameter, $\epsilon$, the degree of likelihood of a takeover prior to the announcement date, and the degree of likelihood of a break-up after the announcement date. Although we use the same symbol, $\epsilon$, the tender and the break-up processes are different and not symmetrical. The presence of a tender offer is noted by the indicator function $\eta$, which is 1 after an announcement, 0 before. Finally, if a stock does make it to termination at acquisition, the stock price will be $S^T$. 
CHAPTER 2

BEHAVIOR WITH INSIDER INFORMATION

Let’s take a look at the dynamics of the acquisition. Assume the announcement happens at time $\tau$, then the stock price will jump from $S_{\tau^-}$ to $S_{\tau^+}$, where $S_{\tau^+}$ is always much larger than $S_{\tau^-}$. Advance information about the takeover announcement can be traded upon in several ways, there are three plans that the inside traders who know the potential takeover normally use. The following examples are given by Mike Lipkin in his talk[9].

2.1 Buy the stock

Since the traders believe that the stock will worth more in the future, they can simply purchase the stock in low price and sell it when price jumps up. However, in this case, the insider traders have high exposure risk to a failed deal.

2.2 Sell long-term premium via Calendar spread

We give an example of the stock XYZ: the current stock price is 32.5; the call option with strike 35 and expiry in June costs $0.16 each, and the call option with strike 35 and expiry in November costs $2.25 each. We believe that the stock price will jump to $45 due to an announcement of takeover, and we duplicate the following position by

Buy one share Call with strike 35 and expiry in June
Sell one share Call with strike 35 and expiry in November

The credit for such a spread is $2.09.

Later, when XYZ goes to $45, the calendar falls to parity (from $2.09). So if the inside trader sell this spread, he can definitely earn the premium when price jumps up. In this case, the implied volatility of long-term option will drop, and the implied volatility of short
term will jump up. This part is highly related to the term structure of the implied volatility, and we will look into it with more details later.

2.3 Do the near term 1-by-many for a credit

Next, we consider doing the 1-by-many near term options of the same stock XYZ. Given that the call option with strike 35 and expiry in June costs $0.16 each, and the call option with strike 32.5 and expiry in November costs $0.82 each, we duplicate the following position by

- Sell one share Call with strike 32.5 and expiry in June
- Buy four shares Call with strike 35 and expiry in June

The traders earn $0.18 by doing this portfolio. Again, when later the price jump up to $45, the lower strike lose $12.5, but the higher strike earns 4 times $10, which will give us a total profit of $27.5, plus the premium. In this case, the implied volatility of at-the-money option will drop below the high near-term strike.

2.4 Consequence

The insider trader who has information can earn huge by purchasing and selling the options, this can’t be seen by looking at the option price since there wouldn’t be any information before takeover announcement. However, we can see the cheating by looking at the implied volatilities.

The traders who have inside information can benefit themselves by using the above strategies. And our goal is to use a math model understand and analyze their behavior before the real announcement happens.
3.1 Our Model

After knowing the fact of the takeover process, we would like to use a mathematical model to simulate the process, and build some analysis based on our model.

We start with the simple model.

Consider the takeover states \( \{N, A\} \), which stand for “null” and “announcement” of the takeover. Next we build up the model where the stock price process \( S_t \) will have the dynamics

\[
dS_t \overset{\text{def}}{=} \chi_{\{X_t = N\}} \left\{ b_0 S_t dt + \sigma S_t dW_t + \gamma_A S_t dJ^A_t \right\} + \chi_{\{X_t = A\}} b_1 S_t dt
\]

where \( b_0, \sigma \) represents the parameters of the dynamic, \( W_t \) is the standard Brownian motion and \( J^A_t \) is the jump process with rate \( \lambda_A \).

The model is a combination of a standard geometric Brownian motion, a continuous time Markov process and a first order ordinary differential equation. When the takeover is in the “null” state, the stock price has a dynamic of a geometric Brownian motion with drift parameter \( b_0 \) and volatility parameter \( \sigma \). Meanwhile, the stock price follows an one-time jump process with hazard rate \( \lambda_A \), and jump rate \( \gamma_A \). This means, the stock price will jump to \( (1 + \gamma_A)S_{\tau_A} \) if the jump happens at time \( \tau_A \). After the jump, the stock price will stay in “announcement” state, and the stock price will increase exponentially with parameter \( b_1 \).

3.2 Background Information

In order to understand the model, we might want to recall some of the definitions first.
3.2.1 Transition rate matrix

**Definition 3.2.1 (Transition rate matrix)**  *Transition rate matrix* is an array of numbers describing the rate a continuous time Markov chain moves between states. The transition matrix should satisfy the following 3 conditions:

- $0 \leq -q_{ii} < \infty$;
- $0 \leq q_{ij}$ for $i \neq j$;
- $\sum_j q_{ij} = 0$ for all $i$.

In our case, as a continuous Markov chain, if we think of the stock in a long time, it will have many states on the time line.

If we think of the $N$ state and $A$ state, then the state looks like

$$
\begin{array}{c}
N \\
\lambda_A
\end{array} \quad \xrightarrow{\lambda_A} 
\begin{array}{c}
A
\end{array}
$$

In that case, our transition rate matrix looks like

$$
\begin{pmatrix}
-\lambda_A & \lambda_A \\
0 & 0
\end{pmatrix}
$$

And in the further discussion, we will also introduce the $F$ failure state. In that case, the space generator is $X \overset{\text{def}}{=} \mathbb{R}_+ \times \{N, A, F\}$, and the states look like

$$
\begin{array}{c}
N \\
\lambda_A
\end{array} \quad \xrightarrow{\lambda_A} 
\begin{array}{c}
A \\
\lambda_F
\end{array} \quad \xrightarrow{\lambda_F} 
\begin{array}{c}
F
\end{array}
$$

In that case, our transition rate matrix looks like

$$
\begin{pmatrix}
-\lambda_A & \lambda_A & 0 \\
0 & -\lambda_F & \lambda_F \\
0 & 0 & 0
\end{pmatrix}
$$

**Note**  The states are not complete, as we mention before, there are many states on the time line of a stock. But now we only think of the states that we will discuss in the paper.
3.2.2 Black-Scholes model and Implied volatility

Options are derivative contracts that give the holder (the "buyer") the right, but not the obligation, to buy or sell the underlying instrument at an agreed-upon price on or before a specified future date. Although the holder of the option is not obligated to exercise the option, the option writer (the "seller") has an obligation to buy or sell the underlying instrument if the option is exercised.

And in order to calculate the option price, we introduce the Black-Scholes model [1]. The Black-Scholes model is a mathematical model of a financial market containing derivative investment instruments. From the model, one can deduce the Black-Scholes formula, which gives a theoretical estimate of the price of European-style options. The Black-Scholes (B-S) formula is the fundamental equation which prices an option in terms of time to expiry, the current stock price, the expiry, and the riskless rate of return. The B-S formula is an idealization, and, moreover, the actual volatility of the stock between the current moment and expiry, is not known.

In financial mathematics, implied volatility is the value of volatility which matches the B-S formula to the traded option price.

In our model, the implied volatility of the acquiree company drops to 0 after announcement. This is because, in the financial market, the implied volatility of the small capital firm is significantly larger than the large capital firm. This phenomenon is known as Small Firm effect. The theory holds that smaller companies have a greater amount of growth opportunities than larger companies. Small cap companies also tend to have a more volatile business environment. Since in most cases, the acquiree companies are small capital firms and the acquirer companies are large capital firms, we can see from the effect that the implied volatility of the acquiree is significant larger than the acquirer before the announcement. And after the announcement, the acquiree company will become part of the acquirer company, which means, the implied volatility of the acquiree will drop to the same level as the acquirer.

3.3 The Option Price

After reviewing some of the important definitions, we notice that, the option price is very important in analyzing the inside trading. Although we build the model on the stock price already, we won’t see the stock price jump up until the announcement happens. However, the option price might show us some useful information, cause the option price is defined at
the time of expiry, which can be later than the announcement.

So next, we use the math calculation to represent the option price.

This model is simple enough that we have a fairly simple representation formula for calls. Letting $C(K, T, S)$ be the price of a call with strike $K$ and expiry $T$ when the current stock prices is $S$, we start the calculation conditioning on “announcement” time $\tau$. Then, we can separate the calculation into two cases: the expiry $T$ is before $\tau$ and after $\tau$. If $T < \tau$, the option price is in “null” state when option expires, the calculation only includes standard geometric Brownian motion part, so we can use the B-S formula; if $T > \tau$, the option price is in “announcement” state when option expires, then we have a more complex calculation. We start with rewrite $C(K, T, S)$ as

$$C(K, T, S) = \mathbb{E} \left[ e^{-rT} (S_T - K)^+ \right] = \mathbb{E} \left[ \mathbb{E} \left[ e^{-rT} (S_T - K)^+ | \tau \right] \right].$$

Letting $\eta$ be a standard Gaussian random variable, we then have that if $t > T$

$$\mathbb{E} \left[ e^{-rT} (S_T - K)^+ | \tau = t \right] = e^{-rT} \mathbb{E} \left[ (S \exp \left[ (r - \lambda \gamma - \frac{1}{2} \sigma^2)T + \sigma W_T \right] - K)^+ \right]$$

$$= e^{-rT} \mathbb{E} \left[ \left( S \exp \left[ (r - \lambda \gamma - \frac{1}{2} \sigma^2)T + \sigma \sqrt{T} \eta \right] - K \right)^+ \right]. \tag{3.2}$$

If $t \leq T$, we have

$$\mathbb{E} \left[ e^{-rT} (S_T - K)^+ | \tau = t \right] = e^{-rT} \mathbb{E} \left[ (S \exp \left[ (r - \lambda \gamma - \frac{1}{2} \sigma^2)t + \sigma W_t \right] (1 + \gamma) e^{r(T-t)} - K)^+ \right]$$

$$= e^{-rT} \mathbb{E} \left[ \left( S \exp \left[ (r - \lambda \gamma - \frac{1}{2} \sigma^2)t + \sigma \sqrt{t} \eta \right] (1 + \gamma) e^{r(T-t)} - K \right)^+ \right]$$

$$= e^{-rT} \mathbb{E} \left[ \left( S \exp \left[ rT - (\lambda \gamma + \frac{1}{2} \sigma^2)t + \sigma \sqrt{t} \eta \right] (1 + \gamma) - K \right)^+ \right]. \tag{3.3}$$

Combining things together, we have that

$$C(K, T, S) = e^{-rT} \mathbb{E} \left[ \left( S \exp \left[ rT - (\lambda \gamma + \frac{1}{2} \sigma^2)T + \sigma \sqrt{T} \eta \right] - K \right)^+ \right] e^{-\lambda T}$$

$$+ \int_{t=0}^{T} e^{-rT} \mathbb{E} \left[ \left( S \exp \left[ rT - (\lambda \gamma + \frac{1}{2} \sigma^2)t + \sigma \sqrt{t} \eta \right] (1 + \gamma) - K \right)^+ \right] \lambda e^{-\lambda t} dt. \tag{3.4}$$

Let’s now expand upon some standard transformations. Define, as usual,

$$\mathcal{N}(x) \overset{\text{def}}{=} \int_{z=-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} z^2 \right] dz.$$
Lemma 3.3.1 Fix constants $A$, $B$, $C$, and $D$, where $A$, $C$, and $D$ are positive. Define

$$X \overset{\text{def}}{=} Ae^{B+C\eta}.$$ 

Then

$$\mathbb{E}[(X - D)^+] = Ae^{B+C^2/2} \Phi \left( \frac{B + \ln A/D}{C} + C \right) - D \mathcal{N} \left( \frac{B + \ln A/D}{C} \right).$$

Let’s write

$$\mathbb{E}[(X - D)^+] = \mathbb{E}[X 1_{\{Ae^{B+C\eta} \geq D\}}] - \mathbb{E}[D 1_{\{Ae^{B+C\eta} \geq D\}}]$$

$$= Ae^B \mathbb{E}[e^{C\eta} 1_{\{\eta \geq (\ln(D/A) - B)/C\}}] - D \mathbb{P}\{\eta \geq (\ln(D/A) - B)/C\}.$$ 

We have that

$$\mathbb{P}\{\eta \geq (\ln(D/A) - B)/C\} = \mathbb{P}\left\{-\eta \leq \frac{B + \ln A/D}{C} \right\} = \Phi \left( \frac{B + \ln A/D}{C} \right).$$

We can also compute that

$$\mathbb{E}[e^{C\eta} 1_{\{\eta \geq (\ln(D/A) - B)/C\}}] = \int_{z=(\ln(D/A) - B)/C} e^{Cz} \exp \left[-\frac{1}{2}z^2 \right] \frac{1}{\sqrt{2\pi}} ds$$

$$= e^{C^2/2} \int_{z=(\ln(D/A) - B)/C} \exp \left[-\frac{1}{2}(z - C)^2 \right] ds$$

$$= e^{C^2/2} \int_{z=(\ln(D/A) - B)/C-C} \exp \left[-\frac{1}{2}z^2 \right] ds$$

$$= e^{C^2/2} \mathbb{P}\{\eta \geq (\ln(D/A) - B)/C - C\}$$

$$= e^{C^2/2} \mathbb{P}\{-\eta \leq C + (B + \ln A/D)/C\}$$

$$= e^{C^2/2} \Phi \left( B + \ln A/D \right) + C.$$ 

Combining things together, we get the result.
Thus if \( t > T \),

\[
\mathbb{E} [e^{-rT}(S_T - K)^+ | \tau = t] = e^{-rT} \left\{ S \exp \left[ \left(r - \lambda \gamma - \frac{1}{2} \sigma^2 \right)T + \frac{1}{2} \sigma^2 T \right] \Phi \left( \frac{r - \lambda \gamma - \frac{1}{2} \sigma^2 T + \ln S/K}{\sigma \sqrt{T}} \right) - K \Phi \left( \frac{r - \lambda \gamma - \frac{1}{2} \sigma^2 T + \ln S/K}{\sigma \sqrt{T}} \right) \right\} = S e^{-\lambda \gamma T} \Phi \left( \frac{r - \lambda \gamma + \frac{1}{2} \sigma^2 T + \ln S/K}{\sigma \sqrt{T}} \right) - K \Phi \left( \frac{r - \lambda \gamma - \frac{1}{2} \sigma^2 T + \ln S/K}{\sigma \sqrt{T}} \right)
\]

while if \( t \leq T \),

\[
\mathbb{E} [e^{-rT}(S_T - K)^+ | \tau = t] = e^{-rT} \left\{ S(1 + \gamma) \exp \left[ rT - (\lambda \gamma + \frac{1}{2} \sigma^2)T + \frac{1}{2} \sigma^2 T \right] \Phi \left( \frac{rT - (\lambda \gamma + \frac{1}{2} \sigma^2)T + \ln S/K}{\sigma \sqrt{T}} \right) - K \Phi \left( \frac{rT - (\lambda \gamma - \frac{1}{2} \sigma^2)T + \ln S/K}{\sigma \sqrt{T}} \right) \right\} = S(1 + \gamma)e^{-\lambda \gamma T} \Phi \left( \frac{rT - (\lambda \gamma - \frac{1}{2} \sigma^2)T + \ln S/K}{\sigma \sqrt{T}} \right) - K \Phi \left( \frac{rT - (\lambda \gamma - \frac{1}{2} \sigma^2)T + \ln S/K}{\sigma \sqrt{T}} \right)
\]

Thus (3.4) becomes

\[
C(K,T,S) = \left\{ S e^{-\lambda \gamma T} \Phi \left( \frac{r - \lambda \gamma + \frac{1}{2} \sigma^2 T + \ln S/K}{\sigma \sqrt{T}} \right) - K \Phi \left( \frac{r - \lambda \gamma - \frac{1}{2} \sigma^2 T + \ln S/K}{\sigma \sqrt{T}} \right) \right\} e^{-\lambda T} + \int_{t=0}^{T} S(1 + \gamma)e^{-\lambda \gamma T} \Phi \left( \frac{rT - (\lambda \gamma - \frac{1}{2} \sigma^2)T + \ln S/K}{\sigma \sqrt{T}} \right) - K \Phi \left( \frac{rT - (\lambda \gamma + \frac{1}{2} \sigma^2)T + \ln S/K}{\sigma \sqrt{T}} \right) \lambda e^{-\lambda dt}.
\]

(3.5)

### 3.4 Properties of the Option Price

The formula of the option price looks similar as the Black-Scholes price, so we want to connect our model price with the standard BS price.
Returning to (3.2) and (3.3), we can decompose $C(K, T, S)$ directly into prices of standard Black-Scholes calls. Define

$$C_{BS}^{\text{BS}}(K, T, S, \sigma, r) \overset{\text{def}}{=} e^{-rT} E \left[ \left( S \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma W_T \right] - K \right)^+ \right]; \quad (3.6)$$

i.e., $C_{BS}^{\text{BS}}(K, T, S, \sigma, r)$ is the standard Black-Scholes price of an option with expiry $T$ and strike price $K$, when the current asset price is $S$ with volatility $\sigma$ and where the riskless rate is $r$.

If $r > \lambda \gamma$, we can then rewrite (3.2) as below:

If $t > T$,

$$E \left[ e^{-rT} (S_T - K)^+ \right] = e^{-\lambda \gamma T} e^{-(r-\lambda \gamma)T} E \left[ \left( S \exp \left[ \left( r - \lambda \gamma - \frac{1}{2} \sigma^2 \right) T + \sigma W_T \right] - K \right)^+ \right]$$

$$= e^{-\lambda \gamma T} C_{BS}^{\text{BS}}(K, T, S, \sigma, r - \lambda \gamma) \quad (3.7)$$

if $t \leq T$,

$$E \left[ e^{-rT} (S_T - K)^+ \right] = e^{-rT} E \left[ \left( S \exp \left[ \left( r - \lambda \gamma - \frac{1}{2} \sigma^2 \right) T + \sigma W_T \right] - K \right)^+ \right]
= e^{-r(T-t)-\lambda \gamma t} e^{-(r-\lambda \gamma)t} E \left[ \left( S(1 + \gamma) e^{r(T-t)} \exp \left[ \left( r - \lambda \gamma - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right] - K \right)^+ \right]
= e^{-r(T-t)-\lambda \gamma t} C_{BS}^{\text{BS}}(K, t, S(1 + \gamma) e^{r(T-t)}, \sigma, r - \lambda \gamma) \quad (3.8)$$

Thus, in addition to (3.5), we can replace (3.4) by

**Theorem 3.4.1** If $r > \lambda \gamma$, the (3.4) can be written as

$$C(K, T, S) = e^{-\lambda (1+\gamma) T} C_{BS}^{\text{BS}}(K, T, S, \sigma, r - \lambda \gamma) + \int_{t=0}^{T} \lambda e^{-r(T-t)-\lambda (1+\gamma)t} C_{BS}^{\text{BS}}(K, t, S(1 + \gamma) e^{r(T-t)}, \sigma, r - \lambda \gamma) dt. \quad (3.9)$$

The equation (3.9) connects our model option price with the Black-Scholes option price, and this will help us understand the model.

As we know the standard BS option price $C(K, T, S, \sigma, r)$ has a lower bound, we want to know the lower bound for our option price as well. Define
\[
M_t \overset{\text{def}}{=} \int_{s=0}^{t} \chi_{\{X_s=N\}} \left\{ \sigma S_s dW_s + \int_{s=0}^{t} \gamma S_s \{dJ_s - \lambda ds\} \right\}
\]

we have that
\[
dS_t = rS_t dt + dM_t
\]

In other words, the stock price is a discounted martingale, so
\[
S_t = S e^{rt} + \int_{s=0}^{t} e^{r(t-s)} dM_s
\]

and then we have
\[
\mathbb{E}[S_t] = S e^{rt}.
\]

Hence
\[
C(K, T, S) = e^{-rT} \mathbb{E}[(S_T - K)^+] \geq e^{-rT} (\mathbb{E}[S_T] - K)^+
\]
\[
= e^{-rT} (S e^{rT} - K)^+ \geq (S - K e^{-rT})^+
\]

This ensures that the implied volatility is of course positive. From this, we also have that
\[
C(K, T, S) \geq (S - K)^+
\]

so, not surprisingly, the option should not be exercised early, and we got the lower bound for our model option price.
CHAPTER 4

TERM STRUCTURE

Next we look into the model term structure. The implied volatility plays an important role in the market, and if we can understand the term structure of the implied volatility, it will help us better understand the model.

4.1 Term Structure when $T \to \infty$

We first look at the term structure when $T \to \infty$ in (3.4), since when $T \to \infty$, we would expect to see the drop of the implied volatility, cause the time that the stock is in “announcement” state becomes longer. This means, the stock will have more expected time to have 0 volatility. This is the first method we use to implement the term structure.

We start with the first part of (3.4).

\[
\lim_{T \to \infty} e^{-\lambda (1+\gamma)T} C_{BS}^{C}(K, T, S, \sigma, r - \lambda \gamma) \\
= \lim_{T \to \infty} e^{-\lambda (1+\gamma)T} e^{-rT} \mathbb{E} \left[ (S \exp \left[ (r - \lambda \gamma - \frac{1}{2} \sigma^2) T + \sigma W_T \right] - K)^+ \right] \\
= \lim_{T \to \infty} e^{-\lambda T} \mathbb{E} \left[ (S \exp \left[ -\frac{1}{2} \sigma^2 T + \sigma W_T \right] - K e^{-(r-\lambda \gamma)T})^+ \right] \\
= \lim_{T \to \infty} e^{-\lambda T} \mathbb{E} \left[ (S \exp \left[ -\frac{1}{2} \sigma^2 T + \sigma W_T \right])^+ \right] \\
= \lim_{T \to \infty} e^{-\lambda T} \mathbb{E} \left[ S \exp \left[ -\frac{1}{2} \sigma^2 T + \sigma W_T \right] \right] \\
= \lim_{T \to \infty} e^{-\lambda T} S \\
= 0.
\]
We then compute the second part

\[
\lim_{T \to \infty} \int_{t=0}^{T} \lambda e^{-r(T-t)-\lambda(1+\gamma)t} C_{BS}(K, t, S(1+\gamma)e^{r(T-t)}, \sigma, r - \lambda \gamma)dt
\]

\[
= \lim_{T \to \infty} \int_{t=0}^{T} \lambda e^{-r(T-t)-\lambda(1+\gamma)t} e^{-(r-\lambda \gamma)t} dt
\times \mathbb{E} \left[ (S(1+\gamma) \exp[r(T-t) + (r - \lambda \gamma - \frac{1}{2} \sigma^2)t + \sigma W_t] - K)^+ \right] dt
\]

\[
= \lim_{T \to \infty} \int_{t=0}^{T} \lambda e^{-\lambda t} e^{-rT} \mathbb{E} \left[ (S(1+\gamma) \exp[rT - (\lambda \gamma + \frac{1}{2} \sigma^2)t + \sigma W_t] - K e^{-rT})^+ \right] dt
\]

\[
= \lim_{T \to \infty} \int_{t=0}^{T} \lambda e^{-\lambda t} \mathbb{E} \left[ (S(1+\gamma) \exp[-(\lambda \gamma + \frac{1}{2} \sigma^2)t + \sigma W_t])^+ \right] dt
\]

\[
= \lim_{T \to \infty} \int_{t=0}^{T} \lambda e^{-\lambda t} \mathbb{E} \left[ (S(1+\gamma) \exp[-(\lambda \gamma + \frac{1}{2} \sigma^2)t + \sigma W_t])^+ \right] dt
\]

\[
= \lim_{T \to \infty} \int_{t=0}^{T} \lambda e^{-\lambda t} \mathbb{E} \left[ S(1+\gamma) \exp[-(\lambda \gamma - \frac{1}{2} \sigma^2)t + \sigma W_t] \right] dt
\]

\[
= \int_{t=0}^{\infty} \lambda e^{-\lambda t} S(1+\gamma)e^{-\lambda \gamma} dt
\]

\[
= S(1+\gamma) \frac{\lambda}{\lambda + \lambda \gamma}
\]

\[
= S.
\]

Summarizing, we have that

**Lemma 4.1.1**

\[
\lim_{T \to \infty} C(K, T, S) = S.
\]

Note that

\[
\lim_{T \to \infty} C_{BS}(K, T, S, \sigma, r) = S
\]

Here we get the consequence, both the model call option price and the BS call option price go to \( S \) when \( T \to \infty \). We can do the numerical calibration by setting the \( T \) to be large, then find out the implied volatility by using the fact that

\[
\lim_{T \to \infty} C(K, T, S) = \lim_{T \to \infty} C_{BS}(K, T, S, \sigma, r) \quad (4.1)
\]
We also want to know their speed of approaching S, then we will have a theoretical term structure of our model. We were trying to calibrate the rates at which the option price approaches S. Unfortunately, the calculation of the term structure when $T \nearrow \infty$ is too complicated, so we use the numerical method below to see some implementations.

Here is the numerical example we have. Take $T = 250$, $S_0 = 50$, $K = 50$, $r = 0.01$, $\gamma = 0.35$ and $\sigma_{\text{model}} = 0.6$ on the left hand of (4.1), and the same $T, S_0, K, r$ on the right hand of (4.1). We increase the $\lambda$ from 0.1 to 6 with 30 steps, and calculate the corresponding model option price for each $\lambda$. Then we use numerical method to calculate the BS implied volatility of the given option price, and plot the $\lambda$ V.S. BS implied volatility in the plot below, so that we can know the relationship between $\lambda$ rate and implied volatility.

Figure 4.1: Relationship between $\lambda$ and BS implied volatility

![Term Structure](image)

The plot shows that, the implied volatility vanishes as the $\lambda$ gets large when $T \nearrow \infty$. This is a result that we are interested in. But due to the failure of the theoretical calculation, we can’t get a representation of the BS implied volatility.

4.2 Term Structure when $\lambda \to \infty$

After trying the method to let $T \nearrow \infty$, we turn to look into the term structure when $\lambda \nearrow \infty$. $\lambda$ is a new parameter that BS model doesn’t have. When $\lambda \nearrow \infty$, the expected time until jump happens will become tiny. Since the stock will either stay in “null” state or jump to “announcement” state, the expected time that the stock stay in “announcement” state will become longer when $\lambda \nearrow \infty$. In that case, we expect to see the BS implied volatility become tiny when $\lambda \nearrow \infty$, cause the time-weighted volatility becomes small in our model, which
would have similar expected result as $T \nearrow \infty$. This is the second method we use to look into term structure.

Let’s think of what happens when $\lambda \nearrow \infty$, so that we can represent the option price formula. We rewrite the (3.5), which is listed below.

$$C(K, T, S) = e^{-rT} \mathbb{E}\left[ \left( S \exp \left[ rT - (\lambda \gamma + \frac{1}{2} \sigma^2)T + \sigma \sqrt{T} \eta \right] - K \right)^+ \right] e^{-\lambda T}$$

$$+ \int_{t=0}^{T} e^{-rT} \mathbb{E}\left[ \left( S \exp \left[ rT - (\lambda \gamma + \frac{1}{2} \sigma^2)t + \sigma \sqrt{t} \eta \right] (1 + \gamma) - K \right)^+ \right] \lambda e^{-\lambda t} dt.$$

We first think of the second part of the (3.4), cause that part is more complicated.

$$\int_{t=0}^{T} \lambda e^{-r(T-t)-\lambda(1+\gamma)t} C^{BS}(K, t, S(1+\gamma)e^{r(T-t)}, \sigma, r - \lambda \gamma) dt$$

$$= \int_{t=0}^{T} \lambda \exp \left[ -r(T - t) - \lambda (1+\gamma)t \right] e^{-(r-\lambda \gamma)t}$$

$$\times \mathbb{E}\left[ \left( S(1+\gamma) \exp \left[ r(T - t) + (r - \lambda \gamma - \frac{1}{2} \sigma^2)t + \sigma W_t \right] - K \right)^+ \right] dt$$

$$= \int_{t=0}^{T} \lambda e^{-\lambda t} e^{-rT} \mathbb{E}\left[ \left( S(1+\gamma) \exp \left[ rT - (\lambda \gamma + \frac{1}{2} \sigma^2)t + \sigma W_t \right] - K \right)^+ \right] dt$$

let $u = \lambda t$, then we have

$$= \int_{t=0}^{\lambda T} \lambda e^{-u} e^{-rT} \mathbb{E}\left[ \left( S(1+\gamma) \exp \left[ rT - (\lambda \gamma + \frac{1}{2} \sigma^2)\frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta \right] - K \right)^+ \right] \frac{1}{\lambda} du$$

$$= \int_{t=0}^{\lambda T} \lambda e^{-u} e^{-\gamma u - \frac{1}{2} \sigma^2 \frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta} \left[ \left( S(1+\gamma) \exp \left[ -\gamma u - \frac{1}{2} \sigma^2 \frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta \right] - K \right)^+ \right] \frac{1}{\lambda} du$$

$$= \int_{t=0}^{\lambda T} e^{-u} \mathbb{E}\left[ \left( S(1+\gamma) \exp \left[ -\gamma u - \frac{1}{2} \sigma^2 \frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta \right] - K \right)^+ \right] du$$

Then we let $\lambda \nearrow \infty$, then $\frac{u}{\lambda} \searrow 0$. We use Taylor expansion of the exponential part in the integral, as when $x \approx 0$

$$e^x \approx 1 + x + \frac{1}{2} x^2$$
Combining the expansion with Itô’s lemma[8], we know when \( \lambda \) large and \( \sigma \) small,

\[
\exp \left[ -\frac{1}{2} \sigma^2 \frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta \right] \\
\approx 1 - \frac{1}{2} \sigma^2 \frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta + \frac{1}{2} \left( \frac{1}{2} \sigma^2 \frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta \right)^2 \\
\approx 1 - \frac{1}{2} \sigma^2 \frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta + \frac{1}{2} \sigma^2 \frac{u}{\lambda} \\
= 1 + \sqrt{\frac{u}{\lambda}} \eta
\]

\[
= \lim_{\lambda \to \infty} \int_{t=0}^{\infty} e^{-u} \mathbb{E} \left[ \left( S(1 + \gamma) \exp \left[ -\gamma u - \frac{1}{2} \sigma^2 \frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta - K \right] \right)^+ \right] du \\
\approx \lim_{\lambda \to \infty} \int_{t=0}^{\infty} e^{-u} \mathbb{E} \left[ \left( S(1 + \gamma) \exp \left[ -\gamma u \right] \left( 1 - \frac{1}{2} \sigma^2 \frac{u}{\lambda} + \sigma \sqrt{\frac{u}{\lambda}} \eta + \frac{1}{2} \sigma^2 \frac{u}{\lambda} \right) - K \right)^+ \right] du \\
= \lim_{\lambda \to \infty} \int_{t=0}^{\infty} e^{-u} \mathbb{E} \left[ \left( S(1 + \gamma) e^{-\gamma u} (1 + \sigma \sqrt{\frac{u}{\lambda}} \eta) - K \right)^+ \right] du \\
= \lim_{\lambda \to \infty} \int_{u=0}^{\infty} e^{-u} S(1 + \gamma) e^{-\gamma u} \mathbb{E} \left[ \left( \sqrt{\frac{u}{\lambda}} \eta - \left( \frac{K e^{\gamma u}}{S(1 + \gamma)} - 1 \right) \right)^+ \right]
\]

Let’s set

\[
\epsilon_{u,\lambda} \overset{\text{def}}{=} \sigma \sqrt{\frac{u}{\lambda}} \quad (4.2) \\
b_u \overset{\text{def}}{=} K \frac{e^{\gamma u}}{S(1 + \gamma)} - 1 \quad (4.3)
\]

And we think about part of the integrand first

\[
\mathbb{E} \left[ \left( \sqrt{\frac{u}{\lambda}} \eta - \left( \frac{K e^{\gamma u}}{S(1 + \gamma)} - 1 \right) \right)^+ \right] = \mathbb{E} \left[ (\epsilon_{u,\lambda} \eta - b_u)^+ \right]
\]

As we defined \( \eta \) as Gaussian, and when \( \epsilon_{u,\lambda} \eta - b_u = 0 \), \( \eta = \frac{b_u}{\epsilon_{u,\lambda}} \), then
\[ 
\mathbb{E} \left[ (\epsilon_{u,\lambda} \eta - b_u)^+ \right] 
\]
\begin{align*}
&= \epsilon_{u,\lambda} \int_{\epsilon_{u,\lambda}}^{\infty} (\omega - b_u) \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} d\omega \\
&= \epsilon_{u,\lambda} \left[ \int_{\epsilon_{u,\lambda}}^{\infty} \omega \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} d\omega - \int_{\epsilon_{u,\lambda}}^{\infty} \frac{b_u}{\epsilon_{u,\lambda}} \frac{1}{\sqrt{2\pi}} e^{-\frac{b_u^2}{2}} d\omega \right] \\
&= \frac{\epsilon_{u,\lambda}}{\sqrt{2\pi}} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}^2}} - b_u \left[ 1 - \Phi \left( \frac{b_u}{\epsilon_{u,\lambda}} \right) \right]
\end{align*}

(4.4) 

(4.5) 

(4.6) 

(4.7) 

And when \( \sigma \) small, \( \epsilon_{u,\lambda} \) is small as well. so \( b_u/\epsilon_{u,\lambda} \) is either positively large or negatively large. Here we can use the asymptotes of the \( \Phi(x) \) when \( x \) large.

**Lemma 4.2.1** Given that \( \epsilon_{u,\lambda} \) is small, from [6] we know that when \( b_u > 0 \),

\[ 
1 - \Phi \left( \frac{b_u}{\epsilon_{u,\lambda}} \right) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}^2}} \left( \frac{\epsilon_{u,\lambda}}{b_u} - \frac{\epsilon_{u,\lambda}^3}{b_u^3} \right)
\]

and when \( b_u < 0 \),

\[ 
1 - \Phi \left( \frac{b_u}{\epsilon_{u,\lambda}} \right) = \Phi \left( -\frac{b_u}{\epsilon_{u,\lambda}} \right) \approx 1 + \frac{1}{\sqrt{2\pi}} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}^2}} \left( \frac{\epsilon_{u,\lambda}}{b_u} - \frac{\epsilon_{u,\lambda}^3}{b_u^3} \right)
\]

(4.8) 

(4.9) 

Use the above lemma, when \( b_u > 0 \), by (4.8)

\[ 
\mathbb{E} \left[ (\epsilon_{u,\lambda} \eta - b_u)^+ \right] 
\]
\begin{align*}
&= \frac{\epsilon_{u,\lambda}}{\sqrt{2\pi}} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}^2}} - \frac{\epsilon_{u,\lambda}}{\sqrt{2\pi}} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}^2}} + \frac{\epsilon_{u,\lambda}^3}{\sqrt{2\pi} b_u^2} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}^2}} \\
&= \frac{\epsilon_{u,\lambda}^3}{\sqrt{2\pi} b_u^2} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}^2}}
\end{align*}

(4.10) 

19
And when \( b_u < 0 \), by (4.9)

\[
\mathbb{E} \left[ (\epsilon_{u,\lambda} \eta - b_u)^+ \right] = \frac{\epsilon_{u,\lambda}}{\sqrt{2\pi}} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}}} - b_u \left( 1 + \frac{1}{\sqrt{2\pi}} b_u e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}}} \left( \frac{\epsilon_{u,\lambda}}{b_u} - \frac{\epsilon_{u,\lambda}^3}{b_u^3} \right) \right) \\
= -b_u + \frac{\epsilon_{u,\lambda}^3}{\sqrt{2\pi} b_u^2} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}}}
\]

Then the initial integral finally becomes

\[
\lim_{\lambda \to \infty} \int_{u=0}^{\infty} e^{-u} S(1 + \gamma) e^{-\gamma u} \mathbb{E} \left[ \left( \sigma \sqrt{\frac{u}{\lambda}} \eta - \left( \frac{Ke^{\gamma u}}{S(1 + \gamma)} - 1 \right) \right) + \right] \\
\approx \lim_{\lambda \to \infty} \left[ \int_{u=0}^{\infty} e^{-u} e^{-\gamma u} S(1 + \gamma) \frac{\epsilon_{u,\lambda}^3}{\sqrt{2\pi} b_u^2} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}}} du - \int_{u=0}^{a} e^{-u} e^{-\gamma u} S(1 + \gamma) b_u du \right]
\]

where \( a \) is the point that makes \( b_u = 0 \).

So if we initialize the condition and set

\[ K \geq S(1 + \gamma) \]

then when \( \lambda \to \infty \), the second part of the price becomes

\[
\int_{u=0}^{\infty} e^{-u} e^{-\gamma u} S(1 + \gamma) \frac{\epsilon_{u,\lambda}^3}{\sqrt{2\pi} b_u^2} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}}} du
\]

Define the price as a function of \( \lambda \).

\[
I(\lambda) \overset{\text{def}}{=} \int_{u=0}^{\infty} e^{-u} e^{-\gamma u} S(1 + \gamma) \frac{\epsilon_{u,\lambda}^3}{\sqrt{2\pi} b_u^2} e^{-\frac{b_u^2}{2\epsilon_{u,\lambda}}} du
\]
Since we have 4.2 and 4.3,

\[
I(\lambda) = \int_{u=0}^{\infty} e^{-u} e^{-\gamma u} S(1 + \gamma) \frac{(\sigma \sqrt{\frac{u}{\lambda}})^3}{\sqrt{2\pi} (K e^{\gamma u} - S(1 + \gamma))^2} e^{\frac{(K e^{\gamma u} - S(1 + \gamma))^2}{2(\sigma \sqrt{\frac{u}{\lambda}})^2}} du
\]

\[
= \int_{u=0}^{\infty} e^{-(\gamma + 1)u} S(1 + \gamma) \frac{\sigma^3}{\lambda^{3/2}} u^{3/2} \frac{1}{\sqrt{2\pi}} \left( \frac{S(1 + \gamma)}{K e^{\gamma u} - S(1 + \gamma)} \right)^2 \times \exp \left[ - \left( \frac{K e^{\gamma u} - S(1 + \gamma)}{S(1 + \gamma)} \right)^2 \frac{\lambda}{2\sigma^2 u} \right] du
\]

\[
= (S(1 + \gamma))^3 \frac{\sigma^3}{\sqrt{2\pi}} \lambda^{-3/2} \int_{u=0}^{\infty} u^{3/2} \exp \left[ - G_\lambda(u) \right] du
\]

where

\[
G_\lambda(u) \overset{\text{def}}{=} (\gamma + 1)u + 2 \log |K e^{\gamma u} - S(1 + \gamma)| + \left( \frac{K e^{\gamma u} - S(1 + \gamma)}{S(1 + \gamma)} \right)^2 \frac{\lambda}{2\sigma^2 u}
\]

We want to use the Cramér asymptotes of the integral given by [3].

Let’s now consider \( G'_\lambda(u) \). Define

\[
C_1 \overset{\text{def}}{=} S(1 + \gamma)
\]

\[
C_2(u) \overset{\text{def}}{=} K e^{\gamma u} - S(1 + \gamma)
\]

Then we have

\[
\frac{dC_2(u)}{du} = \gamma (C_1 + C_2(u))
\]

So

\[
G_\lambda(u) = (\gamma + 1)u + 2 \log C_2(u) + \frac{\lambda}{2u\sigma^2} \left( \frac{C_2(u)}{C_1} \right)^2
\]

\[
G'_\lambda(u) = (\gamma + 1) + \frac{2\gamma}{C_2(u)} (C_1 + C_2(u)) - \frac{\lambda}{2u^2\sigma^2} \frac{C_2(u)^2}{C_1^2} + \frac{1}{u\sigma^2} \frac{\gamma \lambda C_2(u)(C_1 + C_2(u))}{C_1^2}
\]

let \( u^*_\lambda \) solve the equation such that

\[
G'_\lambda(u^*_\lambda) = 0
\]

We can’t solve \( u^* \) explicitly from the equation, but we can use numerical methods to find
the value of $u^*$. Rewrite the equation of $G^{''}_\lambda(u^*_*) = 0$, we get

$$u^* = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{\lambda C_2(u^*^2)}{C_1} - \frac{\gamma\lambda C_2(u^*) (C_1^* + C_2(u^*))}{C_1}$$

(4.10)

Moreover

$$G^{''\prime}_\lambda(u) = -\frac{2\gamma^2 C_1(C_1 + C_2(u))}{C_2(u)^2} + \frac{\lambda}{\sigma^2 u^3} \frac{C_2(u^2)}{C_1^2} - \frac{\lambda}{2u^2 \sigma^2} \frac{C_2(u)(C_1 + C_2(u)) \gamma}{C_1^2}$$

$$+ \frac{\lambda^2}{\sigma^2 - u^2} \frac{C_2(u)}{C_1^2} + \frac{\lambda C_2(u^2)}{\sigma^2 u^3} C_1^2 - 2 \frac{\lambda}{u^2 \sigma^2} \frac{C_2(u)(C_1 + C_2(u)) \gamma}{C_1^2}$$

$$+ \frac{\lambda \gamma (C_1 + 2C_2(u))^2}{u^2 \sigma^2} C_1^2$$

Then we plug $u^*$ into $G^{''}_\lambda(u)$. By using the fact that

$$G^{'}_\lambda(u^*) = 0$$

we will have the following equation

$$-\frac{1}{u^*} \frac{\gamma \lambda C_2(u^*) (C_1 + C_2(u^*))}{C_1^2} = (\gamma + 1) + \frac{2\gamma}{C_2(u^*)} (C_1 + C_2(u^*)) - \frac{\lambda}{2u^2 \sigma^2} \frac{C_2(u)^2}{C_1^2}$$

(4.11)

plug (4.11) into $G^{''}_\lambda(u^*)$,

$$G^{''}_\lambda(u^*) = -\frac{2\gamma^2 C_1(C_1 + C_2(u^*))}{C_2(u^*)^2} + \frac{\lambda}{\sigma^2 u^3} \frac{C_2(u^2)}{C_1^2} - 2 \frac{\lambda}{u^2 \sigma^2} \frac{C_2(u^*) (C_1 + C_2(u^*)) \gamma}{C_1^2}$$

$$+ \frac{\lambda \gamma (C_1 + 2C_2(u^*))^2}{u^* \sigma^2} C_1^2$$

$$= -\frac{2\gamma^2 C_1(C_1 + C_2(u^*))}{C_2(u^*)^2} + \frac{\lambda}{\sigma^2 u^3} \frac{C_2(u^2)}{C_1^2} + \frac{\lambda \gamma (C_1 + 2C_2(u^*))^2}{u^* \sigma^2} C_1^2$$

$$+ \frac{2}{u^*} \left[ (\gamma + 1) + \frac{2\gamma}{C_2(u^*)} (C_1 + C_2(u^*)) - \frac{\lambda}{2u^2 \sigma^2} \frac{C_2(u)^2}{C_1^2} \right]$$

$$= -\frac{2\gamma^2 C_1(C_1 + C_2(u^*))}{C_2(u^*)^2} + \frac{2}{u^*} (\gamma + 1) + \frac{4\gamma}{u^* C_2(u^*)} (C_1 + C_2(u^*)) + \frac{\lambda \gamma (C_1 + 2C_2(u^*))^2}{u^* \sigma^2} C_1^2$$

22
Remembering that we rewrite $u^*$ in (4.10) as a fraction function, we want to find the limit of $u^*$ when $\lambda \to \infty$. However, the equation has $u^*$ on both hand side, it’s hard to see what happens when letting $\lambda \to \infty$. Luckily, we can separate $\lambda$ as a function of $u^*$ as below

$$
\lambda = \frac{u^*[(\gamma + 1) + \frac{2\gamma}{C_2(u^*)} (C_1 + C_2(u^*))]}{\frac{1}{2\sigma^2} \frac{C_2(u^*)^2}{C_1^2} - \frac{\gamma u^*}{\sigma^2} \frac{C_2(u^*) (C_1 + C_2(u^*))}{C_1^2}}
$$

Since we want to know the limit of $u^*$ when $\lambda$ large, it’s an alternative way to see how $u^*$ could perform so that $\lambda$ will become large. For this fraction function of $u^*$ where every single part is continuous, the only cases that can make $\lambda$ large will be $u^* \to 0$, $u^* \to \infty$ or the nominator is 0.

Let’s see what happens when $u^* \to 0$ and $u^* \to \infty$.

When $u^* \to 0$, we know $C_2(u^*) \to K - S(1 + \gamma)$, so

$$
\lambda \approx \frac{u^*^2}{1 - u^*} \approx 0
$$

And when $u^* \to \infty$, $C_2(u^*)$ dominates $u^*$

$$
\lambda \approx \frac{u^*^2}{e^{2\gamma u^*} - u^* e^{2\gamma u^*}} \approx 0
$$

in both cases, $\lambda \to 0$. So the only way that $\lambda$ can be really large is the denominator approaches 0, which is

$$
\frac{1}{2\sigma^2} \frac{C_2(u^*)^2}{C_1^2} - \frac{\gamma u^*}{\sigma^2} \frac{C_2(u^*) (C_1 + C_2(u^*))}{C_1^2} = 0
$$

solving the above equation we get that

$$
u^* = \frac{C_2(u^*)}{2\gamma (C_1 + C_2(u^*))} = \frac{Ke^{\gamma u^*} - S(1 + \gamma)}{2\gamma Ke^{\gamma u^*}}
$$

If we denote the solution of the above equation to be $u^{**}$, it means that

$$
u^{**} = \frac{C_2(u^{**})}{2\gamma (C_1 + C_2(u^{**}))} = \frac{Ke^{\gamma u^{**}} - S(1 + \gamma)}{2\gamma Ke^{\gamma u^{**}}}
$$
and

\[
\lim_{\lambda \to \infty} u^* = \lim_{\lambda \to \infty} \frac{1}{u^*} \left( \frac{\lambda C_2(u^*)^2}{C_1^2} - \frac{\gamma \lambda C_2(u^*)(C_1 + C_2(u^*))}{C_1^2} \right) = u^{**}
\]

**Lemma 4.2.2** \( u^{**} \) is unique when \( K > S(1 + \gamma) \), where \( u^{**} \) is solution of the equation below.

\[
u^{**} = \frac{Ke^{\gamma u^{**}} - S(1 + \gamma)}{2\gamma Ke^{\gamma u^{**}}}
\]

This is simple to show. let’s set

\[
y_1 \overset{\text{def}}{=} \frac{Ke^{\gamma x} - S(1 + \gamma)}{2\gamma Ke^{\gamma x}}
\]

\[
y_2 \overset{\text{def}}{=} x
\]

Since \( y_1 \) can be simplified as

\[
y_1 = \frac{1}{2\gamma} - \frac{S(1 + \gamma)}{2\gamma K} e^{-\gamma x}
\]

Then we know both functions are monotone increasing. What’s more

\[
y_1' = \frac{S(1 + \gamma)}{2K} e^{-\gamma x} < \frac{1}{2}
\]

by assumption, since \( y_2' = 1 \), so \( y_2 - y_1 \) is a monotone increasing function, which means, there will be no more than one solution for \( y_2 = y_1 \).

When \( x = 0 \),

\[
y_1 = \frac{K - S(1 + \gamma)}{2\gamma K} > 0
\]

\[
y_2 = 0
\]

So \( y_2 - y_1 \) will be negative at the initial point, then increase to 0, and keep increasing after that.

Here we show a numerical example. Set \( S = 40, K = 55, \gamma = 0.3 \), and we plot the function \( y_1 \) and \( y_2 \).

And the numerical solution shows that \( u^{**} = 0.168652 \).
So now we plug in the $u^{**}$, when $\lambda \to \infty$,

\[
\lim_{\lambda \to \infty} G_\lambda^*(u^*) \\
= \lim_{\lambda \to \infty} \left[ -\frac{2\gamma^2 C_1(C_1 + C_2(u^*))}{C_2(u^*)^2} + \frac{2}{u^*}(\gamma + 1) + \frac{4\gamma}{u^*C_2(u^*)}(C_1 + C_2(u^*)) + \frac{\lambda\gamma}{u^*\sigma^2} \left( C_1 + 2C_2(u^*) \right)^2 \right] \\
\asymp \frac{\lambda\gamma}{u^{**}\sigma^2} \frac{(C_1 + 2C_2(u^{**}))^2}{C_1^2} \asymp C_3(u^{**})\lambda
\]

Where

\[
C_3(u^{**}) \overset{\text{def}}{=} \frac{\gamma}{u^{**}\sigma^2} \frac{(C_1 + 2C_2(u^{**}))^2}{C_1^2} = \frac{\gamma}{u^{**}\sigma^2} \left( \frac{2Ke^{\gamma u^{**}} - S(1 + \gamma)}{S(1 + \gamma)} \right)^2
\]

And similarly,

\[
\lim_{\lambda \to \infty} G_\lambda(u^*) = \lim_{\lambda \to \infty} \left[ (\gamma + 1)u^* + 2 \log C_2(u^*) + \frac{\lambda}{2u^*\sigma^2} \left( \frac{C_2(u^*)}{C_1} \right)^2 \right] \\
\asymp C_4(u^{**})\lambda
\]
where
\[
C_4(u^{**}) \overset{\text{def}}{=} \frac{1}{2u^{**}\sigma^2} \left( \frac{C_2(u^{**})}{C_1} \right)^2 = \frac{1}{2u^{**}\sigma^2} \left( \frac{Ke^{\gamma u^{**}} - S(1 + \gamma)}{S(1 + \gamma)} \right)^2.
\]

To proceed, we use the Taylor approximation
\[
G_{\lambda}(u) \approx G_{\lambda}(u^*_\lambda) + \frac{1}{2} G''_{\lambda}(u^*_\lambda)(u - u^*_\lambda)^2
\approx C_4(u^{**})\lambda + \frac{1}{2} C_3\lambda(u - u^*)^2
\]

\[
I_1(\lambda) \approx \frac{(S(1 + \gamma))^3}{K_1^2} \sigma^3 \frac{1}{\sqrt{2\pi}} \lambda^{-\frac{3}{2}} \exp\left[-G_{\lambda}(u^*_\lambda)\right] \int_{u=0}^{\infty} u^{3/2} \exp\left[-\frac{1}{2} G''_{\lambda}(u^*_\lambda)(u - u^*_\lambda)^2\right] du.
\]

\[
= \frac{(S(1 + \gamma))^3}{K_1^2} \sigma^3 \frac{1}{\sqrt{2\pi}} \lambda^{-\frac{3}{2}} \exp\left[-C_4(u^{**})\lambda\right] \int_{u=0}^{\infty} u^{3/2} \exp\left[-\frac{1}{2} C_3\lambda(u - u^*_\lambda)^2\right] du.
\]

use \( v = \sqrt{\lambda}(u - u^{**}) \), then \( dv = \sqrt{\lambda} du \), so

\[
I_1(\lambda) \approx \frac{(S(1 + \gamma))^3}{K_1^2} \sigma^3 \frac{1}{\sqrt{2\pi}} \lambda^{-\frac{3}{2}} \exp\left[-C_4(u^{**})\lambda\right] \int_{v=0}^{\infty} \left( \frac{v}{\sqrt{\lambda}} + u^{**}\right)^{3/2} \exp\left[-\frac{1}{2} C_3v^2\right] \frac{1}{\sqrt{\lambda}} dv
\]

cause when \( \lambda \to \infty \), \( \left( \frac{v}{\sqrt{\lambda}} + u^{**}\right)^{3/2} \to (u^{**})^{3/2} \).

So finally we get, when \( \lambda \to \infty \) and \( \sigma \) is small
\[
I(\lambda) \approx \lambda^{-2} \exp\left[-C_4(u^{**})\lambda\right]
\]

where
\[
C_4(u^{**}) = \frac{1}{2u^{**}\sigma^2} \left( \frac{Ke^{\gamma u^{**}} - S(1 + \gamma)}{S(1 + \gamma)} \right)^2.
\]

And now we look into the first part of (3.4) by using the similar method.
\[
e^{-rT} \mathbb{E} \left[ \left( S \exp\left[rT - (\lambda\gamma + \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\eta\right] - K \right)^+ \right] e^{-\lambda T}
\]
By using the Taylor expansion, when $\sigma$ is small,

$$\exp[-\frac{1}{2}\sigma^2 T + \sigma \sqrt{T} \eta] \approx 1 + \sigma \sqrt{T} \eta$$

so

$$e^{-(r+\gamma)T} E \left[ \left( S \exp \left[ r T - (\lambda \gamma + \frac{1}{2} \sigma^2) T + \sigma \sqrt{T} \eta \right] - K \right)^+ \right]$$

$$= e^{-(r+\gamma)T} E \left[ \left( S \exp \left[ r T - (\lambda \gamma) T \right] (1 + \sigma \sqrt{T} \eta) - K \right)^+ \right]$$

$$= S e^{-(r+\gamma)T} e^{(r-\lambda)T} E \left[ \left( 1 + \sigma \sqrt{T} \eta \right) - \left( \frac{K}{S} e^{(\lambda \gamma - r) T} - 1 \right) \right]^+$$

Similarly we define

$$\hat{\epsilon} \overset{\text{def}}{=} \sigma \sqrt{T}$$

$$\hat{b}_\lambda \overset{\text{def}}{=} \frac{K}{S} e^{(\lambda \gamma - r) T} - 1 \gg 0$$

then by using previous lemma (4.8) and letting $\lambda \nearrow \infty$,

$$E \left[ \left( \sigma \sqrt{T} \eta - \left( \frac{K}{S} e^{(\lambda \gamma - r) T} - 1 \right) \right)^+ \right] = E[ (\hat{\epsilon} \eta - \hat{b}_\lambda)^+ ]$$

$$\approx \frac{\hat{\epsilon}^3}{\sqrt{2\pi \hat{b}_\lambda}} e^{-\frac{\hat{\epsilon}^2}{2\hat{b}_\lambda}}$$

Since we know

$$\hat{b}_\lambda = \frac{K}{S} e^{(\lambda \gamma - r) T} - 1$$

goes to $\infty$ exponentially fast when $\lambda \nearrow \infty$, then $e^{-\hat{b}_\lambda}$ goes to zero faster than $I(\lambda)$ in (4.12), when all the parameters are fixed.

Combining two parts together, we finally get that,

**Lemma 4.2.3** When $\lambda \nearrow \infty$ and $\sigma$ is small, the model call option price has the following
asymptotic representation

\[ C(S, K, T) \simeq \lambda^{-2} \exp \left[ -C_4(u^{**}) \lambda \right], \tag{4.13} \]

where

\[ C_4(u^{**}) = \frac{1}{2u^{**}\sigma^2} \left( \frac{Ke^{\gamma u^{**}} - S(1 + \gamma)}{S(1 + \gamma)} \right)^2, \]

and \( u^{**} \) is the numerical solution of the equation below

\[ u^{**} = \frac{Ke^{\gamma u^{**}} - S(1 + \gamma)}{2\gamma Ke^{\gamma u^{**}}} \]
In the previous chapter, the term structure shows the theoretical relationship between the model price and the model $\lambda$ rate when $\lambda \nearrow \infty$. At the same time, we also want to know the usage of the model in the real world. So in this chapter, we will use in-detailed examples to show how our model works in the real trading market when there is a potential takeover.

5.1 One By Many

We start with the case *One by many*. Since the strategy of the *One by many* is not related to the drop of the implied volatility for longterm, we can’t use the term structure model and we introduce a new definition below.

5.1.1 Risk Neutral Density

Construct $\frac{1}{(dK)^2}$ shares of butterfly spread with three call options as mentioned in [4] and [10]

- Long one option at strike $K - dK$;
- Short two options at strike $K$;
- Long one option at strike $K + dK$.

Then the value of this portfolio at time $t$ is

$$V(S_t, K, T, dK) = \frac{C(K - dK) - 2C(K) + C(K + dK)}{(dK)^2}$$
where \( C(K) \) represents the price of call option with strike \( K \) and maturity \( T \). And then we can see that the payoff the portfolio is

\[
V(S_t, K, T, dK) = \begin{cases} 
\frac{S_T - K + dK}{(dK)^2} & \text{if } S_T \in [K - dK, K] \\
\frac{K + dK - S_T}{(dK)^2} & \text{if } S_T \in [K, K + dK] \\
0 & \text{else}
\end{cases}
\]

which can be written further

\[
V(S_t, K, T, dK) = \begin{cases} 
\frac{dK + |S_T - K|}{(dK)^2} & \text{if } S_T \in [K - dK, K + dK] \\
0 & \text{else}
\end{cases}
\]

\[
= \frac{dK + |S_T - K|}{(dK)^2} 1_{S_T \in [K-dK, K+dK]}
\]

Since from the equation we can see

\[
\lim_{dK \to 0} V(S_t, K, T, dK) = \begin{cases} 
\infty & \text{if } S_T = K \\
0 & \text{if } S_T \neq K
\end{cases}
\]

Then we can get

\[
\lim_{dK \to 0} V(S_t, K, T, dK) = \delta(S_T - K)
\]

On the other hand, the portfolio can also be written as the expected payoff under the risk neutral measure \( Q \), discounted to time \( t \),

\[
V(S_t, K, T, dK) = P(t, T) \mathbb{E}_t^Q [V(S_T, K, T, dK)]
\]
where \( P(t, T) \) is the discounting factor. Then we take the limit on both side

\[
\lim_{dK \to 0} V(S_t, K, T, dK) = \lim_{dK \to 0} P(t, T) \mathbb{E}^Q_t[V(S_T, K, T, dK)]
\]

\[
= P(t, T) \mathbb{E}^Q_t[\lim_{dK \to 0} V(S_T, K, T, dK)]
\]

\[
= P(t, T) \mathbb{E}^Q_t[\delta(S_T - K)]
\]

\[
= P(t, T) \int_0^\infty \delta(S_T - K) f_{S_T}(K) dK
\]

\[
= P(t, T) f_{S_T}(K)
\]

On the other hand, the left hand side of the equation is

\[
\lim_{dK \to 0} V(S_t, K, T, dK) = \frac{C(K - dK) - 2C(K) + C(K + dK)}{(dK)^2}
\]

\[
= \frac{C(K - dK) - C(K)}{K} - \frac{C(K) - C(K + dK)}{K}
\]

\[
= \left. \frac{\partial C}{\partial K} \right|_{K+dK} - \left. \frac{\partial C}{\partial K} \right|_K
\]

\[
= \frac{\partial^2 C}{\partial K^2}
\]

Then we can get the result as we what:

\[
P(t, T) f_{S_T}(K) = \frac{\partial^2 C}{\partial K^2}
\]

This means that the discounted risk neutral density is the second derivative of the call price with respect to strike.

5.1.2 Our calibration model

If there’s no inside information, we believe that the risk neutral density will be unimodal, and the peak of the density will be around the current stock price. However, when insider trading exists, due to the high volume of buying and selling, the option price will change and the density is not unimodal anymore. Or, the distribution is unimodal but the peak will shift to the higher price. So we can plot the density of the model and look into the inconsistent point.

We use the example of *Beckman Coulter* [9].
We plot the densities of the call options which will expire in Dec 18th, 2010. The options have different strikes, and we use the high and low of the red dots to represent the density. The yellow dots imply the highest density.

Figure 5.1: Risk neutral density of Beckman Coulter

In the plot we can see, although the plot behaved abnormally sometime before Dec 2nd, 2010, at least the highest densities were around strike price 55. However, the highest density shifted to the option with strike price 57.5 on Dec 3rd, 2010. Then the highest density kept at the strike price 57.5 until the takeover happened on Dec 10th, 2010.

5.2 Calender Spread

Now we calibrate the calender spread. The strategy of the calender spread is highly related to the drop of the implied volatility in the longterm. Although we can use the term structure calculation to calibrate the model, there will be some numerical errors. So we use a more straight forward method to look into the strategy.

We are interested in the effects of volatility dropping to zero (corresponding to a takeover where existing shares are converted to cash or to shares of a name with much less volatility). Assume that the initial volatility of the name of interest is $\sigma^2$, and that the takeover occurs at time $\tau$; we assume that $\tau$ is exponential with rate $\lambda$. 

32
Since in our model SDE (3.1), in “null” state, the model volatility is $\sigma$, and in “announcement” state, the model volatility is 0. Which means, the time-weighted mean of volatility can be considered as $\sigma\sqrt{\frac{\tau \wedge T}{T}}$ if $\tau$ is the announcement time.

For simplicity, let $C_{T,K,S,r}(\sigma)$ be the option price of a call with expiry $T$, strike $K$, current stock price $S$, and interest rate $r$, and (constant) volatility $\sigma$. Roughly, the price of the call with expiry $T$ should then be

$$E\left[C_{T,K,S,r}(\sigma^2 \tau \wedge T \wedge T)\right];$$

the effective volatility will be that given by an average of $\sigma^2$ up to time $\tau$ or expiry (whichever comes first) followed by 0 (after the takeover has occurred).

If we have a calendar spread consisting of options with expiries $\{T_n\}_{n=1}^N$, we then can observe the option prices

$$E\left[C_{T,K,S,r}(\sigma^2 \tau \wedge T \wedge T)\right]$$

for $n \in \{1, 2, \ldots, N\}$. Using these prices, we want to reverse-engineer $\lambda$, or, more generally, the hazard function of $\lambda$.

If the hazard function $\lambda$ is constant, then the option price should roughly be given by something like

$$\int_{t=0}^{\infty} E\left[C_{T,K,S,r}(\sigma^2 t \wedge T \wedge T)\right] \lambda e^{-\lambda t} dt.$$

If the hazard function is in fact time-varying, the option price should be given by

$$\int_{t=0}^{\infty} E\left[C_{T,K,S,r}(\sigma^2 t \wedge T \wedge T)\right] \lambda(t) \exp\left[-\int_{s=0}^{t} \lambda(s) ds\right] dt. \quad (5.1)$$

5.2.1 Multiple Expiries Combination

Since in our model (3.1), we use the hazard rate $\lambda$ to represent the rate of jump, we can then assume that, the higher $\lambda$ means higher jump rate, and the lower $\lambda$ means the lower jump rate. This means, if we use discretization to split $\lambda$ into several periods, and then calibrate the $\lambda$ in each period, we will know the in which period, there will be higher rate that the jump might happen.

Use the idea above, consider the combination which will help us calibrate the $\sigma$. We have two positive expiries, $T_1$ and $T_2$, and two strike price $K_1$ and $K_2$. Suppose that $K_1$ is lower
and $K_2$ is higher. Then we can assume the hazard function to be

$$
\lambda(s) = \sum_{i=1}^{\mathcal{K}} \lambda_i \chi(s \in (T_{i-1}, T_i])
$$

Then consider the option price of the following three options: $C_{\text{option}1}$ expires at $T_1$ with strike $K_1$, $C_{\text{option}2}$ expires at $T_1$ with strike $K_2$, $C_{\text{option}3}$ expires at $T_2$ with strike $K_2$. Then we should have

$$
\int_0^\infty \mathbb{E} \left[ C_{T_1, K_1, S, r} \left( \sigma^2 t \wedge T_1 \right) \right] \lambda(t) \exp \left[ - \int_0^t \lambda(s) ds \right] dt = C_{\text{option}1}.
$$

(5.2)

$$
\int_0^\infty \mathbb{E} \left[ C_{T_1, K_2, S, r} \left( \sigma^2 t \wedge T_1 \right) \right] \lambda(t) \exp \left[ - \int_0^t \lambda(s) ds \right] dt = C_{\text{option}2}.
$$

(5.3)

$$
\int_0^\infty \mathbb{E} \left[ C_{T_2, K_2, S, r} \left( \sigma^2 t \wedge T_2 \right) \right] \lambda(t) \exp \left[ - \int_0^t \lambda(s) ds \right] dt = C_{\text{option}3}.
$$

(5.4)

Since all the other parameters are known except for $\sigma$, $\lambda_1$ and $\lambda_2$, we can calibrate the three parameters by using the three equations above.

5.2.2 Calibration steps

Here I use Python 2.7.8 with distribution IPython 3.1.0 to solve it numerically.

STEP1 build up the general Black Scholes call option price function. The function is given by the formula

$$
\mathbb{C}(S, K, r, T, \sigma) = N(d_1)S - N(d_2)Ke^{-rT}
$$

where

$$
d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) T \right]
$$

$$
d_2 = d_1 - \sigma \sqrt{T}
$$

Here:

- $N(\cdot)$ is the cumulative distribution function of the standard normal distribution
- $T$ is the time to maturity
• $S$ is the spot price of the underlying asset
• $K$ is the strike price
• $r$ is the risk free rate (annual rate, expressed in terms of continuous compounding)
• $\sigma$ is the volatility of returns of the underlying asset

STEP 2 Build up the function $\lambda(t)$ and the corresponding density function $\exp \left[ - \int_{s=0}^{t} \lambda(s)ds \right]$. Since $\lambda(t)$ is just a step function with constant number in each step, we write it

$$
\lambda(t) = \begin{cases} 
\lambda_1 & 0 < t < T_1 \\
\lambda_2 & T_1 < t < T_2 \\
0 & T_2 < t 
\end{cases}
$$

we can then calculate $\exp \left[ - \int_{s=0}^{t} \lambda(s)ds \right]$ explicitly when $t$ is given.

$$
\exp \left[ - \int_{s=0}^{t} \lambda(s)ds \right] = \begin{cases} 
\exp(-\lambda_1 \ast t) & 0 < t < T_1 \\
\exp(\lambda_1 \ast T_1 + \lambda_2(t - T_1)) & T_1 < t < T_2 \\
\exp(\lambda_1 \ast T_1 + \lambda_2(T_2 - T_1)) & T_2 < t 
\end{cases}
$$

STEP 3 Build up the option price function $C_1$, $C_2$, $C_3$ by using the numerical integral. Here we need to make sure that the integrand $\mathbb{E} \left[ C_{T,K,S,r} \left( \sigma^2 \frac{T}{T} \right) \right]$ is integrable with respect to $t$ and density $\lambda(t)$. Actually, if we look into this formula explicitly, we can find that when $t$ is given, the formula is just Black-Scholes $\mathbb{C}(S, K, r, T, \hat{\sigma}(t))$, where $\hat{\sigma}(t) = \sigma^2 \frac{T}{T}$. And we need to make sure that $\mathbb{C}(S, K, r, T, \hat{\sigma}(t))$ is small when $dt$ is small for all $t$. What’s more, although we write the integral on the interval $[0, \infty]$, in reality we only need to integral on $T_2$, since after $T_2$, $\lambda(t) = 0$.

Actually this is obvious, since $\mathbb{C}(S, K, r, T, \hat{\sigma}(t))$ is a monotone increasing function with respect to $\hat{\sigma}(t)$, and we know that $\hat{\sigma}(t)$ is bounded by $\sigma$, then $\mathbb{C}(S, K, r, T, \hat{\sigma}(t))$ is bounded by $\mathbb{C}(S, K, r, T, \sigma)$, which is a constant with respect to $t$. Obviously $\mathbb{C}(S, K, r, T, \sigma)$ is integrable with density $\lambda(t)$, then $\mathbb{C}(S, K, r, T, \hat{\sigma}(t))$ is integrable.

35
Now we can write the integral as

\[
\int_{t=0}^{\infty} E \left[ C_{T,K,S,r} \left( \frac{\sigma^2 T_2 t \wedge T_2}{T_2} \right) \right] \lambda(t) \exp \left[ - \int_{s=0}^{t} \lambda(s)ds \right] dt
\]

\[
= \sum_{t} C(S, K, r, T, \sigma^2 \frac{T_2 t \wedge T}{T}) \lambda(t) \exp \left[ - \int_{s=0}^{t} \lambda(s)ds \right] \Delta t
\]

Take \( \Delta t = 0.01 \) and \( T_2 \) as maximum of year 2, then

\[
\int_{t=0}^{\infty} C_{T,K,S,r} \left( \frac{\sigma^2 T_2 t \wedge T_2}{T_2} \right) \lambda(t) \exp \left[ - \int_{s=0}^{t} \lambda(s)ds \right] dt
\]

\[
= \sum_{i=1}^{200} C(S, K, r, T, \sigma^2 \frac{0.01 \times i \wedge T}{T}) \lambda(t) \exp \left[ - \int_{s=0}^{t} \lambda(s)ds \right] \times 0.01
\]

STEP4 Now we have a system, when given \( S, K_1, K_2, T_1, T_2, \sigma, r, \lambda_1, \lambda_2 \), the system will return us \( C_1, C_2, C_3 \). And then we write the system so that after \( S, K_1, K_2, T_1, T_2, r \) are given, the system is only depending on \( \lambda_1, \lambda_2, \sigma \). We call this function \( \mathbb{C}_{S,K_1,K_2,T_1,T_2,r}(\lambda_1, \lambda_2, \sigma) \) so that

\[
\mathbb{C}_{S,K_1,K_2,T_1,T_2,r}(\lambda_1, \lambda_2, \sigma) = \begin{bmatrix} (5.2) \\ (5.3) \\ (5.4) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}
\]

STEP5 Now we can start solving this numerical system. The data is from Wharton Research Data Services (WRDS). The package I am using from Python is scipy.optimize and the function is called root. The numerical method I use is hybr, which uses a modification of the Powell hybrid method as implemented in MINPACK. I use the default tolerance to solve the numerical system. Once the relative error between two consecutive iterates is at most \( 1.49012e-08 \), the calculation will terminate.

5.2.3 Example of Beckman Coulter

We use the example of Beckman Coulter first. As described before, the company had the announcement on Dec 10th, 2010. When we look into the historical implied volatility, we can see that the implied volatility behaved abnormally on Dec 9th, 2010.

We use the equations above on that day.
\( C_{\text{option}_1} \) represents the option expires on Dec 18th, 2010 with strike price \( K = 50 \), \( C_{\text{option}_1} = 7.25 \).

\( C_{\text{option}_2} \) represents the option expires on Dec 18th, 2010 with strike price \( K = 52.5 \), \( C_{\text{option}_2} = 4.7 \).

\( C_{\text{option}_3} \) represents the option expires on Jan 22nd, 2011 with strike price \( K = 52.5 \), \( C_{\text{option}_3} = 5.3 \).

By using the model, we can get that \( \lambda_1 = 58.8 \), \( \lambda_2 = 15.27 \), and \( \sigma = 0.134 \).

5.2.4 Example of FORE System[9]

Fore system was a computer network switching equipment company based in Pittsburgh, Pennsylvania. GEC announced the acquisition of FORE system on April 26, 1999.

We use the same method before to calibrate it. We found some useful result. We look at the option prices on March 24th, 1999.

\( C_{\text{option}_1} \) represents the option expires on April 17, 1999 with strike price \( K = 7.5 \), \( C_{\text{option}_1} = 10.375 \).

\( C_{\text{option}_2} \) represents the option expires on April 17, 1999 with strike price \( K = 10.0 \), \( C_{\text{option}_2} = 7.875 \).

\( C_{\text{option}_3} \) represents the option expires on Jan 20th, 2001 with strike price \( K = 10.0 \), \( C_{\text{option}_3} = 10.0 \).
By using the model, we can get that $\lambda_1 = 52.52556008$, $\lambda_2 = 282.61960143$, and $\sigma = 1.23287319$.

The result tells us that, the company had a high possibility to have an announcement before April 17th, 1999, furthermore, it had a higher possibility to have an announcement before Jan 20th, 2001, which matches the case.

We also see more calibration result which implies the similar events during this period.
CHAPTER 6
FULL MODEL

Now we extend our model with another state. Consider the takeover states \( \{N, A, F\} \), which stand for “null”, “announcement” and “failure” of the takeover. Then we have a Markov chain with three states.

Now we build up the model where the \( S_t \) will have the dynamics

\[
dS_t = \chi\{X_t = N\} \left\{ b_0 S_t dt + \sigma S_t dW_t + \gamma_A S_t dJ^A_t \right\} \\
+ \chi\{X_t = A\} \left\{ b_1 S_t dt - \gamma_F S_t dJ^F_t \right\} \\
+ \chi\{X_t = F\} \left\{ b_0 S_t dt + \sigma S_t dW_t \right\}
\]

where \( J^A_t \) is the jump process with rate \( \lambda_A \), and \( J^F_t \) is the jump process with rate \( \lambda_F \).

In addition to the (3.1), the full model has the “failure” state, which means the takeover might fail after the announcement, and the stock will return to the initial state.

6.1 Risk neutral model

Since our model should be risk neutral, which means that the discounted expectation of the stock price should be equal to the current stock price, we then have the relation

\[
b_N = b_0 + \gamma_A \lambda_A = r \\
b_A = b_1 - \gamma_A \lambda_F = r
\]

6.2 Calibration of the Jump rate

Next we want to see the jump \( \gamma_A \) and \( \gamma_F \). We believe that, in most cases, the jump \( \gamma_A \) should share a similar rate between companies. We first look at the stock price.
Since we assume

\[ dS_t = \chi_{\{X_t=N\}} \{ b_0 S_t dt + \sigma S_t dW_t + \gamma_A S_t dJ_t^A \} + \chi_{\{X_t=A\}} \{ b_1 S_t dt - \gamma_F S_t dJ_t^F \} + \chi_{\{X_t=F\}} \{ b_0 S_t dt + \sigma S_t dW_t \} \]

then once the announcement occurs, we will see the instant jump of the stock price, which is

\[ dS_t \approx \gamma_A S_t \]

So by gathering the data of the stock prices, in which there is a jump, we can know the jump rate of \( \gamma_A \).

### 6.3 Dynamics of option price

Now we define the option price \( V \) as

\[ V = V(t, S_t, X_t) = \alpha S_t + \beta B \]

The we write the dynamic of \( V \)

\[
dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} \chi_{\{X_t=N\}} \{ b_0 S_t dt + \sigma S_t dW_t \} + \frac{\partial V}{\partial S} \chi_{\{X_t=A\}} b_1 S_t dt + \frac{\partial V}{\partial S} \chi_{\{X_t=F\}} \{ b_0 S_t dt + \sigma S_t dW_t \} + \{ V(t, S_t(1 + \gamma_A), A) - V(t, S_t, N) \} dJ_t^A + \{ V(t, S_t(1 - \gamma_F), F) - V(t, S_t, A) \} dJ_t^F + \frac{1}{2} \sigma^2 S_t \frac{\partial^2 V}{\partial S^2} (\chi_{\{X_t=N\}} + \chi_{\{X_t=F\}}) dt \\
= \alpha(t, S_t, N) \chi_{\{X_t=N\}} \{ b_0 S_t dt + \sigma S_t dW_t + \gamma_A S_t dJ_t^A \} + \alpha(t, S_t, A) \chi_{\{X_t=A\}} b_1 S_t dt - \gamma_F S_t dJ_t^F + \alpha(t, S_t, F) \chi_{\{X_t=F\}} \{ b_0 S_t dt + \sigma S_t dW_t \} + \beta B_t \, dt
\]
Match the terms on both hand side. First when matching $dW_t$, when $X_t = N$ or $X_t = F$,

$$\alpha(t, S_t, N) = \frac{\partial V}{\partial S}$$
$$\alpha(t, S_t, F) = \frac{\partial V}{\partial S}$$

Then we match $dJ_t^A$, we get

Next we match $dJ_t^F$, we get

$$\{V(t, S_t(1 - \gamma_F), F) - V(t, S_t, A)\} = \alpha \chi_{\{X_t = A\}}(\gamma_F S_t)$$

So when $X_t = A$

$$\alpha(t, S_t, A) = \frac{\{V(t, S_t(1 - \gamma_F), F) - V(t, S_t, A)\}}{(-\gamma_F)S_t}$$

And finally we match $d\{V(t, S_t(1 + \gamma_A), A) - V(t, S_t, N)\} = \frac{\partial V}{\partial S} \chi_{\{X_t = N\}} \gamma_A S_t$, When $X_t = N$ or $X_t = F$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r S_t \frac{\partial V}{\partial S} = r V \quad (6.1)$$

And the risk neutral dynamics is

$$dS_t = rS_t dt + \sigma S_t dW_t \quad (6.2)$$

Consider the boundary condition, since we know that the option price is determined as

$$V(S_{T_e}, T_e) = (S_{T_e} - K)^+$$

Then we try to use Feynman Kac

$$V(S_t, t, N) = \mathbb{E}[e^{-r(T_e - t)}(S_{T_e} - K)^+ | X_t = N]$$

$$and \quad V(S_t, t, F) = \mathbb{E}[e^{-r(T_e - t)}(S_{T_e} - K)^+ | X_t = F]$$

41
Similarly, when $X_t = A$

$$\frac{\partial V}{\partial t} + b_1 S_t \frac{\partial V}{\partial S} = \alpha(t, S_t, A)(b_1 - r)S_t + rV$$

$$\frac{\partial V}{\partial t} + b_1 S_t \frac{\partial V}{\partial S} = \frac{\{V(t, S_t(1 - \gamma_F), F) - V(t, S_t, A)\}}{(-\gamma_F)S_t} (b_1 - r)S_t + rV$$

$$\frac{\partial V}{\partial t} + b_1 S_t \frac{\partial V}{\partial S} = \frac{\{V(t, S_t(1 - \gamma_F), F) - V(t, S_t, A)\}}{-\gamma_F} (b_1 - r) + rV$$

The general dynamic of the $S_t$ in risk neutral measure is

$$dS_t = b_1 S_t dt - \gamma_F S_t dJ_t$$

The equation in state A tells us, if we are in risk neutral world, $b_1 = r$

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} = rV$$

This is a linear differential equation, and the solution is

$$V(S_t, t, A) = S_t \times c(t - \frac{\ln S_t}{r})$$
CHAPTER 7
FURTHER DISCUSSION

7.1 CIR PROCESS

Let’s think of the $\lambda(t)$ as a CIR process\cite{2}. Consider the state space $X \overset{\text{def}}{=} \mathbb{R}_+ \times \{N, A, F\}$. Consider next the generator

$$(\mathcal{L} f)(\lambda, s) \overset{\text{def}}{=} \left\{ \frac{1}{2} \sigma(s) \lambda \frac{\partial^2 f}{\partial \lambda^2}(\lambda, s) - \alpha(s)(\lambda - \bar{\lambda}(s)) \frac{\partial f}{\partial \lambda}(\lambda, s) \right\} \chi_{\{s \neq F\}}$$

$$+ \lambda \{f(\lambda^*, A) - f(\lambda, N)\} \chi_{\{s = N\}} + \lambda \{f(\lambda, F) - f(\lambda, A)\} \chi_{\{s = A\}}$$

for some maps $\sigma$, $\alpha$, and $\bar{\lambda}$ from $\{N, A\}$ to $(0, \infty)$. This generates a Markov process which stops when it hits $F$, jumps from $N$ to $A$ and then to $F$. Furthermore, when it jumps from $N$ to $A$, the CIR process jumps to $\lambda^*$. The domain of $\mathcal{L} f$ is the collection of function $f$ in $B(X)$ such that $f(\cdot, s) \in C^2(\mathbb{R}_+)$ for $s \in \{N, A\}$.

We take the initial condition of this Markov process to be $(\lambda_0, N)$. In our previous model, $\lambda_0 = \lambda_A$, and $\lambda^* = \lambda_F - \lambda_A$.

Since the generator depends on the state of $s$, and when $s = N$, $0 \leq t < \tau_A$,

$$(\mathcal{L} f)(\lambda, N) = \left\{ \frac{1}{2} \sigma(N) \lambda \frac{\partial^2 f}{\partial \lambda^2}(\lambda, N) - \alpha(N)(\lambda - \bar{\lambda}(N)) \frac{\partial f}{\partial \lambda}(\lambda, N) \right\}$$

$$+ \lambda \{f(\lambda^*, A) - f(\lambda, N)\}$$

And when $s = A$, $\tau_A \leq t < \tau_F$,

$$(\mathcal{L} f)(\lambda, A) = \left\{ \frac{1}{2} \sigma(A) \lambda \frac{\partial^2 f}{\partial \lambda^2}(\lambda, A) - \alpha(A)(\lambda - \bar{\lambda}(A)) \frac{\partial f}{\partial \lambda}(\lambda, A) \right\}$$

$$+ \lambda \{f(\lambda, F) - f(\lambda, A)\}$$

43
By the generator, the dynamics of \( \lambda \) when \( s = N \) can be written as
\[
d\lambda(t) = -\alpha(N)(\lambda(t) - \bar{\lambda}(N))dt + \sigma(N)\sqrt{\lambda(t)}dW_t
\]
Similiarly when \( s = A \)
\[
d\lambda(t) = -\alpha(A)(\lambda(t) - \bar{\lambda}(A))dt + \sigma(A)\sqrt{\lambda(t)}dW_t
\]
Use previous definition of stopping times, we define
\[
\tau_A \overset{\text{def}}{=} \inf \{ t \geq 0 : X_t = A \}
\]
\[
\tau_F \overset{\text{def}}{=} \inf \{ t \geq \tau_A : X_t = F \}
\]
\[
\tau_C = \tau_A + \tau
\]
\[
\tau_D = \tau_F \wedge \tau_C
\]
The announcement occurs at time \( \tau_A \), the deal fails at time \( \tau_F \), and the deal is completed at time \( \tau_C \). Thus the deal is resolved at time \( \tau_D \), and it is completed if \( \tau_D = \tau_C \leq \tau_F \), and the deal fails if \( \tau_D = \tau_F < \tau_C \). As what we do previously, we still need to calculate the conditional expectation
\[
S_t = S_t^0 \mathbb{E} \left[ \chi_Q | G_t \right] + \mathbb{E} \left[ \exp \left[ -r(\tau_C - T_e) \right] \chi_Q | G_t \right]
\]
Since \( Q_A \) and \( Q_F \) are disjoint, so
\[
\mathbb{E} \left[ \chi_Q | G_t \right] = \mathbb{E} \left[ \chi_Q | G_t \right] + \mathbb{E} \left[ \chi_Q | G_t \right]
\]
To calculate \( \mathbb{E} \left[ \chi_Q | G_t \right] \), we use the fact that \( Q_A \subset \{ \tau_A > t \} \subset \{ \tau_D > t \} \); then
\[
\mathbb{E} \left[ \chi_Q | G_t \right] = \mathbb{E} \left[ \chi_{\{\tau_A > T_e\}} | G_t^0 \right] = \mathbb{E} \left[ \chi_{\{\tau_A > T_e\}} X_{\{\tau_A > T_e\}} | G_t^0 \right] = \mathbb{E} \left[ \chi_{\{\tau_A > T_e\}} X_{\{\tau_A > T_e\}} | G_t^0 \right] \chi_{\{\tau_A > T_e\}}
\]
\[
= \mathbb{E} \left[ \exp \left( - \int_{T_e}^{T_e} \lambda(s) ds \right) | X_t = N, G_t^0 \right] \chi_{\{\tau_A > T_e\}}
\]
After we calculate \( Q_A \), we begin to calculate \( Q_F \), use the partition
\[
Q_F = Q_F^{(1)} \cup Q_F^{(2)} \cup Q_F^{(3)}
\]
where
\[ Q_F^{(1)} \overset{\text{def}}{=} \{ \tau_A > t, \tau_D = \tau_F < \tau_C, \tau_D < T_e \} \]
\[ = \{ \tau_A > t \} \cap \{ \tau_F - \tau_A < T_e - \tau_A \} \]
\[ Q_F^{(2)} \overset{\text{def}}{=} \{ \tau_A \leq t, \tau_D = \tau_F < \tau_C, \tau_D \leq t \} \]
\[ = \{ \tau_A \leq t \} \cap \{ \tau_D < \tau_C \} \cap \{ \tau_D \leq t \} \]
\[ Q_F^{(3)} \overset{\text{def}}{=} \{ \tau_A \leq t, \tau_D = \tau_F < \tau_C, t < \tau_D < T_e \}. \]

Since \( \tau_A \) is a \( \{ \mathcal{G}_t \}_{t \geq 0} \)-stopping time, \( \{ \tau_A \leq t \} \in \mathcal{G}_t \). Then
\[
\mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] = \mathbb{E}[\chi_{Q_F}^{(1)} \mid \mathcal{G}_t] + \mathbb{E}[\chi_{Q_F}^{(2)} \mid \mathcal{G}_t] + \mathbb{E}[\chi_{Q_F}^{(3)} \mid \mathcal{G}_t]
\]
\[
= \mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] \chi_{\{\tau_D \leq t\} \cap \{\tau_D = \tau_F\}} + \mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] \chi_{\{\tau_D \leq t\} \cap \{\tau_D < \tau_F\}}
\]
\[
+ \mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] \chi_{\{\tau_D < t\} \cap \{\tau_D < \tau_F\} \cap \{\tau_D < \tau_F\}},
\]

We first calculate \( \mathbb{E}[\chi_{Q_F}^{(2)} \mid \mathcal{G}_t] \). Note that both \( \{ \tau_D < \tau_C \} \) and \( \{ \tau_D \leq t \} \in \mathcal{G}_t \), so
\[
\mathbb{E}[\chi_{Q_F}^{(2)} \mid \mathcal{G}_t] = \chi_{Q_F}^{(2)}.
\]
Nextly, \( \{ \tau_D = \tau_F \} \) and \( \{ \tau_D < \tau_F \} \) are \( \mathcal{G}_t \)-measurable, thus
\[
\mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] \chi_{\{\tau_D \leq t\} \cap \{\tau_D = \tau_F\}} = \mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] \chi_{\{\tau_D \leq t\} \cap \{\tau_D = \tau_F\}}
\]
\[
\mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] \chi_{\{\tau_D \leq t\} \cap \{\tau_D < \tau_F\}} = \mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] \chi_{\{\tau_D \leq t\} \cap \{\tau_D < \tau_F\}}
\]
What’s more
\[
\{ \tau_D \leq t \} \cap \{ \tau_D = \tau_F \} \subset Q_F \quad \text{and} \quad Q_F \cap \{ \tau_D \leq t \} \cap \{ \tau_D < \tau_F \} = \emptyset
\]
So we get
\[
\mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] \chi_{\{\tau_D \leq t\} \cap \{\tau_D = \tau_F\}} = \chi_{\{\tau_D \leq t\} \cap \{\tau_D = \tau_F\}}
\]
\[
\mathbb{E}[\chi_{Q_F} \mid \mathcal{G}_t] \chi_{\{\tau_D \leq t\} \cap \{\tau_D < \tau_F\}} = 0.
\]
Nextly, we calculate $E[\chi_{Q(3)} | \mathcal{G}_t]$.

$$E[\chi_{Q(3)} | \mathcal{G}_t] = E[\chi_{Q(3)} | \mathcal{G}_t] \chi_{\{\tau_D > t\} \cap \{\tau_A \leq t\}}$$

$$= \mathbb{P} \{\tau_F < \{T_e \wedge (\tau_A + t + T)\} | X_t = A\} \chi_{\{\tau_D > t\} \cap \{\tau_A \leq t\}}$$

$$= E \left[ 1 - \exp \left( - \int_{\tau_A}^{T_e \wedge (\tau_A + T)} \lambda(s) ds \right) | X_t = A, \mathcal{G}_t \right] \chi_{\{\tau_D > t\} \cap \{\tau_A \leq t\}}$$

Finally, $X_t = N$ on $\{\tau_A > t\}$

$$E[\chi_{Q(3)} | \mathcal{G}_t] \chi_{\{\tau_A > t\}} = \mathbb{P} [\tau_F < T_e, \tau_F < \tau_A + T, \tau_A > t | \mathcal{G}_t]$$

Since $\tau_F - \tau_A$ and $\tau_A - 0$ has zero coefficient, so

$$E[\chi_{Q(3)} | \mathcal{G}_t] \chi_{\{\tau_A > t\}} = \mathbb{P} [\tau_A > T_e | \tau_F < \tau_A + T] \mathbb{P} [\tau_A < \tau_F < T_e \wedge (\tau_A + T) | \mathcal{G}_t]$$

$$= \chi_{\{\tau_A > t\}} E \left[ 1 - \exp \left( - \int_{\tau_A}^{T_e \wedge (\tau_A + T)} \lambda(s) ds \right) | X_t = N, \mathcal{G}_t \right]$$

To calculate the conditional expectation (7.2) rigorously, we need to use the method offered by DUFFIE, PAN and SINGLETON.

Define

$$Z(X_t) = e^{\delta(t) + \beta(t)X_t + \int_t^T X_s ds}$$

In order to make $Z^{u(t)}$ as a martingale, $\delta(t)$ and $\beta(t)$ should satisfy the following ODEs with boundary conditions

$$\dot{\beta}(t) = \rho_1 + \alpha(N)\beta(t) - \frac{1}{2} \beta(t)\sigma^2(N)\beta(t) - 0; \quad (7.4)$$

$$\dot{\delta}(t) = \rho_0 - \alpha(N)\lambda(N)\delta(t) - \frac{1}{2} \delta(t) \cdot 0 \cdot \delta(t) - 0; \quad (7.5)$$

$$\beta(T) = u = 0; \quad (7.6)$$

$$\delta(T) = 0; \quad (7.7)$$

By solving the ODEs explicitly, we can calculate that

$$\delta(t) = \frac{\rho_0}{\alpha(N)\lambda(N)} (1 - e^{\alpha(N)\lambda(N)(T-t)})$$
Suppose the two roots of equation

$$\rho_1 + \alpha(N)\beta(t) - \frac{1}{2} \beta(t)\sigma^2(N)\beta(t) = 0$$

are $\beta_1, \beta_2$. If $\beta_1 \neq \beta_2$. Then we can solve that

$$\frac{\beta(t) - \beta_1}{\beta(t) - \beta_2} = Ce^{-C_0t}, \text{ where } C_0 = -\frac{1}{2} \sigma^2(N) (\beta_1 - \beta_2), C = \frac{\beta_1}{\beta_2} e^{C_0T}$$

That’s what we can get for the full model when $\lambda(t)$ is a CIR process.
APPENDIX A

HOW DO THE STRATEGIES WORK

Until now we have a good understanding of the model already, but there is some information that is not related to the model itself, which is also important for this top: how do the strategies work? Why do the portfolios we mention before make many if there is a potential takeover in the market?

And in this chapter we will explain it from the prospect of the potential takeover.

A.1 One by many

We use an example to explain how the strategy of one by many works.

Suppose that the current price of XYZ is $100, and that its implied volatility is 20% (flat volatility surface). We have an at-the-money call with strike $100, and a near-the-money call with strike $105. Suppose that I think that a takeover announcement may occur; I estimate that the stock will jump by 30% with (risk-neutral) $1 - e^{-1} = 0.632$ probability (and with 0.368 probability it will not jump). Statistically, we have

\[
30 \times 0.632 + 0 \times 0.368 = 18.96 \quad \text{mean}
\]
\[
(30 - 18.96)^2 \times 0.632 + (0 - 18.96)^2 \times 0.368 = 14.5^2 \quad \text{variance}
\]

so my belief is that there is an extra 14.5% volatility in the stock. Let’s define \( \hat{\sigma} \defeq \sqrt{\sigma^2 + 14.5^2} \) to give us the new volatility. Let \( C(K, \sigma) \) be the price of an option with strike \( K \) and volatility \( \sigma \).

Let’s now construct a portfolio by using vega[7] and see what happens at expiry. Let’s suppose that we are long \( \hat{\alpha}_{105} \) shares of the $105’s and short \( \hat{\alpha}_{100} \) shares of the $100’s. If everyone knew about the takeover, the price of the options would be \( C(100, \hat{\sigma}) \) and \( C(105, \hat{\sigma}) \), and we want

\[
\hat{\alpha}_{100} C(100, \hat{\sigma}) - \hat{\alpha}_{105} C(105, \hat{\sigma}) = 0 \quad (A.1)
\]
we would have a portfolio which costs nothing to put on, and which should also have zero (risk-neutral) expected value; i.e.,

\[ 0.368 \times \{ \hat{\alpha}_{100} \times 30 - \hat{\alpha}_{105} \times 25 \} + 0.632 \times \{ \hat{\alpha}_{100} \times 0 - \hat{\alpha}_{105} \times 0 \} = 0. \]  

(A.2)

Since \( K \mapsto C(K, \sigma) \) is decreasing, \( C(100, \hat{\sigma}) > C(105, \hat{\sigma}) \), so

\[ \hat{\alpha}_{105} = \frac{\hat{\alpha}_{100} C(100, \hat{\sigma})}{C(105, \hat{\sigma})} > \hat{\alpha}_{100}. \]  

(A.3)

\[ C(100, \hat{\sigma}) \approx C(100, \sigma) + \text{vega}_{100} \Delta \sigma \]

\[ C(105, \hat{\sigma}) \approx C(105, \sigma) + \text{vega}_{105} \Delta \sigma \]

where \( \Delta \sigma \overset{\text{def}}{=} \hat{\sigma} - \sigma \).

Let’s look at our portfolio of (A.1). We have that

\[ \hat{\alpha}_{100} C(100, \sigma) - \hat{\alpha}_{105} C(105, \sigma) = \hat{\alpha}_{100} C(100, \hat{\sigma}) - \hat{\alpha}_{105} C(105, \hat{\sigma}) \]

\[ - \{ \hat{\alpha}_{100} \text{vega}_{100} - \hat{\alpha}_{105} \text{vega}_{105} \} \Delta \sigma \]

\[ = - \{ \hat{\alpha}_{100} \text{vega}_{100} - \hat{\alpha}_{105} \text{vega}_{105} \} \Delta \sigma. \]

Since finally we want

\[ \hat{\alpha}_{100} C(100, \sigma) - \hat{\alpha}_{105} C(105, \sigma) < 0 \]

Then we need to find the condition such that

\[ \hat{\alpha}_{105} \text{vega}_{105} < \hat{\alpha}_{100} \text{vega}_{100} \]

Substitute \( \hat{\alpha}_{105} \) and \( \hat{\alpha}_{100} \) in (A.1), we get that

\[ \text{vega}_{100} C(105, \hat{\sigma}) > \text{vega}_{105} C(100, \hat{\sigma}) \]

Let’s then do with an example. Suppose that on Apr 9th \( T = 0.1 \) and \( \sigma = 0.2 \), \( S_0 = 100 \),

49
then By B-S model, vega is defined as

\[ \text{vega} = S \sqrt{\frac{T}{2\pi}} e^{-\left(\log\left(\frac{S}{K}\right) + \frac{r^2 T}{2} + \frac{\sigma^2 T}{2}\right)^2/(2\sigma^2 T)} \]

then we know

\[ \text{vega}_{100} = 12.60936 \]
\[ \text{vega}_{105} = 9.595307 \]

Now we look at the model price. When \( \gamma = 0.3 \) and \( \lambda_A = 5 \), our model shows

\[ C(100, \hat{\sigma}) = 14.36 \]
\[ C(105, \hat{\sigma}) = 11.04 \]

What’s more, the larger the \( \lambda_A \), the lower the rate \( \frac{C(100, \hat{\sigma})}{C(105, \hat{\sigma})} \). So the inequality

\[ \text{vega}_{100} C(105, \hat{\sigma}) > \text{vega}_{105} C(100, \hat{\sigma}) \]

holds when the \( \lambda_A \) is large, then

\[ \hat{\alpha}_{100} C(100, \sigma) - \hat{\alpha}_{105} C(105, \sigma) < 0 \]

Which means that, by using the strategy, the traders can earn the credit at first, but finally lose nothing if the announcement happens.

A.2 Calender spread

Similarly we analyze the dynamic of the Calender spread. The understanding of the calendar spread is more straightforward. We use the near-term option with strike $105$ which will expire in one month, and the long-term option with strike $105$ which will expire in half year, for example. We buy one share of near-term option and sell one share of long-term option. Since both options have the same strike price, we can earn credit cause the long-term option will have higher price. We believe that there is a potential takeover, so if the takeover happens before the options expire, both option will be deep in the money, and they will have similar price. In that sense, the traders can earn the credit but finally lose nothing.
REFERENCES


