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REALIZABILITY OF FUNDAMENTAL CUT-SET MATRICES OF ORIENTED GRAPHS

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Realizability of Fundamental
Cut-set Matrices of Oriented Graphs

by

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INTRODUCTION

As topology (linear graph theory) has been recognized to be a suitable tool to solve many problems in electrical networks, switching circuits, communication nets, etc., the necessary and sufficient conditions that a matrix be a fundamental cut-set (or circuit) matrix becomes one of the important problems in this field.

If the problem is to find whether a given matrix is a fundamental cut-set matrix of a non-oriented graph, there are four methods[1,2,3,4] of testing such a matrix at present. This paper takes one of these methods and modifies it such that we can test whether a given matrix is a fundamental cut-set matrix of an oriented graph (where every entry in the matrix is +1, -1, or 0).

It is known that the theory of oriented graphs has more applications than that of non-oriented graphs. Also, in many cases, representation of systems by non-oriented graphs is a special case of representation of systems by oriented graphs. For example, topological representation of electrical networks[4,5,6,7,8] and communication nets.[9,10,11]

PRELIMINARY

In order to give a modified theorem in the paper[4] "Necessary and Sufficient Conditions for Realizability of Cut-Set Matrices" so that we can use it to test whether a matrix is a fundamental cut-set matrix at an oriented graph, we will review definitions. Some of these definitions are modified so that it will fit to the problem in this paper.

Definition 1: H-submatrix of matrix $N = [N_{11}U]$ (where $[n_{ij}] = N_{11}$ and $n_{ij} = \pm 1,0$) with respect to row p is a matrix obtained from N by deleting row p and all columns which have non-zero elements at the intersection with row p. For convenience, every row and column of H-submatrices and M-submatrices will be identified by the symbols which are used to identify the rows and columns of a given matrix N such that row p (column q) of H (or M)-submatrix is the row of H (or M) corresponding to row p (column q) of N.

Definition 2: A pair of M-submatrices M_1 and M_2 of a matrix N with respect to row p where H-submatrix of N with respect to row p has the form

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \qquad (1)$$

is a pair of following submatrices of N: (1) M_1 is obtained from N by deleting all rows and columns which belong to H_1 and (2) M_2 is obtained from N by deleting all rows and columns belonging to H_2. Notice that H_1 can be empty.

From a given matrix A, we can obtain a pair of M-submatrices M_1 and M_2 have the form M U , $M_.$ can be considered as a given matrix. Hence, if there exists a row in M, which has not been used to obtain H-submatrix (to obtain M-submatrices), we can obtain M-submatrices M_a and M_b of M, the collection (M_a, M_b, M_2) is also called a set of M-submatrices. Similarly, any of M_a , M_b , and M_2 can be considered as a given matrix. Hence if there exists a row of one of these matrices, say M_2, which has not been used to obtain M-submatrices before, we can obtain a pair of M-submatrices of M_2. Thus we can obtain another set of M-submatrices.

If a matrix M_p in the set of M-submatrices which has been obtained by the above process contains no rows which have not been used to form M-submatrices in the set, M_p is called a "minimum M-submatrix". If every matrix in the set of

M-submatrices which has been obtained by the above process is a minimum M-submatrix, the set is called a "set of minimum M-submatrices".

TO TEST A MATRIX TO BE A FUNDAMENTAL CUT-SET MATRIX OF AN ORIENTED GRAPH

The important theorem for testing whether a matrix is a fundamental cut-set matrix of an oriented graph is given below. Even though this is the modified theorem of a theorem in the paper[4] "Necessary and Sufficient Contitions for Realizability of Cut-Set Matrices", the proof becomes more complicated than that of the original theorem.

Theorem 1: A Matrix $A = [A_{11}U]$, where every entry is ± 1 or 0 and U represents a unit matrix is a fundamental cut-set matrix of an oriented graph if and only if there exists a set of minimum M-submatrices obtained from A such that every matrix in the set becomes an incidence matrix of an oriented graph by multiplying (-1) to some row of the matrix.

Notice that a matrix $C = [C_{ij}]$ where $C_{ij} = \pm 1, 0$ is an incidence matrix if and only if every column of C has either at most one non-zero or two non-zero with opposite signs.

The multiplication of -1 to some row of a M-submatrix is necessary because assigning the sign of each branch in an oriented graph in an incidence set[12] and that of each branch in a cut-set are different. For example, suppose branches a in the graph in Fig. 1 is a branch in the tree corresponding to fundamental cut-set matrix A. Then if we consider $\{a,b\}$ as an incidence set corresponding to a row v in an incidence $\underline{A}$, the intersections of columns corresponding to a and b and row v have -1 and 1 respectively. However, if we consider $\{a,b\}$ as a cut-set in A, the intersections of columns a and b are now representing the cut-set have 1 and -1 respectively. Hence when we form M-submatrices with respect to row p in $A = [A_{11}U]$, the row p in M-submatrices may not represent an incidence set. However, if row p does not represent an incidence set, the multiplication by -1 will make row p to represent an incidence set. Hence the proof of necessary part of the theorem is exactly the same as that of non-oriented case except that the multiplication of $\pm$ 1.

Before proving the sufficient part of theorem 1, we will study the following two theorems:

Theorem 2: If a matrix $M = [M_{11} U]$ where every entry of M_{11} is $\pm$ 1 or 0 and U is a unit matrix is a fundamental cut-set matrix of oriented graph G, then M is also a fundamental cut-set matrix of oriented graph $\bar{G}$ which is obtained from G by reversing the orientation of every branch in G.

Proof: Because of the definition of assigning the sign of elements in a row of M which represents a cut-set, the row of M does not change if the orientation of every branch in G is altered. Hence the theorem is true.

Suppose arrow p of M represents incidence set[3] s but not a cut-set. Let s be consisted of branches $e_1, e_2, \ldots$, and e_s which are incident at vertex p as shown in Fig. 2a. Then reversing the orientation of every branch in G as shown in Fig. 2b makes no longer row p to represent the incidence set s of branches which incident at vertex p because of the definition of assigning the sign of non-zero elements in row p corresponding to incidence set s. However (-1) times

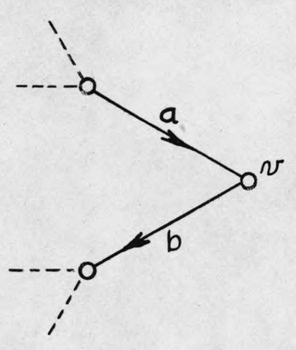

FIGURE 1

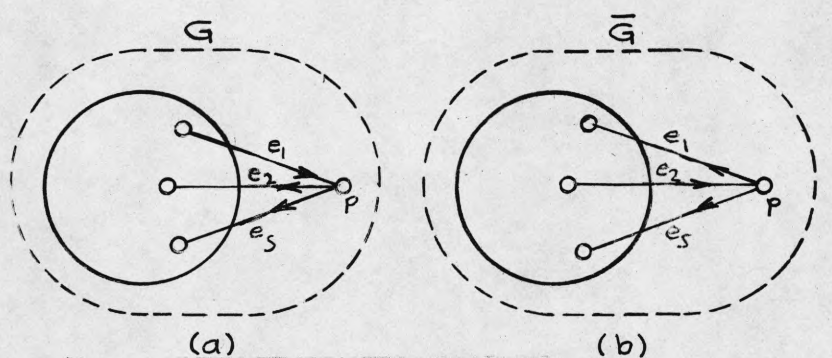

FIGURE 2

row p in M will represent s in $\bar{G}$. Theorem 2 only guarantees that row p represents a cut-set $S = (e_1, \ldots e_i)$ in $\bar{G}$.

Theorem 3: Let a pair of M-submatrices of a matrix M_{i+j} with respect to row p be M_i and M_j. Suppose there exist graphs g_i and g_j such that (1) the fundamental cut-set matrices of g_i and g_j are M_i and M_j respectively, (2) there exists vertex p in g_i such that either row p in M_i or (-1) times row p in M_i represents an incidence set of branches which incident at vertex p and (3) there exists vertex p in g_j such that either row p in M_j or (-1) times row p in M_j represents an incidence set of branches which incident vertex p. Then there exists a graph g_{i+j} such that (a)M_{i+j} is a fundamental cut-set matrix of g_{i+j},(b) for every row q in M_i except row p, which has the property that either row q or (-1) times row q represents an incidence set of branches which incident at vertex q in g_i, there exists row q in M_{i+j} such that either row q or (-1) times row q in M_{i+j} represents an incidence set of branches which incident at vertex q in g_{i+j} and similarly, (c) for every row r in M_j which has the property that either row r or (-1) times row r represents an incidence set of branches which incident at vertex r, there exists row r in M_{i+j} such that either row r or (-1) times row r in M_{i+j} represents an incidence set of branches which incident at vertex r in g_{i+j}.

We will prove theorem 3 by constructing the graph g_{i+j} which satisfies a, b, and c. Since M_i and M_j are a pair of M-submatrices of M_{i+j} with respect to row p, if and only if there exists a non-zero element at the intersection of row p and column e in M_i, there exists non-zero element at the intersection of row p and column e in M_j. Hence, if and only if a branch e_q in g_i is connected on vertex p, there exists branch e_q connected on vertex p in g_j. Also the orientation of e_q in g_i with respect to vertex p is either the same as or opposite to the orientation of e_q in g_j with respect to vertex p.

If branch e_q which is connected on vertex p in g_i has the same orientation with respect to p as e_q which is connected on vertex p in g_j, we alter the orientation of all branches in g_i to form graph $\underline{g}_i$ so that the orientation of branch e_q in $\underline{g}_i$ with respect to vertex p is opposite to the orientation of e_q in g_j with respect to p. M_i is a cut-set matrix of $\underline{g}_i$ by theorem 2. Also, it is clear that if either row q or (-1) times row q represents an incidence set in g_i, either row q or (-1) times row q represents an incidence set in $\underline{g}_i$. Hence, the above operation will produce no alteration to the assumptions and results in theorem 3. If branch e_q which is connected on vertex p in g_i has the opposite orientation as e_q in g_j with respect to vertex p, then we define that $g_i = \underline{g}_i$. Now we construct g_{i+j} whose cut-set matrix is M_{i+j} from $\underline{g}_i$ and g_j as follows:

Let $\underline{g}_i$ and g_j be the graphs shown in Fig. 3a, where the cut-set corresponding to row p of M_i (and M_j) consists of branches e_1, e_2... and e_k.

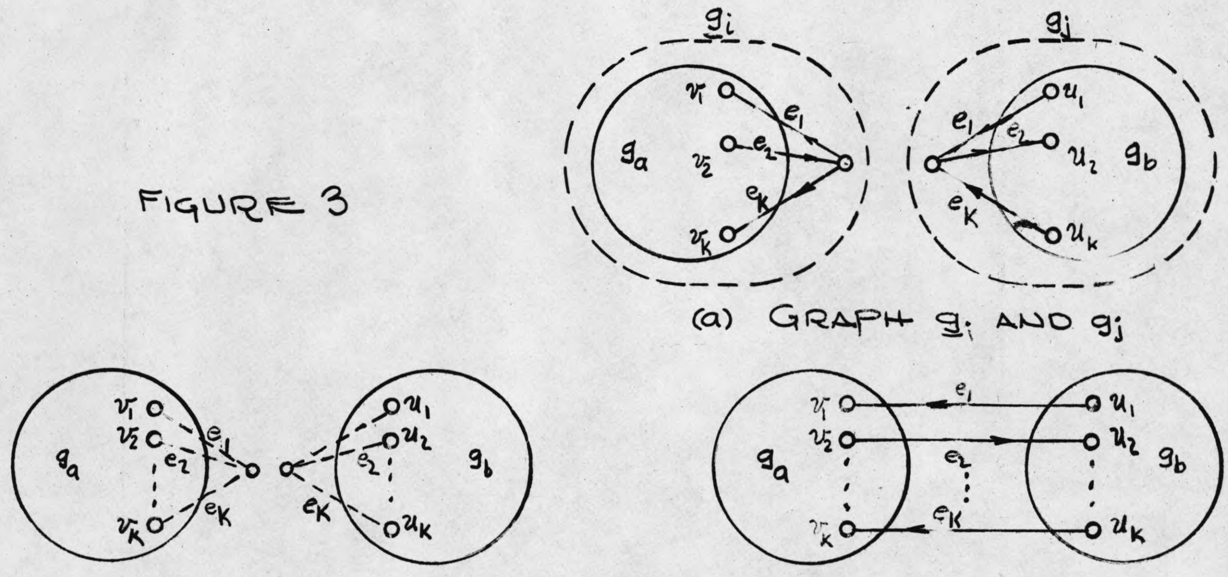

FIGURE 3

(a) GRAPH g_i AND g_j

(b) GRAPHS OF g_a & g_b

(c) GRAPH g_{i+j}

Also let e_w in $\underline{g}_i$ be connected between vertices v_w and p, and e_w in g_j be connected between vertices u_w and p for w = 1, 2, ... k. (Fig. 3a).

(1) Remove all branches $e_1, \ldots,$ and e_k which are connected on vertex p in g_i and g_j, as shown in Fig. 3b.

(2) Connect branch e_w between vertices v_w and u_w and the orientation of e_w is the orientation of e_w in $\underline{g}_i$ for $w = 1, 2, \ldots k$, i.e. if the orientation of e_w in $\underline{g}_i$ is away from v_w, the orientation of e_w in the resultant graph is away from v_w and if the orientation of e_w in $\underline{g}_i$ is toward v_w, the orientation of e_w in the resultant graph is toward v_w, as shown in Fig. 3c.

Because of the first step of the above process, every branch in $\underline{g}_i$ other than $e_1, e_2 \ldots$ and e_k does not be replaced in the resultant graph. Also, as far as the branches in $\underline{g}_i$ are concerned the second step of the above process only replaces the connection of e_w from vertex p to vertex u_w which is in g_j for $w = 1, 2, \ldots k$. Hence, if we coincide all vertices in the resultant graph which are also in g_j, we can obtain $\underline{g}_i$. Likewise, if we coincide all vertices in the resultant graph which are also in $\underline{g}_i$, we can obtain g_j. Because only cut-set $(e_1, e_2, \ldots e_k)$ is in both $\underline{g}_i$ and g_j, M_{i+j} is the fundamental cut-set matrix of the resultant graph with respect to the tree consisting of the branches in the trees of $\underline{g}_i$ and g_j by which the fundamental cut-set matrices M_i and M_j have been obtained. Furthermore, if row r in M_j $(r \neq p)$ represents an incidence set in g_j, row r in M_{i+j} represents an incidence set in the resultant graph. If (-1) times row r in M_j $(r \neq p)$ represents an incidence set in g_j, (-1) times row r represents an incidence set in the resultant graph. If $\underline{g}_i$ is identical with g_i, the above property also holds for g_i. Suppose $\underline{g}_i$ is obtained by reversing the orientations of all branches in g_i, then if row q in M_i $(q \neq p)$ represents an incidence set in g_i, (-1) times row q represents an incidence set in g_i, row q represents an incidence set in the resultant graph. Hence the resultant graph is g_{i+j}, and theorem 3 is proved.

Now we will prove the sufficient part of theorem 1. Consider the process used to obtain set S_v of minimum M-submatrices from a given matrix A. Let this process be $S_1, \ldots S_v$ where $S_1 = (A)$ and S_j $(j = 2, 3, \ldots v)$ is obtained from S_{j-1} by using one matrix M_d in S_{j-1} to form a pair of M-submatrices M_{d_1} and M_{d_2} with respect to row d which has not been used to form a pair of M-submatrices in S_{j-k} (for $k = 1,2,\ldots j-k$) and replace M_d by M_{d_1} and M_{d_2}. Hence number of matrices in S_j is one plus number of matrices in S_{j-1}.

Let distinct rows $p_2, p_3 \ldots$, and p_v be the sequence of rows which are in a given matrix A such that row p_i $(i = 2, 3, \ldots, v)$ is used to obtain a pair of M-submatrices M_{i_1} and M_{i_2} in S_i from a matrix in S_{i-1} to form S_i from S_{i-1}.

By the hypothesis of theorem 1, for each fundamental cut-set matrix M in S_v, there exists an oriented graph g such that either row q or (-1) times row q represents an incidence set in g for all rows in M. (Notice that an M-submatrix of a matrix $[C_{11} U]$ is of the form $[D_{11} U]$ where U is a unit matrix). Hence, we can apply theorem 2 to a pair of M-submatrices M_{v_1} and M_{v_2} with respect to row p_v and can prove that every matrix M_j in S_{v-1} is realizable as a fundamental cut-set matrix of an oriented graph g_j such that either row q or (-1) times row q of M_j represents an incidence set in g_j for all rows in M_j except if row q is row p_v.

If we can apply theorem 2 successively to a pair of M-submatrices M_{i_1} and M_{i_2} in S_i which makes S_i from S_{i-1} for $j = v, v-1, \ldots v-e$ $(e < v-2)$, then we can prove that every matrix in S_{i-1} is realizable as a fundamental cut-set matrix of an oriented graph g such that either row s or (-1) times row s of the matrix represents an incidence set in g for all rows in the matrix except if row s is one of rows p_v, p_{v-1}, ..., and p_{v-e}. The requirements for using theorem 2 to a pair of realizable M-submatrices M_{i_1} and M_{i_2} with respect to row p_i whose oriented graphs are g_{i_1} and g_{i_2} are that either row p_i or (-1) times row p_i in M_i

represents an incidence set in g_i and either row p_i or (-1) times row p_i in M_{i_2} represents an incidence set in g_{i_2}. Because rows $p_2, p_3, \ldots$ and p_v are all different rows in a given matrix A, this requirement will be fulfilled for a pair of M-submatrices in S_i by which S_i is obtained from S_{i-1} for $i = 2,3,\ldots v$ if we apply it starting with a pair of M-submatrices in S_v by which S_v is obtained from S_{v-1}, then to a pair of M-submatrices in S_{v-1} by which S_{v-1} is formed from S_{v-2}, etc. Finally we can apply theorem 2 to a pair of M-submatrices in S_2 by which S_2 is obtained from $S_1 = (A)$ where A is a given matrix which proves the sufficient part of theorem 1.

If H-submatrix H_i of a matrix with respect to a row can be partitioned as

$$H_i = \begin{bmatrix} H_1 & & & & 0 \\ & H_2 & & & \\ & & H_3 & & \\ 0 & & & \ddots & \\ & & & & H_p \end{bmatrix} \qquad (2)$$

then there are 2^{P-1} different pairs of M-submatrices of the matrix with respect to the row. Hence in general there will be many sets of minimum M-submatrices of a given matrix A. However, if one of these sets of minimum M-submatrices of $A = [A_{11} U]$ where the entry in A_{11} is +1, -1, or 0 is satisfied the conditions in theorem 1, A is realizable as a fundamental cut-set matrix of an oriented graph. In other words, unless all possible sets of minimum M-submatrices of A are not satisfied the conditions in theorem 1, we cannot say that A is not realizable as a fundamental cut-set matrix of an oriented graph. On the other hand, there may be a collection U of sets of minimum M-submatrices of A which has the property that if and only if a set S in U satisfies the conditions in theorem 1, any other set S' in U satisfies the conditions in theorem 1. Hence only one of sets in U needs to be tested. The existence of such a collection can be shown as follows: If A is realizable as G_1 in Fig. 4a with row s in A represents a cut-set $S = (e_{11}, e_{12}, \dots e_{1m}, e_{21}, e_{22}, \dots e_{2m})$. Then A can be realizable as G_2 in Figure 4b with either row s or (-1) times row s represents an incidence set S in G_2. In this case H-submatrix of A with respect to row s is of the form in Eq. 1 with $H_1, H_2 \neq \emptyset$ ($H_1 = \emptyset$ means H_1 consists of no row.) Then if the set of minimum M-submatrices of A which is obtained by forming a pair of M-submatrices M_a and M_b of A with respect to row s by letting $H_1 = \emptyset$ and $H_2 = H$ (which is equivalent to saying that $M_a = A$ and M_b is a one row matrix) satisfies the conditions in theorem 1, the set of minimum M-submatrices M_c and M_b of A with respect to row s by $H_1, H_2 \neq \emptyset$ also satisfies the conditions in theorem 1 and vice versa. This is also true when we take an M-submatrix M in S_{j-1} and form a pair of M-submatrices of M to form set S_j of M-submatrices. Thus, whenever H-submatrices of a matrix is of the form in Eq. 1 with $H_1, H_2 \neq \emptyset$ it is not necessary to form a pair of M-submatrices of the matrix with letting $H_1, = \emptyset$ and $H_2 = H$.

<u>Example of Using Theorem 1</u>. The following matrix is not realizable as a fundamental cut-set matrix of an oriented graph because of the following reasons:

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

From H-submatrix of A with respect to row a, which is

$$H = \begin{array}{c} \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

we obtain a pair of M-matrices M_1 and M_2 as

$$M_1 = \begin{array}{c} \\ a \\ c \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

and

$$M_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \end{array}$$

Since M_1 (or M_2) is not realizable as an incidence matrix of an oriented graph by multiplying (-1) to some rows in M_1 and since there is no other way of obtaining a pair of M-submatrices of A except by letting $H_1 = \emptyset$ and $H_2 = H$, A is not realizable as a fundamental cut-set matrix of an oriented graph. Notice that to obtain the set of minimum M-submatrices of A we must obtain a pair of M-submatrices of M_1 with respect to row c and a pair of M-submatrices of M_2 with respect to row b. It is clear that one of the above pair of M-submatrices of M_1 is the same as M_1 and the other is a single row matrix. Similarly, one of the pair of M-submatrices of M_2 with respect to row b is the same as M_2 and

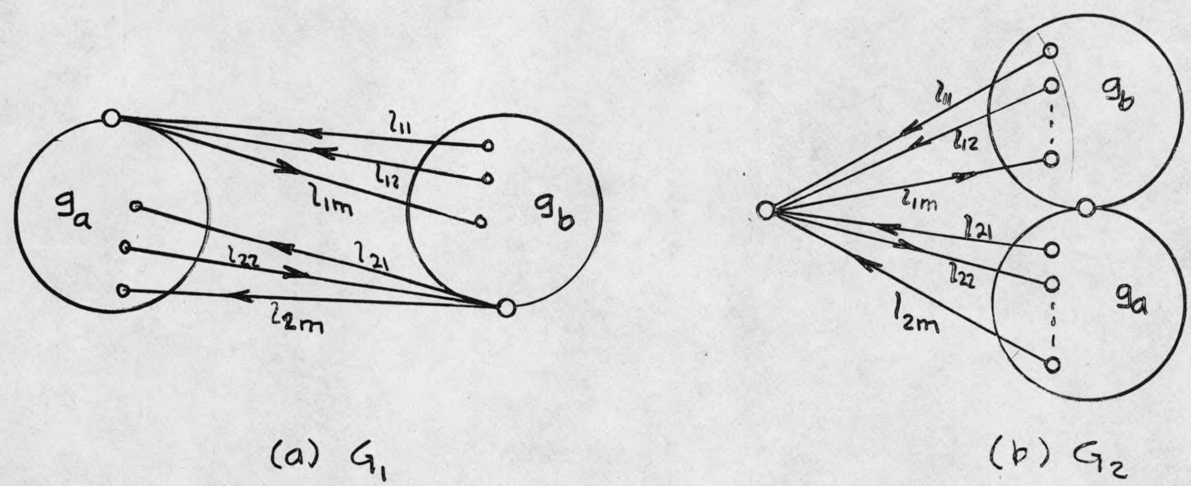

FIGURE 4

the other is a single row matrix. Hence, the set of minimum M-submatrix consists of M_1, M_2, and two single row matrices. Since a single row matrix always satisfies the conditions in theorem 1, it is only necessary to test whether M_1 and M_2 satisfy the conditions in theorem 1 to know whether A can be a fundamental cut-set matrix. It is interesting to notice that if we replace all -1 by +1 in A, then A is realizable as a fundamental cut-set matrix in both oriented and non-oriented graph.

REFERENCES

1. R. L. Gould, "Graphs and Vector Space", J. Math. and Phy. 38, 193-214 (1958).

2. W. T. Tutte, "Matroids and Graphs", Trans. Am. Math. Soc. 90, 525-552 (March, 1959).

3. L. Lofgreen, "Irredundant and Redundant Boolean Branch Networks", IRE-PGCT, CT-6 (Suple.), 158-175.

4. W. Mayeda, "Necessary and Sufficient Conditions for Realizability of Cut-Set Matrices", IRE-PGCT, CT-7, 79-81 (March, 1960).

5. S. Seshu and M. B. Reed, "Linear Graphs and Electrical Networks", Addison-Wesley (1961).

6. C. L. Coates, "General Topological Formulas for Linear Network Functions", IRE-PGCT, CT-5, 42-54.

7. W. Mayeda, "Topological Formulas for Active Networks", Interim Tech. Report No. 8, EERL, University of Illinois (January, 1958); also, "Topological Formulas for Nonreciprocal Networks and Networks with Transformers", Proc. of NEC (October, 1958).

8. W. Mayeda, "Terminal and Branch Capacity Matrices of a Communication Net", IRE-PGCT, CT-7, 260-269 (September, 1960).

9. R. T. Chien, "Synthesis of a Communication Net", IBM J., Vol. 4, 311-320, (July, 1960).

10. D. T. Tang and R. T. Chien, "Analysis and Synthesis Techniques of Oriented Communication Nets", IRE-PGCT, CT-8, 39-43 (March, 1961).

11. O. Wing and R. T. Chien, "Optimal Synthesis of Communication Nets", IRE-PGCT, CT-8, 44-49 (March, 1961).

12. An incidence set is a collection of branches (sometimes called edge or elements) in a graph which are incident at a vertex.