DIMENSIONAL REDUCTION IN NONLINEAR ESTIMATION OF MULTISCALE SYSTEMS

BY

HOONG CHIEH YEONG

DISSERTATION

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Doctoral Committee:

Professor Navaratnam Sri Namachchivaya, Chair, Director of Research
Professor Yuguo Chen
Professor Huck Beng Chew
Professor Nicolas Perkowski, Humboldt Universität
Professor Zoi Rapti
Professor Petros Voulgaris
State or signal estimation of stochastic systems based on measurement data is an important problem in many areas of science and engineering. The true signal is usually hidden, evolving according to its own dynamics, and observations are usually corrupted and possibly incomplete. The goal is to obtain optimal estimates of the signal based on noisy observations. When the dynamical model of the signal is completely known, the theory of filtering provides a recursive algorithm for estimating the conditional density (the filter) of the signal. Particle filters have been well established for the implementation of nonlinear filtering in applications. However, computational issues arise in high dimensions due to large number of particles being required to represent the signal density. The work done in this research attempts to address this issue by combining stochastic averaging with filtering techniques to develop a reduced-dimension particle filtering method for partially observed multiscale diffusion processes. When the dynamical model contains unknown parameters, the parameters need to be estimated along with the hidden states. The parameter estimation problem overlaps with the filtering problem for state estimation. In this research, the theory of maximum likelihood estimation is used to study dimensional reduction in the parameter estimation problem. The main contribution of this work are 1) a theoretical basis for a reduced-dimension filter, 2) a proposed numerical scheme for the reduced-dimension filter, 3) a theoretical basis for reduced-dimension parameter estimation in a special multiscale setting, and 4) a time-varying characterization of the information shared between signal and observations in the reduced-dimension filter.

The results of this research are in the context of slow-fast stochastic systems driven by Brownian motion, in which the timescales of the rates of change of different state/signal components differ by orders of magnitude. The multiscale filtering problem is studied via the Zakai equation that de-
scribes the time evolution of the nonlinear filter. We construct a lower-dimensional Zakai equation for estimation of the slow signal component and show that the solution of the lower dimensional equation converges to that of the original Zakai equation in the wide timescales separation limit. The convergence is shown to be at a rate proportional to the square root of the timescales separation factor (ratio of characteristic timescale of the fast component to that of the slow). A numerical scheme to approximate the reduced-dimension filter (the solution to the lower dimensional Zakai equation) is also constructed. This scheme combines a particle filtering algorithm with an existing multiscale numerical integration scheme. The reduced filter dimension can restore the feasibility of particle filters in certain high dimensional problems and lowers computational costs by appropriately averaging out fast scale components. The particle filtering scheme is adapted to discrete-, sparse-time observations by constructing an optimal importance sampling (proposal) density. In between observation assimilation times, particles are gradually driven towards locations most representative of the next observation by solving a stochastic optimal control problem. This scheme is found to be beneficial especially when the signal dynamics is chaotic, and small errors in estimation can grow at exponentially rates in between observation assimilation times.

The second aspect of nonlinear estimation in this work is in the setting in which stationary, deterministic model parameters are unknown. The theory of maximum likelihood estimation is combined with the reduced-dimension filtering results for the study of parameter estimation in the slow-fast dynamical system setting. Using the nonlinear filters convergence result, a lower-dimensional filtered likelihood function is constructed and shown to converge to the original filtered likelihood function in the wide timescales separation limit. For a special setting in which the slow diffusion is independent of the fast component, the maximum likelihood estimate using the reduced dimension filtered likelihood function is shown to be consistent, i.e. it converges to the true model parameter in the limit of sufficiently large observation set.

The third aspect of this work concerns quantifying the uncertainty in the lower-dimensional state space of the reduced-dimension filter, given observations on the state space of the original multiscale signal. Well-known concepts of entropy and mutual information from information theory are
utilized. Specifically, the time rate of change of uncertainty of the lower-dimensional state given observations is determined. The time rate of change of mutual information between the two then follows. From these, the effects of deterministic signal dynamics, diffusion effects, and information derived from observations on change in uncertainty and/or information over time can be identified and quantified. Uncertainty is found to grow according to the deterministic volumetric growth rate and the square of signal noise amplitude, while decreased by the square of the average information derived from observations.
To my parents and sister.
I would like to express my sincerest gratitude to my graduate research advisor, Prof. Navaratnam Sri Namachchivaya, who has been supportive, patient, and encouraging throughout my years of graduate studies. Prof. Namachchivaya has always encouraged and guided his students to explore different areas of research, and in the process, showed us how to tie them all back to the subset of engineering and mathematics that we are familiar with, and use them in our own research. Prof. Namachchivaya has been a teacher within the university as well as outside of it, being a sharp and perservering intellectual at work, and an easy-going person outside. I would like to thank Prof. Dr. Nicolas Perkowski and Nishanth Lingala, who have accompanied, helped and guided me in our field of study and, indirectly, outside of it. Along with Prof. Namachchivaya, Nishanth and Dr. Perkowski have been a major influence on the development of my technical mindset, leading me towards the end of my dissertation. The nonlinear filtering problems that are at the core of my dissertation are joint work with them as well. During one of his visits to UIUC, Dr. Perkowski gave a short series of lectures on backward stochastic differential equations and stochastic partial differential equations, which I was not familiar with before. The lessons learnt from those lectures and working with Dr. Perkowski had been integral to obtaining the probabilistic estimates for the main results of the filtering problem in the dissertation. Along similar lines, I would like to thank Ryne Beeson and Leo Dostal for the learning and work that we have done together on filtering, parameter estimation, and information theory, and Mirco Thiel for the constructive mathematical discussions. I would also like to thank Prof. Zoi Rapti and Prof. Ryan Sriver, for the discussions that we have had on my area of research and interest. I would like to thank the Department of Aerospace Engineering of the University of Illinois at Urbana-Champaign and the Graduate College, for
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Finally, my gratitude goes to my mother, father, and sister, for their support through the years.
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CHAPTER 1
INTRODUCTION

Multiscale properties are inherent in many dynamical systems in science and engineering, for example, neural networks in neuroscience, global circulation models in geophysical sciences, fracture mechanics in structural and material science, and power distribution and communication networks in electrical and computer engineering. The work for this thesis is on systems with wide timescale separations, i.e. systems with slow-fast dynamics.

Advancements in mathematical modeling and computational capabilities have led to the development and improvement of multiscale models and computational methods. The combination of models and computing tools forms a framework for system state estimation and prediction. However, in practice, exact initial conditions for a model are unknown and the dynamic equations in models can be highly sensitive to perturbations in initial conditions. Hence, additional tools are needed to obtain accurate initial condition estimates for predictions using the dynamic equations.

Advancements in sensing technology, on the other hand, have led to the availability of vast amounts of real observation data, which can be combined with system models to improve estimates of the system states, or the signal. Often, signals are only partially observed and observations are corrupted by noise. While perfect determination of the signal may not be possible using these noisy partial observations, it is possible to obtain probabilistic estimates of the signal, conditioned on available observations. Hence, observations have to be properly and efficiently combined with the multiscale models.

In addition, the multiscale models may be of high dimensions, especially in the case of global circulation models, and may result in complications during data assimilation procedures. While multiscale systems possess complexities due to variables interaction across different timescales, the key in dealing with them is to understand and take advantage of the possible simplifications presented by the multiscale interactions – in particular, how signal
and information interact across scales. The objective of the work here is to develop a mathematical framework for data assimilation in the multiscale estimation problem, along with an efficient data assimilation algorithm that is designed to circumvent dimensionality issues.

1.1 Research motivation: The global circulation model

An example of high-dimensional multiscale data assimilation in scientific applications is in geophysical science, specifically, in the climate estimation and prediction problem. Ocean and atmospheric circulations affect a multitude of climate phenomena, such as long-term temperature fluctuations, polar ice cap variability, and global rainfall pattern. Therefore, a tractable and realistic framework for studying global circulation, and hence climate variability, are coupled ocean-atmosphere models. Motivation for studying global circulation and climate variability extends beyond the scientific – on national and international levels, climate phenomena can affect energy, import-export, and national safety policies.

Ocean and atmosphere coupling can be described via a feedback mechanism: a large-scale anomaly of sea surface temperature (SST) induces diabatic heating or cooling of the atmosphere, which alters atmospheric circulation and hence wind stress and heat fluxes at the ocean surface. In turn, the wind stress variations modify the ocean thermal structure and circulation, giving rise to a series of positive feedbacks that reinforce the initial SST anomaly. Such interaction, along with the effects of Earth’s rotation, contribute to coupled variability on different timescales, for example, the El Ninó-Southern Oscillation (ENSO), Pacific Decadal Oscillation (PDO), Atlantic Multi-decadal Oscillation (AMO), and Atlantic Meridional Overturning Circulation (AMOC). Low-dimensional first order approximation ($\leq 15$) ocean-atmosphere models are a good starting point to gain understanding of these processes. For example, the coupled Lorenz atmosphere ([3]) and Mass ocean ([4]) models that describe the coupling of a two-layer atmospheric model with a basin averaged ocean circulation model. The atmospheric model describes the flow circulation and temperature variation in the two fixed-height layers. The ocean model describes the overturning and density gradient in a rectangular basin. Both models are coupled by wind
shear on the atmosphere-ocean interface. The atmospheric variables undergo fluctuations that are order $10^5$ faster than the oceanic variables ([5]).

![Coupled Lorenz-Maas atmosphere-ocean model](image)

Figure 1.1: Coupled Lorenz-Maas atmosphere-ocean model

Initialized long-range predictions on seasonal and decadal timescales require combination of proper models with real observation data. In addition to conventional data (radiosonde, surface, and dropsonde data, for example), more sophisticated sources are available since the last two decades, for example satellites, radars, and other remote-sensing devices. However, it is still impossible to measure all of the models' degrees of freedom at a given time. In addition, the observations are irregularly distributed in space and time, and have different structures of random error. Therefore, efficient data assimilation methods are required to combine these irregular observations with reliable models to generate the initial conditions on model grid points. This is the motivation for the problem of data assimilation in high-dimensional multiscale systems.

1.2 Multiscale nonlinear filtering

The framework for data assimilation in multiscale systems in this work is based on the theory of nonlinear filtering and stochastic homogenization. Filtering theory is concerned with the problem of estimating an unknown or hidden signal variable based on partial, often corrupted observation data. Signal variables usually represent the state in a dynamic system, for example, the space-averaged momentum component in a global circulation model.
(GCM), or the asset value of a company over time. In order to accurately track and predict the behavior of signal variables over time, exact observations are required. However, physical constraints generally allow only measurement on a subset or indirect measurements of the signal variables in a complex system. For example, the momentum components in a GCM is related to fluid density variations. Fluid density is determined by collecting data on fluid salinity and temperature, hence the estimation of the momentum components depends on data that are not direct observations. Another example is the asset value of a company, which is not directly observed, but can be estimated based on observation of equity traded in a market. In addition, observed data can be inaccurate representations of the actual system state, due to physical disturbances or imperfections in the environment and within a data collection mechanism.

Filtering theory enables the construction of a filter, the conditional expectation of functions of the signal, given observations. In practical applications, a filter is an algorithm that take observation data as input and return the best possible estimate of signal variables as output. The algorithm involves coupling a reliable mathematical model of the signal dynamics with sensor dynamics and optimizing that relation, for example, minimizing a distance function of the two, or obtaining the statistical representation of one based on the other. For linear systems, the Kalman filter is a well-established signal estimation theory and algorithm (see Chapter 2, Sections 2.1 and 2.4). The theoretical framework for nonlinear filtering is also well-established, where we have the well-known results of [6] and [7] that characterize the time evolution of a nonlinear filter (see Chapter 2, Sections 2.2 and 2.3).

In terms of practical implementation, finding the exact nonlinear filter analytically is usually impossible, since nonlinear filtering deals with infinite-dimensional objects – densities, or distributions. Algorithms for nonlinear filtering include linear approximations, such as the extended and ensemble Kalman filters ([8, 9]), and numerical approximation of the complete nonlinear filter by particle methods ([10, 11, 12]). The numerical implementation aspect of the work in this thesis is focused on particle methods. As the name suggests, particle methods use samples of particles to represent the conditional density of the signal, conditioned on observations. Particles are sampled based on the signal’s probability law, hence no assumption of linearity on the model, and thus no Gaussianity assumption on the signal distribution.
is made. This gives particle methods the flexibility of application to a wide range of system models. In addition, there are several rigorous convergence results for the particle approximation of nonlinear filters ([13], [12], [14], see Section 2.6.2). However, in high-dimensional applications, the estimation of continuous distributions suffer from the “curse of dimensionality” (see, for example, [1], [15], [16]), where the computational costs increase exponentially with system dimensions due to the fact that sample size needs to be sufficiently large to properly capture the signal distribution (see Section 2.6, Section 2.6.3).

For multiscale systems with time scales separation on the orders of magnitude, we are often interested only in the slowly-varying dynamics of the system. Additionally, for systems with certain exponential convergence properties in the fast component, stochastic homogenization enables the statistical representation of the slow dynamics by a homogenized process that has the same dimension as the slow component. Hence, in the context of filtering, the data assimilation procedure can be performed by considering a filter for the homogenized process, ignoring fast component dimensions. While convergence of the slow component to the homogenized process via homogenization of stochastic differential equations is an established technique ([17]), convergence of the corresponding filters (conditional expectations), is not trivial. Consider, for example, the signal and observation, $X^\varepsilon = X$ and $Y^\varepsilon = \varepsilon X^\varepsilon$ (for $0 < \varepsilon << 1$), and let $Y = 0$. In the limit $\varepsilon \to 0$, $(X^\varepsilon, Y^\varepsilon) \to (X, Y)$. However, $E[f(X^\varepsilon)|Y^\varepsilon] = f(X^\varepsilon) \to f(X)$ while $E[f(X)|Y] = E[f(X)]$. Hence, $E[f(X^\varepsilon)|Y^\varepsilon]$ does not necessarily converge to $E[f(X)|Y]$, even when $(X^\varepsilon, Y^\varepsilon)$ converges to $(X, Y)$.

1.3 Parameter estimation

In signal estimation via filtering, it is inherently assumed that models describing signal dynamics are completely known. In reality, coefficients that parametrize these models, for example excitation and damping amplitudes, are usually unknown and have to be estimated from observations as well. For example, in a cumulus cloud convection model there are parameters that describe entrainment (ambient air and cloud mixture), cloud convection, and conversion rate of liquid water to rain, which exact values are uncertain ([18]).
In models used for long-range climate prediction, parameter values affect long term behavior significantly ([19]). For determining such parameter values, one approach is to “tune” the parameters such that models match observation data (see, for example, [20]). This involves repeatedly running numerical simulations of a model with varying parameter values to obtain the best fit to observation data. Another approach is by exploring the parameter space in sensitivity experiments (see, for example, [21]). Yet another approach is by “state space augmentation”, where the unknown parameters are augmented to the signal dynamic equations with stationary dynamics and estimation of the new “signal” variables are estimated by conventional filtering methods (see, for example, [18, 22, 23]). An optimization problem approach was also taken in [24], where parameters are determined as the optimal values that minimize a cost function, which quantifies the square of the error between the model and data.

The work for this thesis employs the maximum likelihood approach to the parameter estimation problem in the multiscale setting. Via the maximum likelihood (ML) approach, an unknown parameter is estimated by maximizing the likelihood of the observations or signal, given the parameter. A well-established algorithm for performing ML estimation (MLE) is the Expectation-Maximization (EM) algorithm, introduced in [25] for partially-observed signals. The EM algorithm has been extended to continuous-time, partially observed Markov diffusion processes in [26].

1.4 Information flow rate

The estimation aspects of the thesis problem relies on information from observations to reconstruct the hidden signal process and/or model parameter. The amount and value of information gained from observing the signal is very much dependent on the sensors that are used and the observation strategies. Hence, in addition to developing data assimilation and estimation techniques, we also study how information interacts between signal and observations. Such knowledge can guide the design of sensors and/or observing strategies for collecting data that lead to improved estimation. Information theory provides adequate tools for such study. For example, [27] utilizes the Rényi divergence (a generalized form of the Kullback-Leibler divergence) be-
tween filtering prior and posterior to identify regions of the state space to be targeted by sensors for managing multiple sensor platforms. The general approach in such problems is to quantify the information that can be gained from observations by looking at the distribution of the hidden signal when no observations are available, and how it changes when observations are incorporated, and how to best collect observations such that the resulting posterior distribution most closely resembles the true distribution of the signal. In a series of work in [28, 29, 30], the authors utilize the concept of information entropy and mutual information for the tasks of sensor management in large-scale settings, which includes sensor placement and selection, and continuous path planning for mobile observing platforms. From a purely theoretical aspect, [31] has studied the relations between utilizations of information entropy, mutual information, and Kullback-Leibler divergence in sensor management problems, to justify the goal of the problems as reduction in uncertainty about the signal process. [32] studies the same information theoretic concepts applied to the problem of weather prediction. [33, 34] obtained the information flow between components of a dynamical system using joint information entropy, also in the setting of weather prediction problems.

In the work in this thesis, information flow between signal and observation is studied in the multiscale partially observed diffusion setting. Conditional information entropy and mutual information are utilized to quantify the information content shared between signal and observation and the dynamic behaviour of these quantities are investigated.

1.5 Research goal: Reduced-order filter and parameter estimator

1.5.1 Multiscale nonlinear filtering:

The goal for the multiscale filtering problem is to construct a reduced-order filtering equation based on a homogenized process, and show that the filter of the multiscale signal converges to this homogenized filter, in the wide time scale separation limit. The homogenized filter is of lower dimension than that of the original multiscale signal. For practical applications in filtering high-
dimensional multiscale systems, this reduced-dimension filter can circumvent the dimensionality issue. In an application to a heuristic atmospheric model, the discrete-time, sparse observation setting is also examined and a particle filtering algorithm is adapted by a combination optimal sampling and stochastic control theory.

1.5.2 Multiscale maximum likelihood parameter estimation:
As in the filtering problem, we combine the MLE technique with stochastic homogenization results for multiscale systems for a reduced-order parameter estimator. The idea is similar, in that we propose to use the homogenized process in MLE for determining model parameters, and show that the corresponding estimator is consistent, i.e. it converges to the true parameter value in the wide time scale separation and infinitely many observation limits. For practical purposes, utilizing the homogenized process can reduce computational complexities as filtering and smoothing procedures are required for a parameter estimator.

1.5.3 Multiscale information flow rate:
In this aspect of the research for this thesis, we quantify the information about the signal that is contained in the observation used for estimation in the previous two sections. Specifically, we quantify the how uncertainty about the homogenized process changes given information from observations of the true multiscale signal. This information is not directly of the homogenized process, but of the process which distribution is close to the homogenized process. We utilize information theoretic concepts, starting from the conditional information entropy and obtaining dynamic equations for the rates of
change of uncertainty and information content between the homogenized pro-
cess and true observations. The resulting equations provide an insight into 
quantifying the contribution of different components of the signal-observation 
model to the change in uncertainty and information content of the system.

1.6 Organization of the thesis

The thesis is organized into a first half on state estimation and a second half 
on parameter estimation, but the two are not mutually exclusive. All work 
are in the multiscale diffusion setting. State estimation is based on filtering 
theory while parameter estimation is based on maximum likelihood, which, 
in the diffusion setting, requires filtering and smoothing. Both problems 
consist of a theoretical result aspect and a numerical algorithm development 
and testing/validation aspect. Chapter 2 is a preliminary for the filtering 
problem, describing filtering theory and particle filtering. The latter section 
indicates the problem faced in particle filtering in high dimensions, which is 
one of the main motivations for the development of lower-dimensional non-
linear estimation algorithms in our work. Chapter 3 sets up the multiscale 
filtering problem, describes mathematical tools required for proving the main 
theoretical results, followed by preliminary and the main results. Chapter 4 
presents a lower-dimensional particle filtering algorithm based on the results 
of Chapter 3, with tests on a heuristic model representative of atmospheric 
dynamics. Chapter 5 is a preliminary for the parameter estimation prob-
lem, describing maximum likelihood estimation using the diffusion setting 
and the Expectation-Maximization algorithm, and iterative algorithm for 
estimating unknown system parameters. Chapter 6 sets up the multiscale 
parameter estimation problem and presents preliminary and main results 
for a ML estimator using a lower-dimensional likelihood function based on 
the lower-dimensional filter of Chapter 3. Results from numerical parameter 
estimation experiments on the same heuristic model of atmospheric dynam-
ics are presented at the end of the chapter. Chapter 7 utilizes information 
theoretic concepts with the lower-dimensional filter to determine the rate of 
change of uncertainty of the homogenized signal given observations and the 
information content shared between them. The main results of the thesis are 
summarized in Chapter 8, with discussions on future directions.
Filtering theory is an established field in applied probability and decision and control systems, which is important in many practical applications from inertial guidance of aircrafts and spacecrafts to weather and climate prediction. It provides a recursive algorithm for estimating a signal or state of a random dynamical system based on noisy measurements. More precisely, filtering problems consist of an unobservable signal process \( X \) and an observation process \( Y \) that is a function of \( X \) corrupted by noise. The main objective of filtering theory is to get the best estimate of \( X_t \) based on the information \( Y_t \). This is given by the conditional distribution \( \pi_t \) of \( X_t \) given \( Y_t \) or equivalently, the conditional expectations \( E[\varphi(X_t) | Y_t] \) for a rich enough class of functions. Since this estimate minimizes the mean square error loss, we call \( \pi_t \) the optimal filter. The goal of filtering theory is to characterize this conditional distribution effectively. In simplified problems where the signal and the observation models are linear and Gaussian, the filtering equation is finite-dimensional, and the solution is the well-known Kalman-Bucy filter (see, for example, Chapter 6 of [35], Chapter 7 of [36], Chapter 10 of [37]). In more realistic problems, nonlinearities in the models lead to more complicated equations for \( \pi_t \), defined by [6] and [7], which describe the evolution of the conditional distribution in the space of probability measures (see, for example, Chapter 3 of [35], Chapter 11 of [38], Chapter 8 of [37]).

For the linear, Gaussian case, the explicit Kalman or Kalman-Bucy solution can be implemented directly in practical applications. However, it is impractical to implement a numerical solution to the infinite dimensional stochastic evolution equations of the general nonlinear filtering problem by finite difference or finite element approximations. Therefore, extended Kalman filter algorithms, which use linear approximations to the signal dynamics and observation (see, for example, Chapter 8 of [35], Chapter 9 of [36]), have been
used extensively in applications. These provide essentially a first order approximation to an infinite dimensional problem and can perform quite poorly in problems with strong nonlinearities. In nonlinear settings, particle filters have been well established for numerically approximating nonlinear filters (see, for example, [11], [39], [10]). However, due to dimensionality issues (see, for example, [16]) and computational complexities that arise in representing the signal density using a high number of particles, the problem of particle filtering in high dimensions is still not completely resolved. The particle filtering algorithm and associated dimensionality issue are described in Section 2.6.

In this chapter, we describe filtering theory and the particle filter, a Monte Carlo approximation to the solution of the nonlinear filtering equation. Filtering theory is described in Sections 2.1 to 2.5, in different time settings in state space. The filter for discrete-time signal and observation is described using a Bayesian approach. Combining the Bayesian approach with the Fokker-Planck equation (Kolmogorov forward equation) provides the filter for the continuous signal, discrete observation case. In the continuous-time signal and observation setting, the nonlinear filtering theory for finite-dimensional state space developed by [40] and [6] provide stochastic partial differential equations (SPDEs) for the nonlinear filter. When the signal and observation are linear, the SPDE describes the Kalman-Bucy filter. The particle filter is described in Section 2.6. Sections 2.6.1 and 2.6.2 describes the particle filter in discrete and continuous time, respectively. The connection between particle filtering in discrete and continuous time via particle weights update is described in Section 2.6.2. Finally, Section 2.6.3 describes an obstacle to particle filtering in high dimensions.

2.1 Linear discrete time filtering (Kalman filter)

We follow [41] in describing the Kalman filter for the linear, discrete-time case, using the maximum likelihood approach. There are various approaches to arriving at the Kalman filter solution, for example by the observation innovation with minimum mean square error method ([42], [36]). The choice of the maximum likelihood method here is solely in conjunction with the use of maximum likelihood for the parameter estimation problem in Chapter 5.
Consider the discrete-time signal and observation

Signal: \[ X_{k+1} = A_{k+1}X_k + W_{k+1}, \quad X_0 \in \mathbb{R}^m, \] \hspace{1cm} (2.1a)

Observation: \[ Y_k = H_kX_k + B_k, \quad Y_0 = 0 \in \mathbb{R}^d, \] \hspace{1cm} (2.1b)

where \( X_0 \sim \mathcal{N} (\mu_0, P_0) \) and the signal and observation noises \( \{W_1, \ldots, W_k\} \) and \( \{B_1, \ldots, B_k\} \) are, respectively, \( m \)- and \( d \)-dimensional independent Wiener increments, i.e. \( W_k \sim \mathcal{N}(0, Q_k) \), \( \text{Cov}(W_k, W_j) = Q_k \delta_{jk} \), \( B_k \sim \mathcal{N}(0, R_k) \), \( \text{Cov}(B_k, B_j) = R_k \delta_{jk} \), and \( \text{Cov}(W_k, B_j) = 0 \) for all \( j, k \).

The prior and posterior filtering densities are densities of the signal conditioned on different observation sets. The prior density is conditioned on observations up to the previous time step, while the posterior is conditioned on observations up to the current time step of interest:

prior: \( p(x_k | y_{0:k-1}) \), posterior: \( p(x_k | y_{0:k}) \).

The goal is to determine the posterior density of the signal \( X_k \) given observations up to time \( k \), \( Y_{0:k} \). Using the notation for continuous time in the chapter introduction, the filter is \( \pi_k(\varphi) = \int_{\mathbb{R}^m} \varphi(x_k)p(x_k | y_{0:k})dx_k \), which is the conditional expectation \( \mathbb{E}[\varphi(X_k) | Y_{0:k}] \).

By the maximum likelihood approach, we would like to maximize the conditional density of \( X_k \) given \( Y_{0:k} \). Since the log function is one-to-one, we maximize log of the conditional density, to take advantage of concavity property, i.e. we maximize

\[ L(X_k | Y_{0:k}) \overset{\text{def}}{=} \log p(X_k | Y_{0:k}). \]

Note that the log likelihood notation \( L(\cdot) \) is used differently in Chapter 5, using parameter in place of signal and omitting notation for observations. The sequential Kalman filter equations are in (2.6). The Kalman filter prior is obtained by direct calculation of the conditional first and second moments. The posterior is obtained by applying Bayes’s theorem to the posterior density, using normality in linear setting for each of the densities in the resulting product, and maximizing the log of the posterior density, (2.5).

Bayes’s theorem relates the posterior density to the product of observation
likelihood \( p(Y_k|X_k) \) and prior density \( p(X_k|Y_{0:k-1}) \):

\[
p(X_k|Y_{0:k}) = \frac{p(Y_k|X_k, Y_{0:k-1})}{p(Y_k|Y_{0:k-1})} = \frac{p(Y_k|X_k)p(X_k|Y_{0:k-1})p(Y_{0:k-1})}{p(Y_k|Y_{0:k-1})} = \frac{p(Y_k|X_k)p(X_k|Y_{0:k-1})}{p(Y_k|Y_{0:k-1})}. \tag{2.2}
\]

Under the Gaussian noise assumption, \( Y_k|X_k \sim \mathcal{N}(H_k X_k, R_k) \).

Denote

\[
\begin{align*}
\text{Prior} : \hat{X}_{k|k-1} & \overset{\text{def}}{=} \mathbb{E}[X_k|Y_{0:k-1}], & P_{k|k-1} & \overset{\text{def}}{=} \text{Cov}(X_k|Y_{0:k-1}), \\
\text{Posterior} : \hat{X}_{k|k} & \overset{\text{def}}{=} \mathbb{E}[X_k|Y_{0:k}], & P_{k|k} & \overset{\text{def}}{=} \text{Cov}(X_k|Y_{0:k})
\end{align*}
\]

Due to linearity, \( X_k|Y_{0:k-1} \) is Gaussian too, with mean \( \hat{X}_{k|k-1} \) and covariance \( P_{k|k-1} \). By (2.1a) and the fact that \( W_k \) is independent of \( Y_{0:k-1} \) and has mean zero, iterations for the prior mean and covariance are

\[
\hat{X}_{k|k-1} = \mathbb{E}[X_k|Y_{0:k-1}] = A_k \hat{X}_{k-1|k-1}, \tag{2.3}
\]

and

\[
P_{k|k-1} = \mathbb{E} \left[ (X_k - \hat{X}_{k|k-1})(X_k - \hat{X}_{k|k-1})^* Y_{0:k-1} \right] = A_k P_{k-1|k-1} A_k^* + Q_k, \tag{2.4}
\]

respectively. The prior is an update of the signal density from time \( k - 1 \) to \( k \) based only on signal dynamics, governed by \( A_k \) and \( W_k \).

By the Bayes relation (2.2) and normality of the observation likelihood and prior, the log likelihood is

\[
L(X_k|Y_{0:k}) = -\log(2\pi)^{d/2}|R_k|^{1/2} - \frac{1}{2} (Y_k - H_k X_k)^* R_k^{-1} (Y_k - H_k X_k)
- \log(2\pi)^{m/2}|P_{k|k-1}|^{1/2} - \frac{1}{2} \left( X_k - \hat{X}_{k|k-1} \right)^* P_{k|k-1}^{-1} \left( X_k - \hat{X}_{k|k-1} \right)
- \log p(Y_k|Y_{0:k-1}). \tag{2.5}
\]

Taking derivative w.r.t. \( X_k \) and setting it to zero gives us the maximizer of
the log likelihood function, which is the Kalman filter mean $\hat{X}_{k|k}$:

$$\frac{\partial}{\partial X_k} L(X_k|Y_{0:k-1}) = H_k^* R_k^{-1}(Y_k - H_k X_k) - P_{k|k-1}^{-1}(X_k - \hat{X}_{k|k-1}) = 0$$

$$\implies \hat{X}_{k|k} = (H_k^* R_k^{-1} H_k + P_{k|k-1}^{-1})^{-1}(H_k^* R_k^{-1} Y_k + P_{k|k-1}^{-1} \hat{X}_{k|k-1}).$$

By identities $(AB + I)^{-1} = A(BA + I)^{-1}$ and $(AB + I)^{-1} = I - A(BA + I)^{-1}B$ for matrices $A \in \mathbb{R}^{m \times d}$ and $B \in \mathbb{R}^{d \times m},$

$$\hat{X}_{k|k} = \hat{X}_{k|k-1} + K_k(Y_k - H_k \hat{X}_{k|k-1}),$$

where $K_k \equiv P_{k|k-1} H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1}.$

$Y_k - H_k \hat{X}_{k|k-1}$ is called the innovation, which is the difference between the real observation and that predicted by the prior mean, or the amount of new information gained from the new observation that is not in the prior mean. $K_k$ is called the Kalman gain, which is a factor by which the prior mean is corrected based on new information. The posterior is an update of the signal density based on the current observation $Y_k.$ From a linear feedback control point of view, $K_k(Y_k - H_k \hat{X}_{k|k-1})$ is the control input that steers the prior mean towards the location indicated by observation $Y_k$ for each time step $k.$

Denote the error by $\tilde{X}_{k|k} \equiv X_k - \hat{X}_{k|k}.$ The posterior covariance is obtained as

$$P_{k|k} = \mathbb{E} \left[ \tilde{X}_{k|k} \tilde{X}_{k|k}^* \left| Y_{0:k} \right. \right] = (I - K_k H_k) P_{k|k-1}. $$

By linearity of the signal and observation and Gaussianity of noise, the conditional prior and posterior are always normally distributed, hence the first and second moments are sufficient to characterize the densities. The prior and posterior Kalman filter densities are described by the following means and covariances:

**Prior:**

$$\begin{align*}
\hat{X}_{k|k-1} &= A_k \hat{X}_{k-1|k-1}, \\
P_{k|k-1} &= A_k P_{k-1|k-1} A_k^* + Q_k, \\
\end{align*}$$

(2.6a)

**Posterior:**

$$\begin{align*}
\hat{X}_{k|k} &= \hat{X}_{k|k-1} + K_k(Y_k - H_k \hat{X}_{k|k-1}), \\
P_{k|k} &= (I - K_k H_k) P_{k|k-1}, \\
K_k &= P_{k|k-1} H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1}. \\
\end{align*}$$

(2.6b)
If the initial distribution \( \mathcal{N}(\mu_0, P_0) \), the time-varying matrices \( A_k, H_k \), and noise covariances \( Q_k, R_k \) are all known, then, dimensionality issues of realizing the respective matrices aside, the Kalman filter can be implemented directly using the iterations (2.6) above. If the drift and sensor matrices \( A_k \) and \( H_k \) are time independent, then the prior and posterior error covariances can be computed offline since they are independent of the real observations \( \{Y_1, \ldots, Y_k\} \).

### 2.2 Nonlinear discrete time filtering

Here we consider the discrete-time signal and observation but with nonlinear drift and sensor function:

**Signal:** \( X_{k+1} = b(X_k, k+1) + W_{k+1}, \quad X_0 \in \mathbb{R}^m, \quad (2.7a) \)

**Observation:** \( Y_k = h(X_k, k) + B_k, \quad Y_0 = 0 \in \mathbb{R}^d, \quad (2.7b) \)

where \( b : [0, K] \times \mathbb{R}^m \rightarrow \mathbb{R}^m, h : [0, K] \times \mathbb{R}^m \rightarrow \mathbb{R}^d \) are continuously differentiable, and \( X_0, W_k \) and \( B_k \) are as in the previous section. As in (2.2), we have the posterior density as a product of the observation likelihood and prior density:

\[
p(x_k | y_{0:k}) = \frac{p(y_k | x_k)p(x_k | y_{0:k-1})}{p(y_k | y_{0:k-1})} = \frac{p(y_k | x_k) \int_{\mathbb{R}^m} p(x_k | x_{k-1})p(x_{k-1} | y_{0:k-1}) \, dx_{k-1}}{p(y_k | y_{0:k-1})} . \tag{2.8}
\]

With Gaussian noise in discrete time, the observation likelihood and prior density are again Gaussian,

\[
p(y_k | x_k) = \frac{1}{\sqrt{(2\pi)^d | R_k|}} \exp \left\{ -\frac{1}{2} (y_k - h(x_k, k))^* R_k^{-1} (y_k - h(x_k, k)) \right\}, \quad (2.9)
\]

\[
p(x_k | x_{k-1}) = \frac{1}{\sqrt{(2\pi)^m | Q_k|}} \exp \left\{ -\frac{1}{2} (x_k - b(x_{k-1}, k))^* Q_k^{-1} (x_k - b(x_{k-1}, k)) \right\} . \tag{2.10}
\]
Here, the prior is the conditional density of $X_k$ given $X_{k-1}$.

One method of approximating the posterior density (2.8) is by a sequential Monte Carlo method called particle filtering, using particles to represent possible locations of $X_k$. The basic procedure of importance sampling particle filter is described in Secton 2.6.

2.3 Nonlinear continuous time filtering

Real physical systems are often continuous in time and modeled so, for example ocean-atmospheric models, aircraft and spacecraft dynamical models, and biological systems. In the continuous time setting, the noise-perturbed difference equations of Sections 2.1 and 2.2 are replaced by stochastic differential equations (SDEs). In the work here, we consider only SDEs driven by the Wiener process, or Brownian motion. For detailed construction of the Brownian motion and associated stochastic integral and stochastic differential equation, see, for example, [43] or [44].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ be a filtered probability space that supports a $(k + d)$-dimensional standard Brownian motion $(W, B)$. Consider the continuous time signal and observation governed by the Itô SDEs

\begin{align}
\text{Signal:} & \quad dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 \in \mathbb{R}^m, \\
\text{Observation:} & \quad dY_t = h(X_t)dt + dB_t, \quad Y_0 = 0 \in \mathbb{R}^d,
\end{align}

(2.11a) (2.11b)

where $W \in \mathbb{R}^k$ and $B \in \mathbb{R}^d$ are independent Brownian motions and $b : \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \to \mathbb{R}^{m \times k}$, $h : \mathbb{R}^m \to \mathbb{R}^d$ are finite and Borel-measurable.

Let $\mathcal{Y}_t$ denote the $\sigma$-algebra generated by observations up to time $t$, i.e. $\mathcal{Y}_t \overset{\text{def}}{=} \sigma \{ \omega : Y_s(\omega), s \leq t \}$. In other words, $\mathcal{Y}_t$ contains all the information in the observations up to time $t$. For the filtering problem, we are interested in estimating the posterior probability $\pi_t(A) = Q[X_t \in A | \mathcal{Y}_t]$ for Borel set $A$, or $\pi_t(\varphi) = E_Q [ \varphi(X_t) | \mathcal{Y}_t]$ for Borel-measurable and $C^2_b$ function $\varphi$. The filter $\pi_t(\varphi)$ satisfies a SPDE that is driven by the innovation process ([40]). We provide a heuristic procedure for obtaining the SPDE in this section for completeness. The resulting SPDE, called the Kushner-Stratonovich equation, is (2.16). We will utilize the unnormalized version of the filter to obtain the Kushner-Stratonovich equation. The unnormalized filter, $\rho_t(\varphi)$, satisfies
the Zakai equation (2.15), which is a SPDE driven by the observation process ([6]). \( \rho_t(\varphi) \) is more feasible to work with in practice as well as in mathematical settings. This is because when the signal and sensor noises are uncorrelated, which is the setting that we are working in, under the probability measure that \( \rho_t(\varphi) \) is constructed with, the observation is a Brownian motion that is independent of the signal. [12] provides a rigorous development of a particle filtering method for approximating the solution to the Zakai equation, which is extended to the solution of the Kushner-Stratonovich equation in [14]. These algorithms and their development are also found in Chapter 9 of [35]. For our main result of filter convergence in the multiscale setting (Chapter 3), we utilize a SPDE similar to the Zakai equation for the unnormalized version of a reduced-order filter. In the linear setting, the normalized filter results in the Kalman-Bucy filter (2.18).

### 2.3.1 The Kallianpur-Striebel formula

The filter for (2.11) is

\[
\pi_t(f) = E_Q [\varphi(X_t) | \mathcal{Y}_t] = \int_{\mathbb{X}} \varphi(x) dQ_{(X|Y),t}(x) = \int_{\mathbb{X}} \varphi(x) q_t(x|y) dx,
\]

where \( q_t(x|y) \) is a conditional density. Let

\[
\rho_t(\varphi) = E_P \left[ \varphi(X_t) \frac{dQ}{dP} \bigg| \mathcal{F}_t \right] = \int_{\mathbb{X}} \varphi(x) \frac{dQ}{dP} \bigg| \mathcal{F}_t dP_{(X|Y),t}(x) = \int_{\mathbb{X}} \varphi(x) \frac{dQ}{dP} \bigg| \mathcal{F}_t p_t(x|y) dx.
\]

The Radon-Nikodym derivative/measure change \( \frac{dQ}{dP} \bigg| \mathcal{F}_t \) is a density, and \( p_t(x|y) \) is a conditional density, but their product is not a density (does not integrate to 1). Hence, \( \rho_t(\varphi) \) is called the unnormalized filter and has to be normalized by

\[
\rho_t(1) = E_P \left[ \frac{dQ}{dP} \bigg| \mathcal{F}_t \right] = \int_{\mathbb{X}} \frac{dQ}{dP} \bigg| \mathcal{F}_t p_t(x|y) dx.
\]
Denote the measure change as

\[
D_t \overset{\text{def}}{=} \left. \frac{dP}{dQ} \right|_{\mathcal{F}_t} = \exp \left\{ -\int_0^t h^*(X_s) dB_s - \frac{1}{2} \int_0^t \|h(X_s)\|^2 ds \right\}. \tag{2.12}
\]

\(D_t\) is an exponential martingale (\(\mathcal{F}_t\)-martingale). The proof and properties of the exponential martingale can be found in Chapter 6 of [37]. By Girsanov’s theorem (see, for example, Appendix B.3.1 of [35]), under \(P\),

\[
B_t - \left\langle B_t, -\int_0^t h^*(X_s) dB_s \right\rangle = B_t + \int_0^t h(X_s) ds = Y_t
\]
is a Brownian motion (\(Y_t\) is a \(P\)-Brownian motion). This can be illustrated by an analogy to shifting the mean of a \(N(\mu, \sigma^2)\) random variable: The density of an \(\mathbb{R}\)-valued \(N(\mu, \sigma^2)\) random variable is

\[
q(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2} \right\}.
\]
The probability measure \(Q\) is dominated by the Lebesgue measure on \(\mathcal{B}_{\mathbb{R}}\), so \(q\) is a density of \(Q\) w.r.t. Lebesgue measure, \(dQ(x) = q(x) dx\). Say we would like to shift the mean by \(\gamma\), such that under the new probability measure \(P\), the density is

\[
p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \frac{(x - \mu - \gamma)^2}{\sigma^2} \right\}.
\]

Then,

\[
p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2} - \frac{1}{2} \frac{\gamma^2 + 2\mu\gamma - 2x\gamma}{\sigma^2} \right\}
\]

\[
= q(x) \exp \left\{ -\frac{1}{2} \frac{\gamma^2 + 2\mu\gamma - 2x\gamma}{\sigma^2} \right\}.
\]

Therefore, the measure change/Radon-Nikodym derivative of \(P\) w.r.t. \(Q\) is

\[
\frac{dP(x)}{dQ(x)} = p(x) \frac{1}{q(x)} = \exp \left\{ -\frac{1}{2} \frac{\gamma^2 + 2\mu\gamma - 2x\gamma}{\sigma^2} \right\}.
\]
If \( \mu \equiv 0 \), then
\[
\frac{d\mathbb{P}(x)}{d\mathbb{Q}(x)} = \exp \left\{ \frac{1}{\sigma^2} \left( \gamma x - \frac{1}{2} \gamma^2 \right) \right\}.
\]

Formally, the measure change \( D_t \) in (2.12) can be related to this, where, in order to make the observation \( Y_t = \int_0^t h(X_s)ds + B_t \) a Brownian motion, we have to shift the mean of the Brownian motion \( B_t \) (the \( \mathcal{N}(0, dt) \) Gaussian increment \( dB_t \)) by \(- \int_0^t h(X_s)ds \) (-h(\( X_t \))dt).

The Kallianpur-Striebel formula relates the normalized and unnormalized filters (see, for example, Ch. 3.4 of [35]):
\[
\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)} = \frac{\mathbb{E}_\mathbb{P} \left[ \varphi(X_t)\tilde{D}_t \bigg| \mathcal{Y}_t \right]}{\mathbb{E}_\mathbb{P} \left[ \tilde{D}_t \bigg| \mathcal{Y}_t \right]},
\]
(2.13)
where \( \tilde{D}_t \) \( \overset{\text{def}}{=} \) \( D_t^{-1} \). By finiteness and integrability of sensor function \( h \),
\[
\mathbb{E}_\mathbb{P} \left[ \chi_{\{\tilde{D}_t=0\}} D_t \right] = 0,
\]
where \( \chi \) is the indicator function. So,
\[
0 = \mathbb{E}_\mathbb{P} \left[ \chi_{\{\tilde{D}_t=0\}} D_t \right] = \mathbb{E}_\mathbb{Q} \left[ \chi_{\{\tilde{D}_t=0\}} \right] = \mathbb{Q}[\tilde{D}_t = 0].
\]
This, along with the fact that, \( \tilde{D}_t \) is \( \geq 0 \) because it is an exponential martingale, gives us \( \mathbb{E}_\mathbb{P} \left[ \tilde{D}_t \bigg| \mathcal{Y}_t \right] > 0 \). Therefore, the RHS of (2.13) is well-defined. The Kallianpur-Striebel formula is obtained by application of the tower property to show that, for \( A \subset \mathcal{Y}_t \),
\[
\mathbb{E}_\mathbb{P} \left[ \mathbb{E}_\mathbb{P} \left[ \varphi(X_t)\tilde{D}_t \bigg| \mathcal{Y}_t \right] \chi_A \right] = \mathbb{E}_\mathbb{Q} \left[ \varphi(X_t)\chi_A \right]
\]
and
\[
\mathbb{E}_\mathbb{P} \left[ \pi_t(\varphi) \mathbb{E}_\mathbb{P} \left[ \tilde{D}_t \bigg| \mathcal{Y}_t \right] \chi_A \right] = \mathbb{E}_\mathbb{Q} \left[ \varphi(X_t)\chi_A \right]
\]
(see Appendix 1). Then, since \( \mathbb{E}_\mathbb{P} \left[ \tilde{D}_t \bigg| \mathcal{Y}_t \right] > 0 \), (2.13) holds.
2.3.2 Generator of an Itô diffusion

The generator $\mathcal{L}$ of an Itô diffusion $X_t$ is

$$\mathcal{L} \varphi(x) \overset{\text{def}}{=} \lim_{t \to 0} \frac{\mathbb{E}_Q [\varphi(X_t) | X_0 = x] - \varphi(x)}{t}.$$  

$\mathcal{L}$ describes the average rate of change of a function of an Itô diffusion process. Using Itô’s formula, Itô isometry and properties of the Brownian motion, the generator of an $m$-dimensional Itô diffusion satisfying a SDE of the form (2.11a) can be shown to be

$$\mathcal{L} \varphi(x) = \sum_{i=1}^m b_i(x) \frac{\partial}{\partial x_i} \varphi(x) + \frac{1}{2} \sum_{i,j=1}^m (\sigma \sigma^*)_{ij} \frac{\partial^2}{\partial x_i x_j} \varphi(x). \quad (2.14)$$

By applying integration by parts twice on the inner product $\langle \psi, \mathcal{L} \varphi \rangle$ for $C^2_b$ functions $\psi, \varphi$ to obtain $\langle \mathcal{L}^* \psi, \varphi \rangle$, the adjoint of the generator is

$$\mathcal{L}^* \varphi(x) = -\sum_{i=1}^m \frac{\partial}{\partial x_i} [b_i(x) \varphi(x)] + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2}{\partial x_i x_j} [(\sigma \sigma^*)_{ij} \varphi(x)],$$

The probability density of the Itô diffusion satisfies the Fokker-Planck equation (Kolmogorov forward equation):

$$\frac{d}{dt} p_t(x) = \mathcal{L}^* p_t(x).$$

The conditional density given observation satisfies a similar partial differential equation, except it is driven by a stochastic process corresponding to information from observations.

2.3.3 The Zakai equation

Recall that the unnormalized filter is constructed w.r.t. a new probability measure $\mathbb{P}$ and is related to the original $\mathbb{Q}$ by the measure change $\tilde{D}_t = \frac{d \mathbb{Q}}{d \mathbb{P}} |_{\mathcal{F}_t}$:

$$\rho_t(\varphi) = \mathbb{E}_\mathbb{P} \left[ \varphi(X_t) \tilde{D}_t \right] |_{\mathcal{Y}_t}.$$

The SPDE for $\rho_t(\varphi)$ can be obtained by applications of Itô’s formula. By
the Itô SDE for observation,
\[
\tilde{D}_t = D_t^{-1} = \exp \left\{ \int_0^t h^*(X_s) dB_s + \frac{1}{2} \int_0^t \|h(X_s)\|^2 ds \right\} \\
= \exp \left\{ \int_0^t h^*(X_s) [dY_s - h(X_s)ds] + \frac{1}{2} \int_0^t \|h(X_s)\|^2 ds \right\} \\
= \exp \left\{ \int_0^t h^*(X_s) dY_s - \frac{1}{2} \int_0^t \|h(X_s)\|^2 ds \right\}.
\]

Let \( \Gamma_t := \int_0^t h(X_s)^*dY_s - \frac{1}{2} \int_0^t \|h(X_s)\|^2 ds \). By Taylor series expansion for a function of an Itô diffusion and Itô’s lemma,
\[
d\tilde{D}_t = \tilde{D}_td\Gamma_t + \frac{1}{2} \tilde{D}_td\langle \Gamma \rangle_t = \tilde{D}_th^*(X_t)dY_t,
\]
and
\[
d\varphi(X_t) = \mathcal{L}\varphi(X_t)dt + \nabla\varphi(X_t)dW_t,
\]
for a Borel-measurable, \( \mathcal{C}^2_b \) function \( \varphi : \mathbb{R}^m \to \mathbb{R}^n \), where \( \nabla = \left[ \frac{d}{dx_1} \ldots \frac{d}{dx_m} \right] \).

Using these variations, the variation of \( \rho_t(\varphi) \) is
\[
d\rho_t(\varphi) = \mathbb{E}_P \left[ d\varphi(X_t)\tilde{D}_t \mid \mathcal{Y}_t \right] + \mathbb{E}_P \left[ \varphi(X_t)d\tilde{D}_t \mid \mathcal{Y}_t \right] + \mathbb{E}_P \left[ d\langle \varphi(X_t), \tilde{D}_t \rangle \mid \mathcal{Y}_t \right] \\
= \rho_t(\mathcal{L}\varphi)dt + \rho_t(\varphi h^*)dY_t.
\]

The quadratic variation and \( dW \) in \( d\varphi \) vanish because \( Y \) is a \( \mathbb{P} \)-Brownian motion independent of \( W \). The Zakai equation for the unnormalized filter in the uncorrelated signal and sensor noises setting is
\[
d\rho_t(\varphi) = \rho_t(\mathcal{L}\varphi)dt + \rho_t(\varphi h^*)dY_t, \quad (2.15)
\]
\[
\rho_0(\varphi) = \mathbb{E}_P [\varphi(X_0)].
\]

When the noises are correlated, as would arise in real systems where the observation is directly one or more components of the signal, the \( dW \) term would remain and the quadratic variation results in \( \rho_t(\nabla\varphi\sigma h^*)dt \). In this work, we only consider the uncorrelated noises setting.
2.3.4 The Kushner-Stratonovich equation

By the Kallianpur-Striebel formula, $\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)}$. From the Zakai equation,

$$d\rho_t(1) = \rho_t(h^*)dY_t.$$ The variation of $\pi_t(\varphi)$ is

$$d\pi_t(\varphi) = \frac{1}{\rho_t(1)}d\rho_t(\varphi) - \frac{1}{\rho_t(1)}\pi_t(\varphi)d\rho_t(1) - \frac{1}{\rho_t(1)^2}d\langle \rho_t(\varphi), \rho_t(1) \rangle_t + \frac{\pi_t(\varphi)}{\rho_t(1)^2}d\langle \rho(1) \rangle_t,$$

which results in the Kushner-Stratonovich equation:

$$d\pi_t(\varphi) = \pi_t(L\varphi)dt + [\pi_t(\varphi h^*) - \pi_t(\varphi)\pi_t(h^*)] [dY_t - \pi_t(h)dt], \quad (2.16)$$

$$\pi_0(\varphi) = \mathbb{E}_Q [\varphi(X_0)],$$

where $\nu_t \overset{\text{def}}{=} Y_t - \int_0^t \pi_s(h)ds$ is the innovation process. The innovation process is a $Q$-Brownian motion (see Appendix 2). In the correlated noises setting, there would be additional terms corresponding to those for the Zakai equation mentioned in the previous section. The Kushner-Stratonovich equation for the correlated noises setting can be found in, for example, Chapter 11 of [38].

2.4 Linear continuous time filtering (Kalman-Bucy filter)

When the signal and observations are linear, the Itô SDEs (2.11) are of the form

Signal: $dX_t = A_tX_tdt + \Sigma_t^x dW_t, \quad X_0 \in \mathbb{R}^m$, \quad (2.17a)

Observation: $dY_t = H_tX_tdt + \Sigma_t^y dB_t, \quad Y_0 = 0 \in \mathbb{R}^d$, \quad (2.17b)

where $A_t \in \mathbb{R}^{m \times m}$, $\Sigma_t \in \mathbb{R}^{m \times k}$ and $H_t \in \mathbb{R}^{d \times m}$, and $X_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$. The filter for this system can be obtained by setting $\varphi(x) = x$ in the Kushner-Stratonovich equation (2.16) and rewriting the SPDE using the system (2.17).

In the linear setting, the mean and covariance are sufficient to describe the
posterior distribution. Denote by 

$$\hat{X}_t \defeq \mathbb{E}_Q [X_t | \mathcal{Y}_t] = \pi_t(I_{m \times m}),$$

the mean of the posterior distribution. The Kushner-Stratonovich equation provides the variation of $\hat{X}_t$, which ends up being a SDE driven by the innovation $\nu_t$, which is a $Q$-Brownian motion. Note that we have included sensor noise covariance instead of the sensor noise being just a standard Brownian motion. The Zakai and Kushner-Stratonovich equations described in Section 2.3 still hold, with the inclusion of sensor noise covariance.

Let $P_t$ denote the covariance, where

$$P_{ij} = \mathbb{E}_Q \left[ (X_i^j - \hat{X}_i^j)(X_j^i - \hat{X}_j^i) \bigg| \mathcal{Y}_t \right] = \hat{X}_i^j X_j^i - \hat{X}_i^i \hat{X}_j^j.$$

The resulting equation for $P_t$ is a Riccati equation with initial condition equal to the covariance of the initial distribution. The Kalman-Bucy filter is:

$$d\hat{X}_t = A_t \hat{X}_t dt + P_t H_t^* R_t^{-1} d\nu_t, \quad \hat{X}_0 = \mu_0, \quad (2.18a)$$

$$\dot{P}_t = A_t P_t + P_t A_t^* + Q_t - P_t H_t^* R_t^{-1} H_t P_t, \quad P_0 = \Sigma_0, \quad (2.18b)$$

where $Q_t \defeq \Sigma_t^x (\Sigma_t^x)^*$, $R_t \defeq \Sigma_t^y (\Sigma_t^y)^*$ As in the discrete time case, the posterior mean evolves according to the signal dynamics $A_t$, with a correction $d\nu_t$ based on observations. The Kalman gain for the correction in this case is $P_t H_t^* R_t^{-1}$. The Riccati equation for the posterior covariance contains a part for covariance growth due to signal dynamics, $A_t P_t + P_t A_t^* + Q_t$, and a reduction $P_t H_t^* R_t^{-1} H_t P_t$ that is inversely proportional to the sensor noise covariance $R_t$. When the covariances of the signal and sensor noises and initial distribution are known, the covariance can be computed offline.

### 2.5 Continuous-discrete time filtering

When observations are one or more components of the signal, or sensor measurements are at a high enough frequency, the observation can be modeled as continuous in time. This is not always the case, for example in meteorology or climate science, measurements are usually taken at the end of relatively long intervals. Temperature and wind data may be recorded hourly; ocean
buoy data, ship measurements, and satellite data are collected/transmitted at long intervals, sometimes on the order of days, weeks or months. In such cases, the signal is continuous, but the observations are discrete:

\[
\text{Signal: } dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 \in \mathbb{R}^m, \quad (2.19a)
\]

\[
\text{Observation: } Y_k = h(X_{t_k}) + B_k, \quad Y_0 = 0 \in \mathbb{R}^d, \quad (2.19b)
\]

where \( k = 1, 2, \ldots \), \( b : \mathbb{R}^m \to \mathbb{R}^m \), \( \sigma : \mathbb{R}^m \to \mathbb{R}^{m \times k} \), \( h : \mathbb{R}^m \to \mathbb{R}^d \) are Borel-measurable functions, \( W \) is an \( m \)-dimensional Brownian motion, and \( B_k \sim \mathcal{N}(0, R_k) \).

The prior density \( p_t \) is the density of the Itô diffusion \( X \) with generator \( \mathcal{L} \). \( p_t \) satisfies the Fokker-Planck equation, which is completely determined by the signal dynamics (see Section 2.3.2). At time \( t_k \), when observation is available, the posterior density can be constructed using the product of observation likelihood and prior density by Bayes’s theorem:

\[
p_{t_k}(x|y_{0:k}) = \frac{p(y_k|x)p_{t_k}(x|y_{0:k-1})}{p(y_k|y_{0:k-1})},
\]

where the prior \( p_{t_k}(x|y_{0:k-1}) \) is the solution to the Fokker-Planck equation for time in between observations:

\[
\frac{d}{dt}p_t(x) = \mathcal{L}^*p_t(x), \quad t \in [t_{k-1}, t_k],
\]

\[
p_{t_{k-1}}(x) = p_{t_{k-1}}(x|y_{0:k-1}),
\]

where the initial condition is the posterior from the last observation update.

### 2.6 Particle Filter

Monte Carlo (MC) methods are a class of methods for sampling from probability densities or simulating stochastic processes, sometimes referred to as stochastic simulation ([45]). The Markov-Chain Monte Carlo method is an example for numerically simulating a stochastic process with a specified transition probability and is commonly used in practice, for example in network systems and electrical and computer engineering problems. MC methods involve generating random numbers, which can be a computationally ex-
pensive procedure in high dimensions. The lecture notes of [45] provide a good self-contained overview of MC methods with relevant references.

In the filtering framework, MC methods are used sequentially for sampling from the posterior density (2.8) or approximating the infinite-dimensional solutions to (2.15) and (2.16), when closed-form solutions such as for the linear case is not available. Sequential MC methods for nonlinear filtering are a class of methods called particle filters, first introduced as such in [10]. The Metropolis-Hastings algorithm is a MC method that was introduced earlier by [46], explored more rigorously and generalized by [47]. Although not coined as such by the initial authors, when applied to a partially observed system, the Metropolis-Hastings algorithm can be considered a form of discrete-time nonlinear filtering, for a special case where the system attains an invariant density. Since [10], many variants of particle filters have been developed, mostly based on different resampling techniques to better exploit the use of observations and/or address issues associated with approximating a continuous density using a finite sample. [13], [11] and [39] contain comprehensive, but not exhaustive, survey of particle filtering techniques. [11] and [48] provide a good introduction and theoretical basis for particle filtering in discrete time. Chapter 9 of [35] provides a good theoretical development of a numerical approximation to solutions to the nonlinear filtering equations in continuous time, including convergence results from the papers [12] and [14] by one of the authors. Chapter 10 of [35] develops the discrete-time approximation from the same basis. [13] contains convergence results for discrete- and continuous-time particle filters. The references cited here are based on personal preference, based on preference on rigor, comprehension and exposure, and are not claimed to be the definitive references for theoretical and practical works on particle filtering.

As mentioned in introduction of this chapter, the extended Kalman filter can provide closed form linear approximations in certain nonlinear problems. A sequential MC version called the ensemble Kalman filter has been shown to perform well in a wide class of nonlinear problems (for example, [49], [9] and [50]). A particle filter that combines Bayesian posterior density construction with the Kalman update has also been developed in [51]. Other more recent developments in particle filtering include [52] and [53], [54], [55], and [56] and [57].

Figure 2.1 represents the evolvement of densities from time step $k - 1$ to
Figure 2.1: Evolution of signal density

$k$ for a discrete-time stochastic process (may consider as from $t - \delta t$ to $t$ in continuous time). The left side of each 2.1(a) and 2.1(b) represents time $k - 1$ ($t - \delta t$), the right side represents time $k$ ($t$). The triangle represents location of the true signal, which is one realization from the underlayed gray true density. The true signal is available to the observer only as partial observation that is corrupted by noise. The black trajectory indicates the path taken from one time to the next. The blue density on the left in 2.1(a) represents the initial density, which may be the initial condition, the Fokker-Planck solution, or the posterior from the previous observation update. On the right in the same figure, it represents the prior density – from the state transition density or solution to the Fokker-Planck equation starting from the initial density. Density means are indicated by stars. 2.1(b) is 2.1(a) overlayed with the red posterior density. Green represents the density indicated by observations, which is contaminated by sensor noise. The posterior is obtained by updating the prior using the information from observations, resulting in a mean that is closer to the true signal location. Particle filtering is a computational method for approximating the density evolution in Figure 2.1(a) and observation update in Figure 2.1(b). In this chapter, we provide overviews of particle filters in discrete and continuous time, and cite convergence results from [35] and [13]. At the end, we point out an issue in high dimensional problems by referring to an example of the particle degeneracy issue from [1].
2.6.1 Discrete time

Recall the discrete time stochastic difference equation (2.7) from Section 2.2:

Signal: \( X_{k+1} = b(X_k, k + 1) + W_{k+1}, \ X_0 \in \mathbb{R}^m, \) \hspace{1cm} (2.20a)
Observation: \( Y_k = h(X_k, k) + B_k, \ Y_0 = \mathbf{0} \in \mathbb{R}^d. \) \hspace{1cm} (2.20b)

The posterior density, by Bayes’s theorem, is a product of the observation likelihood and the prior, divided by a normalizing constant:

\[
p(x_{0:k}|y_{0:k}) = \frac{p(y_k|x_k)p(x_k|x_{0:k-1})p(x_{0:k-1}|y_{0:k-1})}{p(y_k|y_{0:k-1})}.
\] \hspace{1cm} (2.21)

Note that here, we consider the joint posterior \( p(x_{0:k}|y_{0:k}) \) instead of the marginal \( p(x_k|y_{0:k}) \) as in Section 2.2 to avoid the integration w.r.t. \( x_{k-1} \) in the prior density. The marginal can be obtained from this posterior by integrating out \( x_{0:k-1} \). However, in implementation, this integration is equivalent to sampling from the posterior, which is a part of the sequential step of particle filtering.

Particle filtering provides the approximation \( \hat{p}^{Ns}(x_{0:k}|y_{0:k}) \) of the posterior (2.21) using a finite sample of \( N_s \) particles that represent possible realizations of \( X \). In short, particle filtering involves

1. sampling from an initial density to construct a sample of particles representing possible locations of the true signal (Figure 2.2(a))
2. propagating the particles forward in time (Figure 2.2(b))
3. construct a posterior density by weighting particles based on their locations relative to information from observations (Figure 2.2(c))

Figure 2.2: Evolution of signal density using particles
Sampling from an initial density can be done by standard sampling techniques, for example the accept-reject algorithm. For the posterior at time \( k \), the initial density is the posterior at time \( k - 1 \), \( p(x_{0:k-1}|y_{0:k-1}) \) in (2.21), which we assume that we already have samples for, from in the iterative procedure. Particles in the initial sample represent possible locations for \( X_{k-1} \), and are propagated forward to possible locations at time \( k \) using the difference equation (2.20a). The sample of particles with new locations represent the prior distribution \( p(x_k|x_{k-1}) \) in (2.21). The conditional is only on \( x_{k-1} \) since \( X_k \) is Markovian based on (2.20a), dependent only on the state at \( k - 1 \).

The posterior is constructed by assigning importance weights to particles, based on how close each particle location is to the location of the true signal. The weights are determined using the observation likelihood \( p(y_k|x_k) \). Let \( x^i_k \) represent location of the \( i \)th particle, \( i = 1, \ldots, N_s \). The weight of particle \( i \) is

\[
\begin{align*}
    w^{i,N_s}_{0:k} &\overset{\text{def}}{=} p(y_k|x^i_k) \bar{w}^{i,N_s}_{0:k-1}, \quad (2.22) \\
\end{align*}
\]

where \( y_k \) is the value of the recorded observation at time \( k \). \( \bar{w}^{i,N_s}_{0:k-1} \) is the cumulative normalized weights from past observation updates, i.e. the weight of particle \( i \) in \( \hat{p}^{N_s}(x_{0:k-1}|y_{0:k-1}) \). Given the system (2.20) with Gaussian noise, the prior is easy to sample from and the observation likelihood can be directly computed, since they are both normal. The weight is

\[
\begin{align*}
    w^{i,N_s}_{0:k} &= \frac{1}{\sqrt{(2\pi)^d|R_k|}} \exp \left\{ -\frac{1}{2} (y_k - h(x^i_k,k))^* R_k^{-1} (y_k - h(x^i_k,k)) \right\} \bar{w}^{i,N_s}_{0:k-1} \\
\end{align*}
\]

When the signal is non-Gaussian and the prior is difficult to sample from, we can use a proposal density \( q(x_k) \) that is easier to sample from (see, for example, [11]). \( q \) has to be absolutely continuous w.r.t. the prior. Then, particle weight includes the ratio of the prior to the proposal, which indicates how “close” the proposal density is to the prior:

\[
\begin{align*}
    w^{i,N_s}_{0:k} &\overset{\text{def}}{=} \frac{p(y_k|x^i_k)p(x_k|x_{k-1})}{q(x_k)} \bar{w}^{i,N_s}_{0:k-1}. \quad (2.23) \\
\end{align*}
\]

The denominator in (2.21) is the same for all \( x^i_k \), since it is a normalizing
constant:

\[ p(y_k|y_{0:k-1}) = \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} p(y_k, x_{0:k}, y_{0:k-1}) \, dx_{0:k} \]

\[ = \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} p(y_k|x_k)p(x_k|x_{0:k-1})p(x_{0:k-1}|y_{0:k-1}) \, dx_{0:k}, \]

which is approximated by the sum of weights \( \sum_{j=1}^{N_s} w_{0:k}^{i,N_s} \). The particle approximation of the posterior density is

\[ \hat{p}^N_s(x_{0:k}|y_{0:k}) = \sum_{i=1}^{N_s} \bar{w}_{0:k}^{i,N_s} \delta_{x_{0:k}^i}(x_{0:k}), \tag{2.24} \]

where we denote by \( \bar{w}_{0:k}^{i,N_s} \) the normalized weight and \( \delta \) is the Dirac delta/indicator function.

Iteratively, the particle filtering algorithm described above is:

```
Draw samples from initial distribution with uniform weights: \{x_{0}^{i}, w_{0}^{i,N_s}\}_{i=1}^{N_s}
where \( w_{0}^{i,N_s} = \frac{1}{N_s} \) for \( i = 1, \ldots, N_s \)

for \( k = 1: \text{number of timesteps} \ K \) do
    for \( i = 1: \text{number of particles} \ N_s \) do
        Propagate \( x_{k-1}^{i} \) forward to \( x_{k}^{i} \) using (2.20a)
        Update weight \( w_{0:k}^{i,N_s} \) using \( y_k \) and \( w_{0:k-1}^{i,N_s} \) by (2.22)
    end for
    Normalize weights \( \bar{w}_{0:k}^{i,N_s} = \frac{w_{0:k}^{i,N_s}}{\sum_{j=1}^{N_s} w_{0:k}^{j,N_s}} \)
    Construct posterior using \( \{x_{0:k}^{i}, w_{0:k}^{i,N_s}\}_{i=1}^{N_s} \) by (2.24)
end for
```

Algorithm 1: SIS particle filter

Note that even though the joint density of \( x_{0:k} \) is constructed at each iteration, only computations for \( x_k \) are performed at iteration \( k \), the samples \( \{x_{0:k-1}^{i}, w_{0:k-1}^{i,N_s}\}_{i=1}^{N_s} \) have been obtained in previous iterations. The particle filtering algorithm is performed sequentially from \( k - 1 \) to \( k \) for \( k = 1, \ldots, K \), and “sampling” of particles from the posterior is achieved by assigning weights, given by observation likelihood. Hence this algorithm is called the Sequential Importance Sampling (SIS) algorithm. As mentioned earlier, if we are only interested in the marginal posterior \( p(x_k|y_{0:k}) \), we can approximate it using \( \{x_{k}^{i}, w_{0:k}^{i,N_s}\}_{i=1}^{N_s} \), without having to integrate the joint posterior.
For finite $N_s$, the approximation $\hat{p}^{N_s}$ is biased, but asymptotically consistent. Denote

$$\hat{\pi}_{0:k}^{N_s}(\phi) \overset{\text{def}}{=} \sum_{i=1}^{N_s} \bar{w}_i^{N_s} \phi(x_{i:k}), \quad \pi_k(\phi) = \int_{\mathbb{R}^m} \ldots \int_{\mathbb{R}^m} \phi(x_{0:k}) dx_{0:k}.$$ 

Asymptotically (in the limit of large sample size), the approximation error amplified by sample size and square root of sample size are ([13])

$$\lim_{N_s \to \infty} N_s (\hat{\pi}_{0:k}^{N_s}(\phi) - \pi_{0:k}(\phi)) = -\int_{\mathbb{R}^m} \ldots \int_{\mathbb{R}^m} (\phi(x_{0:k}) - \pi_{0:k}(\phi)) \frac{p^2(x_{0:k}|y_{0:k})}{p(x_{0:k})} dx_{0:k} \quad \text{(bias)}$$

and

$$\lim_{N_s \to \infty} \sqrt{N_s} (\hat{\pi}_{0:k}^{N_s}(\phi) - \pi_{0:k}(\phi)) \sim \mathcal{N} \left( 0, \int_{\mathbb{R}^m} \ldots \int_{\mathbb{R}^m} (\phi(x_{0:k}) - \pi_{0:k}(\phi))^2 \frac{p^2(x_{0:k}|y_{0:k})}{p(x_{0:k})} dx_{0:k} \right) \quad \text{(asymptotic normality)},$$

respectively. The mean-squared error (MSE) can be written as the sum of bias squared and variance (see Appendix 5). For finite sample size,

$$\text{MSE} = \frac{\text{bias}^2}{\sigma(N_s^{-2})} + \frac{\text{variance}}{\sigma(N_s^{-1})} \quad (2.25)$$

Hence, asymptotically, the bias is negligible compared to the variance.

Resampling

The observation update (2.22) assigns higher weights to particles located closer to the true signal location indicated by observations, as an exponential function of the distance squared. For the SIS algorithm, assuming particles are sampled uniformly from an initial distribution, after a sufficient number of observation updates, weights will end up being concentrated on a small portion of particles within the sample. This issue is called sample impoverishment/particle degeneracy (see, for example, [1], [16], [48]). A large number of $N_s$ particles are carried forward through the iterations of $k$, but only a small
portion contributes to the construction of $\hat{p}^{N_s}$. This phenomena can also be interpreted as the variance of the unnormalized weights increasing with each SIS iteration (see, for example, Proposition 3, page 7 in [58], Theorem on page 285 in [59]). This is inefficient from a computational standpoint and, more importantly, the ability of the sample to represent the randomness of the system is concentrated on a number of particles less than $N_s$, which is unpreferable based on the MSE (2.25).

A suitable measure of particle degeneracy is the effective sample size $N_{\text{eff}}$ introduced in [59] and [60]:

$$N_{\text{eff},k} \overset{\text{def}}{=} \frac{N_s}{1 + \text{Var}(w_{0:k}^*|y_{0:k})},$$

where $w_{0:k}^* \overset{\text{def}}{=} \frac{p(x_{k}|y_{0:k})}{q(x_{k})}$ is the true weight. The true weight cannot be determined exactly, but can be approximated by

$$\hat{N}_{\text{eff},k} \overset{\text{def}}{=} \frac{1}{\sum_{i=1}^{N_s} \left(\bar{w}_{0:k}^i N_s\right)^2}.$$

The particle degeneracy issue can be addressed by performing a resampling procedure after observation update if $\hat{N}_{\text{eff},k}$ falls below a set threshold $N_{\text{thres}}$. This serves to discard particles with insignificant weights and replenish the sample such that all particles have equal or almost equal weights. At iteration $k$, this is done after the weights normalization procedure in Algorithm 1. One algorithm to achieve this is the systematic resampling algorithm described in Algorithm 2. The SIS with resampling is called the Sequential Importance Resampling (SIR) algorithm. For survey and comparison of different resampling algorithms, see, for example, [61] and [62].

### 2.6.2 Continuous time

Recall the diffusion process $X_t$ with partial observation $Y_t$ that satisfy the Itô SDE (2.11) from Section 2.3, with generator $\mathcal{L}$ defined in (2.14). In continuous time, the particle filter approximates the solution to the Kushner-Stratonovich equation (2.16). The algorithm is analogous to the discrete time particle filter, with particle propagation according to the Itô SDE of $X$.
Sample at iteration $k$: \( \{x_{0:k}^i, w_{0:k}^{i,N_s}\}_{i=1}^{N_s} \)

\[ \text{if } \hat{N}_{\text{eff},k} < N_{\text{thres}} \text{ then} \]
\[ \text{Initialize cdf: } c_1 = 0 \]
\[ \text{for } i = 2: \text{number of particles } N_s \text{ do} \]
\[ \text{Construct cdf: } c_i = c_{i-1} + \bar{w}_{0:k}^{i,N_s} \]
\[ \text{end for} \]
\[ \text{Draw uniform } u_1 \sim \mathcal{U}(0, N_s^{-1}) \]
\[ \text{Start at bottom of cdf: Set } i = 1 \]
\[ \text{for } j = 1: \text{number of particles } N_s \text{ do} \]
\[ \text{Move up the cdf: } u_j = u_1 + \frac{j-1}{N_s} \]
\[ \text{while } u_j > c_i \text{ do} \]
\[ \quad i = i + 1 \]
\[ \text{end while} \]
\[ \text{Replicate sample } i: \{ (x_s^*)_{0:k}^j = x_{0:k}^i \} \]
\[ \text{Assign weight: } \bar{w}_{0:k}^{j,N_s} = \frac{1}{N_s} \]
\[ \text{end for} \]
\[ \text{New sample: } \{ (x_s^*)_{0:k}^j, \frac{1}{N_s} \}_{j=1}^{N_s} \]
\[ \text{else} \]
\[ \text{Keep sample } \{ x_{0:k}^i, w_{0:k}^{i,N_s}\}_{i=1}^{N_s} \]
\[ \text{end if} \]

Algorithm 2: Systematic resampling

and particle weights determined according to the measure change \( \hat{D} \) in the Kallianpur-Striebel formula

\[
\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)} = \frac{\mathbb{E}_\mathcal{P}\left[\varphi(X_t)\hat{D}_t \mid \mathcal{Y}_t\right]}{\mathbb{E}_\mathcal{P}\left[\hat{D}_t \mid \mathcal{Y}_t\right]}. \tag{2.26}
\]

The algorithm described here is from Chapter 9.2 of [35], developed in [12] and [14].

Let \( t \in [0, T) \), \( 0 < T \leq \infty \), and \( [0, T) \) be partitioned into \( K \) equal time-steps \( \delta t \). At time \( t = 0 \), a sample of \( N_s \) particles is drawn uniformly from the initial distribution, each particle \( x_s^i \) represents a possible location for \( X_t \) for \( t \in [0, T) \). Each particle is propagated forward in time by the Itô integral

\[
x_s^i = x_0^i + \int_0^t b(x_s^i)ds + \int_0^t \sigma(x_s^i)dW_s,
\]

which can be approximated using stochastic numerical integration methods, for example the Euler-Maruyama scheme (see, for example, [63] for stochastic...
tic integration schemes). The sample \( \{x^i_t\}_{i=1}^{N_s} \) represents the solution to the Fokker-Planck equation at time \( t \) (Figure 2.1(a) and Figure 2.2(b)), equivalently, \( \mathbb{E}_P[\varphi(X_t)] \) is approximated by

\[
\frac{1}{N_s} \sum_{i=1}^{N_s} \delta_{\varphi(x^i_t)} \varphi(x), \quad x \in \mathbb{R}^m. \tag{2.27}
\]

Now consider an interval \([((k-1)\delta t), k\delta t), k = 1, \ldots, K \). Assume that we already have the particle locations and weights \( \{x^i_{(k-1)\delta t}, w^i_{(k-1)\delta t}\}_{i=1}^{N_s} \). For \( t \in \left[ (k-1)\delta t, k\delta t \right) \),

\[
x^i_t = x^i_{(k-1)\delta t} + \int_{(k-1)\delta t}^{t} b(x^i_s)ds + \int_{(k-1)\delta t}^{t} \sigma(x^i_s)dW_s.
\]

Recall that the unnormalized filter is

\[
\rho_t(\varphi) = \mathbb{E}_P\left[ \varphi(X_t) \tilde{D}_t \mid \mathcal{Y}_t \right], \tag{2.28}
\]

where the measure change is

\[
\tilde{D}_t \overset{\text{def}}{=} \frac{dQ}{dP}\big|_{\mathcal{F}_t} = \exp \left\{ \int_0^t h^*(X_s)dY_s - \frac{1}{2} \int_0^t \|h(X_s)\|^2 ds \right\}.
\]

Under \( P \), \( X_t \) and \( Y_t \) are independent, hence the conditional expectation \( \mathbb{E}_P \) can be represented directly by the particles \( \{x^i_t\}_{i=1}^{N_s} \), similar to (2.27). However, the conditional expectation (2.28) involves the measure change \( \tilde{D}_t \). Analogous to the observation likelihood for the discrete time algorithm, \( \tilde{D}_t \) defines the weights for the particle filter algorithm:

\[
w^i_{t,N_s} \overset{\text{def}}{=} \exp \left\{ \int_{(k-1)\delta t}^{t} h^*(x^i_s)dY_s - \frac{1}{2} \int_{(k-1)\delta t}^{t} \|h(x^i_s)\|^2 ds \right\} \tilde{D}_{(k-1)\delta t} \tag{2.29}
\]

\[
\approx \exp \left\{ \sum_{k' = 0}^{K'-1} h^*(x^i_{(k-1)\delta t+k'\delta s}) \left[ Y_{(k-1)\delta t+(k'+1)\delta s} - Y_{(k-1)\delta t+k'\delta s} \right] - \frac{1}{2} \sum_{k' = 0}^{K'-1} \|h(x^i_{(k-1)\delta t+k'\delta s})\|^2 \delta s \right\} w^i_{(k-1)\delta t},
\]

where \( \delta s \overset{\text{def}}{=} \left[ \frac{t-(k-1)\delta t}{K'} \right] \). The particle approximation of the unnormalized
filter is
\[ \hat{\rho}_t^{N_s}(\varphi) \overset{\text{def}}{=} \sum_{i=1}^{N_s} w_t^{i,N_s} \varphi(x_t^i). \]

By the Kallianpur Striebel formula (2.26), the particle approximation of the normalized filter is obtained by normalizing the weights:

\[ \hat{\pi}_t^{N_s}(\varphi) \overset{\text{def}}{=} \frac{1}{N_s} \sum_{i=1}^{N_s} \tilde{w}_t^{i,N_s} \varphi(x_t^i), \quad \text{where} \quad \tilde{w}_t^{i,N_s} \overset{\text{def}}{=} \frac{w_t^{i,N_s}}{\sum_{j=1}^{N_s} w_j^{j,N_s}}. \]

In the numerical procedure over \([0, K\delta t]\), the stochastic integration and weight computation described above is performed sequentially over the intervals \([(k-1)\delta t, k\delta t), k = 1, \ldots, K\].

The MSE of the particle approximation is inversely proportional to \(N_s\), proved in Proposition 9.18 and Theorem 9.19 of [35], for \(\hat{\rho}_t^{N_s}(\varphi)\) and \(\hat{\pi}_t^{N_s}(\varphi)\), respectively: For bounded, Lipschitz \(b, \sigma\) and \(h\), there exist constants \(c_1^T\) and \(c_2^T\) independent of \(N_s\) such that

\[
\mathbb{E}_\mathbb{P} \left[ \sup_{t \in [0,T]} \left( \hat{\rho}_t^{N_s}(\varphi) - \rho_t(\varphi) \right)^2 \right] \leq \frac{c_1^T}{N_s} \| \varphi \|_{2,\infty}^2,
\]

and

\[
\mathbb{E}_\mathbb{P} \left[ \sup_{t \in [0,T]} \left( \hat{\pi}_t^{N_s}(\varphi) - \pi_t(\varphi) \right)^2 \right] \leq \frac{c_2^T}{N_s} \| \varphi \|_{2,\infty}^2,
\]

for \(\mathcal{B}_b^2\) Borel-measurable function \(\varphi\), and \(\| \cdot \|_{2,\infty}\) is a supremum norm. Note that the convergence results are for expected MSE under \(\mathbb{P}\), which can be extended to \(\mathbb{Q}\). The same continuous-time particle approximation and corresponding convergence results are also obtained in [13].

Discrete- and continuous-time particle weights

If we discretize the continuous time system, particle weights evaluated using \(\tilde{D}\) and the observation likelihood \(p(y_k|x_{tk})\) are the same. Consider the time interval \([t - \Delta t, t]\). If we discretize the Itô SDE for the observation, then an
increment of the observation is
\[ \delta Y_t = h(X_{t-\Delta t}) \Delta t + \eta \sqrt{\Delta t}, \]
where \( \eta \sim \mathcal{N}(0,1) \). Given \( X_{t-\Delta t} \), \( \Delta Y_t \) is Gaussian with mean \( h(X_{t-\Delta t}) \Delta t \), variance \( \Delta t \). Also, given \( X \), observation increments are independent over time, so the particle weight in discrete time is
\[ w_i^t = p(\Delta Y_t | X_{t-\Delta t}) w_i^{t-\Delta t}, \]
where the observation likelihood is
\[ p(\Delta Y_t | X_{t-\Delta t}) = \frac{1}{\sqrt{2\pi \Delta t}} \exp \left\{ -\frac{1}{2} \left( \frac{[\Delta Y_t]^2}{\Delta t} - 2 \Delta Y_t h(X_{t-\Delta t}) \Delta t + [h(X_{t-\Delta t}) \Delta t]^2 \right) \right\} \]
\[ = C(\Delta Y_t) \exp \left\{ h(X_{t-\Delta t}) \Delta Y_t - [h(X_{t-\Delta t})]^2 \Delta t \right\}, \quad (2.30) \]
where the constant \( C(\Delta Y_t) = \frac{\exp\{[\Delta Y_t]^2/2\Delta t\}}{\sqrt{2\pi \Delta t}} \) is the same for all particles, so it doesn’t affect weight calculation since weights are normalized anyway.

The discrete form of the measure change \( \tilde{D} \) in the particle weight (2.29) is
\[ \tilde{D}_t = \exp \left\{ h(X_{t-\Delta t}) \Delta Y_t - \frac{1}{2} \left[ h(X_{t-\Delta t}) \right]^2 \Delta t \right\}, \]
which is the same as (2.30), scaled by a constant.

2.6.3 Curse of dimensionality

Particle filtering provide a good finite dimensional numerical approximation to the infinite-dimensional solution of the nonlinear filtering problem. However, despite rigorous convergence results, particle filters can be unstable, as the convergence results are asymptotic properties, in the limit of infinitely large sample size, while implementation can only be performed with finite sample size. For example, a signal with a large initial sample space would
require proportionally large sample size, and following observation updates, particle weights may be highly varying, resulting in the particle degeneracy mentioned in Section 2.6.1, for which resampling is one remedy. In large scale systems, particle filters may still suffer from slow convergence rates. For example, [1] points out a case in [64] where the SIS particle filter collapses to a point mass after just a few observation updates when applied to a large scale geophysical model. Particle weight collapse and the condition for its occurrence have been studies in [1] and [16].

Figure 2.3: Histogram of maximum weight (extracted from [1]); \(d\) varies column wise, \(N_s = n\) varies row-wise, each histogram is for 400 experiments; \(x\)-axis is maximum particle weight \((\in (0,1])\), \(y\)-axis is number of simulations, black line indicates mean.

[1] presents the following simple numerical experiment that illustrates particle degeneracy, where most of the weights in a sample is concentrated on a small number of particles: Let the signal be \(X \sim \mathcal{N}(0, I_{d \times d})\) be a \(d\)-dimensional standard Gaussian and the observation be the signal perturbed by standard Gaussian noise, i.e. \(Y = HX + B\), where \(H = I_{d \times d}\) and \(B \sim \mathcal{N}(0, I_{d \times d})\). Numerical experiments are performed to estimate \(X\) based on \(Y\) by particle filtering for \(d = 10, 50, 100\). For each \(d\), sample size is varied as \(N_s = 10^{2.5}, 50^{2.5}, 100^{2.5}\) and 400 simulations performed for each \(N_s\). Normalized particle weights are calculated using the observation likelihood as described in Section 2.6.1 and the maximum particle weight recorded. Figure 2.3 shows the histogram of maximum particle weight over each 400
experiments. For \( d = 10 \), when \( N_s \) is high, the maximum particle weight mostly fall in the \((0, 1]\) bin, which indicates that particle weights are almost uniform over the sample. For \( d = 100 \), even with high \( N_s \), almost 38\% of the maximum weights fall in the \((0.9, 1]\) bin, indicating that most of or all the weight is on one particle. Such a sample may not be expected to be representative of the true posterior distribution, as the approximation constructed using the sample would almost be a Dirac delta at the location of the particle with maximum weight. A similar experiment is also performed for Cauchy observation noise in [1], with similar result.

[1] finds that, for large \( d \), the maximum weight behaves as

\[
\max w^{i, N_s} \sim \frac{1}{1 + \sum_{i=2}^{N_s} e^{-\sigma \sqrt{d}(Z(n) - Z(1))}}
\]

where \( Z(n) \) is the \( n \)th order statistic from a sample of size \( N_s \). For large \( d \), this approximation is close to unity, hence particle weights are concentrated on a single particle. [1] also shows that a condition for particle degeneracy is \( N_s \) being small compared to \( \exp\{d!^{1/3}\} \). Specifically, the maximum weight converges to unity as \( \frac{\log N_s}{d^{1/3}} \) goes to zero.

Such obstacle in the implementation of particle filtering in high dimensional settings is the motivation behind the work for this thesis. Geophysical models possess both nonlinearity, which presents difficulties in using linear approximation filters, and high dimensions, which present difficulties in implementation of particle filtering for estimation and prediction. However, these models also possess multiple timescales. The models represent processes that occur at different timescales that are sufficiently wide apart such that the slow scale components are almost stationary at the timescale at which variations of the fast components are observed. If we are only interested in estimation and prediction of the slow components, for example in climate prediction for decadal scale processes, and the fast components possess certain invariant properties, then there exist stochastic averaging results that allow representation of only the slow components. By effectively eliminating the fast scale dimensions using such representation, the dimension of the problem is reduced, making the implementation of particle filtering, or any nonlinear filter approximation, more feasible.
CHAPTER 3

DIMENSIONAL REDUCTION IN MULTISCALE NONLINEAR FILTERING

The main result of multiscale nonlinear filtering of the work for this thesis is the convergence of the filter for the multiscale system to a homogenized filter with support on a lower dimensional space, in the wide timescale separation limit. The construction of the homogenized filter is by asymptotic expansion based on the stochastic averaging results of [17]. The convergence proof utilizes the concept of backward stochastic partial differential equations (BSPDEs) and backward doubly-stochastic differential equations (BDSDEs), and some convergence results from [65].

The multiscale nonlinear filtering problem is formulated, with statement of the main result, in Section 3.1. Section 3.2 describes existing and related works. A brief description of homogenization of multiscale ordinary differential equations (ODEs) and the result of [17] for diffusion processes is presented in Section 3.3. Section 3.4 discusses BSPDEs and BDSDEs. Section 3.6 presents some preliminary estimates required for the main theorem. The main result and corresponding proofs are presented in Section 3.7. The results of Sections 3.6 and 3.7 are in the work [66].

3.1 Problem formulation and statement of main result

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})\) be a filtered probability space that supports a \((k + l + d)\)-dimensional standard Brownian motion \((V, W, B)\). Let the signal \((X^\varepsilon, Z^\varepsilon)\) be a two time scale diffusion process with a fast component \(Z^\varepsilon\) and a slow component \(X^\varepsilon\):

\[
\begin{align*}
  dX^\varepsilon_t &= b(X^\varepsilon_t, Z^\varepsilon_t)dt + \sigma(X^\varepsilon_t, Z^\varepsilon_t)dW_t \\
  dZ^\varepsilon_t &= \frac{1}{\varepsilon}f(X^\varepsilon_t, Z^\varepsilon_t)dt + \frac{1}{\sqrt{\varepsilon}}g(X^\varepsilon_t, Z^\varepsilon_t)dV_t,
\end{align*}
\]  

(3.1a)  

(3.1b)
where $X_t^\varepsilon \in \mathbb{R}^m$, $Z_t^\varepsilon \in \mathbb{R}^n$, $W_t \in \mathbb{R}^k$ and $V_t \in \mathbb{R}^l$ are independent standard Brownian motions, $b : \mathbb{R}^{m+n} \to \mathbb{R}^m$, $\sigma : \mathbb{R}^{m+n} \to \mathbb{R}^{m \times k}$, $f : \mathbb{R}^{m+n} \to \mathbb{R}^n$, $g : \mathbb{R}^{m+n} \to \mathbb{R}^{n \times l}$. All the functions above are assumed to be Borel-measurable.

For fixed $x \in \mathbb{R}^m$, define
\[
dZ_t^x = f(x, Z_t^x) dt + g(x, Z_t^x) dV_t.
\]

Assume that for all $x \in \mathbb{R}^m$, $Z^x$ is ergodic and converges rapidly towards its stationary measure $\mu(x, \cdot)$. This will be made precise later.

The $d$-dimensional observation $Y^\varepsilon$ is given by
\[
Y_t^\varepsilon = \int_0^t h(X_s^\varepsilon, Z_s^\varepsilon) ds + B_t
\]
with Borel-measurable $h : \mathbb{R}^{m+n} \to \mathbb{R}^d$. $B$ is assumed to be a $d$-dimensional standard Brownian motion that is independent of $W$ and $V$.

Define $Y_t^\varepsilon = \sigma(Y_s^\varepsilon : 0 \leq s \leq t) \lor \mathcal{N}$, where $\mathcal{N}$ are the $\mathbb{Q}$-negligible sets. For a finite measure $\pi$ on $\mathbb{R}^{m+n}$ and for a bounded measurable function $\varphi$ on $\mathbb{R}^{m+n}$ denote $\pi(\varphi) = \int \varphi(x, z) \pi(dx, dz)$. Then our aim is to calculate the measure-valued process $(\pi_t^\varepsilon, t \geq 0)$ determined by
\[
\pi_t^\varepsilon(\varphi) = \mathbb{E}[\varphi(X_t^\varepsilon, Z_t^\varepsilon)|Y_t^\varepsilon].
\]

Define the Girsanov transform
\[
\frac{d\mathbb{P}^\varepsilon}{d\mathbb{Q}}|_{\mathcal{F}_t} = D_t^\varepsilon = \exp \left( -\int_0^t h(X_s^\varepsilon, Z_s^\varepsilon)^* dB_s - \frac{1}{2} \int_0^t |h(X_s^\varepsilon, Z_s^\varepsilon)|^2 ds \right).
\]

Under $\mathbb{P}^\varepsilon$, the observation process, $Y^\varepsilon$, is a Brownian motion and independent of $(X^\varepsilon, Z^\varepsilon)$. By the Kallianpur-Striebel formula,
\[
\mathbb{E}_{\mathbb{Q}}[\varphi(X_t^\varepsilon, Z_t^\varepsilon)|Y_t^\varepsilon] = \frac{\mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \varphi(X_t^\varepsilon, Z_t^\varepsilon) \frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon}|_{\mathcal{F}_t} \mid Y_t^\varepsilon \right]}{\mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon}|_{\mathcal{F}_t} \mid Y_t^\varepsilon \right]}
\]
with
\[
\frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon}|_{\mathcal{F}_t} = \tilde{D}_t^\varepsilon = \exp \left( \int_0^t h(X_s^\varepsilon, Z_s^\varepsilon)^* dY_s^\varepsilon - \frac{1}{2} \int_0^t |h(X_s^\varepsilon, Z_s^\varepsilon)|^2 ds \right).
\]
So if we define
\[ \rho_t^\varepsilon(\varphi) = \mathbb{E}_{\mathbb{P}_t^\varepsilon} \left[ \varphi(X_t^\varepsilon, Z_t^\varepsilon) \exp \left( \int_0^t h(X_s^\varepsilon, Z_s^\varepsilon)^* dY_s^\varepsilon - \frac{1}{2} \int_0^t |h(X_s^\varepsilon, Z_s^\varepsilon)|^2 ds \right) \right] \]
then
\[ \pi_t^\varepsilon(\varphi) = \frac{\rho_t^\varepsilon(\varphi)}{\rho_t^\varepsilon(1)}. \]

Denote by \( \mathcal{L}^\varepsilon = \frac{1}{\varepsilon} \mathcal{L}_F + \mathcal{L}_S \) the differential operator associated to \((X^\varepsilon, Z^\varepsilon)\). That is,
\[
\mathcal{L}_F = \sum_{i=1}^n f_i(x, z) \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{i,j=1}^n (gg^*)_{ij}(x, z) \frac{\partial^2}{\partial z_i \partial z_j},
\]
\[
\mathcal{L}_S = \sum_{i=1}^m b_i(x, z) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^m (\sigma\sigma^*)_{ij}(x, z) \frac{\partial^2}{\partial x_i \partial x_j},
\]
where \( \cdot^* \) denotes the transpose of a matrix or a vector.

Then the unnormalized measure-valued process, \( \rho^\varepsilon \), satisfies the Zakai equation:
\[
\frac{d\rho_t^\varepsilon(\varphi)}{dt} = \rho_t^\varepsilon(\mathcal{L}^\varepsilon \varphi) dt + \rho_t^\varepsilon (h \varphi) dY_t^\varepsilon \tag{3.4}
\]
\[
\rho_0^\varepsilon(\varphi) = \mathbb{E}_{\mathbb{Q}}[\varphi(X_0^\varepsilon, Z_0^\varepsilon)]
\]
for every \( \varphi \in C_b^k(\mathbb{R}^{m+n}, \mathbb{R}) \) (see, for example, [35]). For \( k \geq 0 \), \( C_b^k \) is the space of \( k \) times continuously differentiable functions \( \varphi \), such that \( \varphi \) and all its partial derivatives up to order \( k \) are bounded.

The theory of stochastic averaging (see, for example, [17]) tells us that under suitable conditions, \( X^\varepsilon \) converges in law to \( X^0 \) as \( \varepsilon \to 0 \), where \( X^0 \) is the solution of a SDE
\[
dX_t^0 = \bar{b}(X_t^0) dt + \bar{\sigma}(X_t^0) dW_t \tag{3.5}
\]
for suitably averaged \( \bar{b} \) and \( \bar{\sigma} \). Denote the generator of \( X^0 \) by \( \mathcal{L} \).

We want to show that as long as we are only interested in estimating the slow component, we can take advantage of this fact. More precisely, we want
to find a homogenized (unnormalized) filter $\rho^0$, such that for small $\varepsilon$, $\rho^{\varepsilon,x}$ which is the $x$-marginal of $\rho^\varepsilon_t$, is close to $\rho^0$. The $x$-marginal of $\rho^\varepsilon_t$ is defined as

$$
\rho^{\varepsilon,x}_t(\varphi) = \int_{\mathbb{R}^{m+n}} \varphi(x) \rho^\varepsilon_t(dx,dz)
$$

for every measurable bounded $\varphi : \mathbb{R}^m \to \mathbb{R}$, and $\rho^0$ is the solution of

$$
d\rho^0_t(\varphi) = \rho^0_t(L\varphi)dt + \rho^0_t(\tilde{h}\varphi)dY^\varepsilon_t
\quad (3.6)
$$

$$
\rho^0_0(\varphi) = \mathbb{E}_Q[\varphi(X^0_0)],
$$

where $\tilde{h}$ is a suitably averaged version of $h$. The measure-valued processes $\pi^0$ and $\pi^{\varepsilon,x}$ are then defined in terms of $\rho^0$ and $\rho^{\varepsilon,x}$ as $\pi^\varepsilon$ was defined in terms of $\rho^\varepsilon$:

$$
\pi^0_t(\varphi) = \frac{\rho^0_t(\varphi)}{\rho^0_0(1)} \quad \text{and} \quad \pi^{\varepsilon,x}_t(\varphi) = \frac{\rho^{\varepsilon,x}_t(\varphi)}{\rho^{\varepsilon,x}_0(\varphi)}.
$$

Note that the homogenized filter is still driven by the real observation $Y^\varepsilon$ and not by a “homogenized observation”, which is practical for implementation of the homogenized filter in applications since such homogenized observation is usually not available. However, should such homogenized observation be available, using it would lead to loss of information for estimating the signal compared to using the actual observation.

The main result is: Under the assumptions stated in Theorem 3.5.1, for every $p \geq 1$ and $T \geq 0$ there exists $C > 0$, such that for every $\varphi \in C^4_b$

$$
\left( \mathbb{E}_Q \left[ \left| \pi^{\varepsilon,x}_T(\varphi) - \pi^0_T(\varphi) \right|^p \right] \right)^{1/p} \leq \sqrt{\varepsilon} C \| \varphi \|_{4,\infty}.
$$

In particular, there exists a metric $d$ on the space of probability measures, such that $d$ generates the topology of weak convergence, and such that for every $T \geq 0$ there exists $C > 0$ such that

$$
\mathbb{E}_Q \left[ d(\pi^{\varepsilon,x}_T, \pi^0_T) \right] \leq \sqrt{\varepsilon} C.
\quad (3.7)
$$
3.2 Existing and related works

Based on (3.4) and (3.6), the filter convergence problem is a problem of homogenization of a SPDE. In [17], homogenization of diffusion processes with periodic structures is done using the martingale problem approach. In [67] and Chapter 2 of [68], limit behavior of stochastic processes is studied using asymptotic analysis. [68] studies linear SPDEs with periodic coefficients and also used a probabilistic approach in Chapter 3. Homogenization in the nonlinear filtering problem framework via asymptotic analysis on the dual representation of the nonlinear filtering equation as we do has been studied in [69] and [70]. The diffusion process considered in [69] is of the form

\[ dX_t^\varepsilon = b\left(\frac{X_t^\varepsilon}{\varepsilon}\right)X_t^\varepsilon dt + \sigma\left(\frac{X_t^\varepsilon}{\varepsilon}\right)dW_t, \quad X_0^\varepsilon = x \in \mathbb{R}^m, \]

\[ Y_t^\varepsilon = \int_0^t h\left(\frac{X_s^\varepsilon}{\varepsilon}\right)X_s^\varepsilon ds + B_t, \quad Y_0^\varepsilon = 0 \in \mathbb{R}^d, \]

where \( b, \sigma, \sigma^*, \) and \( h \) are periodic. The homogenized process satisfies

\[ dX_0^0 = \bar{b}X_0^0 dt + \bar{\sigma}dW_t, \quad X_0^0 = x \in \mathbb{R}^m. \]

The averaged sensor function \( \bar{h} \) is constant and the homogenized filter is constructed using the homogenized observation \( \bar{Y} \). Therefore, the homogenized filter is the Kalman-Bucy filter. The error between the homogenized and true filters is obtained to be \( \mathcal{O}(\varepsilon) \). [70] studies homogenization for Zakai-type SPDEs using two different approaches - the martingale problem approach and BSDE techniques. As far as we are aware, BSDEs is used for studying homogenization of Zakai-type SPDEs for the first time in [70]. Our convergence proof applies BSDE techniques by invoking the dual representation of the filtering equation and using asymptotic analysis to determine the limit behavior of the solution of the backward equation. [65] give precise estimates for the transition function of an ergodic SDE of the type (3.2), and these results are used in our proof. To our knowledge, such method of homogenization for SPDEs combining BSDE and asymptotic methods has not been done before. We are also able to obtain error estimate of \( \mathcal{O}(\sqrt{\varepsilon}) \), which to our knowledge is a first at publication of [66].

[71] obtained the result of (3.7) for a two-dimensional multiscale signal
process with no drift in the fast component SDE. In the work for this thesis, the goal is to obtain the result for an $\mathbb{R}^{m+n}$-dimensional signal process with drift and diffusion coefficients of the fast and slow components dependent on both components. The proof of [71] relied on representing the slow component as a time-changed Brownian motion under a suitable measure, which cannot be extended easily to the multidimensional setting considered here. [71] considered diffusion process of the form

$$dX_\varepsilon^t = \sigma(Z_\varepsilon^t) dW_t, \quad X_0^\varepsilon = x \in \mathbb{R}$$

$$dZ_\varepsilon^t = \frac{1}{\sqrt{\varepsilon}} dV_t, \quad Z_0^\varepsilon = z \in [0, 1]$$

$$Y_\varepsilon^t = \int_0^t h(X_\varepsilon^s, Z_\varepsilon^s) ds + B_t, \quad Y_\varepsilon^t \in \mathbb{R}.$$  

Convergence is obtained via the unnormalized posterior density $u_\varepsilon^t(x, z)$, where $\rho_\varepsilon^{x,z}(\varphi) = \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} u_\varepsilon^t(x, z) dz dx$. $u_\varepsilon^t$ is asymptotically expanded as $u_\varepsilon^t(x, z) = u_0^t(x) + \Phi_\varepsilon^t(x, z) + R_\varepsilon^t(x, z)$ and the correction and remainder terms $\Phi, R$ are shown to $\varepsilon \to 0$. $\Phi$ and $R$ are written as solutions to SPDEs by substituting the expansion into the Zakai equation for $u_\varepsilon^t$:

$$du_\varepsilon^t(x, z) = (L^\varepsilon)^* u_\varepsilon^t(x, z) dt + h(x, z) u_\varepsilon^t(x, z) dY_\varepsilon^t, \quad u_0^t(x, \theta) = p_0(x)p_0(z),$$

and collecting equal order terms. The SPDEs are made more manageable by the fact that $(L^\varepsilon)^* = L^\varepsilon$ in this setup (see Section 2.3.2). To show $\Phi \to 0$, the martingale $X_\varepsilon^t$ is represented as a time-changed Brownian motion (Levy’s theorem). For m-dimensional martingales, the equivalent representation requires covariations to be zero (Knight’s theorem), which is not possible in the setting (3.1) considered here.

To our knowledge, a result presented in Chapter 6 of [72] is the closest to the results of this work. In Theorem 6.3.1 of [72] it is shown that for a fixed test function, the difference of the unnormalized actual and homogenized filters for multiscale jump-diffusion processes converges to zero in distribution. Standard results then give convergence in probability of the fixed time marginals. [72]’s method of proof is by averaging the coefficients of the SDEs for the unnormalized filters and showing that the limits of both filters satisfy the same SDE that possesses a unique solution. We obtain $L^p$ convergence of the measure valued process, not just for fixed test functions, and we are able...
to quantify the rate of convergence, which, to the best of our knowledge, has not been achieved before in homogenization of nonlinear filters at the time of publication of the work [66]. It is worth mentioning that, for the case where the transition density of the fast process is not assumed to converge to an invariant measure, [72] also shows that the homogenized filter is “near optimal”, in the sense that, in the limit, the mean-squared distance between the actual and homogenized filters is minimum over a class of estimators.

In [73], convergence of the nonlinear filter is shown in a very general setting, based on convergence in total variation distance of the law of $(X^\varepsilon, Y^\varepsilon)$. This is then applied to two examples. Since the diffusion matrix of our slow component is allowed to depend on the fast component, our results are not a special case. In the examples of [73], $X^\varepsilon$ converges to $\bar{X}$ in probability, which is no longer the case in our setting. However it might be possible to apply the total variation techniques developed in [73] to obtain convergence in our setting. Only the rate of convergence cannot be determined with these techniques.

The latest work on homogenization in multiscale filtering that we are aware of is in [74] that studied a diffusion process with hidden fast component and completely observed slow component. The diffusion process is of the form

$$
dZ_t^\varepsilon = \frac{1}{\varepsilon} b(Z_t^\varepsilon) dt + \frac{1}{\sqrt{\varepsilon}} \sigma(Z_t^\varepsilon) dW_t, \quad Z_0^\varepsilon = x \in \mathbb{R}^n,
$$

$$
Y_t^\varepsilon = X_t^\varepsilon = \int_0^t h(Z_s^\varepsilon) ds + B_t, \quad Y^\varepsilon = 0 \in \mathbb{R}^d.
$$

The homogenized filter is driven by the real observation/slow component, $\rho^0_t(\varphi) = \mathbb{E}_{P^t} [\varphi(Y_t^\varepsilon)(\tilde{D}_t^\varepsilon)^{-1} \mathcal{Y}^\varepsilon]$, where $\tilde{D}_t^\varepsilon \overset{\text{def}}{=} \exp \left\{ -\tilde{h} Y_t^\varepsilon + \frac{1}{2} \|\tilde{h}\|^2 t \right\}$. In this setting, we have the weak convergence $(h(X_t^\varepsilon) - \tilde{h}) \overset{\text{d}}{\rightarrow} 0$. The multiscale filtering result is the mean square convergence of the homogenized and true filters, which is an intermediate step to the main result of convergence of likelihood functions.
3.3 Averaging and homogenization

In order to illustrate the principle of averaging, consider the following ODE example from Chapter 7.1 of [75]:

\[
\dot{\tilde{X}}_t^\varepsilon = \varepsilon b(\tilde{X}_t^\varepsilon, \xi_t), \quad \tilde{X}_0^\varepsilon = x \in \mathbb{R}^m,
\]

(3.8)

where \(\varepsilon\) is a small parameter. \(b\) is assumed bounded and Lipschitz.

\(\xi\) is fast compared to \(\tilde{X}^\varepsilon\). If components of \(b\) do not increase too fast, then solution to (3.8) converges to \(\tilde{X}^0_0 = x\) as \(\varepsilon \to 0\) on finite time intervals \([0, T]\). However, on intervals of order \(\frac{1}{\varepsilon}\) or higher, we may see significant changes in the behavior of \(\tilde{X}^\varepsilon\) from \(x\). Hence, consider interval \([0, \frac{T}{\varepsilon}]\). To study \(\tilde{X}^\varepsilon\) on interval \([0, \frac{T}{\varepsilon}]\), we can use the process \(\{X_t^\varepsilon, t \in [0, T]\}\), where

\[
X_t^\varepsilon \overset{\text{def}}{=} \tilde{X}_{t/\varepsilon}^\varepsilon, \quad \dot{X}_t^\varepsilon = b(X_t^\varepsilon, \xi_{t/\varepsilon}), \quad X_0^\varepsilon = x \in \mathbb{R}^m.
\]

In the above, \(t \in [0, T]\) is actually the time \(\frac{t}{\varepsilon} \in [0, \frac{T}{\varepsilon}]\) of (3.8).

Consider a small interval \([0, \Delta] \subset [0, T]\) such that \(\Delta \equiv \varepsilon^\alpha\), where \(0 < \alpha < 1\). Then, \(\frac{\varepsilon}{\Delta} = \epsilon^{1-\alpha}\), so \(\frac{\varepsilon}{\Delta} \searrow 0\) and \(\Delta \searrow 0\) as \(\varepsilon \to 0\), even if the convergences are very slow.

\[
X_\Delta^\varepsilon = x + \int_0^\Delta b(X_t^\varepsilon, \xi_{t/\varepsilon})dt \\
= x + \Delta \frac{\varepsilon}{\Delta} \int_0^{\Delta/\varepsilon} [b(X_{\varepsilon s}^\varepsilon, \xi_s) - b(x, \xi_s)] ds + \frac{\varepsilon}{\Delta} \int_0^{\Delta/\varepsilon} b(x, \xi_s)ds.
\]

(3.9)

Assume that the following limit exists:

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau b(x, \xi_s)ds = \bar{b}(x)
\]

(3.10)

for \(x \in \mathbb{R}^m\). Then, \(\bar{b}\) is also bounded and satisfies the Lipschitz condition with the same constant \(K\) as \(b\). Let

\[
\dot{\bar{X}}_t = \bar{b}(\bar{X}_t), \quad \bar{X}_0 = x,
\]

then

\[
|X_\Delta^\varepsilon - \bar{X}_\Delta| \leq \Delta \frac{\varepsilon}{\Delta} \int_0^{\Delta/\varepsilon} |b(X_{\varepsilon s}^\varepsilon, \xi_s) - b(x, \xi_s)| ds
\]

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\[ + \Delta \left| \frac{\varepsilon}{\Delta} \int_0^{\Delta/\varepsilon} b(x, \xi_s) ds - \bar{b}(x) \right| =: \rho^\varepsilon(\Delta) \leq \Delta \frac{\varepsilon}{\Delta} \int_0^{\Delta/\varepsilon} K |X_{\varepsilon s}^\varepsilon - x| ds + \rho^\varepsilon(\Delta) \]
\[ \leq \rho^\varepsilon(\Delta) \exp \left\{ \frac{\Delta \varepsilon}{\Delta} \int_0^{\Delta/\varepsilon} K ds \right\} = \rho^\varepsilon(\Delta) \exp \{ K \Delta \}, \quad (3.11) \]

\( \Delta / \varepsilon \xrightarrow{} \infty \) as \( \varepsilon \to 0 \), so

\[ \lim_{\varepsilon \to 0} \frac{\varepsilon}{\Delta} \int_0^{\Delta/\varepsilon} b(x, \xi_s) ds = \lim_{(\Delta/\varepsilon) \to \infty} \frac{1}{\Delta/\varepsilon} \int_0^{\Delta/\varepsilon} b(x, \xi_s) ds = \bar{b}(x), \]

so \( \rho^\varepsilon(\Delta) \to 0 \) as \( \varepsilon \to 0 \) since \( \Delta \searrow 0 \) as well. Hence,

\[ \lim_{\varepsilon \to 0} |X_{\varepsilon}^\varepsilon - \bar{X}_{\Delta}| = 0, \quad X_{\bar{0}}^\varepsilon = \bar{X}_{\bar{0}} = x \in \mathbb{R}^m. \quad (3.12) \]

If (3.12) holds uniformly for \( x \in \mathbb{R}^m \), then the convergence can be extended to over all finite intervals \([0, T]\), by partitioning \([0, T]\) into intervals of size \( \Delta \). In other words, if \( b \) and \( \xi \) satisfy conditions such that \( b(x, \xi_s) \) can be averaged over \( \xi \) to an \( x \)-dependent function, then \( X^\varepsilon \to \bar{X} \) over timescale of order \( \varepsilon^{-1} \).

The main result utilized in the construction of the homogenized filter is the stochastic averaging result of [17], which we describe here using the setting of (3.1). The result of Theorem 2.1 of [17] includes an intermediate timescale, which is not considered in our work. Consider the diffusion process (3.1). Assume that \( Z^x \) is ergodic i.e. there exists a unique invariant density \( \mu(x, dz) \) that is smooth in \( x \) such that

\[ \lim_{t \to \infty} p_t(x, dz) = \mu(x, dz) \]

uniformly in \( x \), where \( p_t(x, dz) \) is the transition probability density of \( Z_t^x \). The drift and diffusion coefficients of (3.1) are assumed to be smooth, but this assumption can be weakened. Under these conditions, Theorem 2.1 of [17] states that \( X^\varepsilon \) converges weakly in \( \mathcal{C}([0, T], \mathbb{R}^m) \), \( T < \infty \), to a Markov
process $\bar{X}$ generated by $\bar{\mathcal{D}}$. The generator $\bar{L}$ is

$$
\bar{\mathcal{D}} \overset{\text{def}}{=} \int_{\mathbb{R}^n} \mathcal{L}_S \mu(x, dz) + \sum_{i=1}^{m} \int_{\mathbb{R}^m} b_i(x, z) \mu(x, dz) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{m} \int_{\mathbb{R}^m} (\sigma \sigma^\ast)_{ij}(x, z) \mu(x, dz) \frac{\partial^2}{\partial x_i \partial x_j}.
$$

By the smoothness conditions, coefficients of $\bar{\mathcal{D}}$ are smooth and bounded in $x$, so $\bar{\mathcal{D}}$ generates a diffusion Markov process on $\mathbb{R}^m$. If $\mathcal{L}^\varepsilon$ is uniformly elliptic on $\mathbb{R}^m \times \mathbb{R}^n$, then the smoothness conditions can be relaxed such that the diffusion coefficients need to be bounded and continuous and the drift coefficients be bounded and measurable. The proof is by the martingale problem approach (see, for example, Section 8.3 of [43] for the martingale problem).

### 3.4 Backward stochastic partial differential equations and backward doubly-stochastic differential equations

Dual representations of the unnormalized filters $\rho^{\varepsilon,x}(\varphi), \rho^0_x(\varphi)$ are invoked in the proof of the main multiscale filtering result. The duals satisfy backward equations of the form (see Section 3.5)

$$
\psi(\omega, t, x) = \psi(T, x) + \int_t^T \{ \mathcal{L} \psi(\omega, s, x) ds + f(\omega, s, x) \} ds \\
+ \int_t^T \{ g(\omega, s, x) + G(\omega, s, x) \psi(\omega, s, x) \} dB_s, \quad (3.13)
$$

$$
\psi(T, x) = \Psi(\omega, x),
$$

where $\psi : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}, f : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}, g : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^{1 \times d}, G : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^{1 \times d}, \varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ are all jointly measurable, and $(B_t : t \in [0, T])$ is a $d$-dimensional standard Brownian motion under the measure $\mathbb{P}$. The differential operator $\mathcal{L}$ is given by

$$
\mathcal{L} = \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.
$$
for measurable $b : \mathbb{R}^m \to \mathbb{R}^m$ and $a : \mathbb{R}^m \to \mathbb{S}^{m \times m}$ ($\mathbb{S}^{m \times m}$ denotes positive semidefinite symmetric matrices). We will determine probabilistic representation of this equation in terms of BDSDEs as introduced by [76]. Note that for these linear equations it is possible to give a Feynman-Kac type representation without using BDSDEs. This is done, for example, in [77] (“The Method of Stochastic Characteristics”). However the BDSDE-representation has the advantage that it permits us to apply Gronwall’s lemma. This would not be possible with the method of stochastic characteristics.

Backward equation here means that the solution to (3.13) is adapted to the future of $B$: Define

$$F^0_{t,s} \stackrel{\text{def}}{=} \bigcap_{r<t} \sigma(B_u - B_r : r \leq u \leq s)$$

and $F^B_{t,s}$ is the completion of $F^0_{t,s}$. A solution to (3.13) has to satisfy $\psi(t, x) \in F^B_{t,T}$ for all $(t, x)$. Under this interpretation, $dB$ can be interpreted as a backward Itô integral, defined as:

Partition $[0, t]$ into $(t_0, t_1], \ldots, (t_{N-2}, t_{N-1}], (t_{N-1}, t_N]$, where $t_k = t - (N - k) \lfloor t/N \rfloor$. For a simple function $H_s = \sum_{k=1}^N H_k \chi_{[t_k, t_{k+1})}(s)$ with $H_k \in F_{t_k, t}$, the backward Itô integral is defined as

$$\int_0^t H_s dB_s \overset{\text{←}}{=} \lim_{N \to \infty} \sum_{k=1}^N H_k (B_{t_k} - B_{t_{k-1}}).$$

It is defined using the right end-point as opposed to the left end-point of the interval $[t_{k-1}, t_k)$ for the forward integral. It is extended to general locally square integrable, $\mathcal{F}_{t,T}$-predictable $H$ via Itô isometry as for the forward integral. Then, for $s \in (T - t, T]$, $H'_s := H_{T-s}$ and $B'_s := B_T - B_{T-s}$, we have that

$$\int_{T-t}^T H_s dB_s \overset{\text{←}}{=} \int_0^t H'_s dB'_s,$$

where the right side is just the forward Itô integral. This means that we can reverse time in (3.13) to obtain a forward equation. We describe this in the following discussion.

Consider two intervals of equal size $t$, $[0, t]$ and $[T - t, T]$, each partitioned into $N$ subintervals:
The backward stochastic integral \( \int_{T-t}^{T} H_s \, dB_s \) can be written as the infinite sum

\[
\int_{T-t}^{T} H_s \, dB_s = \lim_{N \to \infty} \sum_{s \in \{ T-(N-1) \left| \frac{T}{N} \right., T-(N-2) \left| \frac{T}{N} \right., \ldots, T-\left| \frac{T}{N} \right., T \}} H_s \left( B_s - B_{s-\left| \frac{T}{N} \right.} \right)
\]

\[
= \lim_{N \to \infty} \sum_{s \in \{ T-(N-1) \left| \frac{T}{N} \right., T-(N-2) \left| \frac{T}{N} \right., \ldots, T-\left| \frac{T}{N} \right., T \}} H_s \left( B_T - B_{T-(N-1) \left| \frac{T}{N} \right.} \right) \\
+ \sum_{s \in \{ T-(N-1) \left| \frac{T}{N} \right., T-(N-2) \left| \frac{T}{N} \right., \ldots, T-\left| \frac{T}{N} \right., T \}} H_s \left( B_T - B_{T-(N-2) \left| \frac{T}{N} \right.} \right) \\
+ \ldots + H_s \left( B_T - B_{T-\left| \frac{T}{N} \right.} \right) + H_s \left( B_T - B_{T-\left| \frac{T}{N} \right.} \right)
\]

The forward integral \( \int_{0}^{t} H_s' \, dB'_s \) can be written as

\[
\int_{0}^{t} H_s' \, dB'_s = \int_{0}^{t} H_{T-s} \, d(B_T - B_{T-s})
\]

\[
= \lim_{N \to \infty} \sum_{s \in \{ 0, \ldots, (N-2) \left| \frac{T}{N} \right., (N-1) \left| \frac{T}{N} \right. \}} H_{T-s} \left( B_T - B_{T-(s+\left| \frac{T}{N} \right.)} \right) - [B_T - B_{T-s}] \\
= \lim_{N \to \infty} \sum_{s \in \{ 0, \ldots, (N-2) \left| \frac{T}{N} \right., (N-1) \left| \frac{T}{N} \right. \}} H_{T-s} \left( B_{T-s} - B_{T-(s+\left| \frac{T}{N} \right.)} \right)
\]

\[
= \lim_{N \to \infty} \left\{ H_T \left( B_T - B_{T-\left| \frac{T}{N} \right.} \right) + \sum_{s \in \{ 0, \ldots, (N-2) \left| \frac{T}{N} \right., (N-1) \left| \frac{T}{N} \right. \}} H_{T-s} \left( B_{T-s} - B_{T-(s+\left| \frac{T}{N} \right.)} \right) + \ldots \\
+ H_{T-(N-2) \left| \frac{T}{N} \right.} \left( B_{T-(N-2) \left| \frac{T}{N} \right.} - B_{T-(N-1) \left| \frac{T}{N} \right.} \right) \\
+ H_{T-(N-1) \left| \frac{T}{N} \right.} \left( B_{T-(N-1) \left| \frac{T}{N} \right.} - B_{T-N \left| \frac{T}{N} \right.} \right) \right\},
\]

which is equal to the backward stochastic integral.
The time integrals are related by a variable change \( \tau = T - s \):

\[
\int_0^t H_s' \, ds = \int_0^t H_{T-s} \, ds = - \int_T^T H_\tau \, d\tau = \int_{T-t}^T H_\tau \, d\tau.
\]

Hence, we have the following forward version for (3.13):

Let \( \psi'(t, x) = \psi(T - t, x) \) and \( B' \) be the Brownian motion defined above. Based on the preceding discussions, we have the forward version of (3.13):

\[
\psi'(t, x) = \Psi(x) + \int_0^t \{ \mathcal{L} \psi'(s, x) + f(s, x) \} \, ds + \int_0^t \{ g(s, x) + G(s, x)\psi'(s, x) \} \, dB'_s,
\]

in the classical SPDE sense. Hence, no new theory is required for the backward SPDEs described here, we can solve (3.14) and reverse time (see also, for example, Chapter 1, Section 4.12 in [77]). Alternatively, we can reverse time for forward equations like (3.14) and obtain a backward equation of the form (3.13). Note that the driving noise \( dB' \) does not depend on the space variable, i.e. it is a finite-dimensional noise. Fortunately, this is sufficient for the nonlinear filtering equations.

Now we obtain a BDSDE representation of the solution to the backward equation (3.13). A BDSDE is an integral equation of the form

\[
Y_t = \xi + \int_t^T f(s, \cdot, Y_s, Z_s) \, ds + \int_t^T g(s, \cdot, Y_s, Z_s) \, dB_s - \int_t^T Z_s \, dW_s, \tag{3.15}
\]

where \( f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{1 \times n} \to \mathbb{R} \), \( g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{1 \times n} \to \mathbb{R}^{1 \times l} \), and for fixed \( y \in \mathbb{R}, z \in \mathbb{R}^{1 \times n} \) the processes \( (\omega, t) \mapsto f(t, \omega, x, z) \) and \( (\omega, t) \mapsto g(t, \omega, x, z) \) are \((\mathcal{F}_0^B \vee \mathcal{F}_t^W) \otimes \mathcal{B}(\mathbb{R})\)-measurable, and for every \( t, f(t, \cdot, x, z) \) and \( g(t, \cdot, x, z) \) are \(\mathcal{F}_t\)-measurable. \( B \) and \( W \) are independent Brownian motions.

\((Y, Z)\) will be called solution of (3.15) if \((Y, Z) \in S^2_T(\mathbb{R}) \times H^2_T(\mathbb{R}^{1 \times n})\) and if the couple solves the integral equation. The solution \((Y_t, Z_t)\) will be \(\mathcal{F}^{B}_t \vee \mathcal{F}^W_t\)-measurable.

We will also write the equation in differential form:

\[
-dY_t = f(t, Y_t, Z_t) \, dt + g(t, Y_t, Z_t) \, dB_t - Z_t \, dW_t.
\]
Starting from the notion of BDSDEs, we can define forward-backward doubly stochastic differential equations. Let \( \sigma = a^{1/2} \) and
\[
X^t_{s,x} = x + \int_t^s b(X^t_{s,x}) ds + \int_t^s \sigma(X^t_{s,x}) dW_s \quad \text{for } s \geq t
\]
\[
X^t_{s,x} = x \quad \text{for } s \leq t
\]
We then define the following BDSDE
\[
-dY^t_{s,x} = f(s, X^t_{s,x}) ds + (g(s, X^t_{s,x}) ds + G(s, X^t_{s,x}) Y^t_{s,x}) d\tilde{B}_s - Z^t_{s,x} dW_s
\]
\[
Y^t_{T} = \varphi(X^t_{T,x}).
\]
It turns out that \( Y \) gives a finite-dimensional probabilistic representation for equation (3.13), more precisely we have \( Y^t_{t,x} = \psi(t, x) \). This is not completely covered by [76], because we have random unbounded coefficients, and because we do not assume the diffusion matrix \( a \) to have a smooth square root. On the other side, the equation is of a particularly simple linear type. In the remainder of this section, we give the precise statement and proof for this representation.

We will not be able to get an existence result of the BDSDE representation for classical solutions of the SPDE (3.13) from the theory of BDSDEs: This is due to the fact that for this we would need smoothness properties of a square root of \( a \). But even when \( a \) is smooth, in the degenerate elliptic case it does not need to have a smooth square root (see, for example, [78], Chapter 2.3). Instead we will use the existence result of [77] and only reprove the uniqueness result of [76] in our setting. This will work under Lipschitz continuity of \( a^{1/2} \).

Define for \( 0 \leq t \leq s \leq T \)
\[
\mathcal{F}^{0,B}_{t,s} = \sigma(B_u - B_t : t \leq u \leq s)
\]
and \( \mathcal{F}^{B}_{t,s} \) as the completion of \( \mathcal{F}^{0,B}_{t,s} \) under \( \mathbb{P} \). Introduce the space of adapted random fields of polynomial growth:

**Definition 3.4.1** \( \mathcal{P}_T(\mathbb{R}^m, \mathbb{R}^n) \) is the space of random fields

\[
H : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^n
\]
that are jointly measurable in \((\omega, t, x)\), and for fixed \((t, x)\), \(\omega \mapsto H(\omega, t, x)\) is \(\mathcal{F}^B_{t,T}\)-measurable. Further for fixed \(\omega\) outside a null set, \(H\) has to be jointly continuous in \((t, x)\), and it has to satisfy the following inequality: For every \(p \geq 1\) there is \(C_p > 0, q > 0\), such that for all \(x \in \mathbb{R}^m\)

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |H(t, x)|^p \right] \leq C_p (1 + |x|^q).
\]

We make the following assumptions on the coefficients of the SPDE:

\((S_k)\)  \(f\) and \(g\) are \(k\) times continuously differentiable and the partial derivatives up to order \(k\) are all in \(\mathcal{D}_T\). \(G\) is \((k + 1)\) times continuously differentiable and the partial derivatives up to order \((k + 1)\) are all uniformly bounded in \((\omega, t, x)\). \(\varphi\) is \(k\) times continuously differentiable, and all partial derivatives of order 0 to \(k\) grow at most polynomially.

We make the following assumptions on the coefficients of the differential operator \(\mathcal{L}\):

\((D_k)\)  \(b \in C^k_b(\mathbb{R}^m, \mathbb{R}^m), a \in C^k_b(\mathbb{R}^m, \mathbb{S}^{m \times m})\), and \(a\) is degenerate elliptic: For every \(\xi \in \mathbb{R}^m\) and every \(x \in \mathbb{R}^m\),

\[
\langle a(x)\xi, \xi \rangle = \sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geq 0.
\]

Then we have the following result:

**Proposition 3.4.1**  Assume \((S_k)\) and \((D_k)\) for some \(k \geq 3\). Then the equation (3.13) has a unique classical solution \(\psi\) in the sense that for every fixed \(\omega\) outside a null set, \(\psi(\omega, \cdot, \cdot) \in C^{0,k-1}(\mathbb{R}^d \times [0, T])\), \(\psi\) and its partial derivatives are in \(\mathcal{D}_T(\mathbb{R}^m, \mathbb{R})\), and \(\psi\) solves the integral equation. If \(\tilde{\psi}\) is any other solution of the integral equation, then \(\psi\) and \(\tilde{\psi}\) are indistinguishable. If further \(f, g\) and \(\varphi\) as well as their derivatives up to order \(k\) are uniformly bounded in \((\omega, t, x)\), then for any \(p > 0\) there exist \(C_p, q > 0\) (only depending on \(p\), the dimensions involved, the bounds on \(a, b\) and \(G\), and on \(T\)), such that for all \(|\alpha| \leq k - 1\) and \(x \in \mathbb{R}^m\):

\[
\mathbb{E} \left[ \sup_{t \leq T} |D^\alpha \psi(t, x)|^p \right] \]
\[
\leq C(1 + |x|^\eta) \mathbb{E} \left[ \|\varphi\|_{k,\infty}^p + \sup_{t \leq T} ||f(t, \cdot)||_{k,\infty}^p + \sup_{t \leq T} ||g(t, \cdot)||_{k,\infty}^p \right] .
\]

**Proof** This is a combination of Theorem 4.3.2 and Corollary 4.3.2 of [77] (The claimed bound is only given for the equation in unweighted Sobolev spaces, in Corollary 4.2.2. But from that we can deduce the result for the weighted Sobolev case). The only thing we need to verify is that our polynomial growth assumption on the coefficients is compatible with the Sobolev norm condition there. But if \( \theta \in \mathcal{P}_T(\mathbb{R}^m, \mathbb{R}^n) \), then for any \( p \geq 1 \) there certainly is an \( r < 0 \) such that \( \theta \) takes its values in the weighted \( L^p \)-space with weight \((1 + |x|^2)^{r/2}\):

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \int |\theta(t, x)|^p (1 + |x|^2)^{\frac{r}{2}} dx \right] \leq \mathbb{E} \left[ \int \sup_{0 \leq t \leq T} |\theta(t, x)|^p (1 + |x|^2)^{\frac{r}{2}} dx \right] \\
= \int \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\theta(t, x)|^p \right] (1 + |x|^2)^{\frac{r}{2}} dx \\
\leq \int C_p (1 + |x|^\eta) (1 + |x|^2)^{\frac{r}{2}} dx < \infty
\]

for small enough \( r \). ■

Now we combine this result with the theory of BDSDEs:
Let \( (W_t : t \in [0, T]) \) be an \( n \)-dimensional standard Brownian motion that is independent of \( B \). For \( 0 \leq t \leq s \), \( \mathcal{F}_{t,s}^W \) is defined analogously to \( \mathcal{F}_{t,s}^B \). For \( 0 \leq t \leq T \) we set

\[
\mathcal{F}_t = \mathcal{F}_{t,T}^B \vee \mathcal{F}_t^W .
\]

Note that this is not a filtration, as it is neither decreasing nor increasing in \( t \). Introduce the following notation:

- \( H_t^2(\mathbb{R}^m) \) is the space of measurable \( \mathbb{R}^m \)-valued processes \( Y \) s.t. \( Y_t \) is \( \mathcal{F}_t \)-measurable and
  \[
  \mathbb{E} \left[ \int_0^T |Y_t|^2 dt \right] < \infty .
  \]

- \( S_t^2(\mathbb{R}^m) \) is the space of continuous adapted \( \mathbb{R}^m \)-valued processes \( Y \) s.t. \( Y_t \in \mathcal{F}_t \) and
  \[
  \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] < \infty .
  \]
Observe that with suitable adaptations, all of the following results also hold in the multidimensional case, i.e. for $Y \in \mathbb{R}^m$. We restrict to one-dimensional $Y$ for simplicity and because ultimately we are only interested in that case.

[76] shows that under the following conditions, equation (3.15) has a unique solution:

- $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$
- for any $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times n}$: $f(\cdot, \cdot, y, z) \in H^2_T(\mathbb{R})$ and $g(\cdot, \cdot, y, z) \in H^2_T(\mathbb{R}^{1 \times k})$
- $f$ and $g$ satisfy Lipschitz conditions and $g$ is a contraction in $z$: there exist constants $L > 0$ and $0 < \alpha < 1$ s.t. for any $(\omega, t)$ and $y_1, y_2, z_1, z_2$:
  
  $$|f(t, \omega, y_1, z_1) - f(t, \omega, y_2, z_2)|^2 \leq L(|y_1 - y_2|^2 + |z_1 - z_2|^2)$$
  
  $$|g(t, \omega, y_1, z_1) - g(t, \omega, y_2, z_2)|^2 \leq L|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2$$

(see Appendix 3).

Now we want to associate a diffusion $X$ to the differential operator $\mathcal{L}$. To do so, assume that $(D_k)$ is satisfied for some $k \geq 2$. Then $\sigma := a^{1/2}$ is Lipschitz continuous by Lemma 2.3.3 of [78]. Hence for every $(t, x) \in [0, T] \times \mathbb{R}^m$, there exists a strong solution of the SDE

$$X^{t,x}_s = x + \int_t^s b(X^{t,x}_r)dr + \int_t^s \sigma(X^{t,x}_r)dW_r$$

for $s \geq t$,

$$X^{t,x}_s = x$$

for $s \leq t$.

Associate the following BDSDE to (3.13):

$$-dY^{t,x}_s = f(s, X^{t,x}_s)ds + (g(s, X^{t,x}_s) + G(s, X^{t,x}_s)Y^{t,x}_s)d\tilde{B}_s - Z^{t,x}_s dW_s, \quad (3.16)$$

$$Y^{t,x}_T = \varphi(X^{t,x}_T).$$

Under the assumptions $(S_k)$ and $(D_k)$ for $k \geq 2$, this equation has a unique solution.

**Proposition 3.4.2** Assume $(S_k)$ and $(D_k)$ for some $k \geq 3$. Then the unique classical solution $\psi$ of the SPDE (3.13) is given by $\psi(t, x) = Y^{t,x}_t$, where $(Y^{t,x}, Z^{t,x})$ is the unique solution of the BDSDE (3.16).
We can give exactly the same proof as in [76], Theorem 3.1, taking advantage of the independence of $B$ and $W$. For the reader’s convenience, we include it here.

**Proof** Let $\psi$ be a classical solution of (3.13). It suffices to show that

$$(\psi(s, X^{t,x}_s), D\psi(s, X^{t,x}_s)\sigma(X^{t,x}_s) : t \leq s \leq T)$$

solves the BDSDE (3.16). Here $D\psi$ is the gradient of $\psi$. For this purpose, consider a partition $t = t_0 < t_1 < \cdots < t_n = T$ of $[t, T]$. Then

$$\psi(t, X^{t,x}_t) = \psi(T, X^{t,x}_T) + \sum_{i=0}^{n-1} (\psi(t_i, X^{t,x}_{t_i}) - \psi(t_{i+1}, X^{t,x}_{t_{i+1}}))$$

$$= \varphi(X^{t,x}_T) + \sum_{i=0}^{n-1} (\psi(t_i, X^{t,x}_{t_i}) - \psi(t_{i+1}, X^{t,x}_{t_{i+1}}))$$

and

$$\psi(t_i, X^{t,x}_{t_i}) - \psi(t_{i+1}, X^{t,x}_{t_{i+1}})$$

$$= (\psi(t_i, X^{t,x}_{t_i}) - \psi(t_i, X^{t,x}_{t_{i+1}})) + (\psi(t_i, X^{t,x}_{t_{i+1}}) - \psi(t_{i+1}, X^{t,x}_{t_{i+1}}))$$

$$= -\left( \int_{t_i}^{t_{i+1}} \mathcal{L}\psi(t, X^{t,x}_s)ds + \int_{t_i}^{t_{i+1}} D\psi(t, X^{t,x}_s)\sigma(X^{t,x}_s)dW_s \right)$$

$$+ \int_{t_i}^{t_{i+1}} (\mathcal{L}\psi(s, X^{t,x}_{t_{i+1}}) + f(s, X^{t,x}_{t_{i+1}}))ds$$

$$+ \int_{t_i}^{t_{i+1}} (g(s, X^{t,x}_{t_{i+1}}) + G(X^{t,x}_{t_{i+1}})\psi(s, X^{t,x}_{t_{i+1}}))dB_s.$$ 

This is justified because $X^{t,x}$ and $\psi$ are independent and because $\psi$ grows polynomially, hence we can apply Itô’s formula. We also used the fact that $\psi$ is a classical solution to (3.13). If we let the mesh size tend to 0, then by continuity of $X^{t,x}$ and $\psi$, the result follows. ■

### 3.5 Formal expansion of filtering equations and main result

For a given bounded test function $\varphi$ and terminal time $T$, we follow [79] in introducing the associated dual process $\nu^{\varepsilon,T,\varphi}(x, z)$, which is a dynamic
version of $\mathbb{E}_{\tilde{P}^\varepsilon}[\varphi(X^\varepsilon_T)\tilde{D}^\varepsilon_T|\mathcal{Y}^\varepsilon_T]$:

$$v^\varepsilon_{t,T,\varphi}(x,z) = \mathbb{E}_{\tilde{P}^\varepsilon_{t,x,z}}[\varphi(X^\varepsilon_T)\tilde{D}^\varepsilon_{t,T}|\mathcal{Y}^\varepsilon_{t,T}]$$

where $\mathbb{P}^\varepsilon_{t,x,z}$ is the measure under which $X^\varepsilon$ and $Z^\varepsilon$ are governed by the same dynamics as under $\mathbb{P}^\varepsilon$, but $(X^\varepsilon, Z^\varepsilon)$ stays in $(x,z)$ until time $t$, then it starts to follow the SDE dynamics. $\tilde{D}^\varepsilon_{t,T} = \tilde{D}^\varepsilon_T(\tilde{D}^\varepsilon_t)^{-1}$; and $\mathcal{Y}^\varepsilon_{t,T} = \sigma(Y^\varepsilon_t - Y^\varepsilon_r : t \leq r \leq T) \vee \mathcal{N}$ (recall that $\mathcal{N}$ denotes the $\mathbb{Q}$-negligible sets). From the Markov property of $(X^\varepsilon, Z^\varepsilon)$ it follows that for any $t \in [0,T]$:

$$\rho^\varepsilon_{t,x}(\varphi) = \int_{\mathbb{R}^m \times \mathbb{R}^n} \mathbb{E}_{\tilde{P}^\varepsilon_{t,x}}[\varphi(X^\varepsilon_T)\tilde{D}^\varepsilon_{0,T}|\mathcal{Y}^\varepsilon_{0,T}] \mathbb{P}^\varepsilon(dx,dz) = \rho^\varepsilon_t(v^\varepsilon_{t,T,\varphi}).$$

In particular (because at time 0, $\rho^\varepsilon$ is just the starting distribution of $(X^\varepsilon, Z^\varepsilon)$):

$$\rho^\varepsilon_0(x) = \int v^\varepsilon_{0,T,\varphi}(x,z) Q^\varepsilon_{X^\varepsilon_0,z}(dx,dz).$$

Similarly introduce

$$v^0_{t,T,\varphi}(x) = \mathbb{E}_{\tilde{P}^\varepsilon_{t,x}}[\varphi(X^0_T)\tilde{D}^0_{t,T}|\mathcal{Y}^\varepsilon_{t,T}],$$

where

$$\tilde{D}^0_{t,T} = \exp \left( \int_t^T \bar{h}(X^0_r)^* dY^\varepsilon_r - \frac{1}{2} \int_t^T |\bar{h}(X^0_r)|^2 dr \right)$$

and $\mathbb{P}^\varepsilon_{t,x}$ is the measure under which $X^0$ is governed by the same dynamics as under $\mathbb{P}^\varepsilon$, but stays in $x$ until time $t$. We can also show that for any $t \in [0,T]: \rho^0_t(v^0_{t,T,\varphi}) = \rho^0_T(\varphi)$, so that

$$\rho^0_T(\varphi) = \int v^0_{0,T,\varphi}(x) Q^\varepsilon_{X^\varepsilon_0}(dx).$$

Note that $Q^\varepsilon_{X^\varepsilon_0} = Q^\varepsilon_{X^0}$, because the homogenized process has the same starting distribution as the unhomogenized one.

Now fix $T$ and $\varphi \in C^2_0(\mathbb{R}^m, \mathbb{R})$ and write $v^\varepsilon_t = v^\varepsilon_{t,T,\varphi}$ and $v^0_t = v^0_{t,T,\varphi}$.

Our aim is to show that for nice test functions $\varphi$, and for the dual processes $v^\varepsilon$ and $v^0$ defined above, $\mathbb{E}[(v^\varepsilon_0(x,z) - v^0_0(x))^p]$ is small (in a way that will
depend on $x$ and $z$). Then
\[
\mathbb{E}[^{p}\rho_T^ε(\varphi) - \rho_0^0(\varphi)] = \mathbb{E}[^{p}\left(\int (v_0^ε(x,z) - v_0^0(x))Q_0(x_0^ε, z_0^ε)(dx, dz)\right)]
\leq \mathbb{E}[^{p}\int |v_0^ε(x,z) - v_0^0(x)|Q_0(x_0^ε, z_0^ε)(dx, dz)]
= \int \mathbb{E}[|v_0^ε(x,z) - v_0^0(x)|Q_0(x_0^ε, z_0^ε)(dx, dz)]
\]
will also be small as long as $Q_0(x_0^ε, z_0^ε)$ is well behaved.

Before continuing, let us change notation: For large parts of this article we will only work under $\mathbb{P}^ε$, and the process $Y^ε$ is a Brownian motion under $\mathbb{P}^ε$ which is independent of $(X^ε, Z^ε, X^0)$. Therefore from now on we write $\mathbb{P}$ instead of $\mathbb{P}^ε$ and $B$ instead of $Y^ε$ to facilitate the reading. The distribution and notation for the Markov processes $(X^ε, Z^ε, X^0)$ do not change.

The key point is now that $v^ε$ and $v^0$ solve backward SPDEs (3.17)-(3.18):
\[
-dv_0^ε(x,z) = L^εv_0^ε(x,z)dt + h(x,z)\ast v_0^ε(x,z)dB_t
\]
and
\[
-dv_0^0(x) = \bar{L}v_0^0(x,z)dt + \bar{h}(x)\ast v_0^0(x)dB_t
\]
\[v_0^ε(x,z) = \varphi(x)\]
\[v_0^0(x) = \varphi(x)\]

Here and in all that follows, $dB$ denotes Itô’s backward integral.

We formally expand $v^ε$ as
\[
v_0^ε(x,z) = u_1^0(x,z) + \varepsilon u_1/ε(x,z) + \varepsilon^2 u_2/ε(x,z).
\]

Note that rigorously this does not make any sense, because:

- We work with equations with terminal conditions. But when we send $\varepsilon \to 0$, then $t/\varepsilon$ converges to infinity. So for which time should the terminal condition of e.g. $u^1$ be defined?

- The terms in this expansion will all be stochastic. Then if $u^1$ is adapted to $\mathcal{F}^B$, the stochastic integral $\int_t^T u_1/ε(x,z)dB_s$ a priori does not make any sense for $\varepsilon < 1$. 

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However if we do such a formal asymptotic expansion, and then call
\[ v^0(t, x) = u^0(t, x), \quad \psi^1(t, x, z) = \varepsilon u^1_{t/\varepsilon}(x, z), \quad R(t, x, z) = \varepsilon^2 u^2_{t/\varepsilon}(x, z) \]
(of course all terms except \( v^0 \) depend on \( \varepsilon \), which we omit in the notation to facilitate the reading), then these terms have to solve the following equations:

\[
\begin{align*}
-dv^0_t(x) &= \bar{L}v^0_t(x, z)dt + \bar{h}(x)^* v^0_t(x)dB_t \\
-d\psi^1_t(x, z) &= \frac{1}{\varepsilon} \mathcal{L}_F \psi^1_t(x, z)dt + (\mathcal{L}_S - \mathcal{L})v^0_t(x)dt \\
&\quad + (h(x, z) - \bar{h}(x))^* v^0_t(x)dB_t \\
-dR_t(x, z) &= \mathcal{L}^x R_t(x, z)dt + \mathcal{L}_S \psi^1_t(x, z)dt \\
&\quad + h(x, z)^* (\psi^1_t(x, z) + R_t(x, z)) dB_t
\end{align*}
\]

(3.19) (3.20)

with terminal conditions

\[ v^0(T, x) = \varphi(x), \quad \psi^1(T, x, z) = R(T, x, z) = 0. \]

Note that the equation for \( v^0 \) is exactly the desired equation (3.18). By existence and uniqueness of the solutions to these linear equations, we can apply superposition to obtain that then indeed

\[ v^\varepsilon_t(x, z) = v^0_t(x) + \psi^1_t(x, z) + R_t(x, z). \]

Therefore the problem of showing \( L^p \)-convergence of \( v^\varepsilon \) to \( v^0 \) reduces to showing \( L^p \)-convergence of \( \psi^1 + R \) to 0. To achieve this, we use probabilistic representations of \( \psi^1 \) and \( R \) in terms of backward doubly stochastic differential equations (see Section 3.4). This allows us to apply the existing estimates for the transition function of \( Z^\varepsilon \) from [65].

It will be convenient for us to work with functions that are smoother in their \( x \)-component than they are in their \( z \)-component or vice versa. To do so, introduce the function spaces \( C^{k,l}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^d) \): For \( \theta : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^d, \theta = \theta(x, z) \), write \( \theta \in C^{k,l}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^d) \), if \( \theta \) is \( k \) times continuously differentiable in its \( x \)-components and \( l \) times continuously differentiable in its \( z \)-components. If \( \theta \) as well as its partial derivatives up to order \((k, l)\) are bounded, write \( \theta \in C^{k,l}_b(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^d) \).
Introduce the following assumptions:

(H_{\text{stat}}) For the existence of a stationary distribution $\mu(x, dz)$ for $Z^x$, we suppose that there exist $M_0 > 0, \alpha > 0$, such that for all $|z| \geq M_0$

$$\sup_x \langle f(x, z), z \rangle \leq -C |z|^{\alpha}.$$ 

For the uniqueness of the stationary distribution $\mu(x, dz)$ of $Z^x$, we suppose uniform ellipticity, i.e. that there are $0 < \lambda \leq \Lambda < \infty$, such that

$$\lambda I \leq gg^*(x, y) \leq \Lambda I$$

in the sense of positive semi-definite matrices ($I$ is the unit matrix).

(HF_{k,l}) The coefficients of the fast diffusion satisfy $f \in C^{k,l}_b(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C^{k,l}_b(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^{n \times k})$.

(HS_{k,l}) The coefficients of the slow diffusion satisfy $b \in C^{k,l}_b(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m)$ and $\sigma \in C^{k,l}_b(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^{m \times k})$.

(HO_{k,l}) The observation function $h$ satisfies $h \in C^{k,l}_b(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^d)$.

We will usually write $p_{\infty}(x, dz)$ instead of $\mu(x, dz)$. Also introduce the notation

$$p_t(z, \theta; x) := \int_{\mathbb{R}^n} \theta(x, z') p_t(z, z'; x) dz' := \mathbb{E}_z[\theta(Z^x_t)]$$

where $z$ denotes the starting point of $Z^x$, and $z' \mapsto p_t(z, z'; x)$ is the density of $Z^x_t$ if at time 0 it is started in $z$. Note that the density exists for all $t > 0$ under the condition (H_{\text{stat}}), because of the uniform ellipticity of $gg^*$. Similarly

$$p_{\infty}(\theta; x) = \int_{\mathbb{R}^n} \theta(x, z) p_{\infty}(x, dz).$$

Let the differential operator $\mathcal{L}$ be defined as

$$\mathcal{L} = \sum_{i=1}^m b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^m a_{ij}(x, z) \frac{\partial^2}{\partial x_i \partial x_j}.$$
where $\bar{b}(x) = p_\infty(b; x)$ and $\bar{a} = p_\infty(\sigma \sigma^*; x)$. Also define $\bar{h}(x) = p_\infty(h; x)$.

We introduce the following notation: A multiindex $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^n$ is of order

$$|\alpha| = \alpha_1 + \cdots + \alpha_m.$$ Given such a multiindex, define the differential operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_m^{\alpha_m}}.$$ Finally introduce the following norms for $f \in C_{b}^k(\mathbb{R}^m, \mathbb{R}^n)$:

$$||f||_{k,\infty} = \sum_{|\alpha| \leq k} ||D^\alpha f||_{\infty}$$

where $|| \cdot ||_{\infty}$ is the usual supremum norm.

The main result is

**Theorem 3.5.1** Assume $(H_{stat})$, $(HF_{8,4})$, $(HS_{7,4})$, $(HO_{8,4})$, and that the initial distribution $Q(x_0, z_0)$ has finite moments of every order. Then for every $p \geq 1$ and $T \geq 0$ there exists $C > 0$, such that for every $\varphi \in C^4_b$,

$$\left( \mathbb{E}_Q \left[ |\pi^{\varepsilon,x}_T(\varphi) - \pi^0_T(\varphi)|^p \right] \right)^{1/p} \leq \sqrt{\varepsilon}C||\varphi||_{4,\infty}.$$ In particular, there exists a metric $d$ on the space of probability measures, such that $d$ generates the topology of weak convergence, and such that for every $T \geq 0$ there exists $C > 0$, such that

$$\mathbb{E}_Q \left[ d(\pi^{\varepsilon,x}_T, \pi^0_T) \right] \leq \sqrt{\varepsilon}C.$$ This result will be proven in Section 3.7. In particular we can use Borel-Cantelli to conclude that if $(\varepsilon_n)$ converges quickly enough to 0, then $\pi^{\varepsilon_n}$ will a.s. converge weakly to $\pi^0$.

The ideas are rather simple: We represent the backward SPDEs by finite-dimensional stochastic equations (this will be BDSDEs). The diffusion operators get replaced by the associated diffusions. We are able to solve those finite-dimensional equations explicitly, or at least give explicit estimates up to an application of Gronwall. This allows us to estimate $\psi^1$ and $R$ in terms
of the transition function of the fast diffusion. But [65] proved very precise estimates for this transition function. These estimates allow us to obtain the convergence.

While the ideas are simple, the precise formulation and the actual proofs are quite technical. Probabilistic representation of BSPDEs for the dual, corrector and remainder has presented in the previous section. The next section presents some preliminary estimates. Proof of the main theorem is in the following section.

3.6 Preliminary estimates

The notation $D_x^\alpha$ indicates that the differential operator $D^\alpha$ is only acting on the $x$-variables.

The following result will help us to justify the BDSDE-representations on the deeper levels. Recall that $p_t(z, \theta; x) = \mathbb{E}[\theta(x, Z_t^x) | Z_0^x = z]$.

**Proposition 3.6.1** Assume $(HF_{k,l})$. Let $\theta \in C^{k,l}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$ satisfy for some $C, p > 0$

$$\sum_{|\alpha| \leq k} \sum_{|\beta| \leq l} |D_x^\alpha D_z^\beta \theta(x, z)| \leq C(1 + |x|^p + |z|^p).$$

Then

$$(t, x, z) \mapsto p_t(z, \theta; x) \in C^{0,k,l}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$$

and there exist $C_1, p_1 > 0$, such that for all $(t, x, z) \in [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^n$

$$\sum_{|\alpha| \leq k} \sum_{|\beta| \leq l} |D_x^\alpha D_z^\beta p_t(z, \theta; x)| \leq C_1 e^{C_1 t}(1 + |x|^{p_1} + |z|^{p_1}).$$

If the bound on the derivatives of $\theta$ can be chosen uniformly in $x$, i.e.

$$\sum_{|\alpha| \leq k} \sum_{|\beta| \leq l} \sup_x |D_x^\alpha D_z^\beta \theta(x, z)| \leq C(1 + |z|^p),$$
then the bound on the derivatives of \( p_t(z, \theta; x) \) is also uniform in \( x \):

\[
\sum_{|\alpha| \leq k} \sum_{|\beta| \leq l} \sup_x |D^\alpha_x D^\beta_z p_t(z, \theta; x)| \leq C_1 e^{C_1 t}(1 + |z|^{p_1}).
\]

**Proof** Note that

\[
p_t(z, \theta; x) = \mathbb{E}[\theta(x, Z_t^x)|Z_0^x = z] = \mathbb{E}(\theta(X_t, Z_t)|(X_0, Z_0) = (x, z))
\]

is the solution of Kolmogorov’s backward equation associated to \((X, Z)\), where

\[
X_t = X_0, \\
Z_t = Z_0 + \int_0^t f(X_s, Z_s)ds + \int_0^t g(X_s, Z_s)dW_s.
\]

In this formulation, the first result is standard. Cf. e.g. [78], Corollary 2.2.8. The second statement can be proven in the same way as [78], Corollary 2.2.8. □

Some results from [65] are collected in the following Proposition:

**Proposition 3.6.2** Assume \((H_{stat})\) and \((HF_{k,3})\). Let \( \theta \in C^{k,0}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}) \) satisfy for some \( C, p > 0 \):

\[
\sum_{|\alpha| \leq k} \sup_x |D^\alpha_x \theta(x, z)| \leq C(1 + |z|^{p}).
\]

Then

1. \( x \mapsto p_\infty(\theta; x) \in C_k\left(\mathbb{R}^m, \mathbb{R}\right) \).

2. Assume additionally that \( \theta \) satisfies the centering condition

\[
\int_{\mathbb{R}^n} \theta(x, z)p_\infty(x, dz) = 0
\]

for all \( x \), and that \( \theta \in C^{k,1}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}) \) and

\[
\sum_{|\alpha| \leq k} \sum_{|\beta| \leq 1} \sup_x |D^\beta_z D^\alpha_x \theta(x, z)| \leq C(1 + |z|^{p}).
\]
Then

\[(x, z) \mapsto \int_0^\infty p_t(z, \theta; x) dt \in C^{k,1}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}),\]

and for every \(q > 0\) there exist \(C_1, q_1 > 0\), such that for every \(z \in \mathbb{R}^n\)

\[
\sum_{|\alpha| \leq k} \sum_{|\beta| \leq 1} \int_0^\infty \sup_x |D^\beta z D^\alpha x p_t(z, \theta; x)|^q dt \leq C_1 (1 + |z|^{q_1}).
\]

**Proof** The statements in the Proposition are taken from Theorem 1, Theorem 2 and Proposition 1 of [65]:

1. We get from Theorem 1 of [65], that for any \(q > 0\) there exists \(C_q > 0\), such that for any \((x, z, z') \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n\):

\[
\sum_{|\alpha| \leq k} \sup_x |D^\alpha x p_\infty(z'; x)| \leq \frac{C_q}{1 + |z'|^q}.
\]

So if we choose \(q\) large enough and differentiate \(p_\infty(\theta; x)\) under the integral sign, then we obtain the first claim. (Of course here we have to use the growth constraint on \(\theta\) and its derivatives).

2. This follows from the bounds on the derivatives of \(p_t(z, \theta; x)\) that are given in [65], Theorem 2, formulae (14) and (15): For any \(k > 0\) there exist \(C_k, m_k > 0\), such that for any \((t, x, z) \in [1, \infty) \times \mathbb{R}^m \times \mathbb{R}^n\)

\[
\sum_{|\alpha| \leq k} \sum_{|\beta| \leq 1} \sum_{|\gamma| \leq l} |D^\beta z D^\gamma z p_t(z, \theta; x)| \leq C_k \frac{1 + |z|^{m_k}}{(1 + t)^k}.
\]

We combine this estimate with Proposition 3.6.1, from where we obtain for \((t, x, z) \in \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n\)

\[
\sum_{|\alpha| \leq k} \sum_{|\beta| \leq l} \sup_x |D^\beta z D^\gamma z p_t(z, \theta; x)| \leq C_1 e^{C_1 t} (1 + |z|^{p_1}).
\]

We choose \(k\) such that \(qk > 1\) and use the first estimate on \([1, \infty)\) and the second estimate on \([0, 1)\). The result follows.

We will also need some moment bounds for the diffusions \(X^\varepsilon\) and \(Z^\varepsilon\).
Proposition 3.6.3 Assume \((H_{\text{stat}})\) and that the coefficients \(b\) and \(\sigma\) and \(f\) and \(g\) of the fast and slow motion are bounded and globally Lipschitz continuous. Then for any \(p \geq 1\) there exists \(C_p > 0\), such that

\[
\sup_{(t, \varepsilon, x) \in [0, \infty) \times [0, 1] \times \mathbb{R}^m} \mathbb{E}[|Z_t^\varepsilon|^p](X_t^\varepsilon, Z_t^\varepsilon) = (x, z) \leq C_p(1 + |z|^p).
\]

Also, for every \(T > 0\) and every \(p \geq 1\) there exist \(C(p, T), q > 0\), such that

\[
\sup_{(t, \varepsilon) \in [0, T] \times [0, 1]} \mathbb{E}[|X_t^\varepsilon|^p](X_0^\varepsilon, Z_0^\varepsilon) = (x, z) \leq C(p, T)(1 + |x|^p).
\]

**Proof** The first claim can be proven exactly as in [80]: First write \(Z_t^\varepsilon := Z_t^\varepsilon_2\). Then

\[
d\tilde{Z}^\varepsilon_t = f(X_t^\varepsilon, \tilde{Z}^\varepsilon_t)dt + g(X_t^\varepsilon, \tilde{Z}^\varepsilon_t)d\tilde{W}^\varepsilon_t
\]

where \(\tilde{W}^\varepsilon_t := 1/\varepsilon W^\varepsilon_t\) is a Wiener process. Next, introduce the same time change as in [81], page 1063:

\[
\kappa(x, z) := |g(x, z)z|/|z|, \quad \gamma^\varepsilon(t) := \int_0^t \kappa^2(X_s^\varepsilon, \tilde{Z}_s^\varepsilon)ds, \quad \tau^\varepsilon(t) := (\gamma^\varepsilon)^{-1}(t).
\]

Define \(\tilde{Z}_t^\varepsilon := Z_{\tau^\varepsilon(t)}^\varepsilon\). Then,

\[
d\tilde{Z}^\varepsilon_t = \kappa^{-2}(X_t^\varepsilon, \tilde{Z}_t^\varepsilon)f(X_t^\varepsilon, \tilde{Z}_t^\varepsilon)dt + \kappa^{-1}(X_t^\varepsilon, \tilde{Z}_t^\varepsilon)g(X_t^\varepsilon, \tilde{Z}_t^\varepsilon)d\tilde{W}^\varepsilon_t
\]

with a new standard Brownian motion \(\tilde{W}^\varepsilon\). Now we are in a position to just copy the proof of Lemma 1 in [80] (which we do not do here) to get the first result.

The second claim is obvious, because the coefficients of \(X^\varepsilon\) are bounded. ■

Now we are able to impose conditions on the coefficients of the diffusions that guarantee smoothness of the coefficients of \(\tilde{D}\). Recall that \(\tilde{D}\) was defined as

\[
\tilde{D} = \sum_{i=1}^m \tilde{b}_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^m \tilde{a}_{ij}(x, z) \frac{\partial^2}{\partial x_i \partial x_j}
\]

where \(\tilde{b} = p_\infty(b; x)\) and \(\tilde{a} = p_\infty(\sigma \sigma^*; x)\).

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Proposition 3.6.4 Assume \((HF_{k,3}), (HS_{k,0}),\) and \((HO_{k,0})\). Then
\[
\bar{b} \in C^k_b(\mathbb{R}^m, \mathbb{R}^m), \bar{a} \in C^k_b(\mathbb{R}^m, S^{m \times m}), \bar{h} \in C^k_b(\mathbb{R}^m, \mathbb{R}^k)
\]

**Proof** All the terms of \(\bar{b}, \bar{a}\) and \(\bar{h}\) are of the form \(p_\infty(\theta; x)\). So by Proposition 3.6.2, we only need to verify that the respective \(\theta\) are in \(C^{k,0}\) and satisfy the polynomial bound
\[
\sum_{|\alpha| \leq k} \sup_x |D_x^\alpha \theta(x, z)| \leq C(1 + |z|^p)
\]
for some \(C, p > 0\). But we even assumed them to be in \(C^{k,0}_b\), so the result follows. ■

3.7 Main results

We will find convergence rates for the corrector and remainder terms that are expressed in terms of \(v^0\) and its derivatives. So now we give bounds on \(v^0\) and its derivatives in terms of the test function \(\varphi\). This is necessary, because we do not only want to show convergence of the filter integrating fixed test functions, but with respect to a suitable distance on the space of probability measures.

**Lemma 3.7.1** Let \(k \geq 2\) and assume \(\bar{b}, \bar{a}, \varphi \in C^{k+1}_b\), and \(\bar{h} \in C^{k+2}_b\). Then \(v^0 \in C^{0,k}([0, T] \times \mathbb{R}^m, \mathbb{R})\), and for any \(p \geq 1\) there exist \(C_p, q > 0\), independent of \(\varphi\), such that for all \(x \in \mathbb{R}^m\):
\[
\sum_{|\alpha| \leq k} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |D_x^\alpha v^0_t(x)|^p \right] \leq C_p(1 + |x|^q) ||\varphi||_{k,\infty}^p.
\]

In particular, \(v^0\) and all its partial derivatives up to order \((0, k)\) are in \(\mathcal{P}_T(\mathbb{R}^m, \mathbb{R})\).

**Proof** This is a simple application of Proposition 3.4.1, noting that the equation (3.18) for \(v^0\) is of the type (3.13) with \(f = 0, g = 0,\) and \(G = \bar{h}^*\). ■

We will prove \(L^p\)-convergence of \(\psi^1\) and \(R\) separately.
Lemma 3.7.2 Let $k, l \geq 2$. Assume $(H_{\text{stat}})$, $(HF_{k+1,l+1})$, $(HS_{k+1,l+1})$, and $(HO_{k+1,l+1})$. Also assume $v^0 \in C^{0,k+1}([0,T] \times \mathbb{R}^m, \mathbb{R})$, and that all its partial derivatives in $x$ up to order $k+1$ are in $\mathcal{P}_T(\mathbb{R}^m, \mathbb{R})$. Finally assume $\bar{a}, \bar{b}, \bar{h} \in C^k_b$. Then $\psi^1 \in C^{0,k,l}([0,T] \times \mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$, and $\psi^1$ as well as its partial derivatives up to order $(0,k,l)$ are in $\mathcal{P}_T(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$. For any $p \geq 1$ there exist $C_p, q > 0$, independent of $\varphi$, such that for any $(x, z) \in \mathbb{R}^{m+n}$ and any $\varepsilon \in (0, 1)$

$$\sum_{|\alpha| \leq k-1} \sup_{0 \leq t \leq T} \mathbb{E} \left[ |D^\alpha x \psi^1_t(x, z)|^p \right]$$

$$\leq \varepsilon^{\frac{k}{2}} C_p (1 + |z|^q) \sum_{0 \leq |\alpha| \leq k+1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |D^\alpha x v^0_t(x)|^p \right].$$

**Proof** $\psi^1_t(x, z)$ solves the BSPDE

$$-d\psi^1_t(x, z) = \left[ \frac{1}{\varepsilon} \mathcal{L}_F \psi^1_t(x, z) + (\mathcal{L}_S - \tilde{\mathcal{L}}) v^0_t(x) \right] dt$$

$$+ [h(x, z) - \bar{h}(x)]^* v^0_t(x) dB_t,$$

$$\psi^1_T(x, z) = 0.$$

Existence of the solution $\psi^1$ and its derivatives as well as the polynomial growth all follow from Proposition 3.4.1. Write $Z^{\varepsilon,x,(t,z)}_s$ for the solution of the SDE

$$dZ^{\varepsilon,x,(t,z)}_s = \frac{1}{\varepsilon} f(x, Z^{\varepsilon,x,(t,z)}_s) ds + \frac{1}{\sqrt{\varepsilon}} g(x, Z^{\varepsilon,x,(t,z)}_s) dW_t, \quad s \geq t$$

$$Z^{\varepsilon,x,(t,z)}_s = z, \quad s \leq t.$$

We consider $(x, Z^{\varepsilon,x,(t,z)}_s)$ as a joint diffusion, just as in the proof of Proposition 3.6.1 ($x$ has generator 0). By Proposition 3.4.2, the solution of (3.21) is given by $\theta^{(t,x,z)(1)}_t$, the unique solution to the BDSDE

$$-d\theta^{(t,x,z)(1)}_t = (\mathcal{L}_S(., Z^{\varepsilon,x,(t,z)}_s) - \mathcal{L}) v^0_s(x) ds$$

$$+ (h(x, Z^{\varepsilon,x,(t,z)}_s) - \bar{h}(x))^* v^0_s(x) dB_s + \gamma^{(t,x,z)}_s dW_s,$$

$$\theta^{(t,x,z)(1)}_T = 0.$$

We will drop superscripts $(t, x, z)$ for $\theta^{(t,x,z)(1)}_s$ and write $\theta^*_s$ instead. Similarly,
we write $Z_{s}^{ε,x}$ instead of $Z_{s}^{ε,x,(t,z)}$. $ψ_1^t(x,z)$ is $\mathcal{F}_{t,T}^B$-measurable, hence, so is $θ_1^t$.

We can then write $θ_1^t = \mathbb{E}[θ_1^t|\mathcal{F}_{t,T}^B]$, where

$$
\mathbb{E}[θ_1^t|\mathcal{F}_{t,T}^B] = \mathbb{E}\left[\int_t^T (\mathcal{L}_s - \hat{\mathcal{L}})_s v_0^0(x)ds|\mathcal{F}_{t,T}^B\right] + \mathbb{E}\left[\int_t^T [h(x, Z_{s}^{ε,x}) - \bar{h}(x)]^* v_0^0(x)dB_s|\mathcal{F}_{t,T}^B\right] - \mathbb{E}\left[\int_t^T \gamma_{s}^{t,x,z}dW_s|\mathcal{F}_{t,T}^B\right].
$$

$W$ and $B$ are independent, therefore $W$ is a Brownian motion in the large filtration $(\mathcal{F}_s^W \lor \mathcal{F}_{t,T}^B : s \in [0,T])$, hence $\mathbb{E}\left[\int_t^T \gamma_{s}^{t,x,z}dW_s|\mathcal{F}_{t,T}^W \lor \mathcal{F}_{t,T}^B\right] = 0$, and by the tower property

$$
\mathbb{E}\left[\int_t^T \gamma_{s}^{t,x,z}dW_s|\mathcal{F}_{t,T}^B\right] = 0.
$$

$v_0^0$ is $\mathcal{F}_{t,T}^B$-measurable and $\hat{\mathcal{L}}$ has deterministic coefficients. Thus

$$
\mathbb{E}\left[\int_t^T \hat{\mathcal{L}} v_0^0(x)ds|\mathcal{F}_{t,T}^B\right] = \int_t^T \mathbb{E}[\hat{\mathcal{L}} v_0^0(x)|\mathcal{F}_{s,T}^T] ds = \int_t^T \left\{ \sum_{i=1}^m p_\infty(b_i; x) \frac{∂}{∂x_i} v_0^0(x) + \sum_{i,j=1}^m p_\infty((σσ^*)_{ij}; x) \frac{∂^2}{∂x_i x_j} v_0^0(x) \right\} ds.
$$

Since $Z_{s}^{ε,x}$ is independent of $B$,

$$
\mathbb{E}\left[\int_t^T \mathcal{L}_s(\cdot, Z_{s}^{ε,x})v_0^0(x)ds|\mathcal{F}_{t,T}^B\right] = \int_t^T \mathbb{E}\left[\mathcal{L}_s(\cdot, Z_{s}^{ε,x})v_0^0(x)|\mathcal{F}_{s,T}^B\right] ds
$$

$$
= \int_t^T \left\{ \sum_{i=1}^m \mathbb{E}[b_i(x, Z_{s}^{ε,x})] \frac{∂}{∂x_i} v_0^0(x) + \frac{1}{2} \sum_{i,j=1}^m \mathbb{E}[(σσ^*)_{ij}(x, Z_{s}^{ε,x})] \frac{∂^2}{∂x_i x_j} v_0^0(x) \right\} ds
$$

$$
= \int_t^T \left\{ \sum_{i=1}^m p_{s-t}(z, b_i; x) \frac{∂}{∂x_i} v_0^0(x) + \frac{1}{2} \sum_{i,j=1}^m p_{s-t}(z, (σσ^*)_{ij}; x) \frac{∂^2}{∂x_i x_j} v_0^0(x) \right\} ds.
$$
so

\[
\mathbb{E} \left[ \int_t^T (\mathcal{L}_s - \overline{\mathcal{L}}) v^0_s(x) ds \Big| \mathcal{F}_t^B \right]
\]

\[
= \int_t^T \left\{ \sum_{i=1}^m p_{z,t}(z, b_i - p_\infty(b_i; x); x) \frac{\partial}{\partial x_i} v^0_s(x)
\right.
\]
\[
+ \frac{1}{2} \sum_{i,j=1}^m p_{z,t}(z, (\sigma \sigma^* )_{ij} - p_\infty((\sigma \sigma^* )_{ij}; x); x) \frac{\partial^2}{\partial x_i x_j} v^0_s(x) \bigg\} \, ds
\]

(the \( p_\infty(\cdot; x) \) terms have been brought inside the integral \( p_{z,t}(z, \cdot; x) \)

since they not depend on \( z \))

\[
\leq \varepsilon \sum_{i=1}^m \int_0^{T-t} \left| p_u(z, b_i - p_\infty(b_i; x); x) \frac{\partial}{\partial x_i} v^0_{s+u}(x) \right| du
\]
\[
+ \frac{\varepsilon}{2} \sum_{i,j=1}^m \int_0^{T-t} \left| p_u(z, (\sigma \sigma^* )_{ij} - p_\infty((\sigma \sigma^* )_{ij}; x); x) \frac{\partial^2}{\partial x_i x_j} v^0_{s+u}(x) \right| du
\]

\[
\leq \varepsilon \sum_{i=1}^m \int_0^{\infty} \left| p_u(z, b_i - p_\infty(b_i; x); x) \right| du \sup_{t \leq s \leq T} \left| \frac{\partial}{\partial x_i} v^0_s(x) \right|
\]
\[
+ \frac{\varepsilon}{2} \sum_{i,j=1}^m \int_0^{\infty} \left| p_u(z, (\sigma \sigma^* )_{ij} - p_\infty((\sigma \sigma^* )_{ij}; x); x) \right| du \sup_{t \leq s \leq T} \left| \frac{\partial^2}{\partial x_i x_j} v^0_s(x) \right|
\]

\((f - p_\infty(f; x)\) is centered, so by Proposition 3.6.2, \((2):\)

\[
\leq \varepsilon C_1(1 + |z|^q) \left\{ \sum_{i=1}^m \sup_{t \leq s \leq T} \left| \frac{\partial}{\partial x_i} v^0_s(x) \right| + \sum_{i,j=1}^m \sup_{t \leq s \leq T} \left| \frac{\partial^2}{\partial x_i x_j} v^0_s(x) \right| \right\}
\]

and therefore finally

\[
\mathbb{E} \left[ \mathbb{E} \left[ \int_t^T (\mathcal{L}_s - \overline{\mathcal{L}}) v^0_s(x) ds \Big| \mathcal{F}_t^B \right] \right]^p
\]

\[
\leq \varepsilon^p C_2(1 + |z|^q) \mathbb{E} \left[ \sum_{i=1}^m \sup_{t \leq s \leq T} \left| \frac{\partial}{\partial x_i} v^0_s(x) \right|^p + \sum_{i,j=1}^m \sup_{t \leq s \leq T} \left| \frac{\partial^2}{\partial x_i x_j} v^0_s(x) \right|^p \right].
\]

Next, using again \( v^0_s \in \mathcal{F}_s^B \) and that \( Z^{\varepsilon,x} \) is independent of \( B \),

\[
\mathbb{E} \left[ \int_t^T \left[ h(x, Z^{\varepsilon,x}_s) - \overline{h}(x) \right]^* v^0_s(x) dB_s \Big| \mathcal{F}_t^B \right]
\]

\[
= \int_t^T \mathbb{E} \left[ \left[ h(x, Z^{\varepsilon,x}_s) - \overline{h}(x) \right]^* v^0_s(x) \Big| \mathcal{F}_s^B \right] dB_s
\]

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\[
= \int_t^T p_{\frac{s-t}{\varepsilon}}(z, h - \bar{h}; x)^* v_s^0(x) dB_s.
\]

For \( t \leq r \leq T \), \( r \mapsto \int_r^T p_{\frac{s-t}{\varepsilon}}(z, h - \bar{h}; x)^* v_s^0(x) dB_s \), is a martingale w.r.t. \( (\mathcal{F}_r^T : r \in [t, T]) \) if time is run backwards. Hence by the Burkholder-Davis-Gundy inequality,

\[
\mathbb{E} \left[ \left| \int_t^T p_{\frac{s-t}{\varepsilon}}(z, h - \bar{h}; x)^* v_s^0(x) dB_s \right|^p \right] \\
\leq C_p \mathbb{E} \left[ \left( \int_t^T p_{\frac{s-t}{\varepsilon}}(z, h - \bar{h}; x)^* v_s^0(x) dB_s \right)^\frac{p}{2} \right],
\]

where

\[
\left\{ \int_t^T p_{\frac{s-t}{\varepsilon}}(z, h - \bar{h}; x)^* v_s^0(x) dB_s \right\} = \int_t^T \left| p_{\frac{s-t}{\varepsilon}}(z, h - \bar{h}; x)^* v_s^0(x) \right|^2 ds \\
\leq \varepsilon \int_0^\infty |p_\alpha(z, h - \bar{h}; x)|^2 du \sup_{t \leq s \leq T} |v_s^0(x)|^2 \\
\leq \varepsilon C_3(1 + |z|^q) \sup_{t \leq s \leq T} |v_s^0(x)|^2,
\]

where the last inequality is by Proposition 3.6.2, (2), since \( h - \bar{h} \) is centered. Therefore,

\[
\mathbb{E} \left[ \left| \int_t^T p_{\frac{s-t}{\varepsilon}}(z, h - \bar{h}; x)^* v_s^0(x) dB_s \right|^p \right] \\
\leq \varepsilon^\frac{p}{2} C_4(1 + |z|^q) \mathbb{E} \left[ \sup_{t \leq s \leq T} |v_s^0(x)|^p \right].
\]

Combining (3.22) and (3.23),

\[
\mathbb{E} \left[ |\theta_t^1|^p \right] \leq \varepsilon^p C_4(1 + |z|^q) \sum_{|\alpha| \leq 2} \mathbb{E} \left[ \sup_{t \leq s \leq T} |D_\alpha x v_s^0(x)|^p \right].
\]

Next, consider a first order \( x \)-derivative of \( \theta_t^1 \):

\[
\frac{\partial}{\partial x_k} \theta_t^1 = \frac{\partial}{\partial x_k} \int_t^T \mathbb{E} \left[ \mathcal{L}_S - \mathcal{L}_\bar{S} \right] v_s^0(x) ds \\
+ \frac{\partial}{\partial x_k} \int_t^T \mathbb{E} \left[ h(x, Z_s^x) - \bar{h}(x) \right]^* v_s^0(x) dB_s.
\]

As before, the forward Itô integral term vanished after taking the (condi-
interchanging order of differentiation and integration,

\[
\left| \frac{\partial}{\partial x_k} \int_t^T \mathbb{E} \left[ \mathcal{L}_S - \tilde{\mathcal{L}} \right] v^0_s(x) ds \right| \\
\leq \varepsilon \sum_{i=1}^m \int_0^{T-t} \left\{ \frac{\partial}{\partial x_k} p_u(z, b_i - p_\infty(b_i; x)) \frac{\partial^2}{\partial x_i^2} v^0_{\varepsilon u+t}(x) \right. \\
+ p_u(z, b_i - p_\infty(b_i; x)) \frac{\partial^2}{\partial x_i \partial x_i} v^0_{\varepsilon u+t}(x) \left. \right\} du \\
+ \frac{\varepsilon}{2} \sum_{i,j=1}^{m} \int_0^{T-t} \left\{ \frac{\partial}{\partial x_k} p_u(z, (\sigma \sigma^*)_{ij} - p_\infty((\sigma \sigma^*)_{ij}; x)) \frac{\partial^2}{\partial x_i \partial x_j} v^0_{\varepsilon u+t}(x) \right. \\
+ p_u(z, (\sigma \sigma^*)_{ij} - p_\infty((\sigma \sigma^*)_{ij}; x)) \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} v^0_{\varepsilon u+t}(x) \left. \right\} du \\
\leq \varepsilon \sum_{i=1}^m \int_0^{\infty} \left| \frac{\partial}{\partial x_k} p_u(z, b_i - p_\infty(b_i; x)) \right| du \sup_{t \leq s \leq T} \left| \frac{\partial^2}{\partial x_i^2} v^0_s(x) \right| \\
+ \int_0^{\infty} \left| p_u(z, b_i - p_\infty(b_i; x)) \right| du \sup_{t \leq s \leq T} \left| \frac{\partial^2}{\partial x_k x_i} v^0_s(x) \right| \\
+ \frac{\varepsilon}{2} \sum_{i,j=1}^{m} \int_0^{\infty} \left| \frac{\partial}{\partial x_k} p_u(z, (\sigma \sigma^*)_{ij} - p_\infty((\sigma \sigma^*)_{ij}; x)) \right| du \sup_{t \leq s \leq T} \left| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} v^0_s(x) \right| \\
+ \int_0^{\infty} \left| p_u(z, (\sigma \sigma^*)_{ij} - p_\infty((\sigma \sigma^*)_{ij}; x)) \right| du \sup_{t \leq s \leq T} \left| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} v^0_s(x) \right| \\
\times \sup_{t \leq s \leq T} \left| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} v^0_s(x) \right| \right\}.
\]

Then, from Proposition 3.6.2, (2) again,

\[
\left| \frac{\partial}{\partial x_k} \int_t^T \mathbb{E} \left[ \mathcal{L}_S - \tilde{\mathcal{L}} \right] v^0_s(x) ds \right| \leq \varepsilon C_5 (1 + |z|^{q_6}) \sum_{1 \leq \beta \leq 3} \sup_{t \leq s \leq T} \left| D^\beta_x v^0_s(x) \right|.
\]

since the quantities \( b - \bar{b} \) and \( \sigma \sigma^* - \bar{\sigma} \bar{\sigma}^* \) are centered. Taking expectation,

\[
\mathbb{E} \left[ \left| \frac{\partial}{\partial x_k} \int_t^T \mathbb{E} \left[ \mathcal{L}_S - \tilde{\mathcal{L}} \right] v^0_s(x) ds \right|^p \right] \\
\leq \varepsilon^p C_6 (1 + |z|^{q_6}) \sum_{1 \leq \beta \leq 3} \mathbb{E} \left[ \sup_{t \leq s \leq T} \left| D^\beta_x v^0_s(x) \right|^p \right].
\]

Next, by (HO_{k,l}), we can interchange the order of ordinary differentiation.
and stochastic integration (cf. [82]):

\[
\mathbb{E} \left[ \left| \frac{\partial}{\partial x_k} \left( \int_t^T \mathbb{E} \left[ h(x, Z_s^{\varepsilon,x}) - \bar{h}(x) \right]^* v_s^0(x) dB_s \right)^p \right| \right] \\
= \mathbb{E} \left[ \left| \int_t^T \frac{\partial}{\partial x_k} \left( \mathbb{E} \left[ h(x, Z_s^{\varepsilon,x}) - \bar{h}(x) \right]^* v_s^0(x) \right) dB_s \right|^p \right] \\
\leq C_p \mathbb{E} \left[ \left( \int_t^T \frac{\partial}{\partial x_k} \left( \mathbb{E} \left[ h(x, Z_s^{\varepsilon,x}) - \bar{h}(x) \right]^* v_s^0(x) \right)^2 \right)^{p/2} \right],
\]

where

\[
\int_t^T \left| \frac{\partial}{\partial x_k} \left( \mathbb{E} \left[ h(x, Z_s^{\varepsilon,x}) - \bar{h}(x) \right]^* v_s^0(x) \right) \right|^2 ds \\
= \varepsilon \int_0^T \left| \frac{\partial}{\partial x_k} \left( \mathbb{E} \left[ h(x, Z_s^{\varepsilon,x}) - \bar{h}(x) \right]^* v_s^0(x) \right) \right|^2 du \\
\leq 2\varepsilon \left\{ \int_0^\infty \left| \frac{\partial}{\partial x_k} p_u(z, h - \bar{h}; x) \right|^2 \left| v_{\varepsilon u+t}^0(x) \right|^2 du + \int_0^\infty \left| p_u(z, h - \bar{h}; x) \right|^2 \left| \frac{\partial}{\partial x_k} v_{\varepsilon u+t}^0(x) \right|^2 du \right\} \\
\leq \varepsilon C_7 (1 + |z|^{q_7}) \left\{ \sup_{t \leq s \leq T} \left| v_s^0(x) \right|^2 + \sup_{t \leq s \leq T} \left| \frac{\partial}{\partial x_k} v_s^0(x) \right|^2 \right\}.
\]

The last step follows once again from Proposition 3.6.2, (2). So,

\[
\mathbb{E} \left[ \left| \frac{\partial}{\partial x_k} \left( \int_t^T \mathbb{E} \left[ h(x, Z_s^{\varepsilon,x}) - \bar{h}(x) \right]^* v_s^0(x) dB_s \right)^p \right| \right] \leq \varepsilon \frac{\mathbb{E}}{2} C_8 (1 + |z|^{q_8}) \left\{ \mathbb{E} \left[ \sup_{t \leq s \leq T} \left| v_s^0(x) \right|^p \right] + \mathbb{E} \left[ \sup_{t \leq s \leq T} \left| \frac{\partial}{\partial x_k} v_s^0(x) \right|^p \right] \right\} \quad (3.25)
\]

Combining (3.24) and (3.25)

\[
\mathbb{E} \left[ \left| \frac{\partial}{\partial x_k} \theta_i^l \right|^p \right] \leq \varepsilon \frac{\mathbb{E}}{2} C_9 (1 + |z|^{q_9}) \sum_{\alpha \leq 3} \mathbb{E} \left[ \sup_{t \leq s \leq T} \left| D_{x}^\alpha v_s^0(x) \right|^p \right].
\]
Iterating these arguments for the higher order derivatives of $\theta^1$,

$$
\sum_{|\alpha| \leq k-1} \mathbb{E} \left[ |D^\alpha_x \theta_t^1|^p \right] \leq \varepsilon^{\frac{k}{2}} C_{10}(1 + |z|^{q_0}) \sum_{|\alpha| \leq k+1} \mathbb{E} \left[ \sup_{t \leq s \leq T} |D^\alpha_x \theta_s^0(x)|^p \right].
$$

\textbf{Lemma 3.7.3} Let $k, l \geq 3$. Assume (HF$_{k,l}$), (HS$_{k,l}$), and (HO$_{k+1,l+1}$). Also assume $\psi^1 \in C^{0,k+2,l}([0, T] \times \mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$ and that all its partial derivatives up to order $(0, k + 2, l)$ are in $\mathcal{P}_T([0, T] \times \mathbb{R}^m, \mathbb{R})$. Then for any $p \geq 1$ there exists $C_p > 0$, independent of $\varphi$, such that for any $(x, z) \in \mathbb{R}^{m+n}$, any $\varepsilon \in (0, 1)$, and any $t \in [0, T]$

$$
\mathbb{E} \left[ |R_t(x, z)|^p \right] \leq C_p \sum_{|\alpha| \leq 2} \int_t^T \mathbb{E} \left[ \mathbb{E} \left[ |D^\alpha_x \psi^1_s(x', z')|^p \right]_{(x', z') = (X_s^{t}(t, x), Z_s^{t}(t, z))} \right] ds.
$$

\textbf{Proof} $R_t(x, z)$ solves the BSPDE

$$
\begin{align*}
-dR_t(x, z) &= (\mathcal{L}^\varepsilon R_t(x, z) + \mathcal{L}_S \psi^1_t(x, z)) dt \\
&\quad + h(x, z)^* (\psi^1_t(x, z) + R_t(x, z)) dB_t, \\
R_T(x, z) &= 0.
\end{align*}
$$

Existence of the solution $R$ and its derivatives, as well as the polynomial growth all follow from Proposition 3.4.1. By Proposition 3.4.2, the solution of (3.26) is given by $\theta^{(t, x, z)(2)}_t$, the solution to the BDSDE

$$
\begin{align*}
-d\theta^{(t, x, z)(2)}_s &= \mathcal{L}_S \psi^1_s(X_t^{\varepsilon, (t, x), Z_t^{\varepsilon, (t, z)}) ds \\
&\quad + h(X_t^{\varepsilon, (t, x), Z_t^{\varepsilon, (t, z)})^* \psi^1_s(X_t^{\varepsilon, (t, x), Z_t^{\varepsilon, (t, z)}) dB_s \\
&\quad + h(X_t^{\varepsilon, (t, x), Z_t^{\varepsilon, (t, z)})^* \theta^{(t, x, z)(2)}_s dB_s - \gamma^{t, x, z}_s dW_s - \delta^{t, x, z}_s dV_s \\
\theta^{(t, x, z)(2)}_T &= 0.
\end{align*}
$$

We will drop superscripts $(t, x, z)$ for $\theta^{(t, x, z)(2)}_t$, $(t, z)$ for $Z_t^{\varepsilon, (t, z)}$, and $(t, x)$ for $X_t^{\varepsilon, (t, x)}$.

$R_t(x, z)$ is $\mathcal{F}_{t,T}^B$-measurable, hence, so is $\theta^2_t$. As before, the stochastic integrals over $dV$ and $dW$ vanish when we take conditional expectation with respect to $\mathcal{F}_{t,T}^B$. Thus

$$
\theta^2_t = \mathbb{E} \left[ \int_t^T \mathcal{L}_S \psi^1_s(X_s^t, Z_s^t) dB_s \middle| \mathcal{F}_{t,T}^B \right].
$$

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\begin{align*}
+ \mathbb{E} \left[ \int_t^T h(X_s^e, Z_s^e)^* \psi_1^e(X_s^e, Z_s^e) dB_s \bigg| \mathcal{F}_{t,T} \right] \\
+ \mathbb{E} \left[ \int_t^T h(X_s^e, Z_s^e)^* \theta_2^e dB_s \bigg| \mathcal{F}_{t,T} \right].
\end{align*}

(3.27)

Consider each term separately:

\begin{align*}
&\mathbb{E} \left[ \left| \left| \mathbb{E} \left[ \int_t^T \mathcal{L}_s \psi_1^e(X_s^e, Z_s^e) ds \bigg| \mathcal{F}_{t,T} \right] \right| \right|^p \right] \\
&\leq (T - t)^{p-1} \int_t^T \mathbb{E} \left[ \left| \left( \sum_{i=1}^m b_i(X_s^e, Z_s^e) \frac{\partial}{\partial x_i} \right) + \frac{1}{2} \sum_{i,j=1}^m (\sigma \sigma^*)_{ij}(X_s^e, Z_s^e) \frac{\partial^2}{\partial x_i x_j} \right| \psi_1^e(X_s^e, Z_s^e) \right|^p \right] ds \\
&\leq C_1 \int_t^T \left( ||b||_{\infty} \sum_{i=1}^m \mathbb{E} \left[ \left| \frac{\partial}{\partial x_i} \psi_1^e(X_s^e, Z_s^e) \right|^p \right] \\
&\quad + \frac{1}{2} ||\sigma \sigma^*||_{\infty} \sum_{i,j=1}^m \mathbb{E} \left[ \left| \frac{\partial^2}{\partial x_i x_j} \psi_1^e(X_s^e, Z_s^e) \right|^p \right] \right) ds \\
&\leq C_2 \int_t^T \sum_{1 \leq |\alpha| \leq 2} \mathbb{E} \left[ |D^\alpha_x \psi_1^e(X_s^e, Z_s^e)|^p \right] ds.
\end{align*}

Note that \( Z_s^e \) and \( X_s^e \) are \( \mathcal{F}_s^W \cup \mathcal{F}_s^V \)-measurable, \( \psi_1^e \) is \( \mathcal{F}_{s,t}^B \)-measurable, and \( B \) and \( (V,W) \) are independent. Thus

\begin{align*}
\mathbb{E} \left[ |D^\alpha_x \psi_1^e(X_s^e, Z_s^e)|^p \right] &= \mathbb{E} \left[ \mathbb{E} \left[ |D^\alpha_x \psi_1^e(X_s^e, Z_s^e)|^p \bigg| \mathcal{F}_s^V \cup \mathcal{F}_s^W \right] \right] \\
&= \mathbb{E} \left[ \mathbb{E} \left[ |D^\alpha_x \psi_1^e(x', z')|^p \bigg| (x', z')=(X_s^e, Z_s^e) \right] \right],
\end{align*}

so that

\begin{align*}
&\mathbb{E} \left[ \left| \left| \mathbb{E} \left[ \int_t^T \mathcal{L}_s \psi_1^e(X_s^e, Z_s^e) ds \bigg| \mathcal{F}_{t,T} \right] \right| \right|^p \right] \\
&\leq C_2 \sum_{1 \leq |\alpha| \leq 2} \int_t^T \mathbb{E} \left[ \left| \left| D^\alpha_x \psi_1^e(x', z') \right| \right|^p \bigg| (x', z')=(X_s^e, Z_s^e) \right] ds. \tag{3.28}
\end{align*}

Next, by Jensen’s inequality, the tower property, and the Burkholder-
Davis-Gundy inequality,

\[
\mathbb{E} \left[ \left| \mathbb{E} \left[ \int_t^T h(X^s, Z^s)^* \psi_s^1(X^s, Z^s) \tilde{dB}_s \bigg| \mathcal{F}_{t,T}^B \right] \right|^p \right] \\
\leq \mathbb{E} \left[ \left| \int_t^T h(X^s, Z^s)^* \psi_s^1(X^s, Z^s) \tilde{dB}_s \right|^p \right] \\
\leq C_p \mathbb{E} \left[ \left| \int_t^T h(X^s, Z^s)^* \psi_s^1(X^s, Z^s) \tilde{dB}_s \right|^p \right],
\]

where by H"older's inequality and the Cauchy-Schwarz inequality

\[
\left\langle \int_t^T h(X^s, Z^s)^* \psi_s^1(X^s, Z^s) \tilde{dB}_s \right\rangle^p = \left( \int_t^T |h(X^s, Z^s)^* \psi_s^1(X^s, Z^s)|^2 ds \right)^{p/2} \\
\leq C_3 \int_t^T |h(X^s, Z^s)|^p |\psi_s^1(X^s, Z^s)|^p ds.
\]

So by the same arguments as for the first term,

\[
\mathbb{E} \left[ \left| \mathbb{E} \left[ \int_t^T h(X^s, Z^s)^* \psi_s^1(X^s, Z^s) \tilde{dB}_s \bigg| \mathcal{F}_{t,T}^B \right] \right|^p \right] \\
\leq C_4 \int_t^T \mathbb{E} \left[ |\psi_s^1(x', z')|^p \right]_{(x', z') = (X^s, Z^s)} ds. \quad (3.29)
\]

Finally, using Burkholder-Davis-Gundy in the second line, and Cauchy-Schwarz in the third line

\[
\mathbb{E} \left[ \left| \mathbb{E} \left[ \int_t^T h(X^s, Z^s)^* \theta_s^2 \tilde{dB}_s \bigg| \mathcal{F}_{t,T}^B \right] \right|^p \right] \\
\leq \mathbb{E} \left[ \left| \int_t^T h(X^s, Z^s)^* \theta_s^2 \tilde{dB}_s \right|^p \right] \\
\leq C_p \mathbb{E} \left[ \left( \int_t^T |h(X^s, Z^s)^* \theta_s^2|^2 ds \right)^{p/2} \right] \\
\leq C_p \mathbb{E} \left[ \left( \int_t^T |h(X^s, Z^s)|^2 |\theta_s^2|^2 ds \right)^{p/2} \right] \\
\leq C_5 |\theta_s^2|_\infty \int_t^T \mathbb{E}[|\theta_s^2|^p] ds. \quad (3.30)
\]
Combining (3.27) with (3.28), (3.29), and (3.30)

\[ \mathbb{E} \left[ |\theta_t^2|^p \right] \leq C_6 \sum_{|\alpha| \leq 2} \int_t^T \mathbb{E} \left[ |D_x^\alpha \psi_s^1(x', z')|^p \right] |x' - x|^\alpha \, ds \]

\[ + C_5 ||h||_\infty \int_t^T \mathbb{E}[|\theta_s^2|^p]ds. \]

By Gronwall,

\[ \mathbb{E} \left[ |\theta_t^2|^p \right] \leq C_6 \left( \sum_{|\alpha| \leq 2} \int_t^T \mathbb{E} \left[ |D_x^\alpha \psi_s^1(x', z')|^p \right] \right) \left( e^{(T-t)C_5 ||h||_\infty} \right) \]

\[ \leq C_7 \left( \sum_{|\alpha| \leq 2} \int_t^T \mathbb{E} \left[ |D_x^\alpha \psi_s^1(x', z')|^p \right] \right). \]

where we choose $C_7$ so that the inequality holds for every $t \in [0, T]$ (replace $e^{(T-t)C_5 ||h||_\infty}$ by $e^{TC_5 ||h||_\infty}$). $\blacksquare$

Now we can collect all these results, to obtain the first step towards Theorem 3.5.1.

**Lemma 3.7.4** Assume (H\textsubscript{stat}), (HF\textsubscript{8,4}), (HS\textsubscript{7,4}), (HO\textsubscript{8,4}), and that $\varphi \in C_7^0(\mathbb{R}^m, \mathbb{R})$. Then for every $p \geq 1$ there exists $C, q_1, q_2 > 0$, independent of $\varphi$, such that

\[ \sup_{0 \leq t \leq T} \mathbb{E}[|v_t^\varphi(x, z) - v_t^0(x)|^p] \leq \varepsilon^{p/2}C (1 + |x|^{q_1} + |z|^{q_2}) ||\varphi||_{L_4(\mathbb{R}^m)}^p. \]

**Proof of Theorem 3.5.1** We track the necessary conditions backward from Lemma 3.7.3.

1. For the solution $R$ given in Lemma 3.7.3 to exist and satisfy the stated bound, we need (HF\textsubscript{3,3}), (HS\textsubscript{3,3}), (HO\textsubscript{4,4}), and $\psi^1 \in C_0^{0.5,3}([0, T] \times \mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$. The polynomial growth condition will be satisfied anyways.

2. For $\psi^1$ to be in $C_0^{0.5,3}([0, T] \times \mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$, we need (H\textsubscript{stat}), (HF\textsubscript{6,4}), (HS\textsubscript{6,4}), (HO\textsubscript{6,4}) and $\tilde{a}, \tilde{b}, \tilde{h} \in C_0^5$. We also need $v^0 \in C_0^{0.6}([0, T] \times \mathbb{R}^m, \mathbb{R})$. Again, the polynomial growth condition will be satisfied.

3. For $v^0$ to be in $C_0^{0.6}([0, T] \times \mathbb{R}^m, \mathbb{R})$ we need $\tilde{a}, \tilde{b}, \varphi \in C_0^7$ and $\tilde{h} \in C_0^8$.

4. For $\bar{a}, \bar{b}$ to be in $C_0^7$ we need (HF\textsubscript{7,3}) as well as (HS\textsubscript{7,0}) by Proposition 3.6.4. Similarly we need (HF\textsubscript{8,3}) as well as (HO\textsubscript{8,0}) for $\bar{h}$ to be in $C_0^8$. 75
5. So sufficient conditions are \( (H_{stat}), (HF_{8,4}), (HS_{7,4}), (HO_{8,4}) \). In that case we obtain from Lemma 3.7.1

\[
\sum_{|\alpha| \leq 4} E \left[ \sup_{0 \leq t \leq T} |D^\alpha v^0_t(x)|^p \right] \leq C_1 (1 + |x|^{q_1}) ||\varphi||_{4,\infty}^p. \tag{3.31}
\]

From Lemma 3.7.2 we obtain

\[
\sum_{|\alpha| \leq 2} \sup_{0 \leq t \leq T} E \left[ |D^\alpha \psi^1_t(x,z)|^p \right] \leq \varepsilon \frac{C_2}{2} (1 + |z|^{q_1}) \sum_{|\alpha| \leq 4} E \left[ \sup_{0 \leq t \leq T} |D^\alpha v^0_t(x)|^p \right]. \tag{3.32}
\]

From Lemma 3.7.3 we get

\[
E \left[ |R_t(x,z)|^p \right] \leq C_3 \sum_{|\alpha| \leq 2} \int_t^T E \left[ \left| D^\alpha \psi^1_t(x',z') \right|^p \right] ds. \tag{3.33}
\]

Combining (3.31), (3.33), (3.33), we get for any \( t \in [0, T] \) (by time-homogeneity of \( X^\varepsilon \) and \( Z^\varepsilon \))

\[
E \left[ |R_t(x,z)|^p \right] + E \left[ \left| \psi^1_t(x,z) \right|^p \right]
\leq \varepsilon^{p/2} C_4 \left( 1 + \sup_{0 \leq s \leq T} E \left[ \left| X^\varepsilon_s \right|^{q_1} + \left| Z^\varepsilon_s \right|^{q_2} \right] (X^\varepsilon_0, Z^\varepsilon_0) = (x, z) \right) ||\varphi||_{4,\infty}^p. \tag{3.34}
\]

From Proposition 3.6.3 we obtain

\[
\sup_{0 \leq s \leq T} E \left[ \left| X^\varepsilon_s \right|^{q_1} + \left| Z^\varepsilon_s \right|^{q_2} \right] (X^\varepsilon_0, Z^\varepsilon_0) = (x, z) \leq C_5 (1 + |x|^{q_1} + |z|^{q_4}).
\]

Noting that the right hand side in (3.34) does not depend on \( t \in [0, T] \),

\[
\sup_{0 \leq t \leq T} E \left[ |R_t(x,z)|^p \right] + \sup_{0 \leq t \leq T} E \left[ \left| \psi^1_t(x,z) \right|^p \right]
\leq \varepsilon^{p/2} C_6 (1 + |x|^{q_1} + |z|^{q_4}) ||\varphi||_{4,\infty}^p.
\]

Finally

\[
\sup_{0 \leq t \leq T} E \left[ |v^\varepsilon_t(x,z) - v^0_t(x)|^p \right]
\]
\[ \leq C_7 \left( \sup_{0 \leq t \leq T} \mathbb{E} \left[ |R_t(x, z)|^p \right] + \sup_{0 \leq t \leq T} \mathbb{E} \left[ |v_1^0(x, z)|^p \right] \right) \]
\[ \leq \varepsilon^{p/2} C_8 \left( 1 + |x|^{q_3} + |z|^{q_4} \right) ||\varphi||_{4,\infty}^p, \]

which completes the proof. ■

Now we recall that all the calculations up until now were under the changed measure \( \mathbb{P}^\varepsilon \). We only wrote \( \mathbb{P} \) and \( B \) to facilitate the reading. So let us transfer the results to the original measure \( \mathbb{Q} \).

**Lemma 3.7.5** Assume \( (H_{\text{stat}}), (HF_{8,4}), (HS_{7,4}), (HO_{8,4}) \), and that \( \varphi \in C^b_7(\mathbb{R}^m, \mathbb{R}) \).

Then for every \( p \geq 1 \) there exist \( C, q_1, q_2 > 0 \), independent of \( \varphi \), such that

\[ \sup_{0 \leq t \leq T} \mathbb{E}_Q \left[ |v_t^\varepsilon(x, z) - v_t^0(x)|^p \right] \leq \varepsilon^{p/2} C (1 + |x|^{q_1} + |z|^{q_2}) ||\varphi||_{4,\infty}^p. \]

**Proof** This is a simple application of the Cauchy-Schwarz inequality in combination with Gronwall’s lemma:

\[
\mathbb{E}_Q \left[ |v_t^\varepsilon(x, z) - v_t^0(x)|^p \right] = \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ |v_t^\varepsilon(x, z) - v_t^0(x)|^p \frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon} \right] \\
\leq \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ |v_t^\varepsilon(x, z) - v_t^0(x)|^{2p} \right]^{1/2} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left( \frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon} \right)^2 \right]^{1/2},
\]

so we see that the result is true by Lemma 3.7.4 as long as the second expectation is finite. Recall that we had defined the notation

\[
\frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon} |_{\varphi_t} = \tilde{D}_t^\varepsilon = \exp \left( \int_0^t h(X_s^\varepsilon, Z_s^\varepsilon)^* dY_s^\varepsilon - \frac{1}{2} \int_0^t |h(X_s^\varepsilon, Z_s^\varepsilon)|^2 ds \right).
\]

So \( \tilde{D}^\varepsilon \) satisfies the SDE

\[
d\tilde{D}_t^\varepsilon = \tilde{D}_t^\varepsilon h(X_t^\varepsilon, Z_t^\varepsilon)^* dY_t^\varepsilon, \quad \tilde{D}_0^\varepsilon = 1.
\]

Since under \( \mathbb{P}^\varepsilon \), \( Y^\varepsilon \) is a Brownian motion, we get by Itô-isometry

\[
\mathbb{E}_{\mathbb{P}^\varepsilon} \left[ (\tilde{D}^\varepsilon_t)^2 \right] = \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \int_0^t (\tilde{D}^\varepsilon_s)^2 |h(X_s^\varepsilon, Z_s^\varepsilon)|^2 ds \right] \leq ||h||_\infty^2 \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \int_0^t (\tilde{D}_s^\varepsilon)^2 ds \right],
\]

so that by Gronwall \( \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ (\tilde{D}_T^\varepsilon)^2 \right] < \infty \). ■
Lemma 3.7.6 Assume \((H_{\text{stat}}), (HF_{8,4}), (HS_{7,4}), (HO_{8,4})\), that \(\varphi \in C^7_\mu\), and that the initial distribution \(Q(X_{\delta,0},Z_{\delta,0})\) has finite moments of every order. Then for every \(p \geq 1\) there exists \(C > 0\), independent of \(\varphi\), such that

\[
E_Q[|\rho_T^\varepsilon (x) - \rho_T^0 (\varphi)|^p] \leq \varepsilon^{p/2} C ||\varphi||_{4,\infty}^p.
\]

Proof As we already described in the introduction, we obtain from Lemma 3.7.5

\[
E_Q[|\rho_T^\varepsilon (x) - \rho_T^0 (\varphi)|^p] = E_Q \left[ \int (v_0^\varepsilon (x,z) - v_0^0 (x))Q(X_{\delta,0},Z_{\delta,0})(dx,dz) \right]^p \leq \varepsilon^{p/2} C_1 \int (1 + |x|^q_1 + |z|^q_2) Q(X_{\delta,0},Z_{\delta,0})(dx,dz) ||\varphi||_{4,\infty}^p \leq \varepsilon^{p/2} C_2 ||\varphi||_{4,\infty}^p.
\]

The convergence of the actual filter, i.e. of \(\pi_{\varepsilon,x}^\varepsilon\) to \(\pi_0^0\), now follows exactly as in Chapter 9.4 of [35]. For the sake of completeness, we include the arguments.

Lemma 3.7.7 Let \(p \geq 1\). Then

\[
\sup_{\varepsilon \in (0,1], t \in [0,T]} \{ E_Q[|\rho_t^\varepsilon (1)|^{-p}] + E_Q[|\rho_t^0 (1)|^{-p}] \} < \infty
\]

as long as \(h\) is bounded.

Proof We give the argument for \(E_Q[|\rho_t^\varepsilon (1)|^{-p}]\), \(E_Q[|\rho_t^0 (1)|^{-p}]\) being completely analogue. We have

\[
E_Q[|\rho_t^\varepsilon (1)|^{-p}] = E_{\pi^\varepsilon} \left[ |\rho_t^\varepsilon (1)|^{-p} \frac{dQ}{d\pi^\varepsilon} \right] \leq E_{\pi^\varepsilon} \left[ |\rho_t^\varepsilon (1)|^{-2p} \right]^{1/2} E_{\pi^\varepsilon} \left[ \left( \frac{dQ}{d\pi^\varepsilon} \right)^2 \right]^{1/2}
\]

We showed in the proof of Lemma 3.7.5 that the second expectation is finite.
Note that \( x \mapsto x^{-2p} \) is convex. Therefore by Jensen’s inequality,

\[
\mathbb{E}_{\pi} \left[ |\rho_{t,x}^{\varepsilon}(1)|^{-2p} \right] \\
= \mathbb{E}_{\pi} \left[ \mathbb{E}_{\pi} \left[ \exp \left( \int_0^t h(X_s^{\varepsilon}, Z_s^{\varepsilon})^* dY_s^{\varepsilon} - \frac{1}{2} \int_0^t |\bar{h}(X_s^{\varepsilon}, Z_s^{\varepsilon})|^2 ds \right) |Y_t^{\varepsilon}| \right]^{-2p} \right] \\
\leq \mathbb{E}_{\pi} \left[ \exp \left( \int_0^t h(X_s^{\varepsilon}, Z_s^{\varepsilon})^* dY_s^{\varepsilon} - \frac{1}{2} \int_0^t |\bar{h}(X_s^{\varepsilon}, Z_s^{\varepsilon})|^2 ds \right) \right]^{-2p} \\
\leq \mathbb{E}_{\pi} \left[ \left| \frac{dQ}{dP_{\varepsilon}} \right|^{-2p} \right] = \mathbb{E}_Q \left[ \left| \frac{dP_{\varepsilon}}{dQ} \right|^{2p+1} \right].
\]

The result now follows exactly as in the proof of Lemma 3.7.5, because for \( D_t^{\varepsilon} = \frac{dP_{\varepsilon}}{dQ}|_{\mathcal{F}_t} \), we have

\[
dD_t^{\varepsilon} = -h(X_t^{\varepsilon}, Z_t^{\varepsilon})^* dB_t, \quad D_0^{\varepsilon} = 1
\]

and \( B \) is a Brownian motion under \( Q \). ■

Define for any measurable and bounded test function \( \varphi : \mathbb{R}^m \to \mathbb{R} \)

\[
\pi_t^0(\varphi) = \frac{\rho_{t}^{0}(\varphi)}{\rho_{t}^{0}(1)}.
\]

Recall that \( \pi_t^{\varepsilon,x} \) was defined analogously with \( \rho_t^{\varepsilon,x} \) instead of \( \rho_t^{0} \). We then have

**Lemma 3.7.8** Assume \((H_{stat}), (HF_{8,4}), (HS_{7,4}), (HO_{8,4})\), and that the initial distribution \( Q(X_0^{\varepsilon}, Z_0^{\varepsilon}) \) has finite moments of every order. Let \( p \geq 1 \). Then there exists \( C > 0 \) such that for every \( \varphi \in C_b^7 \)

\[
\mathbb{E}_Q [||\pi_T^{\varepsilon,x}(\varphi) - \pi_T^0(\varphi)||_4]^p \leq \varepsilon^{p/2} C ||\varphi||_{4,\infty}^p.
\]

**Proof** In the third line we use that \( \pi^{\varepsilon,x} \) is a.s. equal to a probability measure.

\[
\mathbb{E}_Q [||\pi_T^{\varepsilon,x}(\varphi) - \pi_T^0(\varphi)||_4]^p \\
= \mathbb{E}_Q \left[ \left| \rho_T^{\varepsilon,x}(\varphi) \rho_T^0(1) - \rho_T^0(\varphi) \rho_T^0(1) \right|^p \right] \\
= \mathbb{E}_Q \left[ \left| \rho_T^{\varepsilon,x}(\varphi) - \rho_T^0(\varphi) \rho_T^0(1) \rho_T^{\varepsilon,x}(1) \rho_T^0(1)^{-1} \right|^p \right].
\]

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\[ \leq C_p \left( \mathbb{E}_Q \left[ \left| \frac{\rho^x_t(\varphi) - \rho^0_t(\varphi)}{\rho^0_t(1)} \right|^p \right] \right) + ||\varphi||_\infty \mathbb{E}_Q \left[ \left| \frac{\rho^x_t(1) - \rho^0_t(1)}{\rho^0_t(1)} \right|^p \right] \]

\[ \leq C_p \left( \mathbb{E}_Q \left[ \left| \rho^0_t(1) \right|^{-2p} \right] \right)^{1/2} \left( \mathbb{E}_Q \left[ \left| \rho^x_t(\varphi) - \rho^0_t(\varphi) \right|^{2p} \right] \right)^{1/2} + ||\varphi||_\infty \mathbb{E}_Q \left[ \left| \rho^x_t(1) - \rho^0_t(1) \right|^{2p} \right]^{1/2} \]

\[ \leq \varepsilon^{p/2} C_1 ||\varphi||_{4,\infty}, \]

where the last step follows from Lemma 3.7.6 and Lemma 3.7.7. ■

Since the bound only depends on \( ||\varphi||_{4,\infty} \), we can replace the assumption \( \varphi \in C^4_b \) by \( \varphi \in C^4_b \). Just approximate \( \varphi \in C^4_b \) by \( \varphi^n \in C^4_b \) in the \( \|\cdot\|_{4,\infty} \)-norm, and take advantage of the fact that \( \pi^x_t \) and \( \pi^0_t \) are a.s. equal to probability measures. Therefore we have

**Corollary 3.7.9** Assume (H\(_\text{stat}\)), (HF\(_{8,4}\)), (HS\(_{7,4}\)), (HO\(_{8,4}\)), and that the initial distribution \( Q_{(X_0, Z_0)} \) has finite moments of every order. Let \( p \geq 1 \). Then there exists \( C > 0 \) such that for every \( \varphi \in C^4_b \),

\[ \mathbb{E}_Q[|\pi^x_t(\varphi) - \pi^0_t(\varphi)|^p] \leq \varepsilon^{p/2} C ||\varphi||_{4,\infty}^p. \]

Now note that there exists a countable algebra \((\varphi_i)_{i \in \mathbb{N}}\) of \( C^4_b \) functions that strongly separates points in \( \mathbb{R}^m \). That is, for every \( x, y \in \mathbb{R}^m \) and \( \delta > 0 \), there exists \( i \in \mathbb{N} \), such that \( \inf_{y : |x - y| > \delta} |\varphi_i(x) - \varphi_i(y)| > 0 \). Take e.g. all functions of the form

\[ \exp \left( -\sum_{j=1}^n q_j(x - x_j)^2 \right) \]

with \( n \in \mathbb{N}, q_j \in \mathbb{Q}_+, x_j \in \mathbb{Q}^m \). By Theorem 3.4.5 of [83], the sequence \((\varphi_i)\) is convergence determining for the topology of weak convergence of probability measures. That is, if \( \mu_n \) and \( \mu \) are probability measures on \( \mathbb{R}^m \), such that \( \lim_{n \to \infty} \mu_n(\varphi_i) = \mu(\varphi_i) \) for every \( i \in \mathbb{N} \), then \( \mu_n \) converges weakly to \( \mu \).

Define the following metric on the space of probability measures on \( \mathbb{R}^m \):

\[ d(\nu, \mu) = d_{(\varphi_i)}(\nu, \mu) = \sum_{i=1}^\infty \frac{|\nu(\varphi_i) - \mu(\varphi_i)|}{2^i}. \]
Because \((\varphi_i)\) is convergence determining, the metric \(d\) generates the topology of weak convergence. Therefore the proof of Theorem 3.5.1 is complete.

Theorem 3.5.1 provides the theoretical basis for development of a lower-dimensional filtering algorithms for state estimation in multiscale systems. It states that the filter of the multiscale system can be replaced by a lower-dimensional homogenized filter that is driven by real observations, at the cost of error on the order of the square root of the timescales separation. The theorem supports numerical algorithms that utilizes stochastically averaged models for less expensive lower-dimensional filtering, such as the algorithms of [84], in the context of homogenized particle filters, and [50], in the context of averaged ensemble Kalman filters.
CHAPTER 4

HOMOGENIZED HYBRID PARTICLE FILTER

Despite the general applicability and rigorous convergence results of particle filters, obstacles remain for implementation of particle filters in high dimensional problems due to the particle degeneracy issue (see Chapter 2, Section 2.6, Section 2.6.3 and references therein). [1] and [16] show that particle weight collapse occurs if sample size is less than exponential of the cube root of the signal dimension, which is a large number in high dimensional systems such as numerical weather and ocean-atmosphere models (for example, if a model has one hidden state variable and 10,000 grid points, then dimension $d = 10,000$, $\exp\{10,000^{1/3}\} \approx 2 \times 10^9$).

Based on Theorem 3.5.1 of Chapter 3, we develop a multiscale nonlinear filtering algorithm, for estimation of the slow scale process. The algorithm combines a homogenization scheme for multiscale numerical integration with the particle filter, and is called the Homogenized Hybrid Particle Filter (HHPF). It was first introduced in [84] based on the results of [71], which has been generalized here in Chapter 3. The HHPF employs the Heterogeneous Multiscale Method (HMM) of [85], [86] for multiscale numerical integration, combined with the SIS particle filter (in discrete time) or the continuous time particle filter of [12], [14] (in continuous time).

The HMM algorithm is described in Section 4.1. The HHPF algorithm is described in Section 4.2. Section 4.3 presents the results of testing the HHPF on a heuristic atmospheric model and its comparison to other nonlinear filters. In the numerical implementation, we considered the sparse, discrete time observations setting representative of real observations. This required modification to the particle filtering algorithm to correct the filter in between observation times. The correction procedure, which is based on probabilistic measure change and stochastic optimal control techniques, is described in Sections 4.4 and 4.5.
4.1 Multiscale numerical integration

The HMM consists of two numerical integration schemes at the fast and slow time scales, respectively. This avoids the burden of computing the slow component at the fast time scale, in which it does not vary significantly (effectively constant) and uses an appropriately homogenized process to approximate the slow dynamics. The scheme was developed by [85] [86] with rigorous convergence results and error estimates, and applied in high-dimensional problems (for example in [87] [50]).

To illustrate the numerical scheme, consider a general SDE with stochastic forcing in the fast component:

\[
\begin{align*}
\dot{X}_t^\varepsilon &= b(X_t^\varepsilon, Z_t^\varepsilon), \quad X_0^\varepsilon = x \in \mathbb{R}^m, \\
\dot{Z}_t^\varepsilon &= \frac{1}{\varepsilon} f(X_t^\varepsilon, Z_t^\varepsilon) + \frac{1}{\sqrt{\varepsilon}} g(X_t^\varepsilon, Z_t^\varepsilon) dV_t, \quad Z_0^\varepsilon = z \in \mathbb{R}^n.
\end{align*}
\]

(4.1a)

(4.1b)

The fast component is assumed to satisfy the Doeblin condition, i.e. it is ergodic and, for fixed \( X^\varepsilon = x \), attains a unique invariant distribution \( \mu(x, dz) \) exponentially fast. By the principle of stochastic averaging (see Chapter 3, Section 3.3), for small \( \varepsilon \), \( X^\varepsilon \) is close to a homogenized process \( X^0 \), that satisfies

\[
dX_t^0 = \bar{b}(X_t^0) dt, \quad X_0^0 = x_0,
\]

(4.2)

where \( \bar{b}(x) = \int_{\mathbb{R}^n} b(x, z) \mu(x, dz) \). It is usually impossible to obtain the invariant distribution \( \mu(x, dz) \) of the fast component of (4.1) analytically. The HMM introduced in [85] approximates the effective dynamics (4.2) by numerical approximation of the invariant distribution of the fast component. This hinges on the property of the fast component that attains its invariant distribution on a time scale much smaller than the time scale needed to evolve the slow component (the Doeblin condition).

The following is the HMM procedure presented in [85]. For simplicity, we use the Euler and Euler-Maruyama scheme for (4.1b) and (4.2): Consider the interval \([0, T]\) discretized into equal timesteps of size \( \Delta t = \frac{T}{N} \) for numerical integration at the slow scale. \( \Delta t \) is called the macro-timestep. Let \( t_k \overset{\text{def}}{=} k\Delta t \) and write \( X_{t_k} \) as \( X_k \). Each macro-timestep interval \([k\Delta t, (k+1)\Delta t]\), \( k = 0, 1, \ldots \) is further divided into micro-timestep intervals \([j\delta t, (j+1)\delta t]\),
For a fixed $k$, the fast process is evolved on micro-timestep intervals according to the micro-solver

$$Z_{k,j+1}^\varepsilon = Z_{k,j}^\varepsilon + \frac{1}{\varepsilon} f(X_k^0, Z_{k,j}^\varepsilon) \delta t + \frac{1}{\sqrt{\varepsilon}} g(X_k^0, Z_{k,j}^\varepsilon) \delta V_{k,j}. \quad (4.3)$$

The averaged coefficient $\bar{b}$ in (4.2) can be approximated by $\tilde{b}$ and $\tilde{h}$ as

$$\tilde{b}(X_k^0) = \frac{1}{MN_m} \sum_{r=1}^{M} \sum_{j=n_T}^{n_T+N_m} b(X_k^0, Z_{k,j}^{\varepsilon,r}),\quad (4.4)$$

where $M$ is the number of replicas of the fast process $Z$ for spatial averaging, $N_m$ is the number of micro-timesteps $\delta t$ for time averaging, and $n_T$ is the number of micro-timesteps skipped to eliminate transient effects. When the homogenized process is a diffusion as in (3.5), we approximate the averaged diffusion by $\tilde{\sigma} = Chol(\tilde{a})$, the Cholesky decomposition of

$$\tilde{a}(X_k^0) = \frac{1}{MN_m} \sum_{r=1}^{M} \sum_{j=n_T}^{n_T+N_m} (\sigma \sigma^*) (X_k^0, Z_{k,j}^{\varepsilon,r}).$$

The homogenized process is integrated forward using the discretization of (4.2) with $\tilde{b}$ in place of $\bar{b}$, called the macro-solver.

The advantage of the HMM method is that, due to the Doeblin condition, $n_T + N_m$ can be selected much smaller than $\lfloor \Delta t / \delta t \rfloor$. Note that a combination of spatial and temporal averaging is used in (4.4). However, by ergodicity of the fast component, spatial and temporal averaging can be interchanged. It is shown in [85] that, based solely on error analysis, the combinations of $M$, $N_m$, and $\delta t$ can be chosen such that no spatial averaging or no spatial and temporal averaging is required. The error of the HMM approximation
is shown to be (Theorem 2.4 of [86]) be the sum of errors due to

- homogenization ($|X^\varepsilon - X^0|, \Theta(\sqrt{\varepsilon})$),
- numerical scheme at slow timescale ($\Theta(\Delta t^k)$, where $k_1$ is order of the numerical integration scheme used) and
- the approximation $\tilde{b}$ (combination of errors due to numerical scheme used at fast timescale and the numerical averaging for $b$).

Detailed explanation and error and efficiency analyses are in [86], [87].

4.2 HHPF algorithm

The HHPF is a numerical approximation to the homogenized filter $\pi^0$ of Chapter 3, via the HMM algorithm and particle filtering. We would like to point out that the term “hybrid” in Homogenized Hybrid Particle Filter is intended to indicate that the homogenized filter is driven by real observation $Y^\varepsilon$, instead of a homogenized version $\bar{Y}$. For a single timescale system, the HHPF is just the SIS or continuous time particle filter described in Section 2.6. For a slow-fast system, the HHPF constructs a sample for the $m$-dimensional homogenized process $X^0$ of 3.5 instead of for the $(m + n)$-dimensional $(X^\varepsilon, Z^\varepsilon)$ of (3.1). For particle weight computation, the HHPF uses an approximation $\tilde{h}$ of the homogenized sensor function $\bar{h}$. The step-by-step algorithm is described below for continuous time, accompanied by the illustration in Figure 4.2. For discrete time, the algorithm is the same, with the stochastic differential equation replaced by the stochastic difference equation and the Radon-Nikodym derivative replaced by observation likelihood in weight calculations.

**initial condition:** Given the initial distribution $\pi^0_0(x, z)$, $(N_s + MN_s)$ particles $\{x^0_{0,i}, \{z^\varepsilon_{0,i,p}\}_{p=1}^M\}_{i=1}^{N_s}$ are drawn. Since we are approximating the filter $\pi^0_t$, we only require $N_s$ independent particles $\{x^{0,i}\}_{i=1}^{N_s}$ to represent possible locations of $X^0$. However, the HMM scheme requires some computation of $Z^\varepsilon$ ($Z^x$, to be precise) for numerical approximation of the averaged drift and diffusion in the SDE of $X^0$, hence for each $x^{0,i}$, we assign $M$ “fast” particles $\{z^{\varepsilon,i,p}\}_{p=1}^M$. We use superscript $^0$ to indicate the $x$ particles represent possible
locations for $X^0$, while superscript $\varepsilon$ indicates that the $z$ particles follow the law of $Z^\varepsilon$ at the fast scale. The initial distributions are the same for the slow component and homogenized processes, $\pi_0^0(x) = \int_{\mathbb{R}^n} \pi_0^\varepsilon(x, z) dz$, so, if the particles are sampled uniformly, then

$$\pi_0^0 \approx \frac{1}{N_s} \sum_{i=1}^{N_s} \delta_{x - x_0^0, i}.$$ 

Note that by ergodicity, as well as the HMM error analysis, we can set $M << N_s$ or even $M = 1$ for the samples of fast particles. When $M = 1$, we have $N_s$ $m$-dimensional particles and $N_s$ $n$-dimensional particles. This is the same as having $N_s$ $(m+n)$-dimensional particles. However, for the multiscale filter, the required sample size is higher because a $(m+n)$-dimensional process needs to be represented.

**prediction:** The prediction and update step is where the HHPF differs from regular particle filters through the incorporation of the homogenization scheme.

In a regular particle filter, particles evolve independently according to the signal diffusion (3.1), over micro-steps $\delta t$. The slow process is integrated every $\delta t$ as well, even though it does not change much in such a small timestep.

In the HHPF particles $\{\bar{x}_k^{0,i}\}_{i=1}^{N_s}$ are propagated independently over macro-timesteps $\Delta t$ using the macro-solver, where the averaged drift is approxi-
mated using (4.4). This approximation employs the fast particles \( \{ z^{\varepsilon,i,p}_{k,j} \}_{p=1}^{M} \). In each macro-timestep interval, fast particles are propagated up to \( n_T + N_m \) micro-timesteps, sufficient for the invariant distribution to be attained. Particles propagation in one macro-timestep interval is illustrated in the "predict" segment of Figure 4.2.

**update:** Particle weights are updated at time-steps when observations are available, illustrated in the update segment of Figure 4.2. Only the sample \( \{ x^0_{k,i} \}_{i=1}^{N_s} \) is updated. The reasoning is that the fast component has attained invariant distribution, represented by the particles \( \{ z^{\varepsilon,i,p}_{k,j} \}_{p=1,i=1}^{M,N_s} \), and it is sufficiently mixing. Hence we can essentially sample the fast particles again for each \( i \) uniformly from the same invariant distribution. Equivalently we keep the samples assigned to each \( i \).

Weight updates are computed using the Radon-Nikodym derivative in the Kallianpur-Striebel formula. By the ergodicity of \( Z^\varepsilon \), the integral \( \int_{k\Delta t}^{(k+1)\Delta t} h(X^\varepsilon_s = x, Z^\varepsilon_s)ds \) for the observation converges to

\[
\bar{h}(x) \overset{\text{def}}{=} \Delta t \int_{\mathbb{R}^n} h(x,z)\mu(x,dz),
\]

and is approximated by \( \tilde{h} \) using the HMM, analogous to (4.4) for the averaged drift \( \bar{b} \). For \( t \in [k\Delta t, (k + K)\Delta t) \),

\[
w^0_{t,i} = \exp \left\{ \int_{k\Delta t}^{(k+K)\Delta t} \bar{h}(\bar{x}^0_{s,i}) \ dY^\varepsilon_s - \frac{1}{2} \int_{k\Delta t}^{(k+K)\Delta t} \| \bar{h}(\bar{x}^0_{s,i}) \|^2 ds \right\}
\approx \exp \left\{ \sum_{k'=0}^{K-1} \bar{h}(\bar{x}^0_{(k+k')\Delta t}) \Delta Y^\varepsilon_{(k+k')\Delta t} - \frac{1}{2} \sum_{k'=1}^{K} \| \bar{h}(\bar{x}^0_{(k+k')\Delta t}) \|^2 \Delta t \right\},
\]

where \( \Delta Y^\varepsilon_{(k+k')\Delta t} \overset{\text{def}}{=} Y^\varepsilon_{(k+k'+1)\Delta t} - Y^\varepsilon_{(k+k')\Delta t} \).

**resampling:** Following weight update, the a resampling procedure is performed if the effective sample size falls below a set threshold (weights concentrated on a small portion of the sample) as in regular particle filters, described in Section 2.6, Section 2.6.1.
4.3 Numerical example: Lorenz ’96 model

The Lorenz ’96 model was originally introduced in [88] to mimic multiscale mid-latitude atmospheric dynamics for an unspecified scalar meteorological quantity. A latitude circle is divided into \( K = 36 \) sectors, and each sector is subdivided into \( J = 10 \) subsectors, with the following dynamics in each sector:

\[
dX^k_t = \left( -X^{k-1}_t(X^{k-2}_t - X^{k+1}_t) - X^k_t + F + \frac{h_x}{J} \sum_{j=1}^{J} Z^{k,j}_t \right) dt + \sigma_x dW_t, \quad k = 1, \ldots, K, \tag{4.6a}
\]

\[
dZ^{k,j}_t = \frac{1}{\varepsilon} \left( -Z^{k,j+1}_t(Z^{k,j+2}_t - Z^{k,j-1}_t) - Z^{k,j}_t + h_z X^k_t \right) dt + \frac{1}{\sqrt{\varepsilon}} \sigma_z dV_t, \quad j = 1, \ldots, J. \tag{4.6b}
\]

\( X^k_t \) represents a slow-scale atmospheric variable at time \( t \) in the \( k^{th} \) sector. (Note: We use superscripts \( k \) and \( j \) to conform with the typical spatial indexing notation used for the Lorenz ’96 model. In sections that follow, subscripts \( k \) and \( j \) will be used as discrete time indices, not to be confused with the spatial indices of the Lorenz model). Each \( X^k_t \) is coupled to its neighbors \( X^{k+1}_t, X^{k-1}_t, \) and \( X^{k-2}_t \) to mimic the westerly wind pattern in midlatitude. (4.6a) applies for all values of \( k \) by letting \( X^{k+K}_t = X^{k-K}_t = X^k_t \). The influence of multiple spatio-temporal scales is incorporated by dividing each sector \( k \) into \( J \) subsectors, and introducing \( Z^{k,j}_t \) in each subsector. The model that we use is slightly different from the one originally introduced in
In the fast scale dynamics in [88], the linear effect is of order $\sqrt{\varepsilon}$ and the slow scale forcing on the fast dynamics is of order $\varepsilon$, hence the nonlinear effects of order $\frac{1}{\varepsilon}$ are dominant. Here, we use the version of the model used in [87] and [50], in which the nonlinear, linear and slow scale effects in the fast dynamics are all of order $\frac{1}{\varepsilon}$. In this setting, [87] showed that (for a lower order version of the Lorenz ’96 model) the fast scale dynamics display ergodic properties such that the averaging technique described in Section 4.1 can be utilized to average out the fast dynamics when we are only interested in the slow dynamics (coarse-grained process). This is taken advantage of to reduced the dimension of the filtering problem in the work presented here.

The dynamics of unresolved modes can be represented by adding forcing in the form of stochastic terms (see, for example, [89, 90]). The use of stochastic terms to represent nonlinear self-interaction effects at short timescales in the unresolved modes is appropriate if we are only interested in the coarse-grained dynamics occurring in the long, slow timescale. This is called stochastic consistency in [90]. $0 < \varepsilon << 1$ in (4.6) is a small timescale separation parameter, hence $Z^{k,j}$ is a fast-scale process. Considering (4.6b), where only quadratic nonlinearity is present, the motivation behind adding stochastic forcing is thus to model higher order self-interaction effects. In (4.6b), this is modeled as $\frac{1}{\sqrt{\varepsilon}} \sigma_z V_t$, where $V$ is a $J$-dimensional standard Brownian motion and $\sigma_z \in \mathbb{R}^{J \times J}$. The effect of $Z^{k,j}$ enters the slow-scale dynamics as forcing $\frac{b}{J} \sum_{j=1}^{J} Z^{k,j}_t$ with larger frequency and smaller amplitude than the linear and quadratic effects in (4.6a). We also incorporate stochastic forcing at the slow scale in the form of $\sigma_x W_t$, where $W$ is a $K$-dimensional standard Brownian motion independent of $V$ and $\sigma_x \in \mathbb{R}^{K \times K}$. $\sigma_x$ is such that the effect of the stochastic forcing is small compared to the linear and quadratic effects in (4.6a).

Based on the preceding discussion, each $X^k_t$ represents a slowly-varying, large amplitude atmospheric quantity, with $J$ fast-varying, low amplitude quantities, $Z^{k,j}_t$, associated with it. In the context of climate modeling, the slow component is also known as the resolved climate modes while the fast-varying component is known as the unresolved non-climate modes. Coupling between neighbors model advection between sectors and subsectors, while coupling between each sector and its subsectors models damping. The model is also subjected to linear external forcing, $F$, on the slow timescale. The two-scale Lorenz ’96 model has been used in [2] to study stochastic parametriza-
tion, in [91] for analyzing targeted observations, and in [92] and several others for analyzing the influence of large-scale spatial patterns on the growth of small perturbations. It has also been used as a testbed for nonlinear filtering algorithms, for example, the reduced-order ensemble Kalman filters, in [93] and [50].

The Lorenz '96 model has chaotic behavior, which results in exponential growth of small initial errors. The HHPF as presented in Section 4.2 was found to be insufficient for application to such high-dimensional chaotic systems. Modifications were made to the HHPF based on the importance sampling principle described in Section 2.6 in order to construct a better (optimal in a certain sense) proposal sampling density using stochastic control techniques. This procedure is derived in the beginning of Section 4.4 for discrete time signal and observation, and the beginning of Section 4.5 for continuous time signal, discrete and sparse in time observation. For comparison of the HHPF with other nonlinear filters that do not utilize homogenization, the optimal importance sampling derivations of Sections 4.4 and 4.5 can be skipped by proceeding directly to the filtering results at the end of Section 4.4 and in Section 4.5.1.

4.4 Optimal importance sampling for discrete time signal and observation

The Lorenz '96 model is a chaotic system, i.e. small deviations of the state grow over time (for discussion on chaotic dynamics in geophysical systems and their implications on predictability and data assimilation, see, for example, Chapter 6 of [94]). In the context of particle filtering, if particle dynamics are based solely on the system model, a large number of particles will be required to properly capture the possible locations in the state space. Over time, the particle sample will suffer from particle collapse, where a large fraction of weights will be concentrated on only a small number of particles (see Section 2.6, Section 2.6.3). Additionally, chaotic behavior may cause the small error between the locations of these particles with significant weights and the true signal to grow.

The issues mentioned above can be overcome by utilizing information from available observations as a guide for the law governing particle dynamics. For
this purpose, we consider the importance sampling particle filtering method described in Section 2.6. In importance sampling, the particle filter draws particles from a proposal density and represents the true conditional density of the signal given observations by weighting particles appropriately. Observations can be used in the construction of the proposal density to more closely represent the true conditional density. Based on this idea, we performed modifications to importance sampling algorithm for construction of an improved proposal density, using tools from optimal control theory. A forcing term is added to the governing equation of the particles, such that many particles have $|Y_t - h(X^*_t)|$ small. This would result in a more even distribution of weights among particles after the observation update. The forcing is interpreted as a control, which can be determined using a stochastic optimal control approach, while making sure not to over do the control — otherwise sample diversity is lost. We first consider the problem from a discrete time signal and observation setting.

Consider the discrete time nonlinear signal with linear observation:

\begin{align}
X_{k+1} &= b(X_k) + \sigma_x W_{k+1}, \quad X_0 = x \in \mathbb{R}^m, \quad (4.7a) \\
Y_{k+1} &= H X_{k+1} + \sigma_y B_{k+1} \quad Y_0 = 0 \in \mathbb{R}^d, \quad (4.7b)
\end{align}

where $W_k, B_k$ are independent standard Gaussian random variables. The goal is to steer particles, prior to updating particle weights, toward locations indicated by observations. We do this by applying a control $u_k : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$ according to

\begin{equation}
X_{k+1}^i = b(x_k^i) + u_k(y_{k+1}, x_k^i) + \sigma_x W_{k+1}.
\end{equation}

(4.8)

In geophysical sciences data assimilation problems, this procedure is called “nudging”. At each timestep $k + 1$, given the value of the new observation $y_{k+1}$, the nudging term $u_k(y_{k+1}, x)$ is determined by minimizing a quadratic cost:

\begin{equation}
J := \mathbb{E}_{k,x} \frac{1}{2} \left[ u_k^*(y, x) Q^{-1} u_k(y, x) + (y - H X^{(k,x)}_{k+1})^* R^{-1} (y - H X^{(k,x)}_{k+1}) \right],
\end{equation}

(4.9)

where $Q, R$ are the signal and observation noise covariance matrices and by $X^{(k,x)}$ we mean that the process $X$ started at time $k$ at the value $x$. 

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The first term in (4.9) represents the control energy and if we allow $u$ to become too big, then heuristically all the particles will coincide with the observation. Then the particles will be a sample from a Dirac distribution, whereas the conditional distribution that we try to simulate is absolutely continuous. The second term represents the distance between $HX_{k+1}$ and the observation, which we aim to minimize. Covariance matrices $Q$ and $R$ in the quadratic terms indicate that dimensions of the signal and observation that have larger noise variance are penalized less by the control. This means that in directions where noise amplitude is large, we allow for more correction by taking $Q^{-1}$, which puts less penalty on the size of the control. The cost due to the second term in (4.9) incurs a penalty for being far away from the signal location indicated by observation. However, in directions where the quality of the observation is not very good, the factor $R^{-1}$ is used to allow our particle to be further away from the observation.

The solution to this linear-quadratic optimal control problem is well known (see, for example, Chap. 12 in [95]). It is given by

$$u_{k}^{\mathrm{opt}}(y, x) = (Q^{-1} + H^* R^{-1} H)^{-1} H^* R^{-1} (y - H b(x)).$$

(4.10)

It can be shown (see, for example, [11]) that, conditioned on the previous location of the particle, the proposal density which keeps the variance of particle weights to a minimum is

$$q_{k}^{\mathrm{opt}}(x_k | x_{k-1}, y_k) = \frac{p(y_k | x_k) p(x_k | x_{k-1})}{\int p(y_k | x_k) p(x_k | x_{k-1}) \, dx_k}.$$  

(4.11)

In the linear observation case of (4.7), we have

$$q_{k}^{\mathrm{opt}}(x_k | x_{k-1}, y_k) = \mathcal{N}(b(x_{k-1}) + \alpha(x_{k-1}, y_k), \hat{Q}),$$

(4.12)

where $\hat{Q} \overset{\text{def}}{=} (Q^{-1} + H^* R^{-1} H)^{-1}$

and $\alpha(x_{k-1}, y_k) \overset{\text{def}}{=} \hat{Q} H^* R^{-1} (y_k - H b(x_{k-1}))$.

This $q_{k}^{\mathrm{opt}}$ is a Gaussian. Once we have particle locations $\{x^i_{k-1}\}_{i=1}^{N_x}$ representing the posterior at time $k-1$, and the observation $y_k$ is recorded, the particles can be evolved according to

$$x^i_{k+1} = b(x^i_k) + \alpha(x^i_k, y_{k+1}) + \sigma x W_{k+1}$$

(4.13)
where \( \hat{\sigma}_x \) is such that \( \hat{\sigma}_x \hat{\sigma}_x^* = \hat{Q} \). Then \( x_k^i \) behaves like a particle sampled from \( q^{opt}(\cdot|x_{k-1}^i, y_k) \). The weights are updated according

\[
w_k^i \propto w_{k-1}^i \exp \left\{ -\frac{1}{2} (y_k - Hb(x_{k-1}^i))^\ast \hat{R}^{-1} (y_k - Hb(x_{k-1}^i)) \right\},
\]

where \( \hat{R} \overset{\text{def}}{=} HQH^* + R \).

The particle control (and optimal proposal density) of (4.8) and (4.13) differ by the noise variances \( Q \) and \( \hat{Q} \). In addition, particle weights for (4.8) are updated using the original sensor noise variance \( R \) instead of \( \hat{R} \). It should be emphasized that the schemes of (4.8) and (4.13) minimize weight variance of each particle (see Section 4.5), conditioned on the previous location of the particle. There are importance sampling schemes that minimize the weight variance over the entire sample, for example, that of [52].

### 4.4.1 Numerical experiments

Numerical experiments are performed on the Lorenz '96 system with 9 slow dimensions and 90 fast dimensions (\( K = 9, J = 10 \)) using different nonlinear filters to estimate the 9-dimensional slow signal component. The time scale separation is set at \( \varepsilon = 0.01 \). The additive slow-scale forcing is \( F = 10 \), the coupling parameters are \( (h_x, h_z) = (-1, 1) \). Based on the homogenization result of Section 3.1, estimation of the slow signal component can be performed using homogenized filters, which have a 9-dimensional state space instead of the (9+90) dimensional of the multiscale system. Numerical integration for the multiscale system is performed using timestep of 0.0005 time units, corresponding to 3.6 seconds in real time. For the homogenized filters, numerical integration is performed using timestep of 0.05 time units, corresponding to 6 hours in real time, as the fast scale process does not need to be realized completely. Additional numerical integration parameters details are described in Section 4.5.1.

Observation is taken as all 9 components of the slow signal component perturbed by standard Gaussian noise. Observations are recorded at every 0.05 time units, i.e. every timestep of the slow scale numerical integration. One realization of the truth with observations is simulated for 20 times units (100 days). The following filters are implemented:
• sequential importance sampling particle filter (PF)
• homogenized hybrid particle filter (HHPF)
• homogenized hybrid particle filter with particle control (HHPF<sub>c</sub>): particles evolve according to (4.13), where the control α is given by (4.10);
• ensemble Kalman filter (enKF) with no homogenization
• homogenized ensemble Kalman filter (henKF); this is the same as the scheme in the wide timescale separation setting in [50]

Each filter is implemented with 30 particles. The results of one experiment are shown in Figures 4.4 and 4.5. The error shown is the normalized root mean square error (RMSE)

$$e_t \overset{\text{def}}{=} \frac{\|X^\text{true}_t - X^\text{filter}_t\|}{\|X^\text{true}_t\|},$$

where $X^\text{filter}$ is computed using the sample mean of the respective filters. The corresponding RMSEs integrated over time and computation time for the filters are in Table 4.1. Using the homogenized filters, computation time can be significantly decreased, as the fast-scale process does not need to be completely realized and numerical integration can be performed at a larger timestep. In terms of estimation error, all the filters perform sufficiently well.

![Figure 4.4: (X^1, X^2, X^3) and RMSE, K = 9, J = 10](image-url)
Figure 4.5: \((X^1, X^2, X^3)\) and RMSE, \(K = 9, J = 10\),

<table>
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<th>Filter</th>
<th>PF</th>
<th>HHPF</th>
<th>henKF</th>
<th>enKF</th>
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<td>3.102</td>
<td>2.187</td>
<td>2.229</td>
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<td>18 s</td>
<td>16 s</td>
<td>107 s</td>
</tr>
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</table>

Table 4.1: RMSE integrated over time and computation time for 1 experiment, \(K = 9, J = 10\)

Figure 4.6: \((X^1, X^2, X^3)\) and RMSE, \(K = 36, J = 10\),

However, when the slow components’ dimension is increased to \(K = 36\), the PF and HHPF are unable to estimate the true signal well with sample size of 30 particles (Figures 4.6(a) and (b)). With application of the control on particles based on (4.13) and (4.10), the HHPF\(_c\) is able to track the
Figure 4.7: \((X^1, X^2, X^3)\) and RMSE, \(K = 36, J = 10\),

Figure 4.8: \(X^2, X^2, X^3\) and RMSE, \(K = 36, J = 10\),

<table>
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<th>Filter</th>
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<th>HHPF</th>
<th>HHPF(_c)</th>
<th>henKF</th>
<th>enKF</th>
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<td>44 s</td>
<td>378 s</td>
</tr>
</tbody>
</table>

Table 4.2: RMSE integrated over time and computation time for 1 experiment, \(K = 36, J = 10\)

truth, at the cost of higher computational effort. The HHPF can be improved by increasing sample size, but it is less efficient than implementation of the HHPF\(_c\). For example, increasing the sample size for the HHPF to 960 results in RMSE of 6.802, but the corresponding computation time is 1368 s. The enKF and henKF perform best, and the henKF has the advantage of lower computational time, without losing much accuracy due to homogenization. Here, observations are recorded every 6 hours, at every slow-scale numerical integration step. In Sections 4.5 and 4.5.1, we study the case when observa-
tions are sparse in time, recorded at close to every error-doubling time of the Lorenz ’96 model.

### 4.4.2 Remarks

The stochastic optimal control approach to modifying particle trajectory described here is similar to the derivation of the 4D-VAR method that is used in geophysical data assimilation (see, for example, Chapter 5.6.3 of [94]). The 4D-VAR method considers the problem of determining the best initial condition at time \( t_0 \) for the forward integration of the model PDEs based on discrete observations collected, up to a finite time \( t_K \), in the future of \( t_0 \). In the 4D-VAR method, the cost function to be minimized with respect to the initial condition \( x(t_0) \) is

\[
J(x(t_0)) = \frac{1}{2} [x(t_0) - x^b(t_0)]^* B_0^{-1} [x(t_0) - x^b(t_0)] \\
+ \frac{1}{2} [H(x(t_K)) - y(t_K)]^* R^{-1} [H(x(t_K)) - y(t_K)],
\]

where \( x^b(t_0) \) was predicted using the model equations from time before \( t_0 \) and \( x(t_K) \) is obtained by integration of the model PDEs using \( x(t_0) \) as initial condition. From this point of view, the stochastic optimal control approach presented here can be viewed as determining the optimal initial condition at every discrete time \( t_k \) using the next available observation at \( t_{k+1} \). The optimal control \( u_k^{opt} \) is the correction made to the state \( x_k \) predicted from \( t_{k-1} \).

[96] uses a different approach that gives the same results for this case. Let \( e^{-G(x_k)} = p(X_k|X_{k-1})p(y_k|X_k) \). A map \( \xi \to x \) is determined such that \( G(x) - \min G(x) = \frac{1}{2} \xi^* \xi \). The value of \( \xi \) is chosen according to a standard Gaussian distribution, and the corresponding \( x \) from the map is chosen as the new location of the particle. As \( \xi \) is a standard Gaussian, the highly likely values of \( \xi \) are in the neighborhood of 0, and these \( \xi \) produce \( X_k \) near the minimum of \( G \), hence a high probability position for the particle. It is shown that this leads to the optimal importance sampling density of (4.12).
4.5 Optimal importance sampling for continuous time signal and sparse-, discrete-in-time observation

Here we consider the continuous time signal with discrete time observation setting that is more representative of real geophysical data assimilation problems, and apply the same importance sampling and stochastic optimal control approach as the previous section to construct an optimal proposal density for the SIS particle filter. The signal and observation are

\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^m, \]  
\[ Y_{t_k} = h(X_{t_k}) + B_{t_k}, \quad Y_0 = 0 \in \mathbb{R}^d, \] 

where \( W \) is a standard Brownian motion and \( B_{t_k} \) is a Gaussian random variable with mean zero and covariance \( R \in \mathbb{R}^{d \times d} \) for \( k = 1, 2, \ldots \). In high dimensions and chaotic systems, the particle filter faces the particle degeneracy issues, as described in the beginning of Section 4.4. To avoid this problem, we introduce an additive “control” in the dynamics of each particle in \( (t_k, t_{k+1}) \) that steers it towards a location most representative of the observation \( Y_{t_{k+1}} \), as in the discrete-time case:

\[ d\hat{X}_t^i = \left( b(\hat{X}_t^i) + u_t^i \right) dt + \sigma(\hat{X}_t^i)dW_t, \quad t \in (t_k, t_{k+1}). \] 

The control \( u_t^i \) is chosen to minimize the cost functional:

\[ J(t_k, x, u) \overset{\text{def}}{=} \mathbb{E}_{t_k, \hat{X}_t^i_k} \left[ \frac{1}{2} \int_{t_k}^{t_{k+1}} u(s)^* Q(\hat{X}_s^i)^{-1} u(s) \, ds + g(Y_{t_{k+1}}, \hat{X}_{t_{k+1}}^i) \right], \]

where \( Q(x) \overset{\text{def}}{=} \sigma\sigma^*(x) \) and

\[ g(y, x) \overset{\text{def}}{=} \frac{1}{2} (y - h(x))^* R^{-1} (y - h(x)). \]

\( E_{t_k, \hat{X}_t^i_k} [\cdot] \) is expectation with respect to the probability measure of the process that starts at the \( \hat{X}_t^i_k \) at time \( t_k \). From here on, we suppress the \( x \)-dependence in the notation for \( Q \), for brevity.

Covariance matrices \( Q \) and \( R \) in the cost indicate that the subspaces of the signal and observation that have larger noise variance contribute less to the
total cost. The matrix $Q^{-1}$ allows for more control in the directions of large signal noise by penalizing the energy of the control less in those directions. The terminal cost $g$ incurs a large cost component when $|Y_{t_{k+1}}^j - h_j(\hat{X}_{t_{k+1}}^i)|$ is large, but $R^{-1}$ reduces the contribution of $|Y_{t_{k+1}}^j - h_j(\hat{X}_{t_{k+1}}^i)|$ to the total cost if quality of observation in direction $j$ is poor, so that particles are controlled less based on information from $Y_{t_{k+1}}^j$.

Optimal control of particles

Following the standard procedure [97], we let $V(t, x)$ be the value function defined by $V(t, x) \defeq \inf_u J(t, x, u)$ for $t \in [t_k, t_{k+1}]$. Then, $V(t, x)$ is the solution of the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\partial V}{\partial t} + H(t, x, D_x V, D_x^2 V) = 0, \quad V(t_{k+1}, x) = g(Y_{t_{k+1}}, x), \quad (4.17)$$

where the Hamiltonian of the associated control problem is

$$H(t, x, p, P) \defeq \sup_u \left[ -(b(x) + u)^* p - \frac{1}{2} u^* Q^{-1} u - \frac{1}{2} \text{tr} (QP) \right] \tag{4.18}$$

$$= \left[ -b(x)^* p + \frac{1}{2} p^* Q p - \frac{1}{2} \text{tr} (QP) \right].$$

(the supremum in the above equation is achieved with $u = -Qp$). The optimal control is

$$u(t) = -Q \nabla_x V(t, \hat{X}_t), \quad (4.19)$$

where $V$ is the solution of (4.17).

Using the form of the optimal control (4.19) in the Hamiltonian, the equation (4.17) can be written as

$$\frac{\partial V}{\partial t} + b(x)^* \nabla_x V + \frac{1}{2} \text{tr} (Q \nabla_x^2 V) - \frac{1}{2} \nabla^* V Q \nabla_x V = 0, \quad t \in [t_k, t_{k+1}], \quad (4.20)$$

$$V(t_{k+1}, x) = g(Y_{t_{k+1}}, x). \quad (4.21)$$

(4.20) is nonlinear due to the $\frac{1}{2} \nabla^*_x V Q \nabla_x V$ term. The nonlinearity can be removed by employing a log-transformation as in [97, 98]: $V(t, x) = \log \left[ \frac{\partial V}{\partial t} + b(x)^* \nabla_x V + \frac{1}{2} \text{tr} (Q \nabla_x^2 V) - \frac{1}{2} \nabla^* V Q \nabla_x V + V(t_{k+1}, x) \right]$. This leads to a linear equation:

$$\frac{\partial \log V}{\partial t} + b(x)^* \nabla_x V + \frac{1}{2} \text{tr} (Q \nabla_x^2 V) = 0, \quad t \in [t_k, t_{k+1}],$$

$$\log V(t_{k+1}, x) = \log g(Y_{t_{k+1}}, x).$$
\[- \log \Phi(t, x). \] The expression for the optimal control (4.19) becomes

\[ u(t, x) = \frac{1}{\Phi(t, \hat{X}_t)} Q \nabla_x \Phi(t, \hat{X}_t), \] (4.22)

where \( \Phi \) satisfies

\[
\begin{align*}
\frac{\partial \Phi}{\partial t} + b(x)^* \nabla_x \Phi + \frac{1}{2} \text{tr} \left( Q \nabla_x^2 \Phi \right) &= \frac{\partial \Phi}{\partial t} + \mathcal{L} \Phi = 0, \quad t \in [t_k, t_{k+1}], \\
\Phi(t_{k+1}, x) &= e^{-g(Y_{t_{k+1}, x})}.
\end{align*}
\] (4.23)

(4.23) is a linear second order PDE. By the Feynman-Kac formula (see, for example, Theorem 4.2 of [44]), the solution to (4.23) can be represented as

\[
\Phi(t, x) = \mathbb{E}_{t, x} \left[ e^{-g(Y_{t_{k+1}, \eta_{t, x}^{t_{k+1}, x}})} \right],
\] (4.24)

where \( \mathbb{E}_{t, x} \) is the expectation with respect to the sample paths generated by the uncontrolled diffusion equation, that is, the probability measure induced by a process \( \eta^{t, x} \) evolving according to

\[
\begin{align*}
d\eta_{s}^{t, x} &= b(\eta_{s}^{t, x}) ds + \sigma(\eta_{s}^{t, x}) d\tilde{W}_s, \quad s \in [t, t_{k+1}], \\
\eta_{t}^{t, x} &= x,
\end{align*}
\] (4.25)

where \( \tilde{W} \) is a standard Brownian motion.

**Remark 4.5.1** The signal dynamics is given by (4.14a). Instead of evolving the particles according to (4.14a), they are evolved according to (4.15) with the optimal control (4.22). However, the optimal control involves the expectation in (4.24), which can be computed using a process \( \eta \) that evolves according to a SDE associated with the generator \( \mathcal{L} \) in (4.23). The SDE is (4.25), which is the same as the original the signal dynamics (4.14a).

For the optimal control (4.22), the gradient of (4.24) is required. In [57], the gradient is obtained using the Clark-Ocone formula in Malliavin calculus [99]. Here, we take advantage of the additive nature of the noise in the Lorenz '96 system of interest. The diffusion coefficient \( \sigma_x \) in (4.6a) is independent of the state \( X \), so in the corresponding optimal control problem, \( \sigma \) in (4.15) and \( Q \) in (4.16) are constant matrices. Let \( \Phi_x := \nabla_x \Phi \). Taking the gradient
of (4.23),
\[
\frac{\partial \Phi_x}{\partial t} + \mathcal{L} \Phi_x + (\nabla_x b(x))^* \Phi_x = 0, \quad t \in [t_k, t_{k+1}], \tag{4.26}
\]
\[
\Phi_x(t_{k+1}, x) = -e^{-g(Y_{t_{k+1}, x})} \nabla_x g(Y_{t_{k+1}, x}).
\]

Using the Feynman-Kac formula,
\[
\Phi_x(t, x) = -\mathbb{E}_{t, x} \left[ e^{-g(Y_{t_{k+1}, \eta^t_{t_{k+1}}})} e^{\int_{t_k}^{t_{k+1}} (\nabla_x b(s))^* ds} \nabla_x g(Y_{t_{k+1}, \eta^t_{t_{k+1}}}) \right], \tag{4.27}
\]
where \(\mathbb{E}_{t, x}\) is expectation with respect to the sample paths \(\eta\) generated by (4.25).

The optimal control using (4.22), (4.24) and (4.27) is
\[
u(t, x) = -Q \mathbb{E}_{t, x} \left[ \hat{w}_{t_{k+1}}(t, x, Y_{t_{k+1}}, \eta^t_{t_{k+1}}) e^{\int_{t_k}^{t_{k+1}} (\nabla_x b(s))^* ds} \nabla_x g(Y_{t_{k+1}, \eta^t_{t_{k+1}}}) \right], \tag{4.28}
\]
where
\[
\hat{w}_{t_{k+1}}(t, x, Y_{t_{k+1}}, \eta^t_{t_{k+1}}) := \frac{e^{-g(Y_{t_{k+1}, \eta^t_{t_{k+1}}})}}{\mathbb{E}_{t, x} \left[ e^{-g(Y_{t_{k+1}, \eta^t_{t_{k+1}}})} \right]}. \tag{4.29}
\]

For particle \(i\) that evolves according to (4.15), the optimal control is
\[
u^i_t = -Q \mathbb{E}_{t, \hat{X}^i_t} \left[ \hat{w}_{t_{k+1}}(t, \hat{X}^i_t, Y_{t_{k+1}}, \eta^t_{t_{k+1}}) e^{\int_{t_k}^{t_{k+1}} (\nabla_x b(s))^* ds} \nabla_x g(Y_{t_{k+1}, \eta^t_{t_{k+1}}}) \right], \tag{4.30}
\]
where
\[
\hat{w}_{t_{k+1}}(t, \hat{X}^i_t, Y_{t_{k+1}}, \eta^t_{t_{k+1}}) := \frac{e^{-g(Y_{t_{k+1}, \eta^t_{t_{k+1}}})}}{\mathbb{E}_{t, \hat{X}^i_t} \left[ e^{-g(Y_{t_{k+1}, \eta^t_{t_{k+1}}})} \right]}. \tag{4.31}
\]

For each \(i\), the optimal control can be approximated using another sample of particles \(\{\eta^t_{t, \hat{X}^i_t}\}_{j=1}^{N'}\), all starting at the position \(\hat{X}^i_t\) of particle \(i\) at time \(t \in [t_k, t_{k+1}]\), as
\[ u^i_t \approx -Q \sum_{j=1}^{N'_s} \hat{w}^{i,j}_{t_{k+1}}(t, \hat{X}^i_t, Y_{t_{k+1}}, \{\eta^{(t,X^i_t)}_{j'}, j' \}_{j' = 1}^{N'_s}) \]

\[ \times e^{\int_{t_k}^{t_{k+1}} \left( \nabla_x b(\eta^{(t,X^i_t)})^{j'} \right)^* \nabla_x g(Y_{t_{k+1}}, \eta^{(t,X^i_t)})} \]

\[ \hat{w}^{i,j}_{t_{k+1}}(t, \hat{X}^i_t, Y_{t_{k+1}}, \{\eta^{(t,X^i_t)}_{j'}, j' \}_{j' = 1}^{N'_s}) := \frac{e^{-g(Y_{t_{k+1}}, \eta^{(t,X^i_t)})}}{\sum_{j'=1}^{N'_s} e^{-g(Y_{t_{k+1}}, \eta^{(t,X^i_t)})}}. \]

\( \hat{w}^{i,j} \) can be interpreted as the weight of the path \( \{\eta^{(t,X^i_t)}_{s}; s \in [t, t_{k+1}]\} \) in determining the optimal control, based on how well its final location agrees with observation \( Y_{t_{k+1}} \).

The optimal control solution (4.28) is the same as the result in [57], when the diffusion coefficient there is state-independent. The perturbation process there then satisfies a linear ordinary differential equation linearized about the trajectory of the uncontrolled diffusion \( \eta \), with identity initial conditions. (4.28) is also the same as the solution to the path integral formulation of the optimal control problem of [100]. The Laplace approximation is used for the discretized transition densities in [100], which requires the noise in the state equation to be small. If applied to the optimal control problem (4.15), (4.16) here, we would require both the signal and observation noises to be small. In the numerical experiments (Section 4.5.1), it is seen that the control of particles are most apparent and work best when both signal and observation noises are small. This is expected, as small sensor noise results in accurate observations, hence particles will be steered towards locations most representative of the truth. Small signal noise along with small sensor noise ensures that the particles still sufficiently obey the true signal dynamics, and not be completely steered based on observations. The optimal control solution (4.28) does not require small noise and applies for the general case. In general, signal and observation noises should be of the same order, to avoid over-penalizing one of the running or terminal costs in minimizing the total cost (4.16). If either the signal or observation noises is small, the running or terminal costs should be scaled accordingly such that both are of the same order in the total cost. If signal noise was larger than sensor noise, then the terminal cost \( g \) in (4.16) would dominate the total cost and control energy is
penalized less in minimizing the cost, resulting in large control being applied on particles. Even though we aim to steer particles towards the best locations indicated by observations, large control is unfavorable, as that means that particles are evolving mostly based on observations instead of the true signal dynamics. Deviation of particles from true dynamics are compensated for by importance sampling when constructing the prior density, described in the next section, but we still would not want particle evolution that is too heavily based on observations.

Updating particle weights

By applying control to the particles, the particle system is deviating from the true signal dynamics. This has to be compensated in the particle weights when constructing the posterior.

Importance sampling is a technique for approximating integrals with respect to one probability distribution using a sample from another. Let $p$ be the target distribution of interest over space $\mathbb{X}$ and $q \gg p$ ($q$ is absolutely continuous with respect to $p$) be the distribution from which sampling is done ($q$ is also called the proposal distribution). Denote by $\mathbb{E}_p[\cdot]$ and $\mathbb{E}_q[\cdot]$ the expectation with respect to the distributions $p$ and $q$, respectively. For any integrable function $\varphi : \mathbb{X} \to \mathbb{R}$, we have

$$
\mathbb{E}_p [\varphi(X)] = \int_{\mathbb{X}} \varphi(x)p(dx) = \int_{\mathbb{X}} \varphi(x)\frac{dp}{dq}(x)q(dx)
= \int_{\mathbb{X}} \varphi(x)w(x)q(dx) = \mathbb{E}_q [w(X)\varphi(X)] ,
$$

where $w \overset{\text{def}}{=} \frac{dp}{dq}$. A collection $\{x^i\}_{i=1}^N$ of $N$ particles can be sampled from $q$ and the particles can be weighted according to $w^i \propto \frac{dp}{dq}(x^i)$ to represent the target distribution $p$ i.e. $p(x) \approx \sum_{i=1}^N w^i \delta_{x^i}(x)$. The weights $w^i$ are normalized such that $\sum w^i = 1$.

We use the principle of importance sampling to determine the particle weight update rule. Note that if the particles were evolved according to the system dynamics (4.14a), then the target and proposal distributions are the same, so weights are updated at each $t_k$ as (see Section 2.6.1 for discrete
observations weights update)

\[ w_{t_{k+1}}^i \propto \exp \left( -g(Y_{t_{k+1}}, X_{t_{k+1}}^i) \right) w_{t_k}^i. \] (4.32)

However, since the particles are evolved with control according to (4.15), the weights at observation times should be updated according to

\[ w_{t_{k+1}}^i \propto \exp \left( -g(Y_{t_{k+1}}, \hat{X}_{t_{k+1}}^i) \right) \frac{d\mu_i}{d\hat{\mu}_i}(t_{k+1}, \hat{X}_{t_k}^i) w_{t_k}^i, \] (4.33)

where \( \frac{d\mu_i}{d\hat{\mu}_i}(\hat{X}^i) \) is the Radon-Nikodym derivative of:

- \( \mu_i \), the measure on the path space \( C([t_k, t_{k+1}], \mathbb{R}^m) \) generated by a process that evolves according to the signal dynamics (4.14a) in \( [t_k, t_{k+1}] \), with starting point \( (t_k, X_{t_k}^i) \),

with respect to

- \( \hat{\mu}_i \), the measure generated by the process that evolves according to the controlled dynamics (4.15), with starting point \( (t_k, \hat{X}_{t_k}^i) \).

According to (4.19), we have \( u(t, \hat{X}_t^i) = -\sigma^* \nabla_x V(t, \hat{X}_t^i) \). Let \( u = \sigma v \), where \( v(t, \hat{X}_t^i) := -\sigma^* \nabla_x V(t, \hat{X}_t^i) \). Then, the particle evolution equation (4.15) becomes

\[ d\hat{X}_t^i = b(\hat{X}_t^i)dt + \sigma \left( dW_t + v(t, \hat{X}_t^i)dt \right). \] (4.34)

Using Girsanov’s theorem to perform a measure change that makes \( B := W + \int v dt \), a Brownian motion under the new measure, we obtain

\[ \frac{d\mu_i}{d\hat{\mu}_i}(t_{k+1}, \hat{X}^i) = \exp \left( -\int_{t_k}^{t_{k+1}} v(s, \hat{X}_s^i)^* dW_s - \frac{1}{2} \int_{t_k}^{t_{k+1}} v(s, \hat{X}_s^i)^* v(s, \hat{X}_s^i) ds \right). \] (4.35)

**Remark 4.5.2** We would like to comment on the optimality of the posterior density that is constructed. The posterior density corresponding to the optimal control problem (4.15), (4.16) is optimal in the sense that, for particles starting from a common location, the weight variance of those particles is minimized. In fact, the variance is zero if those particles have equal weights at the starting time. This is proved in Proposition 3.1 of [57], which we reproduce here for completeness.
Let \( V(s, \hat{X}_s) = -\log \Phi(s, \hat{X}_s) \), a stochastic version of the value function, where \( \hat{X}_s \) is the process governed by (4.34). Let \( \sigma v_s \) be the optimal control as before:

\[
v_s = \sigma^* \nabla V(s, \hat{X}_s) = \frac{1}{\Phi(s, \hat{X}_s)} \sigma^* \nabla \Phi(s, \hat{X}_s).
\]

By Itô’s lemma,

\[
dV(s, \hat{X}_s) = -\frac{1}{\Phi(s, \hat{X}_s)} \left[ \partial \Phi(s, \hat{X}_s) ds + \nabla \Phi(s, \hat{X}_s) b(\hat{X}_s) + \frac{1}{2} tr \left( \sigma \sigma^* \nabla^2 \Phi(s, \hat{X}_s) \right) \right] ds
\]

The terms in the square brackets on the first line of the right hand side is zero, by (4.23). Substituting \( v_s = \frac{1}{\Phi(s, \hat{X}_s)} \sigma^* \nabla \Phi(s, \hat{X}_s) \), we have

\[
dV(s, \hat{X}_s) = -v^*_s dW_s - \frac{1}{2} v^*_s v_s ds.
\] (4.36)

If \( \hat{X} \) starts at a fixed \( x^* \in \mathbb{R}^m \) at time \( t_k \), then \( V(t_k, \hat{X}_{t_k}^x) = -\log \Phi(t_k, x) \) and at time \( t_{k+1} \), \( V(t_{k+1}, \hat{X}_{t_{k+1}}^{x^*}) = g(Y_{t_{k+1}}, \hat{X}_{t_{k+1}}^{x^*}) \). Then, integrating (4.36), exponentiating, and rearranging, we have

\[
\Phi(t_k, x^*) = \exp \left( -g(Y_{t_{k+1}}, \hat{X}_{t_{k+1}}^{x^*}) \right) \exp \left( -\int_{t_k}^{t_{k+1}} v^*_s dW_s - \frac{1}{2} \int_{t_k}^{t_{k+1}} v^*_s v_s ds \right)
\]

\[
= \exp \left( -g(Y_{t_{k+1}}, \hat{X}_{t_{k+1}}^{x^*}) \right) \frac{d\mu(t_{k+1}, \hat{X}_{t_{k+1}}^{x^*})}{d\mu(t_k, \hat{X}_{t_k}^{x^*})}.
\] (4.37)

Therefore, for particles starting at \( x^* \) at time \( t_k \), the particle weight update (4.33) is

\[
w_{t_{k+1}}^i \propto \Phi(t_k, x^*) w_{t_k}^i.
\]

Then, for particles starting at the same location \( x^* \) at time \( t_k \), the variance of their weights is

\[
\text{Var}[w_{t_{k+1}}] = (C \Phi(t_k, x^*))^2 \text{Var}[w_{t_k}]
\]
since $\Phi(t_k, x^*)$ is a deterministic quantity for a given $x^* \in \mathbb{R}^m$. All particles at the same location have equal weights, hence all particles that start at $x^*$ at time $t_k$ have equal weights at time $t_k$, i.e. $\text{Var}[w_{i_k}] = 0$, and, consequently, $\text{Var}[w_{i_{k+1}}] = 0$.

Another particle filtering method using a nudging term to steer particles towards observations has been considered by van Leeuwen [52]. In the interval between available observations $[t_k, t_{k+1}]$, particles are steered by a time exponential function proportional to the signal noise covariance and the distance of the particle locations from the next observation. The method in [52] also includes a procedure to make particle weights almost equal immediately before the time step of the next available observation, which minimizes the weights variance for the entire sample.

The particle filtering procedure

The particle filtering procedure that incorporates the above optimal control is summarized here:

- Approximate $\pi_{t_k}$ with the $\sum_{i=1}^{N} w_{i_k}^i \delta_{\hat{X}_{i_k}^i}$, where $\sum w_{i_k}^i = 1$.
- Evolve each particle in $[t_k, t_{k+1}]$ according to

$$d\hat{X}_i^i(t) = b(\hat{X}_i^i)dt + \sigma \left( dW_t + \sigma v(t, \hat{X}_i^i)dt \right),$$

where $v(t, \hat{X}_i^i) = \left( \Phi(t, \hat{X}_i^i) \right)^{-1} \sigma^* \nabla_x \Phi(t, \hat{X}_i^i)$. $
\Phi(t, \hat{X}_i^i)$ and $\nabla_x \Phi(t, \hat{X}_i^i)$ requires Monte Carlo simulations of

$$d\eta_{t_k, \hat{X}_i^i}^s = b(\eta_{t_k, \hat{X}_i^i})ds + \sigma d\tilde{W}, \text{ for } s \in (t, t_{k+1}],$$

$$\eta_{t, \hat{X}_i^i} = \hat{X}_i^i.$$

$\Phi(t, \hat{X}_i^i)$ and $\nabla_x \Phi(t, \hat{X}_i^i)$ are computed using (4.24) and (4.27), respectively.

- Update weights in between observation times by

$$w_s^i \propto \frac{d\mu_i}{d\mu_i}(s, \hat{X}_i^i)w_{t_k}^i, \text{ for } s \in (t_k, t_{k+1})$$
to construct density $\pi_s \approx \sum_{i=1}^{N} w_i^s \delta_{X_i^s}$.

- Update weights at observation times $t_{k+1}$ according to (4.33) using (4.35) to construct density $\pi_{t_{k+1}} \approx \sum_{i=1}^{N} w_{t_{k+1}}^i \delta_{\hat{X}_{t_{k+1}}^i}$.

Linear Control Strategy

Evaluation of the control strategy for a nonlinear signal based on (4.22) requires, for each particle, directly solving a diffusion process that is exactly like the original signal, on top of solving for the dynamics of the particle. This can be computationally overwhelming. Here, we derive the optimal control explicitly for linear systems. This is presented in Section 3.5 of [57], but is included here for completeness.

Consider the linear signal and observation:

$$dX_t = AX_t dt + \sigma dW_t, \quad (4.38a)$$
$$Y_{tk} = HX_{tk} + B_{tk} \quad (4.38b)$$

We are once again concerned with $t \in [t_k, t_{k+1}]$, between the discrete observations. Let particle location be $x$ at $t$. Let $Q = \sigma \sigma^*$, $R = \text{cov}(B_{tk})$, $\mu(x) = e^{A(t_{k+1}-t)}x$, $\Sigma := \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-s)}Q(e^{A(t_{k+1}-s)})^* ds$, $\hat{\Sigma} := (\Sigma^{-1} + H^* R^{-1} H)^{-1}$ and $R^{-1} := R^{-1}(I - H\hat{\Sigma}H^* R^{-1})$. Then the particle has the distribution $\mathcal{N}(\mu(x), \Sigma)$ at time $t_{k+1}$ and

$$\Phi(t, x) = \mathbb{E}_{t,x} \left[ \exp \left( -\frac{1}{2} (Y_{t_{k+1}} - H\hat{X}_{t_{k+1}}^{t,x})^* R^{-1} (Y_{t_{k+1}} - H\hat{X}_{t_{k+1}}^{t,x}) \right) \right],$$

where the expectation $\mathbb{E}_{t,x}$ is taken with respect to $\mathcal{N}(\mu(x), \Sigma)$, the density of $\hat{X}_{t_{k+1}}^{t,x}$. By completing squares for the integral of the expectation and evaluating the integral,

$$\Phi(t, x) = \sqrt{\frac{\hat{\Sigma}}{\Sigma}} \exp \left( -\frac{1}{2} (Y_{t_{k+1}} - H\mu(x))^* R^{-1} (Y_{t_{k+1}} - H\mu(x)) \right).$$

Taking the gradient with respect to $x$,

$$\nabla_x \Phi(t, x) \quad (107)$$
\[
= (e^{A(t_{k+1}-t)})^*H^*R^{-1}\mathbb{E}[(Y_{t_{k+1}} - Hx) \exp(-\frac{1}{2}(Y_{t_{k+1}} - Hx)^*R^{-1}(Y_{t_{k+1}} - Hx))]
\]
\[
= (e^{A(t_{k+1}-t)})^*H^*R^{-1}\sqrt{\frac{\hat{\Sigma}}{\Sigma}} \exp(-\frac{1}{2}(y - H\mu)^*R^{-1}(y - H\mu)) \times \hat{\mathbb{E}}[(y - Hx)]
\]
\[
= \Phi(t, x)(e^{A(T-t)})^*H^*R^{-1}\hat{\mathbb{E}}[(y - Hx)],
\]
where \(\hat{\mathbb{E}}\) is w.r.t \(\mathcal{N}(\hat{\mu}, \hat{\Sigma})\). Hence

\[
Q \frac{D\Phi}{D\Phi} = Q(e^{A(T-t)})^*H^*R^{-1}[(y - H\hat{\mu})]
\]
\[
= Q(e^{A(T-t)})^*[I + H^*R^{-1}H\Sigma]^{-1}H^*R^{-1}[(y - H\hat{\mu})]
\]

So, for linear systems, the control \(u(t, x)\) is obtained as follows:

\[
u(t, x) = Q(e^{A(t_{k+1}-t)})^*[I + H^*R^{-1}H\Sigma]^{-1}H^*R^{-1}[(Y_{t_{k+1}} - H\mu)]
\] (4.39)

where \(\mu := e^{A(t_{k+1}-t)}x\), and \(\Sigma := \int_t^{t_{k+1}} e^{A(t_{k+1}-s)}Q(e^{A(t_{k+1}-s)})^*ds\) are the mean and variance of the system (4.38) at time \(t_{k+1}\) when it starts at time \(t\) at \(x\).

### 4.5.1 Numerical experiments

Before presenting the results from the Lorenz '96 multiscale model, results from experiments on the 3-dimensional, single scale Lorenz '63 model are discussed, to illustrate just the optimal control aspect.

**Lorenz '63 testbed**

The Lorenz '63 system [101] is given by

\[
d \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ -x_t z_t \\ x_t y_t \end{bmatrix} dt + \sigma_x dW_t,
\] (4.40)
where $\sigma = 10$, $\rho = 28$ and $\beta = \frac{8}{3}$. Here, we have added standard Brownian motion noise $W$ to the system, with $\sigma_x = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}$. Observation is the full state with standard Gaussian noise, i.e.

$$Y_{tk} = H \begin{bmatrix} x_{tk} \\ y_{tk} \\ z_{tk} \end{bmatrix} + B_{tk},$$

where $H = I_{3 \times 3}$ and $B_{tk} \sim \mathcal{N}(0, I_{3 \times 3})$ for all $k = 1, 2, \ldots$ Simulation of (4.40) is taken as the truth, with observations recorded every 0.4 time units.

The deterministic system is chaotic, with one positive Lyapunov exponent $\lambda = 0.9065$. This means that a small initial perturbation $\varepsilon_0$ grows as $\varepsilon_t = \varepsilon_0 e^{\lambda t}$. Setting $\varepsilon_{\tau_d} = 2\varepsilon_0$ and solving for $\tau_d$, we see that estimation error will double every $\tau_d = 0.76$ time units. Therefore, in the experiments shown in this section, the time between observations is slightly more than half the error doubling time.

Numerical integration is performed using a central difference scheme for the deterministic part and Euler-Maruyama scheme for the stochastic part, with a constant timestep of $\delta t = 2^{-8}$. Four filtering methods are implemented:

- standard sequential importance sampling (SIS) particle filter (PF),
- SIS PF with suboptimal linear control (PF$_{c,\text{sub}}$),
- SIS PF with optimal nonlinear control (PF$_{c(N)\prime}$), where $N'_{\text{s}}$ indicates the sample size used to approximate the expected value in (4.30) for the control of each particle, and
- ensemble Kalman filter (enKF).

The suboptimal linear control is computed using the linear coefficient in (4.40) as the $A$ matrix in (4.39). The optimal nonlinear control is computed using (4.30). Each filter is implemented with 10 particles.

The filtering results for one experiment are shown in Figures 4.9, 4.10, 4.11 and 4.12. Solid blue plots are the true states. In Figures (a), broken red plots are the filtered states with $\pm 1 \times$ sample standard deviation error bars. Green error bars are $\pm 2 \times$ sample standard deviation. In Figures (b), red
crosses are particle trajectories and solid greed crosses indicate observations. Observation noise is of the same order as signal noise, but noise amplitudes are small compared to the signal, hence observations are quite accurate. The error plot is the time varying normalized root-mean-squared error (RMSE)

\[
\epsilon_t \overset{\text{def}}{=} \frac{\| X_{\text{true}}^t - X_{\text{filter}}^t \|}{\| X_{\text{true}}^t \|}, \tag{4.41}
\]

where \( X_{\text{filter}} \) is computed using the sample mean of the respective filters.

![Figure 4.9: Standard SIS PF](image)

![Figure 4.10: PF_{c,sub}: SIS PF with suboptimal linear control](image)
Based on Figures (b) of the particle filters, the linear suboptimal and nonlinear optimal control solution is able to steer particles as intended, towards locations indicated by observations. In the experiment presented, by comparing Figures 4.9(b) and 4.10(b), it is not obvious that the suboptimal solution does better for the particles than just the signal dynamics in the interval prior to the first observation. The effects of the suboptimal control is observed more in the following intervals, where the sample variance is smaller about the true trajectory for the controlled particles. The benefit of the the suboptimal control is also observed later in the simulation, after the observation at time 1.6, where the SIS PF fails to track several transitions in Figure 4.9(a), but the PF_{c,sub} does. The sample variance for the PF_{c,sub} does grow at a slightly higher rate compared to that of the PF_{c} further away from the last observation.
At the observation timesteps, particles weights in the PFs are updated using observations and the particles resampled to eliminate particles with low weights and replicate those with large weights. For the enKF, particle locations are corrected based on their respective distances from the observation (innovation). Magnitude of the correction is proportional to the error covariance and inverse to the observation noise covariance (Kalman gain). In between observations, particles in the enKF sample evolve according to the original signal dynamics, so particle trajectories are expected to deviate from the truth as time moves further away from the last observation correction. This is observed especially in the interval after the observations at times 0.4, 1.2, 2.8, and 3.2 in Figure 4.12(b).

40 independent experiments are performed using each filter. For each filter, the RMSE (4.41) integrated over time, averaged over the 40 experiments, are shown in Table 4.3. Let \( \bar{e} \) denote the average RMSEs over 40 experiments. The 99% confidence intervals for the differences in \( \bar{e} \) between the filters are shown in Table 4.4.

<table>
<thead>
<tr>
<th>Filter</th>
<th>PF</th>
<th>PF(_{c,\text{sub}})</th>
<th>PF(_{c,(1)})</th>
<th>PF(_{c,(20)})</th>
<th>PF(_{c,(40)})</th>
<th>enKF</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
<td>3.201</td>
<td>1.174</td>
<td>1.574</td>
<td>1.074</td>
<td>0.935</td>
<td>1.166</td>
</tr>
<tr>
<td>time</td>
<td>0.8 s</td>
<td>5.3</td>
<td>20.3 s</td>
<td>195.5 s</td>
<td>304.3</td>
<td>0.4 s</td>
</tr>
</tbody>
</table>

Table 4.3: RMSE integrated over time, average over 50 experiments, and typical computation time for 1 experiment

<table>
<thead>
<tr>
<th>( \bar{e}<em>{PF} - \bar{e}</em>{PF_{c,\text{sub}}} )</th>
<th>( \bar{e}<em>{enKF} - \bar{e}</em>{PF_{c,\text{sub}}} )</th>
<th>( \bar{e}<em>{PF</em>{c,\text{sub}}} - \bar{e}<em>{PF</em>{c,(20)}} )</th>
<th>( \bar{e}<em>{enKF} - \bar{e}</em>{PF_{c,(20)}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1.140, 2.914]</td>
<td>[-0.258, 0.343]</td>
<td>[-0.077, 0.4264]</td>
<td>[0.139, 0.505]</td>
</tr>
</tbody>
</table>

Table 4.4: 99% confidence interval of difference between average RMSEs of different filters

<table>
<thead>
<tr>
<th>( \bar{e}<em>{PF</em>{c,(1)}} - \bar{e}<em>{PF</em>{c,(20)}} )</th>
<th>( \bar{e}<em>{PF</em>{c,(20)}} - \bar{e}<em>{PF</em>{c,(40)}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.192, 0.824]</td>
<td>[-0.072, 0.0557]</td>
</tr>
</tbody>
</table>

Table 4.5: 99% confidence interval of difference between average RMSEs of PF\(_{c,(N_s')}\) with different \( N_s' \)

At a fixed sample size, nudging particles using the linear sub-optimal or
nonlinear optimal control significantly improves the PF. The choice of incurring the cost of computing the control or increasing sample size to improve the PF estimate would depend on the problem. In the next experiments in which the state space is high-dimensional, we find that we can compute the optimal control at lower cost than the required increase in the sample size to improve the filter. The PF$_{c,sub}$ is similar to the enKF, but has higher computational cost. The PF$_{c,(N)}$ is slightly better than the PF$_{c,sub}$ and enKF, but with significantly higher computational cost. Computation of the optimal nonlinear control here is done by Monte Carlo simulation to approximate the expectation in (4.28) for each particle. This can be computationally overwhelming, especially in high dimensions. Even in 3 dimensions, computing the optimal control using a finite sample size approximation is significantly more computationally intensive than using a linear approximation, as seen in the last row of Table 4.3. In this case, the linear approximation is done using the linear part of the (4.40) and the resulting filter is as good as the enKF. However, it may not always be the case that such linear approximation is sufficiently good. In the Lorenz ’63 model, there is only quadratic nonlinearity in two of the three components, so we can expect the linear approximation to do well. In the the next experiments, we see the same direct linear approximation is not appropriate, as the linear part of the signal drift is just the identity matrix.

Lorenz ’96 testbed

We return to the multiscale Lorenz ’96 system (4.6). For the numerical experiments, a timescale separation of $\varepsilon = 0.01$ is used. The fast- and slow-scale couplings are taken as $(h_x, h_z) = (-1, 10)$, while the slow-scale external forcing is set as $F_x = 10$. The true state is taken as the $\mathbb{R}^{36+360}$ vector

$$
\begin{bmatrix}
[X^\varepsilon]^* \\
[ Z^\varepsilon]^*
\end{bmatrix} = \begin{bmatrix}
X^1 & \ldots & X^{36} \\
Z^{1,1} & \ldots & Z^{1,10} & Z^{2,1} & \ldots & Z^{2,10} & Z^{3,10} & \ldots & Z^{36,10}
\end{bmatrix}^*,
$$

simulated according to (4.6). We have a multiscale system with additive noise of the form

$$
dX^\varepsilon_t = b(X^\varepsilon_t, Z^\varepsilon_t)dt + \sigma_x dW_t, \quad X^\varepsilon \in \mathbb{R}^{36},
$$
\[ d\mathbf{Z}_t^\varepsilon = \frac{1}{\varepsilon} f(X_t^\varepsilon, Z_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}} \sigma_z d\mathbf{V}_t, \quad \mathbf{Z}^\varepsilon \in \mathbb{R}^{360}, \]

where \( \mathbf{W} \in \mathbb{R}^{36} \) and \( \mathbf{V} \in \mathbb{R}^{360} \) are independent standard Brownian motions. The slow-scale signal noise covariance \( \sigma_x \) is taken as a sparse square matrix each with 1 on the diagonal and 0.05 on the first two sub- and super-diagonals. The fast-scale noise covariance \( \sigma_z \) is set similarly. Observation is taken as the complete slow-scale component \( X^\varepsilon \), with standard Gaussian noise, recorded at discrete timesteps:

\[
Y_{t_k}^\varepsilon = H \begin{bmatrix} X_{t_k}^\varepsilon \\ Z_{t_k}^\varepsilon \end{bmatrix} + B_{t_k},
\]

where \( B_{t_k} \sim \mathcal{N}(0_{36 \times 1}, I_{36 \times 36}) \) for all \( k = 1, 2, \ldots \) and \( H = I_{36 \times 36} \).

The system (4.6) is first simulated for 100 days from states randomly perturbed from zero to eliminate transient effects, allowing the system to attain its long term behavior. The resulting states are then taken as initial conditions for a 100-day simulation of (4.6). The states from the second 100-day simulation is taken as the “true signal”. Numerical integration is performed using the Runge-Kutta and the Euler-Maruyama schemes for the deterministic and stochastic parts, respectively. Numerical integration timestep is taken as \( \delta t = 0.0005 \), which is equivalent to 3.6 seconds in real time. Observations are recorded every 600 timesteps, equivalent to 1.5 days in real time. The deterministic version of (4.6) with \( F_x = 10 \) has error doubling time \( \tau_d \approx 1.6 \) days [88]. Hence, observations intervals are approximately equal to the error doubling time. The first three components of \( X^\varepsilon \) with the corresponding discrete observations and fast-scale forcings of one realization of the “true signal” are shown in Figure 4.13.

We are interested only in the coarse-grained dynamics \( X \), hence we can apply the homogenization result described in Section 3.1 and use a reduced dimension (36-dimensional) filter in place of the full (36+360)-dimensional filter. A homogenized ensemble Kalman filter (henKF) and homogenized hybrid particle filter (HHPF) are implemented. For the homogenized filters, the signal is the homogenized system

\[
\begin{align*}
dX_t^0 &= \bar{b}(X_t^0)dt + \sigma_x d\mathbf{W}_t, \\
X^0 &\in \mathbb{R}^{36},
\end{align*}
\]

(4.42)
where the fast-scale effects have been properly averaged in $\bar{b}$. Particles are sampled to represent the state of (4.42). Since there is only one time scale with no small parameter $\epsilon$, numerical integration of the homogenized system can be performed using a larger timestep $\Delta t = 0.05$, which is equivalent to 6 hours in real time.

For the homogenized system, $\bar{b}$ is required for particle evolution and $\bar{H}$ for particle weights update. They are approximated using the Heterogeneous Homogenized Multiscale (HMM) method of [85, 86]: Within each interval $\Delta t$, the slow-scale process is fixed, and the fast-scale process is simulated using a timestep $\delta t$ for sufficiently time so as to eliminate transient behavior and then the steady-state trajectory can be averaged over time. Assuming the fast-scale is ergodic [87], the time average of the fast-scale trajectory can be used to approximate spatial averages for $\bar{b}$ and $\bar{H}$. Note that here, $\bar{H} = H$.

We implement the following filters:

- sequential importance sampling particle filter (PF)
- homogenized hybrid particle filter (HHPF)
- homogenized hybrid particle filter with suboptimal linear particle control (HHPF_{c,sub}): the linear control of Section 4.5 is used, with $A$ matrix taken as Jacobian of the linear part of (4.6a), which is $I_{36 \times 36}$
• homogenized hybrid particle filter with optimal nonlinear particle control (HHPF$_c$): nonlinear control using (4.28); if, for each particle, a sample of size $n_s$ is used to approximate the expectation in (4.28), then we denote the corresponding filter by HHPF$_c(n_s)$.

• ensemble Kalman filter (enKF) with no homogenization

• homogenized ensemble Kalman filter (henKF); this is the same as the scheme in the wide timescale separation setting in [50]

Each filter is implemented with 30 particles. Initial condition for the sample of 30 particles are generated in the same way as for the realization of the true signal. For each particle in the HHPF$_c$, a sample of 30 particles is used to approximate the expectation in (4.30) for the optimal control, i.e. we use a HHPF$_c(30)$.

The filtering results for one experiment are show in Figures 4.14, 4.15, 4.16, 4.17, 4.18 and 4.19. Solid blue plots are the true states. In Figures (a), broken red plots with error bars are filter means with $\pm 1 \times$sample standard deviations. Green error bars are $\pm 2 \times$sample standard deviations. The bottom-most plots are the time varying normalized RMSE (4.41). In Figures (b), red crosses are particle trajectories. Green crosses in Figures (b) indicate the observations. Observation noise is of the same order as signal noise, but noise amplitudes are small compared to the signal, hence observations are quite accurate.

![Figure 4.14: PF](image)

(a) $(X^1, X^2, X^3)$

(b) $(X^1, X^2, X^3)$ particles
Figure 4.15: HHPF

Figure 4.16: HHPF_{c,sub}
Comparing Figures 4.15(b) and 4.16(b), the suboptimal solution does not seem to be better for the particles than just the signal dynamics. This is expected, as the linear approximation is just the Jacobian of the linear part.
Figure 4.20: Effective number of particles, $N_{s,\text{eff}}$ at observation times

Figure 4.21: $(X^1, X^2, X^3)$, HHPF with $N_s = 960$

(a) $(X^1, X^2, X^3)$
(b) $(X^1, X^2, X^3)$ particles

Figure 4.22: HHPF$_{c,(1)}$, observations every 72 hours. Particle trajectories still remain close to true signal trajectory due to control of the drift in (4.6a), which does not capture the significant nonlinear effects. Comparison of RMSEs averaged over 40 experiments in Table 4.6 shows that
Figure 4.23: henKF, observations every 72 hours. Particle trajectories deviate from true signal trajectory in between observation times.

The time integrated RMSE is lower for the HHPF_{c,sub}, so the linear sub-optimal control does contribute to constructing a proposal density that is better than one solely based on signal dynamics. However, the corresponding estimate of the true signal location is still poor. In fact, Figures 4.14(a), 4.15(a) and 4.16(a) show that, with sample size of 30, the PF, HHPF_{c,sub} and HHPF are unable to track the true signal over the 100-day interval. Due to the chaotic nature of the system and observations being sparse in time, a large sample size is required for the HHPF. Increasing the sample size is not observed to improve the performances of the PF, HHPF and HHPF_{c,sub} significantly. Even with sample size of 960, the HHPF does not do well (Figure 4.21). Extending the experiment to a 200-day interval does not result in the filters tracking the true state either. Applying the nonlinear optimal control to particles significantly improves the performance of the HHPF, as seen in Figure 4.17(a). The effective sample size is still small at observation times and is often close to 1, as shown in Figure 4.20. (for sample size N_s, effective sample size at time t is approximated as \( N_{s,eff}(t) \approx \frac{1}{\sum_{i=1}^{N_s}(w_i)^2} \) [60]). However, the optimal control is able to ensure that all particle trajectories stay close to the truth in the intervals in between observations and hence the HHPF_{c(30)} is able to track the true signal well.

The henKF and enKF are both able to track the true signal well. The advantage that of the henKF and enKF over the particle filter is that, even though all particles are weighted equally, the particle locations are corrected at each observation time, as indicated by observations. The error plots of the henKF and enKF display more pronounced peaks, due to error growth in
between observation times, which drop at observation times when new observations are assimilated. Deviation of trajectories of the henKF and enKF particles from the true signal trajectory are observed to grow as time progresses further away from the last observation (Figures 4.18(b), 4.19(b)). These error growths are more significant when observations frequency is decreased. Figures 4.23 and 4.22 show the results from the henKF and HHPF_{c,(1)}, respectively, from an experiment in which observations are recorded every 72 hours instead of every 36 hours.

The HHPF_{c,(30)} and henKF are found to perform sufficiently well with $N_s = 30$, which agrees with the estimate of [16] for particle methods, that $N_s$ needs to be at least larger than $\exp\{36^{1/3}\} \approx 27$.

<table>
<thead>
<tr>
<th>Filter</th>
<th>PF</th>
<th>HHPF</th>
<th>enKF</th>
<th>henKF</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE time</td>
<td>19.074</td>
<td>19.186</td>
<td>5.279</td>
<td>5.450</td>
</tr>
<tr>
<td>Filter</td>
<td>HHPF_{c,sub}</td>
<td>HHPF_{c,(1)}</td>
<td>HHPF_{c,(1)}</td>
<td>HHPF_{c,(30)}</td>
</tr>
<tr>
<td>RMSE time</td>
<td>18.558</td>
<td>4.711</td>
<td>4.558</td>
<td>4.652</td>
</tr>
</tbody>
</table>

Table 4.6: RMSE integrated over time and filter computation time, average over 40 experiments

<table>
<thead>
<tr>
<th>Filter</th>
<th>Obs</th>
<th>PF</th>
<th>HHPF</th>
<th>enKF</th>
<th>henKF</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
<td>4.762</td>
<td>18.748</td>
<td>18.7865</td>
<td>4.345</td>
<td>4.501</td>
</tr>
<tr>
<td>Filter</td>
<td>HHPF_{c,sub}</td>
<td>HHPF_{c,(1)}</td>
<td>HHPF_{c,(1)}</td>
<td>HHPF_{c,(30)}</td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>18.175</td>
<td>4.754</td>
<td>4.446</td>
<td>4.468</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.7: RMSE at observation times

40 independent experiments are performed using each filter and the time integral of the normalized RMSE (4.41) of each filter is computed for each experiment. Average of the RMSEs over the 40 experiments are shown in Table 4.6. Table 4.7 shows the RMSE at observation times compared to the observation RMSE.

Let $\bar{e}$ denote the average of the RMSEs over 20 experiments. The 99% confidence intervals for the differences in $\bar{e}$ between the filters are shown in Table 4.8. The optimal control on particles enables the homogenized particle
filter to track the true signal, and based on Table 4.8, the results are comparable to the homogenized and full ensemble Kalman filters. However, as seen in the computational times in Table 4.6, computation of the nonlinear control is expensive, it completely offsets the dimensional reduction advantage of the HHPF. By increasing $N_s$ for the henKF, estimation error can be lowered with lower computational cost than the HHPF$_c$ (30) and enKF. The optimal control of particles has a theoretically sound basis for application in nonlinear problems, as presented in Section 4.5. However, computation of the expectation in (4.28) incurs a high cost. It remains to be studied whether more efficient schemes can be found for this. In the Lorenz '96 experiments, the HHPF$_c$ is found to work well when the expectation in (4.28) is approximated with just one realization of the process $\eta$ in (4.25), i.e. when the filter HHPF$_c$ (1) is used. The corresponding error and computation time are in Tables 4.6 and 4.8. In this case, the HHPF$_c$ (1) requires computational time comparable to the enKF, but the henKF is still the most efficient.

Note that the metric for comparison of the filters here (4.41) is the RMSE of the sample mean of each filter. This metric is chosen because, in the experiments, the signal and observation noise amplitudes are relatively small compared to variations in the signal, hence it is reasonable to expect the filter means to get close to the location of the true signal at fixed time instances as long as the behavior of the signal is not too “bad”. This is observed in the 99-dimensional experiment presented in Section 4.4, in which observations are recorded at higher frequency (at every slow-scale integration timestep). When steered based on good observations, most of the sample do not stray

<table>
<thead>
<tr>
<th>$\bar{e}<em>{\text{HHPF}} - \bar{e}</em>{\text{HHPF}_c}$</th>
<th>$\bar{e}_{\text{HHPF}<em>c(30)} - \bar{e}</em>{\text{HHPF}_c(1)}$</th>
<th>$\bar{e}<em>{\text{henKF}} - \bar{e}</em>{\text{enKF}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.019, 1.239]</td>
<td>[0.028, 0.160]</td>
<td>[0.042, 0.301]</td>
</tr>
<tr>
<td>$\bar{e}<em>{\text{henKF}} - \bar{e}</em>{\text{HHPF}_c(30)}$</td>
<td>$\bar{e}<em>{\text{enKF}} - \bar{e}</em>{\text{HHPF}_c(30)}$</td>
<td></td>
</tr>
<tr>
<td>[0.674, 0.923]</td>
<td>[0.489, 0.766]</td>
<td></td>
</tr>
<tr>
<td>$\bar{e}<em>{\text{henKF}} - \bar{e}</em>{\text{HHPF}_c(1)}$</td>
<td>$\bar{e}<em>{\text{enKF}} - \bar{e}</em>{\text{HHPF}_c(1)}$</td>
<td></td>
</tr>
<tr>
<td>[0.776, 1.009]</td>
<td>[0.590, 0.852]</td>
<td></td>
</tr>
<tr>
<td>$\bar{e}<em>{\text{henKF}} - \bar{e}</em>{\text{HHPF}_c^{2N_s}}$</td>
<td>$\bar{e}<em>{\text{enKF}} - \bar{e}</em>{\text{HHPF}_c^{2N_s}(1)}$</td>
<td></td>
</tr>
<tr>
<td>[0.604, 0.875]</td>
<td>[0.425, 0.711]</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.8: 99% confidence interval of difference between average RMSEs of different filters
too far away from the truth at all times, so the sample mean provides a good
estimate of the true signal. The downside to this is that particles in the
sample is not distributed well, tending to be clustered about a mean, albeit
close to the truth. This is reflected in the effective sample size being small
and often close to 1 (Figure 4.20). The accurate observations also results in
most weight being concentrated on one particle that is closest to the truth,
and this is seen in all the HHPF variants. When the signal is not too noisy
and observations are accurate, this is not too much of an issue when using
the HHPFₖ, as the sample mean provides a sufficiently accurate estimate of
the true signal. However, the truth can fall outside ±2×sample standard
deviations from the mean, and higher order moments cannot be estimated.
If we are only interested in estimating the true signal, then the sample mean
of the HHPFₖ provides a sufficiently good estimate, and since each particle
is steered towards locations indicated by observations independently of the
rest, we can expect a small sample size to be able to provide a good estimate.
Indeed, Figure 4.24 shows the estimate from one experiment using a sample
size of 2. The time-integrated RMSE is 4.823 and the computation time is
45 s, which is comparable to the henKF. The 99% confidence intervals for
the difference in \( \bar{\epsilon} \) from the henKF and enKF are in the last row of Table
4.8. In further experiments, we find that as the signal and observation noise
amplitudes increase, the accuracy of the HHPFₖ sample mean decreases as
expected, since observations are no longer accurate so particles may not be
steered sufficiently close to the truth. Results of the HHPFₖ,(1) and henKF
when signal and observation noise covariances are increased by a factor of
8 are shown in Figures 4.25 and 4.27, respectively. Tables 4.9 and 4.10
shows the corresponding time averaged RMSEs and RMSEs at observation
times for experiments with noise covariances increased by factors of 4 and 8,
respectively.

Although we have accurate estimate based on sample mean from a con-
centrated sample, as in the discussion in Section 2.6.3, we always desire a
diverse sample in order to be able to properly capture the distribution of the
true signal. A different form of optimal proposal density would be required,
for example, similar to the scheme in [52, 53] that makes all particle weights
almost equal at the observation step. However, when observations are ac-
curate, the particles may again be concentrated. Different schemes remain
to be studied, for example, schemes that utilize the existence of invariant
Figure 4.24: HHPF_{c,(1)}, N_s = 2, observations every 36 hours. Time-integrated RMSE is 4.823 and computation time is 45 s.

Figure 4.25: HHPF_{c,(1)}, observations every 36 hours with signal and observation noise covariances increased by factor of 8.
Figure 4.26: HHPF\textsubscript{c,(1)}, \(N_s = 2\), observations every 36 hours with signal and observation noise covariances increased by factor of 8.

Figure 4.27: henKF, observations every 36 hours with signal and observation noise covariances increased by factor of 8.

<table>
<thead>
<tr>
<th>Filter</th>
<th>henKF</th>
<th>HHPF\textsubscript{c,(1)}</th>
<th>HHPF\textsuperscript{2N_s}\textsubscript{c,(1)}</th>
<th>Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
<td>10.263</td>
<td>8.810</td>
<td>9.024</td>
<td></td>
</tr>
<tr>
<td>RMSE\textsubscript{obs times}</td>
<td>8.699</td>
<td>8.385</td>
<td>8.892</td>
<td>8.878</td>
</tr>
</tbody>
</table>

Table 4.9: RMSE integrated over time and RMSE at observation times when signal and sensor noise covariances increase by factor of 4

<table>
<thead>
<tr>
<th>Filter</th>
<th>henKF</th>
<th>HHPF\textsubscript{c,(1)}</th>
<th>HHPF\textsuperscript{2N_s}\textsubscript{c,(1)}</th>
<th>Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
<td>12.838</td>
<td>12.537</td>
<td>12.832</td>
<td></td>
</tr>
<tr>
<td>RMSE\textsubscript{obs times}</td>
<td>11.074</td>
<td>11.643</td>
<td>12.433</td>
<td>12.038</td>
</tr>
</tbody>
</table>

Table 4.10: RMSE integrated over time and RMSE at observation times when signal and sensor noise covariances increase by factor of 8

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CHAPTER 5
PARAMETER ESTIMATION

In the filtering framework considered so far, models describing signal dynamics are assumed to be completely known. The parameter estimation problem considers the case where models contain parameters with unknown values and need to be estimated along with the signal, using available observations (see Section 1.3 for relevance in geophysical sciences and relevant work). The work here employs the maximum likelihood estimation (MLE) approach. Many methods exist for implementing maximum likelihood estimation in practice, MC methods being a common choice, for example the Metropolis-Hastings algorithm mentioned in the beginning of Chapter 2 (see, for example [102], [103] and references therein). Using the Metropolis-Hastings algorithm as an example, we see the computational task and complexity of performing MLE using MC methods. When the signal is sampled from a desired distribution that depends on model parameters and the parameters are uncertain, the parameter values need to be sampled as well. The bootstrap approach would be to sample different sets of parameter values and simulating signal samples each set of parameter values, calculating the corresponding likelihood function, and identifying the maximum/maxima. The expectation-maximization (EM) algorithm is a procedure for computing the ML parameter value(s) iteratively, introduced in [25]. The EM algorithm does not require sampling parameter values; instead iterations are performed to obtain a sequence of parameter values that converges to the true values. The convergence is fast enough that the number of iterations can be less the number of times parameter values need to be sampled via bootstrapping, but each iteration requires a smoothing procedure to estimate the entire time series of the signal using available observations, which is computationally intensive in the nonlinear setting, where an explicit smoother solution cannot be obtained. The EM algorithm is described in Section 5.5. When using particle methods to approximate the nonlinear smoother, the dimensionality issues faced in filtering
arise. Hence, in Chapter 6, we study the MLE problem in the multiscale setting, with the goal of dimensional reduction as for the filtering problem of Chapter 3.

5.1 Maximum Likelihood Estimation

We describe MLE in the diffusion setting, which is the setting for our work in Chapter 6, where observations generated from a hidden signal that is dependent on unknown or uncertain parameters. In the general setting, observations are a subset of the complete data set that is generated from a distribution with unknown or uncertain parameters. In the case of diffusion processes, this distribution is the solution of the Fokker-Planck equation driven by the generator of the signal process. In our work, we consider only the parametric estimation problem, where the form of the distribution (or diffusion) is known, only the parameters are unknown. We assume that the diffusion coefficient $\sigma$ is completely known, and that observations are available continuously in time.

Consider a data set that is dependent on an unknown deterministic parameter value $\theta \in \Theta \subset \mathbb{R}^p$, and that we are only able to observe a subset of the complete data set. In MLE, the best estimate of the true parameter value is $\hat{\theta} \in \Theta$ that maximizes the likelihood of the complete data set, estimated using the available incomplete data set. Let $X$ denote the complete data and $Y$ the available incomplete data, or observations. $X$ is dependent on the unknown deterministic parameter $\theta \in \Theta \subset \mathbb{R}^p$. Let $(\Omega_{X,Y}, \mathcal{F}^{X,Y})$ be the sample space of the data and corresponding $\sigma$-algebra. For each $\theta \in \Theta$, let $Q^\theta$ be the probability measure induced by $X$ such that $(\Omega_{X,Y}, \mathcal{F}^{X,Y}, Q^\theta)$ is a complete probability space. The goal of the parameter estimation problem is to estimate $\theta$ based on the set of observations, $Y$. Formally, the maximum likelihood estimator for a partially observed signal is

$$\hat{\theta} \overset{\text{def}}{=} \arg \sup_{\theta \in \Theta} q(y; \theta),$$

(5.1)

where $y$ is a given set of observations and $q(y; \theta)$ is the likelihood function. For example, if $q(x, y; \theta)$ is the density of $Q^\theta$ w.r.t. Lebesgue measure, then the likelihood function is $q(y; \theta) = \int_X q(x, y; \theta) dx$, the marginal density of $y$.
given parameter value \( \theta \).

Recall continuity of measures: Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and \(A \subset \Omega\). A measure \(\mu'\) is continuous with respect to \(\mu\) if \(\mu'(A) = 0\) for all \(A \in \mathcal{F}\) for which \(\mu(A) = 0\). Denote the restriction of \(\mu\) on the sub-\(\sigma\)-algebra \(A\) by \(\mu|_A\).

We assume that, for all \(\theta, \theta' \in \Theta\), \(Q_\theta \sim Q_{\theta'}\) i.e. the measures induced by all \(\theta \in \Theta\) are mutually continuous. The change of measure \(Q_\theta\) with respect to another measure \(Q_{\theta'}\) induced by some \(\theta' \in \Theta\) is

\[
\frac{dQ_\theta}{dQ_{\theta'}} = f,
\]

where \(f\) is a non-negative measurable function on \(\Omega_{X,Y}\) and is the density of \(Q_\theta\) w.r.t. \(Q_{\theta'}\) by the Radon-Nikodym theorem (see, for example, Chapter 7 of [37]). Also, we assume the identifiability assumption, i.e. for all \(\theta, \theta' \in \Theta\) such that \(\theta \neq \theta'\), \(\frac{dQ_\theta}{dQ_{\theta'}} \neq 1\) a.s. with respect to \(dQ_{\theta'}\). If the identifiability assumption does not hold, then measures induced by different parameter values are indistinguishable.

By the maximum likelihood approach, we would like to maximize \(Q_\theta(X)\), the likelihood of data set \(X\) given parameter value \(\theta\). Equivalently, we can maximize the log likelihood, \(\log Q_\theta(X)\). The choice of the log function is due to its monotonicity and concavity. The expectation-maximization (EM) algorithm exploits this property to iteratively compute a sequence of parameter estimates \(\{\theta^k\}\) that induces a non-decreasing sequence of log likelihoods as \(k \nearrow\). In the partially observed setting, we only have information from \(Y\), so we consider the measure \(Q_\theta\) restricted to the \(\sigma\)-algebra generated by observations, \(\mathcal{F}_Y\). Define \(Q_\theta|_Y\) as the restriction of \(Q_\theta\) on \(\mathcal{F}_Y\). Now, writing the exact expression for \(Q_\theta\) or \(Q_\theta|_Y\) may be intractable. But by the mutual continuity of induced probability measures assumption, when \(X\) and \(Y\) are diffusion processes, the expression for the change of \(Q_\theta\) with respect to another measure, \(\frac{dQ_\theta}{dQ_{\theta'}|_Y}\), is known. Hence, we can instead maximize \(\log \frac{dQ_\theta}{dQ_{\theta'}|_Y}\) or \(\log \frac{dQ_\theta|_Y}{dQ_{\theta'}|_Y}\) for some \(\theta' \in \Theta\). The choice of \(\theta' \in \Theta\) does not matter: Let \(\mu\) be a dominating measure on \(\mathcal{F}_{X,Y}\). Then, \(Q_\theta, Q_{\theta'}\) are continuous with respect to \(\mu\), and there exist a density \(q\) such that \(dQ_\theta = q(x, y; \theta)d\mu\) and \(dQ_{\theta'} = q(x, y; \theta')d\mu\). Also,

\[
\frac{dQ_\theta}{dQ_{\theta'}} = \frac{q(x, y; \theta)d\mu}{q(x, y; \theta')d\mu} = \frac{q(x, y; \theta)}{q(x, y; \theta')},
\]
which is a ratio of probability densities. The Radon-Nikodym derivative \( \frac{dQ_\theta}{dQ_{\theta'}} \) can be utilized as the likelihood function ([104], [105], [26]), specifically, it is the likelihood ratio. Given observed values of \( Y \), maximizing this Radon-Nikodym derivative/ratio of probability densities w.r.t. \( \theta \) with \( \theta' \) fixed can be interpreted as finding the value \( \theta \neq \theta' \) that maximizes the probability of obtaining those observed values. If such \( \theta \) cannot be found, then \( \theta' \) is the maximizer (at least locally).

Let \( E \in \mathcal{F}^Y \). Then, the probability of \( E \) is equal to the probability measure, restricted to \( \mathcal{F}^Y \), of \( E \), i.e.

\[
Q_\theta(E) = Q_{\theta}|_Y(E). \tag{5.2}
\]

Recall that \( f \) is the Radon-Nikodym derivative \( \frac{dQ_\theta}{dQ_{\theta'}} \). Therefore, for \( E \in \mathcal{F}^Y \),

\[
Q_\theta(E) = \int_E dQ_\theta = \int_E f \, dQ_{\theta'} = \int_E f|_Y \, dQ_{\theta'}|_Y.
\]

Comparing this with (5.2),

\[
Q_{\theta}|_Y(E) = \int_E f|_Y \, dQ_{\theta'}|_Y.
\]

Since \( E \) is arbitrary, we have the following for the restriction of the Radon-Nikodym derivative on \( \mathcal{F}^Y \):

\[
\frac{dQ_\theta|_Y}{dQ_{\theta'}|_Y} = f|_Y. \tag{5.3}
\]

Again, let \( E \in \mathcal{F}^Y \) and let \( E_{Q_{\theta'}} \) denote expectation with respect to \( Q_{\theta'} \), and \( \chi \) be the indicator function. Then,

\[
E_{Q_{\theta'}} \left[ \chi_E E_{Q_{\theta'}} \left[ f| \mathcal{F}^Y \right] \right] = E_{Q_{\theta'}} \left[ E_{Q_{\theta'}} \left[ \chi_E f| \mathcal{F}^Y \right] \right] = E_{Q_{\theta'}} \left[ \chi_E f \right] \quad \text{ (tower property)} = E_{Q_{\theta'}} \left[ \chi_E f|_Y \right] \quad \text{ (since } E \in \mathcal{F}^Y). \]

So, by the way conditional expectation is defined, \( E_{Q_{\theta'}} \left[ f| \mathcal{F}^Y \right] = f|_Y \) and
hence

\[ \mathbb{E}_{Q_\alpha'} \left[ \frac{dQ_\theta}{dQ_{\theta'}} \mid \mathcal{F}^Y \right] = f_\mid_{Y} = \frac{dQ_\theta|_Y}{dQ_{\theta'}|_Y} \]

So, to maximize \( \frac{dQ_\theta|_Y}{dQ_{\theta'}|_Y} \), we can instead maximize the conditional expectation of \( \frac{dQ_\theta}{dQ_{\theta'}} \), conditioned on \( \mathcal{F}^Y \). This is relevant when we consider the practical implementation of the MLE. The (computation of) Radon-Nikodym derivative \( \frac{dQ'}{dQ'} \mid_{Y} \) depends on the state \( X \). Since \( X \) is hidden, we have to estimate the function of \( X \) using observations \( Y \).

In the discussions so far, we have only dealt with probability measure \( Q_\theta \) with density \( q(\cdot; \theta) \) w.r.t a dominating measure, without referring specifically to diffusion processes, so all of the above hold for MLE of a partially observed signal that is dependent on unknown parameters. So, for partially observed systems, the ML estimator can be defined as

\[ \hat{\theta} \overset{def}{=} \begin{array}{c} \arg \sup_{\theta \in \Theta} \mathbb{E}_{Q_\alpha'} \left[ \frac{dQ_\theta}{dQ_{\theta'}} \mid \mathcal{F}^Y \right] \end{array} \]

In the following, we illustrate the MLE principle in two settings: partially observed independent, identically distributed data from a distribution, and partially observed diffusion process. In all that follows, let \( \alpha \in \Theta \) denote the true parameter value. We use \( L(\theta) \) to denote the log likelihood function given parameter value \( \theta \), but the definition varies for different cases.

5.2 Independent, identically distributed complete observations

Let \( \{Y_1, \ldots, Y_N\} \), where \( Y_i \sim q(y; \theta) \) are independent, identically distributed. Given parameter value \( \theta \), the likelihood of observing \( Y_k \) is \( L(Y_k; \theta) \overset{def}{=} q(Y_k; \theta) \).

By the Law of Large Numbers,

\[ \frac{1}{N} \sum_{k=1}^{N} \log q(Y_k; \theta) \to \mathbb{E}_{Q_\alpha} [\log q(Y; \theta)] \quad \text{as} \quad N \to \infty. \]

So, as \( N \to \infty \), the argument that maximizes \( \frac{1}{N} \sum_{k=1}^{N} \log q(Y_k; \theta) \) is the argument that maximizes \( \mathbb{E}_{Q_\alpha} [\log q(Y; \theta)] \). We see that \( \mathbb{E}_{Q_\alpha} [\log q(Y; \theta)] \) is
maximized at $\theta = \alpha$:

\[
\mathbb{E}_{Q_\alpha}[\log q(Y; \theta)] - \mathbb{E}_{Q_\alpha}[\log q(Y; \alpha)]
= \mathbb{E}_{Q_\alpha}[\log \frac{q(Y; \theta)}{q(Y; \alpha)}]
\leq \mathbb{E}_{Q_\alpha}\left[\frac{q(Y; \theta)}{q(Y; \alpha)} - 1\right] \quad \text{(because } x \leq e^{x-1}, \text{ so } \log x \leq x - 1) \\
= \int \left(\frac{q(y; \theta)}{q(y; \alpha)} - 1\right) q(y; \alpha) dy \\
= \int q(y; \theta) dy - \int q(y; \alpha) dy = 1 - 1 = 0.
\]

So, $\mathbb{E}_{Q_\alpha}[\log q(Y; \theta)] - \mathbb{E}_{Q_\alpha}[\log q(Y; \alpha)] \leq 0$ with equality when $\theta = \alpha$.

5.3 Diffusion process with partial, noisy observation

Now, we consider the partially observed diffusion case. Let $(\Omega, \mathcal{F}, Q)$ be a probability space that supports a $k+d$-dimensional Brownian motion $(W, B)$ ($Q$ is the probability measure on $\mathcal{F}$ induced by the Brownian motion $(W, B)$). $(W, B) \in \mathbb{R}^{k+d}$ and is continuous, so here $(\Omega, \mathcal{F}) = (\mathcal{C}([0, T]; \mathbb{R}^{k+d}), \mathcal{B})$, where $\mathcal{B}$ is the Borel $\sigma$-algebra on the set of continuous paths on $\mathbb{R}^{k+d}$, $\mathcal{C}([0, T]; \mathbb{R}^{k+d})$. The signal and observation are

\[
\begin{align*}
    dX_t &= b_\theta(X_t)dt + \sigma(X_t)dW_t, \quad X_0 \in \mathbb{R}^m, \quad (5.4a) \\
    dY_t &= h_\theta(X_t)dt + dB_t, \quad Y_0 \in \mathbb{R}^d. \quad (5.4b)
\end{align*}
\]

Assume that the coefficients of (5.4a) satisfy the conditions for existence of a strong solution (see, for example, Theorem 5.2.1 of [43]). The solution of (5.4) for $t \leq T$ lies in the space $\mathcal{C}_T \overset{dld}{=} \mathcal{C}([0, T]; \mathbb{R}^{m+d})$ with uniform metrics and Borel sigma algebra $\mathcal{B}_T$. Let $Q_\theta^T$ be the probability measure induced by the solution of (5.4) on $(\mathcal{C}_T, \mathcal{B}_T)$. Subscript $\theta$ indicates the probability measure induced by the diffusion $(X, Y)$ with parameter $\theta$ while no subscript indicates the Wiener measure $Q$.

Consider the Radon-Nikodym derivative

\[
\frac{dQ_\theta^T}{d\mathbb{P}_\theta^T} = \exp \left\{ \int_0^T h_\theta(X_t)^* dY_t - \frac{1}{2} \int_0^T \|h_\theta(X_t)\|^2 dt \right\}. \quad (5.5)
\]
By Girsanov’s theorem, \( Y \) is a \( \mathbb{P}_\theta \)-B.M. Recall the normalizer for the nonlinear filter (Section 2.3.1, (2.13)):

\[
\rho^\theta_T(1) = \mathbb{E}^{\mathbb{P}_\theta}_T \left[ \frac{d\mathbb{Q}^T_\theta}{d\mathbb{P}^T_\theta} \right]_{\mathcal{Y}_T},
\]

where \( \mathcal{Y}_T \) is the \( \sigma \)-algebra generated by observations up to time \( T \). Formally, we see that we can choose this as the likelihood function: Partition the time interval \([0,T]\) into \( \{t_0, t_1, \ldots, t_N\} \). For simplicity, let \((X,Y) \in \mathbb{R}^2\). Denote by \( \Delta Y_{t_i} \) the increment \((Y_{t_i} - Y_{t_{i-1}})\). In order to estimate the true parameter value, we can maximize the likelihood of observations \( \{Y_{t_0}, Y_{t_1}, \ldots, Y_{t_N}\} \) given parameter value:

\[
q(Y_{t_0:t_N}; \theta)
\]

\[
= \int_{\mathbb{R}^N} q(Y_{t_0:t_N}, x_{t_0:t_N}; \theta) \, dx_{t_0:t_N}
\]

\[
= \int_{\mathbb{R}^N} \prod_{i=1}^{N} q(Y_{t_i}|Y_{t_{i-1}}, x_{t_{i-1}}; \theta)q(x_{t_0:t_N}; \theta) \, dx_{t_0:t_N}
\]

\[
\propto \int_{\mathbb{R}^N} \prod_{i=1}^{N} \exp \left\{ -\frac{1}{2\Delta t} \left( Y_{t_i} - Y_{t_{i-1}} - h_\theta(X_{t_{i-1}}) \Delta t \right)^2 \right\} q(x_{t_0:t_N}; \theta) \, dx_{t_0:t_N}
\]

\[
\downarrow \quad \text{(see Section 2.6.2, (2.30))}
\]

\[
= \int_{\mathbb{R}^N} \prod_{i=1}^{N} \exp \left\{ h_\theta(X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) - \frac{1}{2} h_\theta(X_{t_{i-1}})^2 \Delta t \right\}
\]

\[
\times c(Y_{t_0:t_N})q(x_{t_0:t_N}; \theta) \, dx_{t_0:t_N}
\]

\[
= \int_{\mathbb{R}^N} \exp \left\{ \sum_{i=1}^{N} h_\theta(X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) - \frac{1}{2} h_\theta(X_{t_{i-1}})^2 \Delta t \right\}
\]

\[
\times c(Y_{t_0:t_N})q(x_{t_0:t_N}; \theta) \, dx_{t_0:t_N}.
\]

As \( N \to \infty \), the quantity inside the exponent goes to the integral in the exponent of the Radon-Nikodym derivative in (5.5). Therefore, from a discrete-time point of view, we see that the likelihood of observations given parameter value is equivalent to the conditional expectation of the Radon-Nikodym derivative (5.6). Below, we see that \( \log \rho^\theta_T(1) \) is maximized at the true parameter value \( \alpha \).

Lemma 3.29 of [35] gives the governing equation for the log likelihood

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function \( \log \rho^\theta_t(1) \) (see Appendix 4):

\[
\log \rho^\theta_t(1) = \int_0^t \pi^\theta_s(h) dY_s - \frac{1}{2} \int_0^t \pi^\theta_s(h)^2 ds,
\]

or

\[
\rho_t(1) = \exp \left\{ \int_0^t \pi_s(h) dY_s - \frac{1}{2} \int_0^t \pi_s(h)^2 ds \right\}.
\] (5.7)

Recall that the innovation process, \( \nu^\alpha_t = Y_t - \int_0^t \pi^\alpha_s(h) ds \), is a \( \mathbb{Q}_\alpha \)-Brownian motion (we would like to point out that the observation is generated by the diffusion induced by the true parameter value \( \alpha \), hence \( \nu^\alpha_t \) is a \( \mathbb{Q}_\alpha \)-B.M., but an innovation process \( \nu^\theta_t = Y_t - \int_0^t \pi^\theta_s(h) ds \) is not a \( \mathbb{Q}_\alpha \)-B.M.). Substituting the innovation in (5.7),

\[
\rho^\theta_T(1) = \exp \left\{ \int_0^T \pi^\theta_t(h) dY_t - \frac{1}{2} \int_0^T \pi^\theta_t(h)^2 dt \right\}
\]

\[
= \exp \left\{ \int_0^T \pi^\theta_t(h) d\nu^\alpha_t + \int_0^T \pi^\theta_t(h) \pi^\alpha_t(h) dt - \frac{1}{2} \int_0^T \pi^\theta_t(h)^2 dt \right\}
\]

\[
= \exp \left\{ \int_0^T \pi^\theta_t(h) d\nu^\alpha_t + \frac{1}{2} \int_0^T \pi^\theta_t(h) \pi^\alpha_t(h) dt
+ \frac{1}{2} \int_0^T \pi^\theta_t(h) \left[ \pi^\alpha_t(h) - \pi^\theta_t(h) \right] dt \right\}.
\]

Let \( L(\theta) \overset{\text{def}}{=} \log \rho^\theta_T(1) \). The log likelihood function divided by observation time \( T \) is

\[
\frac{1}{T} L(\theta) = \frac{1}{T} \left[ \int_0^T \pi^\theta_t(h) d\nu^\alpha_t + \frac{1}{2} \int_0^T \pi^\theta_t(h) \pi^\alpha_t(h) dt
+ \frac{1}{2} \int_0^T \pi^\theta_t(h) \left[ \pi^\alpha_t(h) - \pi^\theta_t(h) \right] dt \right].
\]

The stochastic integral \( \int_0^T \pi^\theta_t(h) d\nu^\alpha_t \) has mean zero under \( \mathbb{Q}_\alpha \) and using Itô isometry, its variance divided by \( T \) goes to 0 as \( T \to \infty \). For the special case where the diffusion process is ergodic and the filter converges to a limit \( \bar{\pi} \) as \( T \to \infty \),

\[
\lim_{T \to \infty} \frac{1}{T} L(\theta) = \lim_{T \to \infty} \frac{1}{T} \left[ \int_0^T \pi^\theta_t(h) d\nu^\alpha_t + \frac{1}{2} \int_0^T \pi^\theta_t(h) \pi^\alpha_t(h) dt
+ \frac{1}{2} \int_0^T \pi^\theta_t(h) \left[ \pi^\alpha_t(h) - \pi^\theta_t(h) \right] dt \right].
\]
\[
+ \frac{1}{2} \int_0^T \pi_t^\theta(h_\theta) \left[ \pi_t^\alpha(h_\alpha) - \pi_t^\theta(h_\theta) \right] dt \\
= \frac{1}{2} \pi^\theta(h_\theta) \bar{\pi}^\alpha(h_\alpha) + \frac{1}{2} \bar{\pi}^\theta(h_\theta) \left[ \bar{\pi}^\alpha(h_\alpha) - \bar{\pi}^\theta(h_\theta) \right] \\
= \frac{1}{2} \bar{\pi}^\alpha(h_\alpha)^2 - \frac{1}{2} \left[ \bar{\pi}^\alpha(h_\alpha) - \bar{\pi}^\theta(h_\theta) \right]^2.
\]

Hence, as \( T \uparrow \infty \), \( \frac{1}{T} \log L(\theta) \) converges to a maximum at \( \frac{1}{2} \bar{\pi}^\theta(h_\theta)^2 \) when \( \theta = \alpha \). In other words, when the observation window \( T \) is infinite, i.e. we have infinite observations, the log likelihood is maximized at the true parameter value. However, we do not have infinite observations in reality. The asymptotic normality property of the ML estimator states that the error, magnified by \( \sqrt{T} \), is normally distributed with mean at the true parameter value and variance inversely proportional to the Fisher information \( I(\alpha) \overset{\text{def}}{=} \mathbb{E}_{\alpha} \left[ \hat{B}(X; \alpha)^2 \right] \), where \( B \) is the bias of the estimator (we will elaborate on this based on a result for ergodic diffusions from [105] in Chapter 6). Then, when we only have observation over finite but sufficiently large \( T \), we can say that the MLE error should be inversely proportional to \( \sqrt{T} \) (the same holds for the partially observed i.i.d. signal case, where \( T \) is replaced by number of observations \( N \)).

### 5.4 Cramér-Rao lower bound

Here, we state the Cramér-Rao inequality for ML estimators (see, for example, Chapter 1.3 of [105]), which states that the lower bound of the mean-squared error is inversely proportional to the Fisher information of the observed data:

\[
I(\theta) \overset{\text{def}}{=} \mathbb{E}_{Q_\theta} \left[ \left( \frac{\partial}{\partial \theta} L(\theta) \right)^2 \right] \\
= \mathbb{E}_{Q_\theta} \left[ \left( \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} \right) \right)^2 \right], \quad \theta, \theta' \in \Theta.
\]

We have written the log likelihood function as the Radon-Nikodym derivative/change of one probability measure w.r.t. another/likelihood ratio. The derivative w.r.t. \( \theta \) indicates sensitivity of the log likelihood to changes in the parameter value. Given an observed data set, the log likelihood represents
the amount of information about the unknown parameter that is contained in the data set. Hence, the Fisher information is a measure of how much information about the unknown parameter that is contained in observations. In the above Radon-Nikodym derivative, $\mathcal{F}$ is the $\sigma$-algebra generated by the complete i.i.d. observations or the observations of the diffusion process up to time $T$, in which case we write $\mathcal{F}_T$. For a continuous function $\varphi$ of $\theta$, the Cramèr-Rao lower bound is (see Appendix 5 for details)

$$
\text{MSE} \geq \frac{\left( \frac{\partial}{\partial \theta} \varphi(\theta) - \frac{\partial}{\partial \theta} B(\theta) \right)^2}{I(\theta)} + B(\theta)^2,
$$

where the MSE, Fisher information and bias are, respectively,

$$
\text{MSE} = \mathbb{E}_{Q_\theta} \left[ (\varphi(\hat{\theta}_T) - \varphi(\theta))^2 \right],
$$

$$
I(\theta) = \mathbb{E}_{Q_\theta} \left[ \left( \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} \bigg| \mathcal{F}_T \right) \right)^2 \right],
$$

$$
B(\theta) = \mathbb{E}_{Q_\theta} \left[ \varphi(\hat{\theta}_T) - \varphi(\theta) \right].
$$

For an unbiased estimator, when $\varphi(\hat{\theta}_T) = \hat{\theta}_T$, we have the lower bound of the MSE to be inversely proportional to the Fisher information:

$$
\mathbb{E}_{Q_\theta} \left[ (\hat{\theta}_T - \theta)^2 \right] \geq I(\theta)^{-1}.
$$

5.5 The Expectation-Maximization Algorithm

As described in the beginning of this chapter, the Expectation-Maximization (EM) algorithm is an iterative algorithm for estimating unknown model parameters from partial observations that is an alternative to bootstrap MC sampling of the parameter space. The EM algorithm was first developed in [25] for partial perfect observations and then for partially observed diffusions in [26]. [25] provides a good illustration of the EM principle using an exponential family example. Here, we describe the EM principle using the partially observed diffusion of (5.4).
5.5.1 EM principle

Consider the partially observed diffusion (5.4),
\[
\begin{align*}
    dX_t &= b_\theta(X_t)dt + \sigma(X_t)dW_t, \quad X_0 \in \mathbb{R}^m, \\
    dY_t &= h_\theta(X_t)dt + dB_t, \quad Y_0 \in \mathbb{R}^d.
\end{align*}
\]

Recall from Section 5.3 that the likelihood function is the normalizer of the nonlinear filter $\rho^\theta_T(1)$. Formally, we can write the normalizer $\rho^\theta_T(1)$ as (we used a discrete-time argument to relate $\rho^\theta_T(1)$ to the likelihood $q_\theta,T(y)$ in Section 5.3)
\[
\begin{align*}
    \rho^\theta_T(1) &\equiv q_\theta,T(y) = q_\theta,T(x,y), \\
    \log \rho^\theta_T(1) &\equiv \log q_\theta,T(x,y) - \log q_\theta,T(x|y).
\end{align*}
\]

Since $\rho^\theta_T(1)$ is $\mathcal{Y}_T$-measurable ($\mathcal{Y}_T$ is the completion of the $\sigma$-algebra generated by observations from time 0 to $T$), taking conditional expectation under measure $Q_{\theta'}$, $\theta' \in \Theta$, $\theta' \neq \theta$, conditioned on $Y_T$,
\[
\begin{align*}
    \log \rho^\theta_T(1) &= \mathbb{E}_{Q_{\theta'}} \left[ \log \rho^\theta_T(1) \mid \mathcal{Y}_T \right] \\
    &= \mathbb{E}_{Q_{\theta'}} \left[ \log q_\theta,T(X,Y) \mid \mathcal{Y}_T \right] - \mathbb{E}_{Q_{\theta'}} \left[ \log q_\theta,T(X|Y) \mid \mathcal{Y}_T \right].
\end{align*}
\]

(Here, we formally write $q_\theta,T(X,Y)$ as the joint density of the path $\{(X_t,Y_t; 0 \leq t \leq T)\}$, similarly for the conditional density.)

Consider the difference between $\log \rho^\theta_T(1)$ and $\log \rho'^\theta_T(1)$:
\[
\begin{align*}
    \log \rho^\theta_T(1) - \log \rho'^\theta_T(1) &= \mathbb{E}_{Q_{\theta'}} \left[ \log \frac{q_\theta,T(X,Y)}{q_{\theta'},T(X,Y)} \mid \mathcal{Y}_T \right] - \mathbb{E}_{Q_{\theta'}} \left[ \log \frac{q_{\theta,T}(X|Y)p_T(Y)}{q_{\theta',T}(X|Y)p_T(Y)} \mid \mathcal{Y}_T \right] \\
    &= \mathbb{E}_{Q_{\theta'}} \left[ \log \frac{q_\theta,T(X,Y)}{q_{\theta'},T(X,Y)} \mid \mathcal{Y}_T \right] - \mathbb{E}_{Q_{\theta'}} \left[ \log \frac{q_{\theta,T}(X|Y)}{q_{\theta',T}(X|Y)} \mid \mathcal{Y}_T \right] \\
    &=: Q(\theta,\theta') \quad \text{and} \quad := H(\theta,\theta').
\end{align*}
\]

In terms of maximizing the log likelihood, we can look at this variation as fixing a value $\theta'$ to estimate the quantities $Q$ and $H$ as functions of $\theta$, and the task is to choose $\theta$ such that the log likelihood under $\theta$ is greater than or equal to the log likelihood under $\theta'$. If we repeat this by using the new
\( \theta \) to perform estimation and eventually converge upon a \( \theta^* \) value, then we will have found a parameter value that maximizes the log likelihood. The quantity \( H(\theta, \theta') \) is always greater than or equal to 0, with equality when \( \theta = \theta' \):

\[
E_{Q_{\theta'}} \left[ \log \frac{q_{\theta,T}(X|Y)}{q_{\theta',T}(X|Y)} \bigg| \mathcal{Y}_T \right] \leq \log E_{Q_{\theta'}} \left[ \frac{q_{\theta,T}(X|Y)}{q_{\theta',T}(X|Y)} \bigg| \mathcal{Y}_T \right] = \log \int_{\mathbb{R}^m} \frac{q_{\theta,T}(x|Y)}{q_{\theta',T}(x|Y)} q_{\theta',T}(x|Y) dx = \log \int_{\mathbb{R}^m} q_{\theta,T}(x|Y) dx = \log(1) = 0,
\]

so \( H(\theta, \theta') \geq 0 \). Since \( H(\theta, \theta') \) is always \( \geq 0 \), in order for \( \log \rho_{\theta}^T(1) - \log \rho_{\theta'}^T(1) \) to be \( \geq 0 \), i.e. the log likelihood to be non-decreasing from \( \theta' \) to \( \theta \), we need \( Q(\theta, \theta') \) to be \( \geq 0 \). We obtain the EM algorithm: maximize \( Q(\theta, \theta') \) w.r.t. \( \theta \), given previous maximizer \( \theta' \), iteratively.

More precisely, begin the algorithm with an initial \( \theta_0 \in \Theta \). At iteration \( i + 1 \), perform the expectation and maximization steps using the previous estimate \( \theta^i \):

- (Expectation step) Compute \( Q(\theta, \theta^i) = E_{Q_{\theta}} \left[ \log \frac{q_{\theta,T}(X|Y)}{q_{\theta',T}(X|Y)} \bigg| \mathcal{Y}_{0:N} \right] \)
- (Maximization step) \( \theta^{i+1} = \arg \sup_{\theta \in \Theta} Q(\theta, \theta^i) \)

To our knowledge, this form of the EM algorithm for partially observed diffusion is first presented in [26]. The algorithm results in a non-decreasing sequence of \( \{ \log \rho_{\theta}^T(1) \} \). The log likelihood function \( \log \frac{q_{\theta,T}(X|Y)}{q_{\theta',T}(X|Y)} \) can be computed using the Radon-Nikodym derivative:

\[
\log \frac{q_{\theta,T}(X,Y)}{q_{\theta',T}(X,Y)} = \log \frac{dQ_{\theta}}{dQ_{\theta'}} \bigg| \mathcal{F}_T = \int_0^T \left[ b_{\theta}(X_s) - b_{\theta'}(X_s) \right] dX_s - \frac{1}{2} \int_0^T \left[ \| b_{\theta}(X_s) \|^2 - \| b_{\theta'}(X_s) \|^2 \right] ds \\
+ \int_0^T \left[ h_{\theta}(X_s) - h_{\theta'}(X_s) \right] dY_s - \frac{1}{2} \int_0^T \left[ \| h_{\theta}(X_s) \|^2 - \| h_{\theta'}(X_s) \|^2 \right] ds \tag{5.8}
\]

It is shown in [25] (Theorem 2), supplemented by [106] and [107], that for a sequence \( \{ \theta^i \} \) of the EM algorithm, if

1. the sequence of log likelihood \( \{ L(\theta^i) \} \) is bounded, and
2. the growth of $Q(\theta^{i+1}; \theta^i)$ is faster than the quadratic growth of $\|\theta^{i+1} - \theta^i\|$, i.e. $Q(\theta^{i+1}; \theta^i) - Q(\theta^i; \theta^i) \geq C\|\theta^{i+1} - \theta^i\|^2$ for some $C > 0$ for all $i$, then the sequence $\{\theta^i\}$ converges to a connected set in $\Theta$. This is shown for the i.i.d. signal with perfect partial observation case, but applies to the partially observed diffusion setting as well, as it only depends on conditions on the log likelihood being bounded and the growth rate of $Q(\theta; \theta')$ in the maximization step.

The convergence rate of the algorithm is quadratic in $\|\theta^{i+1} - \theta^i\|$ (Theorem 3 of [25]): Let $D^{(0)}Q(\theta; \theta')$ denote the $d^{th}$ derivative of $Q(\theta; \theta')$ w.r.t. to the first argument, $\theta$. If $D^{10}Q(\theta^{i+1}; \theta^i) = 0$ for all $i$, then, for all $i$, there exists a $\theta^i_0$ on the line segment joining $\theta^{i+1}$ and $\theta^i$ such that

$$Q(\theta^{i+1}; \theta^i) - Q(\theta^i; \theta^i) = -(\theta^{i+1} - \theta^i)^*D^{20}Q(\theta^i_0; \theta^i)(\theta^{i+1} - \theta^i).$$

Furthermore, if the sequence $\{D^{20}Q(\theta^i_0; \theta^i)\}$ is negative definite with eigenvalues bounded away from zero, and $L(\theta)$ is bounded, then the sequence $\{\theta^i\}$ converges to a connected subset of $\Theta$.

Finally, if the EM algorithm converges to a point $\theta^*$ in the closure of $\Theta$, then Theorem 4 of [25] describes the behavior of the likelihood function at and near $\theta^*$: Let $M : \Theta \rightarrow \Theta$, $M(\theta^i) \overset{\text{def}}{=} \theta^{i+1}$ denote the mapping of the EM algorithm from one iteration to the next. If

1. $\{\theta^i\}$ converges to some $\theta^*$ in the closure of $\Theta$,
2. $D^{10}Q(\theta^{i+1}; \theta^i) = 0$ for all $i$,
3. $D^{20}Q(\theta^{i+1}; \theta^i)$ is negative definite with eigenvalue bounded away from zero for all $i$,

then,

- $DL(\theta^*) = 0$,
- $D^{20}Q(\theta^*; \theta^*)$ is negative definite,
- $DM(\theta^*) = D^{20}H(\theta^*; \theta^*)[D^{20}Q(\theta^*; \theta^*)]^{-1}$.

It can be shown that $\text{Var}\left[\frac{\partial}{\partial \theta} \log k(x|y; \theta)\right] y; \theta] = -D^{20}H(\theta; \theta)$. The second variation of $L$ at $\theta^*$ is:

$$D^{20}L(\theta^*) = D^{20}Q(\theta^*; \theta^*) - D^{20}H(\theta^*; \theta^*)$$
\[
D^{20}Q(\theta^*; \theta^*) - DM(\theta^*)D^{20}Q(\theta^*; \theta^*)
= (I - DM(\theta^*))D^{20}Q(\theta^*; \theta^*),
\]
where \( I \) is the identity matrix. Therefore, \((I - DM(\theta^*))\) describes the behavior of the likelihood function in the neighborhood of the limiting \( \theta^* \).

In the next two sections, we discuss two alternate views to the principle behind the maximization step in the EM algorithm – using an information theoretic approach and a variation of the log likelihood function w.r.t. parameter.

5.5.2 Information theoretic view

We can look at the EM principle from an information theoretic concept. Consider the general setting in which \( X \in \mathbb{R}^m \) is partially observed through \( Y \in \mathbb{R}^d \). We maximize the likelihood of observations:

\[
\hat{\theta} = \arg \sup_{\theta \in \Theta} g(Y; \theta).
\]

If the observation is related to the signal as \( Y = h_\theta(X) \), we can write the density of the observation using density of \( x \):

\[
g(y; \theta) = \int_{h_\theta^{-1}(y)} q(x; \theta) dx,
\]

so

\[
k(x|y; \theta) = \frac{q(x; \theta)}{\int_{h_\theta^{-1}(y)} q(x, y; \theta) dx} = \frac{q(x, y; \theta)}{g(y; \theta)}.
\]

By Bayes’s theorem,

\[
k(x|y; \theta) = \frac{q(x; \theta)}{g(y; \theta)} = \frac{p(y|x; \theta) q(x; \theta)}{g(y; \theta)} = \frac{\delta_{y=h_\theta(x)} q(x; \theta)}{g(y; \theta)}.
\]

Taking the log of above, and since \( y \) is a map from \( \mathcal{X} \) to \( \mathcal{Y} \), we can rearrange as follows:

\[
\log g(y; \theta) = \log q(x; \theta) - \log k(x|y; \theta). \tag{5.9}
\]
Note that the LHS is a function of observation, but if the sensor function \( h_{\theta} \) is known, then it is a function of the signal, i.e. we can write (5.9) as

\[
\log g(h_{\theta}(x); \theta) = \log q(x; \theta) - \log k(x|h_{\theta}(x); \theta).
\]

Consider the variation of \( L(\theta) \overset{\text{def}}{=} \log g(y; \theta) \) w.r.t. \( \theta \): From (5.9) and using the fact that \( L(\theta) \) is \( \mathcal{F}_Y \)-measurable,

\[
L(\theta) - L(\theta') = \log g(y; \theta) - \log g(y; \theta')
= \mathbb{E}_{\theta'}[\log g(y; \theta)] - \mathbb{E}_{\theta'}[\log g(y; \theta')]
= \mathbb{E}_{\theta'}[\log q(x; \theta)] - \mathbb{E}_{\theta'}[\log q(x; \theta')]
- (\mathbb{E}_{\theta'}[\log k(x|y; \theta)] - \mathbb{E}_{\theta'}[\log k(x|y; \theta)])
= Q(\theta; \theta'') - Q(\theta'; \theta'') + [H(\theta) - H(\theta; \theta'')],
\]

where \( Q(\theta; \theta'') \overset{\text{def}}{=} \mathbb{E}_{\theta''}[\log q(x; \theta)] \) and \( H(\theta, \theta'') \overset{\text{def}}{=} \mathbb{E}_{\theta''}[\log k(x|y; \theta)] \). Since \( \theta'' \) is arbitrary, we can set \( \theta'' \) as \( \theta \) so

\[
L(\theta) - L(\theta') = Q(\theta; \theta) - Q(\theta'; \theta) + [H(\theta) - H(\theta'; \theta)].
\]

The variation of \( H \) is

\[
H(\theta; \theta') - H(\theta'; \theta') = -\int_{\mathbb{R}^m} k(x|y; \theta') \log k(x|y; \theta) dx
- \left( -\int_{\mathbb{R}^m} k(x|y; \theta') \log k(x|y; \theta') dx \right)
= -\int_{\mathbb{R}^m} k(x|y; \theta') \log \frac{k(x|y; \theta)}{k(x|y; \theta')} dx.
\]

The relative entropy, or Kullback-Leibler distance, between two probability densities is a non-negative metric: For mutually continuous densities \( p, q \),

\[
D_{KL}(p||q) = -\int_{x} q(x) \log \frac{p(x)}{q(x)} dx = \mathbb{E} \left[ -\log \frac{p(X)}{q(X)} \right]
\geq -\log \mathbb{E} \left[ \frac{p(X)}{q(X)} \right] \quad \text{(since -log is convex)}
= -\log \int_{\mathbb{R}^m} q(x) \frac{p(x)}{q(x)} dx = -\log 1 = 0.
\]
Hence, \([H(\theta; \theta') - H(\theta'; \theta')]\) is always \(\geq 0\), with equality when \(\theta = \theta'\), so if we want to choose \(\theta\) to not decrease the log likelihood from its value under \(\theta'\), it is sufficient to chose it such that \(Q(\theta; \theta')\) is not decreased from \(Q(\theta'; \theta')\).

5.5.3 Variation of log likelihood

We can also look at the variation of the log likelihood w.r.t. \(\theta\). Let \(\theta + \delta \theta\) be a small variation from \(\theta\). Taylor expand \(L(\theta + \delta \theta)\) about \(\theta\):

\[
L(\theta + \delta \theta) = L(\theta) + \frac{\partial}{\partial \theta} L(\theta) \delta \theta + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} L(\theta) \delta \theta + \ldots
= L(\theta) + \frac{\partial}{\partial \theta} Q(\theta; \theta') \delta \theta - \frac{\partial}{\partial \theta} H(\theta; \theta') \delta \theta + O(\delta \theta^2),
\]

so

\[
\frac{\partial}{\partial \theta} L(\theta) = \lim_{\delta \theta \to 0} \frac{L(\theta + \delta \theta) - L(\theta)}{\delta \theta} = \frac{\partial}{\partial \theta} Q(\theta; \theta') - \frac{\partial}{\partial \theta} H(\theta; \theta').
\]

Note that

\[
\frac{\partial}{\partial \theta} H(\theta; \theta') = \frac{\partial}{\partial \theta} \int_X k(x|y; \theta') \log k(x|y; \theta) dx
= \int_{R^n} k(x|y; \theta') \frac{\partial}{\partial \theta} \left[ \log k(x|y; \theta) \right] dx
= \int_{R^n} k(x|y; \theta') \frac{\partial}{\partial \theta} k(x|y; \theta) \cdot \frac{k(x|y; \theta)}{k(x|y; \theta')} dx
\theta' \text{ is arbitrary; if we set } \theta' = \theta, \text{ then }
= \int_{R^n} k(x|y; \theta) \frac{\partial}{\partial \theta} k(x|y; \theta) \cdot \frac{k(x|y; \theta)}{k(x|y; \theta)} dx
= \int_{R^n} \frac{\partial}{\partial \theta} k(x|y; \theta) dx
= \int_{R^n} k(x|y; \theta) dx = \frac{\partial}{\partial \theta} 1 = 0,
\]

so the variation of the log likelihood w.r.t. \(\theta\) only depends on the variation of \(Q\) w.r.t. \(\theta\). So if we want to maximize the log likelihood by varying \(\theta\), it is sufficient to just use \(Q\).
5.5.4 Smoothing

The filtering procedures described in Chapter 2 enables the computation of the conditional estimate \( \mathbb{E} [\varphi(X_t) | \mathcal{Y}] \), i.e. the present estimate given information up to present time. In the expectation step of the EM algorithm, we need to compute the conditional estimate of the log likelihood function using all available information. Based on the Radon-Nikodym derivative form of the log likelihood function (5.8), it is required to compute the conditional estimate at a time based on future information. Consider the discretized approximation of the stochastic and time integrals in (5.8),

\[
\sum_{s=0}^{\lfloor \frac{T}{\Delta t} \rfloor - 1} \left\{ (b_{\theta}(X_{ts}) - b_{\theta'}(X_{ts})) \Delta X_{ts} - \frac{1}{2} \left( \|b_{\theta}(X_{ts})\|^2 - \|b_{\theta'}(X_{ts})\|^2 \right) \Delta t + \ldots \right\}.
\]

The conditional expectation requires the conditional estimate of \( \varphi(X_{ts}), t_s \in [0, T] \) given all information up to \( T \),

\[
\sum_{s=0}^{\lfloor \frac{T}{\Delta t} \rfloor - 1} \{ \mathbb{E}_{Q_{\theta'}} \left[ (b_{\theta}(X_{ts}) - b_{\theta'}(X_{ts})) \Delta X_{ts} | \mathcal{Y}_T \right] \\
- \frac{1}{2} \mathbb{E}_{Q_{\theta'}} \left[ (\|b_{\theta}(X_{ts})\|^2 - \|b_{\theta'}(X_{ts})\|^2) \Delta t | \mathcal{Y}_T \right] + \mathbb{E}_{Q_{\theta'}} [\ldots | \mathcal{Y}_T] \}.
\]

The conditional estimate given future information, \( \mathbb{E} [\varphi(X_t) | \mathcal{Y}_T], t \in [0, T] \) can be computed via smoothing. We describe the smoother in the linear discrete time case and nonlinear discrete time case in the following.

Linear discrete time smoother

As for the linear, discrete-time filter, we follow [41] in describing the Kalman smoother using the maximum likelihood approach. Consider the discrete-time signal and observation

**Signal:** \( X_{t+1} = A_{t+1} X_t + W_{t+1}, \ X_0 \in \mathbb{R}^m \),

**Observation:** \( Y_t = H_t X_t + B_t, \ Y_0 = 0 \in \mathbb{R}^d \),

\( t = 0, 1, \ldots, N, \ W_t \sim \mathcal{N}(0, Q_t), \ B_t \sim \mathcal{N}(0, R_t) \). The goal is to determine \( \hat{X}_{t|N} \) that maximizes the log likelihood \( \log p(X_t | Y_{0:N}) \). Alternatively,
a slightly easier approach (computationally) is to maximize the joint log likelihood, \( \log p(X_t, X_{t+1}|Y_{0:N}), \ t = 0, 1, \ldots, N - 1 \). By Bayes’s Theorem, the joint smoothing density can be written in terms of the filtering density:

\[
p(X_t, X_{t+1}|Y_{0:N}) = \frac{p(Y_{t+1:N}|X_{t+1})p(X_{t+1}|X_t)p(X_t|Y_{0:t})}{p(Y_{t+1:N}|Y_{0:t})}.
\]

The part of the log likelihood that depends on \( X_t \) are from the prior and filtering densities \( p(X_{t+1}|X_t) \) and \( p(X_t|Y_{0:t}) \). In the linear case, the densities in the RHS above are all Gaussian. Using the Gaussian densities, the log likelihood is

\[
\begin{align*}
\log p(X_t, X_{t+1}|Y_{0:N}) &\propto (X_{t+1} - A_{t+1}X_t)^*Q_t^{-1}(X_{t+1} - A_{t+1}X_t) + (X_t - \hat{X}_t|t)^*P_t^*(X_t - \hat{X}_t|t) \\
&+ \{ \text{terms independent of } X_t \}, \\
\end{align*}
\]

where we assume that we already have the filter mean and covariance \( \hat{X}_{t|t} \) and \( P_{t|t} \) for \( t = 1, \ldots, N \). Starting from the filter estimate \( (\hat{X}_{N|N}, P_{N|N}) \), the smoother mean at \( N - 1 \) is

\[
\hat{X}_{N-1|N} = \arg\max_{x \in \mathbb{R}^m} \left\{ (\hat{X}_{N|N} - A_Nx)^*Q_{N-1}^{-1}(\hat{X}_{N|N} - A_Nx) \\
+ (x - \hat{X}_{N-1|N-1})^*P_{N-1|N-1}^*(x - \hat{X}_{N-1|N-1}) \right\}.
\]

Iteratively going backwards in \( t \), the smoother mean for time instance \( t \) is

\[
\hat{X}_{t|N} = \arg\max_{x \in \mathbb{R}^m} \left\{ (\hat{X}_{t+1|N} - A_{t+1}x)^*Q_t^{-1}(\hat{X}_{t+1|N} - A_{t+1}x) \\
+ (x - \hat{X}_{t|t})^*P_{t|t}^*(x - \hat{X}_{t|t}) \right\}.
\]

The maximizer of the RHS can be determined to be

\[
\hat{X}_{t|N} = \hat{X}_{t|t} + P_{t|t}A_{t+1}^*(P_{t+1|t})^{-1} \left[ \hat{X}_{t+1|N} - A_{t+1}\hat{X}_{t|t} \right].
\]

The information update for the Kalman smoother is given by the difference between the smoothed mean one step ahead and that predicted by the filter mean. The error is related to the filter error by

\[
\tilde{X}_{t|N} + P_{t|t}A_{t+1}^*(P_{t+1|t})^{-1} \hat{X}_{t+1|N} = \hat{X}_{t|t} + P_{t|t}A_{t+1}^*(P_{t+1})^{-1}A_{t+1}\hat{X}_{t|t},
\]
from which the smoother error covariance can be determined in terms of the filter covariance to be

\[ P_{t|N} = P_{t|t} + P_{t|t} A_{t+1|t}^{-1} \left( P_{t+1|N} - P_{t+1|t} \right) \left( P_{t|t} A_{t+1}^* \right) \left( P_{t+1|t} \right) \]

The smoothing procedure requires filtering to be performed first and storing the filtered estimates, in the linear case, the mean and covariance, hence it is computationally expensive.

In the continuous time setting, we have the Kalman-Bucy smoother (see, for example, Chapter 5 of [108])

\[
\begin{align*}
\frac{d}{dt} \hat{X}_{t|T} &= A_t \hat{X}_{t|T} + \Sigma_x \left( P_{i|i} \right)^{-1} \left( \hat{X}_{t|T} - \hat{X}_{t|t} \right), \\
\dot{P}_{t|T} &= \left[ A_t + \Sigma_x \left( P_{i|i} \right)^{-1} \right] P_{t|T} + P_{t|T} \left[ A_t^* + \left( P_{t|i} \right)^{-1} \Sigma_x \right] - \Sigma_x,
\end{align*}
\]

which are integrated backwards in time from the filter mean and covariance, \( \hat{X}_{T|T} \) and \( P_{T|T} \), respectively.

Nonlinear discrete time smoother

For the nonlinear signal and observation case, we describe an importance sampling method for sampling from the smoothed density \( p(X_t|Y_0:N) \). Consider the joint conditional density \( p(X_t, X_{t+1}|Y_0:N), t = 0, 1, \ldots, N-1 \). We can write

\[
p(X_t, X_{t+1}|Y_0:N) = \frac{p(X_t|X_{t+1}, Y_0:N)p(X_{t+1}|Y_0:N)p(Y_0:N)}{p(Y_0:N)} = p(X_t|X_{t+1}, Y_0:N)p(X_{t+1}|Y_0:N),
\]

where

\[
p(X_t|X_{t+1}, Y_0:N) = p(X_t|X_{t+1}, Y_t, Y_{t+1:N}) = \frac{p(Y_{t+1:N}|X_t, X_{t+1}, Y_t)p(X_t|X_{t+1}, Y_t)p(X_{t+1}, Y_t)}{p(Y_{t+1:N}|X_{t+1}, Y_t)p(X_{t+1}, Y_t)}.
\]

Assuming present observation only depends on the present state of the signal, \( p(Y_{t+1:N}|X_t, X_{t+1}, Y_t) = p(Y_{t+1:N}|X_{t+1}) \) and \( p(Y_{t+1:N}|X_{t+1}, Y_t) = p(Y_{t+1:N}|X_{t+1}) \),
so

\[ p(X_t|X_{t+1}, Y_{0: N}) = p(X_t|X_{t+1}, Y_t). \]

This states that given information from \( X_{t+1} \) and \( Y_{0: N} \), what we know about \( X_t \) is the same as if we were given information from \( X_{t+1} \) and \( Y_t \). This is because 1) \( Y_t \) is dependent on \( X_t \), and 2) by the Markov property, given \( X_{t+1} \) and observation model, we have all available information contained in \( Y_{t+1:N} \) – \( Y_{t+1:N} \) are dependent on \( \{X_{t+1}, \ldots, X_N\} \), which are described by the joint density \( p(X_{t+1})p(X_{t+2}|X_{t+1}) \ldots p(X_N|X_{N-1}) \).

The last equality can be written as

\[ p(X_t|X_{t+1}, Y_{0: N}) = p(X_t|X_{t+1}, Y_t) = \frac{p(X_{t+1}|X_t)p(X_t|Y_t)}{p(X_{t+1}|Y_t)}. \]

Hence, the joint conditional density can be represented by

\[
p(X_t, X_{t+1}|Y_{0: N}) = p(X_t|X_{t+1}, Y_{0: N})p(X_{t+1}|Y_{0: N})
\]

\[
= \frac{p(X_{t+1}|X_t)}{p(X_{t+1}|Y_t)} \underbrace{p(X_t|Y_t)p(X_{t+1}|Y_{0: N})}_{\text{sampling density}}
\]

\[ (5.10) \]

The smoothing weight, which we will denote \( w(X_t, X_{t+1}) \), consists of the state transition density in the numerator, and the filtering prior density at \( t + 1 \) in the denominator. The weight can be expressed as

\[ w(X_t, X_{t+1}) = \frac{p(X_{t+1}|X_t)}{p(X_{t+1}|Y_t)} = \frac{p(X_{t+1}|X_t)}{\int p(X_{t+1}|X_t)p(X_t|Y_t)dX_t}. \]

[109] and [110] propose sampling algorithms for smoothing by sampling from the joint conditional density according to (5.10). The algorithm is as follows:

- A forward particle filtering procedure is performed from \( t = 0 \) to \( t = N \).
  The smoothing procedure goes backward in time, from \( t = N \) to \( t = 0 \).

- The filtering and smoothing densities at final time \( N \) are the same, \( p(X_N|Y_{0: N}) \). Draw the smoother sample \( \{x_N^j, w_N^j\}_{p=1}^{N_p} \) from the filter sample.

- At time \( t \), we have the smoothed sample \( \{x_{t+1: N}^j, w_{t+1: N}^j\}_{j=1}^{N_s} \). For
each \( j \), draw the associated \( x^j_{t|t} \) from the filter sample that represents 
\( p(X_t|Y_t) \). Note that in order to do this, during the filtering procedure, 
the identity of the parent \( x^j_{t-1|t} \), of each particle \( x^j_{t|t} \) has to also be
stored.

- For each pair \((x^j_{t+1|N}, x^j_{t|t})\), the weight is calculated according to (5.10), 
i.e.

\[
  w^j_{t,t+1|N} = w(x^j_{t|t}, x^j_{t+1|N}) \approx \frac{p(x^j_{t+1|N}|x^j_{t|t})}{p^M_1(x^j_{t+1|N}|Y_t)},
\]

where the filtering prior is approximated using

\[
  p^M_1(x^j_{t+1|N}|Y_t) \overset{\text{def}}{=} \sum_{j_1=1}^{M_1} w^j_{t|t} p(x^j_{t+1|N}|x^j_{t|t}),
\]

where \( w^j_{t|t} \) is the filter weight.

- The sample \( \{x^j_{t|N}, w^j_{t,t+1|N}\}_{j=1}^{N_s} \) is then resampled based on the smoothing weights.

To sample from the marginal conditional density \( p(X_t|Y_0:N) \), we average 
(5.10) over \( X_{t+1} \):

\[
  p(X_t|Y_0:N) = p(X_t|Y_t) \int p(X_{t+1}|X_t) \frac{p(X_{t+1}|Y_0:N)}{p(X_{t+1}|Y_t)} dX_{t+1}.
\]

Hence, the weight of particle \( x^j_{t|N} \) is approximated numerically by ([48])

\[
  w^j_{t|N} = w(x^j_{t|t}) \approx w^j_{t|t} \sum_{j_2=1}^{M_2} w^j_{t+1|N} \left[ \frac{p(x^j_{t+1|N}|x^j_{t|t})}{p^{M_3}(x^j_{t+1|N}|Y_t)} \right],
\]

where

\[
  p^{M_3}(x^j_{t+1|N}|Y_t) \overset{\text{def}}{=} \sum_{j_3=1}^{M_3} w^j_{t|t} \frac{p(x^j_{t+1|N}|x^j_{t|t})}{p^{M_3}(x^j_{t+1|N}|Y_t)}.
\]

If \( M_3 = M_1 \), then sampling for the smoother density this way is \( M_2 \) times 
more expensive than using the joint conditional density.
CHAPTER 6

MULTISCALE PARAMETER ESTIMATION

The main result of multiscale parameter estimation of the work for this thesis is the convergence of an estimator based on a homogenized likelihood function to the true parameter value in the limit of infinite observations and wide timescale separation. The homogenized likelihood function is based on the unnormalized homogenized filter of Chapter 3. Convergence of the likelihood function is a direct consequence of the multiscale filtering result. Convergence of the estimator is obtained for a special case of scalar signal and observation, where the slow component of the signal can be represented by a linear diffusion in the limit of wide timescale separation.

The multiscale parameter estimation problem is formulated, with statement of the main result, in Section 6.1. Existing and related works are discussed in Section 6.2. The results of Chapter 3 are applied to the likelihood function in Section 6.3, and proof of the main result is given in Section 6.4. An application to the Lorenz ’96 model to estimate the external and internal forcing parameters using noisy observations is presented in Section 6.5.

6.1 Problem formulation and statement of main result

The problem setup is the same as the multiscale filtering problem of Chapter 3. We restate it here to include parameter dependence in the slow component SDE and sensor function. Let \((\Omega, \mathcal{F}, Q)\) be a probability space that supports a \(k + l + d\)-dimensional Brownian motion \((W, V, B)\) (Q is the probability measure on \(\mathcal{F}\) induced by the Brownian motion \((W, V, B)\)). \((W, V, B) \in \mathbb{R}^{k+l+d}\) and is continuous, so here \((\Omega, \mathcal{F}) = (C([0, T]; \mathbb{R}^{k+l+d}), \mathcal{B})\), where \(\mathcal{B}\) is the Borel \(\sigma\)-algebra on the set of continuous paths on \(\mathbb{R}^{k+l+d}\), \(C([0, T]; \mathbb{R}^{k+l+d})\).

The signal process consists of fast component \(Z^\varepsilon \in \mathbb{R}^n\) and slow component...
$X^\varepsilon \in \mathbb{R}^m$ governed by

\begin{align}
    dX^\varepsilon_t &= b_\theta(X^\varepsilon_t, Z^\varepsilon_t) \, dt + \sigma_\theta(X^\varepsilon_t, Z^\varepsilon_t) \, dW_t, \\
    dZ^\varepsilon_t &= \frac{1}{\varepsilon} f(X^\varepsilon_t, Z^\varepsilon_t) \, dt + \frac{1}{\sqrt{\varepsilon}} g(X^\varepsilon_t, Z^\varepsilon_t) \, dV_t,
\end{align}

and the observation is $Y^\varepsilon \in \mathbb{R}^d$,

\begin{equation}
    dY^\varepsilon_t = h_\theta(X^\varepsilon_t, Z^\varepsilon_t) \, dt + dB_t, \quad Y^\varepsilon_0 = 0,
\end{equation}

for $t \in [0, T]$. We would like to point out that, for the main result of this chapter, we will restrict ourselves in Section 6.4 to the setting in which the slow diffusion is independent of the fast component. The reason is discussed at the end of Appendix 6. $\theta \in \Theta \subset \mathbb{R}$ is an unknown deterministic parameter. For each $\theta \in \Theta$, in addition to the conditions of in Section 3.5, we impose conditions on the coefficients of (6.1), (6.2) such that a strong solution exists (see, for example, Theorem 5.2.1. of [43]). The solution of (6.1), (6.2) for $t \leq T$ lies in the space $C_T \overset{\text{def}}{=} C([0, T]; \mathbb{R}^{m+n+d})$ with uniform metrics and Borel sigma algebra $\mathcal{B}_T$. We write $Q^\varepsilon_\theta(T)$ for the probability measure on $(C_T, \mathcal{B}_T)$ induced by parameter $\theta$ (induced by $(X^\varepsilon, Z^\varepsilon, Y^\varepsilon)$ of (6.1), (6.2) with parameter $\theta$). Let $Y^\varepsilon_t \overset{\text{def}}{=} \sigma\{Y^\varepsilon_s : 0 \leq s \leq t\} \vee \mathcal{N}$ denote the completion of the $\sigma$-algebra generated by observations up to time $t$.

Define the measure-change

\begin{equation}
    (D^\varepsilon_\theta)^{-1} \overset{\text{def}}{=} \frac{dQ^\varepsilon_\theta(T)}{dP^\varepsilon_\theta(T)} = \exp \left\{ \int_0^t h_\theta(X^\varepsilon_s, Z^\varepsilon_s)^* dY^\varepsilon_s - \frac{1}{2} \int_0^t \|h_\theta(X^\varepsilon_s, Z^\varepsilon_s)\|^2 \, ds \right\}.
\end{equation}

Under $P^\varepsilon_\theta(T)$, the observation $Y^\varepsilon$ is a Brownian motion independent of $(X^\varepsilon, Z^\varepsilon)$. For $Q^\varepsilon_\theta(T)$-, $P^\varepsilon_\theta(T)$-measurable $\mathcal{C}_b$ function $\varphi$, define the unnormalized filter

\begin{equation}
    \rho^\varepsilon_\theta(\varphi) \overset{\text{def}}{=} E_{P^\varepsilon_\theta} \left[ \varphi(X^\varepsilon_t, Z^\varepsilon_t)(D^\varepsilon_\theta)^{-1} \right| Y^\varepsilon_t],
\end{equation}

which satisfies the Zakai equation

\begin{equation}
    d\rho^\varepsilon_\theta(\varphi) = \rho^\varepsilon_\theta(\mathcal{L}^\varepsilon_\theta, \varphi) \, dt + \rho^\varepsilon_\theta(h \varphi) \, dY^\varepsilon_t, \quad \rho^\varepsilon_\theta(\varphi) = E_{Q^\varepsilon_\theta} \left[ \varphi(X^\varepsilon_0, Z^\varepsilon_0) \right],
\end{equation}

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where $\mathcal{L}^{\varepsilon,\theta}$ is the differential operator associated with $(X^\varepsilon, Z^\varepsilon)$:

$$
\mathcal{L}^{\varepsilon,\theta} = \frac{1}{\varepsilon} \mathcal{L}_F + \mathcal{L}_S^{\theta},
$$

$$
\mathcal{L}_F = \sum_{i=1}^n f_i(x, z) \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{i,j=1}^n (gg^*)_{ij}(x, z) \frac{\partial^2}{\partial z_i \partial z_j},
$$

$$
\mathcal{L}_S^{\theta} = \sum_{i=1}^m b_{\theta i}(x, z) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^m (\sigma \sigma^*)_{ij}(x, z) \frac{\partial^2}{\partial x_i \partial x_j}.
$$

For any $\theta \in \Theta$, $Q^{\varepsilon,\theta}_T$ and $P^{\varepsilon,\theta}_T$ are mutually absolutely continuous, and the Radon-Nikodym derivative $(D^{\varepsilon,\theta}_\cdot)^{-1}$ defines a density with respect to $P^{\varepsilon,\theta}_T$ (see, for example, Ch. 7 of [37]), which can be used as the likelihood function for maximum likelihood estimation (MLE) (see Section 5.1).

As discussed in Section 5.3, for partially observed signal, we can estimate the likelihood function that is dependent on the hidden signal based on available observations. We follow [74] in calling the conditional estimate of $(D^{\varepsilon,\theta}_\cdot)^{-1}$,

$$
\rho^{\varepsilon,\theta}_T(1) = \mathbb{E}^{P^{\varepsilon,\theta}_T} \left[ (D^{\varepsilon,\theta}_T)^{-1} | \mathcal{Y}_T \right],
$$

the filtered likelihood function. $\rho^{\varepsilon,\theta}(1)$ is also the normalizer for the nonlinear filter of (6.1), (6.2). The filtered likelihood function is the estimate of the likelihood function based on available observations. The corresponding maximum likelihood (ML) estimator of $\theta$ is, for observations time window $T > 0$,

$$
\hat{\theta}^{\varepsilon}_T \overset{\text{def}}{=} \arg \max_{\theta \in \Theta} \log \rho^{\varepsilon,\theta}_T(1).
$$

Assume that the Doeblin condition is satisfied, i.e. for every fixed $x \in \mathbb{R}^m$, we assume that the solution $Z^x$ of

$$
dZ^x_t = f(x, Z^x_t)dt + g(x, Z^x_t)dW_t
$$

is ergodic and converges exponentially fast to its unique stationary distribution $\mu(x, \cdot)$. By the theory of stochastic averaging (see, for example, [17], also description in Chapter 3, Section 3.3), $X^\varepsilon$ converges in distribution to a
diffusion process \( X^0_t \) governed by a SDE

\[
dX^0_t = \bar{b}_{\theta}(X^0_t)dt + \bar{\sigma}_{\theta}(X^0_t)dV_t, \quad X^0_0 = X^0_0 \in \mathbb{R}^m,
\]

where \( \bar{b}_{\theta}(x) \overset{\text{df}}{=} \int_{\mathbb{R}^n} b_{\theta}(x, z) \mu(x, dz) \), \( \bar{\sigma}_{\theta}(x) \overset{\text{df}}{=} \int_{\mathbb{R}^n} \sigma_{\theta}(x, z) \mu(x, dz) \).

The reduced-order filter is \( \bar{\rho}_{\epsilon,\theta}^0(\varphi) = \mathbb{E}_{\bar{Q}_{\theta}} \left[ \varphi(X^0_0)(\bar{D}_{\epsilon,\theta}^0)^{-1} | Y^\epsilon_t \right] \), which is governed by

\[
d\bar{\rho}_{\epsilon,\theta}^0(\varphi) = \bar{\rho}_{\epsilon,\theta}^0(\mathcal{L}_{\theta})d\varphi + \bar{\rho}_{\epsilon,\theta}^0(\bar{h}_{\theta})dY^\epsilon_t, \quad \bar{\rho}_{\epsilon,\theta}^0(\varphi) = \mathbb{E}_{\bar{Q}_{\theta}} \left[ \varphi(X^0_0) \right],
\]

where \( \bar{D}_{\epsilon,\theta}^0 \) is the same as \( D_{\epsilon,\theta}^0 \), with \( h_{\theta} \) replaced by the homogenized sensor function \( \bar{h}_{\theta}(x) \overset{\text{df}}{=} \int_{\mathbb{R}^n} h_{\theta}(x, z) \mu(x, dz) \):

\[
(\bar{D}_{\epsilon,\theta}^0)^{-1} \overset{\text{df}}{=} \exp \left\{ \int_0^t \bar{h}_{\theta}(X^0_s)^* dY^\epsilon_s - \frac{1}{2} \int_0^t \|\bar{h}_{\theta}(X^0_s)\|^2 ds \right\}.
\]

The equivalent filtered likelihood is \( \bar{\rho}_{\epsilon,\theta}^0(1) \). As in the filtering problem, the homogenized filtered likelihood utilizes the homogenized process \( X^0_t \), with actual observation \( Y^\epsilon \). Note that \( \bar{\rho}_{\epsilon,\theta}^0 \) here is the same as \( \rho_0^\theta \) in Chapter 3. We use an overbar with superscript \( \epsilon \) here to indicate that it is driven by the generator \( \mathcal{L}_{\theta} \) with homogenized coefficients and real observation \( Y^\epsilon \), in order to differentiate from the homogenized filter \( \bar{\rho}_0^\theta \) that is driven by \( Y^0 \), which will be used in Section 6.4.

As in the multiscale filtering problem, the idea is that, under the Doeblin condition assumption on the fast component, if we are interested in estimating only the slow component, then we should make use of the filtered likelihood with the homogenized process \( X^0, \bar{\rho}_{\epsilon,\theta}^0(1) \). The associated ML estimator is, for observations time window \( T > 0 \),

\[
\hat{\theta}_T^\epsilon \overset{\text{df}}{=} \arg \max_{\theta \in \Theta} \frac{1}{T} \log \bar{\rho}_{T,\epsilon}^0(1).
\]

We will be studying the limits of large timescales separation and large observations time window, i.e. \( \epsilon \to 0 \) and \( T \to \infty \), respectively. Therefore, we introduce the additional superscript \( (T) \) for the induced measure \( \bar{Q}_{\theta}^{\epsilon,(T)} \) to indicate finite observations window of size \( T \).

The main result is: Under the setting of the partially observed multiscale diffusion (6.9), (6.10) of Section 6.4 and the conditions in Theorem 6.4.2,
the ML estimator associated with the homogenized filtered likelihood " \hat{p}_{\varepsilon,\theta}^{\varepsilon \theta} (T) " (1) is consistent in the wide timescale separation limit. More precisely, for any \( \delta > 0 \), and \( \alpha \in \Theta \) the true parameter value for the diffusion process,

\[
\lim_{T \to \infty} \lim_{\varepsilon \to 0} Q_{\alpha}^{\varepsilon (T)} \left[ |\hat{\theta}_{\varepsilon,\theta}^{\varepsilon \theta} - \alpha| > \delta \right] = 0.
\]

In other words, in the limit of wide timescale separation and large observations set, the estimator \( \hat{\theta}_{\varepsilon,\theta}^{\varepsilon \theta} \) associated with the homogenized filtered likelihood function for real observations is close to the true parameter value.

6.2 Existing and related works

[105] studies the estimation problem for the ergodic diffusion process in various settings, including nonparametric estimation. Chapter 1.3 of [105] obtains consistency and asymptotic normality of the ML estimator for a completely observed one-dimensional ergodic diffusion process. The asymptotic normality result states that, for large observations time window \( T \), the error of the estimator \( |\hat{\theta}_{\varepsilon,\theta}^{\varepsilon \theta} - \alpha| \) is normally distributed with mean zero and variance inversely proportional to \( TI(\alpha) \), where \( I(\alpha) \) is the Fisher information. Chapter 3.1 of [105] studies the problem of parameteric estimation of a two-dimensional partially observed ergodic diffusion:

\[
\begin{align*}
    dX_t &= -b(\theta)X_t dt + \sigma(\theta) dV_t, \quad X_0 \in \mathbb{R}, \\
    dY_t &= h(\theta)X_t dt + \gamma dB_t, \quad Y_0 \in \mathbb{R}.
\end{align*}
\]

\( b(\theta) \) and \( \sigma(\theta) \) are assumed \( > 0 \) \( \forall \theta \in \Theta \subset \mathbb{R}^p \), and observation is the \( Y \) component, which is completely observed. Since the system is linear, the filter is the Kalman-Bucy filter. The Kalman-Bucy filter is driven by the innovation process, which is a \( Q_{\alpha}^{\varepsilon (T)} \)-Brownian motion (see Section 2.4). In the one dimensional case, the Kalman-Bucy filter solution is an Ornstein-Uhlenbeck process, for which an explicit Gaussian invariant density can be obtained. This property is utilized in showing the consistency and asymptotic normality of the ML estimator. The asymptotic normality of the estimator states that the error \( |\hat{\theta}_{\varepsilon,\theta}^{\varepsilon \theta} - \alpha| \) is normally distributed with mean zero and variance inversely proportional to \( TI(\alpha) \), where \( I(\alpha) \) is a form of the Fisher
information (Theorem 3.1 of [105]). The calculations for consistency of the estimator in Section 6.4 of our work partially follows the procedure of Chapter 3.1 of [105].

As mentioned in Chapter 3, Section 3.2, the work of [74] obtains convergence of the filtered likelihood function. In addition, [74] also determines the asymptotic properties of the corresponding ML estimator. The setting considered in [74] is

$$dZ_t^\varepsilon = \frac{1}{\varepsilon} b_\theta (Z_t^\varepsilon) dt + \frac{1}{\sqrt{\varepsilon}} \sigma_\theta (Z_t^\varepsilon) dW_t, \quad Z_0^\varepsilon = x \in \mathbb{R}^n,$$

$$Y_t^\varepsilon = \int_0^t h_\theta (Z_s^\varepsilon) ds + B_t, \quad Y^\varepsilon = 0 \in \mathbb{R}^d.$$

The homogenized filtered likelihood is driven by the real observation/slow component, $\tilde{\rho}_{t_\varepsilon,\theta} (\varphi) = \mathbb{E}_{P_{\varepsilon,\theta}} \left[ (\tilde{D}_{t_\varepsilon,\theta})^{-1} Y_t^\varepsilon \right]$, where $\tilde{D}_{t_\varepsilon,\theta} \overset{\text{def}}{=} \exp \left\{ -\tilde{h}_\theta Y_t^\varepsilon + \frac{1}{2} \|\tilde{h}_\theta\|^2 t \right\}$.

As mentioned in Section 3.2, the weak convergence $(h_\theta (Z_t^\varepsilon) - \tilde{h}_\theta) \overset{\text{d}}{\rightarrow} 0$ holds in this setting. Additionally, spectral decomposition of the generator of the diffusion process is also utilized in showing the convergence results and asymptotic properties of the ML estimator corresponding to the homogenized filtered likelihood. The estimator is shown to be consistent in the wide timescale separation and infinite observation limits, and, for large $T$, error of the estimator is shown to be normally distributed with mean zero and variance inversely proportional to $T \tilde{h}_\alpha^* \tilde{h}_\alpha$.

Below is the list different settings for the parametric estimation problem and corresponding works that obtain asymptotic properties of the ML estimator.

Completely observed signal:

1) Discrete i.i.d. observations and continuous observations with Gaussian noise (Chapter 2 of [104])

2) Observation is ergodic diffusion process (Chapters 2 and 3 of [104], Chapters 1 and 2 of [105])

3) Observation is scalar null recurrent diffusion process, linear in parameter (Chapter 3.5 of [105])

Partially observed signal:
1) Signal and observation are linear scalar ergodic diffusion processes (Chapter 3.1 of [105])

2) Slow-fast diffusion process where fast component is hidden and slow component is completely observed, with linear homogenized process ([74])

6.3 Convergence of filtered likelihood

For the signal (6.1) and observation (6.2), denote \( \alpha \in \Theta \) as the true parameter value. Filtering is performed using \( \theta \in \Theta \), which may not be equal to \( \alpha \), which is unknown, i.e. we consider filters \( \pi^{(\varepsilon,x),\theta}(\varphi) \), \( \pi^{0,\theta}(\varphi) \). Since the filters are functions of the observation \( Y_{\varepsilon} \) generated from a system with \( \alpha \), we require convergence under \( Q^{\varepsilon,(T)}_{\alpha} \), the measure induced by the true parameter value. This can be obtained by a simple extension of the results of Section 3.7.

**Corollary 6.3.1** Let the conditions of Lemma 3.7.6 hold. In addition, for all \( \theta \in \Theta \), assume

1) \( \|h_{\theta}\|_{\infty} \leq c_{h} < \infty \).

2) \( \mathbb{E}_{Q^{\varepsilon,(T)}_{\alpha}}[\|h_{\theta_{1}} - h_{\theta_{2}}\|^{2p}] \leq K|\theta_{1} - \theta_{2}|^{q} \) for some \( \frac{1}{2} \leq p < \infty \), \( q > 1 \).

Then, for fixed \( \alpha \in \Theta \), for every \( p \geq 1 \) there exists \( C_{1} > 0 \), independent of \( \varphi \), such that for every \( \varphi \in C_{b}^{1} \),

\[
\mathbb{E}_{Q^{\varepsilon,(T)}_{\alpha}}[\left| \rho_{T}^{(\varepsilon,x),\theta}(\varphi) - \bar{\rho}_{T}^{\varepsilon,\theta}(\varphi) \right|^{p}] \leq \varepsilon^{p/2} C_{1} \|\varphi\|_{4,\infty}^{p}.
\]

**Proof** For every \( p \geq 1 \) and any \( t \in [0,T] \),

\[
\mathbb{E}_{Q^{\varepsilon,(T)}_{\alpha}}[\left| \rho_{t}^{(\varepsilon,x),\theta}(\varphi) - \bar{\rho}_{t}^{\varepsilon,\theta}(\varphi) \right|^{p}] = \mathbb{E}_{P^{\varepsilon,(T)}_{\alpha}}\left[ \left( \frac{dQ^{\varepsilon,(T)}_{\alpha}}{dP^{\varepsilon,(T)}_{\alpha}} \right) \left| \rho_{t}^{\varepsilon,\theta}(\varphi) - \bar{\rho}_{t}^{\varepsilon,\theta}(\varphi) \right|^{p} \right]
\]

\[
\leq \left( \mathbb{E}_{P^{\varepsilon,(T)}_{\alpha}}\left[ \left( \frac{dQ^{\varepsilon,(T)}_{\alpha}}{dP^{\varepsilon,(T)}_{\alpha}} \right)^{2} \right] \right)^{1/2} \left( \mathbb{E}_{P^{\varepsilon,(T)}_{\alpha}}[\left| \rho_{t}^{\varepsilon,\theta}(\varphi) - \bar{\rho}_{t}^{\varepsilon,\theta}(\varphi) \right|^{2p}] \right)^{1/2}, \quad (6.4)
\]

where \( P^{\varepsilon,(T)}_{\alpha} \) is the measure under which the observation \( Y_{\varepsilon} \) is a Brownian motion independent of the signal \( (X^{\varepsilon},Z^{\varepsilon}) \) (see (6.3)).
First term in (6.4): Denote \( \tilde{D}^{\varepsilon,\alpha}_T \overset{\text{def}}{=} \frac{dQ^{\varepsilon,(T)}}{dP^{\alpha}} \). By Itô’s formula, \( \tilde{D}^{\varepsilon,\alpha}_t \) satisfies

\[
d\tilde{D}^{\varepsilon,\alpha}_t = \tilde{D}^{\varepsilon,\alpha}_t h_{\alpha}(X^{\varepsilon}_t, Z^{\varepsilon}_t) \, dY^{\varepsilon}_t, \quad \tilde{D}^{\varepsilon,\alpha}_0 = 1.
\]

By condition 1, \( \|h_{\alpha}\|_{\infty} < \infty \). By Itô isometry,

\[
\mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ (\tilde{D}^{\varepsilon,\alpha}_T)^2 \right] = 1 + \mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ \int_0^T (\tilde{D}^{\varepsilon,\alpha}_s)^2 \|h_{\alpha}(X^{\varepsilon}_s, Z^{\varepsilon}_s)\|^2 \, ds \right]
\]

\[
\leq 1 + \|h_{\alpha}\|_{\infty}^2 \mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ \int_0^T (\tilde{D}^{\varepsilon,\alpha}_s)^2 \, ds \right]
\]

\[
\leq \exp \{ \|h_{\alpha}\|_{\infty}^2 T \} \quad \text{(Grönwall’s inequality),} \quad (6.5)
\]

so \( \mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ (\tilde{D}^{\varepsilon,\alpha}_T)^2 \right] \) is finite.

Second term in (6.4):

\[
\mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ \left| \rho^{(\varepsilon,x),\theta}(\varphi) - \bar{\rho}^{\varepsilon,\theta}(\varphi) \right|^{2p} \right]
\]

\[
= \mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ \left| \frac{dP^{\varepsilon,(T)}}{dP^{\theta}} \right| \left| \rho^{(\varepsilon,x),\theta}(\varphi) - \bar{\rho}^{\varepsilon,\theta}(\varphi) \right|^{2p} \right]
\]

\[
\leq \mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ \left| \frac{dP^{\varepsilon,(T)}}{dP^{\theta}} \right| \mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ \left| \rho^{(\varepsilon,x),\theta}(\varphi) - \bar{\rho}^{\varepsilon,\theta}(\varphi) \right|^{4p} \right]^{\frac{1}{2}} \right]. \quad (6.6)
\]

Consider the Radon-Nikodym derivative in (6.6). It is bounded similarly as that in (6.4) by condition 2. Let

\[
\tilde{D}^{\varepsilon,\theta,\alpha}_T \overset{\text{def}}{=} \frac{dP^{\varepsilon,(T)}}{dP^{\theta}} = \exp \left\{ \int_0^T \left( h_{\alpha}(X^{\varepsilon}_s, Z^{\varepsilon}_s) - h_{\theta}(X^{\varepsilon}_s, Z^{\varepsilon}_s) \right)^* dB_s \right\}
\]

\[
+ \frac{1}{2} \int_0^T \left( \|h_{\alpha}(X^{\varepsilon}_s, Z^{\varepsilon}_s)\|^2 - \|h_{\theta}(X^{\varepsilon}_s, Z^{\varepsilon}_s)\|^2 \right) \, ds \right\}.
\]

So,

\[
\mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ \left( \frac{dP^{\varepsilon,(T)}}{dP^{\theta}} \right)^2 \right] = \mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ \left( \tilde{D}^{\varepsilon,\theta,\alpha}_T \right)^2 \right]
\]

\[
= \mathbb{E}^{\varepsilon,(T)}_{\frac{\partial}{\partial x}} \left[ \exp \left\{ \int_0^T \left( h_{\alpha}(X^{\varepsilon}_s, Z^{\varepsilon}_s) - h_{\theta}(X^{\varepsilon}_s, Z^{\varepsilon}_s) \right)^* dB_s \right\}
\]

\[
+ \frac{1}{2} \int_0^T \left( \|h_{\alpha}(X^{\varepsilon}_s, Z^{\varepsilon}_s)\|^2 - \|h_{\theta}(X^{\varepsilon}_s, Z^{\varepsilon}_s)\|^2 \right) \, ds \right\}.
\]
\[ + \frac{1}{2} \int_0^T \left( \|h_\alpha(X_s^\varepsilon, Z_s^\varepsilon)\|^2 - \|h_\theta(X_s^\varepsilon, Z_s^\varepsilon)\|^2 \right) ds \right)^2 \right) \] 

\[ \mathbb{E}_{p_\theta^\varepsilon(T)} \left[ \exp \left\{ \int_0^T \left( h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon) \right) * dY_s^\varepsilon \right. \right. 
\[ \left. \left. - \int_0^T \left( h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon) \right) * h_\theta(X_s^\varepsilon, Z_s^\varepsilon) ds \right. \right. 
\[ \left. \left. + \frac{1}{2} \int_0^T \left( \|h_\alpha(X_s^\varepsilon, Z_s^\varepsilon)\|^2 - \|h_\theta(X_s^\varepsilon, Z_s^\varepsilon)\|^2 \right) ds \right) \right\} \right] \]

\[ \mathbb{E}_{p_\theta^\varepsilon(T)} \left[ \left( 1 + \int_0^T \tilde{D}_s^\varepsilon \alpha \left( h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon) \right) * dY_s^\varepsilon \right) \right. 
\[ \left. + \int_0^T \tilde{D}_s^\varepsilon \alpha \|h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon)\|^2 ds \right)^2 \right] \]

\[ \leq 2 \mathbb{E}_{p_\theta^\varepsilon(T)} \left[ 1 + \left( \int_0^T \tilde{D}_s^\varepsilon \alpha \left( h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon) \right) * dY_s^\varepsilon \right) \right. 
\[ \left. + \left( \int_0^T \tilde{D}_s^\varepsilon \alpha \|h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon)\|^2 ds \right) \right)^2 \]

\[ \leq 2 + 2C_2 \mathbb{E}_{p_\theta^\varepsilon(T)} \left[ \int_0^T \left( \tilde{D}_s^\varepsilon \alpha \right)^2 \|h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon)\|^2 ds \right] 
\[ + 2 \mathbb{E}_{p_\theta^\varepsilon(T)} \left[ \int_0^T \left( \tilde{D}_s^\varepsilon \alpha \right)^2 \|h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon)\|^4 ds \right] \]

by Itô’s Lemma for the fourth equality and the Burkholder-Davis-Gundy and Cauchy-Schwarz inequalities for the second inequality. Resuming the inequality,

\[ \mathbb{E}_{p_\theta^\varepsilon(T)} \left[ \left( \frac{dP_\theta^\varepsilon(T)}{dP_\theta^\varepsilon(T)} \right)^2 \right] = \mathbb{E}_{p_\theta^\varepsilon(T)} \left[ \left( \tilde{D}_T^\varepsilon \alpha \right)^2 \right] \]

\[ \leq 2 \left( 1 + \int_0^T \mathbb{E}_{p_\theta^\varepsilon(T)} \left[ \left( \tilde{D}_s^\varepsilon \alpha \right)^2 \left( C_2 \|h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon)\|^2 
\[ + \|h_\alpha(X_s^\varepsilon, Z_s^\varepsilon) - h_\theta(X_s^\varepsilon, Z_s^\varepsilon)\|^4 \right) \right] ds \right) \]
\[ \leq 2 \left( 1 + \left( C_2 \| h_\alpha - h_\theta \|_\infty^2 + \| h_\alpha - h_\theta \|_4^4 \right) \int_0^T \mathbb{E}_{\hat{\rho}_\theta^{\varepsilon}} \left( (\tilde{D}_s^{\varepsilon,\theta,\alpha})^2 \right) ds \right), \quad (6.7) \]

which is finite by condition 2 and Grönwall’s inequality.

For the second expectation in (6.4), define the duals of \( \rho^{(\varepsilon,x),\theta} \), \( \tilde{\rho}^{\varepsilon,\theta} \) (see Section 3.5):

\[
\rho_T^{(\varepsilon,x),\theta} (\varphi) = \int v_0^{\varepsilon,T,\varphi}(x)Q(X_0,x_0)(dx,dz), \quad \text{and} \\
\tilde{\rho}_T^{\varepsilon,\theta} (\varphi) = \int \tilde{v}_0^{\varepsilon,T,\varphi}(x)Q_x(x)(dx).
\]

Then,

\[
\mathbb{E}_{\hat{\rho}_\theta^{\varepsilon}} \left[ \left| \rho_t^{(\varepsilon,x),\theta} (\varphi) - \tilde{\rho}_t^{\varepsilon,\theta} (\varphi) \right|^{4p} \right] \\
= \mathbb{E}_{\hat{\rho}_\theta^{\varepsilon}} \left[ \left| \int (v_0^{\varepsilon,T,\varphi}(x) - \tilde{v}_0^{\varepsilon,T,\varphi}(x))Q(X_0,x_0)(dx,dz) \right|^{4p} \right] \\
\leq \int \mathbb{E}_{\hat{\rho}_\theta^{\varepsilon}} \left[ \left| v_0^{\varepsilon,T,\varphi}(x) - \tilde{v}_0^{\varepsilon,T,\varphi}(x) \right|^{4p} Q(X_0,x_0)(dx,dz) \right] \\
\leq \varepsilon^{2p} C_1 \| \varphi \|_{1,\infty}^{2p}, \quad (6.8)
\]

The second inequality is by Lemma 3.7.4.

Gathering the estimates (6.5), (6.7) and (6.8), the right hand side of (6.4) is bounded by a constant of order \( \varepsilon^{p/2} \). So, \( \rho^{(\varepsilon,x),\theta} (\varphi) \) and \( \tilde{\rho}^{\varepsilon,\theta} (\varphi) \) are close in \( L^p \)-sense under \( Q^{\varepsilon,\alpha} \) for small \( \varepsilon > 0 \). ■

For the parameter estimation problem considered here, we have dependency on unknown parameters only in the coefficients of the SDEs of the slow component and observation process. Hence, we would like to replace the ML estimator by one that utilizes the homogenized process \( X^0 \), specifically, the likelihood function \( \tilde{\rho}^{\varepsilon,\theta} (1) \). Since \( \rho^{\varepsilon,\theta} (1) \) is the normalizer, it is the same as the marginal \( \rho^{(\varepsilon,x),\theta} (1) \). Therefore, substituting the identity function for \( \varphi \) in the above result, we have that, for small \( \varepsilon > 0 \), \( \tilde{\rho}^{\varepsilon,\theta} (1) \) is close to \( \rho^{(\varepsilon,x),\theta} (1) \) in \( L^p \)-sense under \( Q^{\varepsilon,\alpha} \).
6.4 Main results

We now consider a special case of the multiscale diffusion process and observation, (6.1), (6.2): We consider the case in which the parameter \( \Theta = [\bar{\theta}, \bar{\theta}] \subset \mathbb{R} \), the slow diffusion is independent of the fast component, and the limiting process for the slow component has a unique invariant measure. The signal process is

\[
\begin{align*}
    dX^\varepsilon_t &= b_\theta(X^\varepsilon_t, Z^\varepsilon_t)dt + \sigma_\theta(X^\varepsilon_t)dB_t, \quad X^\varepsilon_0 \in \mathbb{R}^m \\
    dZ^\varepsilon_t &= \frac{1}{\varepsilon}f(X^\varepsilon_t, Z^\varepsilon_t)dt + \frac{1}{\sqrt{\varepsilon}}g(X^\varepsilon_t, Z^\varepsilon_t)dB_t, \quad Z^\varepsilon_0 \in \mathbb{R}^n,
\end{align*}
\]

with observation

\[
    dY^\varepsilon_t = h_\theta(X^\varepsilon_t, Z^\varepsilon_t)dt + dB_t, \quad Y^\varepsilon_0 = 0_{d \times 1} \in \mathbb{R}^d.
\]

Let \( Q^{\varepsilon, (T)}_\theta \) be the probability measure on \( (C([0, T]; \mathbb{R}^{m+n+d}), \mathcal{B}_T) \) induced by parameter \( \theta \) (the probability measure induced by \( (X^\varepsilon, Z^\varepsilon, Y^\varepsilon) \) of (6.9), (6.10) with parameter \( \theta \)). The filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, Q^{\varepsilon, (T)}_\theta) \) supports the \( k + l + d \)-dimensional Brownian motion \( (W, V, B) \). We assume that the initial conditions are independent of \( (W, V, B) \).

The fast component is exponentially mixing and attains a unique invariant density \( \mu(dz; x) \) rapidly for fixed \( X^\varepsilon = x \in \mathbb{R}^m \). As \( \varepsilon \to 0 \), \( X^\varepsilon \) converges in distribution to the homogenized process \( X^0 \) that satisfies the SDE

\[
    dX^0_t = \bar{b}_\theta(X^0_t)dt + \bar{\sigma}_\theta(X^0_t)dB_t, \quad X^0_0 = X^\varepsilon_0 \in \mathbb{R}^m.
\]

The homogenized observation is given by

\[
    dY^0_t = \bar{h}_\theta(X^0_t)dt + dB_t, \quad Y^0_0 = 0_{d \times 1} \in \mathbb{R}^d.
\]

\( Q^{(T)}_\theta \) will be the probability measure induced by \( (X^0, Y^0) \) with parameter \( \theta \).

The averaged drift, sensor function and diffusion are \( \bar{b}_\theta(x) \overset{\text{def}}{=} \int_{\mathbb{R}^n} b_\theta(x, z)\mu(dz) \), \( \bar{h}_\theta(x) \overset{\text{def}}{=} \int_{\mathbb{R}^n} h_\theta(x, z)\mu(dz) \), and, since the slow diffusion is independent of \( Z^\varepsilon \), \( \bar{\sigma}_\theta(x) = \sigma_\theta(x) \). We assume additionally that the limiting \( X^0 \) process is ergodic. Specifically, we assume that the coefficients of (6.11) are such that

\[
\lim_{\|x\| \to \infty} \langle \bar{b}_\theta(x), x \rangle = -\infty
\]
and there exists $0 < c < C < \infty$ such that

$$cI_{m \times m} \leq \sigma\sigma^*(x) \leq CI_{m \times m}.$$ 

Then, the homogenized process $X^0$ has a unique invariant measure ([65, 80]) and the homogenized filter corresponding to $(X^0, Y^0)$ also has a unique invariant measure (see [111] and references therein). We also assume the sensor function $h_{\theta}$ to be bounded, in which case the results from [112] is sufficient, and, in the limit of long time, time average of the completely homogenized filter converges to average w.r.t. the unique invariant measure. This property will be used in showing consistency of the ML estimator (6.12).

From Section 6.3, we know that $|\rho^{\varepsilon, \theta}(1) - \bar{\rho}^{\varepsilon, \theta}(1)| \to 0$ in $L^p$ sense as $\varepsilon \to 0$. We want to utilize the homogenized likelihood function $\bar{\rho}^{\varepsilon, \theta}(1)$ that is driven by $Y^\varepsilon$ in place of the original likelihood function $\rho^{\varepsilon, \theta}(1)$ to estimate the parameter $\theta$:

$$\bar{\rho}^{\varepsilon, \theta}(1) \overset{\text{def}}{=} \exp \left\{ \int_0^T \bar{\pi}^{\varepsilon, \theta}_t(\bar{h}_{\theta})^* dY^\varepsilon_t - \frac{1}{2} \int_0^T \| \bar{\pi}^{\varepsilon, \theta}_t(\bar{h}_{\theta}) \|^2 dt \right\},$$

Define the homogenized ML estimator to be

$$\hat{\theta}^\varepsilon_T \overset{\text{def}}{=} \arg \max_{\theta \in \Theta} \frac{1}{T} \log \bar{\rho}^{\varepsilon, \theta}(1), \quad (6.12)$$

To study asymptotic behavior of $\hat{\theta}^\varepsilon_T$, we will utilize the completely homogenized likelihood function

$$\bar{\rho}^\theta_T(1) \overset{\text{def}}{=} \exp \left\{ \int_0^T \bar{\pi}^\theta_t(\bar{h}_{\theta})^* dY^0_t - \frac{1}{2} \int_0^T \| \bar{\pi}^\theta_t(\bar{h}_{\theta}) \|^2 dt \right\},$$

driven by the homogenized observation $Y^0$.

Using these two preliminary results, we obtain the consistency of the ML estimator (6.12) in the limit as $\varepsilon \to 0$.

### 6.4.1 Consistency of the homogenized ML estimator

**Lemma 6.4.1** For all $\theta \in \Theta$, assume

1) $\| h_{\theta} \|_\infty \leq c_h < \infty$. 

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2) \[ E_{Q_{\alpha}(\nu)} \left[ \| \tilde{h}_{\theta_1} - \tilde{h}_{\theta_2} \|^2 \right] \leq K |\theta_1 - \theta_2|^q \text{ for some } \frac{1}{2} \leq p < \infty, q > 1. \]

Then, for \( T > 0 \), \( \frac{1}{T} \log \bar{\rho}_{\varepsilon, \theta}^T(1) \) converges weakly to \( \frac{1}{T} \log \bar{\rho}_T^\theta(1) \) as \( \varepsilon \to 0 \).

**Proof** of Lemma 6.4.1

First, fix \( \theta \in \Theta \) and, as before, \( \alpha \in \Theta \) is the true parameter value. We would like to obtain weak convergence of \( \frac{1}{T} \log \bar{\rho}_{\varepsilon, \theta}^T(1) \) to \( \frac{1}{T} \log \bar{\rho}_T^\theta(1) \) as \( \varepsilon \to 0 \). We first obtain weak convergence of \( \bar{\rho}_{\varepsilon, \theta}^T(1) \) to \( \bar{\rho}_T^\theta(1) \).

By setting \( \psi = 1 \) in Theorem 4.1 of [69] and performing a measure change as in Section 6.3, we can obtain the convergence in \( L^1 \) of \( \bar{\rho}_{\varepsilon, \theta}^T(1) \) to \( \bar{\rho}_T^\theta(1) \) as \( \varepsilon \to 0 \). Corollary 6.3.1 gives us \( L^1 \) convergence of \( \bar{\rho}_{\varepsilon, \theta}^T(1) \) to \( \bar{\rho}_T^\theta(1) \), and, by an application of the triangle inequality, we can obtain convergence of \( \bar{\rho}_T^\theta(1) \) to \( \bar{\rho}_T^\theta(1) \). However, the result of [69] is for the case where the limiting processes for the slow component and observation are both linear, hence the corresponding limiting filter has a Kalman filter representation.

Proposition 3.3 of [71] gives \( L^1 \) convergence of the filtered likelihoods in a different setting, in which the signal process is diffusion with no drift and the slow diffusion coefficient is only dependent on the fast component. If the slow diffusion in (6.9) is only dependent on the fast component, then for \( m = n = d = k = l = 1 \), an appropriate measure change may be performed to remove the drifts and fast diffusion from (6.9) to match the setting of [71]. However, relating \( \bar{\rho}_T^\theta(1) \) to the unnormalized conditional density for the homogenized signal in [71] is not direct.

In Appendix 6, we show that, for \( p \geq 1 \), \( \bar{\rho}_T^\theta(1) \) converges to \( \bar{\rho}_T^\theta(1) \) in \( L^p \)-sense under \( Q_{\theta}(T) \) as \( \varepsilon \to 0 \), for the setting of (6.9), (6.10), for \( m, n, d, k, l \geq 1 \). By the appropriate measure change as in Section 6.3, we also have \( L^p \) convergence under \( Q_{\alpha}(T) \). Then, \( \bar{\rho}_T^\theta(1) \) also converges weakly to \( \bar{\rho}_T^\theta(1) \) as \( \varepsilon \to 0 \) (see, for example, Theorem 4.1 of [113]).

Consider \( \log \bar{\rho}_T^\theta(1) \). The Continuous Mapping Theorem (see, for example, Corollary 5.1 of [113]) gives us weak convergence of any \( Q_{\varepsilon}(T) \)-a.e. continuous function of \( \bar{\rho}_T^\theta(1) \) as well. The natural log is a continuous function on \( \mathbb{R}^+ \), so we just need to check that we will not encounter \( \log 0 \). Recall that (see Appendix 4)

\[ \bar{\rho}_T^\theta(1) = \exp \left\{ \int_0^T \hat{\pi}_t^\varepsilon(\tilde{h}_\theta)^* dY_t^{\varepsilon} - \frac{1}{2} \int_0^T \| \hat{\pi}_t^\varepsilon(\tilde{h}_\theta) \|^2 dt \right\}. \]

\( \tilde{h}_\theta \) is bounded and the stochastic integral cannot go to infinity in finite \( T \), so...
the exponent cannot be $-\infty$, hence $\tilde{\rho}^\varepsilon_\theta(1) > 0$. Therefore, the Continuous Mapping Theorem gives us weak convergence of the filtered log likelihood

\[
\frac{1}{T} \log \tilde{\rho}^\varepsilon_\theta(1) \to \frac{1}{T} \log \tilde{\rho}^\theta_\theta(1)
\]

for any $\theta \in \Theta$.  

Next, for $\theta_i \in \Theta$, $i = 1, \ldots, k$, consider the vector

\[
\left( \tilde{\rho}^\varepsilon_{\theta_1}(1) - \tilde{\rho}^\theta_{\theta_1}(1), \tilde{\rho}^\varepsilon_{\theta_2}(1) - \tilde{\rho}^\theta_{\theta_2}(1), \ldots, \tilde{\rho}^\varepsilon_{\theta_k}(1) - \tilde{\rho}^\theta_{\theta_k}(1) \right). \tag{6.13}
\]

For a random vector $x \in \mathbb{R}^k$ and $\| \cdot \|$ the Euclidean norm on $\mathbb{R}^k$,

\[
\mathbb{E} \left[ \| x \|^p \right] \leq \mathbb{E} \left[ \left( |x_1| + \ldots + |x_k| \right)^p \right] \leq k^{p-1} \mathbb{E} \left[ |x_1|^p + \ldots + |x_k|^p \right]
\]

for $p \geq 1$. Since we have convergence in $L^p$, $p \geq 1$, of $\tilde{\rho}^\varepsilon_{\theta_1}(1)$ to $\tilde{\rho}^\theta_{\theta_1}(1)$ as $\varepsilon \to 0$ for each $\theta_i \in \Theta$, we also have $L^p$-convergence of the vector (6.13) to zero, i.e.

\[
\left( \tilde{\rho}^\varepsilon_{\theta_1}(1), \tilde{\rho}^\varepsilon_{\theta_2}(1), \ldots, \tilde{\rho}^\varepsilon_{\theta_k}(1) \right) \xrightarrow{L^p} \left( \tilde{\rho}^\theta_{\theta_1}(1), \tilde{\rho}^\theta_{\theta_2}(1), \ldots, \tilde{\rho}^\theta_{\theta_k}(1) \right) \text{ as } \varepsilon \to 0,
\]

and hence weak convergence of the vectors.

The map $(x_1, \ldots, x_k) \to \left( \frac{1}{T} \log x_1, \ldots, \frac{1}{T} \log x_k \right)$ is a continuous map from $(\mathbb{R}^+)^k$ to $\mathbb{R}^k$. By the Continuous Mapping Theorem again, we have weak convergence of the vectors of filtered log likelihoods:

\[
\left( \frac{1}{T} \log \tilde{\rho}^\varepsilon_{\theta_1}(1), \frac{1}{T} \log \tilde{\rho}^\varepsilon_{\theta_2}(1), \ldots, \frac{1}{T} \log \tilde{\rho}^\varepsilon_{\theta_k}(1) \right) \xrightarrow{d} \left( \frac{1}{T} \log \tilde{\rho}^\theta_{\theta_1}(1), \frac{1}{T} \log \tilde{\rho}^\theta_{\theta_2}(1), \ldots, \frac{1}{T} \log \tilde{\rho}^\theta_{\theta_k}(1) \right) \text{ as } \varepsilon \to 0. \tag{6.14}
\]

Next, we check a condition for tightness of measures of $\frac{1}{T} \log \tilde{\rho}^\varepsilon_\theta(1)$ on $\Theta$. For $u, v \in \Theta$ and $p \geq 1$,

\[
\mathbb{E}_{\tilde{Q}^\varepsilon_{\alpha}(T)} \left[ \left| \frac{1}{T} \log \tilde{\rho}^\varepsilon_\theta(1) - \frac{1}{T} \log \tilde{\rho}^\varepsilon_\theta(1) \right|^{2p} \right]
\]

\[
= \mathbb{E}_{\tilde{Q}^\varepsilon_{\alpha}(T)} \left[ \left| \frac{1}{T} \int_0^T \left( \tilde{\pi}^\varepsilon_{\varepsilon-}(\tilde{h}_u) - \tilde{\pi}^\varepsilon_{\varepsilon-}(\tilde{h}_v) \right)^* \, dY_t^\varepsilon 
\right.
\]

\[
- \frac{1}{2T} \int_0^T \left( \| \tilde{\pi}^\varepsilon_{\varepsilon-}(\tilde{h}_u) \|^2 - \| \tilde{\pi}^\varepsilon_{\varepsilon-}(\tilde{h}_v) \|^2 \right) \, dt \right|^{2p} \right]
\]

\[
= \mathbb{E}_{\tilde{Q}^\varepsilon_{\alpha}(T)} \left[ \left| \frac{1}{T} \int_0^T \left( \tilde{\pi}^\varepsilon_{\varepsilon-}(\tilde{h}_u) - \tilde{\pi}^\varepsilon_{\varepsilon-}(\tilde{h}_v) \right)^* \left( h_{\alpha}(X_t^\varepsilon, Z_t^\varepsilon) \, dt + dB_t \right) \right|^{2p} \right].
\]
Then, the time integral terms in (6.15) become

\[- \frac{1}{2T} \int_0^T \| \tilde{\pi}_t^{\varepsilon,u}(\tilde{h}_u) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v) \|^2 \, dt \]

\[- \frac{1}{T} \int_0^T \tilde{\pi}_t^{\varepsilon,u}(\tilde{h}_u) \ast (\tilde{\pi}_t^{\varepsilon,u}(\tilde{h}_u) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v)) \, dt \]

\[\leq C \mathbb{E}_{Q^\alpha}(T) \left[ \frac{1}{T} \int_0^T \left( \tilde{\pi}_t^{\varepsilon,u}(\tilde{h}_u) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v) \right) \ast (h_\alpha(X_t^\varepsilon, Z_t^\varepsilon) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v)) \, dt \right]^{2p} \]

\[+ \frac{1}{2T} \int_0^T \left( \tilde{\pi}_t^{\varepsilon,u}(\tilde{h}_u) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v) \right) \ast d\tilde{B}_t \]

\[\leq C \mathbb{E}_{Q^\alpha}(T) \left[ \frac{1}{T} \int_0^T \left( \tilde{\pi}_t^{\varepsilon,u}(\tilde{h}_u) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v) \right) \ast (h_\alpha(X_t^\varepsilon, Z_t^\varepsilon) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v)) \, dt \right]^{2p} \]

\[+ \frac{1}{2T} \int_0^T \left( \tilde{\pi}_t^{\varepsilon,u}(\tilde{h}_u) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v) \right) \, dt \]

\[+ \mathbb{E}_{Q^\alpha}(T) \left[ \frac{1}{T} \int_0^T \left( \tilde{\pi}_t^{\varepsilon,u}(\tilde{h}_u) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v) \right) \ast d\tilde{B}_t \right]^{2p} \].

We will use the following: For vectors $U, V \in \mathbb{R}^d$,

\[U \ast V = \sum_{i=1}^d U_i V_i \leq \sum_{i=1}^d \max_i |U_i| \max_i |V_i| = d \max_i |U_i| \max_i |V_i| = d \|U\|_\infty \|V\|_\infty.\]

Then, the time integral terms in (6.15) become

\[\mathbb{E}_{Q^\alpha}(T) \left[ \frac{1}{T} \int_0^T \left( \tilde{\pi}_t^{\varepsilon,u}(\tilde{h}_u) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v) \right) \ast (h_\alpha(X_t^\varepsilon, Z_t^\varepsilon) - \tilde{\pi}_t^{\varepsilon,v}(\tilde{h}_v)) \, dt \right]^{2p} \]
\[
\left| \frac{1}{2T} \int_0^T \left( \pi_t^{\varepsilon,u}(\bar{h}_u) - \pi_t^{\varepsilon,v}(\bar{h}_v) \right) dt \right|^{2p} \leq \mathbb{E}_{Q_\alpha} \left[ \frac{d}{T} \int_0^T \| \bar{h}_u - \bar{h}_v \|_\infty \| h_u - h_v \|_\infty dt \right]^{2p} \leq \mathbb{E}_{Q_\alpha} \left[ \frac{2Chd}{T} \int_0^T \| \bar{h}_u - \bar{h}_v \|_\infty dt \right] + \mathbb{E}_{Q_\alpha} \left[ \frac{2Chd}{2T} \int_0^T \| h_u - h_v \|_\infty dt \right]^{2p} \\
\leq \mathbb{E}_{Q_\alpha} \left[ \frac{2Chd}{T} \| \bar{h}_u - \bar{h}_v \|_\infty \int_0^T dt \right] + \mathbb{E}_{Q_\alpha} \left[ \frac{Chd}{T} \| h_u - h_v \|_\infty \int_0^T dt \right]^{2p} \\
= 4^p(Chd)^{2p} \mathbb{E}_{Q_\alpha} \left[ \| \bar{h}_u - \bar{h}_v \|^{2p} + \frac{1}{4p} \| h_u - h_v \|^{2p} \right].
\]

For the stochastic integral w.r.t. Brownian motion \( B \) in (6.15), we use the local martingale moments bound (Burkholder-Davis-Gundy inequality): For \( \mathcal{F}_t \)-martingale \( M_t \),

\[
\mathbb{E} \left[ (M_t)^{2p} \right] \leq C_p \mathbb{E} \left[ (M_t^p) \right].
\]

So, along with Jensen’s inequality, the stochastic integral term is

\[
\mathbb{E}_{Q_\alpha} \left[ \frac{1}{T} \int_0^T \left( \pi_t^{\varepsilon,u}(\bar{h}_u) - \pi_t^{\varepsilon,v}(\bar{h}_v) \right) \ast dBt \right]^{p} \leq \mathbb{E}_{Q_\alpha} \left[ C_p \frac{1}{T^2} \int_0^T \| \pi_t^{\varepsilon,u}(\bar{h}_u) - \pi_t^{\varepsilon,v}(\bar{h}_v) \|^{2} \right]^{2p} \leq \mathbb{E}_{Q_\alpha} \left[ C_p \frac{1}{T^{2p}} \int_0^T \| \pi_t^{\varepsilon,u}(\bar{h}_u) - \pi_t^{\varepsilon,v}(\bar{h}_v) \|^{2p} \right]^{2p} \leq \mathbb{E}_{Q_\alpha} \left[ C_p \frac{T^{1-2p}}{T^{2p}} \| \bar{h}_u - \bar{h}_v \|^{2p} \int_0^T dt \right] \leq C_p T^{1-2p} \mathbb{E}_{Q_\alpha} \left[ \| \bar{h}_u - \bar{h}_v \|^{2p} \right].
\]

Resuming the inequality (6.15),

\[
\mathbb{E}_{Q_\alpha} \left[ \frac{1}{T} \log \bar{p}_T^{\varepsilon,u}(1) - \frac{1}{T} \log \bar{p}_T^{\varepsilon,v}(1) \right]^{2p} \leq 4^p(Chd)^{2p} \mathbb{E}_{Q_\alpha} \left[ \| \bar{h}_u - \bar{h}_v \|^{2p} + \frac{1}{4p} \| h_u - h_v \|^{2p} \right] + C_p T^{1-2p} \mathbb{E}_{Q_\alpha} \left[ \| \bar{h}_u - \bar{h}_v \|^{2p} \right] = 162
\]
\[
\leq \tilde{C} E_{Q^{\xi}(T)} \left[ \| \tilde{h}_u - \tilde{h}_v \|_{\infty}^{2p} \right] \\
\leq \tilde{C} K |u - v|^q
\]

for some \( q > 1 \) by the Hölder continuity for moments assumption (3) of the Lemma.

By Lemma 1.33 of [105] (see also, Theorem 12.3 of [113]), the convergence of finite-dimensional distributions of \( \frac{1}{T} \log \tilde{\rho}_T^\varepsilon,\theta(1) \) (6.14), along with the tightness condition

\[
E_{Q^{\xi}(T)} \left[ \frac{1}{T} \log \tilde{\rho}_T^\varepsilon,u(1) - \frac{1}{T} \log \tilde{\rho}_T^\varepsilon,v(1) \right]^{2p} \leq K |u - v|^q,
\]

implies that the probability measure of \( \frac{1}{T} \log \tilde{\rho}_T^\varepsilon,\theta(1) \) converges to that of \( \frac{1}{T} \log \tilde{\rho}_T^\theta(1) \) as \( \varepsilon \to 0 \) (weak convergence). ■

Using the weak convergence result of Lemma 6.4.1, we can obtain a certain consistency of the homogenized ML estimator. In regular continuous time setting, the ML estimator is said to be consistent if it is close to the true parameter value given a sufficiently large observations set, or as \( T \nearrow \infty \). In our setting with slow-fast timescales, we look at consistency when the timescales separation is large, i.e. when \( \varepsilon \) is small. Specifically, when \( \varepsilon \) is close to zero, we want the ML estimator to be close to the true parameter value as \( T \nearrow \infty \). If it is so, then we say that the estimator is consistent when timescales separation is large. We obtain this in Theorem 6.4.2. Since we want to look at consistency in the large timescales separation setting, in the proof, we first let \( \varepsilon \searrow 0 \), and then look at what happens when \( T \nearrow \infty \). This gives us the desired result of the estimator being close to the true parameter value (with probability 1). Note that, if we reverse the order of taking limits of \( \varepsilon \) and \( T \), we are unable to obtain a similar result. Hence, we have consistency when timescales separation is large, and not a more general consistency result.

**Theorem 6.4.2** Under conditions of Lemma 6.4.1, the ML estimator (6.12) is consistent when the slow-fast timescales separation is large.

**Proof** For \( \delta > 0 \),

\[
Q^{\xi}(T) \left[ \left| \hat{\theta}_T - \alpha \right| > \delta \right] = Q^{\xi}(T) \left[ \max_{|\theta - \alpha| > \delta} \frac{1}{T} \log \tilde{\rho}_T^\varepsilon,\theta(1) > \max_{|\theta - \alpha| \leq \delta} \frac{1}{T} \log \tilde{\rho}_T^\theta(1) \right]
\]
\[ Q_\alpha^0(T) \left[ \max_{|\theta - \alpha| > \delta} \frac{1}{T} \log \tilde{\rho}_T^\theta(1) > \max_{|\theta - \alpha| \leq \delta} \frac{1}{T} \log \tilde{\rho}_T^\theta(1) \right] \rightarrow Q_\alpha \left[ \max_{|\theta - \alpha| > \delta} L(\theta, \alpha) > \max_{|\theta - \alpha| \leq \delta} L(\theta, \alpha) \right] \]

as \( \varepsilon \to 0 \), by the convergence of probability measure of \( \frac{1}{T} \log \tilde{\rho}_T^\varepsilon(1) \) to that of \( \frac{1}{T} \log \tilde{\rho}_T^\theta(1) \) from Lemma 6.4.1.

Next, we let \( T \nearrow \infty \). Recall that the homogenized process \( X^0 \) is ergodic, and by the results of [112], the homogenized filter \( \tilde{\pi}_t^\theta(\tilde{h}_\theta) \) has a unique invariant measure and its time average converges to average w.r.t. the invariant measure. Define the innovation process

\[ \nu_{t,\alpha}^0 \overset{\text{def}}{=} Y_t^0 - \int_0^t \tilde{\pi}_s^\alpha(\tilde{h}_\alpha) ds, \]

which is a \( Q^0,\alpha \)-Brownian motion (see Appendix 2). For fixed \( \theta \in \Theta \),

\[
\frac{1}{T} \log \tilde{\rho}_T^\theta(1) \\
= \frac{1}{T} \int_0^T \tilde{\pi}_t^\theta(\tilde{h}_\theta) dY_t^0 - \frac{1}{2T} \int_0^T \tilde{\pi}_t^\theta(\tilde{h}_\theta)^2 dt \\
= \frac{1}{T} \int_0^T \tilde{\pi}_t^\theta(\tilde{h}_\theta) d\nu_{t,\alpha}^0 - \frac{1}{2T} \int_0^T (\tilde{\pi}_t^\theta(\tilde{h}_\theta) - \tilde{\pi}_t^\alpha(\tilde{h}_\alpha))^2 dt + \frac{1}{2T} \int_0^T \tilde{\pi}_t^\alpha(\tilde{h}_\alpha)^2 dt \\
\rightarrow -\frac{1}{2} \mathbb{E}_{Q_\alpha^0(\infty)} \left[ (\tilde{\pi}^\theta(\tilde{h}_\theta) - \tilde{\pi}^\alpha(\tilde{h}_\alpha))^2 \right] + \frac{1}{2} \mathbb{E}_{Q_\alpha^0(\infty)} \left[ \tilde{\pi}^\alpha(\tilde{h}_\alpha)^2 \right] =: L(\theta, \alpha)
\]

as \( T \to \infty \). Note that \( L(\theta, \alpha) \) is maximized when \( \theta = \alpha \). By the same weak convergence arguments as for the \( \varepsilon \) limit, we have

\[
Q_\alpha^0(T) \left[ \max_{|\theta - \alpha| > \delta} \frac{1}{T} \log \tilde{\rho}_T^\theta(1) > \max_{|\theta - \alpha| \leq \delta} \frac{1}{T} \log \tilde{\rho}_T^\theta(1) \right] \\
\rightarrow Q_\alpha \left[ \max_{|\theta - \alpha| > \delta} L(\theta, \alpha) > \max_{|\theta - \alpha| \leq \delta} L(\theta, \alpha) \right] \\
= \mathbb{1} \left\{ \max_{|\theta - \alpha| > \delta} L(\theta, \alpha) > \max_{|\theta - \alpha| \leq \delta} L(\theta, \alpha) \right\} = 0.
\]

We have that, when \( \varepsilon \) is small, \( Q_\alpha^\varepsilon(T) \left[ |\hat{\theta}_T^\varepsilon - \alpha| > \delta \right] \to 0 \) as \( T \nearrow \infty \).
6.4.2 Simple numerical example

Consider the following signal

\[ dX^\varepsilon_t = \theta (\mu^2 - (Z^\varepsilon_t)^2)X^\varepsilon_t dt + \sigma_x dW_t, \quad X^\varepsilon_0 \in \mathbb{R}, \quad (6.16a) \]

\[ dZ^\varepsilon_t = \frac{1}{\varepsilon} (\mu - Z^\varepsilon_t) dt + \frac{1}{\sqrt{\varepsilon}} g dV_t, \quad Z^\varepsilon_0 \in \mathbb{R}, \quad (6.16b) \]

with observation given by

\[ dY^\varepsilon_t = \theta \left( \frac{\mu}{4} - Z^\varepsilon_t \right) X^\varepsilon_t dt + \sigma_y dB_t, \quad Y^\varepsilon_0 = 0. \quad (6.17) \]

\( Z^\varepsilon \) is an Ornstein-Uhlenbeck process, with stationary distribution \( \mathcal{N} \left( \mu, \frac{g^2}{2} \right) \).

The homogenized drift and sensor functions are

\[ \bar{b}_\theta(x) = \int_{\mathbb{R}} \theta (\mu^2 - z^2) x p_\infty(z) dz = \theta (\mu^2 - \int_{\mathbb{R}} z^2 p_\infty(z) dz) x \]

\[ = -\theta \text{Var}_\infty[Z]x = -\frac{1}{2} \theta g^2 x, \quad (6.18) \]

\[ \bar{h}_\theta(x) = \int_{\mathbb{R}} \theta \left( \frac{\mu}{4} - z \right) x p_\infty(z) dz = \theta \left( \frac{\mu}{4} - \int_{\mathbb{R}} z p_\infty(z) dz \right) x \]

\[ = \theta \left( \frac{\mu}{4} - \mu \right) x = -\frac{3}{4} \theta \mu x, \quad (6.19) \]

respectively. The homogenized filtered log likelihood function is

\[ \log \hat{\rho}^{\varepsilon,\theta}_T(1) = \int_0^T \hat{\pi}_t^{\varepsilon,\theta}(\hat{h}_\theta) dY^\varepsilon_t - \frac{1}{2} \int_0^T \hat{\pi}_t^{\varepsilon,\theta}(\hat{h}_\theta)^2 dt \]

\[ = -\frac{3}{4} \theta \mu \int_0^T \hat{X}_t^{\varepsilon,\theta} dY^\varepsilon_t - \frac{9}{32} \theta^2 \mu^2 \int_0^T \left( \hat{X}_t^{\varepsilon,\theta} \right)^2 dt. \]

Note that in this example, the Hölder continuity condition of Lemma 6.4.1 does not hold, as the homogenized sensor function is linear in \( X^0 \) and we cannot expect it to be bounded, but the ergodicity of the completely homogenized filter (that uses \( Y^0 \)) holds. We use this as an illustrative example, and find that the ML estimator using the homogenized filtered log likelihood does sufficiently well.

The maximum likelihood estimate is determined using the Expectation-Maximization (EM) algorithm. For the EM algorithm, we consider the difference between the log likelihood functions parametrized by two different
parameters. We use the decomposition of the log likelihood function into log of the joint density of signal and observation, minus log of the conditional density of signal given observation, and take conditional expectation given observations. For the original likelihood function, this decomposition is

\[
\log \rho_T^{\epsilon, \theta}(1) - \log \rho_T^{\epsilon, \theta'}(1) = \mathbb{E}_{Q_{\theta'}} \left[ \log \frac{dQ_{\theta}(X^\epsilon, Y^\epsilon)}{dQ_{\theta'}(X^\epsilon, Y^\epsilon)} \bigg| \mathcal{F}_T \right] - \mathbb{E}_{Q_{\theta'}} \left[ \log \frac{dQ_{\theta}(X^\epsilon|Y^\epsilon)}{dQ_{\theta'}(X^\epsilon|Y^\epsilon)} \bigg| \mathcal{F}_T \right].
\]

By Jensen’s inequality, \(H(\theta, \theta')\) can be shown to always be \(\leq 0\). Then, given \(\theta'\), we can find a \(\theta\) such that \(\log \rho_T^{\epsilon, \theta}(1) - \log \rho_T^{\epsilon, \theta'}(1) \geq 0\) by maximizing \(Q(\theta, \theta')\), because the worst case is \(\theta = \theta'\), \(Q(\theta, \theta') = 0\). Since \(H(\theta, \theta')\) is always \(\geq 0\), then \(\log \rho_T^{\epsilon, \theta}(1) - \log \rho_T^{\epsilon, \theta'}(1)\) will be \(\geq 0\).

The Radon-Nikodym derivative in \(Q(\theta, \theta')\) is

\[
\frac{dQ_{\theta}(X^\epsilon, Y^\epsilon)}{dQ_{\theta'}(X^\epsilon, Y^\epsilon)} \bigg| \mathcal{F}_T = \frac{dQ_{\theta}(X^\epsilon, Y^\epsilon)}{d\mathbb{P}_{\epsilon}(X^\epsilon, Y^\epsilon)} \bigg| \mathcal{F}_T \frac{d\mathbb{P}_{\epsilon}(X^\epsilon, Y^\epsilon)}{dQ_{\theta'}(X^\epsilon, Y^\epsilon)} \bigg| \mathcal{F}_T
\]

\[
= \exp \left\{ \int_0^T \left( b_{\theta}(X_t^\epsilon, Z_t^\epsilon) - b_{\theta'}(X_t^\epsilon, Z_t^\epsilon) \right)^* dX_t^\epsilon - \frac{1}{2} \int_0^T \left( \| b_{\theta}(X_t^\epsilon, Z_t^\epsilon) \|^2 - \| b_{\theta'}(X_t^\epsilon, Z_t^\epsilon) \|^2 \right) dt \right.
\]

\[
+ \int_0^T (h_{\theta}(X_t^\epsilon, Z_t^\epsilon) - h_{\theta'}(X_t^\epsilon, Z_t^\epsilon))^* dY_t^\epsilon
\]

\[
- \frac{1}{2} \int_0^T \left( \| h_{\theta}(X_t^\epsilon, Z_t^\epsilon) \|^2 - \| h_{\theta'}(X_t^\epsilon, Z_t^\epsilon) \|^2 \right) dt \right\}.
\]

(6.20)

For the EM algorithm using the reduced-order filtered log likelihood \(\log \tilde{\rho}_{T \epsilon, \theta}(1)\), we consider the Radon-Nikodym derivative as in (6.20), but using the averaged drift and sensor functions \(\bar{b}_{\theta}, \bar{h}_{\theta}\). We define \(\bar{Q}(\theta, \theta')\) as

\[
\bar{Q}(\theta, \theta') = \mathbb{E}_{Q_{\theta'}} \left[ \int_0^T \left( \bar{b}_{\theta}(\bar{X}_t) - \bar{b}_{\theta'}(\bar{X}_t) \right)^* d\bar{X}_t - \frac{1}{2} \int_0^T \left( \| \bar{b}_{\theta}(\bar{X}_t) \|^2 - \| \bar{b}_{\theta'}(\bar{X}_t) \|^2 \right) dt \right.
\]

\[
+ \int_0^T (\bar{h}_{\theta}(\bar{X}_t) - \bar{h}_{\theta'}(\bar{X}_t))^* d\bar{Y}_t^\epsilon - \frac{1}{2} \int_0^T \left( \| \bar{h}_{\theta}(\bar{X}_t) \|^2 - \| \bar{h}_{\theta'}(\bar{X}_t) \|^2 \right) dt \right\].
\]

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The EM algorithm is: Start with \( \theta_0 \). At the \((i + 1)\)th iteration,

i. Expectation step: Compute \( \bar{Q}^\varepsilon(\theta, \theta^i) \)

ii. Maximization step: Set \( \theta^{i+1} = \arg \max_{\theta \in \Theta} \bar{Q}^\varepsilon(\theta, \theta^i) \)

For the system (6.16), (6.17), using the homogenized drift and sensor functions given by (6.18), (6.19), we have

\[
\bar{Q}^\varepsilon(\theta, \theta') = -g^2 (\theta - \theta') \mathbb{E}_{\mathbb{Q}^\varepsilon}[\int_0^T X_t d\bar{X}_t \mid Y_T^\varepsilon] - \frac{g^4}{8}(\theta^2 - (\theta')^2) \int_0^T \mathbb{E}_{\mathbb{Q}^\varepsilon}[\bar{X}_t^2 \mid Y_T^\varepsilon] dt \\
- \frac{3\mu}{4}(\theta - \theta') \int_0^T \mathbb{E}_{\mathbb{Q}^\varepsilon}[\bar{X}_t \mid Y_T^\varepsilon] dY_t^\varepsilon - \frac{9\mu^2}{32}(\theta^2 - (\theta')^2) \int_0^T \mathbb{E}_{\mathbb{Q}^\varepsilon}[\bar{X}_t^2 \mid Y_T^\varepsilon] dt
\]

Setting the first derivative of \( \bar{Q}^\varepsilon(\theta, \theta') \) wrt \( \theta \) to 0, we have

\[
\theta^* = -\left( \frac{g^4}{4} + \frac{9\mu^2}{16} \right) \int_0^T \mathbb{E}_{\mathbb{Q}^\varepsilon}[\bar{X}_t^2 \mid Y_T^\varepsilon] dt \right)^{-1} \\
\times \left( \frac{g^2}{2} \mathbb{E}_{\mathbb{Q}^\varepsilon}[\int_0^T X_t d\bar{X}_t \mid Y_T^\varepsilon] + \frac{3\mu}{4} \int_0^T \mathbb{E}_{\mathbb{Q}^\varepsilon}[\bar{X}_t \mid Y_T^\varepsilon] dY_t^\varepsilon \right)
\]

The second derivative wrt \( \theta \) is

\[
- \left( \frac{g^4}{4} + \frac{9\mu^2}{16} \right) \int_0^T \mathbb{E}_{\mathbb{Q}^\varepsilon}[\bar{X}_t^2 \mid Y_T^\varepsilon] dt,
\]

which is always \( \leq 0 \), hence \( \theta^* \) is a maximizer of \( \bar{Q}^\varepsilon(\theta, \theta') \).

The EM algorithm is implemented using a particle smoother version of the HHPF and a homogenized Kalman smoother. Although the homogenized filter is not linear (it uses \( Y^\varepsilon \) instead of \( Y^0 \); the homogenized filter is linear only if \( Y^0 \) is used), the homogenized Kalman smoother is found to be adequate, and is less computationally intensive. Figure 6.1 shows the EM estimates of \( \theta^* \). The true parameter value is set at 2, and EM iterations are started at random initial guesses and run for 60 and 75 iterations for the homogenized particle and Kalman smoothers, respectively. The EM estimates converge
close to the true parameter value, but still display error, which may be char-
acterized by a study of the asymptotic behavior of the ML estimator error.

Figure 6.1: True parameter value in black, EM iterations in red and blue, respectively

6.5 Numerical example: Lorenz ’96 model

Recall the Lorenz ’96 model from Section 4.3:

\[
\begin{align*}
    dX_t^k &= \left( -X_t^{k-1} (X_t^{k-2} - X_t^{k+1}) - X_t^k + F + h_x \sum_{j=1}^J Z_t^{j,k} \right) dt + \sigma_x dW_t, \\
    k &= 1, \ldots, K, \\
    dZ_t^{j,k} &= \left( -\frac{1}{\varepsilon} Z_t^{j+1,k} (Z_t^{j+2,k} - Z_t^{j-1,k}) - \frac{1}{\sqrt{\varepsilon}} Z_t^{j,k} - h_z X_t^k \right) dt + \frac{1}{\sqrt{\varepsilon}} \sigma_z dV_t, \\
    j &= 1, \ldots, J,
\end{align*}
\]

where \( t \in [0, T] \), with observation

\[
    Y_t = H \begin{bmatrix} X_t \\ Z_t \end{bmatrix} + \sigma_y B_t, \quad Y \in \mathbb{R}^K.
\]

The homogenized signal is

\[
    d\tilde{X}_t^k = \left( -\tilde{X}_t^{k-1} (\tilde{X}_t^{k-2} - \tilde{X}_t^{k+1}) - \tilde{X}_t^k + F + h_x \tilde{b}_2(\tilde{X}_t^k) \right) dt + \sigma_x dW_t,
\]

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with homogenized sensor function \( \bar{H} = H_1.K,1.K \). In discrete time,

\[
\bar{X}_t^k = \bar{X}_{t-1}^k + \left( -\bar{X}_{t-1}^k (\bar{X}_{t-1}^{k-2} - \bar{X}_{t-1}^{k+1}) - \bar{X}_{t-1}^k + \theta_1 + \theta_2 \bar{b}_2(\bar{X}_{t-1}^k) \right) \Delta t
\]

\[
+ \sqrt{\Delta t} \sigma_x \mathcal{N}(0,1),
\]

where \( t \in \{1, \ldots, \lfloor T/N \rfloor \} \) and \( \mathcal{N}(0,1) \) is a standard normal random variable (to reduce cumbersome notations, we abuse notation \( t \) here and use it to represent the discrete time index as well).

Consider the 20-dimensional case with \( K = 4, J = 4, \varepsilon = 0.01 \), and \( h_x = -h_z = h \). We take the external forcing and coupling parameter as unknowns, i.e. \( \theta = (\theta_1, \theta_2) \), \( \theta_1 = F, \theta_2 = h \). For maximum likelihood estimation, consider discrete time. The log likelihood function is

\[
\log p(Y_{0:|T/N|}; \theta)
\]

\[
= \mathbb{E}_{Q_{\theta'}} \left[ \log p(X_{0:|T/N|}, Z_{0:|T/N|}, Y_{0:|T/N|}; Y_{0:|T/N|}) \right]_{Q_{\theta'}}
\]

\[
- \mathbb{E}_{Q_{\theta'}} \left[ \log p(X_{0:|T/N|}, Z_{0:|T/N|}; Y_{0:|T/N|}; Y_{0:|T/N|}) \right]_{Q_{\theta'}}.
\]

By the EM algorithm, we only need to maximize \( Q_{\theta', \theta} \), since because \( H_{\theta, \theta'} \) is always non-increasing. The discrete time setting lets us write the joint density \( p(X_{0:T}, Z_{0:T}, Y_{0:T}; \theta) \) in terms of the product of conditional densities \( p(Y_t|X_t, Z_t)p(X_t, Z_t|X_{t-1}, Z_{t-1}) \):

\[
Q_{\theta', \theta}
\]

\[
= \mathbb{E}_{Q_{\theta'}} \left[ \log \left( p(X_0, Z_0) \prod_{t=1}^{T/N} p(Y_t|X_t, Z_t)p(X_t, Z_t|X_{t-1}, Z_{t-1}) \right) \right]_{Y_{0:|T/N|}}
\]

\[
= \mathbb{E}_{Q_{\theta'}} \left[ \log p(X_0, Z_0)
\right]
\]

\[
+ \sum_{t=1}^{T/N} \left( \log p(Y_t|X_t, Z_t) + \log p(X_t, Z_t|X_{t-1}, Z_{t-1}) \right)_{Y_{0:|T/N|}}
\]

\[
= \mathbb{E}_{Q_{\theta'}} \left[ \log p(X_0, Z_0)
\right]
\]
\[\begin{align*}
&+ \sum_{t=1}^{\lfloor T/N \rfloor} \left( -\frac{K}{2} \log(2\pi) - \frac{1}{2} \log |\Sigma_y| 
\right. \\
&\left. - \frac{1}{2} (Y_t - h(X_t, Z_t))^* \Sigma_y^{-1} (Y_t - h(X_t, Z_t)) 
\right. \\
&\left. - \frac{K(1 + J)}{2} \log(2\pi) - \frac{\Delta t}{2} \log \left| \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_z \end{bmatrix} \right| 
\right. \\
&\left. - \frac{1}{2\Delta t} \left( \begin{bmatrix} X_t \\ Z_t \end{bmatrix} - b_\theta(X_{t-1}, Z_{t-1}, \Delta t) \right)^* 
\right. \\
&\left. \times \left[ \Sigma_x \ 0 \right]^{-1} \left( \begin{bmatrix} X_t \\ Z_t \end{bmatrix} - b_\theta(X_{t-1}, Z_{t-1}, \Delta t) \right) \right| Y_{0: T/N} 
, 
\end{align*}\]

where \( b_\theta(X_{t-1}, Z_{t-1}, \Delta t) \) is \( \begin{bmatrix} X_{t-1} \\ Z_{t-1} \end{bmatrix} \) plus the drift multiplied by discrete time increment \( \Delta t \), and \( \Sigma = \begin{bmatrix} \sigma_x^2 & 0 & \ldots & 0 \\ 0 & \sigma_z^2 & \ldots & 0 \\ 0 & 0 & \ldots & \sigma_z^2 \end{bmatrix} \). Continuing the equality,

\[Q(\theta, \theta')\]

\[= \mathbb{E}_{Q_{\theta'}} \left[ \log p(X_0, Z_0) - \lfloor T/N \rfloor K \left( 1 + \frac{J}{2} \right) \log(2\pi) - \frac{\lfloor T/N \rfloor}{2} \log |\Sigma_y| 
\right. \\
\left. - \frac{\lfloor T/N \rfloor \Delta t}{2} \log \left| \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_z \end{bmatrix} \right| 
\right. \\
\left. - \frac{1}{2} \sum_{t=1}^{\lfloor T/N \rfloor} (Y_t - h(X_t, Z_t))^* \Sigma_y^{-1} (Y_t - h(X_t, Z_t)) 
\right. \\
\left. - \frac{1}{2\Delta t} \sum_{t=1}^{\lfloor T/N \rfloor} \left( \begin{bmatrix} X_t \\ Z_t \end{bmatrix} - b_\theta(X_{t-1}, Z_{t-1}, \Delta t) \right)^* 
\right. \\
\left. \times \left[ \Sigma_x \ 0 \right]^{-1} \left( \begin{bmatrix} X_t \\ Z_t \end{bmatrix} - b_\theta(X_{t-1}, Z_{t-1}, \Delta t) \right) \right| y_{0:T} \]

For the homogenized likelihood function, we use \( \bar{Q}(\theta, \theta') \) in place of \( Q(\theta, \theta') \), where

\[\bar{Q}(\theta, \theta')\]
where \( \bar{b}_\theta \) is the drift of the homogenized signal (6.21).

### 6.5.1 \( \theta_1 \) unknown

Let \( \bar{b}_\theta(\bar{x}_{t-1}, \Delta t) := \bar{b}_\theta(\bar{x}_{t-1}, \Delta t) - \theta_1 \Delta t \), i.e. \( \bar{x}_{t-1} \) plus the part of drift in (6.21) that does not depend on \( \theta_1 \). Considering just the part of \( \bar{Q}(\theta, \theta') \) that depends on \( \theta \) in (6.22),

\[
\mathbb{E}_{\bar{Q}_{\theta'}} \left[ -\frac{1}{2\Delta t} \sum_{t=1}^{[T/N]} (\bar{x}_t - \bar{b}_\theta(\bar{x}_{t-1}, \Delta t))^* \Sigma_x^{-1} (\bar{x}_t - \bar{b}_\theta(\bar{x}_{t-1}, \Delta t)) \bigg| Y_{0:T} \right] 
\]

\[
= \mathbb{E}_{\bar{Q}_{\theta'}} \left[ -\frac{1}{2\Delta t} \sum_{t=1}^{[T/N]} \left( \bar{x}_t - \bar{b}_2(\bar{x}_{t-1}, \Delta t) - \theta_1 \Delta t \right)^* \Sigma_x^{-1} \left( \bar{x}_t - \bar{b}_2(\bar{x}_{t-1}, \Delta t) - \theta_1 \Delta t \right) \bigg| Y_{0:T} \right] 
\]

\[
= \mathbb{E}_{\bar{Q}_{\theta'}} \left[ \frac{1}{2} \sum_{t=1}^{[T/N]} \left( \bar{x}_t^* \Sigma_x^{-1} \theta_1 + \theta_1^* \Sigma_x^{-1} \bar{x}_t - \theta_1^* \Sigma_x^{-1} \bar{b}_2(\bar{x}_{t-1}, \Delta t) - \bar{b}_2(\bar{x}_{t-1}, \Delta t)^* \Sigma_x^{-1} \theta_1 - \Delta t \theta_1^* \Sigma_x^{-1} \theta_1 \right) \bigg| Y_{0:T} \right] 
\]

\[
= \mathbb{E}_{\bar{Q}_{\theta'}} \left[ \frac{1}{2} \sum_{t=1}^{[T/N]} \left( 2\theta_1 \sum_{k=1}^{K} \bar{x}_t^k (\Sigma_x^{-1})^{kk} \right. \right. 
\]

\[
-2\theta_1 \sum_{k=1}^{K} \bar{b}_2^k(\bar{x}_{t-1}, \Delta t) (\Sigma_x^{-1})^{kk} \Delta t \theta_1^2 \text{tr}(\Sigma_x^{-1}) \bigg| Y_{0:T} \right] 
\]

\[
= \mathbb{E}_{\bar{Q}_{\theta'}} \left[ \sum_{t=1}^{[T/N]} \left( \theta_1 \sigma_x^2 \sum_{k=1}^{K} \left( \bar{x}_t^k - \bar{b}_2^k(\bar{x}_{t-1}, \Delta t) \right) - \frac{\Delta t}{2} \theta_1^2 \text{tr}(\Sigma_x^{-1}) \right) \bigg| Y_{0:T} \right] 
\]

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\[
= \mathbb{E}_{\theta'} \left[ \frac{\theta_1}{\sigma_x^2} \sum_{t=1}^{[T/N]} \sum_{k=1}^K \left( \bar{X}_t^k - \tilde{b}_2^k(\bar{X}_{t-1}, \Delta t) \right) - \frac{[T/N]\Delta t}{2} \theta_1^2 \operatorname{tr}(\Sigma_x^{-1}) \right] Y_{0:[T/N]}
\]

\[
= \frac{\theta_1}{\sigma_x^2} \sum_{t=1}^{[T/N]} \sum_{k=1}^K \mathbb{E}_{\theta'} \left[ \left( \bar{X}_t^k - \tilde{b}_2^k(\bar{X}_{t-1}, \Delta t) \right) \right] Y_{0:[T/N]} - \frac{K[T/N]\Delta t}{2\sigma_x^2} \theta_1^2 \tag{6.23}
\]

Setting derivative wrt $\theta_1$ to zero and solving for $\theta_1$:

\[
\hat{\theta}_1^T = \frac{1}{K[T/N]\Delta t} \sum_{t=1}^{[T/N]} \sum_{k=1}^K \mathbb{E}_{\theta'} \left[ \left( \bar{X}_t^k - \tilde{b}_2^k(\bar{X}_{t-1}, \Delta t) \right) \right] Y_{0:[T/N]}
\]

The second derivative of (6.23) is $-\frac{K[T/N]\Delta t}{\sigma_x^2} < 0$, so $\hat{\theta}_1^T$ is a maximizer of $\bar{Q}(\theta, \theta')$. Figure 6.2 shows the EM iterations for the estimates of $\theta_1$ using the homogenized particle smoother at two different forcing values.

![Figure 6.2](image)

Figure 6.2: True parameter value in black, EM iterations in blue; $T=100$ days, 50 EM iterations

6.5.2 $\theta_2$ unknown

Let $\tilde{b}_1(\bar{X}_{t-1}, \Delta t) := \bar{b}_1(\bar{X}_{t-1}, \Delta t) - \theta_2 \tilde{b}_2(\bar{X}_{t-1}) \Delta t$, i.e. $\bar{X}_{t-1}$ plus the part of drift in (6.21) that does not depend on $\theta_2$. Considering just the part of
\( \tilde{Q}(\theta, \theta') \) that depends on \( \theta \) in (6.22),

\[
\mathbb{E}_{Q_{\theta'}} \left[ -\frac{1}{2\Delta t} \sum_{t=1}^{\lfloor T/N \rfloor} \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) \right)^* \bar{b}_2(\bar{X}_{t-1}) \right] \left| Y_0: [T/N] \right]
\]

\[
= \mathbb{E}_{Q_{\theta'}} \left[ -\frac{1}{2\Delta t} \sum_{t=1}^{\lfloor T/N \rfloor} \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) - \theta_2 \bar{b}_2(\bar{X}_{t-1}) \right)^* \right. \\
\times \Sigma_x^{-1} \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) - \theta_2 \bar{b}_2(\bar{X}_{t-1}) \right) \left| Y_0: [T/N] \right]
\]

\[
= \mathbb{E}_{Q_{\theta'}} \left[ -\frac{1}{2\Delta t} \sum_{t=1}^{\lfloor T/N \rfloor} \left( -\theta_2 \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) \right)^* \Sigma_x^{-1} \bar{b}_2(\bar{X}_{t-1}) \Delta t \\
- \theta_2 \Delta t \Sigma_x^{-1} \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) \right) \right. \\
\left. + \theta_2^2 \Delta t^2 \Sigma_x^{-1} \bar{b}_2(\bar{X}_{t-1}) \left| Y_0: [T/N] \right] \right]
\]

\[
= \mathbb{E}_{Q_{\theta'}} \left[ \sum_{t=1}^{\lfloor T/N \rfloor} \left( \theta_2 \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) \right)^* \Sigma_x^{-1} \bar{b}_2(\bar{X}_{t-1}) \right.ight. \\
\left. - \frac{\Delta t^2 \theta_2^2}{2} \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) \right)^* \Sigma_x^{-1} \bar{b}_2(\bar{X}_{t-1}) \right) \left| Y_0: [T/N] \right]
\]

\[
= \frac{1}{\sigma_x^2} \sum_{t=1}^{\lfloor T/N \rfloor} \left( \theta_2 \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) \right)^* \bar{b}_2(\bar{X}_{t-1}) \right)
\]

\[
- \frac{\Delta t^2 \theta_2^2}{2} \left( \bar{b}_2(\bar{X}_{t-1}) \right) \left| Y_0: [T/N] \right]
\]

\[
= \frac{\theta_2}{\sigma_x^2} \sum_{t=1}^{\lfloor T/N \rfloor} \mathbb{E}_{Q_{\theta'}} \left[ \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) \right)^* \bar{b}_2(\bar{X}_{t-1}) \right] \left| Y_0: [T/N] \right]
\]

\[
- \frac{\Delta t^2 \theta_2^2}{2} \sum_{t=1}^{\lfloor T/N \rfloor} \mathbb{E}_{Q_{\theta'}} \left[ \bar{b}_2(\bar{X}_{t-1}) \right] \left| Y_0: [T/N] \right]
\]

(6.24)

Setting the derivative of (6.24) to zero and solving for \( \theta_2 \) gives us

\[
\hat{\theta}_2^T = \left( \Delta t \sum_{t=1}^{\lfloor T/N \rfloor} \mathbb{E}_{Q_{\theta'}} \left[ \bar{b}_2(\bar{X}_{t-1}) \right] \left| Y_0: [T/N] \right] \right)^{-1}
\]

\[
\times \sum_{t=1}^{\lfloor T/N \rfloor} \mathbb{E}_{Q_{\theta'}} \left[ \left( \bar{X}_t - \bar{b}_1(\bar{X}_{t-1}, \Delta t) \right)^* \bar{b}_2(\bar{X}_{t-1}) \right] \left| Y_0: [T/N] \right].
\]

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The second derivative of (6.24) wrt $\theta$ is

$$-rac{\Delta t}{\sigma^2_x} \sum_{t=1}^{T/N} \mathbb{E}_{\mathcal{Q}_{\theta'}} \left[ \bar{b}_2(\bar{X}_{t-1})^* \bar{b}_2(\bar{X}_{t-1}) \left| y_{0:T} \right. \right] < 0,$$

so $\hat{\theta}_2^T$ is a maximizer of $\bar{Q}(\theta, \theta')$. Figure 6.3 shows the EM iterations for the estimates of $\theta_2$ using the homogenized particle smoother at two different forcing values.

![Figure 6.3](image_url)

Figure 6.3: True parameter value in black, EM iterations in blue; $T=5000$ days, 30 EM iterations

In estimation of $\theta_2$, instances where the EM iterations converge to values away from the true parameter value have been observed (see Figure 6.4). The true parameter value $\alpha_2$ is always set to be $< 0$. When the initial condition $\hat{\theta}_2^{0,T}$ is $> 0$, EM iterations can converge to the positive value $|\alpha_2|$ (red curves in Figure 6.4), although it is not a necessary occurrence, as seen in Figure 6.3, where there are initial conditions located on the positive axis that result in convergence to the true parameter value. The parameter $\theta_2$ represents the amplitude of forcing on the slow component due to oscillations at the fast scale. One possible explanation is that it is the variance of the forcing (proportional to square of the forcing amplitude) that is important for estimating the slow dynamics, hence the sign of the forcing is not important. However, there are cases in which the initial condition $\hat{\theta}_2^{0,T}$ is very far off the true parameter value, resulting in an EM estimate that is an order of magnitude off of the true value (magenta in Figure 6.4(a)). In the numerical experiments, the EM iterations initial conditions are chosen randomly. In practice, there
should usually be more knowledge to narrow down initial condition selection to a subset of the parameter space containing the true value such that the initial condition is close. Convergence of the EM iterations are observed to be relatively fast, to be close to the true parameter value within 10 iterations or less, but requires a large observation window.

Figure 6.4: True parameter value in black, EM iterations in blue; $T=5000$ days, 30 EM iterations

The numerical examples in this and the previous sections lead to the question of the error of the reduced-order estimator in the limit of wide timescale separation and infinite observations. The estimates in Figures 6.1 and 6.3 each converge close to the true parameter value, but show a bias, in these cases, an error between the empirical mean of the EM estimates and the true value. This question is discussed further in the conclusions and future work chapter.
CHAPTER 7

INFORMATION FLOW IN MULTISCALE SYSTEMS

In Chapters 3 and 6, the interaction between observation of the multiscale signal and the weak limit of the coarse-grained dynamics is studied using stochastic averaging, filtering theory and maximum likelihood estimation, and reduced-order state and parameter estimators are developed. In this chapter, information theoretic methods are utilized to quantify the time evolution of uncertainty about the signal given observations, and information shared between signal and observations in multiscale systems. Specifically, concepts from information theory are used with the reduced-order filtering results of Chapter 3 to quantify the time evolution of the mutual information between the limiting process of the coarse-grained dynamics and observations of the multiscale signal.

The important concepts of entropy and mutual information from information theory for quantifying uncertainty and information is outlined in Section 7.1. The main results of this chapter are equations for quantifying uncertainty and information of the multiscale system using the reduced-order filter of Chapter 3. They are described in Section 7.2. Section 7.3 describes existing works for single-scale systems. The time evolution equation of entropy and mutual information for the multiscale system based on the reduced-order filter are presented in Section 7.4.

7.1 Shannon entropy, Kullback-Leibler divergence and mutual information

We briefly describe some well-known concepts from information theory. For detailed discussions, see, for example, Chapter 2 of [114].

Definition 7.1.1 Shannon entropy measures the information content of a random variable, or how much memory (in units of bits, for example, if
using log base 2) is required to store the information required to describe the random variable. It can be interpreted as how much uncertainty there is about the random variable. For a continuous random variable, its entropy $H$ is:

$$H(X) = -\int_{\mathbb{R}^m} p(x) \log p(x) dx$$

For example, for a $d$-dimensional normal distribution $X \sim \mathcal{N}(\mu, \Sigma)$, $H(X) = \frac{1}{2} \log((2\pi \exp(1))^d |\Sigma|)$ (see Appendix 7). Entropy is always nonnegative if the density is given by a mass function. We can also define joint and conditional entropy for two random variables:

$$H(X, Y) = -\int_{x \times y} p(x, y) \log p(x, y) dxdy,$$
$$H(X|Y) = -\int_{x \times y} p(x, y) \log p(x|y) dxdy.$$

Note that conditional entropy is not symmetric.

**Definition 7.1.2 Kullback-Leibler divergence** ($D_{KL}$) or relative entropy is a distance measure between two densities. For densities $p$ and $q$,

$$D_{KL}(p||q) = \int_X p(x) \log \left( \frac{p(x)}{q(x)} \right) dx.$$ 

If $p$ is the actual density for a random variable $X$, then $D_{KL}[p||q]$ can be interpreted as the loss of information due to using $q$ instead of $p$ as the density of $X$.

**Definition 7.1.3 Mutual information** between two random variables is the relative entropy between the joint density and the product of the marginal density.

$$I(X; Y) = \int_{x \times y} p(x, y) \log \left( \frac{p(x, y)}{p(x)p(y)} \right) dxdy,$$

which can be checked to be equal to $H(X) - H(X|Y)$, or $H(Y) - H(Y|X)$ by symmetry. Hence, mutual information represents the reduction of entropy (uncertainty) in one random variable due to the knowledge of another random variable.
In the preceding definitions, $X$ is defined to be a random variable, but they can be defined for a stochastic process $X_t$ for each time $t > 0$ as well.

As stated in Definition 7.1.3, mutual information can be interpreted as the difference between entropy and conditional entropy:

$$I(X; Y) = - \int_{X \times Y} p(x, y) \log \frac{p(x)p(y)}{p(x, y)} \, dx \, dy$$

$$= - \int_{X \times Y} p(x, y) \log p(x) \, dx \, dy - \left( - \int_{X \times Y} p(x, y) \log \frac{p(x, y)}{p(y)} \, dx \, dy \right)$$

$$= - \int_{X \times Y} p(x) \log p(x) \, dx - \left( - \int_{X \times Y} p(x, y) \log p(x|y) \, dx \, dy \right)$$

$$= \mathcal{H}(X) - \mathcal{H}(X|Y),$$

or

$$I(X; Y) = - \int_{X \times Y} p(x, y) \log \frac{p(x)p(y)}{p(x, y)} \, dx \, dy$$

$$= - \int_{X \times Y} p(x, y) \log p(y) \, dx \, dy - \left( - \int_{X \times Y} p(x, y) \log \frac{p(x, y)}{p(x)} \, dx \, dy \right)$$

$$= - \int_{X \times Y} p(x) \log p(x|y) \, dy - \left( - \int_{X \times Y} p(x, y) \log p(y|x) \, dx \, dy \right)$$

$$= \mathcal{H}(Y) - \mathcal{H}(Y|X)$$

(here, we take $0 \log \frac{0}{0} = 0$). In other words, mutual information is symmetric. The quantity $\log \frac{p(x)p(y)}{p(x, y)}$ quantifies the “error” in assuming independence of $X$ and $Y$ when they are not independent.

### 7.2 Problem formulation and statement of main results

The problem setting is the same as in Chapter 3, with multiscale signal and observation given by (3.1) and (3.3), which are restated here: Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ be a probability space that supports a $(k+l+d)$-dimensional...
Q-Brownian motion \((W, V, B)\). The multiscale signal is

\[
dX^\varepsilon_t = b(X^\varepsilon_t, Z^\varepsilon_t)dt + \sigma(X^\varepsilon_t, Z^\varepsilon_t)dW_t
\]

\[
dZ^\varepsilon_t = \frac{1}{\varepsilon}f(X^\varepsilon_t, Z^\varepsilon_t)dt + \frac{1}{\sqrt{\varepsilon}}g(X^\varepsilon_t, Z^\varepsilon_t)dV_t,
\]

with observation

\[
Y^\varepsilon_t = \int_0^t h(X^\varepsilon_s, Z^\varepsilon_s)ds + B_t.
\]

As before, assume that the fast component with \(X^\varepsilon = x\) fixed, \(Z^x\), is exponentially mixing and attains its invariant measure \(\mu(x, dz)\) exponentially fast. Assume the following recurrence condition for the fast drift coefficient

\[
\lim_{\|z\| \to \infty} \sup_{x \in \mathbb{R}_m} \langle f(x, z), z \rangle = -\infty
\]

and nondegeneracy of the fast diffusion coefficient, that there exists \(0 < a < A < \infty\) such that

\[
aI_{n \times n} \leq gg^*(x, z) \leq AI_{n \times n}
\]

Conditions (7.3) and (7.4) guarantees the existence and uniqueness, respectively, of \(\mu(x, dz)\) ([65, 80]).

Let \(X^0\) be the solution of a SDE

\[
dx^0_t = \bar{b}(X^0_t)dt + \bar{\sigma}(X^0_t)dW_t,
\]

where \(\bar{b}\) and \(\bar{\sigma}\) are, respectively, the drift and diffusion coefficients of (7.1a) averaged with respect to the invariant measure of \(Z^x\):

\[
\bar{b}(x) = \int_{\mathbb{R}^n} b(x, z)\mu(x, dz), \quad \bar{\sigma \sigma}^*(x) = \int_{\mathbb{R}^n} \sigma\sigma(x, z)\mu(x, dz).
\]

In Section 3.7, it is shown that when \(X^\varepsilon\) converges in law to \(X^0\) as \(\varepsilon \to 0\), the slow component \(X^\varepsilon\) can be estimated using the dynamics of \(X^0\) and the true observation \(Y^\varepsilon\), i.e., a reduced-order filter using \(X^0\) and information from \(Y^\varepsilon\) can be used. In this chapter, we are interested in quantifying the uncertainty and information content of the reduced-order filter. Since the
reduced-order filter estimates $X^0$, we are interested in how uncertainty about $X^0$ changes given information from $Y^\varepsilon$. Specifically, we determine how the conditional entropy of $X^0$ given $Y^\varepsilon$ and the mutual information between $X^0$ and $Y^\varepsilon$ evolves with time.

We will work with densities and conditional densities in this chapter, instead of the measure-valued processes of Chapters 3 and 6. Hence, we restate the filtering quantities and equations of Chapter 3 in terms of densities here.

Denote by $L_\varepsilon = L_F + L_S$ the differential operator associated to $(X^\varepsilon, Z^\varepsilon)$, i.e.

\[
L_F = \sum_{i=1}^n f_i(x, z) \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{i,j=1}^n (gg^*)_{ij}(x, z) \frac{\partial^2}{\partial z_i \partial z_j},
\]

\[
L_S = \sum_{i=1}^m b_i(x, z) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^m (\sigma \sigma^*)_{ij}(x, z) \frac{\partial^2}{\partial x_i \partial x_j}.
\]

Denote by $\bar{L}$

\[
\bar{L} \overset{\text{def}}{=} \int_{\mathbb{R}^n} L_S \mu(x, dz)
\]

\[
= \sum_{i=1}^m \int_{\mathbb{R}^n} b_i(x, z) \mu(x, dz) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^m \int_{\mathbb{R}^m} (\sigma \sigma^*)_{ij}(x, z) \mu(x, dz) \frac{\partial^2}{\partial x_i \partial x_j}
\]

\[
= \sum_{i=1}^m \bar{b}_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^m (\bar{\sigma} \bar{\sigma}^*)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.
\]  \hspace{1cm} (7.6)

$\bar{L}$ generates the diffusion process $X^0$ on $\mathbb{R}^m$.

Let $x = (x, z) \in \mathbb{R}^m \times \mathbb{R}^n$. Let $q_t^\varepsilon$ denote the density of $(X_t^\varepsilon, Z_t^\varepsilon)$ with respect to $m + n$-dimensional Lebesgue measure $dx$ (on $\mathbb{R}^m \times \mathbb{R}^n$). Then, for $\varphi \in C_0^2(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$,

\[
\mathbb{E}_Q[\varphi(X_t^\varepsilon, Z_t^\varepsilon)] = \int_{\mathbb{R}^m \times \mathbb{R}^n} \varphi(x) q_t^\varepsilon(x) dx.
\]

Applying Itô’s lemma, we have the Kolmogorov forward/Fokker-Planck equation

\[
\frac{d}{dt} \mathbb{E}_Q[\varphi(X_t^\varepsilon, Z_t^\varepsilon)] = \mathbb{E}_Q[L^\varepsilon \varphi(X_t^\varepsilon, Z_t^\varepsilon)],
\]

\[
\mathbb{E}_Q[\varphi(X_0^\varepsilon, Z_0^\varepsilon)] = \int_{\mathbb{R}^m \times \mathbb{R}^n} \varphi(x) q_0^\varepsilon(x) dx
\]
Writing in terms of density and integrating by parts,

\[
\frac{d}{dt} \int_{\mathbb{R}^m \times \mathbb{R}^n} \varphi(x) q^\varepsilon_t(x) \, dx = \int_{\mathbb{R}^m \times \mathbb{R}^n} \mathcal{L}^\varepsilon \varphi(x) q^\varepsilon_t(x) \, dx
\]

\[
\int_{\mathbb{R}^m \times \mathbb{R}^n} \varphi(x) \frac{d}{dt} q^\varepsilon_t(x) \, dx = \int_{\mathbb{R}^m \times \mathbb{R}^n} \varphi(x) \mathcal{L}^\varepsilon q^\varepsilon_t(x) \, dx
\]

\[
\implies \frac{d}{dt} q^\varepsilon_t(x) = \mathcal{L}^\varepsilon q^\varepsilon_t(x),
\]

where \( q^i \) denotes the initial density of the state \((X^\varepsilon, Z^\varepsilon)\) and \( \mathcal{L}^\varepsilon^* \) is the adjoint of \( \mathcal{L}^\varepsilon \),

\[
\mathcal{L}^\varepsilon^* = \frac{1}{\varepsilon} \mathcal{L}_F^* + \mathcal{L}_S^*,
\]

\[
\mathcal{L}_F^* \varphi(x) = -\sum_{i=1}^{n} \frac{\partial}{\partial z_i} (f_i(x) \varphi(x)) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial z_i \partial z_j} ((g g^*)_{ij}(x) \varphi(x)),
\]

\[
\mathcal{L}_S^* \varphi(x) = -\sum_{i=1}^{m} \frac{\partial}{\partial x_i} (b_i(x) \varphi(x)) + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\sigma \sigma^*)_{ij}(x) \varphi(x)).
\]

Let \( q^0_t \) denote the density of \( X^0_t \) with respect to \( m \)-dimensional Lebesgue measure \( dx \). \( q^0 \) similarly satisfies

\[
\frac{d}{dt} q^0_t(x) = \mathcal{L}_F^* q^0_t(x),
\]

\[
q^0_0(x) = \int_{\mathbb{R}^n} q^i(x) \, dz,
\]

where

\[
\mathcal{L}^* \varphi(x) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} (b_i(x) \varphi(x)) + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\sigma \sigma^*)_{ij}(x) \varphi(x)).
\]

Let \( \mathcal{Y}_s^\varepsilon = \sigma(Y_s^\varepsilon : 0 \leq s \leq t) \cup \mathcal{N} \), \( \mathcal{N} \) are the \( \mathbb{Q} \)-negligible sets. Let \( \pi^\varepsilon_t \) denote the conditional density of \((X^\varepsilon_t, Z^\varepsilon_t)\) given \( \mathcal{Y}_t^\varepsilon \), i.e. \( \pi^\varepsilon_t(x) = q^\varepsilon_t(x | y_{[0,t]}) \).

Note that in this chapter, we use \( \pi \) to denote density, instead of a measure-valued process as in Chapters 3 and 6. The best estimate of the slow signal component given observations is the conditional expectation (filter)

\[
\mathbb{E}_{\mathbb{Q}} [\varphi(X^\varepsilon_t, Z^\varepsilon_t) | \mathcal{Y}_t^\varepsilon] = \int_{\mathbb{R}^m \times \mathbb{R}^n} \varphi(x) \pi^\varepsilon_t(x) \, dx.
\]
By the Kallianpur Striebel formula (see Appendix 1),

\[ E_Q[\varphi(X_t^\varepsilon, Z_t^\varepsilon)\mid \mathcal{Y}_t^\varepsilon] = \frac{E_{\mathbb{P}^\varepsilon} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon} \Big| \mathcal{F}_t \right] \mathcal{Y}_t^\varepsilon}{E_{\mathbb{P}^\varepsilon} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon} \Big| \mathcal{Y}_t^\varepsilon \right]}, \]

where under \( \mathbb{P}^\varepsilon \), \( Y^\varepsilon \) is a \( d \)-dimensional Brownian motion independent of \( (W, V) \) while the dynamics of \( (X^\varepsilon, Z^\varepsilon) \) remain unchanged, and \( \frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon} \big| \mathcal{F}_t \) is the measure change by Girsanov’s theorem. The unnormalized filter \( E_{\mathbb{P}^\varepsilon} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}^\varepsilon} \Big| \mathcal{Y}_t^\varepsilon \right] \) satisfies the Zakai equation 3.4.

Let \( p^\varepsilon_t \) denote the unnormalized conditional density of \( (X_t^\varepsilon, Z_t^\varepsilon) \) given \( Y_t^\varepsilon \). Then,

\[ E_{\mathbb{P}^\varepsilon} \left[ \varphi(X_t^\varepsilon, Z_t^\varepsilon) \bigg| \mathcal{Y}_t^\varepsilon \right] = \int_{\mathbb{R}^m \times \mathbb{R}^n} \varphi(x) p^\varepsilon_t(x) \, dx. \]

and

\[ \pi^\varepsilon_t(x) = \frac{p^\varepsilon_t(x)}{\int_{\mathbb{R}^m \times \mathbb{R}^n} p^\varepsilon_t(x') \, dx'}. \]

Writing the Zakai equation (3.4) in terms of \( p^\varepsilon \) and integrating by parts gives us the density-valued Zakai equation

\[ dp^\varepsilon_t(x) = \mathcal{L}^\varepsilon p^\varepsilon_t(x) \, ds + p^\varepsilon_t(x) \, h^\varepsilon(x) \, dY^\varepsilon_t, \]

\[ p^\varepsilon_0(x) = q^\varepsilon(x). \]

Let \( p^0_t \) be the density with respect to \( m \)-dimensional Lebesgue measure \( dx \) such that

\[ E_{\mathbb{P}^\varepsilon} \left[ \varphi(X_t^0) \tilde{D}_t^0 \bigg| \mathcal{Y}_t^\varepsilon \right] = \int_{\mathbb{R}^m} \varphi(x) p^0_t(x) \, dx, \]

where

\[ \tilde{D}_{t,T} \overset{\text{def}}{=} \exp \left( \int_t^T \bar{h}(X_r^0)^* dY_r^\varepsilon - \frac{1}{2} \int_t^T \bar{h}(X_r^0)^2 \, dr \right) \]

and the unnormalized homogenized filter \( E_{\mathbb{P}^\varepsilon} \left[ \varphi(X_t^0) \tilde{D}_t^0 \bigg| \mathcal{Y}_t^\varepsilon \right] \) satisfies the
SPDE (3.6) (see Section 3.1). From (3.6), \( p^0_t \) satisfies the SPDE

\[
dp^0_t(x) = \mathcal{L}^* p^0_t(x) ds + p^0_t(x) \hat{h}(x)^* dY^\varepsilon_t, \tag{7.8}
\]

\[
p^0_0(x) = \int_{\mathbb{R}^n} q^1(x) dz.
\]

Using \( p^0 \), we define \( \pi^0 \) as

\[
\pi^0_t(x) = \frac{p^0_t(x)}{\int_{\mathbb{R}^n} p^0_t(x') dx'}.
\]

Let \( \pi^\varepsilon_{x} \) and \( p^\varepsilon_{x} \) denote the \( x \)-marginals of \( q^\varepsilon \), \( \pi^\varepsilon \) and \( p^\varepsilon \), respectively, i.e.

\[
\pi^\varepsilon_{x}(x) \equiv \int_{\mathbb{R}^n} \pi^\varepsilon_{t}(x) dz, \quad p^\varepsilon_{x}(x) \equiv \int_{\mathbb{R}^n} p^\varepsilon_{t}(x) dz.
\]

The results of Section 3.7 state that when \( \varepsilon \) is small, if we are only interested in the distribution of the slow component of (7.1), the densities \( (\pi^0, p^0) \) can be used in place of \( (\pi^\varepsilon_{x}, p^\varepsilon_{x}) \), by using the reduced-order filter specified in Section 3.1. Here, we study how uncertainty about \( X^0 \) changes given information from \( \mathcal{Y}^\varepsilon \). Specifically, we study how uncertainty about \( X^0 \) changes given information from \( \mathcal{Y}^\varepsilon \) changes with time using conditional entropy, and how much of that change is due to information from \( \mathcal{Y}^\varepsilon \) using mutual information.

Denote by \( \mathcal{H}(X^0_t|Y^\varepsilon_{[0,t]}) \) the conditional entropy of \( X^0_t \) given observations \( Y^\varepsilon \) up to time \( t \), i.e.

\[
\mathcal{H}(X^0_t|Y^\varepsilon_{[0,t]}) \equiv -\mathbb{E}_Q \left[ \log \pi^0_t(X^0_t) \right],
\]

where the expectation is over \( X^0_t \) and \( Y^\varepsilon_{[0,t]} \). Denote by \( I(X^0_t; Y^\varepsilon_{[0,t]}) \) the mutual information between \( X^0_t \) and observations \( Y^\varepsilon \) up to time \( t \). Then (see Section 7.1),

\[
I(X^0_t; Y^\varepsilon_{[0,t]}) = \mathcal{H}(X^0_t) - \mathcal{H}(X^0_t|Y^\varepsilon_{[0,t]}). \tag{7.9}
\]

The main results are (Lemma 7.4.2 and Theorem 7.4.3):

The time rate of change of the conditional entropy of \( X^0 \) given \( Y^\varepsilon_{[0,t]} \) is

\[
\frac{d\mathcal{H}_t(X^0_t|Y^\varepsilon_{[0,t]})}{dt} = \mathbb{E}_Q \left[ \text{tr} \left( \nabla b_t(X^0_t) \right) \right] - \frac{1}{2} \mathbb{E}_Q \left[ \text{tr} \left( (\bar{\sigma} \bar{\sigma}^*) (X^0_t) \nabla^2 \log \pi^0_t(X^0_t) \right) \right]
\]

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\[
- \frac{1}{2} \left\{ \mathbb{E}_Q \left[ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \hat{h}_t \|^2 \right] - \mathbb{E}_Q \left[ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \bar{h}(X_t^0) \|^2 \right] \right\}, \]

\[ \mathcal{H}_0(X_0 | Y_0^\varepsilon) = -\mathbb{E}_Q \left[ \log \int_{\mathbb{R}^n} q^i(X_0^0, z) \, dz \right], \]

where \( \hat{h}_t \overset{\text{def}}{=} \mathbb{E}_Q \left[ \bar{h}(X_t^0) \right]. \)

The time rate of change of the mutual information between \( X^0 \) and \( Y_{[0,t]}^\varepsilon \) is

\[
\frac{dI_t(X_t^0; Y_{[0,t]}^\varepsilon)}{dt} = \frac{1}{2} \left\{ \mathbb{E}_Q \left[ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \hat{h}_t \|^2 \right] - \mathbb{E}_Q \left[ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \bar{h}(X_t^0) \|^2 \right] \right\}
\]

\[ + \frac{1}{2} \left\{ \mathbb{E}_Q \left[ \| (\sigma^* \sigma^\top)^{1/2} (X_t^0) \nabla \log \pi_t^0(X_t) \|^2 \right] - \mathbb{E}_Q \left[ \| (\sigma^* \sigma^\top)^{1/2} (X_t^0) \nabla \log q_t^0(X_t^0) \|^2 \right] \right\}, \]

\[ I_0(X_0^0, Y_0^\varepsilon) = 0. \]

### 7.3 Existing and related works

The time rate of change of mutual information between the signal and observation in the single-timescale setting has been obtained in [115]. The problem setting of [115] is

\[
dX_t = b(X_t)dt + \sigma(X_t)dW_t
\]

with observation

\[ Y_t = \int_0^t h(X_s)ds + B_t. \]

The time rate of change of mutual information between \( X_t \) and \( Y_{[0,t]} \) is

\[
\frac{dI_t(X_t; Y_{[0,t]})}{dt} = \frac{1}{2} \left\{ \mathbb{E}_Q \left[ \| h(X_t) - \hat{h}_t \|^2 \right] \right\}
\]

\[ + \frac{1}{2} \left\{ \mathbb{E}_Q \left[ \| (\sigma^* \sigma^\top)^{1/2} (X_t) \nabla \log \pi_t(X_t) \|^2 \right]
\]

\[ - \mathbb{E}_Q \left[ \| (\sigma^* \sigma^\top)^{1/2} (X_t) \nabla \log q_t(X_t) \|^2 \right] \right\}, \]

\[ I_0(X_0, Y_0) = 0. \]
For the multiscale system that is filtered using the limiting homogenized process $X^0$ of (7.5), the rate of change of mutual information is similar, the difference being that uncertainty reduction from filtering is decreased due to the fact that the homogenized process $X^0$ and homogenized sensor function $\bar{h}$ is used in the filter (see discussion following Theorem 7.4.3). The procedure to determine the rate of change of mutual information here is based on that of [115], but differs slightly in that we determine the rates of change of entropy and conditional entropy separately and then use (7.9). Specifically,

\[
I(X_t; Y_{[0,t]}) = \mathcal{H}(X_t) - \mathcal{H}(X_t|Y_{[0,t]}) 
\]

\[
= \mathbb{E}_Q [-\log q_t(X_t)] - \mathbb{E}_Q [-\log \pi_t(X_t)] 
\]

\[
= \mathbb{E}_Q \left[ -\log \frac{q_t(X_t)}{\pi_t(X_t)} \right]. 
\]

[115] determines the rate of change of mutual information by directly evaluating the rate of change of (7.10b) while here, we evaluate the rates of change of each term in (7.10a) separately. The separate equations for entropy and conditional entropy allows us to look at the components of uncertainty growth due to signal dynamics and reduction due to filtering separately (see discussion following Lemma 7.4.2). The contribution of deterministic signal dynamics to uncertainty growth is not seen in the mutual information equation, as it appears in both the rates of change of entropy and conditional entropy. Studying the contribution of deterministic signal dynamics to uncertainty growth can provide insights into improving filtering schemes, for example by studying the “volumetric growth” in the state space due to the deterministic dynamics and identifying different growth rates in different directions.

For the linear system

\[
dX_t = BX_t dt + \sigma dW_t
\]

with observation

\[
Y_t = \int_0^t H X_s ds + B_t.
\]

[116] provides statistical mechanics interpretation of information flow for the Kalman-Bucy filter, using a thermodynamic entropy analog to information
flow between the observations and state spaces.

Entropy and mutual information is used in [32, 33, 34] to quantify uncertainty and information transfer for weather prediction applications. [32] uses mutual information to determine state predictability based on past partial observations in midlatitude atmospheric models. [33, 34] use entropy to quantify the time evolution of information transfer between different components of the system state, or different subsets of the state space.

7.4 Main results

We determine the equations for the rates of change of entropy of $X^0_t$ and conditional entropy of $X^0_t$ given $Y^ε_{[0,t]}$. The rate of change of mutual information between $X^0_t$ and $Y^ε_{[0,t]}$ then follows from (7.9).

For fixed $x ∈ R^m$, define

$$\eta^q_t(x) \overset{\text{def}}{=} -\log q^0_t(x), \quad \eta^p_t(x) \overset{\text{def}}{=} -\log p^0_t(x).$$

The entropy of $X^0_t$ is

$$\mathcal{H}_t(X^0_t) = \mathbb{E}_Q \left[ \eta^q_t(X^0_t) \right],$$

and conditional entropy of $X^0_t$ given $Y^ε_{[0,t]}$ is

$$\mathcal{H}_t(X^0_t | Y^ε_{[0,t]}) = \mathbb{E}_Q \left[ -\log \pi^0_t(X^0_t) \right] = \mathbb{E}_Q \left[ -\log \frac{p^0_t(X^0_t)}{\int_{R^m} p^0_t(\zeta) d\zeta} \right] = \mathbb{E}_Q \left[ \eta^p_t(X^0_t) \right] + \mathbb{E}_Q \left[ \log \int_{R^m} p^0_t(\zeta) d\zeta \right].$$

The normalizer of the conditional density $\int_{R^m} p^0_t(\zeta) d\zeta$, is (see Appendix 4)

$$\int_{R^m} p^0_t(\zeta) d\zeta = \exp \left\{ \int_0^t \hat{h}_s^* dY^ε_s - \frac{1}{2} \int_0^t \| \hat{h}_s \|^2 ds \right\},$$

where

$$\hat{h}_t \overset{\text{def}}{=} \mathbb{E}_Q \left[ \hat{h}(X^0_t) | \mathcal{Y}^ε_t \right].$$
Then, the conditional entropy is

\[
\mathcal{H}_t(X_t^0 | Y_{[0,t]}^\varepsilon) = \mathbb{E}_Q \left[ \eta^0_t(X_t^0) \right] + \mathbb{E}_Q \left[ \int_0^t \hat{h}_s^* dY_s^\varepsilon - \frac{1}{2} \int_0^t \| \hat{h}_s \|^2 ds \right] 
\]

\[
= \mathbb{E}_Q \left[ \eta^0_t(X_t^0) \right] + \mathbb{E}_Q \left[ \int_0^t \hat{h}_s h(X_s^\varepsilon, Z_s^\varepsilon) ds + \int_0^t \hat{h}_s^* dB_s - \frac{1}{2} \int_0^t \| \hat{h}_s \|^2 ds \right] 
\]

\[
= \mathbb{E}_Q \left[ \eta^0_t(X_t^0) \right] + \mathbb{E}_Q \left[ \int_0^t \hat{h}_s h(X_s^\varepsilon, Z_s^\varepsilon) ds - \frac{1}{2} \int_0^t \| \hat{h}_s \|^2 ds \right]. \quad (7.11)
\]

We will always be dealing with \(X^0\) and its density and conditional density given \(Y^\varepsilon\) from here on, so we will drop the 0 superscripts for brevity of notations.

For fixed \(x \in \mathbb{R}^m\), \(q\) and \(p\) satisfy (7.7) and (7.8), respectively, which we restate here without superscripts 0:

\[
\frac{d}{dt} q_t(x) = \hat{\mathcal{L}}^* q_t(x), \quad (7.12)
\]

\[
q_0(x) = \int_{\mathbb{R}^n} q^i(x,z) dz,
\]

and

\[
dp_t(x) = \hat{\mathcal{L}}^* p_t(x) dt + p_t(x) \bar{h}(x)^* dY_t^\varepsilon, \quad (7.13)
\]

\[
p_0(x) = \int_{\mathbb{R}^n} q^i(x,z) dz.
\]

Then,

\[
d\eta^q_t(x) = - \frac{1}{q_t(x)} \hat{\mathcal{L}}^* q_t(x) dt, \quad (7.14)
\]

\[
\eta^q_0(x) = - \log \int_{\mathbb{R}^n} q^i(x,z) dz
\]

and by Itô’s lemma and (7.13),

\[
d\eta^p_t(x) = - \frac{1}{p_t(x)} dp_t(x) + \frac{1}{2p_t(x)^2} d \langle p(x) \rangle_t
\]

\[
= - \frac{1}{p_t(x)} \hat{\mathcal{L}}^* p_t(x) dt - \bar{h}(x)^* dY_t^\varepsilon + \frac{1}{2} \| \bar{h}(x) \|^2 dt
\]

\[
= - \frac{1}{p_t(x)} \hat{\mathcal{L}}^* p_t(x) dt - \bar{h}(x)^* dY_t^\varepsilon + \frac{1}{2} \| \bar{h}(x) \|^2 dt
\]

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\[
= \left\{-\frac{1}{p_t(x)} \mathcal{L}^* q_t(x) + \frac{1}{2} \| \bar{h}(x) \|^2 \right\} dt - \bar{h}(x)^* dY_t^x, \tag{7.15}
\]

\[\eta_0^\eta(x) = - \log \int_{\mathbb{R}^n} q^\eta(x, z) dz.\]

For the entropy and conditional entropy, we require the evolution of \(\eta\) as a function of the stochastic process \(X_t\), not for fixed \(x\), where \(X_t\) given by (7.5) is an \(\mathcal{F}_t\)-semimartingale. The result of [117] provides the extension of Itô’s formula to functions of semimartingales, that are themselves driven by a Wiener process. The result of [117] is the following, which we state without proof:

**Lemma 7.4.1** Let \(\eta\) be a scalar process satisfying

\[\eta_t(x) = \eta_0(x) + \int_0^t a_s(x) ds + \int_0^t c_s^*(x) dB_s,\]

for fixed \(x \in \mathbb{R}^m\), where \(B\) a \(d\)-dimensional \(\mathcal{F}_t\)-Wiener process. If \(X_t \in \mathbb{R}^m\) is an \(\mathcal{F}_t\)-semimartingale, then

\[
\eta_t(X_t) = \eta_0(X_0) + \int_0^t a_s(X_s) ds + \int_0^t c_s^*(X_s) dB_s
\]

\[+ \int_0^t \sum_{i=1}^m \frac{\partial \eta_s(X_s)}{\partial x_i} dX^i_s + \frac{1}{2} \int_0^t \sum_{i,j=1}^m \frac{\partial^2 \eta_s(X_s)}{\partial x_i \partial x_j} d\langle X^i, X^j \rangle_s
\]

\[+ \int_0^t \sum_{i=1}^m \sum_{j=1}^d \frac{\partial c_s^j(X_s)}{\partial x_i} d\langle B^j, X^i \rangle_s.\]

Using Lemma 7.4.1 and (7.14),

\[
d\eta_t^\eta(X_t) = -\frac{1}{q_t(X_t)} \mathcal{L}^* q_t(X_t) dt + \sum_{i=1}^m \frac{\partial \eta_t^\eta(X_t)}{\partial x_i} dX^i_t
\]

\[+ \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 \eta_t^\eta(X_t)}{\partial x_i \partial x_j} d\langle X^i, X^j \rangle_t. \tag{7.16}\]

The first term of (7.16) is

\[- \frac{1}{q_t(X_t)} \mathcal{L}^* q_t(X_t) dt\]

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\[
= \frac{1}{q_t(X_t)} \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left( \bar{b}_i(X_t) q_t(X_t) \right) dt
\]

\[
- \frac{1}{2q_t(X_t)} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} \left( (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) q_t(X_t) \right) dt
\]

\[
= \frac{1}{q_t(X_t)} \sum_{i=1}^{m} \bar{b}_i(X_t) \frac{\partial q_t(X_t)}{\partial x_i} dt + \sum_{i=1}^{m} \frac{\partial \bar{b}_i(X_t)}{\partial x_i} dt
\]

\[
- \frac{1}{2q_t(X_t)} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} \left( (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) q_t(X_t) \right) dt
\]  

(7.17)

The second term of (7.16) is

\[
\sum_{i=1}^{m} \frac{\partial \eta^q_t(X_t)}{\partial x_i} dX_t^i
\]

\[
= -\frac{1}{q_t(X_t)} \sum_{i=1}^{m} \bar{b}_i(X_t) \frac{\partial q_t(X_t)}{\partial x_i} dt - \frac{1}{q_t(X_t)} \sum_{i=1}^{m} \frac{\partial q_t(X_t)}{\partial x_i} \sum_{r=1}^{k} \bar{\sigma}_{ir}(X_t) dW_r^r.
\]

The first term above cancels with the first term of (7.17) in (7.16).

The third term of (7.16) is

\[
\frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 \eta^q_t(X_t)}{\partial x_i \partial x_j} d\langle X^i, X^j \rangle_t
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 \eta^q_t(X_t)}{\partial x_i \partial x_j} \sum_{r,r'=1}^{k} \bar{\sigma}_{ir}(X_t) d\langle W^r, W^{r'} \rangle_t \bar{\sigma}_{jr'}(X_t)
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 \eta^q_t(X_t)}{\partial x_i \partial x_j} \sum_{r,r'=1}^{k} \bar{\sigma}_{ir}(X_t) \bar{\sigma}_{jr'}(X_t) \delta_{rr'} dt
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 \eta^q_t(X_t)}{\partial x_i \partial x_j} \sum_{r=1}^{k} \bar{\sigma}_{ir}(X_t) \bar{\sigma}_{jr}(X_t) dt
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 \eta^q_t(X_t)}{\partial x_i \partial x_j} (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) dt.
\]

Therefore,

\[
d\eta^q_t(X_t) = \sum_{i=1}^{m} \frac{\partial \bar{b}_i(X_t)}{\partial x_i} dt - \frac{1}{2q_t(X_t)} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) q_t(X_t)) dt
\]
\[- \frac{1}{2} \sum_{i,j=1}^{m} (\tilde{\sigma} \tilde{\sigma}^*)_{ij}(X_t) \frac{\partial^2 \log q_t(X_t)}{\partial x_i \partial x_j} dt \]

\[- \frac{1}{q_t(X_t)} \sum_{i=1}^{m} \frac{\partial q_t(X_t)}{\partial x_i} \sum_{r=1}^{k} \hat{\sigma}_{ir}(X_t) dW_t^r. \]

(7.18)

Taking expectation, the stochastic term on the right side vanishes. Assuming the density and its derivatives decay to zero at the extrema, the second term vanishes as well:

\[ \mathbb{E}_Q \left[ \frac{1}{2q_t(X_t)} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\tilde{\sigma} \tilde{\sigma}^*)_{ij}(X_t) q_t(X_t)) \right] \]

\[ = \frac{1}{2} \int_{\mathbb{R}^m} \frac{1}{q_t(x)} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\tilde{\sigma} \tilde{\sigma}^*)_{ij}(x) q_t(x)) q_t(x) \, dx_1 \ldots dx_m \]

\[ = \frac{1}{2} \int_{\mathbb{R}^m} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\tilde{\sigma} \tilde{\sigma}^*)_{ij}(x) q_t(x)) \, dx_1 \ldots dx_m \]

\[ = \frac{1}{2} \sum_{i,j=1}^{m} \int_{\mathbb{R}^m} \frac{\partial^2}{\partial x_i \partial x_j} ((\tilde{\sigma} \tilde{\sigma}^*)_{ij}(x) q_t(x)) \, dx_1 \ldots dx_m. \]

For any \( i, j \in \{1, \ldots, m\}, \)

\[ \int_{\mathbb{R}} \frac{\partial^2}{\partial x_i \partial x_j} ((\tilde{\sigma} \tilde{\sigma}^*)_{ij}(x) q_t(x)) \, dx_j \]

\[ = \frac{\partial}{\partial x_i} ((\tilde{\sigma} \tilde{\sigma}^*)_{ij}(x) q_t(x)) \bigg|_{x_j=-\infty}^{x_j=\infty} - \int_{\mathbb{R}} \frac{\partial}{\partial x_i} ((\tilde{\sigma} \tilde{\sigma}^*)_{ij}(x) q_t(x)) \frac{\partial 1}{\partial x_j} \, dx_j \]

\[ = \frac{\partial}{\partial x_i} ((\tilde{\sigma} \tilde{\sigma}^*)_{ij}(x) q_t(x)) \bigg|_{x_j=-\infty}^{x_j=\infty} \]

\[ = q_t(x) \frac{\partial (\tilde{\sigma} \tilde{\sigma}^*)_{ij}(x)}{\partial x_i} \bigg|_{x_j=-\infty}^{x_j=\infty} + (\tilde{\sigma} \tilde{\sigma}^*)_{ij}(x) \frac{\partial q_t(x)}{\partial x_i} \bigg|_{x_j=-\infty}^{x_j=\infty} = 0 \]

if \( q_t(x) \) and its first order derivatives \( \searrow 0 \) as \( x_j \to \pm \infty. \)

Therefore, taking expectation of (7.18), we have the rate of change of entropy of \( X_t: \)

\[ \frac{d \mathcal{H}_t(X_t)}{dt} = \mathbb{E}_Q \left[ \text{tr} \left( \nabla \hat{b}(X_t) \right) \right] - \frac{1}{2} \mathbb{E}_Q \left[ \text{tr} \left( (\tilde{\sigma} \tilde{\sigma}^*)(X_t) \nabla^2 \log q_t(X_t) \right) \right], \quad (7.19) \]

\[ \mathcal{H}_0(X_0) = -\mathbb{E}_Q \left[ \log \int_{\mathbb{R}^n} q^t(X_0, z) \, dz \right]. \]
Consider the term due to diffusion in (7.19). Here, we perform the calculations to manipulate it into a more illustrative form, such that it can be interpreted as the contribution of diffusion to uncertainty growth (see discussion following Lemma 7.4.2).

\[
- \frac{1}{2} \mathbb{E}_Q \left[ \text{tr} \left( (\bar{\sigma} \bar{\sigma}^*) (X_t) \nabla^2 \log q_t(X_t) \right) \right] = - \frac{1}{2} \mathbb{E}_Q \left[ \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) \frac{\partial^2 \log q_t(X_t)}{\partial x_i \partial x_j} \right] = - \frac{1}{2} \mathbb{E}_Q \left[ \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) \left( - \frac{1}{q_t(X_t)^2} \frac{\partial q_t(X_t)}{\partial x_i} \frac{\partial q_t(X_t)}{\partial x_j} + \frac{1}{q_t(X_t)} \frac{\partial^2 q_t(X_t)}{\partial x_i \partial x_j} \right) \right].
\]

(7.20)

The first term in (7.20) is

\[
- \frac{1}{2} \mathbb{E}_Q \left[ \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) \left( - \frac{1}{q_t(X_t)^2} \frac{\partial q_t(X_t)}{\partial x_i} \frac{\partial q_t(X_t)}{\partial x_j} \right) \right] = \mathbb{E}_Q \left[ \sum_{i,j=1}^{m} \sum_{l=1}^{k} \bar{\sigma}_{il}(X_t) \bar{\sigma}_{jl}(X_t) \frac{1}{q_t(X_t)} \frac{\partial q_t(X_t)}{\partial x_i} \frac{\partial q_t(X_t)}{\partial x_j} \right] = \mathbb{E}_Q \left[ k \sum_{i=1}^{m} \sum_{l=1}^{k} \bar{\sigma}_{il}(X_t) \frac{1}{q_t(X_t)} \frac{\partial q_t(X_t)}{\partial x_i} \left( \sum_{j=1}^{m} \bar{\sigma}_{jl}(X_t) \frac{1}{q_t(X_t)} \frac{\partial q_t(X_t)}{\partial x_j} \right) \right] = \mathbb{E}_Q \left[ \| \bar{\sigma}(X_t)^* \nabla \log q_t(X_t) \|^2 \right].
\]

The second term in (7.20) is

\[
- \frac{1}{2} \mathbb{E}_Q \left[ \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) \frac{1}{q_t(X_t)} \frac{\partial^2 q_t(X_t)}{\partial x_i \partial x_j} \right] = - \frac{1}{2} \int_{\mathbb{R}^m} \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij}(x) \frac{1}{q_t(x)} \frac{\partial^2 q_t(x)}{\partial x_i \partial x_j} q_t(x) \, dx_1 \ldots dx_m = - \frac{1}{2} \sum_{i,j=1}^{m} \int_{\mathbb{R}^m} (\bar{\sigma} \bar{\sigma}^*)_{ij}(x) \frac{\partial^2 q_t(x)}{\partial x_i \partial x_j} \, dx_1 \ldots dx_m.
\]
For any \(i, j \in \{1, \ldots, m\}\),

\[
\int_{\mathbb{R}^2} (\bar{\sigma} \bar{\sigma}^*)_{ij}(x) \frac{\partial^2 q_t(x)}{\partial x_i \partial x_j} \, dx_i \, dx_j
\]

\[
= \int_{\mathbb{R}} (\bar{\sigma} \bar{\sigma}^*)_{ij}(x) \frac{\partial q_t(x)}{\partial x_j} \bigg|_{-\infty}^{\infty} \, dx_j - \int_{\mathbb{R}} \frac{\partial (\bar{\sigma} \bar{\sigma}^*)_{ij}(x)}{\partial x_i} \frac{\partial q_t(x)}{\partial x_j} \bigg|_{-\infty}^{\infty} \, dx_i
\]

\[
+ \int_{\mathbb{R}^2} \frac{\partial^2 (\bar{\sigma} \bar{\sigma}^*)_{ij}(x)}{\partial x_i \partial x_j} q_t(x) \, dx_i \, dx_j
\]

\[
= \mathbb{E}_Q \left[ \frac{\partial^2 (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t)}{\partial x_i \partial x_j} \right]
\]

if the density and its first order derivatives decay to zero at the extrema.

Therefore (7.19) can be written as

\[
\frac{d \mathcal{H}_t(X_t)}{dt} = \mathbb{E}_Q \left[ \text{tr} \left( \nabla b(X_t) \right) \right] - \frac{1}{2} \mathbb{E}_Q \left[ \| \bar{\sigma}(X_t)^* \nabla \log q_t(X_t) \|^2 \right] - \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t)}{\partial x_i \partial x_j},
\]

(7.21)

\[
\mathcal{H}_0(X_0) = -\mathbb{E}_Q \left[ \log \int_{\mathbb{R}^n} q_t^i(X_0, z) \, dz \right].
\]

Next, consider the conditional entropy of \(X_t\) given \(Y_{[0,t]}\), with respect to the unnormalized conditional density \(p\). Using Lemma 7.4.1 and (7.15),

\[
d\eta^p_t(X_t) = \left\{ -\frac{1}{p_t(X_t)} \mathcal{D}^* p_t(X_t) + \frac{1}{2} \| \tilde{h}(X_t) \|^2 \right\} dt - \tilde{h}(X_t)^* dY^c_t
\]

\[
+ \sum_{i=1}^{m} \frac{\partial \eta^p_t(X_t)}{\partial x_i} dX^i_t + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 \eta^p_t(X_t)}{\partial x_i \partial x_j} d \langle X^i, X^j \rangle_t
\]

\[
+ \sum_{i=1}^{m} \sum_{j=1}^{d} \frac{\partial \tilde{h}^i(X_s)}{\partial x_i} d \langle (Y^c)^j, X^i \rangle_t.
\]

(7.22)

As in \(\eta^q_t(X_t)\), the first, fourth and fifth terms of (7.22) are

\[
- \frac{1}{p_t(X_t)} \mathcal{D}^* p_t(X_t) dt + \sum_{i=1}^{m} \frac{\partial \eta^p_t(X_t)}{\partial x_i} dX^i_t + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 \eta^p_t(X_t)}{\partial x_i \partial x_j} d \langle X^i, X^j \rangle_t
\]

\[
= \sum_{i=1}^{m} \frac{\partial \tilde{h}_i(X_t)}{\partial x_i} dt - \frac{1}{2 p_t(X_t)} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t)p_t(X_t)) dt
\]

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\[-\frac{1}{2} \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) \frac{\partial^2 \log p_t(X_t)}{\partial x_i \partial x_j} dt \quad \frac{1}{p_t(X_t)} \sum_{i=1}^{m} \frac{\partial p_t(X_t)}{\partial x_i} \sum_{r=1}^{k} \bar{\sigma}_{ir}(X_t) dW_t^r.\]

Using (7.2), the second and third terms of (7.22) are

\[\frac{1}{2} \| \bar{h}(X_t) \|^2 dt - \bar{h}(X_t)^* dY_t^\varepsilon\]
\[= \frac{1}{2} \| \bar{h}(X_t) \|^2 dt - \bar{h}(X_t)^* h(X_t^\varepsilon, Z_t^\varepsilon) dt - \bar{h}(X_t)^* dB_t\]
\[= \frac{1}{2} \{ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \bar{h}(X_t) \|^2 dt - \| h(X_t^\varepsilon, Z_t^\varepsilon) \|^2 \} dt - \bar{h}(X_t) dB_t.\]

The last term of (7.22) is zero since signal and observation noises are independent:

\[d \langle Y^\varepsilon_j, X^i_t \rangle = d \langle B^j, X^i_t \rangle = d \left( B^j, \sum_{l=1}^{k} \sigma^{ir}(X) W^r \right) = 0.\]

Therefore,

\[d \eta^p_t(X_t)\]
\[= \sum_{i=1}^{m} \frac{\partial \bar{b}_i(X_t)}{\partial x_i} dt - \frac{1}{2p_t(X_t)} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t)p_t(X_t)) dt\]
\[- \frac{1}{2} \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t) \frac{\partial^2 \log p_t(X_t)}{\partial x_i \partial x_j} dt - \frac{1}{p_t(X_t)} \sum_{i=1}^{m} \frac{\partial p_t(X_t)}{\partial x_i} \sum_{r=1}^{k} \bar{\sigma}_{ir}(X_t) dW_t^r\]
\[+ \frac{1}{2} \{ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \bar{h}(X_t) \|^2 dt - \| h(X_t^\varepsilon, Z_t^\varepsilon) \|^2 \} dt - \bar{h}(X_t) dB_t.\]

Taking expectation, the stochastic integral terms vanish and we expect the second term on the right hand side to vanish in a way similar to for \( \eta^p_t(X_t)\): Recall that \( p(x) \) and \( \pi(x) \) are driven by \( Y^\varepsilon \). Let \( \mathbb{E}_Q^{Y^\varepsilon} \) denote expectation over \( Y^\varepsilon_{[0,t]} \).

\[\mathbb{E}_Q \left[ \frac{1}{2p_t(X_t)} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\bar{\sigma} \bar{\sigma}^*)_{ij}(X_t)p_t(X_t)) \right]\]
\[= \frac{1}{2} \mathbb{E}_Q^{Y^\varepsilon} \left[ \int_{\mathbb{R}^m} \frac{1}{p_t(x)} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} ((\bar{\sigma} \bar{\sigma}^*)_{ij}(x)p_t(x)) \pi_t(x) \ dx_1 \ldots \ dx_m \right]\]
\[= \frac{1}{2} \mathbb{E}_Q^{Y^\varepsilon} \left[ \int_{\mathbb{R}^m} \frac{1}{\pi_t(x)} \int_{\mathbb{R}^m} \frac{1}{p_t(\zeta)} \ d\zeta \right] \]

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\[
\sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} (\bar{\sigma} \bar{\sigma}^*)_{ij}(x)p_t(x) \pi_t(x) \, dx_1 \ldots dx_m
\]

\[
\frac{1}{2} \mathbb{E}_Q \left[ \frac{1}{\int_{\mathbb{R}^m} p_t(\xi) \, d\xi} \sum_{i,j=1}^{m} \int_{\mathbb{R}^m} \frac{\partial^2}{\partial x_i \partial x_j} (\bar{\sigma} \bar{\sigma}^*)_{ij}(x)p_t(x) \, dx_1 \ldots dx_m \right] .
\]

If \( p_t(x) \) and its first order derivatives \( \searrow 0 \) as \( x_i \to \pm \infty \) for any \( i \in 1, \ldots, m \), then, integrating by parts, we see that the integral over \( \mathbb{R}^m \) is zero. However, this is only formally. Specifically, in writing the first equality, we formally consider the expectation \( \mathbb{E}_Q \) over \( (X_t, Y_{[0,t]}^\epsilon) \) as integration w.r.t. the joint density of \( (X_t, Y_{[0,t]}^\epsilon) \). The joint density can be written as the product of the conditional density of \( X_t \) given \( Y_{[0,t]}^\epsilon \) and the marginal density of \( Y_{[0,t]}^\epsilon \). However, \( Y_{[0,t]}^\epsilon \) is a path process, and we cannot rigorously construct a density for it. Hence, the argument for the term of interest to vanish is only formal.

We have the rate of change of conditional entropy of \( X_t \) given \( Y_{[0,t]}^\epsilon \) based on the unnormalized conditional density to be

\[
\frac{d\mathbb{E}_Q [\eta_p^p(X_t)]}{dt} = \mathbb{E}_Q \left[ \text{tr} \left( \nabla \bar{b}(X_t) \right) \right] - \frac{1}{2} \mathbb{E}_Q \left[ \text{tr} \left( (\bar{\sigma} \bar{\sigma}^*)(X_t) \nabla^2 \log p_t(X_t) \right) \right]
\]

\[
+ \frac{1}{2} \mathbb{E}_Q \left[ \left\| \bar{h}(X_t, Z_t) - \bar{h}(X_t) \right\|^2 - \left\| h(X_t, Z_t) \right\|^2 \right] , \quad (7.23)
\]

\[
\mathbb{E}_Q [\eta_0^p(X_0)] = -\mathbb{E}_Q \left[ \log \int_{\mathbb{R}^n} q^i(X_0, z) \, dz \right] .
\]

Recall from (7.11) that

\[
\mathcal{H}_t(X_t | Y_{[0,t]}^\epsilon) = \mathbb{E}_Q [\eta_p^p(X_t)] + \mathbb{E}_Q \left[ \int_0^t \bar{h}_s^* h(X_s^\epsilon, Z_s^\epsilon) \, ds - \frac{1}{2} \int_0^t \left\| \bar{h}_s \right\|^2 \, ds \right] ,
\]

where

\[
\bar{h}_t \stackrel{\text{def}}{=} \mathbb{E}_Q \left[ \bar{h}(X_t^0) \right| \mathcal{Y}_t^\epsilon .
\]

Then, using (7.23), the rate of change of conditional entropy of \( X_t \) given \( Y_{[0,t]}^\epsilon \) is

\[
\frac{d\mathcal{H}_t(X_t | Y_{[0,t]}^\epsilon)}{dt} = \frac{d\mathbb{E}_Q [\eta_p^p(X_t)]}{dt} + \mathbb{E}_Q \left[ \bar{h}_t^* h(X_t^\epsilon, Z_t^\epsilon) - \frac{1}{2} \left\| \bar{h}_t \right\|^2 \right]
\]

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As before, the first term in (7.25) is

\[ \mathbb{E}_Q \left[ \text{tr} \left( \nabla \bar{h}(X_t) \right) \right] - \frac{1}{2} \mathbb{E}_Q \left[ \text{tr} \left( (\sigma \sigma^*) (X_t) \nabla^2 \log p_t(X_t) \right) \right] \]

\[ + \frac{1}{2} \mathbb{E}_Q \left[ \|h(X_t, Z_t) - \bar{h}(X_t)\|^2 - \|h(X_t, Z_t)\|^2 \right] , \]

so the diffusion term \( \mathbb{E}_Q \left[ \text{tr} \left( (\sigma \sigma^*) (X_t) \nabla^2 \log p_t(X_t) \right) \right] \) term in (7.24) can be replaced by the equivalent \( \mathbb{E}_Q \left[ \text{tr} \left( (\sigma \sigma^*) (X_t) \nabla^2 \log \pi_t(X_t) \right) \right] \) that uses the normalized conditional density.

This diffusion term in (7.24) can be rewritten into more illustrative quantities as follows (see after Lemma 7.4.2 for discussion): The calculations are similar to those for the entropy equation.

\[ - \frac{1}{2} \mathbb{E}_Q \left[ \text{tr} \left( (\sigma \sigma^*) (X_t) \nabla^2 \log \pi_t(X_t) \right) \right] \]

\[ = - \frac{1}{2} \mathbb{E}_Q \left[ \sum_{i,j=1}^m (\sigma \sigma^*)_{ij} (X_t) \left( - \frac{1}{\pi_t(X_t)^2} \frac{\partial \pi_t(X_t)}{\partial x_i} \frac{\partial \pi_t(X_t)}{\partial x_j} + \frac{1}{\pi_t(X_t)} \frac{\partial^2 \pi_t(X_t)}{\partial x_i \partial x_j} \right) \right] . \]

(7.25)

As before, the first term in (7.25) is

\[ - \frac{1}{2} \mathbb{E}_Q \left[ \sum_{i,j=1}^m (\sigma \sigma^*)_{ij} (X_t) \frac{1}{\pi_t(X_t)^2} \frac{\partial \pi_t(X_t)}{\partial x_i} \frac{\partial \pi_t(X_t)}{\partial x_j} \right] \]

\[ = \mathbb{E}_Q \left[ \|\sigma(X_t)^* \nabla \log \pi_t(X_t)\|^2 \right] . \]
We treat the second term in (7.25) formally, where, as before, we formally write the expectation $E_Q$ over $(X_t, Y^e_{[0,t]})$ as integral w.r.t. the joint density.

$$-\frac{1}{2} E_Q \left[ \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij} (X_t) \frac{1}{\pi_t(X_t)} \frac{\partial^2 \pi_t(X_t)}{\partial x_i \partial x_j} \right]$$

$$= -\frac{1}{2} E^e_Q \left[ \int_{\mathbb{R}^m} \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij} (x) \frac{1}{\pi_t(x)} \frac{\partial^2 \pi_t(x)}{\partial x_i \partial x_j} \pi_t(x) dx_1 \ldots dx_m \right]$$

$$= -\frac{1}{2} E^e_Q \left[ \int_{\mathbb{R}^m} \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij} (x) \frac{\partial^2 \pi_t(x)}{\partial x_i \partial x_j} \pi_t(x) dx_1 \ldots dx_m \right]. \quad (7.26)$$

For any $i, j \in \{1, \ldots, m\}$,

$$\int_{\mathbb{R}^2} (\bar{\sigma} \bar{\sigma}^*)_{ij} (x) \frac{\partial^2 \pi_t(x)}{\partial x_i \partial x_j} dx_i dx_j$$

$$= \int_{\mathbb{R}} (\bar{\sigma} \bar{\sigma}^*)_{ij} (x) \frac{\partial \pi_t(x)}{\partial x_j} \bigg|_{-\infty}^{\infty} dx_j - \int_{\mathbb{R}} \frac{\partial (\bar{\sigma} \bar{\sigma}^*)_{ij} (x)}{\partial x_i} \pi_t(x) \bigg|_{-\infty}^{\infty} dx_i$$

$$+ \int_{\mathbb{R}^2} \frac{\partial^2 (\bar{\sigma} \bar{\sigma}^*)_{ij} (x)}{\partial x_i \partial x_j} \pi_t(x) dx_i dx_j$$

$$= \int_{\mathbb{R}^2} \frac{\partial^2 (\bar{\sigma} \bar{\sigma}^*)_{ij} (x)}{\partial x_i \partial x_j} \pi_t(x) dx_i dx_j$$

if the density and its first order derivatives decay to zero at the extrema. Continuing (7.26),

$$-\frac{1}{2} E_Q \left[ \sum_{i,j=1}^{m} (\bar{\sigma} \bar{\sigma}^*)_{ij} (X_t) \frac{1}{\pi_t(X_t)} \frac{\partial^2 \pi_t(X_t)}{\partial x_i \partial x_j} \right]$$

$$= -\frac{1}{2} E^e_Q \left[ \sum_{i,j=1}^{m} \int_{\mathbb{R}^m} \frac{\partial^2 (\bar{\sigma} \bar{\sigma}^*)_{ij} (x)}{\partial x_i \partial x_j} \pi_t(x) dx_1 \ldots dx_m \right]$$

$$= -\frac{1}{2} \sum_{i,j=1}^{m} \int_{\mathbb{R}^m} \frac{\partial^2 (\bar{\sigma} \bar{\sigma}^*)_{ij} (x)}{\partial x_i \partial x_j} E^e_Q \left[ \pi_t(x) \right] dx_1 \ldots dx_m$$

$$= -\frac{1}{2} \sum_{i,j=1}^{m} \int_{\mathbb{R}^m} \frac{\partial^2 (\bar{\sigma} \bar{\sigma}^*)_{ij} (x)}{\partial x_i \partial x_j} q_t(x) dx_1 \ldots dx_m$$

$$= -\frac{1}{2} E_Q \left[ \sum_{i,j=1}^{m} \frac{\partial^2 (\bar{\sigma} \bar{\sigma}^*)_{ij} (X_t)}{\partial x_i \partial x_j} \right].$$
Therefore (7.24) can be written as

\[
\begin{align*}
\frac{d\mathcal{H}_t(X_t|Y_{[0,t]}^\varepsilon)}{dt} & = \mathbb{E}_Q \left[ \text{tr} \left( \nabla \bar{b}_i(X_t) \right) \right] \\
& + \frac{1}{2} \mathbb{E}_Q \left[ \| \bar{\sigma}(X_t)^* \nabla \log \pi_t(X_t) \|^2 \right] - \frac{1}{2} \mathbb{E}_Q \left[ \sum_{i,j=1}^m \frac{\partial^2 (\bar{\sigma}\bar{\sigma}^*)_{ij}(X_t)}{\partial x_i \partial x_j} \right] \\
& + \frac{1}{2} \mathbb{E}_Q \left[ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \bar{h}(X_t) \|^2 - \| h(X_t^\varepsilon, Z_t^\varepsilon) - \hat{h}_t \|^2 \right], \\
\mathcal{H}_0(X_0|Y_0^\varepsilon) & = -\mathbb{E}_Q \left[ \log \int_{\mathbb{R}^n} q^i(X_0, z) \, dz \right].
\end{align*}
\]

(7.27)

In the following, we reintroduce the superscript 0 for the limiting process \(X^0\). From (7.19) and (7.24) we have the following result:

**Lemma 7.4.2** Under conditions (7.3) and (7.4), the rate of change of entropy of that limiting process is given by

\[
\begin{align*}
\frac{d\mathcal{H}_t(X_0)}{dt} & = \mathbb{E}_Q \left[ \text{tr} \left( \nabla \bar{b}_i(X_0) \right) \right] - \mathbb{E}_Q \left[ \text{tr} \left( (\bar{\sigma}\bar{\sigma}^*)(X_0)^* \nabla^2 \log \pi_0^i(X_0) \right) \right] \\
& - \frac{1}{2} \left\{ \mathbb{E}_Q \left[ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \hat{h}_t \|^2 \right] - \mathbb{E}_Q \left[ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \bar{h}(X_0^\varepsilon) \|^2 \right] \right\}, \\
\mathcal{H}_0(X_0) & = -\mathbb{E}_Q \left[ \log \int_{\mathbb{R}^n} q^i(X_0, z) \, dz \right].
\end{align*}
\]

(7.28)

Under conditions for the filter convergence results of Section 3.7, the rate of change of the conditional entropy of the limiting process given observations history is given by

\[
\begin{align*}
\frac{d\mathcal{H}_t(X_0^0|Y_{[0,t]}^\varepsilon)}{dt} & = \mathbb{E}_Q \left[ \text{tr} \left( \nabla \bar{b}_i(X_0) \right) \right] - \mathbb{E}_Q \left[ \text{tr} \left( (\bar{\sigma}\bar{\sigma}^*)(X_0)^* \nabla^2 \log \pi_0^i(X_0) \right) \right] \\
& - \frac{1}{2} \left\{ \mathbb{E}_Q \left[ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \hat{h}_t \|^2 \right] - \mathbb{E}_Q \left[ \| h(X_t^\varepsilon, Z_t^\varepsilon) - \bar{h}(X_0^\varepsilon) \|^2 \right] \right\}, \\
\mathcal{H}_0(X_0|Y_0^\varepsilon) & = -\mathbb{E}_Q \left[ \log \int_{\mathbb{R}^n} q^i(X_0^0, z) \, dz \right].
\end{align*}
\]

(7.29)

Below, we provide some discussion on the rate of change equations. \(\mathcal{H}_t(X_t^0)\) is the uncertainty in our knowledge of \(X_t^0\) on \(\mathbb{R}^m\). (7.28) describes the time rate of change of this uncertainty. If the signal is deterministic, then \(\bar{\sigma}\bar{\sigma}^* \equiv 0\) and the rate of change is equal to the expected value of the trace of
the gradient of the nonlinear vector field that governs $X^0$. The uncertainty in this case is due to uncertainty in the initial condition, so the expectation is taken with respect to the density that has been propagated forward from the initial density $q^i$ by the generator $\mathcal{L}$, given by (7.6) with $\bar{\sigma}\bar{\sigma}^* \equiv 0$. For a small deviation $\delta X^0$ from an initial $X^0$, the rate of change is given by

$$\delta X^0_t = \nabla b(X^0_t) \delta X^0_t.$$  

Therefore, $\nabla b$ represents the growth rate of an initial error. At a fixed time $t$, $\text{tr} \left( \nabla b(X^0_t) \right)$ is equal to the sum of eigenvalues of $\nabla b(X^0_t)$, which indicates the growth or shrinkage of $\delta X^0_t$ from time $t$ to a small $\delta t$ ahead, a “volumetric change”.

Now consider if the signal is stochastic, i.e. $\bar{\sigma}$ is not equal to zero. In the conventional Fisher information, the second moment of $\nabla \log q^0(X^0)$ represents the sensitivity of $q^0(X^0)$ to changes in $X^0$. The second moment of $\nabla \log q^0(X^0)$ is also equal to negative of the first moment of $\nabla^2 \log q^0(X^0)$. If $\bar{\sigma}$ is not zero, then we see a similar sensitivity term in the entropy equation, in the form of

$$-\frac{1}{2} \mathbb{E}_Q \left[ \text{tr} \left( (\bar{\sigma}\bar{\sigma}^*)(X^0_t) \nabla^2 \log q^0(X^0) \right) \right].$$

in (7.28). It represents the sensitivity of $q^0(X^0)$ to changes in $X^0$, stretched by the diffusion coefficient $\bar{\sigma}$. In (7.21), we have its equivalent in terms of the second moment of the norm of $\bar{\sigma}(X^0)^* \nabla \log p^0(X^0)$. It may be more illustrative to consider (7.21), in which the sensitivity term is decomposed into

$$\frac{1}{2} \mathbb{E}_Q \left[ \|\bar{\sigma}(X^0)^* \nabla \log p^0(X^0)\|^2 \right] - \frac{1}{2} \mathbb{E}_Q \left[ \sum_{i,j=1}^m \frac{\partial^2 (\bar{\sigma}\bar{\sigma}^*)_{ij}(X^0_t)}{\partial x_i \partial x_j} \right].$$

The second moment of $\|\bar{\sigma}(X^0)^* \nabla \log p^0(X^0)\|$ can be interpreted as the sensitivity of $p^0(X^0)$ to $X^0$, amplified by diffusion effects, i.e. the sensitivity of $p^0$ to the stochastic effects of the signal dynamics. It has the same meaning as the conventional Fisher information, with the addition of accounting for diffusion effects. In fact, [115] defines this as the Fisher information in their work. It enters (7.21) as a positive term, hence it contributes to the growth in uncertainty about $X^0$ due to diffusion. In addition, there is also the con-
tribution from sensitivity of \( \bar{\sigma} \bar{\sigma}^* \) to changes in \( X^0 \), which may increase or decrease uncertainty growth, depending on the sign of \( \sum_{i,j=1}^{m} \frac{\partial^2 (\sigma \sigma^*)}{\partial x_i \partial x_j} \).

If \( \bar{\sigma} \) is independent of \( X^0 \), then diffusion effects always contribute to \( \geq 0 \) growth in uncertainty, due to the term.

\[ \mathcal{H}_t(X^0_t|Y^\varepsilon_{[0,t]}) \] is the uncertainty in our knowledge about \( X^0_t \) on \( \mathbb{R}^m \) given information from \( Y^\varepsilon_{[0,t]} \). Its time rate of change given by (7.29) contains the same terms as (7.28) that capture uncertainty growth due to the signal dynamics. In addition, it contains a dissipative term, \( \mathbb{E}_Q \left[ \| h(X^\varepsilon_t, Z^\varepsilon_t) - \tilde{h}_t \| \right] \), due to information from the sensor function. This contributes to a reduction in uncertainty growth. However, this reduction is penalized by the fact that the homogenized sensor function \( \bar{h} \) is used in the reduced-order filter, in the form of the error \( \mathbb{E}_Q \left[ \| h(X^\varepsilon_t, Z^\varepsilon_t) - \bar{h}(X^0_t) \| \right] \).

Next, we obtain the rate of change of mutual information between \( X^0 \) and \( Y^\varepsilon_{[0,t]} \). Recall that mutual information between the signal and observation is equivalent to the reduction in uncertainty about the signal given observations, i.e. the difference between the entropy and conditional entropy given observations (7.9). Hence, using (7.21) and (7.27) for the rates of change of entropy and conditional entropy, Lemma 7.4.2 leads to the following rate of change for mutual information:

**Theorem 7.4.3** Under conditions for the filter convergence results of Section 3.7, the rate of change of the mutual information between the limiting process (coarse grained dynamics) of the signal that is generated by \( \mathcal{L} \) and observations history is

\[
\frac{dI_t(X^0_t, Y^\varepsilon_{[0,t]})}{dt} = \frac{1}{2} \left\{ \mathbb{E}_Q \left[ \| h(X^\varepsilon_t, Z^\varepsilon_t) - \tilde{h}_t \| \right] - \mathbb{E}_Q \left[ \| h(X^\varepsilon_t, Z^\varepsilon_t) - \bar{h}(X^0_t) \| \right] \right\}
+ \frac{1}{2} \left\{ \mathbb{E}_Q \left[ \| \bar{\sigma}(X^0_t)^* \nabla \log \pi^0_t(X^0_t) \| \right] - \mathbb{E}_Q \left[ \| \bar{\sigma}(X^0_t)^* \nabla \log q^0_t(X^0_t) \| \right] \right\},
\]

(7.30)

\[ I_0(X^0_0, Y^\varepsilon_0) = 0. \]

Based on interpretations for (7.28) and (7.29), the right side of (7.30) consists of an information growth rate from the sensor function and a dissipation rate due to sensitivity of the conditional density to stochasticity of signal dynamics. The information growth is penalized by the fact that the
homogenized sensor function $\tilde{h}$ is used in place of the true sensor function $h$. As the filter estimate improves with more observations, the difference between $\hat{h}$ and $h(X^\varepsilon, Z^\varepsilon)$ should decrease, bar the error due to homogenization, and information growth will decrease. As mentioned in the discussion on the work of [115] in Section 7.3, the contribution of deterministic signal dynamics to uncertainty growth is not seen in the mutual information equation, as it appears in both the rates of change of entropy and conditional entropy. Studying the contribution of deterministic signal dynamics to uncertainty growth can provide insights into improving filtering schemes, for example by studying the “volumetric growth” in the state space due to the deterministic dynamics and identifying different growth rates in different directions.
This thesis presented the theoretical basis for the development of a lower-
dimensional estimation algorithms for state estimation in complex multi-
scale systems and a preliminary investigation and theoretical basis for lower-
dimensional parameter estimation algorithms. A reduced-order particle fil-
tering algorithm has been developed, with adaptation to discrete time obser-
vations, tailored for systems with chaotic nature that are inherent in atmo-
spheric models. A combination of the EM algorithm with the reduced-order 
particle filtering and corresponding smoothing algorithms was also imple-
mented for parameter estimation.

The main result in Chapter 3 presents the lower-dimensional filter as 
replacement for the optimal filter for state estimation in multiscale set-
ting. Stochastic homogenization was combined with nonlinear filtering the-
ory to construct a homogenized SPDE that is the approximation of a lower-
dimesional nonlinear filter for the “coarse-grained”/slow process. Conver-
gence of the optimal filter of the “coarse-grained” process to the solution 
of the homogenized filter is shown using BSDEs and asymptotic techniques. 
The homogenized SPDE can be used as the basis for an efficient multi-scale 
particle filtering algorithm for estimating the slow dynamics of the system, 
without directly accounting for the fast dynamics.

For the main result, the conditions on the coefficients are very restrictive 
and exclude, for example, linear models. This is due to the fact that we used 
homogenization of SPDEs to obtain convergence of the filter, and that for 
existence of solutions to the SPDEs, the coefficients need to be bounded and 
sufficiently smooth. Working with weak solutions in place of classical solu-
tions would not improve the conditions much. Using viscosity solutions or 
entirely relying on probabilistic arguments might be a way to get less restric-
tive conditions, however with these methods we do not expect that a rate 
of convergence can be obtained. While we were able to obtain the explicit
rate of convergence $\sqrt{\varepsilon}$, the constant $C$ in Theorem 3.5.1 depends on the terminal time $T$. It would be interesting to find conditions under which this can be avoided. This might be achieved by building on stability results for nonlinear filters, see e.g. [118], Chapter 4, “Stability and asymptotic analysis”. The system considered in (3.1) has decoupled signal and observation noises. In reality, signal and observation noise can be correlated, for example, when sensors are in the same flowfield as the signal, or when there is direct observation of a component or a subset of the components of the signal. A natural extension of the results of Chapter 3 is for the correlated signal and observation noise setting:

$$dX^\varepsilon_t = b(X^\varepsilon_t, Z^\varepsilon_t)dt + \sigma(X^\varepsilon_t, Z^\varepsilon_t)dV_t,$$

$$dZ^\varepsilon_t = \frac{1}{\varepsilon} f(X^\varepsilon_t, Z^\varepsilon_t)dt + \frac{1}{\sqrt{\varepsilon}} g(X^\varepsilon_t, Z^\varepsilon_t)dW_t,$$

where $X^\varepsilon_t \in \mathbb{R}^m$, $Z^\varepsilon_t \in \mathbb{R}^n$, $W_t \in \mathbb{R}^l$ and $V_t \in \mathbb{R}^k$ are independent standard Brownian motions, with $d$-dimensional observation

$$Y^\varepsilon_t = \int_0^t h(X^\varepsilon_s, Z^\varepsilon_s)ds + \sigma_y(X^\varepsilon_t, Z^\varepsilon_t)dV_t + B_t.$$

Another extension is to systems of the type

$$dX^\varepsilon_t = \frac{1}{\varepsilon} b^I(Z^\varepsilon_t, X^\varepsilon_t) + b(Z^\varepsilon_t, X^\varepsilon_t) + \sigma(Z^\varepsilon_t, X^\varepsilon_t) dV_t,$$

$$dZ^\varepsilon_t = \frac{1}{\varepsilon^2} f(Z^\varepsilon_t, X^\varepsilon_t) + \frac{1}{\varepsilon} g(Z^\varepsilon_t, X^\varepsilon_t) dW_t,$$

$$dY^\varepsilon_t = h(Z^\varepsilon_t, X^\varepsilon_t) dt + dB_t,$$

where $W$, $V$ and $B$ are again independent, and there are three distinct timescales. In this setting, even the slow component $X^\varepsilon$ has a fast varying component. This case is important, in particular, for applications in geophysical flows and climate dynamics. The drift term $b$ and the diffusion $\sigma$ cause fluctuations of order order 1, and the drift term $f$ and the diffusion $g$ cause fluctuations of order order $\varepsilon^{-2}$, whereas the drift term $b^I$ causes fluctuations at an intermediate order $\varepsilon^{-1}$. It was found that when the average of $b^I$ with respect to the invariant measure of the fast component $Z^\varepsilon$ (when slow component is fixed) is zero, the limit distribution of the slow component (away from the initial layer) can also be obtained in terms of the solution.
of some auxiliary Poisson equation in the homogenization theory. However, dealing with the $\varepsilon^{-1}$ term in developing a lower-dimensional nonlinear filter for the “coarse-grained” process is still not available.

The HHPF presented in Chapter 4 is a direct result of the filtering results of Chapter 3. The algorithm combines an existing multiscale numerical integration method with the sequential importance sampling particle filter. An improved proposal density construction was incorporated to adapt to discrete time observations, to correct estimates in between observations. The adaptation produces significant estimation enhancement when observation windows are large, especially in chaotic systems, in which errors can grow exponentially with time, as displayed in the numerical experiment of Section 4.5.1. The adaptation involves an optimal control strategy to steer particles such that the proposal density already contains information from the next observation. This is one of various existing particle filter adaptations and optimizations, as referred to in Section 4.4.

The main result in Chapter 6 shows the feasibility of using the likelihood function corresponding to the lower-dimensional filter of Chapter 3 for estimating unknown parameters in the multiscale signal SDEs. The result was obtained for a specific case of scalar slow component, which homogenized representation is ergodic, but is hoped to set a precursor for the more general $m$-dimensional case as for the multiscale filtering problem of Chapter 3. Asymptotic techniques for maximum likelihood estimation of diffusion processes were used, but an additional element of timescale separation had to be dealt with.

The estimator corresponding to the lower-dimensional likelihood function was tested on a simple numerical example that adheres to the ergodicity condition of the main result, but also on a nonlinear high dimensional multiscale system. In both cases, the estimator is found to display bias from the true value. Figure 8.1 shows the distribution of error of the ML estimator for the simple numerical example of Section 6.4.2. The errors are stretched by $\sqrt{T}$, square root of the observation time window, and overlayed with a standard normal density function (in red) for comparison. The errors histogram displays a bell-shaped normal curve offset from mean zero, leading to conjecture of possible bias in the estimator. The natural investigation is to study the limiting distribution of the estimator’s error, whether it satisfies the asymptotic normality property of ML estimators. The numerical
experiments suggest a possible nonzero mean of order $\sqrt{\varepsilon}$ emerging from homogenization, with variance inversely proportional to the Fisher information and observation window length. The next extension is to the general multiscale diffusion setting of Chapter 3, which to our knowledge, has not been done. The parameter estimation problem in that setting is of interest, especially in the geophysical science and climate prediction community, as forcing and coupling parameters are essential in the outcomes of numerical simulation of climate models, as described in the introduction and references cited therein. An example is parameters representative of anthroopogenic forcing, which can affect long term climate state, and may contribute to switching from one stable climate regime to another.

Chapter 7 studies the information content relationship between the homogenized signal and actual observation in the estimation problem of the preceding chapters. Dynamic equations for the information entropy of the homogenized signal given actual observation, and mutual information between the two processes are obtained. The dynamic equations indicate that information entropy/uncertainty of the homogenized signal changes at a rate proportional to the volumetric growth due to deterministic signal dynamics, and increased by diffusion effects. When observations are present, the uncertainty growth is reduced according to information about the signal contained in the sensor function, but the reduction is penalized by the error due to homogenization. So far in this work, no practical application of the dynamic information equations, nor approximations to their solutions has been done. A possible future direction is to utilize the equations for designing dynamic
sensors or dynamic strategies for static sensors to improve information content in the observations over time. For example, the information entropy equation may provide insight into which region of the state space should be targeted more for data collection based on the deterministic volumetric growth of the signal. The mutual information equation may be used to interpret how the Kullback-Leibler divergence between the conditional and unconditioned densities of the signal changes with time, from which sensor functions design can be based off of to construct better filtering posterior densities.
Appendix 1: Kallianpur Striebel formula

The Kallianpur-Striebel formula relates the normalized filter \( \pi \) to the unnormalized filter \( \rho \):

\[
\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)} = \frac{\mathbb{E}_P \left[ \varphi(X_t) \tilde{D}_t \big| \mathcal{Y}_t \right]}{\mathbb{E}_P \left[ \tilde{D}_t \big| \mathcal{Y}_t \right]},
\]

where \( D_t \) is the Radon-Nikodym derivative for the measure change from the original measure \( Q \) to \( P \), where the observation is a \( P \)-Brownian motion (see Section 2.3.1). The formula is obtained by showing that, for \( A \subset \mathcal{Y}_t \),

\[
\mathbb{E}_P \left[ \pi_t(\varphi) \mathbb{E}_P \left[ \tilde{D}_t \big| \mathcal{Y}_t \right] \chi_A \right] = \mathbb{E}_P \left[ \mathbb{E}_Q \left[ \varphi(X_t) \tilde{D}_t \big| \mathcal{Y}_t \right] \chi_A \right].
\]

RHS:

\[
\mathbb{E}_P \left[ \pi_t(\varphi) \mathbb{E}_P \left[ \tilde{D}_t \big| \mathcal{Y}_t \right] \chi_A \right] = \mathbb{E}_P \left[ \mathbb{E}_Q \left[ \varphi(X_t) \tilde{D}_t \big| \mathcal{Y}_t \right] \chi_A \right],
\]

Note the following tower property for expectations:

\[
\int_y f(y) \left( \int_x g(x, y) p(x|y) dx \right) p(y) dy = \int_y \int_{x|y} f(y) g(x, y) \underbrace{p(x|y) p(y) dx dy}_{=p(x,y)}
\]

\[
= \int_y \int_{x|y} f(y) g(x, y) p(x, y) dx dy,
\]

i.e. \( \mathbb{E} [f(Y) \mathbb{E} [g(X, Y)|\mathcal{Y}] = \mathbb{E} [f(Y) g(X, Y)] \). Using this on the LHS:

\[
\mathbb{E}_P \left[ \pi_t(\varphi) \mathbb{E}_P \left[ \tilde{D}_t \big| \mathcal{Y}_t \right] \chi_A \right] = \mathbb{E}_P \left[ \mathbb{E}_Q \left[ \varphi(X_t) \mathbb{E}_P \left[ \tilde{D}_t \big| \mathcal{Y}_t \right] \chi_A \right] \right] = \mathbb{E}_P \left[ \mathbb{E}_Q \left[ \varphi(X_t) \tilde{D}_t \chi_A \right] \left( \mathcal{Y}_t \text{ is } \mathcal{Y}_t\text{-measurable,} \right. \right]
\]

\[
= \mathbb{E}_Q \left[ \varphi(X_t) \big| \mathcal{Y}_t \right] \chi_A \left( \tilde{D} \text{ is the measure change } \frac{dQ}{dP} \right).
\]
\[ E_Q [\varphi(X_t)\chi_A] \quad \text{(tower property)}. \]

So we have the equality (2), and since \( E_P \left[ \tilde{D}_t \mid \mathcal{Y}_t \right] > 0 \) (by finiteness and integrability of sensor function \( h \), see Section 2.3.1), (1) holds.

Appendix 2: Innovation process in filtering

The innovation process

\[ \nu_t \overset{\text{def}}{=} Y_t - \int_0^t \pi_s(h) ds \]

that drives the nonlinear filter (see Section 2.3.4) is a \( \mathcal{Q} \)-Brownian motion:

W.l.o.g., assume \( \nu \) is scalar. Recall that \( Y_t = \int_0^t h(X_s) ds + B_t \). Therefore,

\[ \nu_t = \int_0^t [h(X_s) - \pi_s(h)] ds + B_t. \]

First check that \( \nu_t \) is an \( \mathcal{F}_t \)-martingale: For \( 0 \leq \tau \leq t \),

\[
\mathbb{E}_Q [\nu_t \mid \mathcal{F}_\tau] = \left[ \int_0^\tau \mathbb{E}_Q [h(X_s) - \pi_s(h) \mid \mathcal{F}_s] ds + B_\tau \right]_{\nu_t} + \int_\tau^t \mathbb{E}_Q [h(X_s) - \pi_s(h) \mid \mathcal{F}_s] ds,
\]

where the second integral is zero:

\[
\int_\tau^t \mathbb{E}_Q [h(X_s) - \pi_s(h) \mid \mathcal{F}_s] ds = 0.
\]

Therefore, \( \mathbb{E}_Q [\nu_t \mid \mathcal{F}_\tau] = \nu_\tau \) for \( 0 \leq \tau \leq t \), i.e. \( \nu_t \) is an \( \mathcal{F}_t \)-martingale.

Next, check that the characteristic function of \( \nu_t \) is that of a \( \mathcal{N}(0, t) \) random
variable: Denote $\Lambda_t \overset{\text{def}}{=} \exp \{ i\theta \nu_t \}$ for constant $\theta \in \mathbb{R}$. By Itô’s formula,

$$d\Lambda_t = \Lambda_t \left( i\theta \nu_t - \frac{\theta^2}{2} dt \right) \implies \frac{d\Lambda_t}{\Lambda_t} = i\theta \nu_t - \frac{\theta^2}{2} \, dt.$$ 

Integrating gives us

$$\ln \left( \frac{\Lambda_t}{\Lambda_0} \right) = i\theta \nu_t - \frac{\theta^2}{2} t \implies \Lambda_t = \exp \left\{ i\theta \nu_t - \frac{\theta^2}{2} t \right\}.$$ 

$\exp \{ i\theta \nu_t \}$ is an exponential martingale, so

$$\mathbb{E} \left[ \exp \{ i\theta \nu_t \} \right] = \mathbb{E} \left[ \exp \{ i\theta \nu_t \} | \mathcal{F}_0 \right] = \exp \{ i\theta \nu_0 \} = 1.$$

Therefore, the characteristic function of $\nu_t$ is

$$\mathbb{E} \left[ i\theta \nu_t \right] = \mathbb{E} \left[ \Lambda_t \right] = \mathbb{E} \left[ \exp \{ i\theta \nu_t \} \right] \exp \left\{ -\frac{\theta^2}{2} t \right\} = \exp \left\{ -\frac{\theta^2}{2} t \right\},$$

which is the characteristic function of a $\mathcal{N}(0, t)$ random variable. Therefore, $\nu$ is a $\mathbb{Q}$-Brownian motion.

Appendix 3: Existence and uniqueness of solution to backward doubly stochastic differential equations

The existence and uniqueness of solution to backward doubly-stochastic differential equations (BDSDEs) of the form (3.15) is given by Theorem 1.1. of [76]. The proof is reproduced here with expansions to hopefully aid the reader.

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a complete probability space that supports a $(k + d)$-dimensional Brownian motion $(W, B)$ and

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^B,$$

where, for any process $\eta_t$,

$$\mathcal{F}_{s,t}^\eta := \sigma \{ \eta_r - \eta_s; s \leq r \leq t \} \vee \mathcal{N}, \quad \mathcal{N} \text{ is the set of zero probability,}$$

$$\mathcal{F}_t^\eta := \mathcal{F}_{0,t}^\eta.$$ 

$\mathcal{F}_t$ changes with time, but is not a filtration, as it neither increases or decreases with time.
For any $Y \in \mathbb{R}^d$ and $Z \in \mathbb{R}^{d \times k}$, define

$$
\|Y\| := \sqrt{(Y_1)^2 + (Y_2)^2 + \ldots + (Y_d)^2}, \quad \|Z\| := \sqrt{\text{tr}(ZZ^*)}.
$$

All expectations are with respect to $\mathbb{Q}$ unless stated otherwise. Let $M^2([0,T];\mathbb{R}^d)$ denote the set of $d\mathbb{Q} \times dt$ jointly measurable random processes $\{\varphi_t \in \mathbb{R}^d; t \in [0,T]\}$ that satisfy

i. $\mathbb{E}\left[\int_0^T \|\varphi_t\|^2 dt\right] < \infty$

ii. $\varphi_t$ is $\mathcal{F}_t$-measurable for any $t \in [0,T]$

and $S^2([0,T];\mathbb{R}^d)$ denote the set of continuous random processes $\{\varphi_t \in \mathbb{R}^d; t \in [0,T]\}$ that satisfy

i. $\mathbb{E}\left[\sup_{t \in [0,T]} \|\varphi_t\|^2\right] < \infty$

ii. $\varphi_t$ is $\mathcal{F}_t$-measurable for any $t \in [0,T]$.

Let

$$
\begin{align*}
 f : & \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \to \mathbb{R}^d, \\
 g : & \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \to \mathbb{R}^{d \times k}
\end{align*}
$$

be jointly measurable w.r.t. $d\mathbb{Q} \times dt$ and for any $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d \times k}$,

$$
\begin{align*}
 f(\cdot,\cdot,y,z) & \in M^2([0,T],\mathbb{R}^d), \\
 g(\cdot,\cdot,y,z) & \in M^2([0,T],\mathbb{R}^{d \times k}).
\end{align*}
$$

Assume that there exist constants $c > 0$, $0 < \alpha < 1$ such that for any $(\omega,t) \in \Omega \times [0,T]$, $(y_1,z_1), (y_2,z_2) \in \mathbb{R}^d \times \mathbb{R}^{d \times k}$,

$$
\begin{align*}
\|f(\omega,t,y_1,z_1) - f(\omega,t,y_2,z_2)\|^2 & \leq c \left(\|y_1 - y_2\|^2 + \|z_1 - z_2\|^2\right), \quad (3a) \\
\|g(\omega,t,y_1,z_1) - g(\omega,t,y_2,z_2)\|^2 & \leq c\|y_1 - y_2\|^2 + \alpha \|z_1 - z_2\|^2. \quad (3b)
\end{align*}
$$

From here on, we suppress the $\omega$-dependence in the notations for $f$ and $g$, their randomness on $\omega \in \Omega$ is implied.

For $\mathcal{F}_t$-measurable $\zeta \in \mathbb{R}^d$, $\mathbb{E}\left[\|\zeta\|\right] < \infty$, a BDSDE is of the form

$$
-dY_t = f(t,Y_t,Z_t)dt + g(t,Y_t,Z_t)d\tilde{B}_t - Z_t dW_t, \quad Y_T = \zeta, \quad (4)
$$
or in integral form,

\[ Y_t = \zeta + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) dB_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \]

(5)

where \( dB \) is a “backward Itô integral” while \( dW \) is the standard forward Itô integral. For \( s \in [0, T] \), let \( B'_s := B_T - B_{T-s} \). It can be checked that the backward Itô integral can be written as a standard forward Itô integral with time shifted (see Section 3.4):

\[ \int_{T-t}^T g_s d\widetilde{B}_s = \int_0^t g_{T-s} dB'_s, \quad \text{for} \quad t \in [0, T] \]

(6)

The solution to the BDSDE (4) is a pair process \((Y, Z) \in \mathbb{R}^d \times \mathbb{R}^{d \times k}\). We will check that the solution does exist and is unique.

First, we consider the case where \( f \) and \( g \) are independent of \((Y, Z)\), i.e. the BDSDE of the form

\[-dY_t = f(t) dt + g(t) dB_t - Z_t dW_t, \quad Y_T = \zeta. \]

(7)

The existence and uniqueness of solution to (7) is given by Proposition 1.2 of [76], which is reproduced here with expansions.

**Uniqueness:** Let \((\tilde{Y}, \tilde{Z})\) denote the difference of two solutions \((Y^1, Z^1)\) and \((Y^2, Z^2)\) to (7). Then, for \( t \in [0, T] \),

\[ \tilde{Y}_t = Y^1_t - Y^2_t \]

\[ = \zeta + \int_t^T f(s) ds + \int_t^T g(s) dB_s - \int_t^T Z^1_s dW_s \]

\[ - \left( \zeta + \int_t^T f(s) ds + \int_t^T g(s) dB_s - \int_t^T Z^2_s dW_s \right) \]

\[ = - \int_t^T \tilde{Z}_s dW_s, \]

so

\[ \tilde{Y}_t + \int_t^T \tilde{Z}_s dW_s = 0. \]
Additionally,
\[ \mathbb{E} \left[ \tilde{Y}_t + \int_t^T \tilde{Z}_s dW_s \right] = 0. \]  
(8)

By orthogonality and Itô isometry,
\[ \mathbb{E} \left[ \left( \tilde{Y}_t + \int_t^T \tilde{Z}_s dW_s \right) \left( \tilde{Y}_t + \int_t^T \tilde{Z}_s dW_s \right)^* \right] = 0 \]
\[ \mathbb{E} \left[ \tilde{Y}_t \tilde{Y}_t + \int_t^T \tilde{Z}_s dW_s \left( \int_t^T \tilde{Z}_s^* dW_s \right)^* + 2 \tilde{Y}_t \left( \int_t^T \tilde{Z}_s^* dW_s \right) \right] = 0 \]
\[ \mathbb{E} \left[ \tilde{Y}_t \tilde{Y}_t \right] + \mathbb{E} \left[ \int_t^T \tilde{Z}_s \tilde{Z}_s^* ds \right] = 0. \]  
(9)

By (8) and (9), we have that \((\tilde{Y}, \tilde{Z}) \equiv 0\), so we have uniqueness of solution to (7).

Existence: Let
\[ \mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_T^B \]

(\(\mathcal{G}_t\) can be thought of as a filtration generated by \(W\) up to time \(t\), that contains all information about \(B\) over \([0, T]\)). Define
\[ M_t := \mathbb{E} \left[ \zeta + \int_0^T f(s) ds + \int_0^T g(s) dB_s \middle| \mathcal{G}_t \right]. \]  
(10)

Note that given \(\mathcal{G}_t\), all the information about \(B\) is known from time 0 to \(T\). \(M_t\) is \(\mathcal{G}_t\)-measurable, and \(\mathcal{G}_t\) is an augmentation of the filtration \(\mathcal{F}_t^W\) generated by \(W\). Therefore, \(M_t\) is \(\mathcal{F}_t^W\)-measurable and, for \(s < t\),
\[ \mathbb{E} \left[ M_t \middle| \mathcal{F}_t^W \right] = \mathbb{E} \left[ \mathbb{E} \left[ \zeta + \int_0^T f(s) ds + \int_0^T g(s) dB_s \middle| \mathcal{G}_t \right] \middle| \mathcal{F}_s^W \right] \]
\[ = \mathbb{E} \left[ \mathbb{E} \left[ \zeta + \int_0^T f(s) ds + \int_0^T g(s) dB_s \middle| \mathcal{F}_t^W \vee \mathcal{F}_T^B \right] \middle| \mathcal{F}_s^W \right] \]
\[ = \mathbb{E} \left[ \zeta + \int_0^T f(s) ds + \int_0^T g(s) dB_s \middle| \mathcal{F}_s^W \vee \mathcal{F}_T^B \right] \]
(by tower property, since \(\mathcal{F}_t^W \subset \mathcal{F}_s^W\) for \(s < t\))
\[ = M_s, \]
so $M_t$ is a $\mathcal{F}_t^W$-martingale. Then, by the martingale representation theorem (see, for example, Theorem 3.4.15 of [44]), $\exists$ square-integrable, $\mathcal{G}_s$-measurable $Z_s \in \mathbb{R}^{d \times k}$, $s \in [0, T]$ such that

$$M_t = M_0 + \int_0^t Z_s dW_s \text{ for } t \in [0, T].$$

Then,

$$M_T = M_0 + \int_0^T Z_s dW_s = M_t - \int_0^t Z_s dW_s + \int_t^T Z_s dW_s$$

$$= M_t + \int_t^T Z_s dW_s.$$

Writing the left and right hand sides in terms of the definitions of the $\mathcal{G}_t$-martingales (10):

$$M_T = \mathbb{E} \left[ \zeta + \int_0^T f(s) ds + \int_0^T g(s) d\tilde{B}_s \bigg| \mathcal{G}_T \right]$$

$$= \zeta + \int_0^T f(s) ds + \int_0^T g(s) d\tilde{B}_s \quad \text{since everything is } \mathcal{G}_T\text{-measurable}$$

$$= \zeta + \int_0^t f(s) ds + \int_0^t g(s) d\tilde{B}_s + \int_t^T f(s) ds + \int_t^T g(s) d\tilde{B}_s$$

and

$$M_t + \int_t^T Z_s dW_s$$

$$= \mathbb{E} \left[ \zeta + \int_0^T f(s) ds + \int_0^T g(s) d\tilde{B}_s \bigg| \mathcal{G}_t \right] + \int_t^T Z_s dW_s$$

$$= \mathbb{E} \left[ \zeta + \int_t^T f(s) ds + \int_t^T g(s) d\tilde{B}_s \bigg| \mathcal{G}_t \right] + \mathbb{E} \left[ \int_0^t f(s) ds + \int_0^t g(s) d\tilde{B}_s \bigg| \mathcal{G}_t \right]$$

$$+ \int_t^T Z_s dW_s$$

$$= \mathbb{E} \left[ \zeta + \int_t^T f(s) ds + \int_t^T g(s) d\tilde{B}_s \bigg| \mathcal{G}_t \right] + \int_t^T f(s) ds + \int_t^T g(s) d\tilde{B}_s$$

$$+ \int_t^T Z_s dW_s,$$

since $f(s)$, $g(s)$ are $\mathcal{G}_t$ measurable for $s \in [0, t]$, $t \in [0, T]$, and $B_s$ is $\mathcal{G}_t$-
measurable for all \( s \in [0, T] \), \( t \in [0, T] \). Combining both sides,

\[
\zeta + \int_0^t f(s)ds + \int_0^t g(s)d\tilde{B}_s + \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s = \mathbb{E} \left[ \zeta + \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s \bigg| \mathcal{G}_t \right] + \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s + \int_t^T Z_s dW_s
\]

\[
\implies \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s = \mathbb{E} \left[ \zeta + \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s \bigg| \mathcal{G}_t \right] + \int_t^T Z_s dW_s
\]

\[
\implies \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s - \int_t^T Z_s dW_s = Y_t,
\]

where we have defined \( Y_t := \mathbb{E} \left[ \zeta + \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s \bigg| \mathcal{G}_t \right] \). So, we have the solution pair \((Y, Z)\) for (7).

It remains to show that it is \( F_t \)-measurable for all \( t \in [0, T] \). By definition,

\[
Y_t = \mathbb{E} \left[ \zeta + \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s \bigg| \mathcal{F}_t^W \vee \mathcal{F}_t^B \right]
\]

\[
= \mathbb{E} \left[ \zeta + \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s \bigg| (\mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B) \right]
\]

\[
= \mathbb{E} \left[ \zeta + \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s \bigg| \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \right]
\]

\[
\text{(since } \int_t^T f(s)ds \text{ independent of } B \text{ and } \int_t^T g(s)d\tilde{B}_s \text{ independent of } \mathcal{F}_t^B \text{)}
\]

\[
= \mathbb{E} \left[ \zeta + \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s \bigg| \mathcal{F}_t \right]
\]

so \( Y_t \) is \( \mathcal{F}_t \)-measurable for all \( t \in [0, T] \).

Rearranging (11),

\[
\int_t^T Z_s dW_s = \int_t^T f(s)ds + \int_t^T g(s)d\tilde{B}_s - \underbrace{Y_t}_{\mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B\text{-measurable}}.
\]

So, the entire right side is \( \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \)-measurable. Invoking the martingale representation theorem again, for \( s \in [t, T] \), \( Z_s \) is the \( \mathcal{F}_s^W \vee \mathcal{F}_{s,T}^B \)-measurable
process for the stochastic integral representation of the right side. Then, $Z_s$ must also be $\mathcal{F}_s^W \vee \mathcal{F}_{s,T}^R$-measurable for any $t < s < T$. ■

Before proceeding to the general case of $f(\cdot, y, z)$, $g(\cdot, y, z)$, we need an extension of the Itô’s lemma for backward integrals. This is given in Lemma 1.3 of [76]. Let $\alpha_t$ satisfy

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \delta_s dW_s,$$

where $\alpha \in S^2([0, T]; \mathbb{R}^d)$, $\beta \in M^2([0, T]; \mathbb{R}^d)$, $\gamma \in M^2([0, T]; \mathbb{R}^{d \times d})$ and $\gamma \in M^2([0, T]; \mathbb{R}^{d \times k})$. We wish to determine the integral equation for $\|\alpha_t\|^2$. Consider

$$\|\alpha_{t+1}\|^2 - \|\alpha_t\|^2$$

$$= \|\alpha_{t+1}\|^2 - 2\|\alpha_t\|^2 + \|\alpha_t\|^2 + 2\alpha_{t+1}^* \alpha_{t+1} - 2\alpha_{t+1}^* \alpha_t$$

$$= 2\int_t^{t+1} \beta_s^* ds \alpha_t + 2\int_t^{t+1} \gamma_s dB_s^* \alpha_t + 2\int_t^{t+1} \delta_s dW_s^* \alpha_t$$

$$+ \|\alpha_{t+1} - \alpha_t\|^2$$

$$= 2\int_t^{t+1} \beta_s^* ds \alpha_t + 2\int_t^{t+1} \gamma_s dB_s^* \alpha_{t+1}$$

$$- 2\int_t^{t+1} \gamma_s dB_s^* \alpha_{t+1} - (\alpha_{t+1} - \alpha_{t+1}) + 2\int_t^{t+1} \delta_s dW_s^* \alpha_t$$

$$+ \|\alpha_{t+1} - \alpha_t\|^2.$$  (12)

The third and fifth terms in (12) are, respectively,

$$- 2\int_t^{t+1} \gamma_s dB_s^* \alpha_{t+1} - (\alpha_{t+1} - \alpha_{t+1})$$

$$= -2\left\|\int_t^{t+1} \gamma_s dB_s\right\|^2 - 2\int_t^{t+1} \gamma_s dB_s^* \left(\int_t^{t+1} \beta_s ds + \int_t^{t+1} \delta_s dW_s\right)$$

and

$$\|\alpha_{t+1} - \alpha_t\|^2$$

$$= \left\|\int_t^{t+1} \beta_s ds\right\|^2 + \left\|\int_t^{t+1} \gamma_s dB_s\right\|^2 + \left\|\int_t^{t+1} \delta_s dW_s\right\|^2.$$
\[ + 2 \left( \int_{t_i}^{t_{i+1}} \gamma_s dB_s \right)^* \left( \int_{t_i}^{t_{i+1}} \beta_s ds + \int_{t_i}^{t_{i+1}} \delta_s dW_s \right) \]

Combined, we have

\[
- 2 \left( \int_{t_i}^{t_{i+1}} \gamma_s dB_s \right)^* (\alpha_{t+1} - \alpha_{t+1}) + \|\alpha_{t+1} - \alpha_t\|^2
\]

\[
= - \left\| \int_{t_i}^{t_{i+1}} \gamma_s dB_s \right\|^2 + \left\| \int_{t_i}^{t_{i+1}} \beta_s ds \right\|^2 + \left\| \int_{t_i}^{t_{i+1}} \delta_s dW_s \right\|^2
\]

\[
+ 2 \left( \int_{t_i}^{t_{i+1}} \beta_s ds \right)^* \int_{t_i}^{t_{i+1}} \delta_s dW_s.
\]

Returning to (12), we have

\[
\|\alpha_{t+1}\|^2 - \|\alpha_t\|^2
\]

\[
= 2 \int_{t_i}^{t_{i+1}} \beta_s ds \alpha_t + 2 \left( \int_{t_i}^{t_{i+1}} \gamma_s dB_s \right)^* \alpha_{t+1} + 2 \left( \int_{t_i}^{t_{i+1}} \delta_s dW_s \right)^* \alpha_t
\]

\[
- \left\| \int_{t_i}^{t_{i+1}} \gamma_s dB_s \right\|^2 + \left\| \int_{t_i}^{t_{i+1}} \beta_s ds \right\|^2 + \left\| \int_{t_i}^{t_{i+1}} \delta_s dW_s \right\|^2
\]

\[
+ 2 \left( \int_{t_i}^{t_{i+1}} \beta_s ds \right)^* \int_{t_i}^{t_{i+1}} \delta_s dW_s.
\]

If \( t_{i+1} - t_i \) is a small interval, then

\[
\|\alpha_{t+1}\|^2 - \|\alpha_t\|^2
\]

\[
\approx 2 \alpha_{t_i}^* \beta_{t_i} (t_{i+1} - t_i) + 2 \alpha_{t_{i+1}}^* \gamma_{t_{i+1}} (B_{t_{i+1}} - B_{t_i}) + 2 \alpha_{t_i}^* \delta_{t_i} (W_{t_{i+1}} - W_{t_i})
\]

\[
- \| \gamma_{t_{i+1}} (B_{t_{i+1}} - B_{t_i}) \|^2 + \| \beta_{t_i} \|^2 (t_{i+1} - t_i)^2 + \| \delta_{t_i} (W_{t_{i+1}} - W_{t_i}) \|^2
\]

\[
+ 2 \beta_{t_i}^* \delta_{t_i} (W_{t_{i+1}} - W_{t_i}) (t_{i+1} - t_i)
\]

\[
= 2 \alpha_{t_i}^* \beta_{t_i} (t_{i+1} - t_i) + 2 \alpha_{t_{i+1}}^* \gamma_{t_{i+1}} (B_{t_{i+1}} - B_{t_i}) + 2 \alpha_{t_i}^* \delta_{t_i} (W_{t_{i+1}} - W_{t_i})
\]

\[
- \| \gamma_{t_{i+1}} \|^2 (t_{i+1} - t_i) + \| \beta_{t_i} \|^2 (t_{i+1} - t_i)^2 + \| \delta_{t_i} \|^2 (t_{i+1} - t_i)
\]

\[
+ 2 \beta_{t_i}^* \delta_{t_i} (W_{t_{i+1}} - W_{t_i}) (t_{i+1} - t_i).
\]

Denote \( t_i := i \left\lfloor \frac{t}{N} \right\rfloor \). Using (13),

\[
\|\alpha_t\|^2 - \|\alpha_0\|^2
\]
\[
= \lim_{N \to \infty} \left\| \alpha_N \right\|^2 - \left\| \alpha_{N-1} \right\|^2 + \left\| \alpha_{N-1} \right\|^2 - \left\| \alpha_{N-2} \right\|^2 \\
\quad + \ldots + \left\| \alpha \right\|^2 - \left\| \alpha_0 \right\|^2
\]
\[
= \lim_{N \to \infty} \sum_{i=0}^{N-1} \left\| \alpha_{t_{i+1}} \right\|^2 - \left\| \alpha_{t_i} \right\|^2
\]
\[
= \lim_{N \to \infty} \sum_{i=0}^{N-1} 2\alpha^*_t \beta_i (t_{i+1} - t_i) + 2\alpha^*_t \gamma_{t_{i+1}} (B_{t_{i+1}} - B_{t_i}) \\
\quad + 2\alpha^*_t \delta_t (W_{t_{i+1}} - W_{t_i}) \\
\quad - \left\| \gamma_{t_{i+1}} \right\|^2 (t_{i+1} - t_i) + \left\| \beta_i \right\|^2 (t_{i+1} - t_i)^2 + \left\| \delta_t \right\|^2 (t_{i+1} - t_i) \\
\quad + 2\beta^*_t \delta_t (W_{t_{i+1}} - W_{t_i}) (t_{i+1} - t_i),
\]
where the \((t_{i+1} - t_i)^2\) and \((W_{t_{i+1}} - W_{t_i})(t_{i+1} - t_i)\) terms \(\to 0\) as \(N \to \infty\) \((t_{i+1} - t_i) \to 0\). Then,
\[
\left\| \alpha_t \right\|^2 - \left\| \alpha_0 \right\|^2
\]
\[
= \lim_{N \to \infty} \sum_{i=0}^{N-1} 2\alpha^*_t \beta_i (t_{i+1} - t_i) + 2\alpha^*_t \gamma_{t_{i+1}} (B_{t_{i+1}} - B_{t_i}) \\
\quad + 2\alpha^*_t \delta_t (W_{t_{i+1}} - W_{t_i}) - \left\| \gamma_{t_{i+1}} \right\|^2 (t_{i+1} - t_i) + \left\| \delta_t \right\|^2 (t_{i+1} - t_i) \\
\quad = 2 \int_0^t \alpha^*_s \beta_s ds + 2 \int_0^t \alpha^*_s \gamma_s d\bar{B}_s + 2 \int_0^t \alpha^*_s \delta_s dW_s - \int_0^t \gamma_s \| \| ds + \int_0^t \| \delta_s \|^2 ds,
\]
i.e.
\[
\left\| \alpha_t \right\|^2 = \left\| \alpha_0 \right\|^2 + 2 \int_0^t \alpha^*_s \beta_s ds + 2 \int_0^t \alpha^*_s \gamma_s d\bar{B}_s + 2 \int_0^t \alpha^*_s \delta_s dW_s - \int_0^t \gamma_s \| ^2 ds + \int_0^t \| \delta_s \|^2 ds.
\]
(14)

Consider the forward stochastic integral term: By the Burkholder-Davis-Gundy inequality,
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \alpha^*_s \delta_s dW_s \right| \right] \leq C \mathbb{E} \left[ \left( \int_0^T \alpha^*_s \delta_s dW_s \right)^{\frac{1}{2}} \right]
\]
\[
= C \mathbb{E} \left[ \left( \int_0^T \left( \alpha^*_s \delta_s \right) \right)^{\frac{1}{2}} \right] \leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \alpha_t \right\|^2 \int_0^T \| \delta_s \|^2 ds \right]
\]
\[
\frac{C}{2} \left( \mathbb{E} \left[ \sup_{t \in [0,T]} \| \alpha_t \|^2 \right] + \mathbb{E} \left[ \int_0^T \| \delta_s \|^2 ds \right] \right),
\]

which is bounded since \( \alpha \) is in \( S^2 \) and \( \delta \) is in \( M^2 \). The backward stochastic integral can be bounded similarly. Then, the expected value of the stochastic integrals vanish. We have

\[
\mathbb{E} \left[ \| \alpha_t \|^2 \right] = \mathbb{E} \left[ \| \alpha_0 \|^2 \right] + 2 \mathbb{E} \left[ \int_0^t \alpha_s^* \beta_s ds \right] - \mathbb{E} \left[ \int_0^t \| \gamma_s \|^2 ds \right] + \mathbb{E} \left[ \int_0^t \| \delta_s \|^2 ds \right].
\]  

(15)

Now we prove the existence and uniqueness of the solution \((Y, Z)\) to (4).

**Uniqueness:** Let \((Y^1_t, Z^1_t)\) and \((Y^2_t, Z^2_t)\) be two solutions to (4) and

\[
\tilde{Y}_t := Y^1_t - Y^2_t, \quad \tilde{Z}_t := Z^1_t - Z^2_t.
\]

Then,

\[
-d\tilde{Y}_t = (f(t, Y^1_t, Z^1_t) - f(t, Y^2_t, Z^2_t)) dt + \left( g(t, Y^1_t, Z^1_t) - g(t, Y^2_t, Z^2_t) \right) dB_t \\
- \tilde{Z}_t dW_t,
\]

\( \tilde{Y}_T = 0_{d \times 1} \).

Applying (14),

\[
\| \tilde{Y}_t \|^2 = 2 \int_t^T \tilde{Y}_s^* (f(s, Y^1_s, Z^1_s) - f(s, Y^2_s, Z^2_s)) ds \\
+ \int_t^T \| g(s, Y^1_s, Z^1_s) - g(s, Y^2_s, Z^2_s) \|^2 ds - \int_t^T \| \tilde{Z}_s \|^2 ds \\
+ 2 \int_t^T \tilde{Y}_s^* (g(s, Y^1_s, Z^1_s) - g(s, Y^2_s, Z^2_s)) dB_s \\
- 2 \int_t^T \tilde{Y}_s^* \tilde{Z}_s dW_s.
\]

Taking expected value using (15) and rearranging,

\[
\mathbb{E} \left[ \| \tilde{Y}_t \|^2 \right] + \int_t^T \mathbb{E} \left[ \| \tilde{Z}_s \|^2 \right] ds
\]

(16)
\[
E \left[ \| \tilde{Y}_t \|^2 \right] + \int_t^T E \left[ \| \tilde{Z}_s \|^2 \right] ds \\
\leq 2 \int_t^T E \left[ \frac{1}{2(1 - \gamma)} \| \tilde{Y}_s \|^2 + \frac{1 - \gamma}{2} \| f(s, Y^1_s, Z^1_s) - f(s, Y^2_s, Z^2_s) \|^2 \right] ds \\
+ \int_t^T E \left[ \| g(s, Y^1_s, Z^1_s) - g(s, Y^2_s, Z^2_s) \|^2 \right] ds \\
= \int_t^T E \left[ \frac{1}{(1 - \gamma)} \| \tilde{Y}_s \|^2 + (1 - \gamma) \| f(s, Y^1_s, Z^1_s) - f(s, Y^2_s, Z^2_s) \|^2 \right] ds \\
+ \int_t^T E \left[ \| g(s, Y^1_s, Z^1_s) - g(s, Y^2_s, Z^2_s) \|^2 \right] ds.
\]

Applying the Lipschitz-like condition (3),

\[
E \left[ \| \tilde{Y}_t \|^2 \right] + \int_t^T E \left[ \| \tilde{Z}_s \|^2 \right] ds \\
\leq \int_t^T E \left[ \frac{1}{1 - \gamma} \| \tilde{Y}_s \|^2 + c(1 - \gamma) \left( \| \tilde{Y}_s \| + \| \tilde{Z}_s \| \right) \right] ds \\
+ \int_t^T E \left[ c \| \tilde{Y}_s \|^2 + \alpha \| \tilde{Z}_s \|^2 \right] ds \\
= \frac{1 + (1 - \gamma)c + (1 - \gamma)c}{1 - \gamma} \int_t^T E \left[ \| \tilde{Y}_s \|^2 \right] ds \\
+ ((1 - \gamma)c + \alpha) \int_t^T E \left[ \| \tilde{Z}_s \|^2 \right] ds \\
\implies E \left[ \| \tilde{Y}_t \|^2 \right] \\
\leq \frac{1 + (1 - \gamma)(2 - \gamma)c}{1 - \gamma} \int_t^T E \left[ \| \tilde{Y}_s \|^2 \right] ds.
\]
\[(1 - \gamma) c - (1 - \alpha) \int_t^T \mathbb{E} \left[ \| \tilde{Z}_s \|^2 \right] \, ds.\]

Let \( \gamma = 1 - \frac{1 - \alpha}{2c} \), where \( \alpha \) and \( c \) are the constants in (3). Then,

\[
\mathbb{E} \left[ \| \tilde{Y}_t \|^2 \right] \leq \left( \frac{2c}{1 - \alpha} + c + \frac{1 - \alpha}{2} \right) \int_t^T \mathbb{E} \left[ \| \tilde{Y}_s \|^2 \right] \, ds - \frac{1 - \alpha}{2} \int_t^T \mathbb{E} \left[ \| \tilde{Z}_s \|^2 \right] \, ds.
\]

Let \( C_1(\alpha) := \frac{2c}{1 - \alpha} + c + \frac{1 - \alpha}{2} \) and \( C_2(\alpha) := \frac{1 - \alpha}{2} \). \( \alpha \in (0, 1) \) and \( c > 0 \), so \( C_1(\alpha) > 0 \) and \( C_2(\alpha) < 0 \). Then, we can obtain a looser bound on \( \mathbb{E} \left[ \| \tilde{Y}_t \|^2 \right] \) in (17) by dropping the negative term \( C_2(\alpha) \int_t^T \mathbb{E} \left[ \| \tilde{Z}_s \|^2 \right] \, ds \):

\[
\mathbb{E} \left[ \| \tilde{Y}_t \|^2 \right] \leq C_1(\alpha) \int_t^T \mathbb{E} \left[ \| \tilde{Y}_s \|^2 \right] \, ds.
\]

Applying Grönwall’s lemma, we have

\[
\mathbb{E} \left[ \| \tilde{Y}_t \|^2 \right] \leq 0 \cdot \int_t^T e^{C_1(\alpha)} ds = 0 \cdot (T - t) e^{C_1(\alpha)} = 0.
\]

Therefore, \( \mathbb{E} \left[ \| \tilde{Y}_t \|^2 \right] = 0 \). Returning to (17), and using the fact that \( \mathbb{E} \left[ \| \tilde{Y}_t \|^2 \right] = 0 \), we have

\[
C_2(\alpha) \int_t^T \mathbb{E} \left[ \| \tilde{Z}_s \|^2 \right] \, ds \geq 0.
\]

Since \( C_2(\alpha) < 0 \), then it must be that \( \int_t^T \mathbb{E} \left[ \| \tilde{Z}_s \|^2 \right] \, ds = 0 \) as well.

We have mean-square convergence of \( (\tilde{Y}_t, \tilde{Z}_t; \, t \in [0, T]) \) to zero, which gives us uniqueness of solution \((Y, Z)\) to (3.15).

**Existence:** Existence of solution is shown by Picard iteration. The existence and uniqueness of solution to (7) is used to write the iterative solution to (4), then similar calculations and application of Grönwall as in the uniqueness proof is used to show convergence of the Picard iteration.

Define a recursive sequence:

\[
(Y_i^t, Z_i^t) = (0_{d \times 1}, 0_{d \times k}) \quad \text{for} \quad i = 0,
\]
\[ Y_t^{i+1} = \zeta + \int_t^T f(s, Y_s^i, Z_s^i) ds + \int_t^T g(s, Y_s^i, Z_s^i) dB_s - \int_t^T Z_s^{i+1} dW_s \quad \text{for } i \geq 1. \] (18)

Given \((Y^i, Z^i)\), the solution \((Y^{i+1}, Z^{i+1})\) to (18) exists and is unique by existence and uniqueness of solution to (7). Let

\[ \tilde{Y}_t^{i+1} \overset{\text{def}}{=} Y_t^{i+1} - Y_t^i, \quad \tilde{Z}_t^{i+1} \overset{\text{def}}{=} Z_t^{i+1} - Z_t^i. \]

Then,

\[ \tilde{Y}_t^{i+1} = \int_t^T \left( f(s, Y_s^i, Z_s^i) - f(s, Y_s^{i-1}, Z_s^{i-1}) \right) ds + \int_t^T \left( g(s, Y_s^i, Z_s^i) - g(s, Y_s^{i-1}, Z_s^{i-1}) \right) dB_s - \int_t^T \tilde{Z}_s^{i+1} dW_s, \]

or

\[-d\tilde{Y}_t^{i+1} = \left( f(t, Y_t^i, Z_t^i) - f(t, Y_t^{i-1}, Z_t^{i-1}) \right) dt + \left( g(t, Y_t^i, Z_t^i) - g(t, Y_t^{i-1}, Z_t^{i-1}) \right) dB_t - \tilde{Z}_t^{i+1} dW_t, \]

\[ \tilde{Y}_T^{i+1} = 0_{d \times 1}. \]

Let \(\beta \in \mathbb{R}\). Applying (14) to \(e^{\beta t} \| \tilde{Y}_t^{i+1} \|^2\),

\[
e^{\beta t} \| \tilde{Y}_t^{i+1} \|^2
= -\beta \int_t^T e^{\beta s} \| \tilde{Y}_s^{i+1} \|^2 ds
+ 2 \int_t^T e^{\beta s} \tilde{Y}_s^{i+1}^* (f(s, Y_s^i, Z_s^i) - f(s, Y_s^{i-1}, Z_s^{i-1})) ds
+ \int_t^T e^{\beta s} \| g(s, Y_s^i, Z_s^i) - g(s, Y_s^{i-1}, Z_s^{i-1}) \|^2 ds - \int_t^T e^{\beta s} \| \tilde{Z}_s^{i+1} \|^2 ds
+ 2 \int_t^T e^{\beta s} (\tilde{Y}_s^{i+1})^* (g(s, Y_s^i, Z_s^i) - g(s, Y_s^{i-1}, Z_s^{i-1})) dB_s
- 2 \int_t^T e^{\beta s} (\tilde{Y}_s^{i+1})^* \tilde{Z}_s^{i+1} dW_s.
\]

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Using the Young’s inequality $ab \leq \frac{1}{2(1-\gamma)}a^2 + \frac{1-\gamma}{2}b^2$, $\gamma \in (0, 1)$,

\[
\mathbb{E} \left[ e^{\beta t} \| \tilde{Y}^{i+1} \|^2 \right] + \beta \mathbb{E} \left[ \int_t^T e^{\beta s} \| \tilde{Y}^{i+1}_s \|^2 ds \right] + \mathbb{E} \left[ \int_t^T e^{\beta s} \| \tilde{Z}^{i+1}_s \|^2 ds \right] \\
\leq \mathbb{E} \left[ \int_t^T e^{\beta s} \left( \frac{1}{1-\gamma} \| \tilde{Y}^{i+1}_s \|^2 + (1-\gamma) \| f(s, Y^{i}_s, Z^{i}_s) - f(s, Y^{i-1}_s, Z^{i-1}_s) \|^2 \right) ds \right] \\
+ \mathbb{E} \left[ \int_t^T e^{\beta s} \| g(s, Y^{i}_s, Z^{i}_s) - g(s, Y^{i-1}_s, Z^{i-1}_s) \|^2 ds \right].
\]

Using the Lipschitz-like condition (3),

\[
\mathbb{E} \left[ e^{\beta t} \| \tilde{Y}^{i+1} \|^2 \right] + \left( \beta - \frac{1}{1-\gamma} \right) \mathbb{E} \left[ \int_t^T e^{\beta s} \| \tilde{Y}^{i+1}_s \|^2 ds \right] + \mathbb{E} \left[ \int_t^T e^{\beta s} \| \tilde{Z}^{i+1}_s \|^2 ds \right] \\
\leq \mathbb{E} \left[ \int_t^T e^{\beta s} c(1-\gamma) \left( \| \tilde{Y}^{i}_s \|^2 + \| \tilde{Z}^{i}_s \|^2 \right) ds \right] + \mathbb{E} \left[ \int_t^T e^{\beta s} \left( c \| \tilde{Y}^{i}_s \| + \alpha \| Z^{i}_s \|^2 \right) ds \right] \\
= \mathbb{E} \left[ \int_t^T e^{\beta s} \left( c(2-\gamma) \| \tilde{Y}^{i}_s \|^2 + (c(1-\gamma) + \alpha) \| Z^{i}_s \|^2 \right) ds \right].
\]

Let $\gamma = 1 - \frac{1-\alpha}{2c}$ and $\beta = \bar{c} + \frac{2c}{1-\alpha}$, where $\bar{c} := \frac{2c+1-\alpha}{1-\alpha}$. Then

\[
\mathbb{E} \left[ e^{\beta t} \| \tilde{Y}^{i+1} \|^2 \right] + \left( \bar{c} + \frac{2c}{1-\alpha} - \frac{2c}{1-\alpha} \right) \mathbb{E} \left[ \int_t^T e^{\beta s} \| \tilde{Y}^{i+1}_s \|^2 ds \right] \\
+ \mathbb{E} \left[ \int_t^T e^{\beta s} \| \tilde{Z}^{i+1}_s \|^2 ds \right] \\
\leq \mathbb{E} \left[ \int_t^T e^{\beta s} \left( \frac{2c+1-\alpha}{2} \| \tilde{Y}^{i}_s \|^2 + \frac{1+\alpha}{2} \| \tilde{Z}^{i}_s \|^2 \right) ds \right] \\
\implies \mathbb{E} \left[ e^{\beta t} \| \tilde{Y}^{i+1} \|^2 \right] + \bar{c} \mathbb{E} \left[ \int_t^T e^{\beta s} \| \tilde{Y}^{i+1}_s \|^2 ds \right] + \mathbb{E} \left[ \int_t^T e^{\beta s} \| \tilde{Z}^{i+1}_s \|^2 ds \right] \\
\leq \frac{1+\alpha}{2} \mathbb{E} \left[ \int_t^T e^{\beta s} \left( \bar{c} \| \tilde{Y}^{i}_s \|^2 + \| \tilde{Z}^{i}_s \|^2 \right) ds \right].
\]
\[
\Rightarrow \mathbb{E} \left[ \int_t^T e^{\beta s} \left( \bar{c} \left\| \bar{Y}_{s+1} \right\|^2 + \left\| \bar{Z}_{s+1} \right\|^2 \right) ds \right] \\
\leq \frac{1 + \alpha}{2} \mathbb{E} \left[ \int_t^T e^{\beta s} \left( \bar{c} \left\| \bar{Y}_t \right\|^2 + \left\| \bar{Z}_t \right\|^2 \right) ds \right].
\]

Applying this inequality recursively for \(i, i-1, i-2, \ldots, 2, 1\) on the right side,

\[
\mathbb{E} \left[ \int_t^T e^{\beta s} \left( \bar{c} \left\| \bar{Y}_{s+1} \right\|^2 + \left\| \bar{Z}_{s+1} \right\|^2 \right) ds \right]
\leq \left( \frac{1 + \alpha}{2} \right)^i \mathbb{E} \left[ \int_t^T e^{\beta s} \left( \bar{c} \left\| \bar{Y}_t \right\|^2 + \left\| \bar{Z}_t \right\|^2 \right) ds \right].
\]

Since \(\alpha \in (0, 1)\), so \(\frac{1 + \alpha}{2} < 1\) and \(\left( \frac{1 + \alpha}{2} \right)^i \searrow 0\) as \(i \uparrow \infty\), i.e. \(\{Y^i_t, Z^i_t\}\) is a Cauchy sequence. Therefore, the limit

\[(Y_t, Z_t) = \lim_{i \uparrow \infty} (Y^i_t, Z^i_t)\]
exists. ■

Appendix 4: SDE for \(\rho_t(1)\)

The likelihood function for the partially observed diffusion is the normalizer of the nonlinear filter, \(\rho_\theta(t)\) (see Section 5.3). Lemma 3.29 of [35] gives the governing equation for the log likelihood function \(\log \rho_\theta(t)\) (we suppress parameter dependence in the following): Consider the partially observed diffusion of (5.4). W.l.o.g., let the signal and observation be scalars. We know that the unnormalized filter \(\rho(\varphi)\) satisfies the Zakai equation, (2.15):

\[
\rho_t(\varphi) = \rho_0(\varphi) + \int_0^t \rho_s(\mathcal{L}_s \varphi) ds + \int_0^t \rho_s(h \varphi) dY_s.
\]

\(\rho_0(1) = 1, \mathcal{L}1 = 0\), so

\[
\rho_t(1) = 1 + \int_0^t \rho_s(h) dY_s = 1 + \int_0^t \rho_s(1) \pi_s(h) dY_s.
\]

Next, we determine \(d(\log \rho_t(1))\). We use Itô’s formula, but we cannot use it directly because \(\log\) is undefined at 0 and we do not know a priori that \(\rho(1) > 0\). Instead, we apply Itô’s formula to \(\log \sqrt{\delta + \rho^2}, \delta > 0\), and then
take \( \lim_{\delta \to 0} \). First,
\[
\frac{\partial}{\partial \rho} (\log \sqrt{\delta + \rho^2}) = \frac{1}{\sqrt{\delta + \rho^2}} \cdot \frac{1}{2} \frac{2\rho}{\sqrt{\delta + \rho^2}} = \frac{\rho}{\delta + \rho^2}.
\]
\[
\frac{\partial^2}{\partial^2 \rho} (\log \sqrt{\delta + \rho^2}) = \frac{1}{\delta + \rho^2} - \frac{2\rho^2}{(\delta + \rho^2)^2} = \frac{\delta - \rho^2}{(\delta + \rho^2)^2}.
\]

Applying Itô’s formula to \( \log \sqrt{\delta + \rho^2} \), we have
\[
d(\log \sqrt{\delta + \rho_t(1)^2}) = \frac{\rho_t(1)}{\delta + \rho_t(1)^2} d\rho_t(1) + \frac{1}{2} \frac{\delta - \rho_t(1)^2}{(\delta + \rho_t(1)^2)^2} (d\rho_t(1))_t
\]
\[
= \frac{\rho_t(1)}{\delta + \rho_t(1)^2} \pi_t(h) dY_t + \frac{1}{2} \frac{\delta - \rho_t(1)^2}{(\delta + \rho_t(1)^2)^2} (\rho_t(1) \pi_t(h))^2 dt
\]
\[
\to \pi_t(h) dY_t - \frac{1}{2} \pi_t(h)^2 dt \quad \text{as} \ \delta \to 0.
\]

\( \log \rho_0(1) = \log 1 = 0 \), so
\[
\log \rho_t(1) = \int_0^t \pi_s(h) dY_s - \frac{1}{2} \int_0^t \pi_s(h)^2 ds.
\]

Note that if \( h \in L^2 \), then
\[
\rho_t(1) = \exp \left\{ \int_0^t \pi_s(h) dY_s - \frac{1}{2} \int_0^t \pi_s(h)^2 ds \right\}
\]
is greater than 0 for all \( t \in [0, \infty) \) because the integrals in the exponential cannot go to \(-\infty\) by boundedness of \( h \) and \( h^2 \), and the Wiener integral cannot go to \( \pm \infty \) in finite time.

Appendix 5: Cramér-Rao lower bound

The Cramér-Rao inequality for ML estimators states that the lower bound of the mean-squared error is inversely proportional to the Fisher information of the observed data. Let \( \varphi : \Theta \to \Theta \) be a \( \mathcal{Q} \)-integrable, \( \mathcal{C}(\Theta) \) function. The variance of an estimator \( \hat{\theta}_T \) can be written as the mean-squared error (MSE) minus the bias of the estimator:
\[
\text{Var}_{\mathcal{Q}_\Theta} \left[ \varphi(\hat{\theta}_T) \right] = \mathbb{E}_{\mathcal{Q}_\Theta} \left[ \left( \varphi(\hat{\theta}_T) - \mathbb{E}_{\mathcal{Q}_\Theta} \left[ \varphi(\hat{\theta}_T) \right] \right)^2 \right]
\]
\[
= \mathbb{E}_{\mathcal{Q}_\Theta} \left[ \left( \varphi(\hat{\theta}_T) - \varphi(\hat{\theta}_T) - \left( \mathbb{E}_{\mathcal{Q}_\Theta} \left[ \varphi(\hat{\theta}_T) \right] - \varphi(\theta) \right) \right)^2 \right]
\]

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\[
\begin{align*}
\ &= E_{Q_0} \left[ (\varphi(\hat{\theta}_T) - \varphi(\theta))^2 \right] + \left( E_{Q_0} \left[ \varphi(\hat{\theta}_T) - \varphi(\theta) \right] \right)^2 \\
\ &= 2 \left( E_{Q_0} \left[ \varphi(\hat{\theta}_T) - \varphi(\theta) \right] \right)^2
\end{align*}
\]

(19)

Next we determine an inequality for the variance of the estimator. Using measure change and derivative of the log function, we have

\[
\frac{\partial}{\partial \theta} E_{Q_0} \left[ \varphi(\hat{\theta}_T) \right] = \frac{\partial}{\partial \theta} \int_X \varphi(\hat{\theta}_T) \, dQ_\theta | _{\mathcal{F}_T}
\]

\[
= \int_X \varphi(\hat{\theta}_T) \frac{\partial}{\partial \theta} \left( \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right) dQ_{\theta'} | _{\mathcal{F}_T}
\]

\[
= \int_X \varphi(\hat{\theta}_T) \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right) \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} dQ_{\theta'} | _{\mathcal{F}_T}
\]

\[
= \int_X \varphi(\hat{\theta}_T) \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right) dQ_{\theta} | _{\mathcal{F}_T}
\]

\[
= E_{Q_0} \left[ \varphi(\hat{\theta}_T) \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right) \right]
\]

Note that

\[
E_{Q_0} \left[ \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right) \right] = E_{Q_\theta'} \left[ \frac{\partial}{\partial \theta} \left( \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right) \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right]
\]

\[
= E_{Q_{\theta'}} \left[ \frac{\partial}{\partial \theta} \left( \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right) \right] = \frac{\partial}{\partial \theta} E_{Q_{\theta'}} \left[ \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right]
\]

\[
= \frac{\partial}{\partial \theta} E_{Q_0} [1] = \frac{\partial}{\partial \theta} 1 = 0.
\]

Therefore,

\[
\frac{\partial}{\partial \theta} E_{Q_0} \left[ \varphi(\hat{\theta}_T) \right]
\]

\[
= E_{Q_0} \left[ \varphi(\hat{\theta}_T) \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right) \right]
\]

\[
= E_{Q_0} \left[ \left( \varphi(\hat{\theta}_T) - E_{Q_0} \left[ \varphi(\hat{\theta}_T) \right] \right) \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} | _{\mathcal{F}_T} \right) \right]
\]

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\[
\leq \mathbb{E}_{Q_\theta} \left[ \left( \phi(\hat{\theta}_T) - \mathbb{E}_{Q_\theta} \left[ \phi(\hat{\theta}_T) \right] \right)^2 \right]^{1/2} \mathbb{E}_{Q_\theta} \left[ \left( \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} | \mathcal{F}_T \right) \right)^2 \right]^{1/2},
\]

where we used \( \mathbb{E}_{Q_\theta} \left[ \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} | \mathcal{F}_T \right) \right] = 0 \) in the second equality and Hölder’s inequality in the last inequality. Rearranging the inequality and using (19), we have

\[
\text{Var}_{Q_\theta} \left[ \phi(\hat{\theta}_T) \right] \geq \frac{\left( \frac{\partial}{\partial \theta} \mathbb{E}_{Q_\theta} \left[ \phi(\hat{\theta}_T) \right] \right)^2}{I(\theta)},
\]

and

\[
\text{MSE} \geq \frac{\left( \frac{\partial}{\partial \theta} \phi(\theta) - \frac{\partial}{\partial \theta} B(\theta) \right)^2}{I(\theta)} + B(\theta)^2,
\]

where the MSE, Fisher information and bias are, respectively,

\[
\text{MSE} = \mathbb{E}_{Q_\theta} \left[ \left( \phi(\hat{\theta}_T) - \phi(\theta) \right)^2 \right],
\]

\[
I(\theta) = \mathbb{E}_{Q_\theta} \left[ \left( \frac{\partial}{\partial \theta} \left( \log \frac{dQ_\theta}{dQ_{\theta'}} | \mathcal{F}_T \right) \right)^2 \right],
\]

\[
B(\theta) = \mathbb{E}_{Q_\theta} \left[ \phi(\hat{\theta}_T) \right] - \phi(\theta).
\]

The lower bound for the MSE is the Cramér-Rao bound.

Appendix 6: \( L^p \)-convergence of the filtered likelihood

Consider the signal and observation of Section 6.4, (6.9) and (6.10). We ignore parameter dependence here. For completeness, we rewrite the diffusion equations and filtered likelihood here. Let \((\Omega, \mathcal{F}, Q)\) support a \(k + l + d\)-dimensional Brownian motion \((W, V, B)\), and the signal and observation be

\[
\begin{align*}
dX^\varepsilon_t &= b(X^\varepsilon_t, Z^\varepsilon_t)dt + \sigma(X^\varepsilon_t)dW_t, \quad X^\varepsilon_0 \in \mathbb{R}^n, \\
dZ^\varepsilon_t &= \frac{1}{\varepsilon}f(X^\varepsilon_t, Z^\varepsilon_t)dt + \frac{1}{\sqrt{\varepsilon}}g(X^\varepsilon_t, Z^\varepsilon_t)dV_t, \quad Z^\varepsilon_0 \in \mathbb{R}^n, \\
dY^\varepsilon_t &= h(X^\varepsilon_t, Z^\varepsilon_t)dt + dB_t, \quad Y^\varepsilon \in \mathbb{R}^d,
\end{align*}
\]

Assume (H\text{stat}), (HF\text{\ v}k, 3) and that the coefficients \(b, \sigma, f, g\) and \(h\) of the slow and fast motions and observation are bounded and globally Lipschitz.
conditious.

The weak limit of \((X^\varepsilon, Y^\varepsilon)\) as \(\varepsilon \to 0\), under conditions for existence and uniqueness of invariant measure of \(Z\) at timescale \(\varepsilon t\), \((H_{\text{stat}})\), is

\[
dX_0^0 = \bar{b}(X_0^0) dt + \bar{\sigma}(X_0^0) dW_t, \quad X_0^0 \in \mathbb{R}^m,
\]
\[
dY_0^0 = \bar{h}(X_0^0) dt + dB_t, \quad Y_0^0 \in \mathbb{R}^d,
\]

where \(\bar{\cdot}\) quantities are appropriately averaged w.r.t. the invariant measure of \(Z\).

Define
\[
\tilde{D}_x^\varepsilon \overset{\text{def}}{=} \left. \frac{dP^\varepsilon}{dQ} \right|_{F_T} = \exp \left\{ \int_0^t \bar{h}(X^0_s)^* dY^\varepsilon_s - \frac{1}{2} \int_0^t \| \bar{h}(X^0_s) \|^2 ds \right\},
\]
\[
\tilde{D}_t \overset{\text{def}}{=} \left. \frac{dP^0}{dQ} \right|_{F_T} = \exp \left\{ \int_0^t \bar{h}(X^0_s)^* dY^0_s - \frac{1}{2} \int_0^t \| \bar{h}(X^0_s) \|^2 ds \right\}.
\]

Let \(\mathcal{Y}_{s}^\varepsilon = \sigma \{Y_s^\varepsilon, 0 \leq s \leq t\} \lor \mathcal{N}\) and \(\mathcal{Y}_{s}^0 = \sigma \{Y_s^0, 0 \leq s \leq t\} \lor \mathcal{N}\). The corresponding filtered likelihoods are

\[
\tilde{\rho}_t^\varepsilon(1) \overset{\text{def}}{=} \mathbb{E}_{P^\varepsilon} \left[ \tilde{D}_t \bar{\mathcal{Y}}^\varepsilon_t \right] = 1 + \int_0^t \tilde{\rho}_s^\varepsilon(\bar{h}^*)dY_s^\varepsilon,
\]
\[
\tilde{\rho}_t(1) \overset{\text{def}}{=} \mathbb{E}_{P^0} \left[ \tilde{D}_t \bar{\mathcal{Y}}^0_t \right] = 1 + \int_0^t \tilde{\rho}_s(\bar{h}^*)dY_s^0
\]
(see Appendix 4 for the second equalities).

Here, we show that, for \(T > 0, p \geq 1\),

\[
\mathbb{E}_Q [|| \tilde{\rho}_T^\varepsilon(1) - \tilde{\rho}_T(1) ||^p] \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]

For \(T > 0, p \geq 1\),

\[
\mathbb{E}_Q [|| \tilde{\rho}_T^\varepsilon(1) - \tilde{\rho}_T(1) ||^p] = \mathbb{E}_{P^\varepsilon} \left[ || \tilde{\rho}_T^\varepsilon(1) - \tilde{\rho}_T(1) ||^p \left. \frac{dP^\varepsilon}{dQ} \right|_{F_T} \right] \leq \mathbb{E}_{P^\varepsilon} \left[ || \tilde{\rho}_T^\varepsilon(1) - \tilde{\rho}_T(1) ||^{2p} \right]^{1/2} \frac{1}{2} \mathbb{E}_{P^\varepsilon} \left[ \left( \left. \frac{dP^\varepsilon}{dQ} \right|_{F_T} \right)^2 \right]^{1/2},
\]

where we can check that the Radon-Nikodym derivative term is bounded when \(T\) is finite (see in proof of Lemma 3.7.5).
The difference term is

\[
\mathbb{E}_{P^\varepsilon} \left[ \left| \bar{\rho}^\varepsilon_T(1) - \bar{\rho}_T(1) \right|^{2p} \right]
\]

\[
= \mathbb{E}_{P^\varepsilon} \left[ \left( \int_0^T \bar{\rho}^\varepsilon_t(\bar{h}^*) dY^\varepsilon_t - \int_0^T \bar{\rho}_t(\bar{h}^*) dY^0_t \right)^{2p} \right]
\]

\[
= \mathbb{E}_{P^\varepsilon} \left[ \left( \int_0^T \left( \bar{\rho}^\varepsilon_t(\bar{h}^*) - \bar{\rho}_t(\bar{h}^*) \right) dY^\varepsilon_t + \int_0^T \bar{\rho}_t(\bar{h}^*) (dY^\varepsilon_t - dY^0_t) \right)^{2p} \right]
\]

\[
\leq 2^{2p-1} \mathbb{E}_{P^\varepsilon} \left[ \left( \int_0^T \left( \bar{\rho}^\varepsilon_t(\bar{h}^*) - \bar{\rho}_t(\bar{h}^*) \right) dY^\varepsilon_t \right)^{2p} 
+ \left( \int_0^T \bar{\rho}_t(\bar{h}^*) (h(X^\varepsilon_t, Z^\varepsilon_t) - \bar{h}(X^0_t)) dt \right)^{2p} \right].
\]

Under \( P^\varepsilon \), \( Y^\varepsilon \) is a standard Brownian motion. Using the Burkholder-Davis-Gundy inequality on the term containing the stochastic integral,

\[
\mathbb{E}_{P^\varepsilon} \left[ \left( \int_0^T \left( \bar{\rho}^\varepsilon_t(\bar{h}^*) - \bar{\rho}_t(\bar{h}^*) \right) dY^\varepsilon_t \right)^{2p} \right]
\]

\[
\leq M_p \mathbb{E}_{P^\varepsilon} \left[ \left( \int_0^T \left| \bar{\rho}^\varepsilon_t(\bar{h}) - \bar{\rho}_t(\bar{h}) \right|^2 ds \right)^p \right]
\]

\[
= M_p \mathbb{E}_{P^\varepsilon} \left[ \left( \int_0^T \left| \bar{\rho}^\varepsilon_t(\bar{h}) \right|^2 ds \right)^p \right]
\]

\[
= M_p \mathbb{E}_{P^\varepsilon} \left[ \left( \int_0^T \left| \bar{\rho}^\varepsilon_t(\bar{h}) \right|^2 ds \right)^p \right]
\]

\[
\leq M \left\| h \right\|_{\infty}^{2p} \mathbb{E}_{P^\varepsilon} \left[ \left( \int_0^T \left| \bar{\rho}^\varepsilon_t(1) \right|^2 ds \right)^p \right]
\]

\[
\leq M \left\| h \right\|_{\infty}^{2p} T^{p-1} \mathbb{E}_{P^\varepsilon} \left[ \int_0^T \left| \bar{\rho}^\varepsilon_t(1) \right|^2 ds \right].
\]

where \( M \) can be chosen sufficiently large. Using Grönwall’s inequality,

\[
\mathbb{E}_{P^\varepsilon} \left[ \left| \bar{\rho}^\varepsilon_T(1) - \bar{\rho}_T(1) \right|^{2p} \right]
\]

\[
\leq 2^{2p-1} M \left\| h \right\|_{\infty}^2 T^{p-1} \mathbb{E}_{P^\varepsilon} \left[ \int_0^T \left| \bar{\rho}^\varepsilon_t(1) - \bar{\rho}_t(1) \right|^{2p} ds \right].
\]

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Consider the time integral term. Let \( \Delta := \frac{T}{n} > 0 \), \( t^n_k := k \Delta \) for \( k = 0, \ldots, n \). \( \Delta \) will be chosen properly as a function of \( \varepsilon \) later.

\[
\mathbb{E}_{p^c} \left[ \left| \int_0^T \tilde{p}_t(\tilde{h}^*) \left( h(X_t^\varepsilon, Z_t^\varepsilon) - \tilde{h}(X_t^0) \right) dt \right|^{2p} \right]
\leq 2^{p-1} \mathbb{E}_{p^c} \left[ \left| \int_0^T \tilde{p}_t(\tilde{h}^*) \left( h(X_t^\varepsilon, Z_t^\varepsilon) - \tilde{h}(X_t^0) \right) dt \right|^{2p} \right] \exp \left\{ 2^{p-1} M \| h \|_{\infty}^2 T^p \right\}.
\]

First term in (20): Consider a \( k \)th term in the sum in (20). We bring \( n = \frac{T}{\Delta(\varepsilon)} \) with each term in the following, as \( \varepsilon \) needs to be kept track of and
\( \Delta \) chosen properly as a function of \( \varepsilon \).

\[
\left( \frac{T}{\Delta} \right)^{2p-1} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left| \int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon} \tilde{\rho}_s(\tilde{h}^*)(h - \tilde{h})(X_s^\varepsilon, Z_s^\varepsilon) ds \right|^{2p} \right]
\]

\[= \left( \frac{T}{\Delta} \right)^{2p-1} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left| \int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon} \tilde{\rho}_s(\tilde{h}^*)(h - \tilde{h})(X_s^\varepsilon, Z_s^\varepsilon) ds \right| \right]^{2p} \]

\[+ \int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon} \left\{ \tilde{\rho}_s(\tilde{h}^*)(h - \tilde{h})(X_s^\varepsilon, Z_s^\varepsilon) - \tilde{\rho}_{t_k^\varepsilon}(\tilde{h}^*)(h - \tilde{h})(X_{t_k^\varepsilon}^\varepsilon, Z_{t_k^\varepsilon}^\varepsilon) \right\} ds \]

\[\leq \left( \frac{2T}{\Delta} \right)^{2p-1} \left( \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left| \int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon} \tilde{\rho}_s(\tilde{h}^*)(h - \tilde{h})(X_s^\varepsilon, Z_s^\varepsilon) ds \right|^{2p} \right] \right) \]

\[+ \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left| \int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon} \left\{ \tilde{\rho}_s(\tilde{h}^*)(h - \tilde{h})(X_s^\varepsilon, Z_s^\varepsilon) - \tilde{\rho}_{t_k^\varepsilon}(\tilde{h}^*)(h - \tilde{h})(X_{t_k^\varepsilon}^\varepsilon, Z_{t_k^\varepsilon}^\varepsilon) \right\} ds \right|^{2p} \right]. \tag{21} \]

**First term in (21):** For the expected value in (21), we can condition inside w.r.t. the (random) initial condition at \( t_k^\varepsilon \).

\[
\frac{1}{\Delta^{2p-1}} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left| \int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon} \tilde{\rho}_s(\tilde{h}^*)(h - \tilde{h})(X_s^\varepsilon, Z_s^\varepsilon) ds \right|^{2p} \right]
\]

\[= \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \frac{1}{\Delta^{2p-1}} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left| \int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon} \rho^*(h - \tilde{h})(x, Z_s^\varepsilon(t_k^\varepsilon, z, x)) ds \right|^{2p} \right] \right](\rho, x, z) = \left( \tilde{\rho}_{t_k^\varepsilon}(\tilde{h}), X_{t_k^\varepsilon}^\varepsilon, Z_{t_k^\varepsilon}^\varepsilon \right). \tag{22} \]

where \( Z_s^\varepsilon(t_k^\varepsilon, z, x) \) for \( s \in [t_k^\varepsilon, t_{k+1}^\varepsilon] \) is the process

\[
Z_s^\varepsilon(t_k^\varepsilon, z, x) = z + \frac{1}{\varepsilon} \int_{t_k^\varepsilon}^{s} f(X_r^\varepsilon(t_k^\varepsilon, z, x), Z_r^\varepsilon(t_k^\varepsilon, z, x)) dr + \frac{1}{\sqrt{\varepsilon}} \int_{t_k^\varepsilon}^{s} g(X_r^\varepsilon(t_k^\varepsilon, z, x), Z_r^\varepsilon(t_k^\varepsilon, z, x)) dV_r,
\]

\[
X_s^\varepsilon(t_k^\varepsilon, z, x) = x + \int_{t_k^\varepsilon}^{s} b(X_r^\varepsilon(t_k^\varepsilon, z, x), Z_r^\varepsilon(t_k^\varepsilon, z, x)) dr + \int_{t_k^\varepsilon}^{s} \sigma(X_r^\varepsilon(t_k^\varepsilon, z, x)) dW_r.
\]

We also define a new process \( \hat{Z}_s^\varepsilon(t_k^\varepsilon, z, x) \) that has the same dynamics as \( Z_s^\varepsilon(t_k^\varepsilon, z, x) \) but with the slow component fixed:

\[
\hat{Z}_s^\varepsilon(t_k^\varepsilon, z, x) = z + \frac{1}{\varepsilon} \int_{t_k^\varepsilon}^{s} f(x, Z_r^\varepsilon(t_k^\varepsilon, z, x)) dr + \frac{1}{\sqrt{\varepsilon}} \int_{t_k^\varepsilon}^{s} g(x, Z_r^\varepsilon(t_k^\varepsilon, z, x)) dV_r. \tag{23}
\]
We simplify superscript notations and write $X^\varepsilon := X^{\varepsilon,(t^n_k,z,x)}$, $Z^\varepsilon := Z^{\varepsilon,(t^n_k,z,x)}$, $\hat{Z}^\varepsilon := \hat{Z}^{\varepsilon,(t^n_k,z,x)}$ in the following.

First consider the inner expectation in (22), which is w.r.t. the probability measure with quantities at $t^n_k$ fixed, i.e. we first take expected value over $Z^\varepsilon$ with fixed initial condition $(x,z)$ and the quantities $(\rho^n_k, X^{\varepsilon}_k)$ fixed as $(\rho, x)$.

$$\frac{1}{\Delta^{2p-1}} \mathbb{E}_{P^\varepsilon} \left[ \int_{t^n_k}^{t^{n+1}_k} \rho^s (h - \bar{h})(x, Z^\varepsilon_s) ds \right]^{2p}$$

$$\leq 2^{2p-1} \left( \frac{1}{\Delta^{2p-1}} \mathbb{E}_{P^\varepsilon} \left[ \int_{t^n_k}^{t^{n+1}_k} \rho^s (h - \bar{h})(x, \hat{Z}^\varepsilon_s) ds \right]^{2p} \right)$$

$$+ \frac{1}{\Delta^{2p-1}} \mathbb{E}_{P^\varepsilon} \left[ \int_{t^n_k}^{t^{n+1}_k} \rho^s \left\{ (h - \bar{h})(x, Z^\varepsilon_s) - (h - \bar{h})(x, \hat{Z}^\varepsilon_s) \right\} ds \right]^{2p}$$

(24)

Second term on the RHS of (24): By Cauchy–Schwarz inequality and $h$ being bounded and Lipschitz with Lipschitz constant $K_h$,

$$\frac{1}{\Delta^{2p-1}} \mathbb{E}_{P^\varepsilon} \left[ \int_{t^n_k}^{t^{n+1}_k} \rho^s \left\{ (h - \bar{h})(x, Z^\varepsilon_s) - (h - \bar{h})(x, \hat{Z}^\varepsilon_s) \right\} ds \right]^{2p}$$

$$\leq \frac{1}{\Delta^{2p-1}} \mathbb{E}_{P^\varepsilon} \left[ \int_{t^n_k}^{t^{n+1}_k} \|\rho\| \left\| (h - \bar{h})(x, Z^\varepsilon_s) - (h - \bar{h})(x, \hat{Z}^\varepsilon_s) \right\| ds \right]^{2p}$$

$$\leq \frac{1}{\Delta^{2p-1}} \mathbb{E}_{P^\varepsilon} \left[ \int_{t^n_k}^{t^{n+1}_k} \|h\|_\infty K_h \left\| Z^\varepsilon_s - \hat{Z}^\varepsilon_s \right\| ds \right]^{2p}$$

$$\leq \frac{1}{\Delta^{2p-1}} \left( t^{n+1}_k - t^n_k \right) 2^{p-1} \|h\|_\infty^2 K_h^{2p} \int_{t^n_k}^{t^{n+1}_k} \mathbb{E}_{P^\varepsilon} \left[ \left\| Z^\varepsilon_s - \hat{Z}^\varepsilon_s \right\|^2 ds \right]$$

$$= \|h\|_\infty^{2p} K_h^{2p} \int_{t^n_k}^{t^{n+1}_k} \mathbb{E}_{P^\varepsilon} \left[ \left\| Z^\varepsilon_s - \hat{Z}^\varepsilon_s \right\|^2 ds \right].$$

(25)

We follow the proof of Theorem 9.1 of [75] for $\mathbb{E}_{P^\varepsilon} \left[ \left\| Z^\varepsilon_s - \hat{Z}^\varepsilon_s \right\|^{2p} \right]$ (recall that law of $(X^\varepsilon, Z^\varepsilon)$ is unchanged under $P^\varepsilon$). For $s \in [t^n_k, t^{n+1}_k]$,

$$\mathbb{E}_{P^\varepsilon} \left[ \left\| Z^\varepsilon_s - \hat{Z}^\varepsilon_s \right\|^{2p} \right]$$

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\[
\begin{align*}
\leq 2^{2p-1} & \mathbb{E}_p \left[ \left\| \frac{1}{\varepsilon} \int_{t^n_k}^t \left\{ f(X^\varepsilon_r, Z^\varepsilon_r) - f(x, \hat{Z}^\varepsilon_r) \right\} \, dr \right\|^{2p} \right] \\
+ 2^{2p-1} & \mathbb{E}_p \left[ \left\| \frac{1}{\sqrt{\varepsilon}} \int_{t^n_k}^t \left\{ g(X^\varepsilon_r, Z^\varepsilon_r) - g(x, \hat{Z}^\varepsilon_r) \right\} \, dV_r \right\|^{2p} \right].
\end{align*}
\] (26)

Consider the time integral in (26). By \( f \) being Lipschitz with Lipschitz constant \( K_f \),
\[
\mathbb{E}_p \left[ \left\| \frac{1}{\varepsilon} \int_{t^n_k}^t \left\{ f(X^\varepsilon_r, Z^\varepsilon_r) - f(x, \hat{Z}^\varepsilon_r) \right\} \, dr \right\|^{2p} \right]
\leq \varepsilon^{-2p} (s - t^n_k)^{2p-1} \mathbb{E}_p \left[ \int_{t^n_k}^t \left\| f(X^\varepsilon_r, Z^\varepsilon_r) - f(x, \hat{Z}^\varepsilon_r) \right\|^{2p} \, dr \right]
\leq (\varepsilon^{-1} K_f)^{2p} (2(s - t^n_k))^{2p-1} \mathbb{E}_p \left[ \int_{t^n_k}^t \left\| X^\varepsilon_r - x \right\|^2 \right]
\leq (\varepsilon^{-1} K_f)^{2p} (2(s - t^n_k))^{2p-1}
\times \left( \int_{t^n_k}^t \mathbb{E}_p \left[ \left\| X^\varepsilon_r - x \right\|^2 \right] \, dr \right) + \int_{t^n_k}^t \mathbb{E}_p \left[ \left\| Z^\varepsilon_r - \hat{Z}^\varepsilon_r \right\|^2 \right] \, dr \right).
\]

For \( X^\varepsilon_r, r \in [t^n_k, t^n_{k+1}] \), starting at \( x \) at \( t^n_k \),
\[
\mathbb{E}_p \left[ \left\| X^\varepsilon_r - x \right\|^2 \right]
= \mathbb{E}_p \left[ \left\| \int_{t^n_k}^t b(X^\varepsilon_u, Z^\varepsilon_u) \, du + \int_{t^n_k}^t \sigma(X^\varepsilon_u) \, dW_u \right\|^2 \right]
\leq 2^{2p-1} \left( \mathbb{E}_p \left[ \left\| \int_{t^n_k}^t b(X^\varepsilon_u, Z^\varepsilon_u) \, du \right\|^{2p} \right] + \mathbb{E}_p \left[ \left\| \int_{t^n_k}^t \sigma(X^\varepsilon_u) \, dW_u \right\|^{2p} \right] \right)
\leq 2^{2p-1} \left( (r - t^n_k)^{2p-1} \int_{t^n_k}^t \mathbb{E}_p \left[ \left\| b(X^\varepsilon_u, Z^\varepsilon_u) \right\|^{2p} \right] \, du \right)
\leq 2^{2p-1} \left( (r - t^n_k)^{2p-1} \int_{t^n_k}^t \mathbb{E}_p \left[ \left\| b(X^\varepsilon_u, Z^\varepsilon_u) \right\|^{2p} \right] \, du \right)
\leq 2^{2p-1} \left( (r - t^n_k)^{2p-1} \int_{t^n_k}^t \mathbb{E}_p \left[ \left\| b(X^\varepsilon_u, Z^\varepsilon_u) \right\|^{2p} \right] \, du \right)
\leq 2^{2p-1} \left( (r - t^n_k)^{2p-1} \int_{t^n_k}^t \mathbb{E}_p \left[ \left\| b(X^\varepsilon_u, Z^\varepsilon_u) \right\|^{2p} \right] \, du \right)
\]
inequality is because \((\in \sigma)\) where we have used the Burkholder-Davis-Gundy inequality (here, norm for \((28)^{\circ}\) in \((27)^{\circ}\), we have
\[
E_{(b, \sigma)} \left[ 1 + \|X_u^\varepsilon\|^{q_1} + \|Z_u^\varepsilon\|^{q_2} \right] du
\]
\[
\leq 2^{p-1} \left( (r - t_k^n)^{2p-1} c_b \int_{t_k^n}^r \left[ 1 + \|X_u^\varepsilon\|^{q_1} + \|Z_u^\varepsilon\|^{q_2} \right] du \right.
+ c_2 c_\sigma (r - t_k^n)^{p-1} \int_{t_k^n}^r \left[ 1 + \|X_u^\varepsilon\|^{q_3} + \|Z_u^\varepsilon\|^{q_4} \right] du)
\]
\[
\leq 2^{p-1} \left( (r - t_k^n)^{2p-1} (r - t_k^n) C_1 (1 + \|x\|^{q_1} + \|z\|^{q_2})
\right.
\]
\[
+ (r - t_k^n)^{p-1} (r - t_k^n) C_2 (1 + \|x\|^{q_3} + \|z\|^{q_4})
\]
\[
\leq C_3 (r - t_k^n)^p (1 + \|x\|^{\bar{q}_1} + \|z\|^{\bar{q}_2}),
\]
(28)

where we have used the Burkholder-Davis-Gundy inequality (here, norm for \(\sigma \in \mathbb{R}^{m \times k}\) is the Frobenius norm) and the polynomial growth condition on \((b, \sigma)\). The second to last inequality is by Proposition 3.6.3 and the last inequality is because \((r - t_k^n) < 1\), so \((r - t_k^n)^{2p} < (r - t_k^n)^{p}\). Therefore, using (28) in (27), we have

\[
E_{(b, \sigma)} \left[ \left\| \frac{1}{\varepsilon} \int_{t_k^n}^s \left\{ f(X_u^\varepsilon, Z_u^\varepsilon) - f(x, \hat{Z}_r^\varepsilon) \right\} du \right\|^{2p} \right]
\]
\[
\leq (\varepsilon^{-1} K_f)^{2p} (2(s - t_k^n))^{2p-1}
\]
\[
\times \left( \int_{t_k^n}^s C_3 (s - t_k^n)^{p} (1 + \|x\|^{\bar{q}_1} + \|z\|^{\bar{q}_2}) dr + \int_{t_k^n}^s E_{(b, \sigma)} \left[ \|Z_r^\varepsilon - \hat{Z}_r^\varepsilon\|^{2p} \right] dr \right)
\]
\[
\leq (\varepsilon^{-1} K_f)^{2p} (2(s - t_k^n))^{2p-1}
\]
\[
\times \left( C_3 (s - t_k^n)^{p+1} (1 + \|x\|^{\bar{q}_1} + \|z\|^{\bar{q}_2}) + \int_{t_k^n}^s E_{(b, \sigma)} \left[ \|Z_r^\varepsilon - \hat{Z}_r^\varepsilon\|^{2p} \right] dr \right)
\]
\[
\leq (\varepsilon^{-1} K_f)^{2p} (2\Delta)^{2p-1}
\]
\[
\times \left( C_3 \Delta^{p+1} (1 + \|x\|^{\bar{q}_1} + \|z\|^{\bar{q}_2}) + \int_{t_k^n}^s E_{(b, \sigma)} \left[ \|Z_r^\varepsilon - \hat{Z}_r^\varepsilon\|^{2p} \right] dr \right).
\]

For the stochastic integral in (26),

\[
E_{(b, \sigma)} \left[ \left\| \frac{1}{\sqrt{\varepsilon}} \int_{t_k^n}^s \left\{ g(X_r^\varepsilon, Z_r^\varepsilon) - g(x, \hat{Z}_r^\varepsilon) \right\} dV_r \right\|^{2p} \right]
\]

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\[
\begin{align*}
&\leq \varepsilon^{-p} c_3 \mathbb{E}_\mathbb{P}_\varepsilon \left[ \left| \int_{t_k^n}^s \left| g(X_r^\varepsilon, Z_r^\varepsilon) - g(x, \hat{Z}_r^\varepsilon) \right|^2 \, dr \right| \right] \\
&\leq \varepsilon^{-p} c_3 2^{2p-1} K_g^{2p} (s - t_k^n)^{p-1} \int_{t_k^n}^s \left\{ \mathbb{E}_\mathbb{P}_\varepsilon \left[ \| X_r^\varepsilon - x \|^2 \right] + \mathbb{E}_\mathbb{P}_\varepsilon \left[ \| Z_r^\varepsilon - \hat{Z}_r^\varepsilon \|^2 \right] \right\} \, dr \\
&\leq \varepsilon^{-p} c_3 2^{2p-1} K_g^{2p} (s - t_k^n)^{p-1} \\
&\quad \times \left( C_3 (s - t_k^n)^{p+1} (1 + \| x \| \bar{q}_1 + \| z \| \bar{q}_2) + \int_{t_k^n}^s \mathbb{E}_\mathbb{P}_\varepsilon \left[ \| Z_r^\varepsilon - \hat{Z}_r^\varepsilon \|^2 \right] \, dr \right) \\
&\leq \varepsilon^{-p} c_3 2^{2p-1} K_g^{2p} \Delta^{p-1} \\
&\quad \times \left( C_3 \Delta^{p+1} (1 + \| x \| \bar{q}_1 + \| z \| \bar{q}_2) + \int_{t_k^n}^s \mathbb{E}_\mathbb{P}_\varepsilon \left[ \| Z_r^\varepsilon - \hat{Z}_r^\varepsilon \|^2 \right] \, dr \right)
\end{align*}
\]

Returning to (26),

\[
\begin{align*}
&\mathbb{E}_\mathbb{P}_\varepsilon \left[ \| Z_{s_k}^\varepsilon - \hat{Z}_{s_k}^\varepsilon \|^2 \right] \\
&\leq 4^{2p-1} K_f^{2p} \varepsilon^{-2p} \Delta^{2p-1} \\
&\quad \times \left( C_3 \Delta^{p+1} (1 + \| x \| \bar{q}_1 + \| z \| \bar{q}_2) + \int_{t_k^n}^s \mathbb{E}_\mathbb{P}_\varepsilon \left[ \| Z_r^\varepsilon - \hat{Z}_r^\varepsilon \|^2 \right] \, dr \right) \\
&\quad + 4^{2p-1} c_3 K_g^{2p} \varepsilon^{-p} \Delta^{p-1} \\
&\quad \times \left( C_3 \Delta^{p+1} (1 + \| x \| \bar{q}_1 + \| z \| \bar{q}_2) + \int_{t_k^n}^s \mathbb{E}_\mathbb{P}_\varepsilon \left[ \| Z_r^\varepsilon - \hat{Z}_r^\varepsilon \|^2 \right] \, dr \right) \\
&= 4^{2p-1} C_3 \left( K_f^{2p} \frac{\Delta^{2p-1}}{\varepsilon^{2p}} + c_3 K_g^{2p} \frac{\Delta^{p-1}}{\varepsilon^{p}} \right) \left( \Delta^{p+1} (1 + \| x \| \bar{q}_1 + \| z \| \bar{q}_2) \\
&\quad + 4^{2p-1} C_3 \left( K_f^{2p} \frac{\Delta^{2p-1}}{\varepsilon^{2p}} + c_3 K_g^{2p} \frac{\Delta^{p-1}}{\varepsilon^{p}} \right) \int_{t_k^n}^s \mathbb{E}_\mathbb{P}_\varepsilon \left[ \| Z_r^\varepsilon - \hat{Z}_r^\varepsilon \|^2 \right] \, dr \right)
\end{align*}
\]

By Grönwall’s inequality,

\[
\begin{align*}
&\mathbb{E}_\mathbb{P}_\varepsilon \left[ \| Z_s^\varepsilon - \hat{Z}_s^\varepsilon \|^2 \right] \\
&\leq C_4 \left( \frac{\Delta^{2p-1}}{\varepsilon^{2p}} + \frac{\Delta^{p-1}}{\varepsilon^{p}} \right) \Delta^{p+1} \exp \left\{ C_4 \left( \frac{\Delta^{2p-1}}{\varepsilon^{2p}} + \frac{\Delta^{p-1}}{\varepsilon^{p}} \right) \Delta \right\}
\end{align*}
\]

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\times (1 + \|x\|^{\tilde{q}_1} + \|z\|^{\tilde{q}_2})
\]

\[= C_4 \left( \frac{\Delta_{2p}}{\varepsilon^{2p}} + \frac{\Delta_p}{\varepsilon^p} \right) \Delta^p \exp \left\{ C_4 \left( \frac{\Delta_{2p}}{\varepsilon^{2p}} + \frac{\Delta_p}{\varepsilon^p} \right) \right\} (1 + \|x\|^{\tilde{q}_1} + \|z\|^{\tilde{q}_2}) \]

If we choose \( \Delta \) as

\[\Delta = \Delta(\varepsilon) = \varepsilon \left( \log \varepsilon^{-1} \right)^{\frac{1}{4p}},\]

then as \( \varepsilon \to 0 \),

\[\Delta \to 0, \quad C_4 \left( \frac{\Delta_{2p}}{\varepsilon^{2p}} + \frac{\Delta_p}{\varepsilon^p} \right) \Delta^p \exp \left\{ C_4 \left( \frac{\Delta_{2p}}{\varepsilon^{2p}} + \frac{\Delta_p}{\varepsilon^p} \right) \right\} \to 0.\]

Continuing (25), we have

\[
\frac{1}{\Delta_{2p-1}^p} \mathbb{E}_\rho \left[ \int_{t_k^n}^{t_{k+1}^n} \rho^* \left\{ (h - \bar{h})(x, Z_s^\varepsilon) - (h - \bar{h})(x, \hat{Z}_s^\varepsilon) \right\} ds \right]^{2p} \]

\[
\leq \|h\|_{2p}^p K_h^{2p} \int_{t_k^n}^{t_{k+1}^n} \mathbb{E}_\rho \left[ \|Z_s^\varepsilon - \hat{Z}_s^\varepsilon\|^{2p} \right] ds \]

\[
\leq \|h\|_{2p}^p K_h^{2p} C_4 \left( \frac{\Delta_{2p}}{\varepsilon^{2p}} + \frac{\Delta_p}{\varepsilon^p} \right) \Delta^{p+1} \exp \left\{ C_4 \left( \frac{\Delta_{2p}}{\varepsilon^{2p}} + \frac{\Delta_p}{\varepsilon^p} \right) \right\} \times (1 + \|x\|^{\tilde{q}_1} + \|z\|^{\tilde{q}_2}) \]

(29)

**First term on the RHS of (24):** Let \( \psi^\varepsilon(\rho, x, z) \) be the solution to

\[ -\partial_s \psi^\varepsilon_s(\rho, x, z) = \mathcal{L}_F \psi^\varepsilon_s(\rho, x, z) + \rho^* (h - \bar{h})(x, z), \quad s \in [t_k^n, t_{k+1}^n], \]

\[ \psi^\varepsilon_{t_{k+1}^n} = 0, \]

where

\[ \mathcal{L}_F := \frac{1}{\varepsilon} \left( \sum_{i=1}^n f_i(x, z) \partial_z + \frac{1}{2} \sum_{i,j=1}^n (gg^*)_{ij}(x, z) \partial^2_{zz} \right), \]

which is the generator for the process given by (23). By the Feynman-Kac formula, the solution to the PDE can be given by

\[ \psi^\varepsilon(\rho, x, z) = \int_{s}^{t_{k+1}^n} \mathbb{E} \left[ \rho^* (h - \bar{h})(x, \hat{Z}_r^\varepsilon) \right] dr. \]
By Itô’s formula,

\[
\psi_{t_{k+1}^{n}}^{\varepsilon}(\rho, x, \hat{Z}_{t_{k+1}}^{\varepsilon}) - \psi_{t_{k}^{n}}^{\varepsilon}(\rho, x, \hat{Z}_{t_{k}}^{\varepsilon}) = \int_{t_{k}^{n}}^{t_{k+1}^{n}} \mathcal{L}_{\rho} \psi_{s}^{\varepsilon}(\rho, x, \hat{Z}_{s}^{\varepsilon}) ds + \frac{1}{\sqrt{\varepsilon}} \int_{t_{k}^{n}}^{t_{k+1}^{n}} \partial_x \psi_{s}^{\varepsilon}(\rho, x, \hat{Z}_{s}^{\varepsilon}) g(x, \hat{Z}_{s}^{\varepsilon}) dV_s + \int_{t_{k}^{n}}^{t_{k+1}^{n}} \partial_s \psi_{s}^{\varepsilon}(\rho, x, \hat{Z}_{s}^{\varepsilon}) ds
\]

\[
= \frac{1}{\sqrt{\varepsilon}} \int_{t_{k}^{n}}^{t_{k+1}^{n}} \partial_x \psi_{s}^{\varepsilon}(\rho, x, \hat{Z}_{s}^{\varepsilon}) g(x, \hat{Z}_{s}^{\varepsilon}) dV_s - \int_{t_{k}^{n}}^{t_{k+1}^{n}} \rho^*(h - \tilde{h})(x, \hat{Z}_{s}^{\varepsilon}) ds.
\]

Rearranging and then taking expectation,

\[
\frac{1}{\Delta^{2p-1}} \mathbb{E}_{\rho^{c}} \left[ \int_{t_{k}^{n}}^{t_{k+1}^{n}} \rho^*(h - \tilde{h})(x, \hat{Z}_{s}^{\varepsilon}) ds \right]^{2p}
\]

\[
\leq \left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{\rho^{c}} \left[ \left| \psi_{t_{k}^{n}}^{\varepsilon}(\rho, x, \hat{Z}_{s}^{\varepsilon}) \right|^{2p} \right]^{2p}
\]

\[
+ \left( \frac{2}{\Delta} \right)^{2p-1} \varepsilon^{-p} \mathbb{E}_{\rho^{c}} \left[ \int_{t_{k}^{n}}^{t_{k+1}^{n}} \partial_x \psi_{s}^{\varepsilon}(\rho, x, \hat{Z}_{s}^{\varepsilon}) g(x, \hat{Z}_{s}^{\varepsilon}) dV_s \right]^{2p},
\]

the LHS which is the first term on the RHS of (24).

Using the Feynman-Kac solution, the first term in (30) is

\[
\left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{\rho^{c}} \left[ \left| \psi_{t_{k}^{n}}^{\varepsilon}(\rho, x, \hat{Z}_{s}^{\varepsilon}) \right|^{2p} \right]
\]

\[
= \left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{\rho^{c}} \left[ \int_{t_{k}^{n}}^{t_{k+1}^{n}} \mathbb{E} \left[ \rho^*(h - \tilde{h})(x, \hat{Z}_{r}^{\varepsilon}) \right] \hat{Z}_{r=t_{k}^{n}}^{\varepsilon} \hat{Z}_{r-t_{k}^{n}}^{\varepsilon} \left( \hat{Z}_{r}^{\varepsilon}, \rho^*(h - \tilde{h}); x \right) dr \right]^{2p}
\]

\[
= \left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{\rho^{c}} \left[ \int_{0}^{t_{k+1}^{n}} \mathbb{E} \left[ \rho^*(h - \tilde{h})(x, \hat{Z}_{r}^{\varepsilon}) \right] p_{r-t_{k}^{n}}^{\varepsilon} \left( \hat{Z}_{r}^{\varepsilon}, \rho^*(h - \tilde{h}); x \right) dr \right]^{2p}
\]

\[
\leq \left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{\rho^{c}} \left[ \varepsilon \int_{0}^{t_{k+1}^{n}} \mathbb{E} \left[ \rho^*(h - \tilde{h})(x, \hat{Z}_{r}^{\varepsilon}) \right] p_{r-t_{k}^{n}}^{\varepsilon} \left( \hat{Z}_{r}^{\varepsilon}, \rho^*(h - \tilde{h}); x \right) dr \right]^{2p}
\]

Continuing the above inequality using Proposition 3.6.2 and then Proposition
3.6.3,

\[
\left(\frac{2}{\Delta}\right)^{2p-1} \mathbb{E}_p \left[ \left| \psi_{i_k}^\varepsilon (\rho, x, \hat{Z}_{s}^\varepsilon) \right|^{2p} \right] \leq \left(\frac{2}{\Delta}\right)^{2p-1} \mathbb{E}_p \left[ \varepsilon C_5 (1 + \| \hat{Z}_{r}^\varepsilon \|_{q_6})^{2p} \right] \\
\leq \left(\frac{2}{\Delta}\right)^{2p-1} \varepsilon^{2p} C_5 \left(1 + \mathbb{E}_p \left[ \| \hat{Z}_{r}^\varepsilon \|_{q_6} \right] \right) \\
\leq \left(\frac{2}{\Delta}\right)^{2p-1} \varepsilon^{2p} C_5 \left(1 + \| z \|_{q_6} \right) \\
= \left(\frac{2\varepsilon}{\Delta}\right)^{2p-1} \varepsilon C_5 \left(1 + \| z \|_{q_6} \right),
\]

where the factor \( C_5 \) and exponent \( q_5 \) changes from line to line.

The second term in (30) is

\[
\left(\frac{2}{\Delta}\right)^{2p-1} \varepsilon^{-p} \mathbb{E}_p \left[ \left| \int_{t_n^k}^{t_{n+1}^k} \partial_z \psi_{s}^\varepsilon (\rho, x, \hat{Z}_{s}^\varepsilon) g(x, \hat{Z}_{s}^\varepsilon) dV_s \right|^{2p} \right] \\
\leq \left(\frac{2}{\Delta}\right)^{2p-1} \varepsilon^{-p} C_6 \mathbb{E}_p \left[ \left| \int_{t_n^k}^{t_{n+1}^k} \partial_z \psi_{s}^\varepsilon (\rho, x, \hat{Z}_{s}^\varepsilon) g(x, \hat{Z}_{s}^\varepsilon) dV_s \right|^{2p} \right],
\]

where

\[
\partial_z \psi_{s}^\varepsilon (\rho, x, z) = \partial_z \int_{s}^{t_{n+1}^k} \mathbb{E} \left[ \rho^*(h - \bar{h})(x, \hat{Z}_{s}^\varepsilon) \right] d \bar{z} \\
= \partial_z \int_{s}^{t_{n+1}^k} p_{x-z} (z, \rho^*(h - \bar{h}); x) d \bar{z} \\
= \varepsilon \partial_z \int_{0}^{t_{n+1}^k} p_u (z, \rho^*(h - \bar{h}); x) d u \\
= \varepsilon \int_{0}^{t_{n+1}^k} \partial_z p_u (z, \rho^*(h - \bar{h}); x) d u \\
\leq \varepsilon \int_{0}^{\infty} |\partial_z p_u (z, \rho^*(h - \bar{h}); x)| d u \\
\leq \varepsilon C_7 (1 + \| z \|_{q_6})
\]

by Propositions 3.6.3 and 3.6.2. Therefore,

\[
\left(\frac{2}{\Delta}\right)^{2p-1} \varepsilon^{-p} \mathbb{E}_p \left[ \left| \int_{t_n^k}^{t_{n+1}^k} \partial_z \psi_{s}^\varepsilon (\rho, x, \hat{Z}_{s}^\varepsilon) g(x, \hat{Z}_{s}^\varepsilon) dV_s \right|^{2p} \right]
\]
\[
\leq \left( \frac{2}{\Delta} \right)^{2p-1} \varepsilon^{-p} C_6 \mathbb{E}_{p^c} \left[ \left| \int_{t^n_k}^{t^{n+1}_k} \left| \varepsilon C_7 (1 + \| \hat{Z}_s^\varepsilon \|^{q_6}) \right|^2 \| g(x, \hat{Z}_s^\varepsilon) \|^2 ds \right|^{p} \right] \\
\leq \left( \frac{2}{\Delta} \right)^{2p-1} \varepsilon^p C_6 \Delta^{p-1} \int_{t^n_k}^{t^{n+1}_k} \mathbb{E}_{p^c} \left[ (1 + \| x \|^{q_7} + \| \hat{Z}_s^\varepsilon \|^{q_6}) \right] ds \\
= 2^{2p-1} \left( \frac{\varepsilon}{\Delta} \right)^{p} C_6 \int_{t^n_k}^{t^{n+1}_k} \mathbb{E}_{p^c} \left[ (1 + \| x \|^{q_7} + \| \hat{Z}_s^\varepsilon \|^{q_6}) \right] ds \\
\leq 2^{2p-1} \left( \frac{\varepsilon}{\Delta} \right)^{p} \int_{t^n_k}^{t^{n+1}_k} (1 + \| x \|^{q_7} + C_8(1 + \| z \|^{q_6})) ds \\
= 2^{2p-1} \left( \frac{\varepsilon}{\Delta} \right)^{p} \Delta C_6 (1 + \| x \|^{q_7} + C_8(1 + \| z \|^{q_6})) \\
= 2^{2p-1} \left( \frac{\varepsilon}{\Delta} \right)^{p-1} \varepsilon C_6 (1 + \| x \|^{q_7} + C_8(1 + \| z \|^{q_6})),
\]

by polynomial growth of \( g \) and Proposition 3.6.3. The factor \( C_6 \) and exponent \( q_6 \) change from line to line.

Therefore, (30) is bounded as

\[
\frac{1}{\Delta^{2p-1}} \mathbb{E}_{p^c} \left[ \left| \int_{t^n_k}^{t^{n+1}_k} \rho^*(h - \bar{h})(x, \hat{Z}_s^\varepsilon) ds \right|^{2p} \right] \\
\leq \left( \frac{\varepsilon}{\Delta} \right)^{2p-1} \varepsilon C_5 (1 + \| z \|^{q_5}) + \left( \frac{\varepsilon}{\Delta} \right)^{p-1} \varepsilon C_6 (1 + \| x \|^{q_7} + C_8(1 + \| z \|^{q_6})),
\]

(31)

where we have absorbed the factor \( 2^{2p-1} \) into \( C_5 \) and \( C_6 \).

**Bound on first term in (21):** Collecting (31) and (29) for (24), we have

\[
\frac{1}{\Delta^{2p-1}} \mathbb{E}_{p^c} \left[ \left| \int_{t^n_k}^{t^{n+1}_k} \rho^*(h - \bar{h})(x, Z_s^\varepsilon) ds \right|^{2p} \right] \\
\leq \left( \frac{\varepsilon}{\Delta} \right)^{2p-1} \varepsilon C_5 (1 + \| z \|^{q_5}) \\
+ \left( \frac{\varepsilon}{\Delta} \right)^{p-1} \varepsilon C_6 (1 + \| x \|^{q_7} + C_8(1 + \| z \|^{q_6})) \\
+ \| h \|_\infty^{2p} C_4^2 \varepsilon^{2p} \left( \frac{\Delta_2^p}{\varepsilon^p} + \frac{\Delta^p}{\varepsilon^p} \right) \Delta^{p+1} \exp \left\{ C_4 \left( \frac{\Delta_2^p}{\varepsilon^p} + \frac{\Delta^p}{\varepsilon^p} \right) \right\} \\
\times (1 + \| x \|^{q_1} + \| z \|^{q_2}).
\]

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Taking expectation over initial conditions at $t^n_k$ gives us the bound for (22):

$$
\frac{1}{\Delta^{2p-1}} \mathbb{E}_{p^c} \left[ \left\| \int_{t^n_k}^{t^{n+1}_k} \tilde{\rho}_{t^n_k}(\tilde{h}^*) (h - \tilde{h})(X^\varepsilon_{t^n_k}, Z^\varepsilon_{t^n_k}) ds \right\|^2 \right]^{2p} \leq \mathbb{E}_{p^c} \left[ \left( \frac{\varepsilon}{\Delta} \right)^{2p-1} \varepsilon C_5 \left( 1 + \|Z^\varepsilon_{t^n_k}\|^q_5 \right) \right. \\
+ \left. \left( \frac{\varepsilon}{\Delta} \right)^{p-1} \varepsilon C_6 (1 + \|X^\varepsilon_{t^n_k}\|^q + C_8 (1 + \|Z^\varepsilon_{t^n_k}\|^q_6)) \right) \\
+ \|h\|_{2p}^2 K^2 h^{2p} C_4 \left( \frac{\Delta^{2p}}{\varepsilon^{2p}} + \frac{\Delta^p}{\varepsilon^p} \right) \Delta^{p+1} \exp \left\{ C_4 \left( \frac{\Delta^{2p}}{\varepsilon^{2p}} + \frac{\Delta^p}{\varepsilon^p} \right) \right\} \\
\times (1 + \|X^\varepsilon_{t^n_k}\|^q_1 + \|Z^\varepsilon_{t^n_k}\|^q_2) .
$$

Second term in (21):

$$
\frac{1}{\Delta^{2p-1}} \mathbb{E}_{p^c} \left[ \left\| \int_{t^n_k}^{t^{n+1}_k} \left\{ \tilde{\rho}_s(\tilde{h}^*) (h - \tilde{h})(X^\varepsilon_s, Z^\varepsilon_s) - \tilde{\rho}_{t^n_k}(\tilde{h}^*) (h - \tilde{h})(X^\varepsilon_{t^n_k}, Z^\varepsilon_{t^n_k}) \right\} ds \right\|^2 \right]^{2p} = \frac{1}{\Delta^{2p-1}} \mathbb{E}_{p^c} \left[ \left\| \int_{t^n_k}^{t^{n+1}_k} \left\{ \left( \tilde{\rho}_s(\tilde{h}^*) - \tilde{\rho}_{t^n_k}(\tilde{h}^*) \right) (h - \tilde{h})(X^\varepsilon_s, Z^\varepsilon_s) \\
+ \tilde{\rho}_{t^n_k}(\tilde{h}^*) \left( (h - \tilde{h})(X^\varepsilon_s, Z^\varepsilon_s) - (h - \tilde{h})(X^\varepsilon_{t^n_k}, Z^\varepsilon_{t^n_k}) \right) \right\} ds \right\|^2 \right]^{2p} \leq \left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{p^c} \left[ \left\| \int_{t^n_k}^{t^{n+1}_k} \left( \tilde{\rho}_s(\tilde{h}^*) - \tilde{\rho}_{t^n_k}(\tilde{h}^*) \right) (h - \tilde{h})(X^\varepsilon_s, Z^\varepsilon_s) ds \right\|^2 \right]^{2p} \\
+ \left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{p^c} \left[ \left\| \int_{t^n_k}^{t^{n+1}_k} \tilde{\rho}_{t^n_k}(\tilde{h}^*) \left( (h - \tilde{h})(X^\varepsilon_s, Z^\varepsilon_s) - (h - \tilde{h})(X^\varepsilon_{t^n_k}, Z^\varepsilon_{t^n_k}) \right) ds \right\|^2 \right]^{2p} .
$$

First term in (33):

$$
\left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{p^c} \left[ \left\| \int_{t^n_k}^{t^{n+1}_k} \left( \tilde{\rho}_s(\tilde{h}^*) - \tilde{\rho}_{t^n_k}(\tilde{h}^*) \right) (h - \tilde{h})(X^\varepsilon_s, Z^\varepsilon_s) ds \right\|^2 \right] \leq \left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{p^c} \left[ \Delta^{2p-1} \int_{t^n_k}^{t^{n+1}_k} \left\| \tilde{\rho}_s(\tilde{h}^*) - \tilde{\rho}_{t^n_k}(\tilde{h}^*) \right\|^2 (h - \tilde{h})(X^\varepsilon_s, Z^\varepsilon_s)^2 ds \right] \\
\leq \left( \frac{2\Delta}{\Delta} \right)^{2p-1} \mathbb{E}_{p^c} \left[ \int_{t^n_k}^{t^{n+1}_k} \left\| \tilde{\rho}_s(\tilde{h}^*) - \tilde{\rho}_{t^n_k}(\tilde{h}^*) \right\|^2 \left\| (h - \tilde{h})(X^\varepsilon_s, Z^\varepsilon_s) \right\|^{2p} ds \right].
$$
\[
\left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{P} \left[ \int_{t_k^n}^{t_{k+1}^n} \left( \bar{\rho}_s(\bar{h}^*) - \bar{\rho}_{t_k^n}(\bar{h}^*) \right) \left( h - \bar{h} \right) (X_{s}^{\varepsilon}, Z_{s}^{\varepsilon}) ds \right]^{2p} \\
\leq 2^{2p-1} M \|h\|^2_{\infty} \int_{t_k^n}^{t_{k+1}^n} C_0 \|h\|_{\infty}^{2p} \Delta^{p} ds \\
= 2^{2p-1} M \|h\|^{6p}_{\infty} \Delta^{p+1}.
\]

The second term in (33):
\[
\left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{P} \left[ \int_{t_k^n}^{t_{k+1}^n} \left( \bar{\rho}_{t_k^n}(\bar{h}^*) \left( (h - \bar{h}) (X_{s}^{\varepsilon}, Z_{s}^{\varepsilon}) - (h - \bar{h}) (X_{t_k^n}^{\varepsilon}, Z_{t_k^n}^{\varepsilon}) \right) ds \right]^{2p} \\
\leq \left( \frac{2}{\Delta} \right)^{2p-1} \mathbb{E}_{P} \left[ \Delta^{2p-1} \int_{t_k^n}^{t_{k+1}^n} \left| \bar{\rho}_{t_k^n}(\bar{h}^*) \left( (h - \bar{h}) (X_{s}^{\varepsilon}, Z_{s}^{\varepsilon}) - (h - \bar{h}) (X_{t_k^n}^{\varepsilon}, Z_{t_k^n}^{\varepsilon}) \right) \right|^{2p} ds \right] \\
\leq \left( \frac{2\Delta}{\Delta} \right)^{2p-1}.
\]
\[ \times \mathbb{E}_{\mathbb{P}^e} \left[ \int_{t_{k}^{n}}^{t_{k+1}^{n}} \| \tilde{\rho}_s(\tilde{h}^*) \|^2 \| (h - \tilde{h})(X_s^\varepsilon, Z_s^\varepsilon) - (h - \tilde{h})(X_{t_k}^\varepsilon, Z_{t_k}^\varepsilon) \|^2 \, ds \right] \]

\[ \leq 2^{2p-1} K_h^{2p} \| h \|^2_{\infty} \mathbb{E}_{\mathbb{P}^e} \left[ \int_{t_{k}^{n}}^{t_{k+1}^{n}} \left| \left| X_s^\varepsilon - X_{t_k}^\varepsilon \right| \right| ^2 \, ds \right] \]

\[ = 2^{2p-1} K_h^{2p} \| h \|^2_{\infty} \int_{t_{k}^{n}}^{t_{k+1}^{n}} \mathbb{E}_{\mathbb{P}^e} \left[ \left| \left| X_s^\varepsilon - X_{t_k}^\varepsilon \right| \right| ^2 \right] \, ds \]

\[ \leq 2^{2p-1} K_h^{2p} \| h \|^2_{\infty} \int_{t_{k}^{n}}^{t_{k+1}^{n}} \mathbb{E}_{\mathbb{P}^e} \left[ \left| \left| X_s^\varepsilon - X_{t_k}^\varepsilon \right| \right| ^2 \right] \, ds \]

\[ \leq 2^{2p-1} K_h^{2p} \| h \|^2_{\infty} \int_{t_{k}^{n}}^{t_{k+1}^{n}} C_3 (s - t_k^n)^p \left( 1 + \mathbb{E}_{\mathbb{P}^e} \left[ \| X_{t_k}^\varepsilon \|^{\tilde{q}_1} \right] + \mathbb{E}_{\mathbb{P}^e} \left[ \| Z_{t_k}^\varepsilon \|^{\tilde{q}_2} \right] \right) \, ds \]

\[ \leq 2^{2p-1} K_h^{2p} \| h \|^2_{\infty} \int_{t_{k}^{n}}^{t_{k+1}^{n}} C_3 \Delta^{p+1} \left( 1 + \mathbb{E}_{\mathbb{P}^e} \left[ \| X_{t_k}^\varepsilon \|^{\tilde{q}_1} \right] + \mathbb{E}_{\mathbb{P}^e} \left[ \| Z_{t_k}^\varepsilon \|^{\tilde{q}_2} \right] \right) \, ds \]

using (28).

**Bound on second term in (21):** Collecting the bounds for (33), the second term in (21) is bounded as

\[ \frac{1}{\Delta^{2p-1}} \mathbb{E}_{\mathbb{P}^e} \left[ \int_{t_{k}^{n}}^{t_{k+1}^{n}} \left\{ \tilde{\rho}_s(\tilde{h}^*)(h - \tilde{h})(X_s^\varepsilon, Z_s^\varepsilon) - \tilde{\rho}_{t_k}(\tilde{h}^*)(h - \tilde{h})(X_{t_k}^\varepsilon, Z_{t_k}^\varepsilon) \right\} \, ds \right]^{2p} \]

\[ \leq 2^{2p-1} \Delta^{p+1} \times \left( M \| h \|_{\infty}^{6p} + 2^{2p-1} K_h^{2p} \| h \|_{\infty}^{2p} C_3 \left( 1 + \mathbb{E}_{\mathbb{P}^e} \left[ \| X_{t_k}^\varepsilon \|^{\tilde{q}_1} \right] + \mathbb{E}_{\mathbb{P}^e} \left[ \| Z_{t_k}^\varepsilon \|^{\tilde{q}_2} \right] \right) \right). \]

**Bound on (21):** Collecting (32) and (35), (21) is bounded as

\[ \left( \frac{T}{\Delta} \right)^{2p-1} \mathbb{E}_{\mathbb{P}^e} \left[ \int_{t_{k}^{n}}^{t_{k+1}^{n}} \tilde{\rho}_s(\tilde{h}^*)(h - \tilde{h})(X_s^\varepsilon, Z_s^\varepsilon) \, ds \right]^{2p} \]

\[ \leq (2T)^{2p-1} \left\{ \left( \frac{\varepsilon}{\Delta} \right)^{2p-1} \varepsilon C_5 \left( 1 + \mathbb{E}_{\mathbb{P}^e} \left[ \| Z_{t_k}^\varepsilon \|^{\tilde{q}_1} \right] \right) \right\}^{2p} \]

\[ + \left( \frac{\varepsilon}{\Delta} \right)^{p-1} \varepsilon C_6 \left( 1 + \mathbb{E}_{\mathbb{P}^e} \left[ \| X_{t_k}^\varepsilon \|^{\tilde{q}_1} \right] + C_8 \left( 1 + \mathbb{E}_{\mathbb{P}^e} \left[ \| Z_{t_k}^\varepsilon \|^{\tilde{q}_2} \right] \right) \right) \]

\[ + \| h \|_{\infty}^{2p} K_h^{2p} C_4 \left( \frac{\Delta^{2p}}{\varepsilon^{2p}} + \frac{\Delta^{p}}{\varepsilon^{p}} \right) \Delta^{p+1} \exp \left\{ C_4 \left( \frac{\Delta^{2p}}{\varepsilon^{2p}} + \frac{\Delta^{p}}{\varepsilon^{p}} \right) \right\} \]

\[ \times \left( 1 + \mathbb{E}_{\mathbb{P}^e} \left[ \| X_{t_k}^\varepsilon \|^{\tilde{q}_1} \right] + \mathbb{E}_{\mathbb{P}^e} \left[ \| Z_{t_k}^\varepsilon \|^{\tilde{q}_2} \right] \right) \]

\[ + 2^{2p-1} \Delta^{p+1} \times \left( M \| h \|_{\infty}^{6p} + 2^{2p-1} K_h^{2p} \| h \|_{\infty}^{2p} C_3 \left( 1 + \mathbb{E}_{\mathbb{P}^e} \left[ \| X_{t_k}^\varepsilon \|^{\tilde{q}_1} \right] + \mathbb{E}_{\mathbb{P}^e} \left[ \| Z_{t_k}^\varepsilon \|^{\tilde{q}_2} \right] \right) \right). \]
For each \( k \in \{0, \ldots, \lceil t/\Delta \rceil \} \) and \( q \geq 1 \),

\[
\mathbb{E}_p \left[ \|X^\varepsilon_{t_k} \|^q \right] \\
= \int_{\mathbb{R}^m} \mathbb{E}_p \left[ \|X^\varepsilon_{t_k} \|^q \right] (X^\varepsilon_0, Z_0) = (x, z) \ d\mathbb{P}^\varepsilon_{(x_0, z_0)}(x) \\
\leq \int_{\mathbb{R}^m} \sup_{(t^\varepsilon_{k}, x) \in [0, T] \times (0, 1)} \mathbb{E}_p \left[ \|X^\varepsilon_{t^\varepsilon_{k}} \|^q \right] (X^\varepsilon_0, Z_0) = (x, z) \ d\mathbb{P}^\varepsilon_{(x_0, z_0)}(x) \\
\leq \int_{\mathbb{R}^m} C_{17}(1 + \|x\|q) d\mathbb{P}^\varepsilon_{(x_0, z_0)}(x) = \int_{\mathbb{R}^m} C_{17}(1 + \|x\|q) dQ_{(x_0, z_0)}(x) \\
= C_{17}(1 + \mathbb{E}_Q[\|X^\varepsilon_0 \|^q])
\]

by Proposition 3.6.3, and similarly for each \( Z^\varepsilon_{t^\varepsilon_{k}} \). Continuing (36),

\[
\left( \frac{T}{\Delta} \right)^{2p-1} \mathbb{E}_p \left[ \left| \int_{t_{k}^\varepsilon}^{t_{k+1}^\varepsilon} \bar{\rho}_s(h - \bar{h})(X^\varepsilon_s, Z^\varepsilon_s) \ ds \right|^{2p} \right] \\
\leq T^{2p-1} \left\{ \left( \frac{\varepsilon}{\Delta} \right)^{2p-1} \mathbb{E}_Q[\|Z^\varepsilon_0 \|^{\bar{q}_1}] + C_{19} \mathbb{E}_Q[\|Z^\varepsilon_0 \|^{\bar{q}_2}] + \mathbb{E}_Q[\|X^\varepsilon_0 \|^{\bar{q}_1}] + C_{22} \mathbb{E}_Q[\|Z^\varepsilon_0 \|^{\bar{q}_2}] + C_{23} \mathbb{E}_Q[\|X^\varepsilon_0 \|^{\bar{q}_1}] + C_{25} \mathbb{E}_Q[\|Z^\varepsilon_0 \|^{\bar{q}_2}] \right\}
\]

where we have absorbed the factor \( 2^{2p-1} \) in \( C_{18}, C_{20}, C_{23} \) and \( C_{26} \).

**Bound on first term in (20):** Define \( F_1(\varepsilon, \Delta) \) as the expression in the curly braces on the RHS of the inequality in (37). The sum in (20) is bounded as

\[
\sum_{k=0}^{n-1} \left( \frac{T}{\Delta} \right)^{2p-1} \mathbb{E}_p \left[ \left| \int_{t_{k}^\varepsilon}^{t_{k+1}^\varepsilon} \bar{\rho}_s(h - \bar{h})(X^\varepsilon_s, Z^\varepsilon_s) \ ds \right|^{2p} \right] \\
\leq \sum_{k=0}^{n-1} T^{2p-1} F_1(\varepsilon, \Delta) \\
= T^{2p-1} F_1(\varepsilon, \Delta) \sum_{k=0}^{n-1} 1 = T^{2p-1} F_1(\varepsilon, \Delta) \ n = T^{2p} \frac{F_1(\varepsilon, \Delta)}{\Delta}.
\]

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For $\frac{F_1(\varepsilon, \Delta)}{\Delta}$, the $\Delta^p$ and $C_4 \left( \frac{\Delta^{2p}}{\varepsilon^{2p}} + \frac{\Delta^p}{\varepsilon^p} \right)$ terms go to zero as $\varepsilon \to 0$. The $\left( \frac{\Delta}{\varepsilon} \right)^p$, $p \geq 1$, terms become $\frac{1}{(-\log \varepsilon)^{\frac{p+1}{p}}}$, which also goes to zero as $\varepsilon \to 0$. Hence,

$$\sum_{k=0}^{n-1} \left( \frac{T}{\Delta} \right)^{2p-1} \mathbb{E}_{\varepsilon} \left[ \left| \int_{t_k}^{t_{k+1}} \tilde{\rho}_s(h) \left( h - \bar{h} \right) (X_s^\varepsilon, Z_s^\varepsilon) ds \right|^{2p} \right] \to 0 \quad \text{as} \quad \varepsilon \to 0.$$  

Second term in (20):

$$\mathbb{E}_{\varepsilon} \left[ \left| \int_0^T \tilde{\rho}_t(h^*) \left( h(X_t^\varepsilon) - \bar{h} \right) dt \right|^{2p} \right] \leq \mathbb{E}_{\varepsilon} \left[ T^{2p-1} \int_0^T \left| \tilde{\rho}_t(h^*) \left( h(X_t^\varepsilon) - \bar{h} \right) \right|^{2p} dt \right] \leq T^{2p-1} \mathbb{E}_{\varepsilon} \left[ \int_0^T \left( \tilde{\rho}_t(h^*) \right)^{2p} \left| h(X_t^\varepsilon) - \bar{h} \right|^{2p} dt \right] \leq T^{2p-1} ||h||_{\infty}^{2p} K_h^{2p} \mathbb{E}_{\varepsilon} \left[ \int_0^T \left| X_t^\varepsilon - X_t^0 \right|^{2p} dt \right] = T^{2p-1} ||h||_{\infty}^{2p} K_h^{2p} \int_0^T \mathbb{E}_{\varepsilon} \left[ \left| X_t^\varepsilon - X_t^0 \right|^{2p} \right] dt. \quad (38)$$

For $t \in [0, T]$,

$$\mathbb{E}_{\varepsilon} \left[ \left| X_t^\varepsilon - X_t^0 \right|^{2p} \right] = \mathbb{E}_{\varepsilon} \left[ \left| \int_0^t \left\{ b(X_s^\varepsilon, Z_s^\varepsilon) - \bar{b}(X_s^0) \right\} ds + \int_0^t \left\{ \sigma(X_s^\varepsilon) - \bar{\sigma}(X_s^0) \right\} dW_s \right|^{2p} \right] \leq 4^{2p-1} \mathbb{E}_{\varepsilon} \left[ \left| \int_0^t \left\{ b(X_s^\varepsilon, Z_s^\varepsilon) - \bar{b}(X_s^\varepsilon) \right\} ds \right|^{2p} + \left| \int_0^t \left\{ \sigma(X_s^\varepsilon) - \bar{\sigma}(X_s^\varepsilon) \right\} dW_s \right|^{2p} \right] \leq 4^{2p-1} \mathbb{E}_{\varepsilon} \left[ \left| \int_0^t \left\{ b(X_s^\varepsilon, Z_s^\varepsilon) - \bar{b}(X_s^\varepsilon) \right\} ds \right|^{2p} + \left| \int_0^t \left\{ \sigma(X_s^\varepsilon) - \bar{\sigma}(X_s^\varepsilon) \right\} dW_s \right|^{2p} \right]. \quad (39)$$

Since the slow diffusion is independent of the fast component, $\bar{\sigma} \equiv \sigma$ and the
third term in (39) vanishes. By Lipschitz property of the drift and diffusion of $X^\varepsilon$, the second and fourth terms in (39) are

$$
\mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| \int_0^t \left\{ \bar{b}(X_s^\varepsilon) - \bar{b}(X_0^0) \right\} ds \right\|^{2p} \right] \leq t^{2p-1} \int_0^t \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| \bar{b}(X_s^\varepsilon) - \bar{b}(X_0^0) \right\|^{2p} \right] ds
$$

$$
\leq t^{2p-1} K_b^{2p} \int_0^t \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| X_s^\varepsilon - X_0^0 \right\|^{2p} \right] ds
$$

and

$$
\mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| \int_0^t \left\{ \sigma(X_s^\varepsilon) - \sigma(X_0^0) \right\} dW_s \right\|^{2p} \right] \leq C_{10} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| \sigma(X_s^\varepsilon) - \sigma(X_0^0) \right\|^{2p} \right]
$$

$$
\leq t^{p-1} C_{10} K_\sigma^{2p} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \int_0^t \left\| X_s^\varepsilon - X_0^0 \right\|^{2p} ds \right].
$$

Continuing (39) and using Grönwall’s inequality,

$$
\mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| X_t^\varepsilon - X_t^0 \right\|^{2p} \right]
$$

$$
\leq 4^{2p-1} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| \int_0^t (b - \bar{b}) (X_s^\varepsilon, Z_s^\varepsilon) ds \right\|^{2p} \right]
$$

$$
+ 4^{2p-1} \left( t^{2p-1} K_b^{2p} + t^{p-1} C_{10} K_\sigma^{2p} \right) \int_0^t \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| X_s^\varepsilon - X_0^0 \right\|^{2p} \right] ds
$$

$$
\leq 4^{2p-1} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| \int_0^t (b - \bar{b}) (X_s^\varepsilon, Z_s^\varepsilon) ds \right\|^{2p} \right] \exp \left\{ 4^{2p-1} t \left( t^{2p-1} K_b^{2p} + t^{p-1} C_{10} K_\sigma^{2p} \right) \right\}
$$

$$
\leq 4^{2p-1} \mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| \int_0^t (b - \bar{b}) (X_s^\varepsilon, Z_s^\varepsilon) ds \right\|^{2p} \right] \exp \left\{ 2T^p \left( T^p K_b^{2p} + C_{10} K_\sigma^{2p} \right) \right\}.
$$

(40)

Time integral in (40): We discretize the interval $[0,t]$ and estimate time integral in (40) similarly to the sum in (20), without $\bar{\rho} (\bar{h})$ and with $(b - \bar{b})$ in place of $(h - \bar{h})$. We use the same discretization $\Delta(\varepsilon)$ as before and the estimate (36):

$$
\mathbb{E}_{\mathbb{P}^\varepsilon} \left[ \left\| \int_0^t (b - \bar{b}) (X_s^\varepsilon, Z_s^\varepsilon) ds \right\|^{2p} \right]
$$


\[ \sum_{k=0}^{\frac{t}{\Delta}} \int_{t_k}^{t_{k+1}} (b - \bar{b})(X^\varepsilon_s, Z^\varepsilon_s)ds + \int_{\frac{t}{\Delta}}^{t} (b - \bar{b})(X^\varepsilon_s, Z^\varepsilon_s)ds \] 

\[ \leq 2^{2p-1} E_{\mathbb{P}c} \left[ \sum_{k=0}^{\frac{t}{\Delta}} \int_{t_k}^{t_{k+1}} (b - \bar{b})(X^\varepsilon_s, Z^\varepsilon_s)ds \right] + \phi(\Delta^p) 

\[ = \left( 2 \left\lfloor \frac{t}{\Delta} \right\rfloor \right) 2^{p-1} \sum_{k=0}^{\frac{t}{\Delta}} \frac{t}{\Delta} \left[ \left\lfloor \frac{t}{\Delta} \right\rfloor \right] + \phi(\Delta^p) \]

\[ \leq \left( 4 \left\lfloor \frac{t}{\Delta} \right\rfloor \right) 2^{p-1} \sum_{k=0}^{\frac{t}{\Delta}} \left\{ \varepsilon^{2p} C_{11} \left( 1 + E_{\mathbb{P}c} \left[ \|Z^\varepsilon_{t_k}\|^q_k \right] \right) + (\varepsilon\Delta)^p C_{12} \left( 1 + E_{\mathbb{P}c} \left[ \|X^\varepsilon_{t_k}\|^{2p} \right] \right) + \Delta \left( 1 + E_{\mathbb{P}c} \left[ \|Z^\varepsilon_{t_k}\|^{14} \right] \right) \right\} 

\[ + \left( M \|h\|_\infty^{6p} + 2^{2p-1} K_{\bar{b}}^{2p} \|h\|_\infty^{2p} C_{15} \left( 1 + E_{\mathbb{P}c} \left[ \|X^\varepsilon_{t_k}\|^{13} \right] + E_{\mathbb{P}c} \left[ \|Z^\varepsilon_{t_k}\|^{4} \right] \right) \right) \] 

As before, for each \( k \in \{0, \ldots, \left\lfloor t/\Delta \right\rfloor \} \) and \( q \geq 1, \)

\[ E_{\mathbb{P}c} \left[ \|X^\varepsilon_{t_k}\|^q \right] \leq C_{17} \left( 1 + E_{\mathbb{Q}} \left[ \|X^\varepsilon_0\|^q \right] \right) \]

by Proposition 3.6.3, and similarly for each \( Z^\varepsilon_{t_k} \). Using such bounds in (41), terms in the curly braces are independent of the summation index \( k \), and the summation becomes over ones. Using \( \left\lfloor \frac{t}{\Delta} \right\rfloor \leq \frac{T}{\Delta} \) for \( t \in [0, T] \), continuing
(41) gives us

\[
E_{p^C} \left[ \left\| \int_0^t (b - \bar{b})(X_s^\varepsilon, Z_s^\varepsilon) ds \right\|^{2p} \right] \\
\leq 4^{2p-1} \left( \frac{T}{\Delta} \right)^{2p} \left\{ \varepsilon^{2p} C_{29} \left( 1 + C_{30} E_Q \left[ \| Z_0^\varepsilon \|^{q_8} \right] \right) + (\varepsilon \Delta)^p C_{31} \left( 1 + C_{32} E_Q \left[ \| X_0^\varepsilon \|^{q_9} \right] + C_{33} E_Q \left[ \| Z_0^\varepsilon \|^{q_{10}} \right] \right) \right\} \\
+ \left( \frac{\Delta}{\varepsilon} \right)^p \varepsilon^{2p} C_{34} \left( 1 + C_{35} E_Q \left[ \| X_0^\varepsilon \|^{q_11} \right] + C_{36} E_Q \left[ \| Z_0^\varepsilon \|^{q_{12}} \right] \right) \\
+ \Delta^p C_{37} \left( 1 + C_{38} E_Q \left[ \| X_0^\varepsilon \|^{q_{13}} \right] + C_{39} E_Q \left[ \| Z_0^\varepsilon \|^{q_{14}} \right] \right) + o(\Delta^p) \\
= T^{2p} \left\{ \varepsilon^{2p} \frac{(\varepsilon \Delta)^p}{\Delta^p} C_{29} \left( 1 + C_{30} E_Q \left[ \| Z_0^\varepsilon \|^{q_8} \right] \right) \right\} \\
+ \frac{\varepsilon^p}{\Delta^p} C_{31} \left( 1 + C_{32} E_Q \left[ \| X_0^\varepsilon \|^{q_9} \right] + C_{33} E_Q \left[ \| Z_0^\varepsilon \|^{q_{10}} \right] \right) \\
+ \left( \frac{\Delta}{\varepsilon} \right)^p \varepsilon^{2p} C_{34} \left( 1 + C_{35} E_Q \left[ \| X_0^\varepsilon \|^{q_11} \right] + C_{36} E_Q \left[ \| Z_0^\varepsilon \|^{q_{12}} \right] \right) \\
+ \Delta^p C_{37} \left( 1 + C_{38} E_Q \left[ \| X_0^\varepsilon \|^{q_{13}} \right] + C_{39} E_Q \left[ \| Z_0^\varepsilon \|^{q_{14}} \right] \right) + o(\Delta^p),
\]

(42)

where we have absorbed the factor \(4^{2p-1}\) in \(C_{29}, C_{31}, C_{34}\) and \(C_{37}\).

**Bound on second term in (20):** Define \(F_2(\varepsilon, \Delta(\varepsilon))\) as the expression inside the curly braces on the RHS of the equality in (42). Then, using (40) and (42) in (38), the second term in (20) is bounded as

\[
E_{p^C} \left[ \left\| \int_0^T \tilde{\rho}_t (\tilde{h}^e) \left( \tilde{h}(X_t^\varepsilon) - \tilde{h}(X_t^0) \right) dt \right\|^{2p} \right] \\
\leq T^{2p-1} \| h \|_{\infty}^{2p} K_h^{2p} \int_0^T E_{p^C} \left[ \| X_t^\varepsilon - X_t^0 \|^{2p} \right] dt \\
\leq T^{2p-1} \| h \|_{\infty}^{2p} K_h^{2p} \\
\times \int_0^T 4^{2p-1} E_{p^C} \left[ \left\| \int_0^t (b - \bar{b})(X_s^\varepsilon, Z_s^\varepsilon) ds \right\|^{2p} \right] \exp \left\{ 2T^p \ (T^p K_b^{2p} + C_{16} K_{\sigma}^{2p}) \right\} dt \\
\leq T^{2p-1} \| h \|_{\infty}^{2p} K_h^{2p} \\
\times \int_0^T \left( 4^{2p-1} T^{2p} F_2(\varepsilon, \Delta) + o(\Delta^p) \right) \exp \left\{ 2T^p \ (T^p K_b^{2p} + C_{16} K_{\sigma}^{2p}) \right\} dt
\]
\[ = \left(4^{2p-1}T^{4p} \|h\|_\infty^{2p}K_h^{2p}F_2(\varepsilon, \Delta) + \sigma(\Delta^p T^{2p})\right) \exp\left\{2T^p \left(T^p K_b^{2p} + C_{10} K_\sigma^{2p}\right)\right\} \]

Recall that we chose \( \Delta(\varepsilon) = \varepsilon (\log \varepsilon)^{\frac{1}{4}} \). Then, the \( \Delta^p \) and \( C_{14} \left( \Delta^p s + \frac{\Delta s}{\varepsilon} \right) \Delta^p \exp\left\{C_{14} \left( \Delta^p s + \frac{\Delta s}{\varepsilon} \right)\right\} \) terms in \( F_2(\varepsilon, \Delta) \) go to zero as \( \varepsilon \to 0 \). The \( \left( \frac{\varepsilon}{\Delta}\right)^p, p \geq 1 \), terms become \( \frac{1}{(-\log \varepsilon)^{\frac{1}{4}}} \), which also goes to zero as \( \varepsilon \to 0 \).

Hence,

\[ \mathbb{E}_{\varepsilon^p} \left[ \left| \int_0^T \mu(X^\varepsilon_t) \left( \tilde{h}(X^\varepsilon_t) - \tilde{h}(X^0_t) \right) dt \right|^{2p^p} \right] \to 0 \quad \text{as} \quad \varepsilon \to 0. \]

Note that we are able to obtain \( L^p \) convergence with the slow diffusion being independent of the fast component, when \( \bar{\sigma} \equiv \sigma \). In general, when \( \sigma \) is dependent on the fast component, \( \bar{\sigma} \sigma^*(x) = \int_{\mathbb{R}^n} \sigma \sigma^*(x, z) \mu(x, dz) \), hence \( \bar{\sigma} \) is not unique. While \( X^\varepsilon \) converges weakly to \( X^0 \), it may not be possible to expect a strong convergence. In the calculations here, difficulty would arise in dealing with \( \mathbb{E}_{\varepsilon^p} \left[ \left\| \int_0^T \sigma(X^\varepsilon_s, Z^\varepsilon_s) - \bar{\sigma}(X^\varepsilon_s) \right\|_{2p^p}^2 \right] \) in (39). Another setting in which it is possible to obtain stronger than weak convergence of \( X^\varepsilon \) to \( X^0 \) is in the scalar case when the slow diffusion depends only on the fast component, as in [71]. Lemma 5.1 of [71] obtains convergence in probability of \( X^\varepsilon \) to \( X^0 \) by application of Levy’s theorem to represent the diffusions

\[ X^\varepsilon_t = \int_0^t \sigma(Z^\varepsilon_s) dW_s, \quad X^0_t = \bar{\sigma} W_t \]

as time-changed Brownian motions \( W_{\int_0^t \sigma(Z^\varepsilon_s)^2 ds} \) and \( W_{\bar{\sigma}^2 t} \), respectively.

Appendix 7: Shannon entropy of a Gaussian

Let \( X \sim \mathcal{N}(\mu, \Sigma) \). The Shannon entropy of \( X \) is

\[ \mathcal{H}(X) \]

\[ = - \int_X p(x) \log p(x) \, dx \]

\[ = - \int_X \frac{1}{\sqrt{(2\pi)^n |\Sigma|^\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \left(x - \mu\right)^\ast \Sigma^{-1} \left(x - \mu\right) \right\} \]

\[ \times \left[ -\frac{1}{2} \log (2\pi)^n |\Sigma| - \frac{1}{2} \left(x - \mu\right)^\ast \Sigma^{-1} \left(x - \mu\right) \right] \, dx \]

\[ = - \left( -\frac{1}{2} \log (2\pi)^n |\Sigma| \right) \]

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\[-\frac{1}{2} \int x \left(\frac{1}{(2\pi)^{\frac{m}{2}}|\Sigma|^\frac{1}{2}}\right) \exp \left\{ -\frac{1}{2} (x - \mu)^* \Sigma^{-1} (x - \mu) \right\} (x - \mu)^* \Sigma^{-1} (x - \mu) dx \]

\[= - \left( -\frac{1}{2} \log(2\pi)^m |\Sigma| - \frac{1}{2} \text{tr}(\Sigma \Sigma^{-1}) \right) \]

\[= - \left( -\frac{1}{2} \log(2\pi)^m |\Sigma| - \frac{1}{2} m \right) \]

\[= - \left( -\frac{1}{2} \log(2\pi e)^m |\Sigma| \right) = \frac{1}{2} \log(2\pi e)^m |\Sigma|.\]
REFERENCES


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