THE DISTRIBUTION OF K-FREE NUMBERS AND INTEGERS WITH FIXED NUMBER OF PRIME FACTORS

BY

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DISSERTATION

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Abstract

This thesis includes four chapters. In Chapter 1, we briefly introduce the history and the main results of the topics of this thesis: the distribution of k-free numbers and the derivative of the Riemann zeta-function, the generalization of Chebyshev's bias to products of any $k \geq 1$ primes, and the distribution of integers with prime factors from specific arithmetic progressions.

In Chapter 2, for any $k \geq 2$, we study the distribution of k-free numbers. It is known that the number of k-free numbers up to x is $\widetilde{M_k}(x) \sim \frac{x}{\zeta(k)}$, where $\zeta(s)$ is the Riemann zeta-function. In this chapter, we focus on the distribution of the error term $M_k(x) := \widetilde{M_k}(x) - \frac{x}{\zeta(k)}$. Under the Riemann Hypothesis, we prove an equivalent relation between a mean square of the error term $M_k(x)$ and the negative moments of $|\zeta'(\rho)|$ as ρ runs over the zeros of $\zeta(s)$. Under some reasonable conjectures, we show that $M_k(x) \ll x^{\frac{1}{2k}}(\log x)^{\frac{1}{2}-\frac{1}{2k}+\epsilon}$ for all $\epsilon > 0$ except on a set of finite logarithmic measure, and that $e^{-\frac{y}{2k}} M_k(e^y)$ has a limiting distribution. Finally, based on the analysis of the tail of the limiting distribution, we make a precise conjecture on the maximal order of the error term.

In Chapter 3, we generalize the Chebyshev's bias and the so-called prime race problems to the distribution of products of any $k \geq 1$ primes in different arithmetic progressions. For any $k \geq 1$, we derive a formula for the difference between the number of integers $n \leq x$ with $\omega(n) = k$ or $\Omega(n) = k$ in two different arithmetic progressions, where $\omega(n)$ is the number of distinct prime factors of n and $\Omega(n)$ is the number of prime factors of n counted with multiplicity. Under the extended Riemann Hypothesis (ERH) and the Linear Independence Conjecture (LI) for Dirichlet L-functions, we show that, if k is odd, the integers

with $\Omega(n) = k$ have preference for quadratic non-residue classes; and if k is even, such integers have preference for quadratic residue classes. This result confirms a conjecture of Richard Hudson. However, the integers with $\omega(n) = k$ always have preference for quadratic residue classes. Moreover, as k increases, the biases decrease for both cases. For large k, we also give asymptotic formulas for the logarithmic densities of the sets on which the corresponding difference functions have a given sign.

In Chapter 4, we prove an asymptotic formula for the number of integers $\leq x$ which can be written as the product of k (≥ 2) distinct primes $p_1 \cdots p_k$ with each prime factor from a fixed arithmetic progression $p_j \equiv a_j \bmod q$, $(a_j, q) = 1$ $(q \geq 3, 1 \leq j \leq k)$. For any $A > 0$, our result is uniform for $2 \leq k \leq A \log\log x$. Moreover, we show that, there are large biases toward certain arithmetic progressions $\boldsymbol{a} = (a_1, \cdots, a_k)$, and that such biases have connections with Mertens' theorem and the least prime in arithmetic progressions. Unlike the previous two topics, all results in this chapter are unconditional.

To Father and Mother.

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List of Abbreviations

RH	The Riemann Hypothesis.
LI	The Linear Independence Conjecture: the positive imaginary ordinates of the zeros of $\zeta(s)$ are linearly independent over $\mathbb{Q}$.
$\mathrm{ERH_q}$	The Extended Riemann Hypothesis for Dirichlet L-functions modulo q.
$\mathrm{LI_q}$	The Linear Independence conjecture for the imaginary parts of the zeros of all Dirichlet L-functions modulo q.
$\mathrm{SH_q}$	Simplicity Hypothesis: $\forall\chi \neq \chi_0 \bmod q$, $L(\frac{1}{2},\chi) \neq 0$ and the zeros of $L(s,\chi)$ are simple.

List of Symbols

$\mu_k(n)$	Characteristic function of k-free numbers.
$\mu(n)$	Möbius function.
$\omega(n)$	The number of distinct prime factors of n.
$\Omega(n)$	The number of prime factors of n counted with multiplicity.
$\zeta(s)$	The Riemann zeta-function.
χ	Dirichlet character.
$L(s,\chi)$	Dirichlet L-function.
ρ	Generic zero of $\zeta(s)$ or $L(s,\chi)$.
γ	Imaginary part of a zero ρ.
$\delta(S)$	The logarithmic density of a set S.
$\phi(n)$	Euler's totient function.

Chapter 1

Introduction and Results

In this chapter, we summarize the history and main results of the topics covered in this thesis, which include better understandings of the error terms in the counting function of k-free numbers and subtle inequalities in the distribution of certain restricted integers among different arithmetic progressions. We will give their detailed proofs in subsequent chapters.

1.1 The distribution of k-free numbers and the first derivative of the Riemann zeta-function

Let $\mu_k(n)$ be the characteristic function of k-free numbers, where $k \geq 2$. Let $\widetilde{M}_k(x) = \sum_{n\leq x} \mu_k(n)$ be the number of k-free integers $\leq x$, and $M_k(x) = \widetilde{M}_k(x) - \frac{x}{\zeta(k)}$. Using elementary arguments, one can derive

$$\widetilde{M}_k(x) = \frac{x}{\zeta(k)} + O(x^{\frac{1}{k}}).$$

Many authors have worked to improve the error term. The best unconditional result is due to Walfisz [61],

$$M_k(x) \ll x^{\frac{1}{k}} \exp\{-ck^{-\frac{8}{5}} \log^{\frac{3}{5}} x \log\log^{-\frac{1}{5}} x\},$$

where $c > 0$ is an absolute constant. In the opposite direction, Evelyn and Linfoot [12]

Most of the results in Section 1.1 and Chapter 2 appear in the paper [42]. In this thesis, we give detailed proofs of some results and a careful analysis of the maximal order of $M_k(x)$.

proved that

$$M_k(x) = \Omega(x^{\frac{1}{2k}}).$$

Under the Riemann Hypothesis, Montgomery and Vaughan [48] showed that

$$M_k(x) \ll x^{\frac{1}{k+1}+\epsilon}, \quad \forall \epsilon > 0.$$

Later, several authors made contributions to the improvement of the error term using the estimates of some exponential sums under the Riemann Hypothesis, such as Graham [18], Baker and Pintz [3], Jia [28], [29], Graham and Pintz [19], and Baker and Powell [4], etc. They improved Montgomery and Vaughan's result for various k. Their bounds have the shape $M_k(x) \ll x^{E(k)}$, where $E(k) \sim \frac{1}{k}$ as $k \to \infty$. For $k = 2$, the best result under the Riemann Hypthosis is due to Jia [29] in which the exponent is $\frac{17}{54} + \epsilon$. For a nice survey, see [53] or [57]. However, there is still a large gap to the conjectured result,

$$M_k(x) \ll x^{\frac{1}{2k}+\epsilon}, \quad \forall \epsilon > 0.$$

In the first part of my thesis, we take a different approach to studying the distribution of k-free numbers by connecting it to the analytic properties of $\zeta(s)$. Define

$$J_{-l}(T) = \sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|^{2l}},$$

where $l \in \mathbb{R}$, ρ is a zero of $\zeta(s)$, and γ is the imaginary part of ρ. For the existence of the above sum, we implicitly assume that the zeros of $\zeta(s)$ are simple. Gonek [16] and Hejhal [23] independently conjectured that

$$J_{-l}(T) \asymp T(\log T)^{(l-1)^2}. \tag{1.1.1}$$

For $l = 1$, Gonek [16] proved that $J_{-1}(T) \gg T$ subject to the Riemann Hypothesis and the simplicity of zeros. Moreover, he conjectured in [17] that

$$J_{-1}(T) \sim \frac{3}{\pi^2} T.$$

Hughes, Keating, and O'Connell [26] used a different heuristic method based on random matrix theory and made the conjecture

$$J_{-l}(T) \sim C_l \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{(l-1)^2} \quad \text{for} \quad l < \frac{3}{2},$$

where C_l is a constant depending on l.

Recently, Ng [51] connected the summatory function of the Möbius function, $M(x) = \sum_{n \le x} \mu(n)$, with the behavior of $J_{-1}(T)$. He showed that, under the Riemann Hypothesis, $J_{-1}(T) \ll T$ implies the so called weak Mertens conjecture, which asserts

$$\int_2^X \left(\frac{M(x)}{x} \right)^2 dx \ll \log X. \tag{1.1.2}$$

He also proved that $e^{-y/2} M(e^y)$ has a limiting distribution under the assumptions of the Riemann Hypothesis and $J_{-1}(T) \ll T$, and studied the tail of the distribution using Montgomery's probabilistic methods. On the other hand, under the Riemann Hypothesis, Titchmarsh [59] (Chapter XIV, pages 376-380) showed that the weak Mertens conjecture implies the simplicity of zeros of $\zeta(s)$, the estimate $\frac{1}{\zeta'(\rho)} = O(|\rho|)$, and the convergence of the series $\sum_\rho \frac{1}{|\rho \zeta'(\rho)|^2}$.

Motivated by Ng's work and Titchmarsh's argument, we connect the estimation of $J_{-1}(T)$ to the distribution of k-free numbers. In our work, we don't require as strong an assumption $J_{-1}(T) \ll T$ as that used in [51]. Under the Riemann Hypothesis, we show that

$$J_{-1}(T) = \sum_{0 < \gamma \le T} \frac{1}{|\zeta'(\rho)|^2} \ll_\epsilon T^{1+\epsilon} \quad \forall \epsilon > 0, \tag{1.1.3}$$

and

$$\int_1^X \left(\frac{M_k(x)}{x^{\frac{1}{2k}}}\right)^2 \frac{dx}{x} \ll_k \log X \quad \forall k \geq 2, \tag{1.1.4}$$

are equivalent. We can view (1.1.4) as an analogue of the weak Mertens conjecture.

Theorem 1.1. *Under the Riemann Hypothesis, (1.1.3) and (1.1.4) are equivalent.*

Theorem 1.2. *Assume the Riemann Hypothesis and (1.1.3). Then, for any $k \geq 2$ and any $\epsilon > 0$, we have*

$$M_k(x) \ll_{k,\epsilon} x^{\frac{1}{2k}} (\log x)^{\frac{1}{2} - \frac{1}{2k} + \epsilon},$$

except on a set of finite logarithmic measure.

Remark 1. From (1.1.4), we can trivially show the weaker result that $M_k(x) \ll_k x^{\frac{1}{2k}} (\log x)^{\frac{1}{2}+\epsilon}$ except on a set of finite logarithmic measure.

Theorem 1.3. *Assume the Riemann Hypothesis and (1.1.3). Then, for any $k \geq 2$, we have*

$$\int_1^X \left(\frac{M_k(x)}{x^{\frac{1}{2k}}}\right)^2 \frac{dx}{x} \sim \beta_k \log X,$$

where

$$\beta_k = \sum_{\gamma > 0} \frac{2|\zeta(\frac{\rho}{k})|^2}{|\rho \zeta'(\rho)|^2}.$$

The main part of our proof is to show that $\phi(y) = e^{-\frac{y}{2k}} M_k(e^y)$ is a B^2-*almost periodic* function, which means, for any $\epsilon > 0$, there exists a real valued trigonometric polynomial $P_{N(\epsilon)}(y) = \sum_{n=1}^{N(\epsilon)} r_n(\epsilon) e^{i\lambda_n(\epsilon) y}$, such that

$$\limsup_{Y \to \infty} \frac{1}{Y} \int_0^Y |\phi(y) - P_{N(\epsilon)}(y)|^2 dy < \epsilon^2. \tag{1.1.5}$$

By the work of Besicovitch (see [5], Chapter II of [6], or Theorem 1.14 of [2]), a Parseval type identity is true for B^2-almost periodic functions. Moreover, B^2-almost periodic functions possess limiting distributions (see Theorem 2.9 of [2]). Thus, we get the following result.

Theorem 1.4. *Assume the Riemann Hypothesis and (1.1.3). Then, for any* $k \geq 2$, $e^{-\frac{y}{2k}} M_k(e^y)$ *has a limiting distribution* $\nu = \nu_k$ *on* $\mathbb{R}$*, that is*

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y f(e^{-\frac{y}{2k}} M_k(e^y)) dy = \int_{-\infty}^{\infty} f(x) d\nu(x)$$

for all bounded Lipschitz continuous functions f *on* $\mathbb{R}$.

The Linear Independence Conjecture (LI) states that the positive imaginary ordinates of the zeros of $\zeta(s)$ are linearly independent over $\mathbb{Q}$. If we add the assumption of LI, we can show the following corollaries. The first corollary is similar to Theorem 1.9 of [2]. We omit the proof.

Corollary 1.4.1. *Assume the Riemann Hypothesis, (1.1.3), and LI. Then the Fourier transform* $\hat{\nu}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} d\nu(t)$ *exists and equals*

$$\widehat{\nu}(\xi) = \prod_{\gamma > 0} \widetilde{J}_0 \left(\frac{2|\zeta(\frac{\rho}{k})|\xi}{|\rho \zeta'(\rho)|} \right),$$

where $\widetilde{J}_0(z)$ *is the ordinary Bessel function* $\widetilde{J}_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{2m}}{(m!)^2}$.

Corollary 1.4.2. *Assume the Riemann Hypothesis, (1.1.3), and LI. Then, for any* $k \geq 2$ *and any* $\epsilon > 0$*, we have*

$$\exp\left(-\widetilde{c}_1 V^{\frac{2k}{k-1}+\epsilon}\right) \leq \nu([V, \infty)) \leq \exp\left(-\widetilde{c}_2 V^{\frac{2k}{k-1}-\epsilon}\right),$$

for some constants $\widetilde{c}_1, \widetilde{c}_2 > 0$ *depending on* k *and* ϵ.

Under the Riemann Hypothesis, (1.1.1) (for $l < \frac{3}{2}$), and LI, with a refined analysis, we can prove a more precise upper bound for the tail of the limiting distribution. Further analysis of the bounds for $\nu([V, \infty))$ suggests the following conjecture.

Large Deviation Conjecture. There exist positive constants c_1', c_2' such that for large

V,

$$\exp\left(-c_2'\frac{V^{\frac{2k}{k-1}}}{(\log V)^{\frac{1}{2(k-1)}}}\right) \ll \nu([V,\infty)) \ll \exp\left(-c_1'\frac{V^{\frac{2k}{k-1}}}{(\log V)^{\frac{1}{2(k-1)}}}\right). \tag{1.1.6}$$

Theorem 1.5. *Assume the Riemann Hypotheis, (1.1.1) (for $l < \frac{3}{2}$), and LI. Then, there exists a constant $c_1'' > 0$ such that*

$$\nu([V,\infty)) \ll \exp\left(-c_1''\frac{V^{\frac{2k}{k-1}}}{(\log V)^{\frac{1}{2(k-1)}+o(1)}}\right). \tag{1.1.7}$$

Remark 2. Bounds for the tail of the probabilistic measure can be used to heuristically estimate the maximal variation of $M_k(x)$.

With the above Large Deviation Conjecture and heuristic argument similar to Section 4.3 of [51], we make the following conjecture.

Conjecture. For any $k \geq 2$, there exists a number $C_k > 0$, such that

$$\overline{\lim}_{x\to\infty}\frac{M_k(x)}{x^{\frac{1}{2k}}(\log\log x)^{\frac{k-1}{2k}}(\log\log\log x)^{\frac{1}{4k}}} = \pm C_k. \tag{1.1.8}$$

For the proof of Theorem 1.5, we need a result on the moments of the Riemann zeta-function.

Theorem 1.6. *Assume the Riemann Hypothesis. For any fixed integer $l \geq 1$ and $0 < w < 1$,*

$$\sum_{0<\gamma\leq T}|\zeta(1-w\rho)|^{2l} = C_{w,l}T\log T + O_{w,l}(T(\log T)^{\frac{1}{2}}), \tag{1.1.9}$$

and

$$\sum_{0<\gamma\leq T}\frac{1}{|\zeta(1-w\rho)|^{2l}} = C_{w,l}'T\log T + O_{w,l}(T(\log T)^{\frac{1}{2}}), \tag{1.1.10}$$

where

$$C_{w,l} = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{d_l^2(n)}{n^{2-w}},\ C_{w,l}' = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\tilde{d}_l^2(n)}{n^{2-w}},$$

$d_l(n)$ denotes the number of ways n may be written as a product of l factors, and $\tilde{d}_l(n) =$

$\underbrace{(\mu * \mu * \cdots * \mu}_{l \text{ times}})(n)$.

Remark 3. Using our techniques, we may replace the assumption of $J_{-1}(T) \ll T$ in [51] by $J_{-1}(T) \ll T^{2-\epsilon}$ for any $\epsilon > 0$, and still obtain the conclusions in Theorem 1, parts ii), iii) and iv), Theorem 2, and Theorem 3 of [51]. Letting $k = 1$ and $\theta = 2 - \epsilon$ in our Lemma 2.5, we get

$$\int_{\log Z}^{\log Z+1} \left| \sum_{T \le \gamma \le X} \frac{1}{\rho \zeta'(\rho)} e^{i\gamma y} \right|^2 dy \ll \frac{1}{T^{\epsilon/2}}.$$

With this bound and an argument similar to Theorems 1.1, 1.2, 1.3, and 1.4 we have

Theorem 1.7. *The Riemann Hypothesis and $J_{-1}(T) \ll T^{2-\epsilon}$ for any fixed $\epsilon > 0$ imply*

1) *the weak Mertens conjecture (1.1.2);*

2) $M(x) = \sum_{n \le x} \mu(n) \ll x^{1/2} (\log \log x)^{3/2}$ *except on a set of finite logarithmic measure;*

3)
$$\int_1^X \left(\frac{M(x)}{x^{1/2}} \right)^2 \frac{dx}{x} \sim \beta \log X,$$

where $\beta = \sum_{\gamma > 0} \frac{2}{|\rho \zeta'(\rho)|^2}$;

4) $e^{-y/2} M(e^y)$ *has a limiting distribution on* $\mathbb{R}$.

1.2 Chebyshev's bias for products of k primes

First, we consider products of k primes in arithmetic progressions. Let

$$\pi_k(x; q, a) = |\{n \le x : \omega(n) = k,\ n \equiv a \bmod q\}|,$$

and

$$N_k(x; q, a) = |\{n \le x : \Omega(n) = k,\ n \equiv a \bmod q\}|,$$

where $\omega(n)$ is the number of distinct prime divisors of n, and $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. For $k = 1$, $N_1(x; q, a)$ is the number of primes $\pi(x; q, a)$ in the arithmetic progression $a \bmod q$.

Dirichlet (1837) [9] showed that, for any a and q with $(a, q) = 1$, there are infinitely many primes in the arithmetic progression $a \bmod q$. In fact (see [8]), we have, for any $(a, q) = 1$,

$$\pi(x; q, a) \sim \frac{x}{\phi(q) \log x},$$

where ϕ is Euler's totient function. Analogous asymptotic formulas are available for products of k primes. Landau (1909) [34] showed that, for each fixed integer $k \geq 1$,

$$N_k(x) := |\{n \leq x : \Omega(n) = k\}| \sim \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!}.$$

The same asymptotic is also true for the function $\pi_k(x) := |\{n \leq x : \omega(n) = k\}|$. For more precise formulas, see [58] (II. 6, Theorems 4 and 5). Using similar methods as in [8] and [58], one can show that, for any fixed residue class $a \bmod q$ with $(a, q) = 1$,

$$N_k(x; q, a) \sim \pi_k(x; q, a) \sim \frac{1}{\phi(q)} \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!}.$$

For the case of primes (i.e. $\Omega(n) = 1$), Chebyshev (1853) [7] observed that there seem to be more primes in the progression 3 mod 4 than in the progression 1 mod 4. That is, it appears that $\pi(x; 4, 3) \geq \pi(x; 4, 1)$. In general, for any $a \not\equiv b \bmod q$ and $(a, q) = (b, q) = 1$, one can study the behavior of the functions

$$\Delta_{\omega_k}(x; q, a, b) := \pi_k(x; q, a) - \pi_k(x; q, b),$$

$$\Delta_{\Omega_k}(x; q, a, b) := N_k(x; q, a) - N_k(x; q, b).$$

The results in Section 1.2 and Chapter 3 appear in the paper [43].

Denote $\Delta(x;q,a,b) := \Delta_{\Omega_1}(x;q,a,b)$. Littlewood [39] proved that $\Delta(x;4,3,1)$ changes sign infinitely often. Actually, $\Delta(x;4,3,1)$ is negative for the first time at $x = 26,861$ [38]. Starting with [32], Knapowski and Turán generalized Littlewood's theorem and indicated a large number of problems related to the sign changes and extreme values of the functions $\Delta(x;q,a,b)$. Such problems are colloquially known today as "prime race problems". A tendency for $\Delta(x;q,a,b)$ to be of one sign is known as "Chebyshev's bias". For a nice survey of such results, see [14] and [20].

Chebyshev's bias can be well understood in the sense of *logarithmic density*. We say a set S of positive integers has *logarithmic density*, if the following limit exists:

$$\delta(S) = \lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{n\le x\\ n\in S}} \frac{1}{n}.$$

Let $\delta_{f_k}(q;a,b) = \delta(P_{f_k}(q;a,b))$, where $P_{f_k}(q;a,b)$ is the set of integers with $\Delta_{f_k}(n;q,a,b) > 0$, and $f = \Omega$ or ω. In order to study the Chebyshev's bias and the existence of the logarithmic density, we need the following assumptions:

1) the Extended Riemann Hypothesis ($\mathrm{ERH_q}$) for Dirichlet L-functions modulo q;

2) the Linear Independence Conjecture ($\mathrm{LI_q}$), which states that the imaginary parts of the zeros of all Dirichlet L-functions modulo q are linearly independent over $\mathbb{Q}$.

Under these two assumptions, Rubinstein and Sarnak [56] showed that, for Chebyshev's bias for primes (i.e. $\Omega(n) = 1$), the logarithmic density $\delta_{\Omega_1}(q;a,b)$ exists, and in particular, $\delta_{\Omega_1}(4;3,1) \approx 0.996$ which indicates a strong bias for primes in the arithmetic progression 3 mod 4. Recently, using the same assumptions, Ford and Sneed [15] studied Chebyshev's bias for products of two primes (i.e. $\Omega(n) = 2$) by transforming this problem into manipulations of some double integrals. They connected $\Delta_{\Omega_2}(x;q,a,b)$ with $\Delta(x;q,a,b)$, and showed that $\delta_{\Omega_2}(q;a,b)$ exists and the bias is in the opposite direction compared to the case of primes, in particular, $\delta_{\Omega_2}(4;3,1) \approx 0.10572$, which indicates a strong bias for the arithmetic progression 1 mod 4.

Neither the method of Rubinstein and Sarnak [56] nor the method of Ford and Sneed [15] readily generalizes to the cases of more prime factors ($k \geq 3$). From the point of view of L-functions, the most natural sum to consider is

$$\sum_{\substack{n_1 \cdots n_k \leq x \\ n_1 \cdots n_k \equiv a \bmod q}} \Lambda(n_1) \cdots \Lambda(n_k) - \sum_{\substack{n_1 \cdots n_k \leq x \\ n_1 \cdots n_k \equiv b \bmod q}} \Lambda(n_1) \cdots \Lambda(n_k).$$

However, estimates for $\Delta_{\Omega_k}(x; q, a, b)$ or $\Delta_{\omega_k}(x; q, a, b)$ cannot be recovered from such analogues by partial summation.

Ford and Sneed [15] overcome this obstacle in the case $k = 2$, after passing from arithmetic progressions to characters, by means of the 2-dimensional integral

$$\int_0^\infty \int_0^\infty \sum_{p_1 p_2 \leq x} \frac{\chi(p_1 p_2) \log p_1 \log p_2}{p_1^{u_1} p_2^{u_2}} du_1 du_2.$$

Analysis of an analogous k-dimensional integral leads to an explosion of cases, depending on the relative sizes of the variables u_j, and becomes increasingly messy as k increases.

We take an entirely different approach, working directly with the unweighted sums. We express the associated Dirichlet series in terms of products of the logarithms of Dirichlet L-functions, then apply Perron's formula, and use Hankel contours to avoid the zeros of $L(s, \chi)$ and the point $s = \frac{1}{2}$. Using the above assumptions $\mathrm{ERH_q}$ and $\mathrm{LI_q}$, we show that, for any $k \geq 1$, both $\delta_{\Omega_k}(q; a, b)$ and $\delta_{\omega_k}(q; a, b)$ exist. Moreover, we show that, as k increases, if a is a quadratic non-residue and b is a quadratic residue, the bias oscillates with respect to the parity of k for the case $\Omega(n) = k$, but $\delta_{\omega_k}(q; a, b)$ is less than $\frac{1}{2}$ and increases monotonically.

For some of our results, we need only a much weaker substitute for the condition $\mathrm{LI_q}$, which we call the Simplicity Hypothesis ($\mathrm{SH_q}$): For all $\chi \neq \chi_0 \bmod q$, we have that $L(\frac{1}{2}, \chi) \neq 0$ and the zeros of $L(s, \chi)$ are simple.

By orthogonality of Dirichlet characters, we have

$$\Delta_{\Omega_k}(x;q,a,b)=\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0 \bmod q}(\overline{\chi}(a)-\overline{\chi}(b))\sum_{\substack{n\leq x\\ \Omega(n)=k}}\chi(n), \tag{1.2.1}$$

and

$$\Delta_{\omega_k}(x;q,a,b)=\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0 \bmod q}(\overline{\chi}(a)-\overline{\chi}(b))\sum_{\substack{n\leq x\\ \omega(n)=k}}\chi(n). \tag{1.2.2}$$

Then, using the weaker assumptions $\mathrm{SH_q}$ and $\mathrm{ERH_q}$, we prove the following theorems.

Theorem 1.8. *Assume ERH_q and SH_q. Then, for any fixed $k\geq 1$, and fixed large T_0,*

$$\Delta_{\Omega_k}(x;q,a,b)=\frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}\Bigg\{\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}(\overline{\chi}(a)-\overline{\chi}(b))\sum_{\substack{|\gamma_\chi|\leq T_0\\ L(\frac12+i\gamma_\chi,\chi)=0}}\frac{x^{i\gamma_\chi}}{\frac12+i\gamma_\chi}$$
$$+\frac{N(q,a)-N(q,b)}{2^{k-1}\phi(q)}+\Sigma_k(x;q,a,b,T_0)\Bigg\},$$

where $N(q,a)$ is the number of $u \bmod q$ such that $u^2\equiv a \bmod q$, and

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_1^Y|\Sigma_k(e^y;q,a,b,T_0)|^2\,dy\ll\frac{\log^2 T_0}{T_0}.$$

Since $\Delta_{\Omega_1}(x;q,a,b)=\Delta(x;q,a,b)$, we get the following corollary.

Corollary 1.8.1. *Assume ERH_q and SH_q. Then, for any fixed $k\geq 2$,*

$$\frac{\Delta_{\Omega_k}(x;q,a,b)\log x}{\sqrt{x}(\log\log x)^{k-1}}=\frac{(-1)^{k+1}}{(k-1)!}\left(1-\frac{1}{2^{k-1}}\right)\frac{N(q,a)-N(q,b)}{\phi(q)}$$
$$+\frac{(-1)^{k+1}}{(k-1)!}\frac{\Delta(x;q,a,b)\log x}{\sqrt{x}}+\Sigma'_k(x;q,a,b),$$

where, as $Y\to\infty$,

$$\frac{1}{Y}\int_1^Y|\Sigma'_k(e^y;q,a,b)|^2dy=o(1).$$

In the above theorem, the constant $\frac{(-1)^k}{2^{k-1}}\frac{N(q,a)-N(q,b)}{\phi(q)}$ represents the bias in the distribution of products of k primes counted with multiplicity. Richard Hudson conjectured that, as k increases, the bias would change directions according to the parity of k. Our result above confirms his conjecture (under $\mathrm{ERH_q}$ and $\mathrm{SH_q}$). Figures 1.1 and 1.2 show the graphs corresponding to $(q,a,b) = (4,3,1)$ for $\frac{2\log x}{\sqrt{x}(\log\log x)^2}\Delta_{\Omega_3}(x;4,3,1)$ and $\frac{6\log x}{\sqrt{x}(\log\log x)^3}\Delta_{\Omega_4}(x;4,3,1)$, plotted on a logarithmic scale from $x = 10^3$ to $x = 10^8$. In these graphs, the functions do not appear to be oscillating around $\frac{1}{4}$ and $-\frac{1}{8}$ respectively as predicted in our theorem. This is caused by some terms of order $\frac{1}{\log\log x}$ and even lower order terms, and $\log\log 10^8 \approx 2.91347$ and $\frac{1}{\log\log 10^8} \approx 0.343233$. However, we can still observe the expected direction of the bias through these graphs.

For the distribution of products of k primes counted without multiplicity, we have the following theorem. In this case, the bias will be determined by the constant $\frac{N(q,a)-N(q,b)}{2^{k-1}\phi(q)}$ in the theorem below.

Theorem 1.9. *Assume ERH_q and SH_q. Then, for any fixed $k \geq 1$, and fixed large T_0,*

$$\Delta_{\omega_k}(x;q,a,b) = \frac{1}{(k-1)!}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}\left\{\frac{(-1)^k}{\phi(q)}\sum_{\chi\neq\chi_0}(\overline{\chi}(a)-\overline{\chi}(b))\sum_{\substack{|\gamma_\chi|\leq T_0\\ L(\frac{1}{2}+i\gamma_\chi,\chi)=0}}\frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi}\right.$$
$$\left.+\frac{N(q,a)-N(q,b)}{2^{k-1}\phi(q)}+\widetilde{\Sigma}_k(x;q,a,b,T_0)\right\},$$

where $N(q,a)$ is the number of $u \bmod q$ such that $u^2 \equiv a \bmod q$, and

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_1^Y\left|\widetilde{\Sigma}_k(e^y;q,a,b,T_0)\right|^2 dy \ll \frac{\log^2 T_0}{T_0}.$$

Corollary 1.9.1. *Assume ERH_q and SH_q. Then, for any fixed $k \geq 1$,*

$$\frac{\Delta_{\omega_k}(x;q,a,b)\log x}{\sqrt{x}(\log\log x)^{k-1}} = \left(\frac{1}{2^{k-1}}+(-1)^{k+1}\right)\frac{N(q,a)-N(q,b)}{(k-1)!\phi(q)}$$

$$+\frac{(-1)^{k+1}}{(k-1)!}\frac{\Delta(x;q,a,b)\log x}{\sqrt{x}}+\widetilde{\Sigma}_k'(x;q,a,b),$$

where, as $Y\to\infty$,

$$\frac{1}{Y}\int_1^Y|\widetilde{\Sigma}_k'(e^y;q,a,b)|^2dy=o(1).$$

For the distribution of $\Delta(x;q,a,b)$, Rubinstein and Sarnak [56] showed the following theorem. This is the version from [15].

Theorem RS. *Assume* ERH_q *and* LI_q. *For any* $a\not\equiv b \bmod q$ *and* $(a,q)=(b,q)=1$, *the function*

$$\frac{u\Delta(e^u;q,a,b)}{e^{u/2}}$$

has a probabilistic distribution. This distribution i) has mean $\frac{N(q,b)-N(q,a)}{\phi(q)}$, *ii) is symmetric with respect to its mean, and iii) has a continuous density function.*

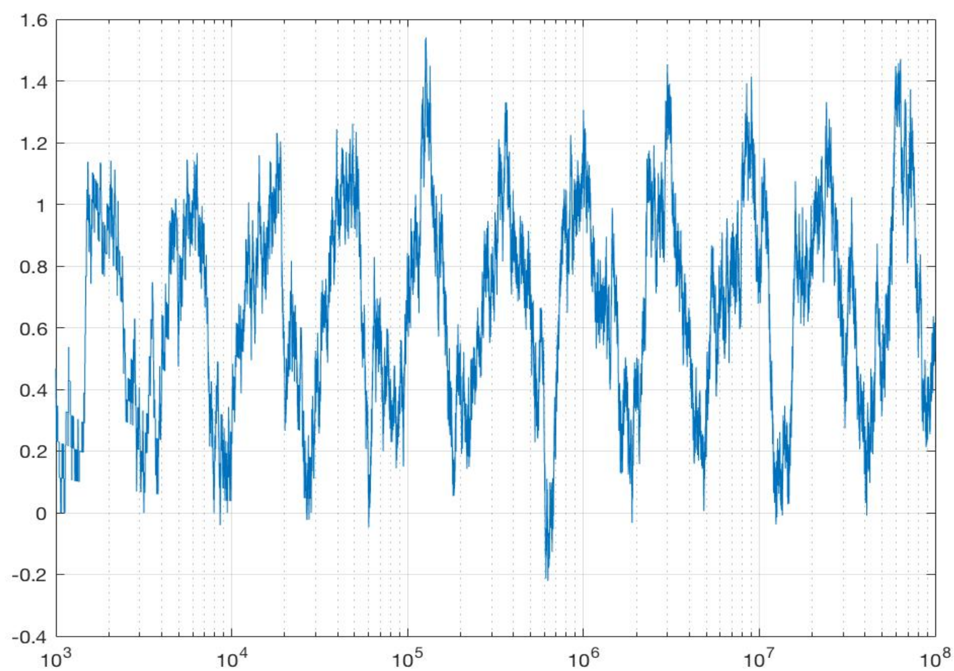

Figure 1.1: $\frac{2\log x}{\sqrt{x}(\log\log x)^2}\Delta_{\Omega_3}(x;4,3,1)$

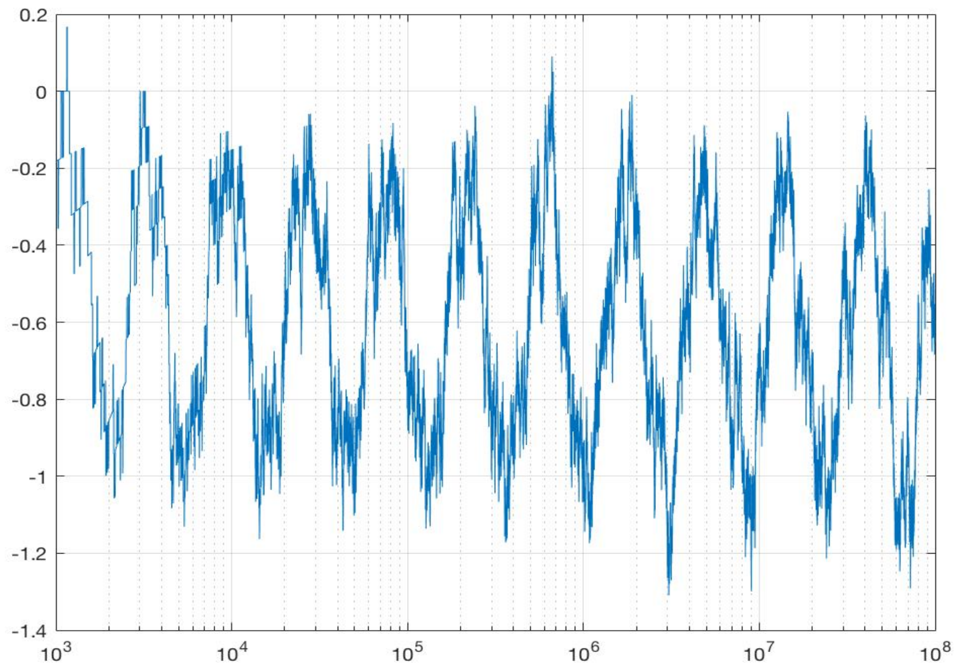

Figure 1.2: $\frac{6\log x}{\sqrt{x}(\log\log x)^3}\Delta_{\Omega_4}(x;4,3,1)$

Corollaries 1.8.1, 1.9.1, and Theorem RS imply the following result.

Theorem 1.10. *Let $a \not\equiv b \bmod q$ and $(a,q)=(b,q)=1$. Assuming ERH_q and LI_q, for any $k \geq 1$, $\delta_{\Omega_k}(q;a,b)$ and $\delta_{\omega_k}(q;a,b)$ exist. More precisely, if a and b are both quadratic residues or both quadratic non-residues, then $\delta_{\Omega_k}(q;a,b)=\delta_{\omega_k}(q;a,b)=\frac{1}{2}$. Moreover, if a is a quadratic non-residue and b is a quadratic residue, then, for any $k\geq 1$,*

$$1-\delta_{\Omega_{2k-1}}(q;a,b)<\delta_{\Omega_{2k}}(q;a,b)<\frac{1}{2}<\delta_{\Omega_{2k+1}}(q;a,b)<1-\delta_{\Omega_{2k}}(q;a,b),$$

$$\delta_{\omega_k}(q;a,b)<\delta_{\omega_{k+1}}(q;a,b)<\frac{1}{2}.$$

$$\delta_{\Omega_{2k}}(q;a,b)=\delta_{\omega_{2k}}(q;a,b),\qquad \delta_{\Omega_{2k-1}}(q;a,b)+\delta_{\omega_{2k-1}}(q;a,b)=1.$$

Remark 1. The above results confirm a conjecture of Hudson proposed years ago in his communications with Ford. Borrowing the methods from [56] (Section 4), we are able to calculate $\delta_{\Omega_k}(q;a,b)$ and $\delta_{\omega_k}(q;a,b)$ precisely for special values of q, a, and b. In particular, we record in Tables 1.1 and 1.2 the logarithmic densities up to products of 10 primes for two cases: $q=3$, $a=2$, $b=1$, and $q=4$, $a=3$, $b=1$.

$q=3,\ a=2,\ b=1$		
k	$\delta_{\Omega_k}(3;2,1)$	$\delta_{\omega_k}(3;2,1)$
1	0.99906, [56]	0.00094
2	0.069629	0.069629
3	0.766925	0.233075
4	0.35829	0.35829
5	0.571953	0.428047
6	0.463884	0.463884
7	0.518075	0.481925
8	0.49096	0.49096
9	0.50452	0.49548
10	0.49774	0.49774

Table 1.1: $\delta_{\Omega_k}(3;2,1)$ and $\delta_{\omega_k}(3;2,1)$

$q=4,\ a=3,\ b=1$		
k	$\delta_{\Omega_k}(4;3,1)$	$\delta_{\omega_k}(4;3,1)$
1	0.9959, [56]	0.0041
2	0.10572, [15]	0.10572
3	0.730311	0.269689
4	0.380029	0.380029
5	0.56061	0.43939
6	0.469616	0.469616
7	0.515202	0.484798
8	0.492398	0.492398
9	0.503801	0.496199
10	0.498099	0.498099

Table 1.2: $\delta_{\Omega_k}(4;3,1)$ and $\delta_{\omega_k}(4;3,1)$

For fixed q and large k, we give asymptotic formulas for $\delta_{\Omega_k}(q;a,b)$ and $\delta_{\omega_k}(q;a,b)$.

Theorem 1.11. *Assume ERH_q and LI_q. Let $A(q)$ be the number of real characters* mod q. *Let a be a quadratic non-residue and b be a quadratic residue, and $(a,q)=(b,q)=1$. Then, for any nonnegative integer K, and any $\epsilon>0$,*

$$\delta_{\Omega_k}(q;a,b)=\frac{1}{2}+\frac{(-1)^{k-1}}{2\pi}\sum_{j=0}^{K}\left(\frac{1}{2^{k-1}}\right)^{2j+1}\frac{(-1)^j A(q)^{2j+1}C_j(q;a,b)}{(2j+1)!}+O_{q,K,\epsilon}\left(\frac{1}{(2^{k-1})^{2K+3-\epsilon}}\right), \tag{1.2.3}$$

$$\delta_{\omega_k}(q;a,b)=\frac{1}{2}-\frac{1}{2\pi}\sum_{j=0}^{K}\left(\frac{1}{2^{k-1}}\right)^{2j+1}\frac{(-1)^j A(q)^{2j+1}C_j(q;a,b)}{(2j+1)!}+O_{q,K,\epsilon}\left(\frac{1}{(2^{k-1})^{2K+3-\epsilon}}\right), \tag{1.2.4}$$

where $C_j(q;a,b)$ is a constant depending on j, q, a, and b, $C_j(q;a,b)=\int_{-\infty}^{\infty}x^{2j}\Phi_{q;a,b}(x)dx$,

$$\Phi_{q;a,b}(z)=\prod_{\chi\neq\chi_0}\prod_{\substack{\gamma_\chi>0\\ L(\frac{1}{2}+i\gamma_\chi)=0}}J_0\left(\frac{2|\chi(a)-\chi(b)|z}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right),$$

and $J_0(z)$ *is the Bessel function,* $J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{2m}}{(m!)^2}$. *In particular, for* $K = 0$,

$$\delta_{\Omega_k}(q; a, b) = \frac{1}{2} + (-1)^{k-1} \frac{A(q) C_0(q; a, b)}{2^k \pi} + O_{q,\epsilon}\left(\frac{1}{(2^k)^{3-\epsilon}}\right),$$
$$\delta_{\omega_k}(q; a, b) = \frac{1}{2} - \frac{A(q) C_0(q; a, b)}{2^k \pi} + O_{q,\epsilon}\left(\frac{1}{(2^k)^{3-\epsilon}}\right).$$

Remark 2. Numerically, $C_0(3; 2, 1) \approx 3.66043$ and $C_0(4; 3, 1) \approx 3.08214$. When q is large, using the method in [13] (Section 2), we can find asymptotic formulas for $C_j(q; a, b)$,

$$C_j(q; a, b) = \frac{(2j-1)!!\sqrt{2\pi}}{V(q; a, b)^{j+\frac{1}{2}}} + O_j\left(\frac{1}{V(q; a, b)^{j+\frac{3}{2}}}\right),$$

where $(2j-1)!! = (2j-1)(2j-3)\cdots 3 \cdot 1$, $(-1)!! = 1$, and

$$V(q; a, b) = \sum_{\chi \bmod q} |\chi(b) - \chi(a)|^2 \sum_{\substack{\gamma_\chi \in \mathbb{R} \\ L(\frac{1}{2} + i\gamma_\chi, \chi) = 0}} \frac{1}{\frac{1}{4} + \gamma_\chi^2}.$$

By Proposition 3.6 in [13], under $\mathrm{ERH_q}$, $V(q; a, b) \sim 2\phi(q) \log q$.

1.3 Large bias for integers with prime factors from arithmetic progressions

In this section, we consider another type of restricted integers different from the one studied in Section 1.2. All results in this section are unconditional.

For any $k \geq 2$, $q \geq 3$, and integers $(a_j, q) = 1$ $(1 \leq j \leq k)$, we consider the number of integers $\leq x$ which can be written as product of k distinct primes $p_1 p_2 \cdots p_k$ with $p_j \equiv a_j \bmod q$ $(1 \leq j \leq k)$. Here when we count the number of such integers, we allow any ordering of the prime factors.

The results in this Section 1.3 and Chapter 4 appear in the paper [44].

In the previous section, we mentioned that Ford and Sneed [15] investigated subtle biases in the distribution of the product of two primes in different arithmetic progressions subject to the Extended Riemann Hypothesis ($\mathrm{ERH_q}$) and the Linear Independence conjecture ($\mathrm{LI_q}$) for Dirichlet L-functions. By the results in Section 1.2, for each fixed k and q, different arithmetic progressions contain virtually the same number of products of k primes below x, indeed, under the $\mathrm{ERH_q}$, the number equals $\frac{1}{\phi(q)}\frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!} + O(x^{1/2+o(1)})$.

It is reasonable to expect that these integers break up very evenly, with errors of size $O(x^{1/2+o(1)})$, when one specifies further which arithmetic progression modulo q each prime factor lies in. However, this is not the case. Dummit, Granville and Kisilevsky [10] showed that there is a very large bias for the odd integers $p_1p_2 \le x$ with two prime factors satisfying $p_1 \equiv p_2 \equiv 3 \pmod 4$. More precisely, they showed that

$$\frac{\#\{p_1p_2 \le x : p_1 \equiv p_2 \equiv 3 \bmod 4\}}{\frac{1}{4}\#\{p_1p_2 \le x\}} = 1 + \frac{c + o(1)}{\log\log x},$$

for some positive constant c. The authors exhibit a similar bias for products of 2 primes, where $\chi_q(p_1) = \chi_q(p_2) = \eta$, χ_q is a quadratic Dirichlet character with fixed conductor q, and $\eta \in \{-1, 1\}$. If q is allowed to grow with x, they further conjecture that the bias may be a bit larger. Recently, Hough [25] confirmed their conjecture and showed that, for $\eta \in \{-1, 1\}$, there exist many $q \le x$ for which

$$\frac{\#\{p_1p_2 \le x : \chi_q(p_1) = \chi_q(p_2) = \eta\}}{\frac{1}{4}\#\{p_1p_2 \le x : (p_1p_2, q) = 1\}} \text{ is at least as large as } 1 + \frac{\log\log\log x + O(1)}{\log\log x}.$$

On the other hand, Moree [50] considered all the integers with every prime factor from the same arithmetic progression $a \bmod q$, and proved that there is a large bias towards certain residue classes $a \bmod q$.

In this section, we generalize the large bias results found in [10] to products of any $k \ge 2$ primes and any fixed modulus $q \ge 3$, and prove uniform estimates in a large range of k. For any fixed $A > 0$ and fixed $q \geq 3$, we prove an asymptotic formula uniformly for $2 \leq k \leq A \log\log x$ for the number of integers $p_1 \cdots p_k \leq x$ with $p_j \equiv a_j \bmod q$ $(1 \leq j \leq k)$. We show that there are large biases for some arithmetic progressions $(a_1 \bmod q, \ldots, a_k \bmod q)$, and that this phenomenon has connections with Mertens theorem and the least prime in arithmetic progressions.

Let $\mathbf{a} := (a_1, a_2, \ldots, a_k) \in (\mathbb{Z}/q\mathbb{Z})^k$ with $(a_j, q) = 1$ for all $1 \leq j \leq k$. One may regard the vector $\mathbf{a}$ as an unordered k-tuple, or as a multiset. Denote

$$M_k(x; \mathbf{a}) := \#\{n \leq x : n = p_1 p_2 \cdots p_k, p_j \text{ distinct primes}, p_j \equiv a_j \bmod q, (a_j, q) = 1, 1 \leq j \leq k\},$$

where the prime factors p_j can be in any order, and

$$S_k(x) := \#\{n \leq x : n = p_1 p_2 \cdots p_k, (p_1 p_2 \cdots p_k, q) = 1, p_j \text{ distinct primes}, 1 \leq j \leq k\}.$$

Let χ be a Dirichlet character modulo q, and χ_0 be the principal character modulo q. Denote

$$C(q, a) := \lim_{x \to \infty} \left(\phi(q) \sum_{\substack{p \leq x \\ p \equiv a \bmod q}} \frac{1}{p} - \sum_{\substack{p \leq x \\ (p,q)=1}} \frac{1}{p} \right) = \sum_{\chi \neq \chi_0} \bar{\chi}(a) \sum_{p} \frac{\chi(p)}{p}.$$

We will see in our theorems that this constant $C(q, a)$ reflects the bias in our problem.

Our first result is for the special case when $a_1 = a_2 = \cdots = a_k = a$. In other words, all the k prime factors are from the same residue class $a \bmod q$.

Theorem 1.12. *Let $q \geq 3$ be fixed, and $\mathbf{a} = (a, a, \ldots, a)$, $(a, q) = 1$. We have the following results.*

i) For fixed $k \geq 2$,

$$\frac{M_k(x; \mathbf{a})}{\frac{1}{\phi^k(q)} S_k(x)} = 1 + \frac{(k-1)C(q, a)}{\log\log x} + O_{q,k}\left(\frac{1}{(\log\log x)^2} \right).$$

ii) If $k = o(\log\log x)$, as $x \to \infty$,

$$\frac{M_k(x;\mathbf{a})}{\frac{1}{\phi^k(q)}S_k(x)} = 1 + \frac{(k-1)(C(q,a)+o(1))}{\log\log x}.$$

iii) For fixed $A > 0$ and $k \sim A\log\log x$, we have, as $x \to \infty$,

$$\frac{M_k(x;\mathbf{a})}{\frac{1}{\phi^k(q)}S_k(x)} \sim \prod_p \frac{1 + \frac{A\phi(q)\mathbb{1}_{p\equiv a \bmod q}(p)}{p}}{1 + \frac{A\chi_0(p)}{p}}.$$

Remark 1. If k is fixed, by Lemmas 4.6 and 4.7, $M_k(x;\mathbf{a})$ and $\frac{1}{\phi^k(q)}S_k(x)$ have main terms of the same order which is $\frac{1}{\phi^k(q)}\frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}$ with different secondary terms and hence the bias is determined by the constant $C(q,a)$. Thus we see that, as k increases, the bias will become larger and larger.

For $k \sim A\log\log x$, the main terms of $M_k(x;\mathbf{a})$ and $\frac{1}{\phi^k(q)}S_k(x)$ have the same order of magnitude but with different coefficients. One may compare this with the result of Moree [50] who showed that the counting function $N(x;q,a) := \#\{n \le x : p|n \Rightarrow p \equiv a \bmod q\}$ satisfies $N(x;q,a) \sim B_{q,a}x/(\log x)^{1-1/\phi(q)}$ for some positive constant $B_{q,a}$ depending on q and a, and in particular, $N(x;4,3) \ge N(x;4,1)$ holds for all x.

For the general case, assume there are l distinct values $b_1,\ldots,b_l$ in the coordinates of $\mathbf{a}$. Fix l, for each $1 \le j \le l$, let k_j be the number of prime factors congruent to b_j mod q. Then $\sum_{j=1}^{l} k_j = k$.

Theorem 1.13. *Let $q \ge 3$ be fixed. Then, for fixed $k \ge 2$,*

$$\frac{M_k(x;\mathbf{a})}{\frac{1}{\phi^k(q)}\frac{k!}{k_1!k_2!\cdots k_l!}S_k(x)} = 1 + \frac{k-1}{\log\log x}\frac{1}{k}\sum_{j=1}^{k} C(q,a_j) + O_{q,k,l}\left(\frac{1}{(\log\log x)^2}\right).$$

Moreover, for fixed l and fixed $A > 0$, assume $k = \sum_{j=1}^{l} k_j \sim A\log\log x$ and $e_j :=$

$\lim_{x\to\infty} \frac{k_j}{\log\log x}$ *exists for every* $1 \le j \le l$*. Then as* $x \to \infty$*,*

$$\frac{M_k(x;\mathbf{a})}{\frac{1}{\phi^k(q)}\frac{k!}{k_1!k_2!\cdots k_l!}S_k(x)} \sim \prod_p \frac{\prod_{j=1}^l \left(1+\frac{\phi(q)e_j \mathbb{1}_{p\equiv b_j \bmod q}(p)}{p}\right)}{1+\frac{A\chi_0(p)}{p}}, \tag{1.3.1}$$

where $\sum_{j=1}^l e_j = A$*.*

Remark 2. In the general case $\mathbf{a} = (a_1, \dots, a_k)$, there are $\frac{k!}{k_1!k_2!\cdots k_l!}$ orderings of the numbers $a_1, \dots, a_k$.

Remark 3. For $k \sim A\log\log x$, if the coordinates of $\mathbf{a}$ cover all the reduced residue classes modulo q and all the e_j's are the same, then the right side of (1.3.1) is exactly 1.

1.3.1 Mertens theorem and the least prime in arithmetic progressions

The constant $C(q,a)$, which affects the biases in our theorems, is related to the classical Mertens theorem ([21], §22.8) and the Mertens theorem [45] for arithmetic progressions, that

$$\sum_{p\le x}\frac{1}{p} = \log\log x + \gamma + B + O\left(\frac{1}{\log x}\right), \tag{1.3.2}$$

and if $(a,q)=1$,

$$\sum_{\substack{p\le x\\ p\equiv a \bmod q}}\frac{1}{p} = \frac{\log\log x}{\phi(q)} + M(q,a) + O\left(\frac{1}{\log x}\right), \tag{1.3.3}$$

where γ is Euler's constant, $B := \sum_p \left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)$ is the Mertens constant, and $M(q,a)$ is a number depending on q and a. Languasco and Zaccagnini [35] investigated the value of $M(q,a)$ and other related constants. By (1.3.2), (1.3.3), and the orthogonality of Dirichlet characters, letting $x\to\infty$, we get

$$C(q,a) = \phi(q)M(q,a) - \gamma - B + \sum_{p|q}\frac{1}{p}, \tag{1.3.4}$$

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} M(q,a) = \gamma + B - \sum_{p|q} \frac{1}{p}. \tag{1.3.5}$$

Hence the value of $M(q,a)$ determines how the bias behaves.

In particular, with the values of $M(q,a)$ calculated by Languasco and Zaccagnini [36], by (1.3.4), we have

$$C(3,2) \approx 0.641945, \quad C(3,1) \approx -0.641945;$$

$$C(4,3) \approx 0.334981, \quad C(4,1) \approx -0.334981;$$

$$C(7,2) \approx 1.83747, \quad C(7,5) \approx 0.159006, \quad C(7,6) \approx -0.946269;$$

$$C(13,3) \approx 2.68478, \quad C(13,6) \approx -0.846522, \quad C(13,8) \approx -1.31962.$$

Here the interesting phenomenon is that 2 is a quadratic residue modulo 7, while 5 and 6 are quadratic non-residues modulo 7; 3 is a quadratic residue modulo 13, while 6 and 8 are quadratic non-residues modulo 13. There is no consistent preference for either quadratic non-residue classes or quadratic residue classes modulo q.

The above phenomenon is different from the biases among products of k primes studied in [15] and Section 1.2 (or [43]). Using a similar method as in Section 1.2, one can show that, under the $\mathrm{ERH_q}$ and $\mathrm{LI_q}$, the integers $n = p_1 \cdots p_k$, which are products of exactly k distinct primes, have preference for either quadratic non-residues or quadratic residues, depending on the parity of k.

The biases in Theorems 1.12 and 1.13 ultimately stem from the fact that $M(q,a)$ is heavily dependent on the least prime $p(q,a)$ in the arithmetic progression a mod q. Pomerance [55] and Norton [52] independently showed that

$$\sum_{\substack{p \le x \\ p \equiv a \bmod q}} \frac{1}{p} - \frac{\log\log x}{\phi(q)} = \frac{1}{p(q,a)} + O\left(\frac{\log 2q}{\phi(q)}\right), \tag{1.3.6}$$

where the implied constant is uniform for all q, a, and $x \ge q$.

In Theorem 1.13, we allow any ordering of the primes p_j $(1 \le j \le k)$, and hence the constant $\frac{1}{k}\sum_{j=1}^{k} C(q, a_j)$ represents the bias. One may ask when this constant is 0? Trivially, by (1.3.4) and (1.3.5), if $\mathbf{a}$ covers every element of the reduced residue class modulo q the same number of times, $\frac{1}{k}\sum_{j=1}^{k} C(q, a_j) = 0$. But we don't know if the converse is true. Alternatively, by (1.3.4) and (1.3.5), we may consider the distribution of the values of $M(q, a_j)$ $(1 \le j \le k)$. By (1.3.6), it is reasonable to conjecture that all the $M(q, a_j)$'s are distinct and that, except in the trivial case, they are linearly independent over $\mathbb{Q}$. Hence, we conjecture that the trivial case is the only case for which $\frac{1}{k}\sum_{j=1}^{k} C(q, a_j) = 0$.

Chapter 2

The distribution of k-free numbers and the derivative of the Riemann zeta-function

In this Chapter, we give the proof of the results Section 1.1.

2.1 Main Lemmas and Proofs

Since (1.1.3) implies that the zeros of $\zeta(s)$ are simple, in this section, we implicitly assume the simplicity of zeros of $\zeta(s)$. The implicit constants in our estimates may depend only on k or ϵ, unless otherwise specified.

Lemma 2.1. *Assume $J_{-1}(T) \ll T^{\theta}$ for some $\theta \geq 1$. Then, for $a > \frac{\theta+1}{2}, b > \theta$ and any $\epsilon > 0$, we have*

$$\sum_{\gamma > T} \frac{1}{\gamma^{a}|\zeta'(\rho)|} \ll_{a,\theta,\epsilon} \frac{1}{T^{a-\frac{\theta+1}{2}-\epsilon}}, \quad \textit{and} \quad \sum_{\gamma > T} \frac{1}{\gamma^{b}|\zeta'(\rho)|^{2}} \ll_{b,\theta,\epsilon} \frac{1}{T^{b-\theta}}.$$

Proof. Let $N(T)$ be the number of zeros of $\zeta(s)$ in the region $0 < \sigma < 1$, $0 < t \leq T$. By Theorem 1.7 in [27],

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{2.1.1}$$

By the Cauchy-Schwarz inequality, and (2.1.1),

$$J_{-\frac{1}{2}}(T) = \sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|} \ll \left(\sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|^{2}}\right)^{\frac{1}{2}} \left(\sum_{0<\gamma\leq T} 1\right)^{\frac{1}{2}} \ll T^{\frac{\theta+1}{2}}(\log T)^{\frac{1}{2}}.$$

Let $f(t) = \frac{1}{t^a}$. Then, $f'(t) = -at^{-a-1}$, and by partial summation,

$$\sum_{\gamma > T} \frac{1}{\gamma^a |\zeta'(\rho)|} = f(t) J_{-\frac{1}{2}}(t) |_T^\infty - \int_T^\infty J_{-\frac{1}{2}}(t) f'(t) dt \ll \frac{1}{T^{a - \frac{\theta+1}{2} - \epsilon}}.$$

Similarly, using $J_{-1}(T) \ll T^\theta$ and partial summation, we can prove the second formula. □

We also need the following lemma from [51].

Lemma 2.2 ([51], Lemma 3)**.** *Assume the Riemann Hypothesis. There exists a sequence of numbers $\mathcal{T} = \{T_n\}_{n=0}^\infty$ which satisfies*

$$n \leq T_n \leq n+1 \quad \text{and} \quad \frac{1}{\zeta(\sigma + iT_n)} = O(T_n^\epsilon) \quad (-1 \leq \sigma \leq 2).$$

One can use classical tools, like Perron's formula, contour integration, and the functional equation, to prove the following lemma.

Lemma 2.3. *Assume the Riemann Hypothesis and that all zeros of $\zeta(s)$ are simple. For $T \in \mathcal{T}$,*

$$\widetilde{M}_k(x) = \sum_{n \leq x} \mu_k(n) = \frac{x}{\zeta(k)} + \sum_{|\gamma| < T} \frac{\zeta(\frac{\rho}{k})}{\rho \zeta'(\rho)} x^{\frac{\rho}{k}} + \widetilde{E}(x, T),$$

where

$$\widetilde{E}(x, T) \ll_{k,\epsilon} \frac{x \log x}{T} + \frac{x}{T^{\frac{1}{2} - \epsilon}} + 1.$$

The two lemmas below are the main lemmas we use to prove Theorems 1.1, 1.2, 1.3, and 1.4.

Lemma 2.4. *Assume the Riemann Hypothesis and (1.1.3). For $x \geq 2$, $T \geq 2$, and any $\epsilon > 0$,*

$$\widetilde{M}_k(x) = \frac{x}{\zeta(k)} + \sum_{|\gamma| < T} \frac{\zeta(\frac{\rho}{k})}{\rho \zeta'(\rho)} x^{\frac{\rho}{k}} + E(x, T),$$

where

$$E(x, T) \ll_{k,\epsilon} \frac{x \log x}{T} + \frac{x}{T^{\frac{1}{2} - \epsilon}} + \frac{x^{\frac{1}{2k}}}{T^{\frac{1}{2k} - \epsilon}} + 1.$$

Lemma 2.5. *Let $k \geq 1$ be an integer. Assume the Riemann Hypothesis and $J_{-1}(T) \ll T^{\theta}$ for some $1 \leq \theta < 1 + \frac{1}{k}$. Then for any $\epsilon > 0$, $Z \geq 0$ and $0 < T < X$,*

$$\int_{\log Z}^{\log Z+1} \left| \sum_{T \leq \gamma \leq X} \frac{u_k(\rho)}{\rho \zeta'(\rho)} e^{\frac{i\gamma y}{k}} \right|^2 dy \ll_{k,\epsilon} \frac{1}{T^{1+\frac{1}{k}-\theta-\epsilon}},$$

where $u_k(\rho) = \zeta(\frac{\rho}{k})$ if $k \geq 2$, $u_k(\rho) = 1$ if $k = 1$.

2.1.1 Asymptotic formula for the counting function: proof of Lemma 2.3

By Perron's formula ([54], pp. 376-379), with $c = 1 + \frac{1}{\log x}$, we have

$$\widetilde{M}_k(x) = \sum_{n \leq x} \mu_k(n) = \frac{1}{2\pi i} \int_{c-\frac{iT}{k}}^{c+\frac{iT}{k}} \frac{\zeta(s)}{\zeta(ks)} \frac{x^s}{s} ds + O\left(\frac{x \log x}{T} + 1\right).$$

Let U be an odd positive integer. Then, by the residue theorem,

$$\begin{aligned} \widetilde{M}_k(x) &= \frac{1}{\zeta(k)} x + \sum_{|\gamma|<T} \frac{\zeta(\frac{\rho}{k})}{\rho \zeta'(\rho)} x^{\frac{\rho}{k}} + \sum_{1 \leq n \leq \frac{U}{2}} \frac{\zeta(-\frac{2n}{k})}{(-2n)\zeta'(-2n)} x^{-\frac{2n}{k}} \\ &\quad + \frac{1}{2\pi i} \left(\int_I + \int_{II} + \int_{III} \right) \frac{\zeta(s)}{\zeta(ks)} \frac{x^s}{s} ds + O\left(\frac{x \log x}{T} + 1\right), \end{aligned} \tag{2.1.2}$$

where I represents the straight line path from $c - \frac{iT}{k}$ to $-U - \frac{iT}{k}$, II is the straight line path from $-U - \frac{iT}{k}$ to $-U + \frac{iT}{k}$, and III is the straight line path from $-U + \frac{iT}{k}$ to $c + \frac{iT}{k}$.

Now we estimate these integrals. Let

$$I_2 := \int_{II} \frac{\zeta(s)}{\zeta(ks)} \frac{x^s}{s} ds = \int_{-U-\frac{iT}{k}}^{-U+\frac{iT}{k}} \frac{\zeta(s)}{\zeta(ks)} \frac{x^s}{s} ds.$$

By the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s), \tag{2.1.3}$$

we get

$$I_2 = \int_{1+U-\frac{iT}{k}}^{1+U+\frac{iT}{k}} \frac{\zeta(1-s)}{\zeta(k(1-s))}\frac{x^{1-s}}{1-s}ds$$
$$= \int_{1+U-\frac{iT}{k}}^{1+U+\frac{iT}{k}} \frac{2^{1-s}\pi^{1-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\zeta(s)}{2^{k-ks}\pi^{k-1-ks}\sin\left(\frac{\pi k(1-s)}{2}\right)\Gamma(ks-k+1)\zeta(ks-k+1)}\frac{x^{1-s}}{1-s}ds. \qquad (2.1.4)$$

By the Gauss Multiplication Formula (see [1], p. 256),

$$\prod_{l=0}^{k-1}\Gamma\left(z+\frac{l}{k}\right) = (2\pi)^{\frac{k-1}{2}}k^{\frac{1}{2}-kz}\Gamma(kz).$$

Taking $z = s - \frac{k-1}{k}$, then we have

$$\frac{\Gamma(s)}{\Gamma(ks-k+1)} = \frac{(2\pi)^{\frac{k-1}{2}}k^{k-\frac{1}{2}-ks}}{\prod_{l=1}^{k-1}\Gamma\left(s-\frac{l}{k}\right)}. \qquad (2.1.5)$$

Then, by (2.1.4),

$$I_2 = \int_{1+U-\frac{iT}{k}}^{1+U+\frac{iT}{k}} \frac{2^{1-s}\pi^{-s}(2\pi)^{\frac{k-1}{2}}k^{k-\frac{1}{2}-ks}}{2^{k-ks}\pi^{k-1-ks}}\cdot\frac{1}{\prod_{l=1}^{k-1}\Gamma\left(s-\frac{l}{k}\right)}\cdot\frac{\sin\left(\frac{\pi(1-s)}{2}\right)}{\sin\left(\frac{k\pi(1-s)}{2}\right)}\cdot\frac{\zeta(s)}{\zeta(ks-k+1)}\cdot\frac{x^{1-s}}{1-s}ds.$$

By Stirling's formula (see A. Karatsuba [30], Chapter III, Theorem 5, p. 44),

$$\frac{1}{|\Gamma(\sigma+it)|} \ll e^{\sigma-(\sigma-\frac{1}{2})\log\sigma+\frac{\pi|t|}{2}}, \quad (\sigma \geq \frac{1}{2}) \qquad (2.1.6)$$

we have

$$\frac{1}{\prod_{l=1}^{k-1}\Gamma\left(s-\frac{l}{k}\right)} \ll e^{\frac{(k-1)\pi|t|}{2}+\sum_{l=1}^{k-1}\left(1+U-\frac{l}{k}-\left(1+U-\frac{l}{k}-\frac{1}{2}\right)\log\left(1+U-\frac{l}{k}\right)\right)}.$$

Also

$$\left|\sin\left(\frac{\pi(1-s)}{2}\right)\right| \ll e^{\frac{\pi|t|}{2}}, \text{ and } \quad \frac{1}{\left|\sin\left(\frac{k\pi(1-s)}{2}\right)\right|} \ll e^{-\frac{k\pi|t|}{2}}. \tag{2.1.7}$$

Thus, by (2.1.4),

$$I_2 \ll_k \int_{-\frac{T}{k}}^{\frac{T}{k}} \left(\frac{(2\pi)^{k-1}}{k^k x}\right)^{1+U} e^{\sum_{l=1}^{k-1}\left(1+U-\frac{l}{k}-\left(1+U-\frac{l}{k}-\frac{1}{2}\right)\log\left(1+U-\frac{l}{k}\right)\right)} \frac{1}{T}dt.$$

The right-hand side of the above formula goes to 0 as U goes to ∞. By the functional equation (2.1.3),

$$\zeta(-\frac{2n}{k}) = 2^{-\frac{2n}{k}}\pi^{-\frac{2n}{k}-1}\sin\left(-\frac{2n}{k}\right)\Gamma\left(1+\frac{2n}{k}\right)\zeta\left(1+\frac{2n}{k}\right).$$

Taking the derivative on both sides of (2.1.3), we get

$$\zeta'(-2n) = \frac{(-1)^n}{2(2\pi)^{2n}}(2n)!\zeta(2n+1).$$

Thus,

$$\sum_{n\geq 1} \frac{\zeta(-\frac{2n}{k})}{(-2n)\zeta'(-2n)} x^{-\frac{2n}{k}} = O(1).$$

Then by (2.1.2), we have

$$\widetilde{M}_k(x) = \frac{1}{\zeta(k)}x + \sum_{|\gamma|<T} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} + \frac{1}{2\pi i}\left(\int_{I'} + \int_{III'}\right) \frac{\zeta(s)}{\zeta(ks)}\frac{x^s}{s}ds + O\left(\frac{x\log x}{T}+1\right), \tag{2.1.8}$$

where I' represents the integral over the straight line path from $c-\frac{iT}{k}$ to $-\infty-\frac{iT}{k}$, and III' is the straight line path from $-\infty+\frac{iT}{k}$ to $c+\frac{iT}{k}$.

We only need to estimate the integral on path III' since I' is similar to III'. Let

$$\begin{aligned} I_3 := \int_{-\infty+\frac{iT}{k}}^{c+\frac{iT}{k}} \frac{\zeta(s)}{\zeta(ks)}\frac{x^s}{s}ds &= \int_{-\infty+\frac{iT}{k}}^{0+\frac{iT}{k}} \frac{\zeta(s)}{\zeta(ks)}\frac{x^s}{s}ds + \int_{0+\frac{iT}{k}}^{c+\frac{iT}{k}} \frac{\zeta(s)}{\zeta(ks)}\frac{x^s}{s}ds \\ &=: I_1' + I_2'. \end{aligned} \tag{2.1.9}$$

By the functional equation (2.1.3) and the properties of Gamma function (2.1.5),

$$I_1' = \int_{1-\frac{iT}{k}}^{\infty-\frac{iT}{k}} \frac{k^{k-\frac{1}{2}}}{(2\pi)^{\frac{k-1}{2}}} \cdot \left(\frac{(2\pi)^{k-1}}{k^k}\right)^s \cdot \frac{1}{\prod_{l=1}^{k-1}\Gamma\left(s-\frac{l}{k}\right)} \cdot \frac{\sin\left(\frac{\pi(1-s)}{2}\right)}{\sin\left(\frac{k\pi(1-s)}{2}\right)} \cdot \frac{\zeta(s)}{\zeta(ks-k+1)} \cdot \frac{x^{1-s}}{1-s}ds.$$

By (2.1.6) and (2.1.7), and the estimates (see (14.2.5) and (14.2.6) in [59]),

$$\frac{1}{|\zeta(ks-k+1)|} \ll_{k,\epsilon} T^{\epsilon}, \quad |\zeta(s)| \ll T^{\epsilon}, \text{ for every } \sigma > \frac{1}{2}, \tag{2.1.10}$$

we get

$$\begin{aligned} I_1' &\ll T^{\epsilon}\int_1^{\infty} \frac{x}{T}\left(\frac{(2\pi)^{k-1}}{k^k x}\right)^{\sigma} e^{\sum_{l=1}^{k-1}\left(\sigma-\frac{l}{k}-\left(\sigma-\frac{l}{k}-\frac{1}{2}\right)\log\left(\sigma-\frac{l}{k}\right)\right)}d\sigma \\ &\ll_{k,\epsilon} \frac{x}{T^{1-\epsilon}}. \end{aligned} \tag{2.1.11}$$

Now we estimate I_2'. Under the Riemann Hypothesis, by Section 13.1 in [59],

$$|\zeta(\sigma+it)| \ll (|t|+2)^{\frac{1}{2}-\sigma+\epsilon}, \text{ for } 0 \le \sigma \le \frac{1}{2}. \tag{2.1.12}$$

Thus, by (2.1.10) and Lemma 2.2, for $T \in \mathcal{T}$,

$$\begin{aligned} I_2' &= \int_{0+\frac{iT}{k}}^{c+\frac{iT}{k}} \frac{\zeta(s)}{\zeta(ks)}\frac{x^s}{s}ds \\ &\ll_{k,\epsilon} \int_0^{\frac{1}{2}} \frac{T^{\frac{1}{2}-\sigma+\epsilon}}{T}x^{\sigma}d\sigma + \int_{\frac{1}{2}}^{c}\frac{T^{\epsilon}}{T}x^{\sigma}d\sigma \\ &\ll_{k,\epsilon} \frac{x}{T^{\frac{1}{2}-\epsilon}}. \end{aligned} \tag{2.1.13}$$

Combining (2.1.9), (2.1.11) and (2.1.13), we get

$$I_3 \ll_{k,\epsilon} \frac{x}{T^{\frac{1}{2}-\epsilon}}. \tag{2.1.14}$$

By (2.1.8) and (2.1.14), we get the desired result in Lemma 2.3. □

2.1.2 Proof of Lemma 2.4: a proper truncation on the sum over zeros

Let $T \geq 2$ and $n \leq T \leq n+1$. Without loss of generality, assume $n \leq T_n \leq T \leq n+1$. Then, by Lemma 2.3,

$$\widetilde{M}_k(x) = \frac{x}{\zeta(k)} + \sum_{|\gamma| \leq T} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} - \sum_{T_n \leq |\gamma| \leq T} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} + \widetilde{E}(x,T).$$

By the Cauchy-Schwarz inequality,

$$\left| \sum_{T_n \leq \gamma \leq T} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} \right| \leq x^{\frac{1}{2k}} \left(\sum_{T_n \leq \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \right)^{\frac{1}{2}} \cdot \left(\sum_{T_n \leq \gamma \leq T} \left| \frac{\zeta(\frac{\rho}{k})}{\rho} \right|^2 \right)^{\frac{1}{2}}.$$

Under the Riemann Hypothesis, by Section 13.1 in [59],

$$|\zeta(\sigma + it)| \ll (|t|+2)^{\frac{1}{2}-\sigma+\epsilon}, \text{ for } 0 \leq \sigma \leq \frac{1}{2}. \tag{2.1.15}$$

By (2.1.15), we have

$$\left|\zeta(\frac{\rho}{k})\right| \ll_{k,\epsilon} |\gamma|^{\frac{1}{2}-\frac{1}{2k}+\epsilon}, \tag{2.1.16}$$

Thus, by (1.1.3), (2.1.16), and (2.1.1),

$$\left| \sum_{T_n \leq \gamma \leq T} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} \right| \ll_{k,\epsilon} x^{\frac{1}{2k}} T^{\frac{1}{2}+\epsilon} \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}-\epsilon}} \ll_{k,\epsilon} \frac{x^{\frac{1}{2k}}}{T^{\frac{1}{2k}-\epsilon}}.$$

The desired result follows. □

2.1.3 Proof of Lemma 2.5: key mean square result

In this lemma, we assume the bound

$$J_{-1}(T) \ll T^{\theta}, \tag{2.1.17}$$

for some $1 \le \theta < 1 + \frac{1}{k}$, and this implies, by the Cauchy-Schwarz inequality, that

$$J_{-\frac{1}{2}}(T) = \sum_{0<\gamma\le T} \frac{1}{|\zeta'(\rho)|} \ll T^{\frac{\theta+1}{2}} (\log T)^{\frac{1}{2}}. \tag{2.1.18}$$

We have

$$\begin{aligned} \int_{\log Z}^{\log Z+1} \left| \sum_{T\le\gamma\le X} \frac{u_k(\rho)}{\rho\zeta'(\rho)} e^{\frac{i\gamma y}{k}} \right|^2 dy &= \sum_{T\le\gamma\le X} \sum_{T\le\gamma'\le X} \frac{u_k(\rho)\overline{u_k(\rho')}}{\rho\zeta'(\rho)\overline{\rho'\zeta'(\rho')}} \int_{\log Z}^{\log Z+1} e^{\frac{i(\gamma-\gamma')y}{k}} dy \\ &\ll_k \sum_{T\le\gamma\le X} \sum_{T\le\gamma'\le X} \left| \frac{u_k(\rho)\overline{u_k(\rho')}}{\rho\zeta'(\rho)\overline{\rho'\zeta'(\rho')}} \right| \min\left(1, \frac{1}{|\gamma-\gamma'|}\right). \end{aligned}$$

Note that $\rho = \frac{1}{2} + i\gamma$ and $\rho' = \frac{1}{2} + i\gamma'$ denote zeros of $\zeta(s)$. We break the last sum into two parts:

$$\Sigma_1 : \text{ the sum over } |\gamma-\gamma'| \le 1, \quad \text{and} \quad \Sigma_2 : \text{ the sum over } |\gamma-\gamma'| > 1.$$

In the following proof, we always assume $T \le \gamma' \le X$. Without loss of generality, we assume $0 < \epsilon < \frac{1+\frac{1}{k}-\theta}{10}$. By (2.1.16), for any $\epsilon > 0$,

$$|u_k(\rho)| \ll |\gamma|^{\frac{1}{2}-\frac{1}{2k}+\epsilon}. \tag{2.1.19}$$

For the first sum, since $1 - \frac{1}{k} - 2\epsilon > \theta$, by the Cauchy-Schwarz inequality, (2.1.17), and

Lemma 2.1,

$$
\begin{aligned}
\Sigma_1 &\ll \sum_{T\le\gamma\le X}\frac{|u_k(\rho)|}{|\rho\zeta'(\rho)|}\sum_{\gamma-1\le\gamma'\le\gamma+1}\frac{|u_k(\rho')|}{|\rho'\zeta'(\rho')|}\\
&\ll \left(\sum_{T\le\gamma\le X}\frac{|u_k(\rho)|^2}{|\rho\zeta'(\rho)|^2}\right)^{\frac{1}{2}}\left(\sum_{T\le\gamma\le X}\left(\sum_{\gamma-1\le\gamma'\le\gamma+1}\frac{|u_k(\rho')|}{|\rho'\zeta'(\rho')|}\right)^2\right)^{\frac{1}{2}}\\
&\ll \left(\sum_{T\le\gamma\le X}\frac{|u_k(\rho)|^2}{|\rho\zeta'(\rho)|^2}\right)^{\frac{1}{2}}\left(\sum_{T\le\gamma\le X}\left(\sum_{\gamma-1\le\gamma'\le\gamma+1}\frac{|u_k(\rho')|^2}{|\rho'\zeta'(\rho')|^2}\right)\cdot\log\gamma\right)^{\frac{1}{2}}\\
&\ll \left(\sum_{T\le\gamma\le X}\frac{1}{|\gamma|^{1+\frac{1}{k}-2\epsilon}|\zeta'(\rho)|^2}\right)^{\frac{1}{2}}\left(\sum_{T\le\gamma\le X}\sum_{\gamma-1\le\gamma'\le\gamma+1}\frac{|u_k(\rho')|^2}{|\rho'\zeta'(\rho')|^2}\log\gamma'\right)^{\frac{1}{2}}\\
&\ll \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}-\frac{\theta}{2}-\epsilon}}\left(\sum_{T\le\gamma'\le X}m(\gamma')\frac{|u_k(\rho')|^2}{|\rho'\zeta'(\rho')|^2}\cdot\log\gamma'\right)^{\frac{1}{2}},
\end{aligned}
$$

where $m(\gamma') = \#\{\gamma : \gamma-1\le\gamma'\le\gamma+1\} \ll \log\gamma'$ by (2.1.1). Since $1+\frac{1}{k}-3\epsilon > \theta$, by Lemma 2.1 and (2.1.19),

$$
\Sigma_1 \ll \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}-\frac{\theta}{2}-\epsilon}}\left(\sum_{T\le\gamma'\le X}\frac{1}{|\gamma'|^{1+\frac{1}{k}-3\epsilon}|\zeta'(\rho')|^2}\right)^{\frac{1}{2}} \ll \frac{1}{T^{1+\frac{1}{k}-\theta-3\epsilon}}. \tag{2.1.20}
$$

We write Σ_2 as follows,

$$
\Sigma_2 = \sum_{T\le\gamma\le X}\frac{|u_k(\rho)|}{|\rho\zeta'(\rho)|}\sum_{\substack{T\le\gamma'\le X\\|\gamma-\gamma'|>1}}\frac{|u_k(\rho')|}{|\rho'\zeta'(\rho')||\gamma-\gamma'|}. \tag{2.1.21}
$$

Take $N = [\frac{1}{\epsilon}]+2$, and let

$$
1 > a_1 > a_2 > a_3 > \cdots > a_N = 0.
$$

Then, by (2.1.21) we have

$$
\Sigma_2 = \sum_{l=1}^{2N+1}\sigma_l, \tag{2.1.22}
$$

where

$$\sigma_l = \sum_{T \le \gamma \le X} \frac{|u_k(\rho)|}{|\rho\zeta'(\rho)|} \sum_{\gamma' \in L_l} \frac{|u_k(\rho')|}{|\rho'\zeta'(\rho')||\gamma - \gamma'|},$$

and

$$\begin{aligned}
L_1 &: T \le \gamma' < \gamma - \gamma^{a_1}, \\
L_2 &: \gamma - \gamma^{a_1} \le \gamma' < \gamma - \gamma^{a_2}, \\
&\vdots \qquad\qquad \vdots \\
L_{N-1} &: \gamma - \gamma^{a_{N-2}} \le \gamma' < \gamma - \gamma^{a_{N-1}}, \\
L_N &: \gamma - \gamma^{a_{N-1}} \le \gamma' < \gamma - 1, \\
L_{N+1} &: \gamma + 1 \le \gamma' < \gamma + \gamma^{a_{N-1}}, \\
L_{N+2} &: \gamma + \gamma^{a_{N-1}} \le \gamma' < \gamma + \gamma^{a_{N-2}}, \\
&\vdots \qquad\qquad \vdots \\
L_{2N-1} &: \gamma + \gamma^{a_2} \le \gamma' < \gamma + \gamma^{a_1}, \\
L_{2N} &: \gamma + \gamma^{a_1} \le \gamma' < 2\gamma, \\
L_{2N+1} &: 2\gamma \le \gamma'.
\end{aligned}$$

Note that, some of the L_l's might be empty for those γ's which are close to T, in which case the estimation will be trivial. Hence, we can assume each L_l is not empty.

We take

$$a_1 = 1 - \frac{1}{N}, \quad a_2 = 1 - \frac{2}{N}, \quad \ldots, \quad a_{N-1} = \frac{1}{N}, \quad \text{and} \quad a_N = 0.$$

Using the Cauchy-Schwarz inequality and (2.1.19), we find that

$$\sigma_1 = \sum_{T \le \gamma \le X} \frac{|u_k(\rho)|}{|\rho\zeta'(\rho)|} \sum_{\gamma' \in L_1} \frac{|u_k(\rho')|}{|\rho'\zeta'(\rho')||\gamma - \gamma'|} \ll \sum_{T \le \gamma \le X} \frac{1}{|\gamma|^{\frac{1}{2} + \frac{1}{2k} + a_1 - \epsilon}|\zeta'(\rho)|} \sum_{\gamma' \in L_1} \frac{1}{|\gamma'|^{\frac{1}{2} + \frac{1}{2k} - \epsilon}|\zeta'(\rho')|}.$$

Then, by partial summation and (2.1.18),

$$\sum_{\gamma'\in L_1} \frac{1}{|\gamma'|^{\frac{1}{2}+\frac{1}{2k}-\epsilon}|\zeta'(\rho')|} \ll \sum_{0<\gamma'<\gamma} \frac{1}{|\gamma'|^{\frac{1}{2}+\frac{1}{2k}-\epsilon}|\zeta'(\rho')|} \ll \gamma^{\frac{\theta}{2}-\frac{1}{2k}+2\epsilon}.$$

Since $0<\epsilon<\frac{1+\frac{1}{k}-\theta}{10}$ and $N=[\frac{1}{\epsilon}]+2$, $\frac{3}{2}+\frac{1}{k}-\frac{\theta}{2}-\frac{1}{N}-3\epsilon>\frac{\theta+1}{2}$, Lemma 2.1 applies. Thus, by (2.1.17) and Lemma 2.1, we obtain

$$\sigma_1 \ll \sum_{T\le\gamma\le X} \frac{\gamma^{\frac{\theta}{2}-\frac{1}{2k}+2\epsilon}}{|\gamma|^{\frac{1}{2}+\frac{1}{2k}+a_1-\epsilon}|\zeta'(\rho)|} \ll \sum_{T\le\gamma\le X} \frac{1}{|\gamma|^{\frac{3}{2}+\frac{1}{k}-\frac{\theta}{2}-\frac{1}{N}-3\epsilon}|\zeta'(\rho)|} \ll \frac{1}{T^{1+\frac{1}{k}-\theta-\frac{1}{N}-4\epsilon}} \ll \frac{1}{T^{1+\frac{1}{k}-\theta-5\epsilon}}. \tag{2.1.23}$$

For $2\le l\le 2N$, noting that $1+\frac{1}{k}-2\epsilon>\theta$, by the Cauchy-Schwarz inequality, (2.1.19) and Lemma 2.1, we have

$$\begin{aligned}
\sigma_l &\ll \left(\sum_{T\le\gamma\le X} \frac{|u_k(\rho)|^2}{|\rho\zeta'(\rho)|^2}\right)^{\frac{1}{2}} \left(\sum_{T\le\gamma\le X}\left(\sum_{\gamma'\in L_l}\frac{|u_k(\rho')|}{|\rho'\zeta'(\rho')||\gamma-\gamma'|}\right)^2\right)^{\frac{1}{2}} \\
&\ll \left(\sum_{T\le\gamma\le X} \frac{1}{\gamma^{1+\frac{1}{k}-2\epsilon}|\zeta'(\rho)|^2}\right)^{\frac{1}{2}} \left(\sum_{T\le\gamma\le X}\left(\sum_{\gamma'\in L_l}\frac{|u_k(\rho')|^2}{|\rho'\zeta'(\rho')|^2|\gamma-\gamma'|^2}\right)N(L_l)\right)^{\frac{1}{2}} \\
&\ll \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}-\frac{\theta}{2}-\epsilon}} \left(\sum_{T\le\gamma\le X}\left(\sum_{\gamma'\in L_l}\frac{|u_k(\rho')|^2}{|\rho'\zeta'(\rho')|^2|\gamma-\gamma'|^2}\right)N(L_l)\right)^{\frac{1}{2}},
\end{aligned}$$

where $N(L_l)$ is the number of γ''s in L_l. By (2.1.1), $N(L_l)\ll\gamma^{a_{l-1}+\epsilon}$. Then, for $2\le l\le N$, since $\gamma'\asymp\gamma$ for $\gamma'\in L_l$, we have

$$\frac{N(L_l)}{|\gamma-\gamma'|^2} \ll \frac{1}{\gamma^{2a_l-a_{l-1}-\epsilon}} = \frac{1}{\gamma^{1-\frac{l+1}{N}-\epsilon}} \ll \frac{1}{(\gamma')^{1-\frac{l+1}{N}-\epsilon}}.$$

Then, we have

$$\sigma_l \ll \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}-\frac{\theta}{2}-\epsilon}} \left(\sum_{T\le\gamma\le X}\sum_{\gamma'\in L_l}\frac{1}{|\gamma'|^{2+\frac{1}{k}-\frac{l+1}{N}-3\epsilon}|\zeta'(\rho')|^2}\right)^{\frac{1}{2}}$$

$$\ll \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}-\frac{\theta}{2}-\epsilon}} \left(\sum_{T \le \gamma' \le X} m_l(\gamma') \frac{1}{|\gamma'|^{2+\frac{1}{k}-\frac{l+1}{N}-3\epsilon} |\zeta'(\rho')|^2} \right)^{\frac{1}{2}},$$

by swapping summation and where $m_l(\gamma') = \#\{\gamma : \gamma - \gamma^{a_{l-1}} \le \gamma' < \gamma - \gamma^{a_l}\}$. By (2.1.1), $m_l(\gamma') = \#\{\gamma : \gamma' + \gamma^{a_l} < \gamma \le \gamma' + \gamma^{a_{l-1}}\} \ll \#\{\gamma : \gamma' + (\gamma')^{a_l} < \gamma \le \gamma' + (C\gamma')^{a_{l-1}} \text{ for some } C > 0\} \ll (\gamma')^{1-\frac{l-1}{N}+\epsilon}$.

Since $N = [\frac{1}{\epsilon}] + 2$ and $0 < \epsilon < \frac{1+\frac{1}{k}-\theta}{10}$, $1 + \frac{1}{k} - \frac{2}{N} - 4\epsilon > \theta$, Lemma 2.1 applies. Thus, for $2 \le l \le N$, we get,

$$\sigma_l \ll \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}-\frac{\theta}{2}-\epsilon}} \left(\sum_{T \le \gamma' \le X} \frac{1}{|\gamma'|^{1+\frac{1}{k}-\frac{2}{N}-4\epsilon} |\zeta'(\rho')|^2} \right)^{\frac{1}{2}} \ll \frac{1}{T^{1+\frac{1}{k}-\theta-\frac{1}{N}-3\epsilon}} \ll \frac{1}{T^{1+\frac{1}{k}-\theta-4\epsilon}}, \tag{2.1.24}$$

Similarly, we have

$$\sigma_l \asymp \sigma_{2N+1-l}, \text{ for } N+1 \le l \le 2N. \tag{2.1.25}$$

Finally, we calculate σ_{2N+1},

$$\sigma_{2N+1} \ll \sum_{T \le \gamma \le X} \frac{|u_k(\rho)|}{|\rho \zeta'(\rho)|} \left(\sum_{m=1}^{\infty} \sum_{2^m \gamma \le \gamma' \le 2^{m+1}\gamma} \frac{|u_k(\rho')|}{|\rho' \zeta'(\rho')| |\gamma - \gamma'|} \right).$$

By (2.1.17), (2.1.19), and Lemma 2.1, the inner sum is

$$\begin{aligned}
&\ll \sum_{m=1}^{\infty} \frac{1}{(2^m - 1)\gamma} \left(\sum_{2^m \gamma \le \gamma' \le 2^{m+1}\gamma} \frac{1}{|\rho' \zeta'(\rho')|^2} \right)^{\frac{1}{2}} \left(\sum_{2^m \gamma \le \gamma' \le 2^{m+1}\gamma} |u_k(\rho')|^2 \right)^{\frac{1}{2}} \\
&\ll \sum_{m=1}^{\infty} \frac{1}{(2^m - 1)\gamma} \left(\frac{1}{2^m \gamma^{2-\theta}} \right)^{\frac{1}{2}} \left(2^{m+1}\gamma\right)^{\frac{1}{2}-\frac{1}{2k}+\epsilon} \left(2^{m+1}\gamma \log(2^{m+1}\gamma)\right)^{\frac{1}{2}} \\
&\ll \sum_{m=1}^{\infty} \frac{1}{2^{(\frac{1}{2}+\frac{1}{2k}-\epsilon)m}} \frac{1}{\gamma^{1+\frac{1}{2k}-\frac{\theta}{2}-\epsilon}} \ll \frac{1}{\gamma^{1+\frac{1}{2k}-\frac{\theta}{2}-\epsilon}}.
\end{aligned}$$

Noting that $\frac{3}{2}+\frac{1}{k}-\frac{\theta}{2}-2\epsilon > \frac{\theta+1}{2}$, by Lemma 2.1 and (2.1.19), we find that

$$\sigma_{2N+1} \ll \sum_{T\le\gamma\le X} \frac{|u_k(\rho)|}{|\rho\zeta'(\rho)|}\frac{1}{\gamma^{1+\frac{1}{2k}-\frac{\theta}{2}-\epsilon}} \ll \sum_{T\le\gamma\le X} \frac{1}{\gamma^{\frac{3}{2}+\frac{1}{k}-\frac{\theta}{2}-2\epsilon}|\zeta'(\rho)|} \ll \frac{1}{T^{1+\frac{1}{k}-\theta-3\epsilon}}. \tag{2.1.26}$$

Combining all the estimates (2.1.22)-(2.1.26), we have

$$\Sigma_2 \ll_{k,\epsilon} \frac{1}{T^{1+\frac{1}{k}-\theta-5\epsilon}}. \tag{2.1.27}$$

Therefore, by (2.1.20) and (2.1.27), we deduce that, under the assumption (2.1.17), for sufficiently small ϵ,

$$\int_{\log Z}^{\log Z+1} \left| \sum_{T\le\gamma\le X} \frac{u_k(\rho)}{\rho\zeta'(\rho)} e^{\frac{i\gamma y}{k}} \right|^2 dy \ll_{k,\epsilon} \frac{1}{T^{1+\frac{1}{k}-\theta-\epsilon}}.$$

The conclusion of Lemma 2.5 follows. □

2.2 Equivalent relations

2.2.1 Proof of Theorem 1.1: (1.1.3) implies (1.1.4)

By Lemma 2.4, for $X \le x \ll X$, taking $T = X^2$, we have

$$M_k(x) = \sum_{|\gamma|<X^2} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} + O(X^\epsilon).$$

Then,

$$M_k^2(x) \ll \left| \sum_{|\gamma|<X^2} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} \right|^2 + O(X^{2\epsilon}).$$

Since the imaginary part of the first zero of $\zeta(s)$ is > 14, we let $x = e^y$, take $\theta = 1+\epsilon$, $T = 14$, $Z = X$, and replace X by X^2 in Lemma 2.5. Using change of variable $x = e^y$, we deduce that

$$\int_X^{eX} \left(\frac{M_k(x)}{x^{\frac{1}{2k}}}\right)^2 \frac{dx}{x} \ll \int_X^{eX} \left| \sum_{14<\gamma<X^2} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} \right|^2 \frac{1}{x^{\frac{1}{k}}} \frac{dx}{x} + O(X^{-\frac{1}{k}+2\epsilon}) \ll 1. \tag{2.2.1}$$

So we get

$$\int_2^X \left(\frac{M_k(x)}{x^{\frac{1}{2k}}}\right)^2 \frac{dx}{x} \ll \sum_{l=1}^{[\log(\frac{X}{2})]+1} \int_{\frac{X}{e^{l-1}}}^{\frac{X}{e^l}} \left(\frac{M_k(x)}{x^{\frac{1}{2k}}}\right)^2 \frac{dx}{x} \ll_k \log X.$$

Here, it is easy to get a weaker result than Theorem 1.3. By (2.2.1), we have

$$\int_X^{eX} \left(\frac{M_k(x)}{x^{\frac{1}{2k}}}\right)^2 dx \ll X.$$

Substituting $\frac{X}{e}$, $\frac{X}{e^2}$, $\ldots$, for X in the above formula, we obtain

$$\int_2^X \left(\frac{M_k(x)}{x^{\frac{1}{2k}}}\right)^2 dx \ll X.$$

2.2.2 Proof of Theorem 1.1: (1.1.4) implies (1.1.3)

We need the following lemmas.

Lemma 2.6. *Assume the Riemann Hypothesis. The formula (1.1.4) implies the zeros of $\zeta(s)$ on the critical line are simple, that*

$$\frac{\zeta(\frac{\rho}{k})}{\zeta'(\rho)} = O_k(|\rho|), \quad \forall k \geq 2,$$

and

$$\frac{1}{\zeta'(\rho)} = O_\epsilon\left(|\rho|^{\frac{1}{2}+\epsilon}\right), \quad \forall \epsilon > 0.$$

Using the above lemma and a similar argument to Theorem 14.29 (B) of [59], we prove the following lemma.

Lemma 2.7. *Assume the Riemann Hypothesis. If (1.1.4) is true, then for any* $k \geq 2$ *the series*

$$\sum_{\rho} \frac{|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2}$$

is convergent.

Before giving the proofs of these two lemmas, we show how Lemma 2.7 implies the desired result.

For any $\epsilon > 0$, under the Riemann Hypothesis, we know that ([59], (14.2.6), p. 337)

$$\frac{1}{\zeta(\sigma+it)} = O(|t|^{\frac{\epsilon}{4}}), \text{ for every fixed } \sigma > \frac{1}{2}.$$

So by the functional equation, we have

$$|\zeta(\frac{\rho}{k})| \gg_k |\rho|^{\frac{1}{2}-\frac{1}{2k}-\frac{\epsilon}{4}}. \tag{2.2.2}$$

Take $k = \left[\frac{2}{\epsilon}\right] + 2$,

$$|\zeta(\frac{\rho}{k})| \gg_\epsilon |\rho|^{\frac{1}{2}-\frac{\epsilon}{2}}.$$

Then, by Lemma 2.7, we have

$$\sum_{0<\gamma\leq T} \frac{1}{\gamma^{1+\epsilon}|\zeta'(\rho)|^2} \ll \sum_{0<\gamma\leq T} \frac{|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2} \ll_\epsilon 1.$$

Hence, by partial summation, we derive that, for any $\epsilon > 0$,

$$\sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|^2} = \sum_{0<\gamma\leq T} \frac{1}{\gamma^{1+\epsilon}|\zeta'(\rho)|^2}\gamma^{1+\epsilon} \ll_\epsilon T^{1+\epsilon} + \int_1^T t^\epsilon dt \ll_\epsilon T^{1+\epsilon}.$$

This proves (1.1.3). □

Proof of Lemma 2.6. By considering Mellin transforms, for $\Re s > 1$,

$$\frac{\zeta(s)}{\zeta(ks)} - \frac{\zeta(s)}{\zeta(k)} = \sum_{n=1}^{\infty} \frac{\mu_k(n) - \frac{1}{\zeta(k)}}{n^s} = s\int_1^\infty \frac{M_k(x) + \frac{\{x\}}{\zeta(k)}}{x^{s+1}}dx = s\int_1^\infty \frac{M_k(x)}{x^{s+1}}dx + \frac{s}{\zeta(k)}\int_1^\infty \frac{\{x\}}{x^{s+1}}dx. \tag{2.2.3}$$

We show that the right-hand side of (2.2.3) can be analytically continued to the half-plane $\Re s > \frac{1}{2k}$. Letting $s = \sigma + it$ with $\sigma > \frac{1}{2k}$, we have

$$\begin{aligned}
\left| s\int_1^\infty \frac{M_k(x)}{x^{s+1}}dx \right| &\leq |s| \int_1^\infty \frac{|M_k(x)|}{x^{\sigma+1}}dx = |s| \int_1^\infty \frac{|M_k(x)|}{x^{\frac{1}{2}\sigma + \frac{1}{2} + \frac{1}{4k}}} \frac{1}{x^{\frac{1}{2}\sigma + \frac{1}{2} - \frac{1}{4k}}}dx \\
&\leq |s| \left(\int_1^\infty \frac{M_k^2(x)}{x^{\sigma+1+\frac{1}{2k}}}dx \right)^{\frac{1}{2}} \left(\int_1^\infty \frac{1}{x^{\sigma+1-\frac{1}{2k}}}dx \right)^{\frac{1}{2}} \\
&\leq \frac{|s|}{\sqrt{\sigma - \frac{1}{2k}}} \left(\int_1^\infty \frac{M_k^2(x)}{x^{\sigma+1+\frac{1}{2k}}}dx \right)^{\frac{1}{2}}. \qquad (2.2.4)
\end{aligned}$$

Let

$$f(X) = \int_1^X \frac{M_k^2(x)}{x^{1+\frac{1}{k}}}dx.$$

By assumption (1.1.4), $f(X) \ll \log X$. Then using an integration by parts,

$$\begin{aligned}
\int_1^\infty \frac{M_k^2(x)}{x^{\sigma+1+\frac{1}{2k}}}dx &= \int_1^\infty \frac{f'(x)}{x^{\sigma-\frac{1}{2k}}}dx = \left(\sigma - \frac{1}{2k}\right) \int_1^\infty \frac{f(x)}{x^{\sigma+1-\frac{1}{2k}}}dx \\
&= O\left(\left(\sigma - \frac{1}{2k}\right) \int_1^\infty \frac{\log x}{x^{\sigma+1-\frac{1}{2k}}}dx \right) = O\left(\int_1^\infty \frac{1}{x^{\sigma+1-\frac{1}{2k}}}dx \right) \\
&= O\left(\frac{1}{\sigma - \frac{1}{2k}} \right). \qquad (2.2.5)
\end{aligned}$$

Then, by (2.2.4) and (2.2.5), we get

$$\left| s\int_1^\infty \frac{M_k(x)}{x^{s+1}}dx \right| = O\left(\frac{|s|}{\sigma - \frac{1}{2k}} \right),$$

which shows that the integral converges uniformly and absolutely for $\sigma \geq \frac{1}{2k} + \delta$, for any fixed $\delta > 0$. The second integral on the right-hand side of (2.2.3) is also uniformly and absolutely convergent for $\sigma \geq \frac{1}{2k}$, and

$$\left|\frac{s}{\zeta(k)}\int_1^\infty \frac{\{x\}}{x^{s+1}}dx\right| \ll 1.$$

Thus, the formula (2.2.3) can be analytically continued to $\Re s > \frac{1}{2k}$, and we get

$$\frac{\zeta(s)}{\zeta(ks)} - \frac{\zeta(s)}{\zeta(k)} = O\left(\frac{|s|}{\sigma - \frac{1}{2k}}\right) + O(1). \tag{2.2.6}$$

Let ρ be a nontrivial zero of $\zeta(s)$ and $s = \frac{\rho}{k} + \frac{h}{k}$, where $h > 0$ is a small real positive number. Then by (2.2.6),

$$\frac{\zeta(\frac{\rho}{k}+\frac{h}{k})}{\zeta(\rho+h)} = O\left(\frac{|\frac{\rho}{k}+\frac{h}{k}|}{\frac{h}{k}}\right) + O\left(\frac{|\zeta(\frac{\rho}{k}+\frac{h}{k})|}{|\zeta(k)|}+1\right) = O\left(\frac{|\rho+h|}{h}\right) + O\left(\frac{|\zeta(\frac{\rho}{k}+\frac{h}{k})|}{|\zeta(k)|}+1\right). \tag{2.2.7}$$

This would be false for $h \to 0$ if ρ is not a simple zero. Multiplying by h on both sides of (2.2.7), and letting $h \to 0$, we get

$$\frac{\zeta(\frac{\rho}{k})}{\zeta'(\rho)} = O(|\rho|),$$

where the constant in big O depends on k.

For any $\epsilon > 0$, by (2.2.2), $\zeta(\frac{\rho}{k}) \gg |\rho|^{\frac{1}{2}-\frac{1}{2k}-\frac{\epsilon}{2}}$. Then, we have $\frac{1}{\zeta'(\rho)} = O\left(|\rho|^{\frac{1}{2}+\frac{1}{2k}+\frac{\epsilon}{2}}\right)$. Taking $k = \left[\frac{1}{\epsilon}\right] + 2$, hence we get $\frac{1}{\zeta'(\rho)} = O\left(|\rho|^{\frac{1}{2}+\epsilon}\right)$. □

Proof of Lemma 2.7. By the symmetry of $\zeta(s)$, the sum over the zeros in the following formula is actually real. We have

$$\begin{aligned} 0 \;&\leq\; \int_1^X \left(\frac{M_k(x)}{x^{\frac{1}{2k}}} - \sum_{|\gamma|<T}\frac{\zeta(\frac{\rho}{k})x^{\frac{\rho}{k}-\frac{1}{2k}}}{\rho\zeta'(\rho)}\right)^2\frac{dx}{x} \\ &=\; \int_1^X \left(\frac{M_k(x)}{x^{\frac{1}{2k}}}\right)^2\frac{dx}{x} + \sum_{|\gamma|,|\gamma'|<T}\frac{\zeta(\frac{\rho}{k})\zeta(\frac{\rho'}{k})}{\rho\rho'\zeta'(\rho)\zeta'(\rho')}\int_1^X x^{\frac{\rho+\rho'-1}{k}}\frac{dx}{x} \\ &\quad -2\sum_{|\gamma|<T}\frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)}\int_1^X M_k(x)x^{\frac{\rho-1}{k}}\frac{dx}{x}. \end{aligned} \tag{2.2.8}$$

In the first sum of (2.2.8), the terms with $\rho' = 1-\rho$ contribute

$$\sum_{|\gamma|<T} \frac{\zeta(\frac{\rho}{k})\zeta(\frac{1-\rho}{k})}{\rho(1-\rho)\zeta'(\rho)\zeta'(1-\rho)} \int_1^X \frac{dx}{x} = \log X \sum_{|\gamma|<T} \frac{|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2}. \tag{2.2.9}$$

For the remaining terms, with $\rho = \frac{1}{2}+i\gamma$, $\rho' = \frac{1}{2}+i\gamma'$, and $\gamma' \neq -\gamma$,

$$\int_1^X x^{\frac{\rho+\rho'-1}{k}} \frac{dx}{x} = \frac{k(X^{\frac{\rho+\rho'-1}{k}}-1)}{\rho+\rho'-1} = O\left(\frac{k}{|\gamma+\gamma'|}\right). \tag{2.2.10}$$

Thus, the sum of these terms is less than a constant $K_1 = K_1(k,T)$.

In the last sum of (2.2.8),

$$\int_1^X M_k(x) x^{\frac{\rho-1}{k}} \frac{dx}{x} = \int_1^X M_k(x) x^{\frac{\rho-1}{k}} \left(1-\frac{x}{X}\right)\frac{dx}{x} + \frac{1}{X}\int_1^X M_k(x) x^{\frac{\rho-1}{k}} dx. \tag{2.2.11}$$

By the assumption (1.1.4), the last term is

$$\begin{aligned} \frac{1}{X}\int_1^X M_k(x) x^{\frac{\rho-1}{k}} dx &= O\left(\frac{1}{X}\int_1^X |M_k(x)| x^{-\frac{1}{2k}} dx\right) = O\left(\frac{1}{X}\int_1^X \frac{|M_k(x)|}{x^{\frac{1}{2k}}}\frac{1}{\sqrt{x}}\sqrt{x}dx\right) \\ &= O\left(\frac{1}{X}\left(\int_1^X \left(\frac{M_k(x)}{x^{\frac{1}{2k}}}\right)^2 \frac{dx}{x}\right)^{\frac{1}{2}} \left(\int_1^X x dx\right)^{\frac{1}{2}}\right) = O(\sqrt{\log X}). \end{aligned} \tag{2.2.12}$$

For the first term in (2.2.11),

$$\begin{aligned} \int_1^X M_k(x) x^{\frac{\rho-1}{k}} \left(1-\frac{x}{X}\right)\frac{dx}{x} &= \int_1^X \left(\sum_{n\le x}\mu_k(n) - \frac{x}{\zeta(k)}\right) x^{\frac{\rho-1}{k}} \left(1-\frac{x}{X}\right)\frac{dx}{x} \\ &= \int_1^X \left(\sum_{n\le x}\mu_k(n)\right) x^{\frac{\rho-1}{k}} \left(1-\frac{x}{X}\right)\frac{dx}{x} - \frac{1}{\zeta(k)}\int_1^X \left(1-\frac{x}{X}\right) x^{\frac{\rho-1}{k}} dx. \end{aligned} \tag{2.2.13}$$

For the last integral,

$$\frac{1}{\zeta(k)}\int_1^X \left(1-\frac{x}{X}\right) x^{\frac{\rho-1}{k}} dx = \frac{k^2}{\zeta(k)} \frac{X^{\frac{\rho-1}{k}+1}-1}{(\rho-1+k)(\rho-1+2k)} - \frac{k}{\zeta(k)(\rho-1+2k)}\left(1-\frac{1}{X}\right). \tag{2.2.14}$$

For the first integral in (2.2.13), we show that

$$\int_1^X \left(\sum_{n\le x}\mu_k(n)\right) x^{\frac{\rho-1}{k}} \left(1-\frac{x}{X}\right)\frac{dx}{x} = \frac{1}{2\pi i}\int_{2k-i\infty}^{2k+i\infty} k^2 \frac{\zeta(\frac{w}{k})}{\zeta(w)} \frac{X^{\frac{w+\rho-1}{k}}-1}{w(w+\rho-1+k)(w+\rho-1)} dw. \tag{2.2.15}$$

To prove this, we use the following formula, for $a > \frac{1}{2}$, $y > 0$,

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{y^s}{s(s+\rho-1+k)(s+\rho-1)} ds = \begin{cases} \frac{1}{k}\left(\frac{y^{1-\rho}-1}{1-\rho} + \frac{y^{1-\rho-k}-1}{\rho-1+k}\right), & \text{for } y > 1; \\ 0, & \text{for } 0 < y \le 1. \end{cases} \tag{2.2.16}$$

In fact, for $y > 1$, we shift the path of the integration to the left as far as possible and calculate the residues at $s = 0$, $s = 1-\rho$, and $s = 1-\rho-k$. Then we get the first result. For $0 < y < 1$, we shift the path of the integration to the right as far as possible. Since there are no poles to the right of the line $\Re s = a > \frac{1}{2}$ and $0 < y < 1$, the integral is zero. For $y = 1$, by the continuity of y on both sides of (2.2.16), we see that the integral equals zero.

By the abosolute convergence of $\frac{\zeta(\frac{w}{k})}{\zeta(w)} = \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^{\frac{w}{k}}}$, we substitute it on the right hand side of (2.2.15) and integrate term by term. By (2.2.16), taking $y = \left(\frac{X}{n}\right)^{\frac{1}{k}}$, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\mu_k(n)}{2\pi i}\int_{2k-i\infty}^{2k+i\infty} \frac{k^2}{n^{\frac{w}{k}}} \frac{X^{\frac{w+\rho-1}{k}}}{w(w+\rho-1+k)(w+\rho-1)} dw \\ &= \sum_{n\le X}\mu_k(n)\left(k\frac{X^{\frac{\rho-1}{k}}-n^{\frac{\rho-1}{k}}}{\rho-1} - \frac{k}{\rho-1+k}\frac{X^{\frac{\rho-1}{k}+1}-n^{\frac{\rho-1}{k}+1}}{X}\right) \\ &= \sum_{n\le X}\mu_k(n)\int_n^X \left(\frac{x^{\frac{\rho-1}{k}}}{x} - \frac{x^{\frac{\rho-1}{k}}}{X}\right) dx \\ &= \int_1^X \left(\sum_{n\le x}\mu_k(n)\right) x^{\frac{\rho-1}{k}}\left(1-\frac{x}{X}\right)\frac{dx}{x}. \end{aligned}$$

And taking $y = \left(\frac{1}{n}\right)^{\frac{1}{k}}$, we get, for all $n \geq 1$,

$$\int_{2k-i\infty}^{2k+i\infty} \frac{k^2}{n^{\frac{w}{k}}} \frac{1}{w(w+\rho-1+k)(w+\rho-1)} dw = 0.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{\mu_k(n)}{2\pi i} \int_{2k-i\infty}^{2k+i\infty} \frac{k^2}{n^{\frac{w}{k}}} \frac{X^{\frac{w+\rho-1}{k}} - 1}{w(w+\rho-1+k)(w+\rho-1)} dw = \int_1^X \left(\sum_{n \leq x} \mu_k(n) \right) x^{\frac{\rho-1}{k}} \left(1 - \frac{x}{X}\right) \frac{dx}{x},$$

which is the left-hand side of (2.2.15).

Let $U > T$ such that U is not an ordinate of a zero of $\zeta(s)$. Then the right-hand side of (2.2.15) is equal to

$$\frac{1}{2\pi i} \left(\int_{2k-i\infty}^{2k-iU} + \int_{2k-iU}^{\frac{1}{4}-iU} + \int_{\frac{1}{4}-iU}^{\frac{1}{4}+iU} + \int_{\frac{1}{4}+iU}^{2k+iU} + \int_{2k+iU}^{2k+i\infty} \right) + \text{sum of residues in } -U < \Im w < U. \tag{2.2.17}$$

Let ρ'' be a generic zero of $\zeta(s)$ with $|\gamma''| < U$. Since $U > T$, $w = 1 - \rho$ is a pole with residue

$$\frac{\zeta(\frac{1-\rho}{k})}{(1-\rho)\zeta'(1-\rho)} \log X. \tag{2.2.18}$$

There is also a pole at $w = k$ with residue

$$\frac{k^2}{\zeta(k)} \frac{X^{\frac{\rho-1}{k}+1} - 1}{(\rho-1+k)(\rho-1+2k)}. \tag{2.2.19}$$

The residue at other ρ'' is

$$\text{Res}(\rho'') = k^2 \frac{\zeta(\frac{\rho''}{k})}{\zeta'(\rho'')} \frac{X^{\frac{\rho''+\rho-1}{k}} - 1}{\rho''(\rho''+\rho-1+k)(\rho''+\rho-1)}.$$

By Lemma 2.6,

$$\frac{\zeta(\frac{\rho''}{k})}{\zeta'(\rho'')} = O_k(|\rho''|).$$

Then,

$$\operatorname{Res}(\rho'') = O_k\left(\frac{1}{|(\rho''+\rho-1+k)(\rho''+\rho-1)|}\right) = O_k\left(\frac{1}{|\gamma+\gamma''|^2}\right). \tag{2.2.20}$$

Thus, since $|\gamma| < T$,

$$\sum_{\substack{-U<\gamma''<U\\ \gamma''\neq-\gamma}} \frac{1}{|\gamma''+\gamma|^2} \le \sum_{\gamma''\neq-\gamma} \frac{1}{|\gamma''+\gamma|^2} < K_2(T). \tag{2.2.21}$$

In the following, we estimate those five integrals in (2.2.17). First, we have

$$\begin{aligned}
&\int_{2k+iU}^{2k+i\infty} k^2 \frac{\zeta(\frac{w}{k})}{\zeta(w)} \frac{X^{\frac{w+\rho-1}{k}}-1}{w(w+\rho-1+k)(w+\rho-1)}dw \\
&= O_k\left(X^2\int_U^\infty \frac{dv}{v(v+\gamma)^2}\right) = O_k\left(\frac{X^2}{U(U+\gamma)}\right) = O_k\left(\frac{X^2}{U(U-T)}\right). \qquad (2.2.22)
\end{aligned}$$

Similarly, we get the same estimate for the integral over $(2k-i\infty, 2k-iU)$. Next,

$$\begin{aligned}
&\int_{\frac14-iU}^{\frac14+iU} k^2 \frac{\zeta(\frac{w}{k})}{\zeta(w)} \frac{X^{\frac{w+\rho-1}{k}}-1}{w(w+\rho-1+k)(w+\rho-1)}dw \\
&= \left(\int_{\frac14-iT}^{\frac14+iT} + \int_{\frac14-iU}^{\frac14-iT} + \int_{\frac14+iT}^{\frac14+iU}\right) k^2 \frac{\zeta(\frac{w}{k})}{\zeta(w)} \frac{X^{\frac{w+\rho-1}{k}}-1}{w(w+\rho-1+k)(w+\rho-1)}dw \\
&= K_3'(k,T) + O\left(\int_T^U \frac{dv}{v^{\frac12}(v+\gamma)^2}\right) = K_3'(k,T) + O(1) \le K_3(k,T). \qquad (2.2.23)
\end{aligned}$$

By Lemma 2.2, we choose $n \le U = U_n \le n+1$ so that

$$\frac{1}{\zeta(\sigma+iU)} = O(|U|^\epsilon), \text{ for } \frac14 \le \sigma \le 2.$$

and we have $\zeta(\frac{\sigma+iU}{k}) = O(U^{\frac{1}{2}+\epsilon})$. Then, we get

$$\int_{\frac{1}{4}+iU}^{2k+iU} k^2 \frac{\zeta(\frac{w}{k})}{\zeta(w)} \frac{X^{\frac{w+\rho-1}{k}}-1}{w(w+\rho-1+k)(w+\rho-1)} dw = O_k\left(\frac{X^2}{U^{\frac{1}{2}-\epsilon}(U+\gamma)^2}\right) = O_k\left(\frac{X^2}{U^{\frac{1}{2}-\epsilon}(U-T)^2}\right). \tag{2.2.24}$$

We have similar estimates for the integral over $(2k - iU, \frac{1}{4} - iU)$.

Combining (2.2.13)-(2.2.24), and making $U \to \infty$, we get

$$\int_1^X M_k(x) x^{\frac{\rho-1}{k}} \left(1 - \frac{x}{X}\right) \frac{dx}{x} = \frac{\zeta(\frac{1-\rho}{k})}{(1-\rho)\zeta'(1-\rho)} \log X + R, \tag{2.2.25}$$

where $|R| < K_4(k,T)$ if $|\gamma| < T$.

By (2.2.8)-(2.2.12) and (2.2.25), and by assumption (1.1.4), we deduce that

$$0 \le A_k \log X + \log X \sum_{|\gamma|<T} \frac{|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2} - 2\log X \sum_{|\gamma|<T} \frac{|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2} + A'_k\sqrt{\log X} + K(k,T),$$

where A_k and A'_k are constants depending only on k. Thus,

$$\sum_{|\gamma|<T} \frac{|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2} \le A_k + \frac{A'_k}{\sqrt{\log X}} + \frac{K(k,T)}{\log X}.$$

Making $X \to \infty$, we have $\sum_{|\gamma|<T} \frac{|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2} \le A_k$. Since the right-hand side of the above result is independent of T, we get the convergence of $\sum_\rho \frac{|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2}$. □

2.3 Proof of Theorems 1.2 and 1.3

2.3.1 Proof of Theorem 1.2: upper bound for $M_k(x)$

By Lemma 2.4, for $X \le x \ll X$, taking $T = X^2$, we have

$$M_k(x) = \sum_{|\gamma|<X^2} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} + O(X^\epsilon). \tag{2.3.1}$$

By Lemma 2.5, we have for $T < X^2$,

$$\int_X^{eX} \left| \sum_{T<|\gamma|<X^2} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} \right|^2 \frac{1}{x^{\frac{1}{k}}} \frac{dx}{x} \ll_{k,\epsilon} \frac{1}{T^{\frac{1}{k}-\epsilon}}.$$

For any $\epsilon > 0$, consider the set

$$S = \left\{ x \geq 2 : \left| \sum_{T<|\gamma|<X^2} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} \right| \geq x^{\frac{1}{2k}} (\log x)^{\frac{1}{2}-\frac{1}{2k}+\epsilon} \right\}.$$

Then,

$$(\log X)^{1-\frac{1}{k}+2\epsilon} \int_{S\cap[X,eX]} \frac{dx}{x} \leq \int_X^{eX} \left| \sum_{T<|\gamma|<X^2} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} \right|^2 \frac{1}{x^{\frac{1}{k}}} \frac{dx}{x} \ll \frac{1}{T^{\frac{1}{k}-\epsilon}}.$$

Taking $T = \log X$, we get

$$\int_{S\cap[X,eX]} \frac{dx}{x} \ll \frac{1}{T^{1+\epsilon}}.$$

Choosing $X = e^l$, $l = 2, 3, \ldots$, we have

$$\int_{S\cap[e^2,\infty]} \frac{dx}{x} \ll \sum_{l=2}^{\infty} \frac{1}{l^{1+\epsilon}} < \infty.$$

Thus, the set S has finite logarithmic measure. By (2.1.18), $J_{-\frac{1}{2}}(T) = \sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|} \ll T^{1+\epsilon}$. For $X \leq x \leq eX$, let $f(t) = \left(\frac{1}{t}\right)^{\frac{1}{2}+\frac{1}{2k}-\epsilon}$. By (2.1.16) and partial summation,

$$\begin{aligned} \left| \sum_{0<|\gamma|\leq T} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} \right| &\ll X^{\frac{1}{2k}} \sum_{0<|\gamma|\leq T} \frac{1}{\gamma^{\frac{1}{2}+\frac{1}{2k}-\epsilon}|\zeta'(\rho)|} \ll X^{\frac{1}{2k}} \left(\frac{J_{-\frac{1}{2}}(T)}{T^{\frac{1}{2}+\frac{1}{2k}-\epsilon}} + \int_1^T f'(t) J_{-\frac{1}{2}}(t) dt \right) \\ &\ll X^{\frac{1}{2k}} T^{\frac{1}{2}-\frac{1}{2k}+\epsilon}. \end{aligned}$$

Thus, by (2.3.1),

$$M_k(x) = \left| \sum_{T<|\gamma|\leq X^2} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} x^{\frac{\rho}{k}} \right| + O\left(X^{\frac{1}{2k}} (\log X)^{\frac{1}{2}-\frac{1}{2k}+\epsilon} \right).$$

Define the set

$$S_H = \left\{ x \geq 2 : |M_k(x)| \geq H x^{\frac{1}{2k}} (\log x)^{\frac{1}{2} - \frac{1}{2k} + \epsilon} \right\}.$$

For $x \in S_H \cap [X, eX]$, we have

$$\begin{aligned} \left| \sum_{T < |\gamma| < X^2} \frac{\zeta(\frac{\rho}{k})}{\rho \zeta'(\rho)} x^{\frac{\rho}{k}} \right| &\geq |M_k(x)| - O\left(X^{\frac{1}{2k}} (\log X)^{\frac{1}{2} - \frac{1}{2k} + \epsilon} \right) \\ &\geq H x^{\frac{1}{2k}} (\log x)^{\frac{1}{2} - \frac{1}{2k} + \epsilon} - O\left(X^{\frac{1}{2k}} (\log X)^{\frac{1}{2} - \frac{1}{2k} + \epsilon} \right) \\ &\geq x^{\frac{1}{2k}} (\log x)^{\frac{1}{2} - \frac{1}{2k} + \epsilon}, \end{aligned}$$

as long as X is sufficiently large and H is larger than the constant in big O. Thus, $S_H \cap [X, eX] \subset S \cap [X, eX]$ for sufficiently large X, and hence S_H has finite logarithmic measure.

□

2.3.2 Proof of Theorem 1.3: B^2-almost periodic function

We define $\phi(y) = e^{-\frac{y}{2k}} M_k(e^y) = \phi^{(T)}(y) + \epsilon^{(T)}(y)$, where

$$\phi^{(T)}(y) = \sum_{|\gamma| \leq T} \frac{\zeta(\frac{\rho}{k})}{\rho \zeta'(\rho)} e^{\frac{i\gamma y}{k}},$$

and, let $Y = \log X$,

$$\epsilon^{(T)}(y) = \sum_{T < |\gamma| \leq e^{2Y}} \frac{\zeta(\frac{\rho}{k})}{\rho \zeta'(\rho)} e^{\frac{i\gamma y}{k}} + e^{-\frac{y}{2k}} E(e^y, e^{2Y}), \tag{2.3.2}$$

where $E(x, T)$ is defined in Lemma 2.4. Note that

$$\int_{\log 2}^{Y} |e^{-\frac{y}{2k}} E(e^y, e^{2Y})|^2 dy \ll \int_{\log 2}^{Y} \frac{y^2 e^{(2-\frac{1}{k})y}}{e^{4Y}} + \frac{e^{(2-\frac{1}{k})y}}{e^{2(1-\epsilon)Y}} + \frac{1}{e^{4(\frac{1}{2k}-\epsilon)Y}} + \frac{1}{e^{\frac{y}{k}}} dy \ll 1.$$

For $k \geq 2$, taking $\theta = 1+\epsilon$ in Lemma 2.5, and by (2.3.2), we have

$$\limsup_{Y\to\infty} \frac{1}{Y}\int_0^Y \left|\phi(y) - \sum_{|\gamma|\leq T} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} e^{\frac{i\gamma y}{k}}\right|^2 dy = \limsup_{Y\to\infty}\frac{1}{Y}\left(\int_0^{\log 2} + \int_{\log 2}^Y\right)|\epsilon^{(T)}(y)|^2 dy$$
$$\ll \limsup_{Y\to\infty}\frac{1}{Y}\int_{\log 2}^Y \left|\sum_{T\leq\gamma\leq e^{2Y}} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} e^{\frac{i\gamma y}{k}}\right|^2 dy + \lim_{Y\to\infty}\frac{1}{Y}\int_{\log 2}^Y |e^{-\frac{y}{2k}}E(e^y, e^{2Y})|^2 dy$$
$$\ll \limsup_{Y\to\infty}\frac{1}{Y}\sum_{j=0}^{[Y]}\int_{\log 2+j}^{\log 2+j+1}\left|\sum_{T\leq\gamma\leq e^{2Y}} \frac{\zeta(\frac{\rho}{k})}{\rho\zeta'(\rho)} e^{\frac{i\gamma y}{k}}\right|^2 dy \ll \frac{1}{T^{\frac{1}{k}-\epsilon}}.$$

Then, by (1.1.5), we see that $\phi(y) = e^{-\frac{y}{2k}}M_k(e^y)$ is a B^2-almost periodic function. Thus, by the work of Besicovitch (see [5], Chapter II of [6], or Theorem 1.14 of [2]), we get the conclusion of our theorem. □

2.4 Applications of LI: estimates on the tail of the limiting distribution

In this section, we assume the Linear Independence conjecture, and give the proof of Corollary 1.4.2.

Let X be a random variable on the infinite torus $\mathbb{T}^\infty$,

$$X(\boldsymbol{\theta}) = \sum_{l=1}^{\infty} r_l \sin 2\pi\theta_l,$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots) \in \mathbb{T}^\infty$ and $r_l \in \mathbb{R}$ for $l \geq 1$. If we assume $\sum_{l=1}^\infty r_l^2 < \infty$, then X converges almost everywhere by Komolgorov's theorem. Let $\mathbf{P}$ be the canonical probability measure on $\mathbb{T}^\infty$. Define

$$\nu_X(x) = \mathbf{P}(X^{-1}(-\infty, x)).$$

Montgomery [46] proved the following result. It also follows from Hoeffding's inequality

[24].

Lemma 2.8. *Let* $X(\boldsymbol{\theta}) = \sum_{l=1}^{\infty} r_l \sin 2\pi\theta_l$ *where* $\sum_{l=1}^{\infty} r_l^2 < \infty$. *For any integer* $K \geq 1$,

$$\mathbf{P}\left(X(\boldsymbol{\theta}) \geq 2\sum_{l=1}^{K} r_l \right) \leq \exp\left(-\frac{3}{4} \left(\sum_{l=1}^{K} r_l \right)^2 \left(\sum_{l>K} r_l^2 \right)^{-1} \right).$$

The linear independence assumption implies that the limiting distribution ν obtained in Theorem 1.4 equals ν_X, where X is the random variable

$$X(\boldsymbol{\theta}) = \sum_{\gamma>0} \frac{2|\zeta(\frac{\rho}{k})|}{|\rho\zeta'(\rho)|} \sin(2\pi\theta_\gamma).$$

Let $r_\gamma = \frac{2|\zeta(\frac{\rho}{k})|}{|\rho\zeta'(\rho)|}$. Define

$$A(T) := \sum_{0<\gamma<T} r_\gamma = \sum_{0<\gamma<T} \frac{2|\zeta(\frac{\rho}{k})|}{|\rho\zeta'(\rho)|}, \quad \text{and} \quad B(T) := \sum_{\gamma\geq T} r_\gamma^2 = \sum_{\gamma\geq T} \frac{4|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2}.$$

By (1.134) in [27] (p. 45), (14.2.6) and (14.5.1) in [59], and the functional equation,

$$\left|\zeta(\frac{\rho}{k})\right| \gg |\gamma|^{\frac{1}{2}-\frac{1}{2k}-\epsilon}, \quad \text{and} \quad |\zeta'(\rho)| \ll |\gamma|^{\epsilon}.$$

So, we deduce that, for $\gamma < T$,

$$r_\gamma = \frac{2|\zeta(\frac{\rho}{k})|}{|\rho\zeta'(\rho)|} \gg_k \frac{1}{|\gamma|^{\frac{1}{2}+\frac{1}{2k}+\epsilon}} \gg \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}+\epsilon}}. \tag{2.4.1}$$

Then, by (2.1.1),

$$A(T) \gg \sum_{0<\gamma<T} \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}+\epsilon}} \gg T^{\frac{1}{2}-\frac{1}{2k}-\epsilon}.$$

Thus, by partial summation, (1.1.3), and the Riemann Hypothesis, we get

$$T^{\frac{1}{2}-\frac{1}{2k}-\epsilon} \ll A(T) \ll T^{\frac{1}{2}-\frac{1}{2k}+\epsilon}, \quad \text{and} \quad \frac{1}{T^{\frac{1}{k}+\epsilon}} \ll B(T) \ll \frac{1}{T^{\frac{1}{k}-\epsilon}}. \tag{2.4.2}$$

Let V be a large parameter. We want to use the above estimates to find upper and lower bounds for the tail of the distribution,

$$\nu([V,\infty)) := \int_V^\infty d\nu(x) = \mathbf{P}(X(\boldsymbol{\theta}) \geq V).$$

2.4.1 The upper bound

Choose T such that $A(T^-) < V \leq A(T)$. We have the inequalities,

$$T^{\frac{1}{2}-\frac{1}{2k}-\epsilon} \ll A(T^-) < V \leq A(T) \ll T^{\frac{1}{2}-\frac{1}{2k}+\epsilon}.$$

From this, we see that

$$V^{\frac{2k}{k-1}-\epsilon} \ll T \ll V^{\frac{2k}{k-1}+\epsilon}. \tag{2.4.3}$$

Then, by Lemma 2.8, (2.4.2) and the above formulas,

$$\begin{aligned}\mathbf{P}\left(X(\boldsymbol{\theta}) \geq c_1 V\right) \leq \mathbf{P}(X(\boldsymbol{\theta}) \geq 2A(T)) &\leq \exp\left(-\frac{3}{4}A(T)^2 B(T)^{-1}\right)\\ &\leq \exp\left(-c_2 V^2 T^{\frac{1}{k}-\epsilon}\right) \leq \exp\left(-c_3 V^{\frac{2k}{k-1}-\epsilon}\right).\end{aligned}$$

2.4.2 The lower bound

Consider the expectation of $e^{\lambda X(\boldsymbol{\theta})}$, $\mathbf{E}(e^{\lambda X(\boldsymbol{\theta})}) = \int e^{\lambda X(\boldsymbol{\theta})} d$. By the definition of $X(\boldsymbol{\theta})$, we know that

$$\mathbf{E}(e^{\lambda X(\boldsymbol{\theta})}) = \prod_{\gamma>0} I(\lambda r_\gamma),$$

where $I(r) = \int_0^1 e^{r \sin 2\pi\theta} d\theta$. Montgomery [46] (formulas (2) and (3), p. 17) showed that

$$I(r) \leq \begin{cases} e^r \\ e^{\frac{r^2}{4}} \end{cases} \quad \text{for all} \quad r \geq 0, \tag{2.4.4}$$

and

$$I(r) > \begin{cases} 2e^{\frac{r}{2}}, & r \geq 7, \\ e^{\frac{r^2}{19}}, & 0 < r \leq 7. \end{cases} \tag{2.4.5}$$

Now we want to find a $\lambda > 0$ so that

$$\mathbf{E}(e^{\lambda X(\boldsymbol{\theta})}) = 2\exp\left(\frac{\lambda}{2}\sum_{0<\gamma<T} r_\gamma\right). \tag{2.4.6}$$

We show that such λ exists. In fact, the two sides of (2.4.6) are continuous functions of λ; for $\lambda = 0$ the left-hand side is 1 while the right-hand side is 2. Moreover, by (2.4.5), if $\lambda > \frac{7}{r_\gamma}$ for all r_γ ($\gamma < T$), then

$$\mathbf{E}(e^{\lambda X(\boldsymbol{\theta})}) \geq \prod_{0<\gamma<T} I(\lambda r_\gamma) \geq 2^{N(T)} \exp\left(\frac{\lambda}{2}\sum_{0<\gamma<T} r_\gamma\right) \geq 2\exp\left(\frac{\lambda}{2}\sum_{0<\gamma<T} r_\gamma\right),$$

where $N(T)$ is the number of zeros of $\zeta(s)$ for $0 < \gamma < T$. Thus, there is such a λ. By (2.4.1), we have, for $\gamma < T$, $r_\gamma \gg \frac{1}{T^{\frac{1}{2}+\frac{1}{2k}+\epsilon}}$. Thus,

$$\lambda \leq c_4 T^{\frac{1}{2}+\frac{1}{2k}+\epsilon}, \tag{2.4.7}$$

for some constant $c_4 > 0$.

To get the lower bound, we need an inequality from [46] (formula (4), p. 19). For any non-negative random variable F,

$$\mathbf{P}\left(F \geq \frac{1}{2}\mathbf{E}(F)\right) \geq \frac{\mathbf{E}(F)^2}{4\mathbf{E}(F^2)}. \tag{2.4.8}$$

Let $F = e^{\lambda X(\boldsymbol{\theta})}$. By (2.4.4),

$$\mathbf{E}(F^2) \leq \exp\left(2\lambda \sum_{0<\gamma<T} r_\gamma + \lambda^2 \sum_{\gamma \geq T} r_\gamma^2\right).$$

So, by the above formula, (2.4.6), and (2.4.8), we have

$$\mathbf{P}\left(X(\boldsymbol{\theta}) \geq \frac{1}{2}\sum_{0<\gamma<T} r_\gamma\right) \geq \frac{\exp(\lambda \sum_{0<\gamma<T} r_\gamma)}{\mathbf{E}(F^2)} \geq \exp\left(-\lambda \sum_{0<\gamma<T} r_\gamma - \lambda^2 \sum_{\gamma\geq T} r_\gamma^2\right).$$

Then, by (2.4.2), (2.4.3), and (2.4.7), we get

$$\mathbf{P}\left(X(\boldsymbol{\theta}) \geq \frac{1}{2}\sum_{0<\gamma<T} r_\gamma\right) \geq \exp\left(-\lambda A(T) - \lambda^2 B(T)\right) \geq \exp\left(-c_5 T^{1+\epsilon}\right) \geq \exp\left(-c_6 V^{\frac{2k}{k-1}+\epsilon}\right).$$

Hence, for any $\epsilon > 0$,

$$\exp\left(-\widetilde{c}_1 V^{\frac{2k}{k-1}+\epsilon}\right) \leq \nu([V,\infty)) \leq \exp\left(-\widetilde{c}_2 V^{\frac{2k}{k-1}-\epsilon}\right),$$

for some constants $\widetilde{c}_1, \widetilde{c}_2 > 0$ depending on k and ϵ.

2.5 Large Deviation Conjecture and the order of $M_k(x)$

In this section, we examine the tail of the limiting distribution more carefully, and heuristically derive a Large Deviation Conjecture and a conjecture about the maximal order of $M_k(x)$.

First, we give the proof of Theorem 1.5. For any integer $k \geq 2$ and any integer $l \geq 1$, taking $w = \frac{1}{k}$ in Theorem 1.6, and by the functional equation, we get

$$\sum_{0<\gamma\leq T} |\zeta(\frac{\rho}{k})|^{2l} \asymp T^{2l(\frac{1}{2}-\frac{1}{2k})} T \log T. \tag{2.5.1}$$

By Hölder's inequality,

$$\sum_{0<\gamma\le T}\frac{|\zeta(\frac{\rho}{k})|}{|\zeta'(\rho)|}\le\left(\sum_{0<\gamma\le T}|\zeta(\frac{\rho}{k})|^{2l}\right)^{\frac{1}{2l}}\left(\sum_{0<\gamma\le T}\left|\frac{1}{\zeta'(\rho)}\right|^{\frac{2l}{2l-1}}\right)^{\frac{2l-1}{2l}}.$$

Since $l\ge 1$, $\frac{l}{2l-1}<\frac{3}{2}$. Thus, by (2.5.1) and (1.1.1),

$$\sum_{0<\gamma\le T}\frac{|\zeta(\frac{\rho}{k})|}{|\zeta'(\rho)|}\ll_l T^{\frac{1}{2}-\frac{1}{2k}}\cdot T(\log T)^{\frac{l}{4l-1}}. \tag{2.5.2}$$

Also, for any $0<\delta<1$, by Hölder's inequality, we have

$$\sum_{\delta T\le\gamma\le T}\frac{1}{|\zeta'(\rho)|^{\frac{2l}{2l+1}}}=\sum_{\delta T\le\gamma\le T}\left|\frac{\zeta(\frac{\rho}{k})}{\zeta'(\rho)}\right|^{\frac{2l}{2l+1}}\cdot\frac{1}{|\zeta(\frac{\rho}{k})|^{\frac{2l}{2l+1}}}\le\left(\sum_{\delta T\le\gamma\le T}\frac{|\zeta(\frac{\rho}{k})|}{|\zeta'(\rho)|}\right)^{\frac{2l}{2l+1}}\left(\sum_{\delta T\le\gamma\le T}\frac{1}{|\zeta(\frac{\rho}{k})|^{2l}}\right)^{\frac{1}{2l+1}}. \tag{2.5.3}$$

By (1.1.1), there exists a small enough $\delta>0$ such that

$$\sum_{\delta T\le\gamma\le T}\frac{1}{|\zeta'(\rho)|^{\frac{2l}{2l+1}}}\asymp T(\log T)^{\left(\frac{l+1}{2l+1}\right)^2}.$$

For such δ, by Theorem 1.6 and the functional equation,

$$\sum_{\delta T\le\gamma\le T}\frac{1}{|\zeta(\frac{\rho}{k})|^{2l}}\asymp\frac{T\log T}{T^{(\frac{1}{2}-\frac{1}{2k})\cdot 2l}}.$$

So, by (2.5.3), we deduce that

$$\sum_{0<\gamma\le T}\frac{|\zeta(\frac{\rho}{k})|}{|\zeta'(\rho)|}\ge\sum_{\delta T\le\gamma\le T}\frac{|\zeta(\frac{\rho}{k})|}{|\zeta'(\rho)|}\gg_l T\cdot T^{\frac{1}{2}-\frac{1}{2k}}(\log T)^{\frac{l}{2(2l+1)}}. \tag{2.5.4}$$

Thus, by (2.5.2) and (2.5.4), we get

$$T^{\frac{1}{2}-\frac{1}{2k}}\cdot T(\log T)^{\frac{1}{4}-o(1)}\ll\sum_{0<\gamma\le T}\frac{|\zeta(\frac{\rho}{k})|}{|\zeta'(\rho)|}\ll T^{\frac{1}{2}-\frac{1}{2k}}\cdot T(\log T)^{\frac{1}{4}+o(1)}.$$

Similarly, we have

$$\sum_{0<\gamma\le T} \frac{|\zeta(\frac{\rho}{k})|^2}{|\zeta'(\rho)|^2} \ll_l T^{1-\frac{1}{k}} \cdot T(\log T)^{\frac{1}{l-1}} \ll T^{1-\frac{1}{k}} \cdot T(\log T)^{o(1)}.$$

Then, by partial summation,

$$T^{\frac{1}{2}-\frac{1}{2k}} \cdot (\log T)^{\frac{1}{4}-o(1)} \ll A(T) = \sum_{0<\gamma<T} \frac{2|\zeta(\frac{\rho}{k})|}{|\rho\zeta'(\rho)|} \ll T^{\frac{1}{2}-\frac{1}{2k}} (\log T)^{\frac{1}{4}+o(1)}, \tag{2.5.5}$$

and

$$B(T) = \sum_{\gamma\ge T} \frac{4|\zeta(\frac{\rho}{k})|^2}{|\rho\zeta'(\rho)|^2} \ll \frac{(\log T)^{o(1)}}{T^{\frac{1}{k}}}. \tag{2.5.6}$$

We choose T such that $A(T^-) < V \le A(T)$. Then, by (2.5.5),

$$T^{\frac{1}{2}-\frac{1}{2k}} (\log T)^{\frac{1}{4}-o(1)} \ll V \ll T^{\frac{1}{2}-\frac{1}{2k}} (\log T)^{\frac{1}{4}+o(1)}.$$

So, we have

$$\left(\frac{V}{(\log V)^{\frac{1}{4}+o(1)}}\right)^{\frac{2k}{k-1}} \ll T \ll \left(\frac{V}{(\log V)^{\frac{1}{4}-o(1)}}\right)^{\frac{2k}{k-1}}.$$

Then, by (2.5.5), (2.5.6), and Lemma 2.8, we get (1.1.7). Hence, Theorem 1.5 follows.

By (2.5.5) and (2.5.6), the conjectured formulas are

$$A(T) \asymp T^{\frac{1}{2}-\frac{1}{2k}} (\log T)^{\frac{1}{4}}, \quad \text{and} \quad B(T) \asymp \frac{1}{T^{\frac{1}{k}}}. \tag{2.5.7}$$

Then, by Lemma 2.8, we get

$$\nu([V,\infty)) \ll \exp\left(-c_1' \frac{V^{\frac{2k}{k-1}}}{(\log V)^{\frac{1}{2(k-1)}}}\right). \tag{2.5.8}$$

In Remark 2.4 of [26], the authors mentioned that: if one redefines $J_{-l}(T)$ to exclude these rare points, where $|\zeta'(\frac{1}{2}+i\gamma_n)|$ is very close to zero, then the Random Matrix Theory should still predict the universal behavior. Thus, if we let $\{r_{\gamma'}\}$ be the decreasing sequence after reordering the sequence $\{r_\gamma\}$, we conjecture that we still have similar estimates like (2.5.7), i.e.

$$A'(T) := \sum_{0<\gamma'<T} r_{\gamma'} \asymp T^{\frac{1}{2}-\frac{1}{2k}} (\log T)^{\frac{1}{4}}, \quad \text{and} \quad B'(T) := \sum_{\gamma' \geq T} r_{\gamma'}^2 \asymp \frac{1}{T^{\frac{1}{k}}}. \tag{2.5.9}$$

Moreover, Hattori and Matsumoto ([22], Theorem 4) proved an equivalent condition in terms of A' and B' for the existence of the lower bound of Montgomery type (the type of lower bound in Theorem 1 of [46]). Hence, by Theorem 1 in [46], Theorem 4 in [22], and (2.5.9), we conjecture that the upper bound (2.5.8) gives the correct order of $\nu([V,\infty))$, i.e. the Large Deviation Conjecture (1.1.6).

2.5.1 The maximal order of $M_k(x)$

In this section, we use heuristic analysis similar to Section 4.3 of [51] to derive our conjecture (1.1.8).

Assuming RH, (1.1.3) and LI, we get

$$\lim_{Y\to\infty} \frac{1}{Y} \text{meas} \left\{ y \in [0,Y] \mid M_k(e^y) \geq e^{\frac{y}{2k}} V \right\} = \nu([V,\infty)).$$

For V sufficiently large, there exists a function $f(V)$ such that, if $Y \geq f(V)$,

$$\frac{1}{Y} \text{meas} \left\{ y \in [0,Y] \mid M_k(e^y) \geq e^{\frac{y}{2k}} V \right\} \geq \exp\left(-c_2' \frac{V^{\frac{2k}{k-1}}}{(\log V)^{\frac{2v}{k-1}}} \right).$$

Let

$$V = \alpha (\log Y)^{\frac{k-1}{2k}} (\log_2 Y)^{\frac{v}{k}},$$

where

$$\alpha = \left(\frac{k-1}{2k} \right)^{\frac{v}{k}} \left(\frac{\delta}{c_2'} \right)^{\frac{k-1}{2k}},$$

for $0<\delta<1$. Then, for large Y,

$$\operatorname{meas}\left\{y\in[0,Y]\mid M_k(e^y)e^{-\frac{y}{2k}}\geq\alpha(\log Y)^{\frac{k-1}{2k}}(\log_2 Y)^{\frac{v}{k}}\right\}\gg\exp\left(\log Y-\delta\log Y\cdot b(Y)\right), \tag{2.5.10}$$

where

$$b(Y)=\frac{1}{\left(1+\frac{2v}{k-1}\frac{\log_3 Y}{\log_2 Y}\right)^{\frac{2v}{k-1}}}.$$

Since $0<\delta<1$, and $b(Y)<1$, the left-hand side of (2.5.10) goes to ∞ as $Y\to\infty$. Then, there exists an increasing sequence $\{y_m\}$, $y_m\to\infty$, such that

$$M_k(e^{y_m})e^{-\frac{y_m}{2k}}\geq\alpha(\log y_m)^{\frac{k-1}{2k}}(\log_2 y_m)^{\frac{v}{k}}.$$

Indeed, suppose the above inequality is false. There exists a u_0, such that, for all $u_0\leq y\leq Y$,

$$M_k(e^y)e^{-\frac{y}{2k}}<\alpha(\log y)^{\frac{k-1}{2k}}(\log_2 y)^{\frac{v}{k}}\leq\alpha(\log Y)^{\frac{k-1}{2k}}(\log_2 Y)^{\frac{v}{k}}.$$

Then,

$$\begin{aligned}&\operatorname{meas}\left\{y\in[0,Y]\mid M_k(e^y)e^{-\frac{y}{2k}}\geq\alpha(\log Y)^{\frac{k-1}{2k}}(\log_2 Y)^{\frac{v}{k}}\right\}\\&\quad=\operatorname{meas}\left\{y\in[0,u_0]\mid M_k(e^y)e^{-\frac{y}{2k}}\geq\alpha(\log Y)^{\frac{k-1}{2k}}(\log_2 Y)^{\frac{v}{k}}\right\}\leq u_0.\end{aligned}$$

By (2.5.10) and the above formula,

$$\exp\left(\log Y-\delta\log Y\cdot b(Y)\right)\leq u_0\ll 1,$$

which contradicts the previous assumption that the left-hand side goes to infinity as $Y\to\infty$.

Thus, we obtain

$$\limsup_{y\to\infty} \frac{M_k(e^y)}{e^{\frac{y}{2k}}(\log y)^{\frac{k-1}{2k}}(\log\log y)^{\frac{v}{k}}} \geq \alpha = \left(\frac{k-1}{2k}\right)^{\frac{v}{k}} \left(\frac{\delta}{c_2'}\right)^{\frac{k-1}{2k}}.$$

Letting $\delta \to 1$, we get

$$\limsup_{y\to\infty} \frac{M_k(e^y)}{e^{\frac{y}{2k}}(\log y)^{\frac{k-1}{2k}}(\log\log y)^{\frac{v}{k}}} \geq \left(\frac{k-1}{2k}\right)^{\frac{v}{k}} \left(\frac{1}{c_2'}\right)^{\frac{k-1}{2k}}.$$

Now we consider the upper bound in (1.1.6). We have

$$\nu([V,\infty)) = \mathbf{P}(\boldsymbol{\theta} \in \mathbb{T}^\infty \mid X(\boldsymbol{\theta}) \geq V) \ll \exp\left(-c_1' \frac{V^{\frac{2k}{k-1}}}{(\log V)^{\frac{2v}{k-1}}}\right). \tag{2.5.11}$$

For $\delta' > 1$, define the event

$$A_n = \{\boldsymbol{\theta} \in \mathbb{T}^\infty \mid X(\boldsymbol{\theta}) \geq \beta(\log n)^{\frac{k-1}{2k}}(\log\log n)^{\frac{v}{k}}\},$$

where

$$\beta = \left(\frac{k-1}{k}\right)^{\frac{v}{k}} \left(\frac{\delta'}{c_1'}\right)^{\frac{k-1}{2k}}.$$

Then, by (2.5.11), for sufficiently large n_0,

$$\sum_{n=n_0}^{\infty} \mathbf{P}(A_n) \ll \sum_{n=n_0}^{\infty} \frac{1}{n^{\delta'}} \ll 1.$$

By the Borel-Cantelli lemma,

$$\mathbf{P}(A_n \text{ infinitely often}) = 0.$$

Thus,

$$\limsup_{y\to\infty} \frac{M_k(e^y)}{e^{\frac{y}{2k}}(\log y)^{\frac{k-1}{2k}}(\log\log y)^{\frac{v}{k}}} \leq \beta = \left(\frac{k-1}{k}\right)^{\frac{v}{k}} \left(\frac{\delta'}{c_1'}\right)^{\frac{k-1}{2k}}.$$

Letting $\delta' \to 1$,

$$\limsup_{y\to\infty} \frac{M_k(e^y)}{e^{\frac{y}{2k}}(\log y)^{\frac{k-1}{2k}}(\log\log y)^{\frac{v}{k}}} \le \left(\frac{k-1}{k}\right)^{\frac{v}{k}} \left(\frac{1}{c_1'}\right)^{\frac{k-1}{2k}}.$$

Hence, (1.1.6) suggests that

$$\left(\frac{k-1}{2k}\right)^{\frac{v}{k}} \left(\frac{1}{c_2'}\right)^{\frac{k-1}{2k}} \le \limsup_{y\to\infty} \frac{M_k(e^y)}{e^{\frac{y}{2k}}(\log y)^{\frac{k-1}{2k}}(\log\log y)^{\frac{v}{k}}} \le \left(\frac{k-1}{k}\right)^{\frac{v}{k}} \left(\frac{1}{c_1'}\right)^{\frac{k-1}{2k}},$$

where $v = \frac{1}{4}$.

Therefore, we can make the following conjecture.

Conjecture. There exists a number $C = C_k > 0$, such that

$$\overline{\underline{\lim}}_{x\to\infty} \frac{M_k(x)}{x^{\frac{1}{2k}}(\log\log x)^{\frac{k-1}{2k}}(\log\log\log x)^{\frac{1}{4k}}} = \pm C_k.$$

2.6 Moments of $\zeta(1-w\rho)$ for $0 < w < 1$

In this section, we give the proof of Theorem 1.6.

Let $l \ge 1$ be an integer. Let $c = \frac{1+w}{4w} > \frac{1}{2}$. By the residue theorem,

$$\begin{aligned}
&\sum_{0<\gamma\le T} |\zeta(1-w\rho)|^{2l} \\
&= \frac{1}{2\pi i}\left(\int_{c+i}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+i} + \int_{1-c+i}^{c+i}\right) \zeta^l(1-ws)\zeta^l(1-w(1-s))\frac{\zeta'}{\zeta}(s)ds \\
&=: J_1 + J_2 + J_3 + J_4. \qquad (2.6.1)
\end{aligned}$$

By [8] (page 108), we may assume that T satisfies

$$|\gamma - T| \gg \frac{1}{\log T}, \quad \text{for all ordinates } \gamma$$

and

$$\frac{\zeta'}{\zeta}(\sigma+iT) \ll (\log T)^2, \quad \text{uniformly for all } 1-c \le \sigma \le c.$$

For general T, since $w<1$, under the Riemann Hypothesis, and by (2.1.1), the error is

$$\ll \sum_{T\le\gamma\le T+1} |\zeta(1-w\rho)|^{2l} \ll_{w,l} T^{\epsilon}.$$

Since $0<w<1$ and $c=\frac{1+w}{4w}$, for $1-c\le\sigma\le c$, we have $1-w\sigma \ge 1-wc = 1-\frac{1+w}{4} > \frac{1}{2}$ and $1-w(1-\sigma)\ge 1-wc > \frac{1}{2}$. So under the Riemann Hypothesis,

$$J_2 = \frac{i}{2\pi}\int_{1-c}^{c} \zeta^l(1-w\sigma-iwT)\zeta^l(1-w(1-\sigma)+iwT)\frac{\zeta'}{\zeta}(\sigma+iT)d\sigma \ll_{w,l} T^{\epsilon}. \qquad (2.6.2)$$

Similarly, $J_4 \ll_{w,l} 1$. We relate J_3 to J_1,

$$\begin{aligned} J_3 &= \frac{1}{2\pi}\int_T^1 \zeta^l(1-w(1-c)-iwt)\zeta^l(1-wc+iwt)\frac{\zeta'}{\zeta}(1-c+it)dt \\ &= -\frac{1}{2\pi}\overline{\int_1^T \zeta^l(1-w(1-c)+iwt)\zeta^l(1-wc-iwt)\frac{\zeta'}{\zeta}(1-c-it)dt}. \end{aligned}$$

By the functional equation $\zeta(s)=\chi(s)\zeta(1-s)$, where $\chi(s)=2^s\pi^{s-1}\Gamma(1-s)\sin(\frac{\pi s}{2})$, we find that

$$\begin{aligned} J_3 = &-\frac{1}{2\pi}\overline{\int_1^T \zeta^l(1-w(1-c)+iwt)\zeta^l(1-wc-iwt)\frac{\chi'}{\chi}(1-c-it)dt} \\ &+\frac{1}{2\pi}\overline{\int_1^T \zeta^l(1-w(1-c)+iwt)\zeta^l(1-wc-iwt)\frac{\zeta'}{\zeta}(c+it)dt}, \end{aligned}$$

By Stirling's formula,

$$-\frac{\chi'}{\chi}(1-c-it) = \log\left(\frac{|t|}{2\pi}\right)\left(1+O\left(\frac{1}{|t|}\right)\right).$$

The term $O\left(\frac{1}{|t|}\right)$ contributes to J_3 an amount of $O_{w,l}(T)$. Let

$$K = \frac{1}{2\pi}\int_1^T \zeta^l(1-w(1-c)+iwt)\zeta^l(1-wc-iwt)\log(\frac{t}{2\pi})dt.$$

Then,

$$J_3 = K + \overline{J_1} + O_{w,l}(T). \tag{2.6.3}$$

First, by the residue theorem, we calculate

$$\begin{aligned}
I(T) &:= \int_1^T \zeta^l(1-w(1-c)+iwt)\zeta^l(1-wc-iwt)dt \\
&= \frac{1}{i}\int_{c+i}^{c+iT} \zeta^l(1-w(1-s))\zeta^l(1-ws)ds. \\
&= \frac{1}{i}\left(\int_{c+i}^{\frac{1}{2}+i} + \int_{\frac{1}{2}+i}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{c+iT}\right)\zeta^l(1-w(1-s))\zeta^l(1-ws)ds \\
&= \int_1^T \zeta^l(1-\frac{w}{2}+iwt)\zeta^l(1-\frac{w}{2}-iwt)dt + O_{w,l}(T^\epsilon) \\
&= \frac{1}{w}\int_w^{wT}\left|\zeta(1-\frac{w}{2}+it)\right|^{2l}dt + O_{w,l}(T^\epsilon).
\end{aligned}$$

For $\frac{1}{2} < \sigma < 1$ and $\frac{x}{2} < t < x$, taking $T = x^3$ in the proof of Theorem 13.3 in [59] (page 330), we get

$$\zeta^l(s) = \sum_{n<x}\frac{d_l(n)}{n^s} + O(x^{-\epsilon}).$$

Then, by Montgomery and Vaughan's mean value theorem for Dirichlet polynomials (Lemma 1 in [60], originally due to [47], Corollary 3 to Theorem 2), we deduce that

$$I(T) = \left(\sum_{n=1}^{\infty}\frac{d_l^2(n)}{n^{2-w}}\right)T + O_{w,l}(T^{1-\epsilon}). \tag{2.6.4}$$

By partial summation, we get

$$K = \frac{1}{2\pi}\log(\frac{T}{2\pi})I(T) - \frac{1}{2\pi}\int_1^T\frac{I(t)}{t}dt = \frac{1}{2\pi}\left(\sum_{n=1}^{\infty}\frac{d_l^2(n)}{n^{2-w}}\right)T\log T + O_{w,l}(T). \tag{2.6.5}$$

Now, we estimate J_1. By Theorem 14.5 in [59] (page 341),

$$\frac{\zeta'}{\zeta}(s) = O((\log t)^{2-2\sigma}) \quad \text{uniformly for} \quad \frac{1}{2} < \sigma_0 \leq \sigma \leq \sigma_1 < 1.$$

Thus, by (2.6.4) and Hölder's inequality,

$$|J_1| \ll_{w,l} T(\log T)^{\frac{1}{2}}. \tag{2.6.6}$$

Combining (2.6.1), (2.6.2), (2.6.3), (2.6.5), and (2.6.6), we get (1.1.9). By Theorem 14.25 (A) in [59], under the Riemann Hypothesis, the series $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ is convergent, and its sum is $\frac{1}{\zeta(s)}$, for every s with $\sigma > \frac{1}{2}$. Hence, the same proof also works for negative powers, and we have (1.1.10). □

Chapter 3

Chebyshev's bias for products of k primes

3.1 Formulas for the associated Dirichlet series and origin of the bias

In this section, we study the following Dirichlet series and express them in terms of the Dirichlet L-functions. Let χ be a Dirichlet character, and denote

$$F_{f_k}(s,\chi) := \sum_{f(n)=k} \frac{\chi(n)}{n^s},$$

where $f = \omega$ or Ω, $\omega(n)$ is the number of distinct prime factors of n and $\Omega(n)$ is the number of prime factors of n counted with multiplicity.

3.1.1 Symmetric functions

Let x_1, x_2, $\ldots$ be an infinite collection of indeterminates. We say a formal power series $P(x_1, x_2, \ldots)$ with bounded degree is a *symmetric function* if it is invariant under all finite permutations of the variables x_1, x_2, $\ldots$.

The n-th *elementary symmetric function* $e_n = e_n(x_1, x_2, \ldots)$ is defined by the generating function $\sum_{n=0}^{\infty} e_n z^n = \prod_{i=1}^{\infty}(1 + x_i z)$. Thus, e_n is the sum of all square-free monomials of degree n. Similary, the n-th *homogeneous symmetric function* $h_n = h_n(x_1, x_2, \ldots)$ is defined by the generating function $\sum_{n=0}^{\infty} h_n z^n = \prod_{i=1}^{\infty} \frac{1}{1-x_i z}$. We see that, h_n is the sum of all possible monomials of degree n. And the n-th *power symmetric function* $p_n = p_n(x_1, x_2, \ldots)$

is defined to be $p_n = x_1^n + x_2^n + \cdots$.

The following result is due to Newton or Girard (see [40], Chapter 1, (2.11) and (2.11'), page 23, or [41], Chapter 2, Theorems 2.8 and 2.9).

Lemma 3.1. *For any integer $k \geq 1$,*

$$kh_k = \sum_{n=1}^{k} h_{k-n} p_n, \tag{3.1.1}$$

$$ke_k = \sum_{n=1}^{k} (-1)^{n-1} e_{k-n} p_n. \tag{3.1.2}$$

3.1.2 Formula for $F_{\Omega_k}(s, \chi)$

Let

$$F(s, \chi) := \sum_{p} \frac{\chi(p)}{p^s},$$

the sum being over all prime p. Since

$$\log L(s, \chi) = \sum_{m=1}^{\infty} \sum_{p \text{ prime}} \frac{\chi(p^m)}{mp^{ms}}, \tag{3.1.3}$$

we then have

$$F(s, \chi) = \log L(s, \chi) - \frac{1}{2} \log L(2s, \chi^2) + G(s), \tag{3.1.4}$$

where $G(s)$ is absolutely convergent for $\Re(s) \geq \sigma_0$ for any fixed $\sigma_0 > \frac{1}{3}$. Henceforth, σ_0 will be a fixed abscissa $> \frac{1}{3}$, say $\sigma_0 = 0.34$.

For any complex number s with $\Re(s) \geq \sigma_0 > \frac{1}{3}$, let $x_p = \frac{\chi(p)}{p^s}$ if p is a prime, 0 otherwise. Then, by (3.1.1) in Lemma 3.1, we have the following relation

$$kF_{\Omega_k}(s, \chi) = \sum_{n=1}^{k} F_{\Omega_{k-n}}(s, \chi) F(ns, \chi^n). \tag{3.1.5}$$

For example, for $k = 2$,

$$2F_{\Omega_2}(s, \chi) = F^2(s, \chi) + F(2s, \chi^2).$$

For $k = 3$,

$$\begin{aligned}
3!F_{\Omega_3}(s, \chi) &= 2F_{\Omega_2}(s, \chi)F(s, \chi) + 2F(s, \chi)F(2s, \chi^2) + 2F(3s, \chi^3) \\
&= F^3(s, \chi) + 3F(s, \chi)F(2s, \chi^2) + 2F(3s, \chi^3).
\end{aligned}$$

For $k = 4$,

$$\begin{aligned}
4!F_{\Omega_4}(s, \chi) &= 3!F_{\Omega_3}(s, \chi)F(s, \chi) + 3!F_{\Omega_2}(s, \chi)F(2s, \chi^2) + 3!F(s, \chi)F(3s, \chi^2) + 3!F(4s, \chi^4) \\
&= F^4(s, \chi) + 6F^2(s, \chi)F(2s, \chi^2) + 8F(s, \chi)F(3s, \chi^3) + 6F(4s, \chi^4) \\
&\quad + 3F^2(2s, \chi^2).
\end{aligned}$$

For any integer $l \geq 1$, we define the set

$$S_{m,l}^{(k)} := \{(n_1, \cdots, n_l) \mid n_1 + \cdots + n_l = k - m, 2 \leq n_1 \leq n_2 \leq \cdots \leq n_l, n_j \in \mathbb{N}(1 \leq j \leq l)\}$$

Let $S_m^{(k)} = \bigcup_{l \geq 1} S_{m,l}^{(k)}$. Thus any element of $S_m^{(k)}$ is a partition of $k - m$ with each part ≥ 2. For any $\boldsymbol{n} = (n_1, n_2, \cdots, n_l) \in S_m^{(k)}$, denote

$$F(\boldsymbol{n}s, \chi) := \prod_{j=1}^{l} F(n_j s, \chi^{n_j}).$$

Hence, by (3.1.5) and induction on k, we deduce that

$$k!F_{\Omega_k}(s, \chi) = F^k(s, \chi) + \sum_{m=0}^{k-2} F^m(s, \chi)F_{\boldsymbol{n_m}}(s, \chi), \tag{3.1.6}$$

where $F_{\boldsymbol{n_m}}(s, \chi) = \sum_{\boldsymbol{n} \in S_m^{(k)}} a_m^{(k)}(\boldsymbol{n})F(\boldsymbol{n}s, \chi)$ for some $a_m^{(k)}(\boldsymbol{n}) \in \mathbb{N}$.

3.1.3 Formula for $F_{\omega_k}(s,\chi)$

By definition, we have

$$F_{\omega_k}(s,\chi) = \sum_{\substack{p_1<p_2<\cdots<p_k \\ p_i \text{ prime}}} \prod_{n=1}^{k} \left(\sum_{j=1}^{\infty} \frac{\chi(p_n^j)}{p_n^j} \right).$$

Denote

$$\widetilde{F}(s,\chi) := \sum_{p \text{ prime}} \left(\frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots \right),$$

and for any $u \in \mathbb{N}^+$,

$$\widetilde{F}(s,\chi;u) := \sum_{p \text{ prime}} \left(\frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots \right)^u = \sum_{p \text{ prime}} \sum_{j=u}^{\infty} \left(D_u(j) \frac{\chi(p^j)}{p^{js}} \right),$$

where $D_u(j) = \binom{j-1}{u-1}$ is the number of ways of writing j as sum of u ordered positive integers.

By (3.1.3), we have

$$\widetilde{F}(s,\chi) = \widetilde{F}(s,\chi;1) = \sum_{p \text{ prime}} \sum_{j=1}^{\infty} \frac{\chi(p^j)}{p^{js}} = \log L(s,\chi) + \frac{1}{2} \log L(2s,\chi^2) + \widetilde{G}_1(s), \qquad (3.1.7)$$

and

$$\widetilde{F}(s,\chi;2) = \sum_{p \text{ prime}} \sum_{j=2}^{\infty} (j-1) \frac{\chi(p^j)}{p^{js}} = \log L(2s,\chi^2) + \widetilde{G}_2(s), \qquad (3.1.8)$$

where $\widetilde{G}_1(s)$ and $\widetilde{G}_2(s)$ are absolutely convergent for $\Re(s) \geq \sigma_0$. Moreover, for any fixed $u \geq 3$, $\widetilde{F}(s,\chi;u)$ is absolutely convergent for $\Re(s) \geq \sigma_0$.

For any complex number s with $\Re(s) \geq \sigma_0$, take $x_p = \sum_{j=1}^{\infty} \frac{\chi(p^j)}{p^{js}}$ if p is a prime, 0 otherwise. Then by (3.1.2) in Lemma 3.1, we get the following formula,

$$kF_{\omega_k}(s,\chi) = F_{\omega_{k-1}}(s,\chi)\widetilde{F}(s,\chi) - \sum_{n=2}^{k} (-1)^n F_{\omega_{k-n}}(s,\chi)\widetilde{F}(s,\chi;n). \qquad (3.1.9)$$

For example, for $k = 2$,

$$2F_{\omega_2}(s,\chi) = \widetilde{F}^2(s,\chi) - \widetilde{F}(s,\chi;2).$$

For $k = 3$,

$$\begin{aligned}3!F_{\omega_3}(s,\chi) &= 2F_{\omega_2}(s,\chi)\widetilde{F}(s,\chi) - 2F_{\omega_1}(s,\chi)\widetilde{F}(s,\chi;2) + 2\widetilde{F}(s,\chi;3)\\ &= \widetilde{F}^3(s,\chi) - 3\widetilde{F}(s,\chi)\widetilde{F}(s,\chi;2) + 2\widetilde{F}(s,\chi;3).\end{aligned}$$

For $k = 4$,

$$\begin{aligned}4!F_{\omega_4}(s,\chi) &= 3!F_{\omega_3}(s,\chi)\widetilde{F}(s,\chi) - 3!F_{\omega_2}(s,\chi)\widetilde{F}(s,\chi;2) + 3!\widetilde{F}(s,\chi)\widetilde{F}(s,\chi;3) - 3!\widetilde{F}(s,\chi;4)\\ &= \widetilde{F}^4(s,\chi) - 6\widetilde{F}^2(s,\chi)\widetilde{F}(s,\chi;2) + 8\widetilde{F}(s,\chi)\widetilde{F}(s,\chi;3) - 6\widetilde{F}(s,\chi;4) + 3\widetilde{F}^2(s,\chi;2).\end{aligned}$$

Hence, by (3.1.9) and induction on k, we get

$$k!F_{\omega_k}(s,\chi) = \widetilde{F}^k(s,\chi) + \sum_{m=0}^{k-2} \widetilde{F}^m(s,\chi)\widetilde{F}_{\boldsymbol{n_m}}(s,\chi), \tag{3.1.10}$$

where $\widetilde{F}_{\boldsymbol{n_m}}(s,\chi) = \sum_{\boldsymbol{n}\in S_m^{(k)}} b_m^{(k)}(\boldsymbol{n})\widetilde{F}(\boldsymbol{n}s,\chi)$ for some $b_m^{(k)}(\boldsymbol{n}) \in \mathbb{Z}$, and for any $\boldsymbol{n} = (n_1, \cdots, n_l) \in S_m^{(k)}$, $\widetilde{F}(\boldsymbol{n}s,\chi) := \prod_{j=1}^{l} \widetilde{F}(s,\chi;n_j)$.

3.1.4 Origin of the bias

In this section, we heuristically explain the origin of the bias in our theorems.

In order to get formulas for $\Delta_{\Omega_k}(x;q,a,b)$ and $\Delta_{\omega_k}(x;q,a,b)$, our strategy is to apply Perron's formula to the associated Dirichlet series $F_{\Omega_k}(s,\chi)$ and $F_{\omega_k}(s,\chi)$, then we choose special contours to avoid the singularities of these Dirichlet series. See Section 3.2 for the details.

First, we have a look at the case of counting primes in arithmetic progressions. If we only count primes, by (3.1.4), we have

$$F_{\Omega_1}(s,\chi) = F(s,\chi) = \sum_p \frac{\chi(p)}{p^s} = \log L(s,\chi) - \frac{1}{2}\log L(2s,\chi^2) + G(s).$$

The main contributions for $\Delta_{\Omega_1}(x;q,a,b)$ are from the first two terms,

$$\log L(s,\chi) - \frac{1}{2}\log L(2s,\chi^2).$$

The first term $\log L(s,\chi)$ counts all the primes with weight 1 and prime squares with weight $\frac{1}{2}$. The higher order powers of primes are negligible since they only contribute $O(x^{\frac{1}{3}})$. The singularities of $\log L(s,\chi)$, i.e. the zeros of $L(s,\chi)$, on the critical line contribute the oscillating terms in our result. In our proof, we use special Hankel contours to avoid the singularities of $\log L(s,\chi)$ and extract these oscillating terms (Lemma 3.9). See Sections 3.2 and 3.3 for the details of how to handle these singularities. The second term $-\frac{1}{2}\log L(2s,\chi^2)$ counts the prime squares with weight $-\frac{1}{2}$ and contributes the bias term. When χ is a real character, the point $s = \frac{1}{2}$ is a pole of $L(2s,\chi^2)$, and hence the integration of $-\frac{1}{2}\log L(2s,\chi^2)$ over the Hankel contour around $s = \frac{1}{2}$ contributes a bias term with order of magnitude $\frac{\sqrt{x}}{\log x}$. Using the orthogonality of Dirichlet characters, and the formula $\sum_{\chi \text{ real}}(\overline{\chi}(a) - \overline{\chi}(b)) = N(q,a) - N(q,b)$, we get the expected size of the bias.

If we count all the prime powers with the same weight 1, by (3.1.7), we have

$$F_{\omega_1}(s,\chi) = \widetilde{F}(s,\chi) = \log L(s,\chi) + \frac{1}{2}\log L(2s,\chi^2) + \widetilde{G}_1(s).$$

In this case, the bias is from the second term $\frac{1}{2}\log L(2s,\chi^2)$ for real character χ which counts the prime squares with positive weight $\frac{1}{2}$. This is why the bias is opposite to the case of counting only primes.

For the general case, when we derive the formula for $\Delta_{\Omega_k}(x;q,a,b)$ using analytic methods, by (3.1.6), the main contributions for $F_{\Omega_k}(s,\chi)$ will be from $\frac{1}{k!}F^k(s,\chi)$, which is essentially

$$\frac{1}{k!}\left(\log L(s,\chi) - \frac{1}{2}\log L(2s,\chi^2)\right)^k.$$

In the expansion of the above formula, the term $\frac{1}{k!}\log^k L(s,\chi)$ contributes the oscillating terms (see (3.3.7) and (3.3.11))

$$\frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}\sum_{L(\frac{1}{2}+i\gamma_\chi,\chi)=0}\frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi}.$$

When χ is real, the term

$$\frac{1}{k!}\left(-\frac{1}{2}\log L(2s,\chi^2)\right)^k = \frac{(-1)^k}{k!2^k}\left(\log L(2s,\chi^2)\right)^k$$

contributes a bias term (see (3.3.8) and (3.3.12))

$$\frac{1}{(k-1)!}\frac{(-1)^k}{2^{k-1}}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}.$$

Then summing over all the real characters, we get the expected bias term in our formula for $\Delta_{\Omega_k}(x;q,a,b)$. The factor $\frac{(-1)^k}{2^{k-1}}$ explains why the bias has different directions depending on the parity of k and why the bias decreases as k increases. Other terms with factors of the form $\log^{k-j} L(s,\chi)\log^j(2s,\chi^2)$ for $1\le j\le k-1$ only contribute oscillating terms with lower orders of $\log\log x$ which can be put into the error term in our formula (see Lemma 3.12).

Similarly, for the case of $\Delta_{\omega_k}(x;q,a,b)$, by (3.1.10), the main contributions for $F_{\omega_k}(s,\chi)$ are from

$$\frac{1}{k!}\widetilde{F}^k(s,\chi) = \frac{1}{k!}\left(\log L(s,\chi) + \frac{1}{2}\log L(2s,\chi^2) + \widetilde{G}_1(s)\right)^k.$$

The main terms are from the contributions of the terms $\frac{1}{k!}\log^k L(s,\chi)$ and $\frac{1}{k!}\left(\frac{1}{2}\log L(2s,\chi^2)\right)^k$. Thus, the main oscillating terms are the same as that of $\Delta_{\Omega_k}(x;q,a,b)$, and the bias term has the same size without direction change.

Through the above analysis, we see that the biases are mainly affected by the powers of

$\pm\frac{1}{2}\log L(2s,\chi^2)$ for real characters which count the products of prime squares.

3.2 Use of Hankel contours and main lemmas

Let γ be the imaginary part of a zero of $L(s,\chi)$ in the critical strip.

Lemma 3.2 ([15], Lemma 2.2)**.** *Let χ be a Dirichlet character modulo q. Let $N(T,\chi)$ denote the number of zeros of $L(s,\chi)$ with $0<\Re(s)<1$ and $|\Im(s)|<T$. Then*

1) $N(T,\chi)=O(T\log(qT))$ for $T\geq 1$.

2) $N(T,\chi)-N(T-1,\chi)=O(\log(qT))$ for $T\geq 1$.

3) Uniformly for $s=\sigma+it$ and $\sigma\geq -1$,

$$\frac{L'(s,\chi)}{L(s,\chi)}=\sum_{|\gamma-t|<1}\frac{1}{s-\rho}+O(\log q(|t|+2)). \tag{3.2.1}$$

Lemma 3.3 ([15], Lemma 2.4)**.** *Assume $L(\frac{1}{2},\chi)\neq 0$. For $A\geq 0$ and real $l\geq 0$,*

$$\sum_{\substack{|\gamma_1|,|\gamma_2|\geq A\\ |\gamma_1-\gamma_2|\geq 1}}\frac{\log^l(|\gamma_1|+3)\log^l(|\gamma_2|+3)}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|}\ll_l\frac{(\log(A+3))^{2l+3}}{A+1}.$$

Let

$$\psi_{f_k}(x,\chi):=\sum_{\substack{n\leq x\\ f(n)=k}}\chi(n),$$

where $f=\Omega$ or ω. By Perron's formula ([30], Chapter V, Theorem 1), we have the following lemma.

Lemma 3.4. *For any $T\geq 2$,*

$$\psi_{f_k}(x,\chi)=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}F_{f_k}(s,\chi)\frac{x^s}{s}ds+O\left(\frac{x\log x}{T}+1\right),$$

where $c=1+\frac{1}{\log x}$, and $f=\Omega$ or ω.

Lemma 3.5. *Assume ERH_q. Then, for any $0 < \delta < \frac{1}{6}$ and for all $\chi \neq \chi_0 \bmod q$, there exists a sequence of numbers $\mathcal{T} = \{T_n\}_{n=0}^{\infty}$ satisfying $n \le T_n \le n+1$ such that, for $T \in \mathcal{T}$,*

$$F_{f_k}(\sigma + iT) = O\left(\log^k T\right), \ (\frac{1}{2} - \delta < \sigma < 2)$$

where $f = \Omega$ or ω.

Proof. Using the similar method as in [59] (Theorem 14.16), one can show that, for any $\epsilon > 0$ and for all $\chi \neq \chi_0 \bmod q$, there exists a sequence of numbers $\mathcal{T} = \{T_n\}_{n=0}^{\infty}$ satisfying $n \le T_n \le n+1$ such that, $T_n^{-\epsilon} \ll |L(\sigma + iT_n, \chi)| \ll T_n^{\delta+\epsilon}$, $(\frac{1}{2} - \delta < \sigma < 2)$. Hence, by formulas (3.1.4), (3.1.6), (3.1.7), (3.1.8), and (3.1.10), we get the conclusion of this lemma. □

Let ρ be a zero of $L(s, \chi)$, Δ_ρ be the distance of ρ to the nearest other zero, and $D_\gamma := \min_{T \in \mathcal{T}}(|\gamma - T|)$. For each zero ρ, and $X > 0$, let $\mathcal{H}(\rho, X)$ denote the truncated Hankel contour surrounding the point $s = \rho$ with radius $0 < r_\rho \le \min(\frac{1}{x}, \frac{\Delta_\rho}{3}, \frac{D_\gamma}{2})$, which includes the circle $|s - \rho| = r_\rho$ excluding the point $s = \rho - r_\rho$, and the half-line $(\rho - X, \rho - r]$ traced twice with arguments $+\pi$ and $-\pi$ respectively. Let $\mathcal{H}(\frac{1}{2}, X)$ denote the corresponding Hankel contour surrounding $s = \frac{1}{2}$ with radius $r_0 = \frac{1}{x}$.

Take $\delta = \frac{1}{10}$. By Lemma 3.4, we pull the contour to the left to the line $\Re(s) = \frac{1}{2} - \delta$ using the truncated Hankel contour $\mathcal{H}(\rho, \delta)$ to avoid the zeros of $L(s, \chi)$ and using $\mathcal{H}(\frac{1}{2}, \delta)$ to avoid the point $s = \frac{1}{2}$. See Figure 3.1.

Then we have the following lemma.

Lemma 3.6. *Assume ERH_q, and $L(\frac{1}{2}, \chi) \neq 0$ $(\chi \neq \chi_0)$. Then, for any fixed $k \ge 1$, and $T \in \mathcal{T}$,*

$$\psi_{f_k}(x, \chi) = \sum_{|\gamma| \le T} \frac{1}{2\pi i} \int_{\mathcal{H}(\rho, \delta)} F_{f_k}(s, \chi) \frac{x^s}{s} ds + a(\chi) \frac{1}{2\pi i} \int_{\mathcal{H}(\frac{1}{2}, \delta)} F_{f_k}(s, \chi) \frac{x^s}{s} ds$$
$$+ O\left(\frac{x \log x}{T} + \frac{x (\log T)^k}{T} + x^{\frac{1}{2} - \delta} (\log T)^{k+1}\right),$$

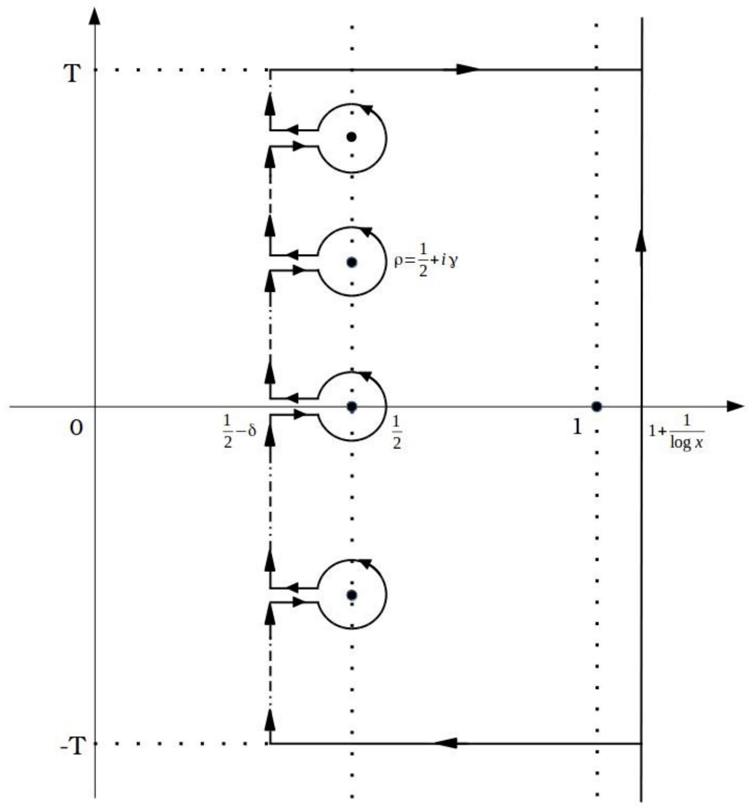

Figure 3.1: Integration contour

where $a(\chi) = 1$ if χ is real, 0 otherwise, and $f = \omega$ or Ω.

Proof. By formulas (3.1.6) and (3.1.10), if χ is not real, $s = \frac{1}{2}$ is not a singularity of $F_{f_k}(s, \chi)$. Hence the second term is zero if χ is not real. By Lemma 3.5, the integral on the horizontal line is

$$I_{ho} \ll (\log T)^k \int_{\frac{1}{2}-\delta}^{c} \frac{x^\sigma}{|\sigma + iT|} d\sigma \ll \frac{x^c (\log T)^k}{T} \ll \frac{x(\log T)^k}{T}. \tag{3.2.2}$$

Under the assumption $\mathrm{ERH_q}$, the integral on the vertical line $\Re(s) = \frac{1}{2} - \delta$ is

$$I_{ve} \ll \int_{-T}^{T} \frac{x^{\frac{1}{2}-\delta} \log^k(|t| + 2)}{|\frac{1}{2} - \delta + it|} dt \ll x^{\frac{1}{2}-\delta} (\log T)^{k+1}. \tag{3.2.3}$$

By (3.2.2), (3.2.3), and Lemma 3.4, we get the desired error term in this lemma. □

Let $\mathcal{H}(0,X)$ be the truncated Hankel contour surrounding 0 with radius r.

Lemma 3.7 ([37], Lemma 5)**.** *For $X>1$, $z\in\mathbb{C}$ and $j\in\mathbb{Z}^+$, we have*

$$\frac{1}{2\pi i}\int_{\mathcal{H}(0,X)} w^{-z}(\log w)^j e^w dw = (-1)^j\frac{d^j}{dz^j}\left(\frac{1}{\Gamma(z)}\right)+E_{j,z}(X),$$

where

$$|E_{j,z}(X)|\le\frac{e^{\pi|\Im(z)|}}{2\pi}\int_X^\infty\frac{(\log t+\pi)^j}{t^{\Re(z)}e^t}dt.$$

For simplicity, we denote

$$\frac{1}{\Gamma_j(u)}:=\left[\frac{d^j}{dz^j}\left(\frac{1}{\Gamma(z)}\right)\right]_{z=u}.$$

Lemma 3.8. *For any integers $k\ge1$ and $m\ge0$, we have*

$$\int_0^\delta|(\log\sigma-i\pi)^k-(\log\sigma+i\pi)^k|\sigma^m x^{-\sigma}d\sigma\ll_{m,k}\frac{(\log\log x)^{k-1}}{(\log x)^{m+1}}.$$

Lemma 3.9. *Let $\mathcal{H}(a,\delta)$ be the truncated Hankel contour surrounding a complex number a ($\Re(a)>2\delta$) with radius $0<r\ll\frac{1}{x}$. Then, for any integer $k\ge1$,*

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathcal{H}(a,\delta)}\log^k(s-a)\frac{x^s}{s}ds\\&=\frac{(-1)^k x^a}{a\log x}\left\{k(\log\log x)^{k-1}+\sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}+O_k\left(\frac{|x^{a-\delta/3}|}{|a|}\right)\\&\quad+O_k\left(\frac{|x^a|}{|a|^2\log^2 x}(\log\log x)^{k-1}\right)+O_k\left(\frac{|x^a|}{|a|^2|\Re(a)-\delta|}\frac{(\log\log x)^{k-1}}{(\log x)^3}\right).\end{aligned}$$

Remark. By (3.5.3) in the proof of Lemma 3.8, one can easily show that

$$\left|\frac{1}{\Gamma_j(0)}\right|\ll\Gamma(j+1).\qquad(3.2.4)$$

Lemma 3.10. *For any integers $N, j \geq 1$, and $0 < |\delta_n| \leq 1$, we have*

$$\int_0^\delta \left|\sum_{n=1}^N \log(\sigma + i\delta_n)\right|^j x^{-\sigma} d\sigma \ll_j \frac{1}{\log x}\left\{\min\left(N \log\log x, \log\frac{1}{\Delta_N}\right) + N\pi\right\}^j,$$

where $\Delta_N = \prod_{n=1}^N |\delta_n|$.

Lemmas 3.8-3.10 are the main lemmas we used, we will give their proof in Section 3.5.

3.3 Proof of Theorems 1.8 and 1.9: products of k primes among different arithmetic progressions

3.3.1 Notations and outline of the proof

By (3.1.4) and (3.1.7), and the assumptions of our theorems, on each truncated Hankel contour $\mathcal{H}(\rho, \delta)$, we write

$$F(s, \chi) = \log(s - \rho) + H_\rho(s), \tag{3.3.1}$$

$$\widetilde{F}(s, \chi) = \log(s - \rho) + \widetilde{H}_\rho(s). \tag{3.3.2}$$

By integration from formula (3.2.1) in Lemma 3.2, we have

$$H_\rho(s) = \sum_{0<|\gamma'-\gamma|\leq 1} \log(s - \rho') + O(\log|\gamma|),$$

$$\widetilde{H}_\rho(s) = \sum_{0<|\gamma'-\gamma|\leq 1} \log(s - \rho') + O(\log|\gamma|).$$

If χ is real, $s = \frac{1}{2}$ is a pole of $L(2s, \chi^2)$. So, by (3.1.4) and (3.1.7), on the truncated Hankel contour $\mathcal{H}(\frac{1}{2}, \delta)$, for a real character χ, we write

$$F(s, \chi) = \frac{1}{2}\log\left(s - \frac{1}{2}\right) + H_B(s), \tag{3.3.3}$$

$$\widetilde{F}(s,\chi) = -\frac{1}{2}\log\left(s-\frac{1}{2}\right) + \widetilde{H}_B(s), \tag{3.3.4}$$

where $H_B(s) = O(1)$ and $\widetilde{H}_B(s) = O(1)$ on $\mathcal{H}(\frac{1}{2},\delta)$.

Denote

$$I_\rho(x) := \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} k! F_{\Omega_k}(s,\chi)\frac{x^s}{s}ds,$$
$$I_B(x) := \frac{1}{2\pi i}\int_{\mathcal{H}(\frac{1}{2},\delta)} k! F_{\Omega_k}(s,\chi)\frac{x^s}{s}ds,$$

and

$$\widetilde{I}_\rho(x) := \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} k! F_{\omega_k}(s,\chi)\frac{x^s}{s}ds,$$
$$\widetilde{I}_B(x) := \frac{1}{2\pi i}\int_{\mathcal{H}(\frac{1}{2},\delta)} k! F_{\omega_k}(s,\chi)\frac{x^s}{s}ds.$$

We define a function $T(x)$ as follows: for $T_{n'} \in \mathcal{T}$ satisfying $e^{2^{n+1}} \leq T_{n'} \leq e^{2^{n+1}} + 1$, let $T(x) = T_{n'}$ for $e^{2^n} \leq x \leq e^{2^{n+1}}$. In particular, we have

$$x \leq T(x) \leq 2x^2 \quad (x \geq e^2).$$

Thus, by Lemma 3.6, for $T = T(x)$,

$$\psi_{\Omega_k}(x,\chi) = \frac{1}{k!}\sum_{|\gamma|\leq T} I_\rho(x) + \frac{a(\chi)}{k!}I_B(x) + O\left(x^{\frac{1}{2}-\frac{\delta}{2}}\right), \tag{3.3.5}$$
$$\psi_{\omega_k}(x,\chi) = \frac{1}{k!}\sum_{|\gamma|\leq T} \widetilde{I}_\rho(x) + \frac{a(\chi)}{k!}\widetilde{I}_B(x) + O\left(x^{\frac{1}{2}-\frac{\delta}{2}}\right). \tag{3.3.6}$$

By (3.1.6) and (3.3.1), we have

$$I_\rho(x) = \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^k \frac{x^s}{s}ds + \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} \sum_{j=1}^{k}\binom{k}{j}(\log(s-\rho))^{k-j}(H_\rho(s))^j\frac{x^s}{s}ds$$

$$+\frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)}\sum_{m=0}^{k-2}F^m(s,\chi)F_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}ds$$

$$=: I_{M_\rho}(x)+E_{M_\rho}(x)+E_{R_\rho}(x), \tag{3.3.7}$$

and by (3.1.6) and (3.3.3),

$$\begin{aligned} I_B(x) &= \frac{1}{2^k}\frac{1}{2\pi i}\int_{\mathcal{H}(\frac{1}{2},\delta)}\left(\log\left(s-\frac{1}{2}\right)\right)^k\frac{x^s}{s}ds \\ &\quad+\frac{1}{2\pi i}\int_{\mathcal{H}(\frac{1}{2},\delta)}\sum_{j=1}^{k}\binom{k}{j}\left(\frac{1}{2}\log\left(s-\frac{1}{2}\right)\right)^{k-j}(H_B(s))^j\frac{x^s}{s}ds \\ &\quad+\frac{1}{2\pi i}\int_{\mathcal{H}(\frac{1}{2},\delta)}\sum_{m=0}^{k-2}F^m(s,\chi)F_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}ds \\ &=: B_M(x)+E_B(x)+E_R(x). \end{aligned} \tag{3.3.8}$$

Similarly, by (3.1.10) and (3.3.2), we have

$$\begin{aligned} \widetilde{I}_\rho(x) &= \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)}(\log(s-\rho))^k\frac{x^s}{s}ds+\frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)}\sum_{j=1}^{k}\binom{k}{j}(\log(s-\rho))^{k-j}(\widetilde{H}_\rho(s))^j\frac{x^s}{s}ds \\ &\quad+\frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)}\sum_{m=0}^{k-2}\widetilde{F}^m(s,\chi)\widetilde{F}_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}ds \\ &=: \widetilde{I}_{M_\rho}(x)+\widetilde{E}_{M_\rho}(x)+\widetilde{E}_{R_\rho}(x), \end{aligned} \tag{3.3.9}$$

and by (3.1.10) and (3.3.4),

$$\begin{aligned} \widetilde{I}_B(x) &= \frac{(-1)^k}{2^k}\frac{1}{2\pi i}\int_{\mathcal{H}(\frac{1}{2},\delta)}\left(\log\left(s-\frac{1}{2}\right)\right)^k\frac{x^s}{s}ds \\ &\quad+\frac{1}{2\pi i}\int_{\mathcal{H}(\frac{1}{2},\delta)}\sum_{j=1}^{k}\binom{k}{j}\left(-\frac{1}{2}\log\left(s-\frac{1}{2}\right)\right)^{k-j}\left(\widetilde{H}_B(s)\right)^j\frac{x^s}{s}ds \\ &\quad+\frac{1}{2\pi i}\int_{\mathcal{H}(\frac{1}{2},\delta)}\sum_{m=0}^{k-2}\widetilde{F}^m(s,\chi)\widetilde{F}_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}ds \\ &=: \widetilde{B}_M(x)+\widetilde{E}_B(x)+\widetilde{E}_R(x). \end{aligned} \tag{3.3.10}$$

Applying Lemma 3.9, we have

$$I_{M_\rho}(x) = \widetilde{I}_{M_\rho}(x) = \frac{(-1)^k\sqrt{x}}{\log x}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}$$
$$+ O\left(\frac{1}{|\gamma|^2}\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right) + O\left(\frac{x^{\frac{1}{2}-\frac{\delta}{3}}}{|\gamma|}\right), \tag{3.3.11}$$

$$B_M(x) = \frac{(-1)^k\sqrt{x}}{2^{k-1}\log x}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}$$
$$+ O\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right) + O\left(x^{\frac{1}{2}-\frac{\delta}{3}}\right), \tag{3.3.12}$$

and

$$\widetilde{B}_M(x) = (-1)^k B_M(x). \tag{3.3.13}$$

Then, by (3.3.7), (3.3.9), and (3.3.11), and Lemma 3.2, for $T = T(x)$,

$$\sum_{|\gamma|\le T} I_\rho(x) = \frac{(-1)^k\sqrt{x}}{\log x}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}$$
$$+ \sum_{|\gamma|\le T} E_{M_\rho}(x) + \sum_{|\gamma|\le T} E_{R_\rho}(x) + O\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{\log^2 x}\right), \tag{3.3.14}$$

and

$$\sum_{|\gamma|\le T} \widetilde{I}_\rho(x) = \frac{(-1)^k\sqrt{x}}{\log x}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}$$
$$+ \sum_{|\gamma|\le T} \widetilde{E}_{M_\rho}(x) + \sum_{|\gamma|\le T} \widetilde{E}_{R_\rho}(x) + O\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{\log^2 x}\right). \tag{3.3.15}$$

For $T = T(x)$, denote

$$\Sigma_1(x;\chi) := \frac{\log x}{\sqrt{x}} \sum_{|\gamma|\le T} E_{M_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} E'_{M_\rho}(x), \tag{3.3.16}$$

$$\Sigma_2(x;\chi) := \frac{\log x}{\sqrt{x}} \sum_{|\gamma|\le T} E_{R_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} E'_{R_\rho}(x), \tag{3.3.17}$$

where $E'_{M_\rho}(x) = \frac{E_{M_\rho}(x)}{x^\rho}$, and $E'_{R_\rho}(x) = \frac{E_{R_\rho}(x)}{x^\rho}$. Similarly, denote

$$\widetilde{\Sigma}_1(x;\chi) := \frac{\log x}{\sqrt{x}} \sum_{|\gamma|\le T} \widetilde{E}_{M_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} \widetilde{E}'_{M_\rho}(x), \tag{3.3.18}$$

$$\widetilde{\Sigma}_2(x;\chi) := \frac{\log x}{\sqrt{x}} \sum_{|\gamma|\le T} \widetilde{E}_{R_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} \widetilde{E}'_{R_\rho}(x), \tag{3.3.19}$$

where $\widetilde{E}'_{M_\rho}(x) = \frac{\widetilde{E}_{M_\rho}(x)}{x^\rho}$, and $\widetilde{E}'_{R_\rho}(x) = \frac{\widetilde{E}_{R_\rho}(x)}{x^\rho}$. Moreover, let

$$S_1(x;\chi) := (-1)^k \sum_{j=2}^{k} \binom{k}{j} (\log\log x)^{k-j} \frac{1}{\Gamma_j(0)} \sum_{|\gamma|\le T} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}, \tag{3.3.20}$$

and for fixed large T_0,

$$S_2(x,T_0;\chi) := \sum_{|\gamma|\le T} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} - \sum_{|\gamma|\le T_0} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}. \tag{3.3.21}$$

In the following subsections, we prove these lemmas below.

Lemma 3.11. *For the bias terms,*

$$I_B(x) = \frac{(-1)^k k}{2^{k-1}} \frac{\sqrt{x}}{\log x} (\log\log x)^{k-1} + O\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right),$$
$$\widetilde{I}_B(x) = \frac{k}{2^{k-1}} \frac{\sqrt{x}}{\log x} (\log\log x)^{k-1} + O\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right).$$

Lemma 3.12. *For the error terms, we have*

$$\int_2^Y \left(|\Sigma_1(e^y;\chi)|^2 + |\Sigma_2(e^y;\chi)|^2\right) dy = o\left(Y(\log Y)^{2k-2}\right),$$
$$\int_2^Y \left(\left|\widetilde{\Sigma}_1(e^y;\chi)\right|^2 + \left|\widetilde{\Sigma}_2(e^y;\chi)\right|^2\right) dy = o\left(Y(\log Y)^{2k-2}\right).$$

Lemma 3.13. *We have*

$$\int_2^Y |S_1(e^y;\chi)|^2\, dy = o\left(Y(\log Y)^{2k-2}\right),$$

and for fixed large T_0*,*

$$\int_2^Y |S_2(e^y, T_0;\chi)|^2 dy \ll Y\frac{\log^2 T_0}{T_0} + \log Y \frac{\log^3 T_0}{T_0} + \log^5 T_0.$$

Combining Lemmas 3.11, 3.12, and 3.13 with (3.3.5), (3.3.6), (3.3.14), and (3.3.15), we get, for fixed large T_0,

$$\psi_{\Omega_k}(x,\chi) = \frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}\left(\sum_{|\gamma|\le T_0}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \Sigma(x,T_0;\chi)\right) + a(\chi)\frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}, \tag{3.3.22}$$

where

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_1^Y |\Sigma(e^y,T_0;\chi)|^2\, dy \ll \frac{\log^2 T_0}{T_0}.$$

Also,

$$\psi_{\omega_k}(x,\chi) = \frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}\left(\sum_{|\gamma|\le T_0}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \widetilde{\Sigma}(x,T_0;\chi)\right) + a(\chi)\frac{1}{(k-1)!}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}, \tag{3.3.23}$$

where

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_1^Y\left|\widetilde{\Sigma}(e^y,T_0;\chi)\right|^2dy \ll \frac{\log^2 T_0}{T_0}.$$

Note that $\sum_{\chi\neq\chi_0}(\overline{\chi}(a)-\overline{\chi}(b))a(\chi) = N(q,a)-N(q,b)$. Hence, combining (3.3.22) and (3.3.23) with (1.2.1) and (1.2.2), we get the conclusions of Theorem 1.8 and Theorem 1.9. □

3.3.2 The bias term

In this section, we give the proof of Lemma 3.11. First, we prove the following result.

Lemma 3.14. *Assume the function $f(s)=O(1)$ on $\mathcal{H}(\frac{1}{2},\delta)$. Then, for any integer $m\geq 0$,*

$$\left|\int_{\mathcal{H}(\frac{1}{2},\delta)}\left(\log\left(s-\frac{1}{2}\right)\right)^m f(s)\frac{x^s}{s}ds\right| \ll \frac{\sqrt{x}(\log\log x)^{m-1}}{\log x}.$$

Proof of Lemma 3.14. Since the left-hand side is 0 when $m=0$, we assume $m\geq 1$ in the following proof. By Lemma 3.8, we have

$$\begin{aligned}
&\left|\int_{\mathcal{H}(\frac{1}{2},\delta)}\left(\log\left(s-\frac{1}{2}\right)\right)^m f(s)\frac{x^s}{s}ds\right| \\
&\leq \left|\int_{r_0}^{\delta}\left((\log\sigma-i\pi)^m-(\log\sigma+i\pi)^m\right)f\left(\frac{1}{2}-\sigma\right)\frac{x^{\frac{1}{2}-\sigma}}{\frac{1}{2}-\sigma}d\sigma\right| \\
&\quad + O\left(\int_{-\pi}^{\pi}\frac{\left(\log\frac{1}{r_0}+\pi\right)^m x^{\frac{1}{2}+r_0}}{\frac{1}{2}-r_0}r_0 d\alpha\right) \\
&\ll \sqrt{x}\left(\int_0^{\delta}\left|(\log\sigma-i\pi)^m-(\log\sigma+i\pi)^m\right|x^{-\sigma}d\sigma+\frac{(\log x+\pi)^m}{x}\right) \\
&\ll \frac{\sqrt{x}(\log\log x)^{m-1}}{\log x}. \qquad (3.3.24)
\end{aligned}$$

This completes the proof of this lemma. □

Proof of Lemma 3.11. Since $H_B(s) = O(1)$, by (3.3.8), (3.3.10), and Lemma 3.14,

$$|E_B(x)| \ll \sum_{j=1}^{k} \left| \int_{\mathcal{H}(\frac{1}{2},\delta)} \left(\log\left(s-\frac{1}{2}\right)\right)^{k-j} (H_B(s))^j \frac{x^s}{s} ds \right| \ll \frac{\sqrt{x}}{\log x} \sum_{j=1}^{k} (\log\log x)^{k-j-1} \ll \frac{\sqrt{x}}{\log x} (\log\log x)^{k-2}. \tag{3.3.25}$$

Similarly,

$$|\widetilde{E}_B(x)| \ll \sum_{j=1}^{k} \left| \int_{\mathcal{H}(\frac{1}{2},\delta)} \left(\log\left(s-\frac{1}{2}\right)\right)^{k-j} \left(\widetilde{H}_B(s)\right)^j \frac{x^s}{s} ds \right| \ll \frac{\sqrt{x}}{\log x} (\log\log x)^{k-2}. \tag{3.3.26}$$

In the following, we estimate $E_R(x)$ in (3.3.8) and $\widetilde{E}_R(x)$ in (3.3.10). If χ is not real, $E_R(x) = \widetilde{E}_R(x) = 0$. If χ is real, by (3.1.4), on $\mathcal{H}(\frac{1}{2}, \delta)$, we write

$$F(2s, \chi^2) = -\log\left(s-\frac{1}{2}\right) + H_2(s). \tag{3.3.27}$$

On $\mathcal{H}(\frac{1}{2}, \delta)$, $|H_2(s)| = O(1)$. By (3.1.6), we have

$$|E_R(x)| \ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n} \in S_m^{(k)}} \left| \int_{\mathcal{H}(\frac{1}{2},\delta)} F^m(s,\chi) F(\boldsymbol{n}s, \chi) \frac{x^s}{s} ds \right|. \tag{3.3.28}$$

For each $0 \le m \le k-2$, we write

$$F^m(s,\chi)F(\boldsymbol{n}s,\chi) = F^m(s,\chi)F^{m'}(2s,\chi^2)G_{\boldsymbol{n}}(s),$$

where $m + 2m' \le k$, and $G_{\boldsymbol{n}}(s) = O(1)$ on $\mathcal{H}(\frac{1}{2}, \delta)$. Thus, by (3.3.3), (3.3.27), and Lemma 3.14,

$$\left| \int_{\mathcal{H}(\frac{1}{2},\delta)} F^m(s,\chi) F(\boldsymbol{n}s, \chi) \frac{x^s}{s} ds \right|$$

$$\ll \left| \int_{\mathcal{H}(\frac{1}{2},\delta)} \left(\log\left(s-\frac{1}{2}\right) + H_B(s) \right)^m \left(\log\left(s-\frac{1}{2}\right) - H_2(s) \right)^{m'} G_{\boldsymbol{n}}(s) \frac{x^s}{s} ds \right|$$
$$\ll \sum_{j_1=0}^{m} \sum_{j_2=0}^{m'} \left| \int_{\mathcal{H}(\frac{1}{2},\delta)} \left(\log\left(s-\frac{1}{2}\right) \right)^{m+m'-j_1-j_2} (H_B(s))^{j_1} (H_2(s))^{j_2} G_{\boldsymbol{n}}(s) \frac{x^s}{s} ds \right|$$
$$\ll \sum_{j_1=0}^{m} \sum_{j_2=0}^{m'} \frac{\sqrt{x}}{\log x} (\log\log x)^{m+m'-j_1-j_2-1} \ll \frac{\sqrt{x}}{\log x} (\log\log x)^{k-2}. \tag{3.3.29}$$

In the last step, we used the conditions $0 \le m \le k-2$ and $m + 2m' \le k$.

Combining (3.3.28) and (3.3.29), we deduce that

$$|E_R(x)| \ll \frac{\sqrt{x}}{\log x} (\log\log x)^{k-2}. \tag{3.3.30}$$

Similarly, if χ is real, by (3.1.8), we write

$$\widetilde{F}(s,\chi;2) = -\log\left(s-\frac{1}{2}\right) + \widetilde{H}_2(s), \tag{3.3.31}$$

where $\widetilde{H}_2(s) = O(1)$ on $\mathcal{H}(\frac{1}{2},\delta)$. Using a similar argument as above, by (3.3.4), (3.3.31), and Lemma 3.14, we have

$$|\widetilde{E}_R(x)| \ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n}\in S_m^{(k)}} \left| \int_{\mathcal{H}(\frac{1}{2},\delta)} \widetilde{F}^m(s,\chi) \widetilde{F}(\boldsymbol{n}s,\chi) \frac{x^s}{s} ds \right| \ll \frac{\sqrt{x}}{\log x} (\log\log x)^{k-2}. \tag{3.3.32}$$

By (3.3.8), (3.3.12), (3.3.25), and (3.3.30), we get

$$I_B(x) = \frac{(-1)^k \sqrt{x}}{2^{k-1}\log x} \left\{ k(\log\log x)^{k-1} + \sum_{j=2}^{k} \binom{k}{j} (\log\log x)^{k-j} \frac{1}{\Gamma_j(0)} \right\} + O\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x} \right).$$

Then, by (3.2.4),

$$\left| \sum_{j=2}^{k} \binom{k}{j} (\log\log x)^{k-j} \frac{1}{\Gamma_j(0)} \right| \ll (\log\log x)^{k-2}.$$

Hence,

$$I_B(x) = \frac{(-1)^k k}{2^{k-1}} \frac{\sqrt{x}}{\log x}(\log\log x)^{k-1} + O\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right). \tag{3.3.33}$$

Similarly, by (3.3.10), (3.3.13), (3.3.26), and (3.3.32), we have

$$\widetilde{I}_B(x) = \frac{k}{2^{k-1}} \frac{\sqrt{x}}{\log x}(\log\log x)^{k-1} + O\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right). \tag{3.3.34}$$

This completes the proof of Lemma 3.11. □

3.3.3 Proof of Lemma 3.12

In this section, we give the proof of Lemma 3.12. We need the following lemma.

Lemma 3.15. *Let ρ be a zero of $L(s,\chi)$. Assume the function $g(s) \ll (\log|\gamma|)^c$ on $\mathcal{H}(\rho,\delta)$ for some constant $c \geq 0$, and*

$$H_\rho(s) = \sum_{0<|\gamma'-\gamma|\leq 1} \log(s-\rho') + O\left(\log|\gamma|\right) \quad \text{on } \mathcal{H}(\rho,\delta). \tag{3.3.35}$$

For any integers $m, n \geq 0$, denote

$$E(x;\rho) := \int_{\mathcal{H}(\rho,\delta)} \left(\log(s-\rho)\right)^m \left(H_\rho(s)\right)^n g(s) \frac{x^{s-\rho}}{s} ds.$$

Then, for $T = T(x)$, we have

$$\int_2^Y \left| y \sum_{|\gamma|\leq T(e^y)} e^{i\gamma y} E(e^y;\rho)\right|^2 dy = o\left(Y(\log Y)^{2m+2n-2}\right).$$

Before giving the proof of the above lemma, we use it to prove Lemma 3.12 first.

Proof of Lemma 3.12. By (3.3.16), we have

$$|\Sigma_1(x;\chi)|^2 = \left|\log x \sum_{|\gamma|\le T} x^{i\gamma} E'_{M_\rho}(x)\right|^2 \ll \sum_{j=1}^{k} \left|\log x \sum_{|\gamma|\le T} x^{i\gamma} E_{\rho,j}(x)\right|^2, \tag{3.3.36}$$

where

$$E_{\rho,j}(x) = \int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^{k-j} (H_\rho(s))^j \frac{x^{s-\rho}}{s} ds.$$

By Lemma 3.15, taking $m = k-j$, $n = j$, and $g(s) \equiv 1$, (i.e. $c = 0$), we get

$$\int_2^Y \left| y \sum_{|\gamma|\le T(e^y)} e^{i\gamma y} E_{\rho,j}(e^y)\right|^2 dy = o\left(Y(\log Y)^{2k-2}\right).$$

Thus,

$$\int_2^Y |\Sigma_1(e^y;\chi)|^2 dy = o\left(Y(\log Y)^{2k-2}\right). \tag{3.3.37}$$

By definition (3.3.7) and (3.3.17), we have

$$\begin{aligned} |\Sigma_2(x;\chi)|^2 &\ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n}\in S_m^{(k)}} \left|\log x \sum_{|\gamma|\le T} x^{i\gamma} \int_{\mathcal{H}(\rho,\delta)} F^m(s,\chi) F(\boldsymbol{n}s,\chi) \frac{x^{s-\rho}}{s} ds\right|^2 \\ &\ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n}\in S_m^{(k)}} \sum_{j=0}^{m} \left|\log x \sum_{|\gamma|\le T} x^{i\gamma} E_{m,j}(x,\chi;\boldsymbol{n})\right|^2, \end{aligned} \tag{3.3.38}$$

where

$$E_{m,j}(x,\chi;\boldsymbol{n}) = \int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^{m-j} (H_\rho(s))^j F(\boldsymbol{n}s,\chi) \frac{x^{s-\rho}}{s} ds.$$

Since on $\mathcal{H}(\rho,\delta)$, we know $F(\boldsymbol{n}s,\chi) = O\left((\log|\gamma|)^{\frac{k-m}{2}}\right)$, by Lemma 3.15, we get

$$\int_2^Y \left| y \sum_{|\gamma|\le T(e^y)} e^{i\gamma y} E_{m,j}(e^y,\chi;\boldsymbol{n})\right|^2 dy = o\left(Y(\log Y)^{2m-2}\right).$$

Hence, by (3.3.38), we deduce that

$$\int_2^Y |\Sigma_2(e^y;\chi)|^2\,dy = o\left(Y(\log Y)^{2k-2}\right). \tag{3.3.39}$$

Combining (3.3.37) and (3.3.39), we get the first formula in Lemma 3.12.

For $\widetilde{\Sigma}_1(x;\chi)$ and $\widetilde{\Sigma}_2(x,\chi)$, by (3.3.18), using a similar argument with Lemma 3.15,

$$\int_2^Y \left|\widetilde{\Sigma}_1(e^y;\chi)\right|^2 dy \ll \sum_{j=1}^k \int_2^Y \left|y\sum_{|\gamma|\le T(e^y)} e^{i\gamma y}\widetilde{E}_{\rho,j}(e^y)\right|^2 dy = o\left(Y(\log Y)^{2k-2}\right), \tag{3.3.40}$$

where

$$\widetilde{E}_{\rho,j}(x) = \int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^{k-j}(\widetilde{H}_\rho(s))^j \frac{x^{s-\rho}}{s}ds.$$

Similarly, by (3.3.19) and Lemma 3.15,

$$\int_2^Y \left|\widetilde{\Sigma}_2(e^y;\chi)\right|^2 dy \ll \sum_{m=0}^{k-2}\sum_{\boldsymbol{n}\in S_m^{(k)}}\sum_{j=0}^m \int_2^Y \left|y\sum_{|\gamma|\le T(e^y)} e^{i\gamma y}\widetilde{E}_{m,j}(e^y,\chi;\boldsymbol{n})\right|^2 dy = o\left(Y(\log Y)^{2k-2}\right), \tag{3.3.41}$$

where $\widetilde{E}_{m,j}(x,\chi;\boldsymbol{n}) = \int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^{m-j}\left(\widetilde{H}_\rho(s)\right)^j \widetilde{F}(\boldsymbol{n}s,\chi)\frac{x^{s-\rho}}{s}ds$.

Combining (3.3.40) and (3.3.41), we get the second formula in Lemma 3.12. □

Proof of Lemma 3.15. If $m=0$, $E(x;\rho)=0$ and hence the integral is 0. In the following, we assume $m\ge 1$. Let Γ_ρ represent the circle in the Hankel contour $\mathcal{H}(\rho,\delta)$. Then,

$$\begin{aligned}
E(x;\rho) &= \int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^m (H_\rho(s))^n g(s)\frac{x^{s-\rho}}{s}ds \\
&= \int_{r_\rho}^{\delta} ((\log\sigma - i\pi)^m - (\log\sigma + i\pi)^m)\left(H_\rho\left(\frac{1}{2}-\sigma+i\gamma\right)\right)^n g\left(\frac{1}{2}-\sigma+i\gamma\right) \\
&\qquad \times \frac{x^{-\sigma}}{\frac{1}{2}-\sigma+i\gamma}d\sigma + \int_{\Gamma_\rho} (\log(s-\rho))^m (H_\rho(s))^n g(s)\frac{x^{s-\rho}}{s}ds. \\
&=: E_h(x;\rho) + E_r(x;\rho).
\end{aligned} \tag{3.3.42}$$

For the second integral in (3.3.42), since $r_\rho \leq \frac{1}{x}$, by Lemma 3.2,

$$\begin{aligned}
|E_r(x;\rho)| &\ll \frac{(\log|\gamma|)^c r_\rho x^{r_\rho}}{|\gamma|}\left(\log\frac{1}{r_\rho}+\pi\right)^m\left(\sum_{0<|\gamma-\gamma'|\leq 1}\log\left(\frac{1}{|\gamma'-\gamma|-r_\rho}\right)+O(\log|\gamma|)\right)^n \\
&\ll \frac{(\log|\gamma|)^c r_\rho x^{r_\rho}}{|\gamma|}\left(\log\frac{1}{r_\rho}+\pi\right)^m(\log|\gamma|)^n\left(\log\left(\frac{1}{r_\rho}\right)+O(1)\right)^n \\
&\ll \frac{(\log|\gamma|)^{n+c}}{|\gamma|}\frac{(\log(1/r_\rho)+\pi)^{m+n}}{1/r_\rho} \ll \frac{(\log|\gamma|)^{n+c}}{|\gamma|}\frac{1}{x^{1-\epsilon}}. \qquad (3.3.43)
\end{aligned}$$

Denote

$$\Sigma(x;\mathrm{g}) := \left|\sum_{|\gamma|\leq T} x^{i\gamma}E(x;\rho)\right|^2 \ll \left|\sum_{|\gamma|\leq T} x^{i\gamma}E_h(x;\rho)\right|^2 + \left|\sum_{|\gamma|\leq T} x^{i\gamma}E_r(x;\rho)\right|^2. \qquad (3.3.44)$$

By (3.3.43) and $T(x) \ll x^2$, we get

$$\left|\sum_{|\gamma|\leq T} x^{i\gamma}E_r(x;\rho)\right|^2 \ll \frac{1}{x^{2-\epsilon}}\left(\sum_{|\gamma|\leq T(x)}\frac{(\log|\gamma|)^{n+c}}{|\gamma|}\right)^2 \ll \frac{1}{x^{2-\epsilon}}. \qquad (3.3.45)$$

For the first sum in (3.3.44),

$$\begin{aligned}
\left|\sum_{|\gamma|\leq T} x^{i\gamma}E_h(x;\rho)\right|^2 &= \left(\sum_{\substack{|\gamma_1-\gamma_2|\leq 1\\ |\gamma_1|,|\gamma_2|\leq T}} + \sum_{\substack{|\gamma_1-\gamma_2|>1\\ |\gamma_1|,|\gamma_2|\leq T}}\right) x^{i(\gamma_1-\gamma_2)}E_h(x;\rho_1)E_h(x;\overline{\rho}_2) \\
&=: \Sigma_1(x;\mathrm{g}) + \Sigma_2(x;\mathrm{g}).
\end{aligned}$$

By (3.3.42),

$$|E_h(x;\rho)| \ll \frac{(\log|\gamma|)^c}{|\gamma|}\sum_{j=1}^m\int_0^\delta|\log\sigma|^{m-j}\left|H_\rho\left(\frac{1}{2}-\sigma+i\gamma\right)\right|^n x^{-\sigma}d\sigma. \qquad (3.3.46)$$

Let

$$S_j(x) := \int_0^\delta |\log\sigma|^{m-j} \left| H_\rho\left(\frac{1}{2} - \sigma + i\gamma\right)\right|^n x^{-\sigma} d\sigma$$
$$\leq \left(\int_0^\delta |\log\sigma|^{2(m-j)} x^{-\sigma} d\sigma\right)^{\frac{1}{2}} \left(\int_0^\delta \left| H_\rho\left(\frac{1}{2} - \sigma + i\gamma\right)\right|^{2n} x^{-\sigma} d\sigma\right)^{\frac{1}{2}}.$$

By (3.5.5) in the proof of Lemma 3.8,

$$\int_0^\delta |\log\sigma|^{2(m-j)} x^{-\sigma} d\sigma \ll \frac{(\log\log x)^{2(m-j)}}{\log x}. \tag{3.3.47}$$

By condition (3.3.35) and the Cauchy-Schwarz inequality,

$$\left| H_\rho\left(\frac{1}{2} - \sigma + i\gamma\right)\right|^{2n} \ll \left| \sum_{0<|\gamma'-\gamma|\leq 1} \log(\sigma + i(\gamma' - \gamma))\right|^{2n} + (\log|\gamma|)^{2n}.$$

Then, by Lemma 3.10,

$$\int_0^\delta \left| H_\rho\left(\frac{1}{2} - \sigma + i\gamma\right)\right|^{2n} x^{-\sigma} d\sigma \ll \frac{(M_\gamma(x))^{2n} + (\log|\gamma|)^{2n}}{\log x}, \tag{3.3.48}$$

where $M_\gamma(x) = \min\left(N(\gamma)\log\log x, \log\frac{1}{\Delta_{N(\gamma)}}\right)$, $N(\gamma)$ is the number of zeros γ' in the range $0 < |\gamma' - \gamma| \leq 1$, and $\Delta_{N(\gamma)} = \prod_{0<|\gamma'-\gamma|\leq 1} |\gamma' - \gamma|$.

Thus, by (3.3.47) and (3.3.48),

$$S_j(x) \ll \frac{(\log\log x)^{m-j}}{\log x} \left((M_\gamma(x))^n + (\log|\gamma|)^n\right).$$

Substituting this into (3.3.46), we get

$$|E_h(x;\rho)| \ll \frac{(\log|\gamma|)^c}{|\gamma|} \sum_{j=1}^m \frac{(\log\log x)^{m-j}}{\log x} \left((M_\gamma(x))^n + (\log|\gamma|)^n\right)$$
$$\ll \frac{(\log|\gamma|)^c}{|\gamma|} \frac{(\log\log x)^{m-1}}{\log x} \left((M_\gamma(x))^n + (\log|\gamma|)^n\right). \tag{3.3.49}$$

Then, by Lemma 3.2, we have

$$\begin{aligned}
|\Sigma_1(x;\mathrm{g})| &\ll \sum_{|\gamma|\le T} \log(|\gamma|) \left(\max_{|\gamma'-\gamma|<1} |E_h(x;\rho')| \right)^2 \\
&\ll \frac{(\log\log x)^{2(m-1)}}{\log^2 x} \sum_{\gamma} \frac{(\log|\gamma|)^{2c}}{|\gamma|^2} \left((M_\gamma(x))^{2n} + (\log|\gamma|)^{2n} \right) \\
&= \frac{(\log\log x)^{2m+2n-2}}{\log^2 x} o(1).
\end{aligned}$$

Thus, for each positive integer l,

$$\int_{2^l}^{2^{l+1}} \Sigma_1(e^y;\mathrm{g})dy = o\left(\frac{l^{2m+2n-2}}{2^l} \right). \tag{3.3.50}$$

In the following, we examine $\Sigma_2(x;\mathrm{g})$. By (3.3.42),

$$\Sigma_2(x;\mathrm{g}) = \sum_{\substack{|\gamma_1-\gamma_2|>1 \\ |\gamma_1|,|\gamma_2|\le T}} x^{i(\gamma_1-\gamma_2)} E_h(x;\rho_1) E_h(x;\overline{\rho}_2). \tag{3.3.51}$$

For $e^{2^l} \le x \le e^{2^{l+1}}$, $T = T(x) = T_{l'}$ is a constant, and so we define

$$J(x;\mathrm{g}) := \sum_{\substack{|\gamma_1-\gamma_2|>1 \\ |\gamma_1|,|\gamma_2|\le T_{l'}}} x^{i(\gamma_1-\gamma_2)} \int_{r_{\rho_1}}^{\delta} \int_{r_{\overline{\rho}_2}}^{\delta} R_{\rho_1}(\sigma_1;x) R_{\overline{\rho}_2}(\sigma_2;x) \frac{d\sigma_1 d\sigma_2}{i(\gamma_1-\gamma_2)-(\sigma_1+\sigma_2)}, \tag{3.3.52}$$

where

$$R_\rho(\sigma;x) = \left((\log\sigma - i\pi)^m - (\log\sigma + i\pi)^m \right) H_\rho^n\left(\frac{1}{2} - \sigma + i\gamma \right) \frac{g\left(\frac{1}{2}-\sigma+i\gamma\right) x^{-\sigma}}{\frac{1}{2}-\sigma+i\gamma}.$$

Thus,

$$\int_{e^{2^l}}^{e^{2^{l+1}}} \sum_{\substack{|\gamma_1-\gamma_2|>1 \\ |\gamma_1|,|\gamma_2|\le T_{l'}}} x^{i(\gamma_1-\gamma_2)} E_h(x;\rho_1) E_h(x;\overline{\rho}_2) \frac{dx}{x} = J(e^{2^{l+1}};\mathrm{g}) - J(e^{2^l};\mathrm{g}). \tag{3.3.53}$$

By (3.3.52), (3.3.46), and (3.3.49), and Lemma 3.3, for $e^{2^l} \le x \le e^{2^{l+1}}$

$$
\begin{aligned}
|J(x;\mathrm{g})| &\ll \sum_{|\gamma_1-\gamma_2|>1} \frac{(\log|\gamma_1|)^c(\log|\gamma_2|)^c}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|}\left(\frac{(\log\log x)^{m-1}}{\log x}\right)^2 \\
&\quad \times \left((M_{\gamma_1}(x))^n + (\log|\gamma_1|)^n\right)\left((M_{\gamma_2}(x))^n + (\log|\gamma_2|)^n\right) \\
&\ll \frac{(\log\log x)^{2m+2n-2}}{\log^2 x} \sum_{|\gamma_1-\gamma_2|>1} \frac{(\log|\gamma_1|)^{n+c}(\log|\gamma_2|)^{n+c}}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \ll \frac{(\log\log x)^{2m+2n-2}}{\log^2 x}.
\end{aligned}
\tag{3.3.54}
$$

Hence, by (3.3.51), (3.3.53), and (3.3.54), we get, for any positive integer l,

$$
\int_{2^l}^{2^{l+1}} \Sigma_2(e^y;\mathrm{g})dy = o\left(\frac{l^{2m+2n-2}}{2^l}\right). \tag{3.3.55}
$$

Therefore, by (3.3.45), (3.3.50) and (3.3.55),

$$
\begin{aligned}
\int_2^Y \left| y \sum_{|\gamma|\le T(e^y)} e^{i\gamma y} E(e^y;\rho)\right|^2 dy &\ll \sum_{l\le \frac{\log Y}{\log 2}+1} 2^{2l} \int_{2^l}^{2^{l+1}} \Sigma(e^y;\mathrm{g})dy \\
&\ll 1 + \sum_{l\le \frac{\log Y}{\log 2}+1} 2^{2l} \int_{2^l}^{2^{l+1}} \left(\Sigma_1(e^y;\mathrm{g}) + \Sigma_2(e^y;\mathrm{g})\right) dy = o\left(Y(\log Y)^{2m+2n-2}\right).
\end{aligned}
$$

This completes the proof of Lemma 3.15. □

3.3.4 Proof of Lemma 3.13

By (3.3.20),

$$
\int_2^Y |S_1(e^y;\chi)|^2\, dy \ll \sum_{j=2}^k (\log Y)^{2k-2j} \int_2^Y \left|\sum_{|\gamma|\le T(e^y)} \frac{e^{i\gamma y}}{\frac{1}{2}+i\gamma}\right|^2 dy.
$$

For the inner integral, by Lemma 3.2 and Lemma 3.3, and the definition of $T = T(x)$,

$$\int_2^Y \left| \sum_{|\gamma| \le T(e^y)} \frac{e^{i\gamma y}}{\frac{1}{2} + i\gamma} \right|^2 dy \le \sum_{l \le \frac{\log Y}{\log 2}+1} \int_{2^l}^{2^{l+1}} \left(\sum_{\substack{|\gamma_1-\gamma_2| \le 1 \\ |\gamma_1|,|\gamma_2| \le T_{l'}}} + \sum_{\substack{|\gamma_1-\gamma_2| > 1 \\ |\gamma_1|,|\gamma_2| \le T_{l'}}} \right) \frac{e^{i(\gamma_1-\gamma_2)y}}{(\frac{1}{2}+i\gamma_1)(\frac{1}{2}-i\gamma_2)} dy$$

$$\ll \sum_{l \le \frac{\log Y}{\log 2}+1} \left(2^l \sum_{\gamma} \frac{\log|\gamma|}{|\gamma|^2} + \sum_{\gamma_1,\gamma_2} \frac{1}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \right) \ll Y.$$

Thus,

$$\int_2^Y |S_1(e^y;\chi)|^2 \, dy \ll \sum_{j=2}^{k} Y(\log Y)^{2k-2j} = o(Y(\log Y)^{2k-2})).$$

Next, we examine $S_2(x, T_0; \chi)$. For fixed T_0, let X_0 be the largest x such that $T = T(x) \le T_0$. Since $x \le T(x) \le 2x^2$, $\log X_0 \asymp \log T_0$. By Lemma 3.2 and Lemma 3.3,

$$\int_2^Y |S_2(e^y, T_0; \chi)|^2 dy \le \int_2^{\log X_0} \left| \sum_{|\gamma| \le T_0} \frac{1}{|\gamma|} \right|^2 dy + \int_{\log X_0}^Y \left| \sum_{T_0 \le |\gamma| \le T(e^y)} \frac{e^{i\gamma y}}{\frac{1}{2}+i\gamma} \right|^2 dy$$

$$\ll \log^5 T_0 + \sum_{\frac{\log\log X_0}{\log 2} \le l \le \frac{\log Y}{\log 2}+1} \int_{2^l}^{2^{l+1}} \left(\sum_{\substack{|\gamma_1-\gamma_2| \le 1 \\ T_0 \le |\gamma_1|,|\gamma_2| \le T_{l'}}} + \sum_{\substack{|\gamma_1-\gamma_2| > 1 \\ T_0 \le |\gamma_1|,|\gamma_2| \le T_{l'}}} \right) \frac{e^{i(\gamma_1-\gamma_2)y}}{(\frac{1}{2}+i\gamma_1)(\frac{1}{2}-i\gamma_2)} dy$$

$$\ll \log^5 T_0 + \sum_{\frac{\log\log X_0}{\log 2} \le l \le \frac{\log Y}{\log 2}+1} \left(2^l \sum_{|\gamma| \ge T_0} \frac{\log|\gamma|}{|\gamma|^2} + \sum_{|\gamma_1|,|\gamma_2| \ge T_0} \frac{1}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \right)$$

$$\ll Y \frac{\log^2 T_0}{T_0} + \log Y \frac{\log^3 T_0}{T_0} + \log^5 T_0.$$

This completes the proof of this lemma. □

3.4 Asymptotic formulas for the logarithmic densities

In this section, we give the proof of Theorem 1.11.

For large q, Fiorilli and Martin [13] gave an asymptotic formula for $\delta_{\Omega_1}(q; a, b)$. Lamzouri

[33] also derived such an asymptotic formula using another method. Here, we want to derive asymptotic formulas for $\delta_{\Omega_k}(q;a,b)$ and $\delta_{\omega_k}(q;a,b)$ for fixed q and large k.

Let a be a quadratic non-residue $\bmod q$ and b be a quadratic residue $\bmod q$, and $(a,q) = (b,q) = 1$. Letting $\lambda_k = \frac{1}{2^{k-1}}$, similar to formula (2.10) of [13], we have, under the assumptions $\mathrm{ERH_q}$ and $\mathrm{LI_q}$,

$$\delta_{\Omega_k}(q;a,b) = \frac{1}{2} + \frac{(-1)^k}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\lambda_k(N(q;a)-N(q;b))x}{x}\Phi_{q;a,b}(x)dx.$$

Noting that $N(q,a) - N(q,b) = -A(q)$,

$$\delta_{\Omega_k}(q;a,b) = \frac{1}{2} + \frac{(-1)^{k-1}}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)dx. \tag{3.4.1}$$

For any $\epsilon > 0$,

$$\int_{-\infty}^{\infty} \frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)dx = \left(\int_{-\infty}^{\frac{1}{\lambda_k^\epsilon}} + \int_{-\frac{1}{\lambda_k^\epsilon}}^{\frac{1}{\lambda_k^\epsilon}} + \int_{\frac{1}{\lambda_k^\epsilon}}^{\infty}\right)\frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)dx. \tag{3.4.2}$$

By Proposition 2.17 in [13], $|\Phi_{q;a,b}(t)| \le e^{-0.0454\phi(q)t}$ for $t \ge 200$. So for large enough k,

$$\int_{\frac{1}{\lambda_k^\epsilon}}^{\infty} \frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)dx \ll \lambda_k \int_{\frac{1}{\lambda_k^\epsilon}}^{\infty} e^{-0.0454\phi(q)x}dx \ll_{q,J,\epsilon} \lambda_k^J, \text{ for any } J > 0. \tag{3.4.3}$$

The integral for $x \le -\frac{1}{\lambda_k^\epsilon}$ is also bounded by λ_k^J.

By Lemma 2.22 in [13], for each nonnegative integer K and real number $C > 1$, we have, uniformly for $|z| \le C$,

$$\frac{\sin z}{z} = \sum_{j=0}^{K} (-1)^j \frac{z^{2j}}{(2j+1)!} + O_{C,K}\left(|z|^{2K+2}\right).$$

Thus, the second integral in (3.4.2) is equal to

$$\begin{aligned}
&\lambda_k A(q) \int_{-\frac{1}{\lambda_k^\epsilon}}^{\frac{1}{\lambda_k^\epsilon}} \frac{\sin(\lambda_k A(q)x)}{\lambda_k A(q)x} \Phi_{q;a,b}(x)dx \\
&= \sum_{j=0}^{K} \lambda_k^{2j+1} \frac{(-1)^j A(q)^{2j+1}}{(2j+1)!} \int_{-\frac{1}{\lambda_k^\epsilon}}^{\frac{1}{\lambda_k^\epsilon}} x^{2j} \Phi_{q;a,b}(x)dx + O_{q,K}\left(\lambda_k^{2K+3-\epsilon}\right) \\
&= \sum_{j=0}^{K} \lambda_k^{2j+1} \frac{(-1)^j A(q)^{2j+1}}{(2j+1)!} \int_{-\infty}^{\infty} x^{2j} \Phi_{q;a,b}(x)dx + O_{q,K,\epsilon}\left(\lambda_k^{2K+3-\epsilon}\right). \qquad (3.4.4)
\end{aligned}$$

Combining (3.4.1), (3.4.3), and (3.4.4), we get the asymptotic formula (1.2.3) for $\delta_{\Omega_k}(q;a,b)$. Similarly, or by the results in Theorem 1.10, we have the asymptotic formula (1.2.4) for $\delta_{\omega_k}(q;a,b)$. □

3.5 Proof of main lemmas in Section 3.2

3.5.1 Proof of Lemma 3.8

Let I represent the integral in the lemma. Then, we have

$$I \le 2\sum_{j=1}^{k} \binom{k}{j} \pi^j \int_0^\delta |\log\sigma|^{k-j} \sigma^m x^{-\sigma} d\sigma \ll_k \sum_{j=1}^{k} \int_0^\delta |\log\sigma|^{k-j} \sigma^m x^{-\sigma} d\sigma \qquad (3.5.1)$$

Using a change of variable, $\sigma \log x = t$, we have

$$\begin{aligned}
&\int_0^\delta |\log\sigma|^{k-j}\sigma^m x^{-\sigma} d\sigma \le \frac{1}{(\log x)^{m+1}} \int_0^{\delta\log x} |\log t - \log\log x|^{k-j} t^m e^{-t} dt \\
&\le \frac{1}{(\log x)^{m+1}} \sum_{l=0}^{k-j} \binom{k-j}{l} (\log\log x)^{k-j-l} \int_0^{\delta\log x} |\log t|^l t^m e^{-t} dt \\
&\ll_k \frac{1}{(\log x)^{m+1}} \sum_{l=0}^{k-j} (\log\log x)^{k-j-l} \int_0^{\delta\log x} |\log t|^l t^m e^{-t} dt. \qquad (3.5.2)
\end{aligned}$$

Next, we estimate

$$\int_0^{\delta \log x} |\log t|^l t^m e^{-t} dt \le \left(\int_0^1 + \int_1^\infty \right) |\log t|^l t^m e^{-t} dt =: I_{l_1} + I_{l_2}. \tag{3.5.3}$$

For the first integral in (3.5.3),

$$I_{l_1} = \int_0^1 |\log t|^l t^m e^{-t} dt \le \int_0^1 |\log t|^l dt \overset{t \to \frac{1}{e^t}}{=} \int_0^\infty \frac{t^l}{e^t} dt = \Gamma(l+1).$$

For the second integral in (3.5.3),

$$I_{l_2} = \int_1^\infty \frac{t^m (\log t)^l}{e^t} dt \overset{t \to e^t}{=} \int_0^\infty \frac{t^l}{e^{e^t - (m+1)t}} dt \ll_m \Gamma(l+1). \tag{3.5.4}$$

Then, by (3.5.2)-(3.5.4), we have

$$\int_0^\delta |\log \sigma|^{k-j} \sigma^m x^{-\sigma} d\sigma \ll_k \frac{1}{(\log x)^{m+1}} \sum_{l=0}^{k-j} (\log \log x)^{k-j-l} O_{m,l}(1) \ll_{m,k} \frac{(\log \log x)^{k-j}}{(\log x)^{m+1}}. \tag{3.5.5}$$

Thus, by (3.5.1),

$$I \ll_{m,k} \frac{(\log \log x)^{k-1}}{(\log x)^{m+1}}.$$

Hence, we get the conclusion of this lemma. □

3.5.2 Proof of Lemma 3.9

We have the equality

$$\frac{1}{s} = \frac{1}{a} + \frac{a-s}{a^2} + \frac{(a-s)^2}{a^2 s}.$$

With the above equality, we write the integral in the lemma as

$$\frac{1}{2\pi i}\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\left(\frac{1}{a}+\frac{a-s}{a^2}+\frac{(a-s)^2}{a^2 s}\right)x^s ds =: I_1+I_2+I_3.$$

For I_3, using Lemma 3.8, we get

$$\begin{aligned}
&\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\frac{(a-s)^2}{a^2 s}x^s ds\\
&\le \left|\int_r^\delta \left((\log\sigma - i\pi)^k-(\log\sigma+i\pi)^k\right)\sigma^2 x^{-\sigma}\frac{x^a}{a^2(a-\sigma)}d\sigma\right|\\
&\quad+\int_{-\pi}^{\pi} x^{\Re(a)+r}\left(\log\frac{1}{r}+\pi\right)^k\frac{r^2}{|a|^2|\Re(a)-r|}rd\alpha\\
&\ll \frac{|x^a|}{|a|^2|\Re(a)-\delta|}\left(\int_0^\delta |(\log\sigma-i\pi)^k-(\log\sigma+i\pi)^k|\sigma^2x^{-\sigma}d\sigma+\frac{(\log\frac{1}{r}+\pi)^k}{(1/r)^3}\right)\\
&\ll_k \frac{|x^a|}{|a|^2|\Re(a)-\delta|}\left(\frac{(\log\log x)^{k-1}}{(\log x)^3}+\frac{1}{x^{3-\epsilon}}\right)\ll_k \frac{|x^a|}{|a|^2|\Re(a)-\delta|}\frac{(\log\log x)^{k-1}}{(\log x)^3}. \qquad (3.5.6)
\end{aligned}$$

We estimate I_2 similarly, by Lemma 3.8,

$$\begin{aligned}
&\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\frac{a-s}{a^2}x^s ds\\
&\le \left|\int_r^\delta \left((\log\sigma - i\pi)^k-(\log\sigma+i\pi)^k\right)\sigma x^{-\sigma}\frac{x^a}{a^2}d\sigma\right|+\int_{-\pi}^{\pi} x^{\Re(a)+r}\left(\log\frac{1}{r}+\pi\right)^k\frac{r}{|a|^2}rd\alpha\\
&\ll \frac{|x^a|}{|a|^2}\left(\int_0^\delta |(\log\sigma-i\pi)^k-(\log\sigma+i\pi)^k|\sigma x^{-\sigma}d\sigma+\frac{(\log\frac{1}{r}+\pi)^k}{(1/r)^2}\right)\\
&\ll_k \frac{|x^a|}{|a|^2}\left(\frac{(\log\log x)^{k-1}}{(\log x)^2}+\frac{1}{x^{2-\epsilon}}\right)\ll_k \frac{|x^a|}{|a|^2}\frac{(\log\log x)^{k-1}}{(\log x)^2}. \qquad (3.5.7)
\end{aligned}$$

For I_1, using change of variable $(s-a)\log x = w$, by Lemma 3.7, we deduce that

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\frac{1}{\log x}\int_{\mathcal{H}(0,\delta\log x)}(\log w-\log\log x)^k\frac{x^a e^w}{a}dw\\
&= \frac{x^a}{a\log x}(-1)^k(\log\log x)^k\frac{1}{2\pi i}\int_{\mathcal{H}(0,\delta\log x)} e^w dw\\
&\quad+(-1)^{k-1}k\frac{x^a}{a\log x}(\log\log x)^{k-1}\frac{1}{2\pi i}\int_{\mathcal{H}(0,\delta\log x)} e^w\log w dw
\end{aligned}$$

$$+\frac{x^a}{a\log x}\sum_{j=2}^{k}\binom{k}{j}\frac{1}{2\pi i}\int_{\mathcal{H}(0,\delta\log x)}(-\log\log x)^{k-j}(\log w)^j e^w dw$$

$$=\frac{(-1)^k x^a}{a\log x}\left\{k(\log\log x)^{k-1}+\sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}$$

$$+\frac{x^a}{a\log x}\sum_{j=1}^{k}\binom{k}{j}E_{j,0}(\delta\log x)(-\log\log x)^{k-j}. \tag{3.5.8}$$

By Lemma 3.7,

$$|E_{j,0}(\delta\log x)|\le\frac{1}{2\pi}\int_{\delta\log x}^{\infty}\frac{(\log t+\pi)^j}{e^t}dt\ll_j e^{-\frac{\delta\log x}{2}}\int_{\frac{\delta\log x}{2}}^{\infty}\frac{(\log t)^j}{e^{t/2}}dt\ll_j x^{-\frac{\delta}{2}}.$$

Hence, we get

$$\left|\frac{x^a}{a\log x}\sum_{j=1}^{k}\binom{k}{j}E_{j,0}(\delta\log x)(-\log\log x)^{k-j}\right|\ll_k\frac{x^{\Re(a)}}{|a|\log x}\sum_{j=1}^{k}x^{-\frac{\delta}{2}}(\log\log x)^{k-j}\ll_k\frac{|x^{a-\delta/3|}}{|a|}. \tag{3.5.9}$$

Combining (3.5.6), (3.5.7), (3.5.8), and (3.5.9), we get the conclusion of this lemma. □

3.5.3 Proof of Lemma 3.10

Let I denote the integral in the lemma. We consider two cases: $\Delta_N\ge\left(\frac{1}{\log x}\right)^N$, and $\Delta_N<\left(\frac{1}{\log x}\right)^N$.

1) If $\Delta_N\ge\left(\frac{1}{\log x}\right)^N$, we have

$$I\ll\left(\log\frac{1}{\Delta_N}+N\pi\right)^j\int_0^\delta x^{-\sigma}d\sigma\ll\frac{1}{\log x}\left(\log\frac{1}{\Delta_N}+N\pi\right)^j\ll\frac{1}{\log x}(N\log\log x+N\pi)^j. \tag{3.5.10}$$

2) If $\Delta_N<\left(\frac{1}{\log x}\right)^N$, we write

$$I=\left(\int_0^{(\Delta_N)^{\frac{1}{N}}}+\int_{(\Delta_N)^{\frac{1}{N}}}^{\frac{1}{\log x}}+\int_{\frac{1}{\log x}}^{\delta}\right)\left|\sum_{n=1}^{N}\log(\sigma+i\delta_n)\right|^j x^{-\sigma}d\sigma=:I_1+I_2+I_3. \tag{3.5.11}$$

First, we estimate I_1,

$$I_1 \ll \left(\log \frac{1}{\Delta_N} + N\pi\right)^j \int_0^{(\Delta_N)^{\frac{1}{N}}} x^{-\sigma} d\sigma \ll (\Delta_N)^{\frac{1}{N}} \left(\log \frac{1}{\Delta_N} + N\pi\right)^j. \tag{3.5.12}$$

For $0 < t < 1$, consider the function $f(t) = t^{\frac{1}{N}} \left(\log \frac{1}{t} + N\pi\right)^j$. Since the critical point of $f(t)$ is $t = e^{N(\pi-1)} > 1$, by (3.5.12), we have

$$I_1 \ll f\left(\frac{1}{(\log x)^N}\right) = \frac{1}{\log x}(N \log\log x + N\pi)^j \ll \frac{1}{\log x}\left(\log \frac{1}{\Delta_N} + N\pi\right)^j. \tag{3.5.13}$$

Next, we estimate I_3. Using the change of variable $\sigma \log x = t$, we get

$$\begin{aligned} I_3 &\ll \int_{\frac{1}{\log x}}^{\delta} \left(N \log \frac{1}{\sigma} + N\pi\right)^j x^{-\sigma} d\sigma \\ &= \frac{1}{\log x} \int_1^{\delta \log x} (N \log\log x - N \log t + N\pi)^j e^{-t} dt \\ &= \frac{N^j}{\log x} \sum_{l=0}^{j} \binom{j}{l} (\log\log x + \pi)^{j-l} \int_1^{\delta \log x} (-\log t)^l e^{-t} dt \\ &\ll_j \frac{N^j}{\log x} \sum_{l=0}^{j} (\log\log x + \pi)^{j-l} \int_1^{\infty} \frac{t^l}{e^t} dt \\ &\ll_j \frac{(N \log\log x + N\pi)^j}{\log x} \ll \frac{1}{\log x}\left(\log \frac{1}{\Delta_N} + N\pi\right)^j. \end{aligned} \tag{3.5.14}$$

For I_2, similar to I_3, using the change of variable $\sigma \log x = t$, we get

$$\begin{aligned} I_2 &\ll \int_{(\Delta_N)^{\frac{1}{N}}}^{\frac{1}{\log x}} \left(N \log \frac{1}{\sigma} + N\pi\right)^j x^{-\sigma} d\sigma \\ &= \frac{1}{\log x} \int_{(\Delta_N)^{\frac{1}{N}} \log x}^{1} (N \log\log x - N \log t + N\pi)^j e^{-t} dt \\ &= \frac{N^j}{\log x} \sum_{l=0}^{j} \binom{j}{l} (\log\log x + \pi)^{j-l} \int_{(\Delta_N)^{\frac{1}{N}} \log x}^{1} (-\log t)^l e^{-t} dt \quad (t \to \frac{1}{e^t}) \\ &\ll_j \frac{N^j}{\log x} \sum_{l=0}^{j} (\log\log x + \pi)^{j-l} \int_0^{\infty} \frac{t^l}{e^t} dt \end{aligned}$$

$$\ll_j \quad \frac{(N \log\log x + N\pi)^j}{\log x} \ll \frac{1}{\log x}\left(\log\frac{1}{\Delta_N} + N\pi\right)^j. \tag{3.5.15}$$

Combining (3.5.13), (3.5.14), (3.5.15), with (3.5.11), we get

$$I \ll_j \frac{(N \log\log x + N\pi)^j}{\log x} \ll_j \frac{1}{\log x}\left(\log\frac{1}{\Delta_N} + N\pi\right)^j. \tag{3.5.16}$$

By (3.5.10) and (3.5.16), we get the conclusion of this lemma. □

Chapter 4

Large bias for integers with prime factors from arithmetic progressions

4.1 Lemmas and Preparations

Lemma 4.1 ([30], Chapter IX, §2, Theorem 2, [8], page 96, (12)). *The Dirichlet L-function $L(s,\chi)$ has no zeros in the domain*

$$\Re(s) = \sigma \geq 1 - \frac{c_1}{\log q(|t|+2)},$$

for some constant $c_1 > 0$, except a possible simple real zero close to 1 when χ is real, which is called a Siegel zero. If χ is real, there exists an effective constant $c_2 > 0$ such that $L(\sigma,\chi) \neq 0$ in the range

$$\sigma > 1 - \frac{c_2}{\sqrt{q}\log^2 q}.$$

We need the following terminologies (Part II. 5.2, [58]).

Defintion 4.1. *Let $z \in \mathbb{C}$, $c_0 > 0$, $0 < \delta \leq 1$, $M > 0$. We say that a Dirichlet series $F(s)$ has the property $\mathcal{P}(z; c_0, \delta, M)$ if the Dirichlet series $G(s;z) := F(s)\zeta(s)^{-z}$ can be analytically continued to the region $\sigma \geq 1 - c_0/(\log(|t|+2))$, and in this region, $|G(s;z)| \leq M(1+|t|)^{1-\delta}$.*

Defintion 4.2. *We say $F(s)$ has type $\mathcal{T}(z, w; c_0, \delta, M)$, if $F(s) = \sum_{n\geq 1} a_n/n^s$ has property $\mathcal{P}(z; c_0, \delta, M)$, and there exists a sequence of non-negative real numbers $\{b_n\}_{n=1}^{\infty}$ such that $|a_n| \leq b_n$, and the series $\sum_{n\geq 1} b_n/n^s$ satisfies $\mathcal{P}(w; c_0, \delta, M)$ for some complex number w,*

Lemma 4.2 ([58], Part II, Theorem 5.2). *Let $F(s) := \sum_{n\geq 1} a_n/n^s$ be a Dirichlet series of* *type $\mathcal{T}(z, w; c_0, \delta, M)$. For $x \geq 3$, $N \geq 0$, $A > 0$, $|z| \leq A$, and $|w| \leq A$, we have*

$$\sum_{n \leq x} a_n = x(\log x)^{z-1} \left\{ \sum_{0 \leq n \leq N} \frac{u_n(z)}{(\log x)^n} + O(MR_n(x)) \right\},$$

where

$$u_n(z) := \frac{1}{\Gamma(z-n)} \sum_{l+j=n} \frac{1}{l!j!} G^{(l)}(1; z)\gamma_j(z),$$

$$G^l(s; z) := \frac{\partial^l}{\partial s^l} G(s, z), \quad \gamma_j(z) := \frac{d^j}{ds^j} \left(\frac{\{(s-1)\zeta(s)\}^z}{s} \right),$$

and

$$R_N(x) = e^{-c_1\sqrt{\log x}} + \left(\frac{c_2 N + 1}{\log x} \right)^{N+1}, \tag{4.1.1}$$

for some constants c_1 and c_2 depending at most on c_0, δ, and A.

Lemma 4.3. *Let $a_z(n)$ be an arithmetic function depending on a complex parameter z and $a_z(n) = \sum_{k=0}^{\infty} c_k(n) z^k$ in the disk $|z| \leq A$. Suppose there exist $N+1$ ($N \geq 0$) functions $h_j(z)$ ($0 \leq j \leq N$) holomorphic for $|z| \leq A$, and a quantity $R_N(x)$, independent of z, such that, for $x \geq 3$ and $|z| \leq A$, we have*

$$\sum_{n \leq x} a_z(n) = x(\log x)^{z-1} \left\{ \sum_{0 \leq j \leq N} \frac{z h_j(z)}{(\log x)^j} + O_A(R_N(x)) \right\}.$$

Then, uniformly for $x \geq 3$ and $1 \leq k \leq A \log\log x$, we have

$$C_k(x) := \sum_{n \leq x} c_k(n) = \frac{x}{\log x} \left\{ \sum_{0 \leq j \leq N} \frac{Q_{j,k}(\log\log x)}{(\log x)^j} + O_A\left(\frac{(\log\log x)^k}{k!} R_N(x) \right) \right\},$$

where

$$Q_{j,k}(X) := \sum_{m+l=k-1} \frac{1}{m!l!} h_j^{(m)}(0) X^l.$$

If in addition, we suppose that $|h_0''(z)| \leq B$ for $|z| \leq A$, then uniformly for $x \geq 3$, $1 \leq k \leq$

$A \log\log x$, *we have*

$$C_k(x) = \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}\left\{h_0\left(\frac{k-1}{\log x}\right) + O_A\left(\frac{B(k-1)}{(\log\log x)^2} + \frac{\log\log x}{k}R_0(x)\right)\right\}. \quad (4.1.2)$$

If we suppose $|h_0^{(4)}(z)| \leq B_2$ *for* $|z| \leq A$*, then uniformly for* $x \geq 3$*,* $3 \leq k \leq A\log\log x$*, we have*

$$\begin{aligned} C_k(x) = \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}\Bigg\{&h_0(0) + \frac{k-1}{\log\log x}h_0'(0) + \frac{(k-1)(k-2)}{(\log\log x)^2}g\left(\frac{k-3}{\log\log x}\right) \\ &+ O_A\left(\frac{B_2(k-1)(k-2)(k-3)}{(\log\log x)^4} + \frac{\log\log x}{k}R_0(x)\right)\Bigg\}, \quad (4.1.3)\end{aligned}$$

where

$$g(z) = \int_0^1 h_0''(tz)(1-t)dt.$$

Proof. Formula (4.1.2) is a special case of Theorem 6.3 Part II in [58].

For all $r \leq A$, the main term in (4.1.3) is from

$$I := \frac{x}{\log x}\frac{1}{2\pi i}\oint_{|z|=r} h_0(z)\frac{e^{z\log\log x}}{z^k}dz = \frac{x}{\log x}\frac{1}{2\pi i}\oint_{|z|=r}(h_0(0) + zh_0'(0) + z^2 g(z))\frac{e^{z\log\log x}}{z^k}dz.$$

When $k \leq A\log\log x$, choose $r_j = \frac{k-j}{\log\log x}$ $(1 \leq j \leq 3)$, we see that

$$\begin{aligned} I =& \frac{x}{\log x}\frac{1}{2\pi i}\oint_{|z|=r_1} h_0(0)\frac{e^{z\log\log x}}{z^k}dz + \frac{x}{\log x}\frac{1}{2\pi i}\oint_{|z|=r_2} h_0'(0)\frac{e^{z\log\log x}}{z^{k-1}}dz \\ &+ \frac{x}{\log x}\frac{1}{2\pi i}\oint_{|z|=r_3} g(z)\frac{e^{z\log\log x}}{z^{k-2}}dz \\ =& \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}\left\{h_0(0) + \frac{k-1}{\log\log x}h_0'(0)\right\} + \frac{x}{\log x}\frac{1}{2\pi i}\oint_{|z|=r_3} g(z)\frac{e^{z\log\log x}}{z^{k-2}}dz. \quad (4.1.4)\end{aligned}$$

Next, we examine the last integral in (4.1.4). Since we assume $|h_0^{(4)}(z)| \leq B_2$ for $|z| \leq A$, we have

$$
\begin{aligned}
g(z) =& g(r_3) + (z-r_3)g'(r_3) + (z-r_3)^2 \int_0^1 (1-t)g''(r_3+t(z-r_3))dt \\
=& g(r_3) + (z-r_3)g'(r_3) + O\left(B_2|z-r_3|^2\right).
\end{aligned}
$$

Thus, the last integral in (4.1.4) equals

$$
\begin{aligned}
&\frac{x}{\log x}\Bigg\{\frac{g(r_3)}{2\pi i}\oint_{|z|=r_3} \frac{e^{z\log\log x}}{z^{k-2}}dz + \frac{1}{2\pi i}\oint_{|z|=r_3}(z-r_3)\frac{e^{z\log\log x}}{z^{k-2}}dz \\
&\qquad + O\left(B_2\int_0^{2\pi} |e^{i\alpha}-1|^2 e^{r_3\log\log x\cos\alpha} r_3^{5-k} d\alpha\right)\Bigg\} \\
&= \frac{x}{\log x}\left\{g(r_3)\frac{(\log\log x)^{k-3}}{(k-3)!} + \frac{(\log\log x)^{k-4}}{(k-4)!} - r_3\frac{(\log\log x)^{k-3}}{(k-3)!} + O\left(B_2\frac{(\log\log x)^{k-5}}{(k-4)!}\right)\right\} \\
&= \frac{x}{\log x}\frac{(\log\log x)^{k-3}}{(k-3)!}\left\{g\left(\frac{k-3}{\log\log x}\right) + O\left(\frac{B_2(k-3)}{(\log\log x)^2}\right)\right\}. \qquad (4.1.5)
\end{aligned}
$$

The error term $O\left(R_0(x)\log\log x/k\right)$ is the same as that in the proof of (4.1.2). Combing (4.1.4) and (4.1.5), we get the desired result. □

We need some results for holomorphic functions of several variables [11].

Defintion 4.3. *Let* $\mathbb{R}^l_{>0} := \{\mathbf{y} = (y_1, \dots, y_l) \in \mathbb{R}^l \mid y_j > 0$ *for all* $j\}$, $\mathbf{r} = (r_1, \dots, r_l) \in \mathbb{R}^l_{>0}$, $\mathbf{a} \in \mathbb{C}^l$. *Then,* $\Delta_{\mathbf{r}}(\mathbf{a}) := \{\mathbf{z} \in \mathbb{C}^l \mid |z_j - r_j| < r_j, 1 \le j \le l\}$ *is called the polycylinder around* $\mathbf{a}$ *with (poly-)radius* $\mathbf{r}$. *The boundary of the closure of* $\Delta_{\mathbf{r}}(\mathbf{a})$ *contains an n-dimensional torus* $T_{\mathbf{r}}(\mathbf{a}) := \{\mathbf{z} \in \mathbb{C}^l \mid |z_j - a_j| = r_j, 1 \le j \le l\}$.

In order to simplify the expression in the following lemma, we introduce multiindices. Let v_j, $1 \le j \le l$, be nonnegative integers, $\mathbf{z} = (z_1, \dots, z_l) \in \mathbb{C}^l$. Denote $\mathbf{v} := (v_1, \dots, v_l)$, $|\mathbf{v}| = v_1 + \cdots + v_l$, $\mathbf{v}! := v_1! \cdots v_n!$, $\mathbf{z}^{\mathbf{v}} := z_1^{v_1} \cdots z_l^{v_l}$, and

$$
D^{\mathbf{v}} f = \frac{\partial^{|\mathbf{v}|}}{\partial z_1^{v_1} \cdots \partial z_l^{v_l}}.
$$

We have the following result.

Lemma 4.4 ([11], Chapter 2, Propositions 2.7 and 2.11). *Let* $U \subset \mathbb{C}^l$ *be open and* $f : U \to \mathbb{C}$ *holomorphic. Furthermore, let* $\mathbf{w} \in U$ *and* $\Delta := \Delta_{\mathbf{r}}(\mathbf{w})$ *be a polycylinder around* $\mathbf{w}$ *with* $\bar{\Delta} \subset U$, $T = T_{\mathbf{r}}(\mathbf{w})$. *Then* f *can be expanded as a power series*

$$f(\mathbf{z}) = \sum_{\mathbf{v}=0}^{\infty} a_{\mathbf{v}}(\mathbf{z}-\mathbf{w})^{\mathbf{v}} = \sum_{v_1 \geq 0, \cdots, v_l \geq 0} a_{\mathbf{v}}(z_1 - w_1)^{v_1} \cdots (z_l - w_l)^{v_l}$$

in a neighborhood of $\mathbf{w}$, *with coefficients*

$$a_{\mathbf{v}} = \frac{D^{\mathbf{v}} f(\mathbf{w})}{\mathbf{v}!} = \frac{1}{\mathbf{v}!} \frac{\partial^{v_1+\cdots+v_l} f}{\partial z_1^{v_1} \cdots \partial z_l^{v_l}}(\mathbf{w}) = \left(\frac{1}{2\pi i}\right)^l \int_T \frac{f(\boldsymbol{\zeta})}{(\zeta_1 - w_1)^{v_1+1} \cdots (\zeta_l - w_l)^{v_l+1}} d\boldsymbol{\zeta}.$$

4.2 Associated Dirichlet series

Let $(a, q) = 1$. We define a function $\lambda_a(n)$ in the following way,

$$\lambda_a(n) = \begin{cases} 1, & \text{if } n \text{ square-free, } p|n \Rightarrow p \equiv a \bmod q, p \text{ is a prime,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.1}$$

We consider the Dirichlet series

$$F(s; a, z) := \sum_{n=1}^{\infty} \frac{(z\lambda_a(n))^{\omega(n)}}{n^s} = \prod_p \left(1 + \frac{z\lambda_a(p)}{p^s}\right), \quad (\Re(s) > 1), \tag{4.2.2}$$

where $\omega(n)$ is the number of distinct prime factors of n. Letting χ_0 be the principal character modulo q, we denote

$$F(s; z) := \sum_{n=1}^{\infty} \frac{\mu^2(n)(z\chi_0(n))^{\omega(n)}}{n^s} = \prod_p \left(1 + \frac{z\chi_0(p)}{p^s}\right), \quad (\Re(s) > 1).$$

where $\mu(n)$ is the Möbius function.

Then we have the following lemma.

Lemma 4.5. *For any $A > 0$, $|z| \le A$, and $\Re(s) > 1$,*

$$F(s;a,z) = (L(s,\chi_0))^{\frac{z}{\phi(q)}} \prod_{\chi\neq\chi_0} (L(s,\chi))^{\frac{\bar{\chi}(a)z}{\phi(q)}} G_1(s;a,z),$$

and

$$F(s;z) = (L(s,\chi_0))^{z} G_2(s;z),$$

where χ is a Dirichlet character modulo q, and $G_1(s;a,z)$ and $G_2(s;z)$ are absolutely convergent for $\Re(s) \ge \sigma_0 > \frac{1}{2}$.

Proof. By (4.2.2), the function

$$\begin{aligned} G_1(s;a,z) &:= F(s;a,z)(L(s,\chi_0))^{-\frac{z}{\phi(q)}} \prod_{\chi\neq\chi_0} (L(s,\chi))^{-\frac{\bar{\chi}(a)z}{\phi(q)}} \\ &= \prod_p \left(1 + \frac{z\lambda_a(p)}{p^s}\right)\left(1 - \frac{\chi_0(p)}{p^s}\right)^{\frac{z}{\phi(q)}} \prod_{\chi\neq\chi_0}\left(1 - \frac{\chi(p)}{p^s}\right)^{\frac{\bar{\chi}(a)z}{\phi(q)}} \end{aligned}$$

is expandable as a Dirichlet series $G_1(s;a,z) = \sum_{n\ge1} \frac{b_z(n)}{n^s}$, where

$$1 + \sum_{v\ge1} b_z(p^v)\xi^v = (1 + z\lambda_a(p)\xi)(1 - \chi_0(p)\xi)^{\frac{z}{\phi(q)}} \prod_{\chi\neq\chi_0} (1 - \chi(p)\xi)^{\frac{\bar{\chi}(a)z}{\phi(q)}}, \quad (|\xi| < 1). \quad (4.2.3)$$

By the orthogonality of Dirichlet characters, for any prime p, $\lambda_a(p) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a)\chi(p)$. Thus, by (4.2.3), $b_z(p) = 0$. By Cauchy's inequality, for $v \ge 2$ and $|z| \le A$,

$$|b_z(p^v)| \le M2^{v/2}, \quad (4.2.4)$$

with

$$M = M(A,q) := \sup_{p,|z|\le A,|\xi|\le 1/\sqrt{2}} \left|(1 + z\lambda_a(p)\xi)(1 - \chi_0(p)\xi)^{\frac{z}{\phi(q)}} \prod_{\chi\neq\chi_0} (1 - \chi(p)\xi)^{\frac{\bar{\chi}(a)z}{\phi(q)}}\right|.$$

Since $\lambda_a(p), \chi_0(p) = 0$ or 1, and there are only $\phi(q)$ possible values for $\chi(p)$, $M(A,q)$ exists.

Hence, by (4.2.4), for $\Re(s) \geq \sigma_0 > \frac{1}{2}$,

$$\sum_p \sum_{v \geq 1} \frac{|b_z(p^v)|}{p^{v\sigma_0}} \leq 2M \sum_p \frac{1}{p^{\sigma_0}(p^{\sigma_0} - \sqrt{2})} \leq \frac{cM}{\sigma_0 - 1/2}$$

for some absolute constant $c > 0$. Therefore, $G_1(s; a, z) \ll_{A,q} 1$ for $\Re(s) \geq \sigma_0 > \frac{1}{2}$.

The proof of the expression for $F(s; z)$ is similar. □

Given $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, assume there are l distinct values $b_1, \ldots, b_l$ in the coordinates of $\mathbf{a}$. We assume b_i $(1 \leq i \leq l)$ appears $k_i(> 0)$ times in $\mathbf{a}$ with $k_1 + k_2 + \cdots + k_l = k$. Let $\mathbf{k}(\mathbf{a}) := (k_1, k_2, \ldots, k_l)$, $\mathbf{b}(\mathbf{a}) := (b_1, \ldots, b_l)$, and $\mathbf{z} = (z_1, z_2, \ldots, z_l)$. Denote

$$F(s; \mathbf{a}, \mathbf{z}) := \prod_{j=1}^{l} F(s; b_j, z_j) = \prod_{j=1}^{l} \prod_p \left(1 + \frac{z_j \lambda_{b_j}(p)}{p^s}\right). \tag{4.2.5}$$

Let $\mathbf{n} = (n_1, \ldots, n_l) \in \mathbb{Z}^l$ $(n_j > 0, 1 \leq j \leq l)$. We write the Dirichlet series $F(s; \mathbf{a}, \mathbf{z}) = \sum_{\mathbf{n} > \mathbf{0}} \frac{a(\mathbf{n};\mathbf{z})}{P^s(\mathbf{n})}$ with $P(\mathbf{n}) = \prod_{1 \leq j \leq l} n_j$. Then,

$$a(\mathbf{n}; \mathbf{z}) = \sum_{\substack{\mathbf{k} = (k_1, \cdots, k_l) \\ k_j \geq 0}} c(\mathbf{k}, \mathbf{n}) z_1^{k_1} \cdots z_l^{k_l},$$

for some $c(\mathbf{k}, \mathbf{n}) \in \mathbb{Z}^+$. Thus, for given $\mathbf{a}$, by Lemma 4.4,

$$M_k(x; \mathbf{a}) = \sum_{P(\mathbf{n}) \leq x} c(\mathbf{k}(\mathbf{a}), \mathbf{n}) = \left(\frac{1}{2\pi i}\right)^l \oint_{|z_l| = r_l} \cdots \oint_{|z_1| = r_1} \left(\sum_{P(\mathbf{n}) \leq x} a(\mathbf{n}; \mathbf{z})\right) \frac{dz_1}{z_1^{k_1+1}} \cdots \frac{dz_l}{z_l^{k_l+1}}. \tag{4.2.6}$$

4.3 A Uniform Result

First, we prove the following result.

Theorem 4.4. *For any $A > 0$, fixed $q \geq 3$ and fixed $l \geq 1$, uniformly for $2 \leq k \leq A \log\log x$, $N \geq 0$, we have*

$$M_k(x; \mathbf{a}) = \frac{x}{\log x} \left\{ \frac{1}{\phi(q)} \sum_{0 \leq j \leq N} \frac{Q_{j,\mathbf{k}}(\frac{\log\log x}{\phi(q)})}{(\log x)^j} + O_{A,q,l} \left(\frac{1}{\phi^k(q)} \frac{(\log\log x)^k}{k_1! \cdots k_l!} R_N(x) \right) \right\},$$

where $Q_{j,\mathbf{k}}(X)$ is a polynomial of degree at most $k-1$ ($k = k_1 + \cdots + k_l$), and

$$R_N(x) = e^{-c_1\sqrt{\log x}} + \left(\frac{c_2 N + 1}{\log x} \right)^{N+1},$$

for some constants c_1 and c_2 depending on A and q. In particular, the coefficient of the term $\frac{x}{\log x}(\log\log x)^{k-1}$ is $\frac{1}{\phi^k(q)} \frac{k}{k_1! k_2! \cdots k_l!}$, and the coefficient of $\frac{x}{\log x}(\log\log x)^{k-2}$ is

$$\frac{1}{\phi^k(q)} \frac{k(k-1)}{k_1! k_2! \cdots k_l!} \left(\gamma + B + \frac{1}{k} \sum_{j=1}^{k} C(q, a_j) - \sum_{p|q} \frac{1}{p} \right),$$

where γ is Euler's constant, and $B := \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right)$ is Mertens' constant.

Proof of Theorem 4.4. By Lemma 4.5 and (4.2.5), we have

$$\begin{aligned} F(s; \mathbf{a}, \mathbf{z}) &= (L(s, \chi_0))^{\frac{z_1 + \cdots + z_l}{\phi(q)}} \prod_{\chi \neq \chi_0} (L(s, \chi))^{\frac{\bar{\chi}(b_1) z_1 + \cdots + \bar{\chi}(b_l) z_l}{\phi(q)}} \prod_{j=1}^{l} G_1(s; b_j, z_j) \\ &= (\zeta(s))^{\frac{z_1 + \cdots + z_l}{\phi(q)}} H(s; \mathbf{a}, \mathbf{z}), \end{aligned} \tag{4.3.1}$$

where

$$\begin{aligned} H(s; \mathbf{a}, \mathbf{z}) &= \prod_{p|q} \left(1 - \frac{1}{p^s}\right)^{\frac{z_1 + \cdots + z_l}{\phi(q)}} \prod_{\chi \neq \chi_0} (L(s, \chi))^{\frac{\bar{\chi}(b_1) z_1 + \cdots + \bar{\chi}(b_l) z_l}{\phi(q)}} \prod_{j=1}^{l} G_1(s; b_j, z_j) \\ &= \prod_{p} \left(1 - \frac{1}{p^s}\right)^{\frac{z_1 + \cdots + z_l}{\phi(q)}} \prod_{j=1}^{l} \left(1 + \frac{z_j \lambda_{b_j}(p)}{p^s}\right) \end{aligned}$$

Let $\sigma = \Re(s)$. Kolesnik [31] showed that, for $\frac{1}{2} \le \sigma \le 1$,

$$|L(s,\chi)| \ll (|t|+2)^{\frac{35}{108}(1-\sigma)} q^{1-\sigma} \log^3 q(|t|+2). \tag{4.3.2}$$

Let q be fixed. By Lemma 4.1 and (4.3.2), for any $A > 0$, $|z_j| \le A$ $(1 \le j \le l)$, and $0 < \delta < 1$, we can choose $c_0 = c_0(A,\delta)$ such that, $L(s,\chi)$ has no zeros in the region $\sigma \ge 1 - c_0/(\log(|t|+2))$, and by Theorem 11.4 in [49], in this region, $|H(s;\mathbf{a},\mathbf{z})| \ll_{q,A,\delta} (|t|+2)^{1-\delta}$. Thus, by Definitions 4.1 and 4.2, $F(s;\mathbf{a},\mathbf{z})$ is in $\mathcal{T}(\frac{z_1+\cdots+z_l}{\phi(q)}, w; c_0, \delta, M)$. By (4.3.1) and following the same proof of Lemma 4.2 ([58], Part II, Theorem 5.2), we deduce that (the difference to Lemma 4.2 is the expansion of $H(s;\mathbf{a},\mathbf{z})$),

$$\sum_{n_1\cdots n_l \le x} a(\mathbf{n};\mathbf{z}) = x(\log x)^{\frac{z_1+\cdots+z_l}{\phi(q)}-1} \left\{ \sum_{0\le m\le N} \frac{u_m(\mathbf{a};\mathbf{z})}{(\log x)^m} + O_A(R_N(x)) \right\}, \tag{4.3.3}$$

where $R_N(x)$ is defined by (4.1.1),

$$u_m(\mathbf{a};\mathbf{z}) = \frac{1}{\Gamma\left(\frac{z_1+\cdots+z_l}{\phi(q)} - m\right)} \sum_{l+j=m} \frac{1}{l!j!} H^{(l)}(1;\mathbf{a},\mathbf{z})\gamma_j(z),$$

$$H^{(l)}(s;\mathbf{a},\mathbf{z}) := \frac{\partial^l}{\partial s^l} H(s;\mathbf{a},\mathbf{z}).$$

In particular, we write

$$u_0(\mathbf{a};\mathbf{z}) = \frac{z_1+\cdots+z_l}{\phi(q)} u(\mathbf{a};\mathbf{z}), \quad \text{with} \quad u(\mathbf{a};\mathbf{z}) := \frac{H(1;\mathbf{a},\mathbf{z})}{\Gamma\left(\frac{z_1+\cdots+z_l}{\phi(q)}+1\right)}. \tag{4.3.4}$$

By (4.2.6), (4.3.3), and Lemma 4.4, we have

$$M_k(x;\mathbf{a}) = \frac{x}{\log x} \left\{ \frac{1}{\phi(q)} \sum_{0\le j\le N} \frac{Q_{j,\mathbf{k}}(\frac{\log\log x}{\phi(q)})}{(\log x)^j} + \widetilde{R}_N(x) \right\},$$

where $Q_{j,\mathbf{k}}(X)$ is a polynomial of degree at most $k-1$ ($k = k_1 + \cdots + k_l$), and in particular,

$$Q_{0,\mathbf{k}}(X) := \Bigg\{ \sum_{m_1+j_1=k_1-1} \sum_{m_2+j_2=k_2} \cdots \sum_{m_l+j_l=k_l} + \sum_{m_1+j_1=k_1} \sum_{m_2+j_2=k_2-1} \cdots \sum_{m_l+j_l=k_l} + \cdots + \sum_{m_1+j_1=k_1} \cdots \sum_{m_{l-1}+j_{l-1}=k_{l-1}} \sum_{m_l+j_l=k_l-1} \Bigg\} \frac{1}{m_1!j_1!\cdots m_l!j_l!} \frac{\partial^{m_1+\cdots+m_l}}{\partial z_1^{m_1}\cdots \partial z_l^{m_l}} u(\mathbf{a};(0,\cdots,0)) X^{j_1+\cdots+j_l}, \tag{4.3.5}$$

and

$$\widetilde{R}_N(x) \ll_A \frac{R_N(x)}{(2\pi)^l} \prod_{j=1}^{l} \oint_{|z_j|=r_j} (\log x)^{\frac{\Re(z_j)}{\phi(q)}} \frac{|dz_j|}{|z_j|^{k_j+1}}. \tag{4.3.6}$$

Taking $r_j = \frac{\phi(q)k_j}{\log\log x}$, we have

$$\begin{aligned}
\oint_{|z_j|=r_j} (\log x)^{\frac{\Re(z_j)}{\phi(q)}} \frac{|dz_j|}{|z_j|^{k_j+1}} &= \left(\frac{\log\log x}{\phi(q)k_j}\right)^{k_j} \int_0^{2\pi} e^{k_j\cos\theta} d\theta \\
&\leq \left(\frac{\log\log x}{\phi(q)k_j}\right)^{k_j} \left(2\int_0^{\frac{\pi}{2}} e^{k_j\cos\theta} d\theta + \pi\right) \\
&= \left(\frac{\log\log x}{\phi(q)k_j}\right)^{k_j} \left(2\int_0^{1} e^{k_j t} \frac{dt}{\sqrt{1-t^2}} + \pi\right) \\
&\leq \left(\frac{\log\log x}{\phi(q)k_j}\right)^{k_j} \left(2e^{k_j}\int_0^{1} e^{-k_j(1-t)} \frac{dt}{\sqrt{1-t}} + \pi\right) \\
&\leq \left(\frac{\log\log x}{\phi(q)k_j}\right)^{k_j} \left(2\Gamma\left(\frac{1}{2}\right) e^{k_j} k_j^{-\frac{1}{2}} + \pi\right).
\end{aligned} \tag{4.3.7}$$

Since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, substitute (4.3.7) into (4.3.6),

$$\widetilde{R}_N(x) \ll_{A,l} \frac{1}{\phi^k(q)} \frac{(\log\log x)^k}{k_1!\cdots k_l!} R_N(x).$$

Theorem 4.4 follows. □

Remark 4.1. Similar to the proof of Lemma 4.3, we write

$$u(\mathbf{a},\mathbf{z}) = u(\mathbf{a},\mathbf{r}) + \sum_{|\mathbf{v}|=1} D^{\mathbf{v}} u(\mathbf{a},\mathbf{r}) + \sum_{|\boldsymbol{\beta}|=2} (\mathbf{z}-\mathbf{r})^{\boldsymbol{\beta}} R_{\boldsymbol{\beta}}(\mathbf{z}),$$

where

$$R_{\boldsymbol{\beta}}(\mathbf{z}) = \frac{|\boldsymbol{\beta}|}{\boldsymbol{\beta}!} \int_0^1 (1-t) D^{\boldsymbol{\beta}} u(\mathbf{a},\mathbf{r}+t(\mathbf{z}-\mathbf{r}))dt.$$

Then, by (4.2.6) and (4.3.3), using a similar proof as in Lemma 4.3, we have

$$M_k(x;\mathbf{a}) = \frac{1}{\phi^k(q)} \frac{k}{k_1!k_2!\cdots k_l!} \frac{x}{\log x} (\log\log x)^{k-1} \left\{ g\left(\frac{\phi(q)}{\log\log x};\mathbf{k}\right) + O_{A,q}\left(\frac{k}{(\log\log x)^2}\right)\right\}, \tag{4.3.8}$$

where $g(z;\mathbf{k}) := \sum_{j=1}^{l} \frac{k_j}{k} u(\mathbf{a};(k_1 z,\ldots,k_{j-1}z,k'_j z,k_{j+1}z,\ldots,k_l z))$ with $k'_j = k_j - 1$. Moreover, if $|kz| \le A$, then $|g(z,\mathbf{k})| = O_{A,q}(1)$.

4.4 Proof of Theorems 1.12 and 1.13

For $\mathbf{a} = (a,\ldots,a)$, $(a,q)=1$, this is a special case of Theorem 4.4. Denote

$$H(s;a,z) := F(s;a,z)(\zeta(s))^{-\frac{z}{\phi(q)}} = \prod_p \left(1-\frac{1}{p^s}\right)^{\frac{z}{\phi(q)}} \left(1+\frac{z\lambda_a(p)}{p^s}\right),$$

and

$$h(a;z) := \frac{H(1;a,z)}{\Gamma\left(\frac{z}{\phi(q)}+1\right)}. \tag{4.4.1}$$

Hence, by Theorem 4.4 and (4.3.8), we get the following result.

Lemma 4.6. *For $\mathbf{a} = (a,\ldots,a)$ and any $A>0$, uniformly for $2 \le k \le A\log\log x$, we have*

$$M_k(x;\mathbf{a}) = \frac{x}{\log x}\left\{ \frac{1}{\phi(q)} \sum_{0\le j\le N} \frac{P_{j,k}(\frac{\log\log x}{\phi(q)})}{(\log x)^j} + O_A\left(\frac{1}{\phi^k(q)} \frac{(\log\log x)^k}{k!} R_N(x)\right)\right\},$$

where $P_{j,k}(X)$ *is a polynomial of degree at most* $k-1$*, and in particular,*

$$P_{0,k} := \sum_{m+l=k-1} \frac{1}{m!l!} h^{(m)}(a;0) X^l,$$

and $R_N(x) = R_N(x; c_1, c_2) = e^{-c_1\sqrt{\log x}} + \left(\frac{c_2 N+1}{\log x}\right)^{N+1}$ *for some constants* c_1 *and* c_2 *depending on* A *and* q*. Moreover, under the same assumptions, we have*

$$\begin{aligned} M_k(x;\mathbf{a}) =& \frac{1}{\phi^k(q)} \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \Bigg\{ 1 + \frac{k-1}{\log\log x} C_{a,q} \\ &+ \frac{(k-1)(k-2)\phi^2(q)}{(\log\log x)^2} \widetilde{h}\left(a; \frac{(k-3)\phi(q)}{\log\log x}\right) + O_{A,q}\left(\frac{k^2}{(\log\log x)^3}\right) \Bigg\}, \end{aligned}$$

where

$$C_{a,q} := \phi(q) h'(a,0) = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{\phi(q)\lambda_a(p)}{p} \right),$$

$\gamma \approx 0.57722$ *is Euler's constant, and*

$$\widetilde{h}(a;z) = \int_0^1 h''(a,tz)(1-t)dt.$$

Remark 4.2. Notice that, for $|z| \le A$, the function $|h''(a,z)| = O_{q,A}(1)$.

We also require a formula for $S_k(x)$. By Lemma 4.5, and Definitions 4.1 and 4.2, $F(s;z)$ is in $\mathcal{T}(z,w;c_0,\delta,M)$. Denote

$$G(s;z) := F(s;z)(\zeta(s))^{-z} = \prod_p \left(1 - \frac{1}{p^s}\right)^z \left(1 + \frac{z\chi_0(p)}{p^s}\right),$$

and

$$g(z) := \frac{G(1;z)}{\Gamma(z+1)}. \tag{4.4.2}$$

Then, applying Lemma 4.2 and Lemma 4.3 successively, we get the following lemma.

Lemma 4.7. *For any $A > 0$, uniformly for $2 \le k \le A \log\log x$, we have*

$$S_k(x) = \frac{x}{\log x}\left\{\frac{\widetilde{P}_{j,k}(\log\log x)}{(\log x)^j} + O_A\left(\frac{(\log\log x)^k}{k!}R_N(x)\right)\right\},$$

where $\widetilde{P}_{j,k}(X)$ is a polynomial of degree at most $k-1$, and in particular,

$$\widetilde{P}_{0,k}(X) := \sum_{m+l=k-1} \frac{1}{m!l!} g^{(m)}(0) X^l.$$

Moreover, under the same assumptions, we have

$$\begin{aligned} S_k(x) =& \frac{x}{\log x}\frac{(\log\log x)^k}{(k-1)!}\left\{1 + \frac{k-1}{\log\log x} g'(0) + \frac{(k-1)(k-2)}{(\log\log x)^2}\widetilde{g}\left(\frac{k-3}{\log\log x}\right)\right. \\ &\left. + O_{A,q}\left(\frac{k^2}{(\log\log x)^3}\right)\right\}, \end{aligned}$$

where $g'(0) = \gamma + B - \sum_{p|q} \frac{1}{p}$, γ is Euler's constant, $B = \sum_p \left(\log\left(1-\frac{1}{p}\right) + \frac{1}{p}\right)$ is the Mertens constant in (1.3.2), and

$$\widetilde{g}(z) = \int_0^1 g''(tz)(1-t)dt.$$

Remark 4.3. Here for $|z| \le A$, the function $|g''(z)| = O_{q,A}(1)$.

Proof of Theorem 1.12. By Lemmas 4.6 and 4.7, we get

$$\begin{aligned} M_k(x, \mathbf{a}) - \frac{1}{\phi^k(q)}S_k(x) = & \frac{1}{\phi^k(q)}\frac{x}{\log x}\frac{(\log\log x)^{k-2}}{(k-2)!}\left\{C(q,a) + \frac{k-2}{\log\log x}\phi^2(q)\widetilde{h}\left(a; \frac{(k-3)\phi(q)}{\log\log x}\right)\right. \\ &\left. - \frac{k-2}{\log\log x}\widetilde{g}\left(\frac{k-3}{\log\log x}\right) + O_{A,q}\left(\frac{k}{(\log\log x)^2}\right)\right\}. \end{aligned}$$

For the cases of fixed k and $k = o(\log\log x)$, by Remarks 4.2 and 4.3, and Lemma 4.7, we immediately get the conclusions in Theorem 1.12 using the equality

$$\frac{M_k(x,\mathbf{a})}{\frac{1}{\phi^k(q)}S_k(x)} = 1 + \frac{M_k(x,\mathbf{a}) - \frac{1}{\phi^k(q)}S_k(x)}{\frac{1}{\phi^k(q)}S_k(x)}.$$

For any fixed $A > 0$, if $k \sim A\log\log x$, by Lemmas 4.6 and 4.7, and (4.3.8), as $x \to \infty$, the above quotient will approach

$$\frac{h(a, A\phi(q))}{g(A)} = \prod_p \frac{1 + \frac{A\phi(q)\mathbb{1}_{p\equiv a \bmod q}(p)}{p}}{1 + \frac{A\chi_0(p)}{p}},$$

where $h(a,z)$ and $g(z)$ are defined in (4.4.1) and (4.4.2) respectively. □

Proof of Theorem 1.13. For fixed k, by Theorem 4.4 and Lemma 4.7, we have

$$\begin{aligned} &M_k(x;\mathbf{a}) - \frac{1}{\phi^k(q)}\frac{k!}{k_1!k_2!\cdots k_l!}S_k(x) \\ &\quad = \frac{1}{\phi^k(q)}\frac{k(k-1)}{k_1!k_2!\cdots k_l!}\frac{x}{\log x}(\log\log x)^{k-2}\left\{\frac{1}{k}\sum_{j=1}^{k} C(q,a_j) + O_{k,q}\left(\frac{1}{\log\log x}\right)\right\}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{M_k(x;\mathbf{a})}{\frac{1}{\phi^k(q)}\frac{k!}{k_1!k_2!\cdots k_l!}S_k(x)} &= 1 + \frac{M_k(x;\mathbf{a}) - \frac{1}{\phi^k(q)}\frac{k!}{k_1!k_2!\cdots k_l!}S_k(x)}{\frac{1}{\phi^k(q)}\frac{k!}{k_1!k_2!\cdots k_l!}S_k(x)} \\ &= 1 + \frac{k-1}{\log\log x}\frac{1}{k}\sum_{j=1}^{k} C(q,a_j) + O_{q,k}\left(\frac{1}{(\log\log x)^2}\right). \end{aligned}$$

For any fixed $A > 0$, if $k \sim A\log\log x$ and $e_j := \lim_{x\to\infty}\frac{k_j}{\log\log x}$ exists, by (4.3.8) and Lemma 4.7, as $x \to \infty$, the above quotient will approach

$$\frac{u(\mathbf{a};(\phi(q)e_1,\cdots,\phi(q)e_l))}{g(A)} = \prod_p \frac{\prod_{j=1}^{l}\left(1 + \frac{\phi(q)e_j\mathbb{1}_{p\equiv b_j \bmod q}(p)}{p}\right)}{1 + \frac{A\chi_0(p)}{p}},$$

where $u(\mathbf{a};\mathbf{z})$ and $g(z)$ are defined in (4.3.4) and (4.4.2) respectively. □

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