SOME RESULTS ON SYMMETRIC SIGNINGS

BY

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THESIS

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ABSTRACT

In this work, we investigate several natural computational problems related to identifying symmetric signings of symmetric matrices with specific spectral properties. We show NP-completeness for verifying whether an arbitrary matrix has a symmetric signing that is positive semi-definite, is singular, or has bounded eigenvalues. We exhibit a stark contrast between invertibility and the above-mentioned spectral properties by presenting a combinatorial characterization of matrices with invertible symmetric signings and an efficient algorithm using this characterization to verify whether a given matrix has an invertible symmetric signing. Finally, we give efficient algorithms to verify and find invertible and singular symmetric signing for matrices whose support graph is bipartite.

To my mom and dad who let me do me.

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CHAPTER 1

INTRODUCTION

For a real symmetric $n \times n$ matrix M and a $n \times n$ matrix s taking values in $\{\pm 1\}$—which we refer to as a *signing*—we define the *signed matrix* $M(s)$ to be the matrix obtained by taking entry-wise products of M and s. We say that s is a *symmetric* signing if s is a symmetric matrix and an *off-diagonal* signing if s takes value $+1$ on the diagonal.

Signed adjacency matrices (respectively, Laplacians) can be interpreted as the adjacency matrix (respectively, Laplacian) of a *signed graph*. That is, a graph where edges are assigned a positive or negative *weights*. Signed graphs and their associated matrices have been used as early as 1953 by Harary [2] to model social relations such as disliking, indifference, and liking. They remain a regular tool for modeling such systems as well as an object of independent interest in many areas of combinatorics and computer science [3, 4, 5].

In this work, we present a study of the spectra and invertibility of symmetric signings of matrices. We consider several natural spectral properties and address the computational problems of finding/verifying the existence of symmetric signings with these properties given a symmetric matrix. We recall that a real symmetric matrix is *positive semi-definite* (psd) if all its eigenvalues are non-negative. We study the following computational problems:

BoundedEvalueSigning: Given a real symmetric matrix M and a real number λ, verify if there exists an off-diagonal symmetric signing s such that the largest eigenvalue $\lambda_{\max}(M(s))$ is at most λ.

PsdSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is positive semi-definite.

SingularSigning: Given a real symmetric matrix M, verify if there exists an off-diagonal symmetric signing s such that $M(s)$ is singular.

InvertibleSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is invertible (that is, non-singular).

The main motivation behind the study of BoundedEvalueSigning is its relation to constructing optimal expanders. The Alon-Boppana bound [6] shows that $\lambda_2(G) \geq 2\sqrt{d-1} - o(1)$ for every d-regular graph G. We say that a graph G is *Ramanujan* if $\lambda_2(G) \leq 2\sqrt{d-1}$. Such graphs are optimal expanders and are of great interests in many areas of mathematics and computer science research.

The construction of Ramanujan graphs by Lubotzky, Philips, and Sarnak [7] was a landmark achievement. However, their construction only produced Ramanujan graphs of certain degrees. Thus, the *efficient* construction of Ramanujan graphs of *all* degrees remains an important open problem. A combinatorial approach to this problem, initiated by Friedman [8], is to obtain larger Ramanujan graphs from a smaller ones by taking *lifts* of the smaller graph. This operation preserves degree and hence its repeated application produces an Ramanujan graphs of specific degree and arbitrary size.

A 2-*lift* H of G is obtained by replacing each vertex v of G with two copies of itself, say v_1 and v_2, in H, and for each edge $\{u, v\}$ in G, introduce either $\{u_1, v_2\}, \{u_2, v_1\}$ or $\{u_1, v_1\}, \{u_2, v_2\}$ as edges of H.

It is easy to see that there exists a natural bijection between 2-lifts of a graph G and symmetric signings of its adjacency matrices. Moreover, the eigenvalues of the adjacency matrix of a 2-lift H are given by the union of the eigenvalues of the adjacency matrix of the base graph G (also called the "old" eigenvalues) and the signed adjacency matrix of G that corresponds to the 2-lift (the "new" eigenvalues).

A result of Marcus, Spielman, and Srivastava [9] shows that there is always a 2-lift of every d-regular bipartite graph that is Ramanujan. This immediately suggests an iterative algorithm that can construct bipartite Ramanujan graphs of arbitrary degree d and size given a small d-regular Ramanujan graph. We recall that it is easy to construct small d-regular Ramanujan graphs (consider $K_{d,d}$). Hence, to efficiently construct bipartite Ramanujan graphs of arbitrary degree and size we only need an efficient way to find 2-lifts of d-regular bipartite Ramanujan graphs with minimum λ_2. This naturally motivates BoundedEvalueSigning since an efficient algorithm to solve BoundedEvalueSigning would immediately suggest an efficient algorithm to construct bipartite Ramanujan graphs of any degree and size.

We note that Cohen [10] gave an efficient algorithm to find bipartite Ramanujan *multi-graphs*. However, it remains open to find an efficient construction of bipartite Ramanujan *simple* graphs of all degrees. We also note that Cohen's algorithm is not based on the above-mentioned lifting operation.

Another motivation of this work is the long history of research studying the determinant of signed adjacency matrices of graphs as it relates to several fundamental questions concerning graphs and linear systems [11, 12, 13, 14]. We mention two of these questions:

Pólya's scheme: Given an adjacency matrix A, is there a signing of A such that the permanent of A equals the determinant of the signed matrix?

Sign solvability: Given a real square matrix, is every real matrix with the same sign pattern (that is, the corresponding entries either have the same sign, or are both zero) invertible?

Both these questions are known to be equivalent and in particular, closely related to the problem of counting the number of perfect matchings in a given bipartite graph (see the survey by Thomas [13]).

In this work we also study the relationship between symmetric signings and manipulating the determinant of symmetric matrices. Namely, we investigate the complexity of SingularSigning and InvertibleSigning. However, we note that the signings studied in the related works mentioned above are not necessarily symmetric.

Intriguingly, the complexity of BoundedEvalueSigning has not been studied in the literature even though it is widely believed to be a difficult problem in the graph sparsification community. We shed light on this problem by showing that it is NP-complete. Owing to the close connection between the maximum eigenvalue, positive semi-definiteness, and singularity (by suitable translations), we obtain that PsdSigning and SingularSigning are also NP-complete.

Theorem 1.1. BoundedEvalueSigning*,* PsdSigning*, and* SingularSigning *are NP-complete.*

We remark that the hard instances generated by our proof of Theorem 1.1 are real symmetric matrices with non-zero diagonal entries and hence, it does not completely resolve the computational complexity of the problem of finding a signing of a given *adjacency matrix* that minimizes its largest eigenvalue. However, it gives some indication that the task of making the result by Marcus *et al.*[9] constructive would require techniques that are specific to graphs and graph-related matrices.

In contrast, we next show that SingularSigning and its search variant admit efficient algorithms when the input matrix corresponds to the adjacency matrix of a bipartite graph. This result provides some evidence that an efficient algorithm to solve the NP-complete problems appearing in Theorem 1.1 for *graph-related* matrices may exist.

Theorem 1.2. *There exists a polynomial-time algorithm to verify if the adjacency matrix A_G of a given bipartite graph G has a symmetric signing s such that $A_G(s)$ is singular, and if so, find such a signing.*

We also show a stark difference in complexity between SingularSigning and InvertibleSigning. In contrast to SingularSigning which is NP-complete for arbitrary input matrices (Theorem 1.1), we show that InvertibleSigning is solvable in polynomial time for arbitrary input matrices.

Theorem 1.3. *There exists a polynomial-time algorithm to solve* InvertibleSigning*.*

Our algorithm for solving InvertibleSigning is based on a novel graph-theoretic characterization of symmetric matrices M for which every symmetric signed matrix $M(s)$ is singular. We believe that our characterization might be of independent interest. We describe the characterization now.

The *support graph* of a real symmetric $n \times n$ matrix M is an undirected graph G where the vertex set of G is $\{1, \ldots, n\}$, and the edge set of G is $\big\{\{u, v\} \mid M[u, v] \neq 0\big\}$. We note that G could have self-loops depending on the diagonal entries of M.

A *perfect 2-matching* in a graph G with edge set E is an assignment $x : E \to \{0, 1, 2\}$ of values to the edges such that $\sum_{e \in \delta(v)} x_e = 2$ holds for every vertex v in G (where $\delta(v)$ denotes the set of edges incident to v).

We show the following characterization from which Theorem 1.3 follows immediately.

Theorem 1.4. *Let M be a symmetric $n \times n$ matrix and let G be the support graph of M. The following are equivalent:*

1. *The signed matrix $M(s)$ is singular for every symmetric signing s.*

2. *The support graph G does not contain a perfect 2-matching.*

Moreover, there exists a polynomial-time algorithm to verify whether the signed matrix $M(s)$ is singular for every symmetric signing s.

We remark that Theorem 1.4 can also be stated with respect to *non-expanding independent sets*. For a subset S of vertices in a graph G, let $N_G(S)$ be the *non-inclusive neighborhood* of S, that is,

$$N_G(S) := \{u \in V \setminus S \mid \{u, v\} \text{ is an edge of } G \text{ for some } v \text{ in } S\}.$$

A subset S of vertices is said to be *independent* if there are no edges between any pair of vertices in S. A subset S of vertices is said to be *expanding* in G if $|N_G(S)| \geq |S|$.

Tutte [15] showed that the existence of a *non-expanding independent set* is equivalent to the absence of perfect 2-matchings in the graph, which in turn has been used in the study of independent sets [16, 17, 18].

Thus, Theorem 1.4 can be interpreted as a *spectral* characterization of graphs with non-expanding independent sets: a graph contains a non-expanding independent set if and only if every symmetric signed adjacency matrix of the graph is singular.

Our final result focuses on the search variant of InvertibleSigning. We mention that our proof of Theorem 1.4 is non-constructive; that is, even if the support graph of the given matrix has a perfect 2-matching, our proof does not lead to an efficient algorithm to find an invertible signing. While we do not have an efficient algorithm for the search problem for arbitrary symmetric matrices, we obtain an efficient algorithm for those whose support graph is bipartite. This may be evidence that the search variants of PsdSigning and SingularSigning are also solvable efficiently we restricted to matrices with bipartite support.

Theorem 1.5. *There exists a polynomial-time algorithm to verify if a given symmetric matrix M, whose support graph is bipartite, has a symmetric signing s such that $M(s)$ is invertible, and if so, find such a signing.*

1.1 Organization

In Section 1.2, we review definitions, notations, and results relevant to this work. In Chapter 2, we focus on our results related to invertible signings. This includes Section 2.1 which focuses on a combinatorial characterization of matrices with invertible signings (Theorem 1.4) and Section 2.2 which gives an algorithm to find an invertible signing of adjacency matrices of bipartite graphs (Theorem 1.5). In Chapter 3, we turn our focus to results related to singular signings. This includes an efficient algorithm to find a singular signing of adjacency matrices of bipartite graphs (Theorem 1.2) in Section 3.1, and a proof of NP-completeness of SingularSigning (Lemma 3.1) in Section 3.2. Finally, in Chapter 4 we complete Theorem 1.1 by showing that PsdSigning and BoundedEvalueSigning are also NP-complete. We conclude by discussing open questions and potential avenues for future research in Chapter 5.

1.2 Preliminaries

In this section we introduce definitions, notation, and theorems used throughout this work. We also discuss some related results. We assume the reader is familiar with basic graph theory and linear algebra.

1.2.1 Matchings

A *matching* in a graph G is a vertex-disjoint subset of the edge set E. A *perfect matching* in a graph G is a matching such that every vertex is incident to an edge.

A *perfect 2-matching* in a graph G is an assignment $x : E \to \{0, 1, 2\}$ of values to the edges such that $\sum_{e \in \delta(v)} x_e = 2$ holds for every vertex v in G (where $\delta(v)$ denotes the set of edges incident to v). We note that a perfect 2-matching in G can also be described a collection of vertex-disjoint edges, cycles and self-loops.

It is useful to note some key differences between bipartite graphs and general graphs with respect to perfect 2-matching. Namely, it is easy to see that a bipartite graph contains a perfect 2-matching if and only if it also contains a perfect matching. Thus, our main theorem immediately suggests that a bipartite graph contains no perfect matching if and only if every symmetric signing of its adjacency matrix is singular. This fact will later be exploited in Sections 2.2 and 3.1 to produce efficient algorithms to find invertible and singular signings of adjacency matrices of bipartite graphs.

1.2.2 Graph Theory and Linear Algebra

The *support graph* of a real symmetric $n \times n$ matrix M is an undirected graph G where the vertex set of G is $\{1, \ldots, n\}$, and the edge set of G is $\big\{\{u, v\} \mid M[u, v] \neq 0\big\}$. We note that G could have self-loops depending on the diagonal entries of M.

The *adjacency matrix* of a n-vertex graph G—denoted as A_G—is a $n \times n$ symmetric matrix where $A_G[u, v] = 1$ if $\{u, v\}$ is an edge in G and 0 otherwise. We note that A_G may have non-zero entries on its diagonal if the graph G has self-loops.

Let M be a $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. We recall that M is positive semi-definite if $\lambda_i \geq 0$ for all i and positive definite if the inequality is strict for all i. Since the determinant of a matrix is equal to the product of its eigenvalues, it follows that a matrix is positive definite only if its determinant is strictly greater than zero.

We use the notion of *Schur complement* repeatedly. The following lemma summarizes the definition and relevant properties of the Schur complement used in this work.

Lemma 1.1 (Horn and Johnson [19])**.** *Let D be a symmetric matrix whose blocks are of the following form (with appropriate dimensions):*

$$D = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

Suppose A is invertible. Then the Schur complement *of C in matrix D is defined to be*

$$D_C := C - BA^{-1}B^T.$$

We have the following properties:

(*i*) *Suppose A is positive definite. Then, D is positive semi-definite if and* *only if the Schur complement of C in D, namely D_C, is positive semi-definite.*

(ii) $\det(D) = \det(A) \cdot \det(D_C)$.

Let G be a graph on n vertices with edge set E. The *Tutte matrix* of G is a $n \times n$ matrix A such that

$$A[i,j] = \begin{cases} -x_{ij}, & \text{if } \{i,j\} \in E \text{and} i < j \\ x_{ij}, & \text{if } \{i,j\} \in E \text{and} j < i \\ 0, & \text{otherwise} \end{cases}.$$

Such matrices where $A[i,j] = -A[j,i]$ for all i and j are called *skew-symmetric*.

There are several known results that relate the existence of perfect matchings in a graph to the determinant of adjacency like matrices not being identically zero. One such result is thanks to Tutte [20] which shows that the determinant of the Tutte matrix of a graph G is identically zero if and only if G does not contain a perfect matching.

1.2.3 Matrix Signings

Unless otherwise specified, all matrices are symmetric and take values over the reals. We recall that for a real symmetric $n \times n$ matrix M, a *signing* of M is a $n \times n$ matrix s taking values in $\{\pm 1\}$. Moreover, we define the *signed matrix* $M(s)$ to be the matrix obtained by taking entry-wise products of M and s. For simplicity, in the rest of this work will use the term *signing* to refer to a symmetric signing.

Let S_n denote the set of permutations of n elements. Then, the *permutation expansion* of the determinant of a signed matrix $M(s)$ is given by

$$\det M(s) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \prod_{i=1}^{n} M(s)[i, \sigma(i)].$$

For ease of presentation, we define $M_\sigma(s) \coloneqq \mathrm{sgn}(\sigma) \cdot \prod_i M(s)[i, \sigma(i)]$ and $M_\sigma \coloneqq M_\sigma(J)$, where J is the signing corresponding to all entries being $+1$.

Then the permutation expansion can be written as

$$\det M(s) = \sum_{\sigma \in S_n} M_\sigma(s).$$

We recall that a permutation σ in S_n has a unique cycle decomposition that corresponds to a vertex disjoint union of directed cycles and self-loops on n vertices. Removing the orientation gives a vertex-disjoint union of cycles of length at least three, matching edges, and self-loops. Let the collection of edges in the cycle components, matching components, and self-loop components in the resulting undirected graph be denoted by Cycles(σ), Matchings(σ), and Loops(σ) respectively. We observe that sgn(σ) is the *parity* of the sum of the number of matching edges and the number of even-length cycles (cycles with an even number of edges) in the undirected subgraph induced by the edges in Cycles(σ) $\cup$ Matchings(σ). For a matrix M and a signing s, we define

$$M_{\text{Cycles}}(\sigma, s) := \left(\prod_{\{u,v\} \in \text{Cycles}(\sigma)} M(s)[u,v] \right),$$

$$M_{\text{Matchings}}(\sigma, s) := \left(\prod_{\{u,v\} \in \text{Matchings}(\sigma)} M(s)[u,v]^2 \right), \text{ and}$$

$$M_{\text{Loops}}(\sigma, s) := \left(\prod_{\{u,u\} \in \text{Loops}(\sigma)} M(s)[u,u] \right).$$

We use the convention that a product over an empty set is equal to 1. With this notation, we have

$$M_\sigma(s) = \text{sgn}(\sigma) \cdot M_{\text{Cycles}}(\sigma, s) \cdot M_{\text{Matchings}}(\sigma, s) \cdot M_{\text{Loops}}(\sigma, s).$$

Hence, the parity of $M_\sigma(s)$ is completely determined by the set of cycle and loop edges of σ.

CHAPTER 2

MATRICES WITH INVERTIBLE SIGNINGS

In this chapter, we focus on invertible signings. First, in Section 2.1 we prove Theorem 1.4. Next, in Section 2.2 we present an algorithm to find an invertible signing of the adjacency matrix of a given bipartite graph. This completes the proof of Theorem 1.5.

2.1 Invertibility Characterization

To prove Theorem 1.4 we introduce the following lemma which relates the existence of a invertible signing to the existence of non-zero terms in the determinant.

Lemma 2.1. *Let M be a symmetric $n \times n$ matrix. Then $M_\sigma = 0$ for every permutation σ in S_n if and only if $M(s)$ is singular for all signings s.*

Proof. We first show the forward implication which follows almost immediately from the definition of the permutation expansion. Suppose $M_\sigma = 0$ for every permutation σ in S_n. Then $M_\sigma(s) = 0$ for every permutation σ in S_n and every signings s. That is, every term in the permutation expansion of the determinant of $M(s)$ is zero for every signings s.

To complete the proof we now show the contrapositive of the reverse implication. Suppose that there is a non-empty subset of permutations Σ such that $M_\sigma = 0$ holds for all $\sigma \in \Sigma$. Let τ be a permutation in Σ with the fewest number of cycle and loop edges and $\mathrm{T} \subseteq \Sigma$ be the set of permutations with the same set of cycle and loop edges as τ. We recall that $M_\sigma(s) = M_\tau(s)$ for all permutations $\sigma \in \mathrm{T}$ and signings s since. We also note $\mathrm{Cycles}(\sigma) \cup \mathrm{Loops}(\sigma) \setminus \mathrm{Cycles}(\tau) \cup \mathrm{Loops}(\tau) \neq \emptyset$ for all $\sigma \in \Sigma \setminus \mathrm{T}$.

Let Q be the set of signings s such that $s_{ij} = 1$ for $(i, j) \in \mathrm{Cycles}(\tau)$. It follows that for $\sigma \in \Sigma \setminus \mathrm{T}$, the number of signings s in Q where $M_\sigma(s)$ is positive is equal to the number of signings s in Q where $M_\sigma(s)$ is negative. Moreover, for all signings s in Q the parity of $M_\tau(s)$ is the same. Hence,

$$\begin{aligned}\sum_{s\in Q}\sum_{\sigma\in S_n} M_\sigma(s) &= \sum_{s\in Q}\sum_{\sigma\in\Sigma} M_\sigma(s)\\ &= \sum_{s\in Q}\sum_{\sigma\in\mathrm{T}} M_\sigma(s) + \sum_{s\in Q}\sum_{\sigma\in\Sigma\setminus\mathrm{T}} M_\sigma(s)\\ &= \pm 2^{|Q|}|\mathrm{T}|.\end{aligned}$$

Thus, there must exist a signing s in Q where $\det M(s)$ is not zero. □

The author is aware of several proofs for Lemma 2.1. Among them include a proof using the DeMillo-Lipton-Schwartz-Zippel lemma [21, 22, 23] by exploiting the low-degree nature of the multivariate determinant polynomial and a similar proof to the one provided that uses a probabilistic argument. All known proofs are non-constructive but this proof is presented for its simplicity.

To complete the proof of Theorem 1.4, we use the following lemma about the complexity of verifying the existence of a perfect 2-matching in a given graph. The lemma follows from a well-known reduction to the perfect matching problem in bipartite graphs.

Lemma 2.2 (Tutte [15]; Lovász and Plummer [24, Corollary 6.1.5])**.** *There exists a polynomial-time algorithm to verify if a given graph (possibly with loops) has a perfect* 2*-matching.*

We now have everything required to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. By Lemma 2.1, the signed matrix $M(s)$ is singular for every signing s if and only if $M_\sigma = 0$ holds for every permutation σ in S_n. The existence of a perfect 2-matching in the support graph of M is equivalent to the fact that $M_\sigma \neq 0$ for some σ in S_n, and therefore we have that $M_\sigma = 0$ for every σ in S_n if and only if the support graph of M has no perfect 2-matchings. Moreover, Lemma 2.2 immediately gives us a polynomial-time algorithm to verify whether the signed matrix $M(s)$ is singular for every signing s. □

2.2 Finding Invertible Signings of Bipartite Graphs

In this section we present an algorithm to find an invertible signing of the adjacency matrix of a given bipartite graph. We first need to define one additional concept in regards to matrix signings.

We say that a signing s' *extends* another signing s *on entry* (u, v) if $s'[i, j] = s[i, j]$ for every entry $(i, j) \notin \{(u, v), (v, u)\}$. Thus, if s' extends a signing s on entry (u, v), then s' could be s or it differs from s only in the entry corresponding to u'th row and v'th column (and by symmetry, the entry corresponding to v'th row and u'th column). We now have the ingredients to show that incrementing a signing while preserving invertibility is possible.

Lemma 2.1 (Incremental Signing)**.** *Let G be a bipartite graph with bipartition (L, R) of the vertex set, and let A_G be the adjacency matrix of G. Suppose there exists a signing s such that $A_G(s)$ is invertible. Let $\ell \in L, r \in R$ be vertices in G such that $e := \{\ell, r\}$ is not an edge of G. Then there exists a signing s' that extends s on (ℓ, r) such that $A_{G+e}(s')$ is invertible, where $G + e$ is the graph obtained by adding the edge e to G.*

Proof. Let n be the number of vertices in G. Let s be a signing such that $A_G(s)$ is invertible. Let s' be an $n \times n$ matrix where $s'[i, j] = s[i, j]$ for all pairs of (i, j) besides (ℓ, r) and (r, ℓ), and set $s'[\ell, r]$ (and thus by symmetry, $s'[r, \ell]$) to be a variable x. Let b_ℓ and b_r be vectors of length $n - 2$ such that $b_\ell[i] = A_G(s')[\ell, i]$ and $b_r[i] = A_G(s')[r, i]$ for every i not equal to ℓ or r. For a subgraph G' of G with adjacency matrix $A_{G'}$, let $A_{G'}(s)$ denote the signed adjacency matrix of G' obtained by the entry-wise product of $A_{G'}$ and the signing obtained by projecting s to the edges of G'.

Consider the matrix $A_{G+e}(s')$ obtained by taking entry-wise product of A_{G+e} and s'. Let H be the graph obtained by removing vertices r and ℓ from G, and let A_H be the adjacency matrix of H. In the notation defined, we have

$$A_{G+e}(s') = \begin{bmatrix} 0 & x & b_\ell \\ x & 0 & b_r \\ b_\ell^T & b_r^T & A_H(s) \end{bmatrix}$$

with the first and second rows (by symmetry, columns) corresponding to vertices ℓ and r respectively. Let $f(x) := \det(A_{G+e}(s'))$. We have that

$$f(x) = -\det\left(A_H(s)\right)x^2 - \det\left(\begin{bmatrix} 0 & b_r \\ b_\ell^T & A_H(s) \end{bmatrix}\right)x + \det\left(A_G(s)\right).$$

We note that $f(x)$ is a quadratic function of x. Now suppose for the sake of contradiction that the matrix $A_{G+e}(s')$ is singular for both $x = \pm 1$. Then $f(1) = f(-1) = 0$ and hence the following holds.

$$\det\left(\begin{bmatrix} 0 & b_r \\ b_\ell^T & A_H(s) \end{bmatrix}\right) = 0 \tag{2.1}$$

$$\det(A_H(s)) = \det(A_G(s)) \tag{2.2}$$

We recall that $\det(A_G(s)) \neq 0$ and hence $\det(A_H(s)) \neq 0$ by equation (2.2). Since $\det(A_H) \neq 0$, we use the property of the Schur complement (Lemma 1.1) to obtain that

$$\begin{aligned} \det(A_G(s)) &= \det(A_H(s)) \cdot \det\left(0 - \begin{bmatrix} b_\ell \\ b_r \end{bmatrix} A_H(s)^{-1} \begin{bmatrix} b_\ell^T & b_r^T \end{bmatrix}\right) \\ &= \det(A_H(s)) \cdot \det\left(\begin{bmatrix} b_\ell A_H(s)^{-1} b_\ell^T & b_\ell A_H(s)^{-1} b_r^T \\ b_r A_H(s)^{-1} b_\ell^T & b_r A_H(s)^{-1} b_r^T \end{bmatrix}\right). \end{aligned}$$

Using equation (2.2), we thus have

$$\det\left(\begin{bmatrix} b_\ell A_H(s)^{-1} b_\ell^T & b_\ell A_H(s)^{-1} b_r^T \\ b_r A_H(s)^{-1} b_\ell^T & b_r A_H(s)^{-1} b_r^T \end{bmatrix}\right) = 1. \tag{2.3}$$

Let $G - \ell$ and $G - r$ be the graphs obtained by removing vertices ℓ and r from G respectively. Then by applying the Schur complement on $A_{G-r}(s)$ (Lemma 1.1), we have that

$$\det(A_{G-r}(s)) = \det\left(\begin{bmatrix} 0 & b_r \\ b_r^T & A_H(s) \end{bmatrix}\right) = \det(A_H(s)) \cdot \det(0 - b_r A_H(s)^{-1} b_r^T), \tag{2.4}$$

and hence

$$\det(A_{G-r}(s)) = -\det(A_H(s)) \cdot b_r A_H(s)^{-1} b_r^T. \tag{2.5}$$

Similarly, we also have

$$\det(A_{G-\ell}(s)) = -\det(A_H(s)) \cdot b_\ell A_H(s)^{-1} b_\ell^T. \tag{2.6}$$

Moreover, by equation (2.1) and the property of Schur complement (Lemma 1.1), we have that

$$0 = \det\left(\begin{bmatrix} 0 & b_r \\ b_\ell^T & A_H(s) \end{bmatrix}\right) = \det(A_H(s)) \cdot \det(0 - b_r A_H(s)^{-1} b_\ell^T).$$

Hence,

$$b_r A_H(s)^{-1} b_\ell^T = 0. \tag{2.7}$$

Similarly, we also have

$$b_\ell A_H(s)^{-1} b_r^T = 0. \tag{2.8}$$

Thus, using equations (2.5), (2.6), (2.7), and (2.8), we have

$$\det\left(\begin{bmatrix} b_\ell A_H(s)^{-1} b_\ell^T & b_\ell A_H(s)^{-1} b_r^T \\ b_r A_H(s)^{-1} b_\ell^T & b_r A_H(s)^{-1} b_r^T \end{bmatrix}\right) = \frac{\det(A_{G-r}(s))}{\det(A_H(s))} \cdot \frac{\det(A_{G-\ell}(s))}{\det(A_H(s))}. \tag{2.9}$$

However, since G is bipartite and has a perfect 2-matching, the subgraphs $G-r$ and $G-\ell$ must be bipartite and have an odd number of vertices. Hence, the subgraphs $G-r$ and $G-\ell$ have no perfect 2-matching. Thus, by Lemma 2.2 and the backward direction of Theorem 1.4, we have $\det(A_{G-r}(s)) = \det(A_{G-\ell}(s)) = 0$ which together with equation (2.9) contradicts equation (2.3). $\square$

Lemma 2.1 suggests a natural algorithm to find an invertible signing of the adjacency matrix of a given bipartite graph in polynomial time that is presented in Figure 2.1. The correctness of the algorithm follows from Lemma 2.1. It can be implemented to run in polynomial time since a perfect matching in a bipartite graph can be found efficiently and moreover, Step 4.2 only requires us to consider the determinant of the signed adjacency matrix of $H+e$ for the two possible signings s' that extend s on (ℓ, r) (where the two extensions are obtained by signing the edge e as ± 1). This completes the proof of Theorem 1.5. Our algorithm also gives an alternative constructive

FINDINVERTIBLESIGNINGBIPARTITE(G): *Input*: A bipartite graph G.

1. Find a perfect matching M in G.
2. Let H be the subgraph of G with edge set M.
3. Let s be the all-one signing.
4. While $G \neq H$:
   - 4.1. Let $e := \{\ell, r\}$ be an edge in G but not in H.
   - 4.2. Find a signing s' that extends s on (ℓ, r) such that $A_{H+e}(s')$ is invertible.
   - 4.3. Update $s \leftarrow s'$ and $H \leftarrow H + e$.
5. Return s.

Figure 2.1: The algorithm FINDINVERTIBLESIGNINGBIPARTITE(G).

proof of Theorem 1.4 for matrices whose support graph is bipartite.

CHAPTER 3

SINGULAR MATRICES

In this chapter, we focus on singular signings. First we give an efficient algorithm to find a singular signing of adjacency matrices of bipartite graphs in Section 3.1. This completes the proof of Theorem 1.2. Next, in Section 3.2 we prove that SingularSigning is NP-complete—which will be used to complete the proof of Theorem 1.1 in Chapter 4, and Theorem 1.2.

3.1 Finding Singular Signings of Bipartite Graphs

In this section, we characterize bipartite graphs whose signed adjacency matrix is invertible for all signings. We use this characterization to prove Theorem 1.2. We will use the following results by Little [25] for our characterization. (Lemma 3.1 is a slight extension to the original result by Little.

Lemma 3.1 (Little [25])**.** *Let G be a graph with adjacency matrix A_G. Then $\det(A_G(s))$ is even for all signings s if and only if G has an even number of perfect matchings.*

Theorem 3.2 (Little [25])**.** *Let G be a graph. Then G has an even number of perfect matchings if and only if there is a set $S \subseteq V(G)$ such that every vertex in G has even number of neighbors in S. Moreover, if G has an even number of perfect matchings, then such a set S can be found in polynomial time.*

We now have the ingredients to characterize bipartite graphs whose signed adjacency matrix is invertible for all signings.

Lemma 3.3. *Let G be a bipartite graph and let A_G be the adjacency matrix of G. Then G has an odd number of perfect matchings if and only if $\det(A_G(s)) \neq 0$ for all signings s.*

Proof. Suppose G has an odd number of perfect matchings. By Lemma 3.1, we have that $\det(A_G(s)) \neq 0$ for all signings s.

Now suppose that G has an even number of perfect matchings. By Theorem 3.2, there exists a set $S \subseteq V(G)$ such that $|N_G(v) \cap S|$ is even for all $v \in V(G)$. We observe that the subgraph $G[S]$ induced by S is bipartite with every vertex having even degree. Thus, any closed walk on $G[S]$ has even number of edges and every connected component in $G[S]$ has an Eulerian tour with even number of edges. Let C be a connected component of $G[S]$ with m edges and let $T := (e_1, e_2, \ldots, e_m)$ be an ordering of the edges that represents an Eulerian tour of C. Then we sign edge e_i to be positive if i is even and negative otherwise. Every vertex $v \in V(G) \setminus S$ has even number of edges between v and vertices in S. We partition the edges incident to v into two arbitrary parts of equal size and sign all the edges in one part to be positive and the rest of the edges in the other part to be negative. Let $\hat{s}$ denote the resulting signing.

Under the signing $\hat{s}$ every vertex v of G has an equal number of positive and negative edges to vertices in S. Thus, the sum of the column vectors corresponding to the vertices in S will be zero and hence $\det(A_G(\hat{s})) = 0$. □

We note that the proof of Lemma 3.3 is constructive since we can find a set S for which every vertex has even number of neighbors in S in polynomial time by Theorem 3.2. Thus, Theorem 1.2 follows from Theorem 3.2 and Lemma 3.3.

3.2 Hardness of Singular Signing Problem

In this section we prove that SINGULARSIGNING is NP-complete. In order to show this result, we reduce from the partition problem, which is a well-known NP-complete problem [26]. We recall the problem below:

PARTITION: Given an n-dimensional vector b of non-negative integers, determine if there is a ± 1-signing vector z such that the inner product $\langle b, z \rangle$ equals zero.

Lemma 3.1. SINGULARSIGNING *is NP-complete.*

Proof. SINGULARSIGNING is in NP since if there is an (off-diagonal) signing of the given matrix that is positive semi-definite or singular, then this signing gives the witness. In particular, we can verify if a given (off-diagonal) symmetric signed matrix is positive semi-definite or singular in polynomial time by computing its spectrum [27].

We show NP-hardness of SINGULARSIGNING by reducing from PARTITION. Let the n-dimensional vector $b := (b_1, \ldots, b_n)^T$ be the input to PARTITION, where each b_i is a non-negative integer. We construct a matrix M as an instance of SINGULARSIGNING as follows: Consider the following $(n+2) \times (n+2)$-matrix

$$M := \begin{bmatrix} I_n & b & \mathbf{1}_n \\ b^T & \langle b, b \rangle & 0 \\ \mathbf{1}_n^T & 0 & n \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n-dimensional column vector of all ones. Claim 3.2 proves the correctness of the reduction to SINGULARSIGNING. □

Claim 3.2. *The matrix M has a symmetric off-diagonal signing s such that $M(s)$ is singular if and only if there is a vector $z \in \{\pm 1\}^n$ such that the inner product $\langle b, z \rangle$ is zero.*

Proof. Construct the Schur complement M'_C of C of M' as in Claim 4.2. Using property (ii) of Lemma 1.1, we have that

$$\det M' = \det(I_n) \cdot \det(M'_C) = \det(I_n) \cdot \det\left(\begin{bmatrix} 0 & -\langle \hat{b}, z \rangle \\ -\langle \hat{b}, z \rangle & 0 \end{bmatrix}\right) = -\langle \hat{b}, z \rangle^2.$$

Therefore, $\det M' = 0$ if and only if $\langle \hat{b}, z \rangle = 0$. We note that $\langle \hat{b}, z \rangle = 0$ if and only if there is a ± 1-vector z' such that $\langle b, z' \rangle = 0$. □

CHAPTER 4

HARDNESS OF EIGENVALUE PROBLEMS

In this chapter we prove that PsdSigning and BoundedEvalueSigning are NP-complete. Together with Lemma 3.1 this completes the proof of Theorem 1.1.

4.1 Hardness of Positive Semi-definite Signing Problem

In order to show the NP-completeness of PsdSigning, we again reduce from Partition [26]. The proof has a similar outline to the NP-completeness proof of SingularSigning (Lemma 3.1).

Lemma 4.1. PsdSigning *is NP-complete.*

Proof. PsdSigning is in NP since if there is an (off-diagonal) signing of the given matrix that is positive semi-definite, then this signing gives the witness. In particular, we can verify if a given (off-diagonal) symmetric signed matrix is positive semi-definite in polynomial time by computing its spectrum [27].

We show NP-hardness of PsdSigning by reducing from Partition. Let the n-dimensional vector $b := (b_1, \ldots, b_n)^T$ be the input to the Partition problem, where each b_i is a non-negative integer. We construct a matrix M as an instance of PsdSigning as follows: Consider the following $(n+2) \times (n+2)$-matrix

$$M := \begin{bmatrix} I_n & b & \mathbf{1}_n \\ b^T & \langle b, b \rangle & 0 \\ \mathbf{1}_n^T & 0 & n \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n-dimensional column vector of all ones. Claim 4.2 proves the correctness of the reduction to PsdSigning. $\square$

Claim 4.2. *The matrix M has a signing s such that $M(s)$ is positive semi-definite if and only if there is a ± 1-vector z such that the inner product $\langle b, z\rangle$ is zero.*

Proof. We may assume that any signed matrix $M(s)$ that is positive semi-definite may not have negative entries in the diagonal because a positive semi-definite matrix will not have negative entries on its diagonal. Hence, we will only consider symmetric off-diagonal signing s of the matrix M of the following form:

$$M' := M(s) = \begin{bmatrix} I_n & \hat{b} & z \\ \hat{b}^T & \langle b, b\rangle & 0 \\ z^T & 0 & n \end{bmatrix},$$

where the n-dimensional vector z takes values in $\{\pm 1\}^n$ and $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_n)^T$, where $\hat{b}_i$ takes value in $\{\pm b_i\}$ for every i. Let

$$\begin{aligned} A &:= I_n, \\ B &:= \begin{bmatrix} \hat{b} & z \end{bmatrix}, \text{ and} \\ C &:= \begin{bmatrix} \langle b, b\rangle & 0 \\ 0 & n \end{bmatrix}. \end{aligned}$$

Since $A = I_n$ is invertible, the Schur complement of C in M' is well-defined and is given by

$$\begin{aligned} M'_C &= \begin{bmatrix} \langle b, b\rangle & 0 \\ 0 & n \end{bmatrix} - \begin{bmatrix} \hat{b}^T \\ z^T \end{bmatrix} I_n^{-1} \begin{bmatrix} \hat{b} & z \end{bmatrix} \\ &= \begin{bmatrix} \langle b, b\rangle & 0 \\ 0 & n \end{bmatrix} - \begin{bmatrix} \langle \hat{b}, \hat{b}\rangle & \langle \hat{b}, z\rangle \\ \langle \hat{b}, z\rangle & \langle z, z\rangle \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\langle \hat{b}, z\rangle \\ -\langle \hat{b}, z\rangle & 0 \end{bmatrix}, \end{aligned}$$

where the last equation follows because we have $\langle \hat{b}, \hat{b}\rangle = \langle b, b\rangle$ and $\langle z, z\rangle = n$.

We note that $A = I_n$ is positive definite. Therefore, by property (1) of Lemma 1.1, the matrix M' is positive semi-definite if and only if M'_C is positive semi-definite. Therefore, M' is positive semi-definite if and only if $\langle \hat{b}, z\rangle = 0$. Finally, we note that $\langle \hat{b}, z\rangle = 0$ if and only if there is a ± 1-vector

z' such that $\langle b, z' \rangle = 0$. □

4.2 Hardness of Bounded Eigenvalue Signing Problem

To prove that BoundedEvalueSigning is NP-complete, we consider the following problem that is closely related to PsdSigning:

NsdSigning: Given a real symmetric matrix M, verify if there exists a signing s such that $M(s)$ is negative semi-definite.

We observe that a real symmetric $n \times n$ matrix is positive semi-definite if and only if $-M$ is negative semi-definite. Lemma 4.1 and this observation lead to the following corollary.

Corollary 4.1. NsdSigning *is NP-complete.*

We next reduce NsdSigning to BoundedEvalueSigning which also completes the proof of Theorem 1.1.

Lemma 4.2. BoundedEvalueSigning *is NP-complete.*

Proof. BoundedEvalueSigning is in NP since if there is an off-diagonal signing of a given matrix that has all eigenvalues bounded above by a given real number λ, then this signing gives the witness. We can verify if all eigenvalues of a given off-diagonal symmetric signed matrix are at most λ in polynomial time by computing the spectrum of the matrix.

We show NP-hardness of BoundedEvalueSigning by reducing from NsdSigning which is NP-complete by Corollary 4.1. Let the real symmetric $n \times n$ matrix M be the input to the NsdSigning problem. We construct an instance of BoundedEvalueSigning by considering $\lambda = 0$ and the matrix M' obtained from M as follows (where $|a|$ denotes the magnitude of a):

$$M'[i,j] = \begin{cases} M[i,j] & \text{if } i \neq j, \\ -\big|M[i,j]\big| & \text{if } i = j. \end{cases}$$

We observe that every negative semi-definite signing of M has to necessarily have negative values on the diagonal. Hence, there is a signing s such that

that $M(s)$ is negative semi-definite if and only if there is an off-diagonal signing t such that $\lambda_{\max}(M'(t)) \leq \lambda = 0$. □

CHAPTER 5

DISCUSSION

The complexities of the four problems that we studied in this work are still open and are of special interest when we restrict the input to be the adjacency matrix of simple graphs. That is, symmetric matrices with zero in the diagonal entries. We still do not know if PsdSigning and SingularSigning are efficiently solvable or NP-complete for any nontrivial class of graphs besides bipartite graphs. Moreover, we also do not know the complexity of BoundedEvalueSigning for bipartite graphs since the natural reduction from BoundedEvalueSigningto PsdSigning invalidates the diagonal entries of the matrix. With respect to our original motivation, what is of perhaps more interest is the search variant of the four problems for graph-related matrices.

REFERENCES

[1] C. Carlson, K. Chandrasekaran, H. Chang, and A. Kolla, "Invertibility and largest eigenvalue of symmetric matrix signings," *Submitted*, 2017.

[2] F. Harary, "On the notion of balance of a signed graph," *Michigan Math. J.*, vol. 2, no. 2, pp. 143–146, 1953.

[3] Y. Hou, J. Li, and Y. Pan, "On the laplacian eigenvalues of signed graphs," *Linear and Multilinear Algebra*, vol. 51, no. 1, pp. 21–30, 2003. [Online]. Available: http://dx.doi.org/10.1080/0308108031000053611

[4] J. Gallier, "Spectral theory of unsigned and signed graphs. applications to graph clustering: a survey," *CoRR*, vol. abs/1601.04692, 2015.

[5] T. Zaslavsky, "Signed graphs," *Discrete Applied Mathematics*, vol. 4, no. 1, pp. 47 – 74, 1982. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0166218X82900336

[6] A. Nilli, "On the second eigenvalue of a graph," *Discrete Math*, vol. 91, no. 2, pp. 207–210, 1991.

[7] A. Lubotzky, R. Phillips, and P. Sarnak, "Ramanujan graphs," *Combinatorica*, vol. 8, no. 3, pp. 261–277, 1988.

[8] J. Friedman, "Relative expanders or weakly relatively Ramanujan graphs," *Duke Math. J.*, vol. 118, no. 1, pp. 19–35, 2003.

[9] A. Marcus, D. Spielman, and N. Srivastava, "Interlacing families I: Ramanujan graphs of all degrees," *Annals of Mathematics*, vol. 182, no. 1, pp. 307–325, 2015.

[10] M. B. Cohen, "Ramanujan graphs in polynomial time," in *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science*, 2016, pp. 276–281.

[11] N. Robertson, P. Seymour, and R. Thomas, "Permanents, Pfaffian orientations, and even directed circuits," *Annals of Mathematics*, vol. 150, no. 3, pp. 929–975, 1999.

[12] M. Plummer, "Extending matchings in graphs: A survey," *Discrete Mathematics*, vol. 127, no. 1, pp. 277–292, 1994.

[13] R. Thomas, "A survey of Pfaffian orientations of graphs," in *Proceedings of the International Congress of Mathematicians*, vol. 3, 2006, pp. 963–984.

[14] V. Vazirani and M. Yannakakis, "Pfaffian orientations, 0-1 permanents, and even cycles in directed graphs," *Discrete Applied Mathematics*, vol. 25, no. 1, pp. 179–190, 1989.

[15] W. Tutte, "The 1-factors of oriented graphs," *Proceedings of the American Mathematical Society*, vol. 4, no. 6, pp. 922–931, 1953.

[16] J. Brown, R. Nowakowski, and D. Rall, "The ultimate categorical independence ratio of a graph," *SIAM Journal on Discrete Mathematics*, vol. 9, no. 2, pp. 290–300, 1996.

[17] N. Alon and E. Lubetzky, "Independent sets in tensor graph powers," *Journal of Graph Theory*, vol. 54, no. 1, pp. 73–87, 2007.

[18] Á. Tóth, "On the ultimate categorical independence ratio," *Journal of Combinatorial Theory, Series B*, vol. 108, pp. 29–39, 2014.

[19] R. Horn and C. Johnson, *Matrix Analysis*, 2nd ed. Cambridge University Press, 2012.

[20] W. T. Tutte, "The factorization of linear graphs," *Journal of the London Mathematical Society*, vol. s1-22, no. 2, pp. 107–111, 1947. [Online]. Available: http://dx.doi.org/10.1112/jlms/s1-22.2.107

[21] J. T. Schwartz, "Fast probabilistic algorithms for verification of polynomial identities," *Journal of the ACM*, vol. 27, no. 4, pp. 701–717, 1980.

[22] R. Zippel, "Probabilistic algorithms for sparse polynomials," *Symbolic and algebraic computation*, pp. 216–226, 1979.

[23] R. A. DeMillo and R. J. Lipton, "A probabilistic remark on algebraic program testing," *Information Processing Letters*, vol. 7, no. 4, pp. 193–195, 1978.

[24] L. Lovász and M. Plummer, *Matching Theory*, ser. AMS Chelsea Publishing Series. American Mathematical Soc., 2009.

[25] C. Little, "The parity of the number of 1-factors of a graph," *Discrete Math.*, vol. 2, no. 2, pp. 179–181, May 1972.

[26] R. Karp, "Reducibility among combinatorial problems," in *Complexity of Computer Computations*, 1972, pp. 85–103.

[27] G. Golub and C. Van Loan, *Matrix Computations*. Johns Hopkins University Press, 2013.