The Mathematics of Poker: Extending the Nash-Shapley Model

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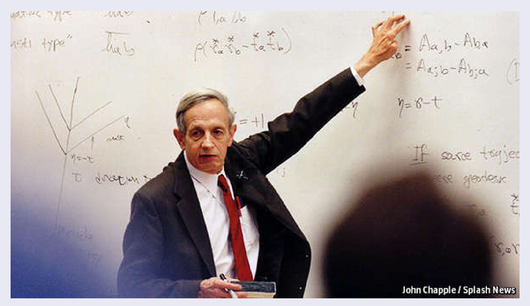
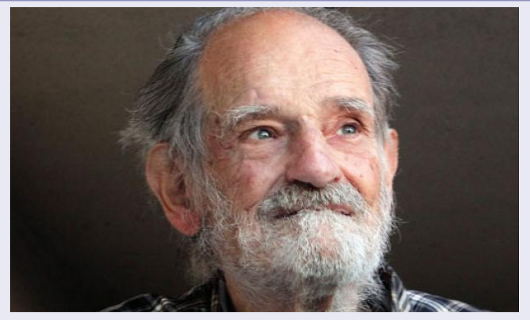
John Nash and Lloyd Shapley

- Mathematicians who made fundamental contributions to game theory.
- Received the Nobel Prize in Economics in 1994 and 2012, respectively.
- Nash was portrayed in the movie *A Beautiful Mind*.

The Nash Equilibrium

- A **Nash Equilibrium** is a stable state in the game in which no player can increase their expected profit by changing their strategy.
- Nash and Shapley found the Nash-Equilibrium strategies for this game by solving a system of equations involving 24 variables.
- We call the Nash Equilibrium strategy **optimal**. However, it is only optimal when playing against skilled players. As we will see, these "optimal" strategies are far from optimal against certain player types.

The Optimal Strategy

The strategy Nash and Shapley found is nearly identical to the naive strategy with a few exceptions:

- **Player 1**: Sandbags with a certain probability when given a low card.
- **Player 2**: When Player 1 passes and Player 2 has a high card, Player 2 will sandbag with the same probability as Player 1.
- **Player 3**: Bluffs with a certain probability when given a low card and Players 1 and 2 both pass.

Player Profiles

- **Naive**: Player always bets on high cards and always passes on low cards.
- **Random**: Player bets or passes with equal probability, regardless of the value of his or her card or the actions of other players.
- **Loose (Aggressive)**: Player always bets when holding a high card and bets with a given probability when given a low card.
- **Tight (Conservative)**: Player does not always bet when holding a high card and never bets on a low card.

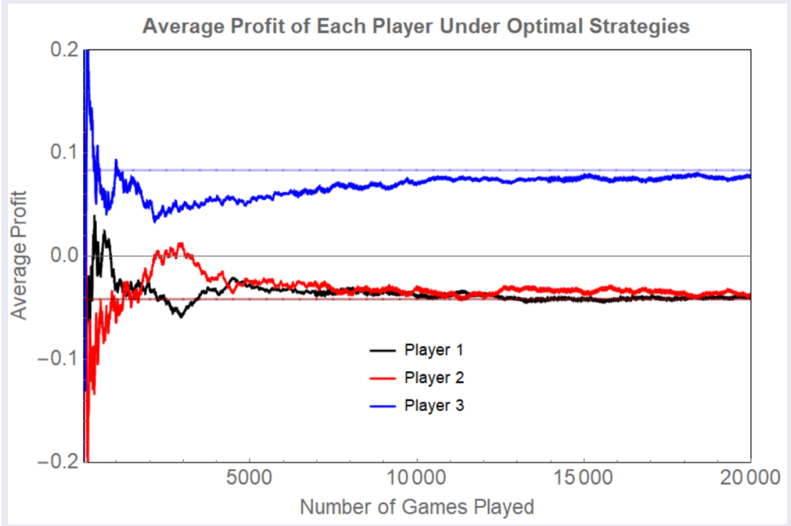
The Nash-Shapley Poker Model

- Simplified model with three players and only two types of cards: a high card and a low card.
- Each player pays a fixed ante a and either bets b, or passes in a game lasting up to five rounds. (We considered the case where $a = 1$ and $b = 2$).

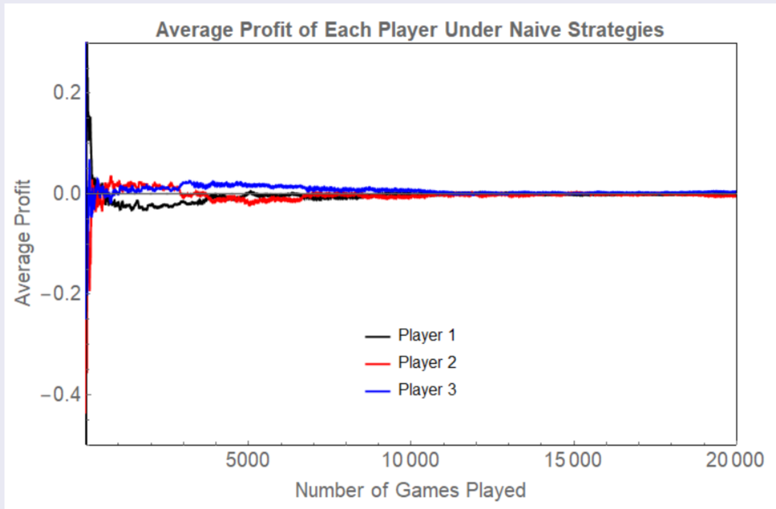
Motivating Questions

- What is the effect of player position on expected profit?
- What is the best response to different player profiles, given a player's position?
- Under what circumstances is cooperation favored over competition?
- How close do simulations approximate theoretical values?

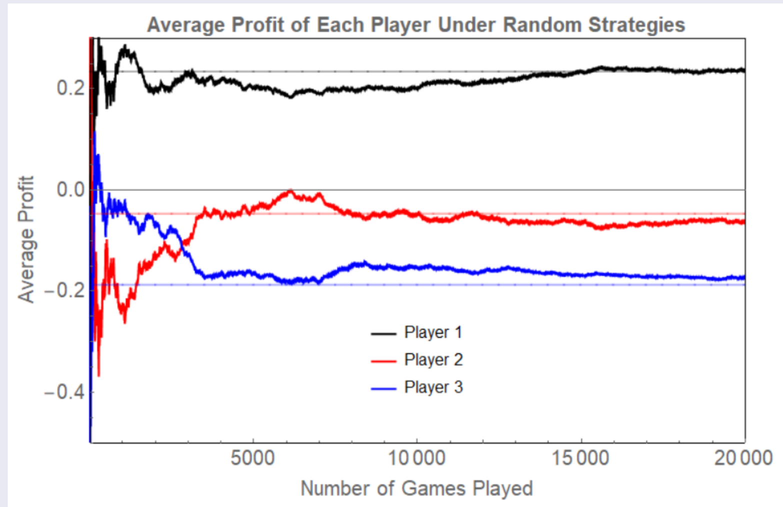
Effect of Player Position



- When playing optimally, Players 1 and 2 follow the same strategy and eventually receive the same negative profit.



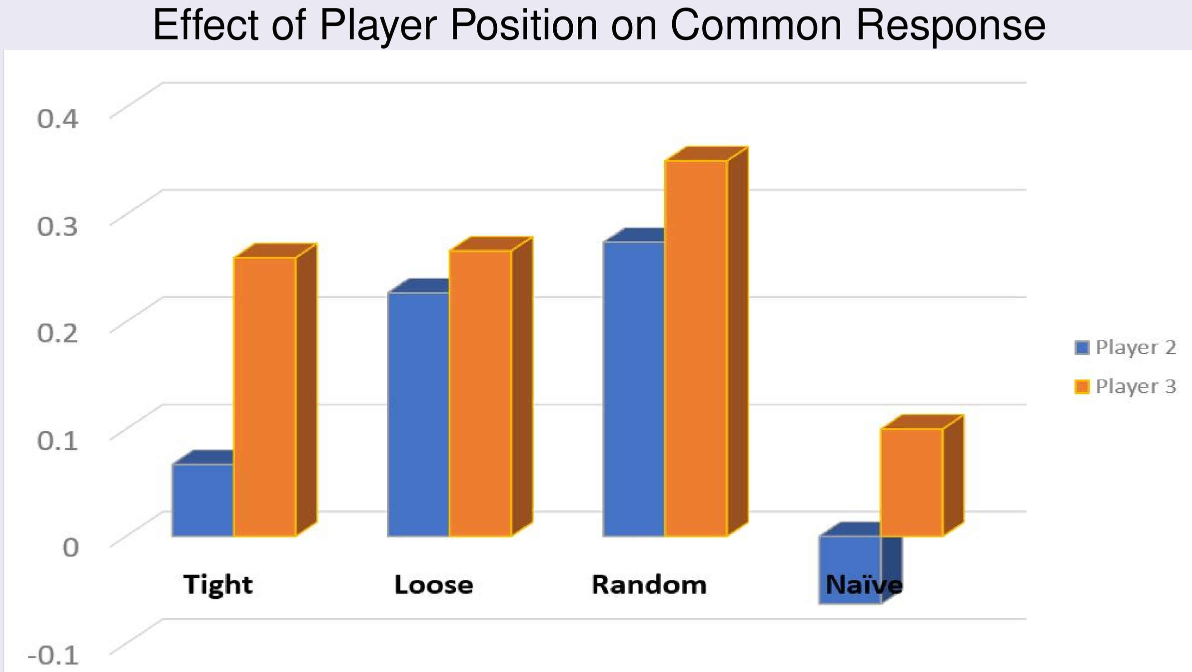
- The expected profit of all players is 0 when they each follow the naive strategy.



- The fourth and fifth rounds give Player 1 an advantage when all players play randomly.

Best Response to Different Player Profiles

Effect of Player Position on Common Response



- Profits for Players 2 and 3 assuming Optimal Play under different strategies for Player 1.

Player 1 Best Response and Expected Profit

P2 \ P3	Optimal	Random	Naïve	Loose	Tight
Optimal	-0.0417	0.3489	0.0364	0.1432	0.2421
Random	0.2520	0.7188	0.3125	0.4688	0.5654
Naïve	-0.0605	0.3125	0	0.1250	0.1875
Loose	0.1115	0.4688	0.1250	0.2813	0.3125
Tight	0.0968	0.5625	0.1875	0.3125	0.4492

Key:
Very Loose
Optimal
Naïve

- On average, Naive tends to be the best response strategy for Player 1.
- We see that Optimal is a highly specialized strategy.

Cooperation vs. Competition

Player 1 Profit with Player 3 Optimal

P1 \ P2	Random	Naïve	Optimal	Loose	Tight
Random	-0.264	-0.650	-0.624	-0.554	-0.361
Naïve	0.252	**-0.060**	-0.042	0.095	0.097
Optimal	0.252	-0.063	**-0.042**	0.111	0.078
Loose	0.016	-0.445	-0.426	-0.310	0.078
Tight	-0.029	-0.265	-0.240	-0.148	-0.145

Player 2 Profit with Player 3 Optimal

P1 \ P2	Random	Naïve	Optimal	Loose	Tight
Random	-0.436	0.283	0.274	0.016	-0.169
Naïve	-0.603	**-0.060**	-0.063	-0.320	-0.343
Optimal	-0.591	-0.042	**-0.042**	-0.338	-0.295
Loose	-0.593	0.189	0.178	-0.107	-0.297
Tight	-0.446	0.034	0.033	-0.196	-0.216

- Under the assumption Player 3 plays optimally, Player 2 has a strictly dominating strategy of Naive, regardless of what Player 1 will choose.
- Assuming common rationality, Player 1 will know this and will therefore choose Naive to optimize their expected profit. Thus, this leads us to one of our equilibrium in a dynamic game setting.
- Comparing the Naive and Optimal strategy profits of Players 1 and 2, they are better off both playing optimally (cooperation), than playing naively (competition).

Additional Findings

- Given that all three players use the same strategy, Player 2's expected profit is always bounded by Player 1 and 3's.
- If we change the Nash-Shapley model from a static game, to a dynamic one (introduce the ability to change strategies), we encounter two equilibria. One stable (Naive, Naive, Optimal) and one unstable (Optimal, Optimal, Optimal).

Reference

- Nash, J. F., & Shapley, L. S. (1952). *Simple Three-person Poker Game.* Princeton University Press.

Acknowledgements

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