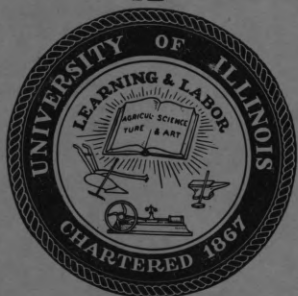


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SOME APPLICATIONS OF
LINEAR GRAPHS

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PART I
ON THE MODIFICATIONS OF FLOW GRAPHS

1. Introduction

The association of a topological structure with a set of linear algebraic equations was introduced by Mason [1,2] and is called the signal flow graph. Coates [3] invented a different linear graph, called a flow graph, for the same association and gave a rigorous, systematic development of some of the topological formulas. However, these formulas for the graph gains (for both the signal flow graphs and the flow graphs) are not very efficient because of the existence of a large number of cancellation terms. The purpose of this paper is to modify the (Mason or Coates) flow graphs a little bit (only the weights associated with the self-loops) so that the more efficient formulas can be obtained.¹

2. Definitions

A directed (linear) graph G consists of a set V of elements called nodes together with a set E of ordered pairs of the form (i,j) , i and $j \in V$, called the edges of the graph; the node i is called the initial node, and node j the terminal node. For any $i \in V$ the symbols $\rho(i)$ and $\rho^*(i)$ will be used to denote the cardinals of the sets of edges of G having i as initial and terminal nodes, respectively. They

¹Seshu and Reed list this problem as one of the research problems in the appendix of their book [4] (p. 297, problem 18).

are called the outgoing and incoming degrees of G at i . A node not incident to any edge is called an isolated node. Two subgraphs are disjoint when they have no edges and nodes in common. If S is a subgraph of G and each node of G is a node of S , then S is a spanning subgraph of G .

A directed graph G is regular of degree k if $\rho(i) = \rho^*(i) = k$ for each $i \in V$. A directed circuit of G is a regular subgraph of degree 1. A directed circuit is of length m if the number of edges contained in the directed circuit is m . A directed circuit of length 1 is called a directed self-loop (or simply a self-loop).

A directed path P_{ij} is a sequence of the form

$$P_{ij} = (i, k_1)(k_1, k_2)(k_2, k_3) \cdots (k_m, j)$$

where i, j and k_t , $t = 1, \dots, m$ are nodes in V . It is not required that all the nodes of P_{ij} shall be distinct. If they are, P_{ij} is said to be a directed simple path. Again node i is called the initial node of P_{ij} and node j the terminal node. Both nodes i and j are referred to as end nodes.

To every directed linear graph G there is an associated undirected graph G_u whose edges are the same as those in G but with directions omitted and parallel edges combined. G_u is said to be connected if, for any two nodes i and j , there exists an undirected path in G_u with these two nodes as end nodes. A component of G_u is a maximal connected subgraph of G_u . G is connected if G_u is connected. A subgraph H of G is called a component of G if H is a maximal connected

subgraph of G . An even component is a component which contains an even number of edges. An isolated node is considered as an even component.

An n -factor of G is a spanning subgraph of G which is regular of degree n . More specifically, a 1-factor is a set of directed disjoint circuits which include all nodes of G . Sometimes 1-factors are also referred to as connections in literature [3,5].

For the sake of later analysis the following operations and notation will be used without any further explanation.

$A_1 \vee A_2$ (union) = elements contained either in A_1 or A_2 or both

$A_1 \wedge A_2$ (intersection) = elements contained both in A_1 and A_2

$A_1 - A_2$ (minus) = elements contained in A_1 but not in A_2

$\mu(A_1)$ = the number of elements contained in A_1

where A_1 and A_2 are subsets of some set A .

3. Modifications of Flow Graphs and Network Determinants

It is always possible to associate a directed linear graph, called a flow graph [3,5], with a given square matrix $A = [a_{ij}]$ of order n . Each row (or column) is represented by a node and is labelled by one of the integers from 1 to n such that the node labelled k is associated with k^{th} row (or k^{th} column). If $a_{ji} \neq 0$, there is an edge (i,j) directed from i to j with associated weight a_{ji} . For a more compact description of a flow graph, the notation of 3-tuple $G(V,E,f)$ is used where V is a set of nodes; E is a set of directed edges; and f is a mapping function from E to the complex field such that $f((i,j)) = a_{ji}$ for all $i,j \in V$. Also it is convenient to extend the mapping

function f from a single edge $(i,j) \in E$ to any subgraph R of $G(V,E,f)$ such that

$$f(R) = \prod f((t,k))$$

where the product is taken over all $(t,k) \in R$.

Coates has given a topological formula for the graph gain of a flow graph [3,5]. The reason for its inefficiency in the calculation of the electrical network gain is mainly due to the fact that there exists a large number of cancellations. This is best illustrated by the following example. Consider the network shown in Figure 1(a). The corresponding flow graph is shown in Figure 1(b).

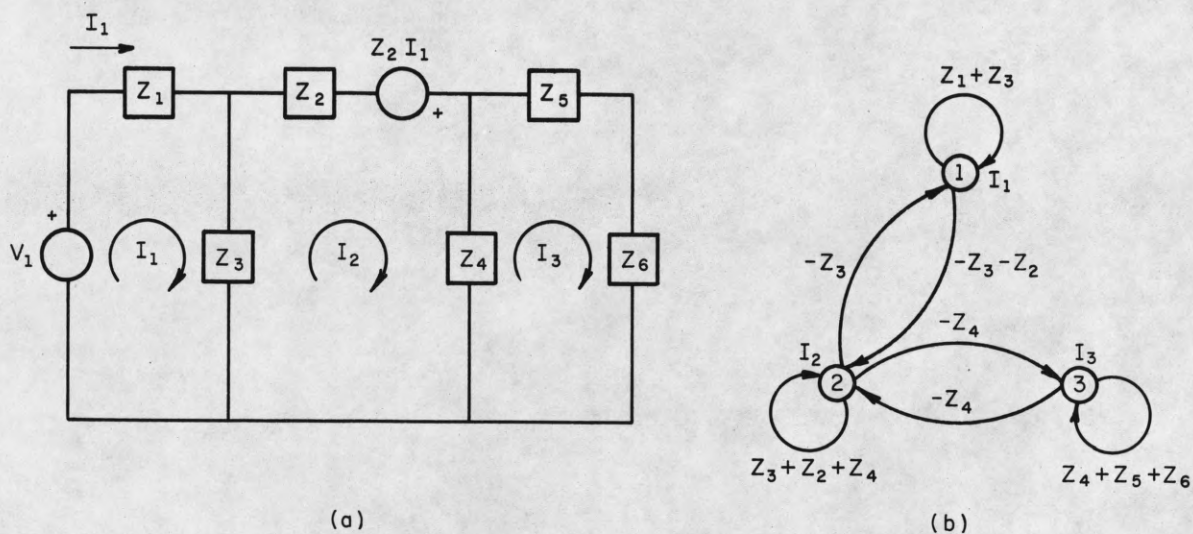


Figure 1. The Illustrative Example.

The corresponding network determinant Δ is given by

$$\begin{aligned}\Delta &= (Z_1+Z_3)(Z_3+Z_2+Z_4)(Z_4+Z_5+Z_6) - (-Z_3)(-Z_3-Z_2)(Z_4+Z_5+Z_6) \\ &\quad - (-Z_4)(-Z_4)(Z_1+Z_3) \\ &= Z_1(Z_2+Z_3)(Z_4+Z_5+Z_6) + (Z_1+Z_3)Z_4(Z_5+Z_6).\end{aligned}$$

At this point one readily realizes that the main reason for the existence of such cancellations is because the weights associated with the self-loops contain some of the weights of the edges incident to that particular node. In order to obtain a more efficient formula, the modified flow graph $G'(V', E', f')$ of a given flow graph $G(V, E, f)$ is obtained as follows:

$$\begin{aligned}V' &= V \\ E' &= E \\ f'((i, j)) &= f((i, j)) \quad \text{for } i \neq j \\ &= \sum_{k=1}^n f((k, j)) \quad i = j.\end{aligned}$$

This amounts to the same thing that the weights associated with the edges (i, i) , $i = 1, 2, \dots, n$ have been changed to the sums of weights originally incident to the node i , $i = 1, 2, \dots, n$. In Figure 1(b), the weights associated with the self-loops $(1, 1)$, $(2, 2)$, $(3, 3)$ have been changed to Z_1 , 0 , Z_5+Z_6 , respectively.

Definition 1: A semi-factor R of a flow graph $G(V, E, f)$ is a spanning subgraph of $G(V, E, f)$ which does not contain any directed circuits of length ≥ 2 , and such that $\rho^*(i) = 1$ for all nodes i in R .

The following theorem is a direct consequence of the above definition.

Theorem 1: Each component of a semi-factor R of a flow graph $G(V, E, f)$ contains exactly one self-loop. Furthermore, the number of edges in R is $\mu(V)$.

The following theorem will be proved.

Theorem 2: Suppose $G'(V', E', f')$ is the modified flow graph of a given flow graph $G(V, E, f)$. Then the determinant (denoted by $\det G$) of the matrix associated with $G(V, E, f)$ can be obtained by

$$\det G = \sum_{R'} (-1)^{q'} f'(R')$$

where R' is a semi-factor of $G'(V', E', f')$; q' is the number of the even components in R' ; and the summation is over all possible $R' \in G'(V', E', f')$.

The proof of this theorem is contained in Appendix II.

It is interesting to note that if one associates each component of R' either a plus or a minus sign, according to whether the component is odd or even, then the resulting product will give the same sign as $(-1)^{q'}$.

The modified flow graph $G'(V', E', f')$ of Figure 1(b) is shown in Figure 2. The semi-factors of $G'(V', E', f')$ are shown in Figure 3. By Theorem 2, one gets

$$\begin{aligned} \det G = \Delta &= Z_1(Z_3+Z_2)Z_4 + Z_3Z_4(Z_5+Z_6) \\ &+ Z_1(Z_5+Z_6)Z_4 + Z_1(Z_5+Z_6)(Z_3+Z_2) \\ &= Z_1(Z_2+Z_3)(Z_4+Z_5+Z_6) + (Z_1+Z_3)Z_4(Z_5+Z_6). \end{aligned}$$

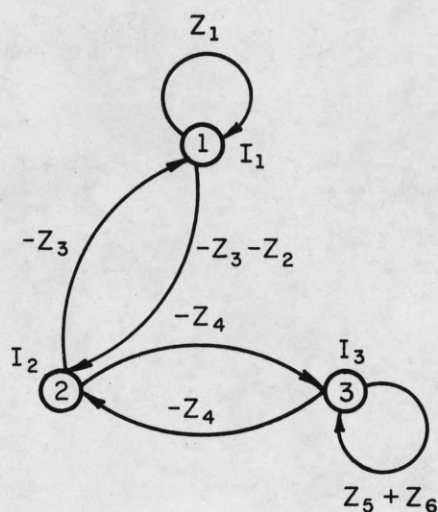


Figure 2. The Modified Flow Graph $G'(V', E', f')$ of Figure 1(b).

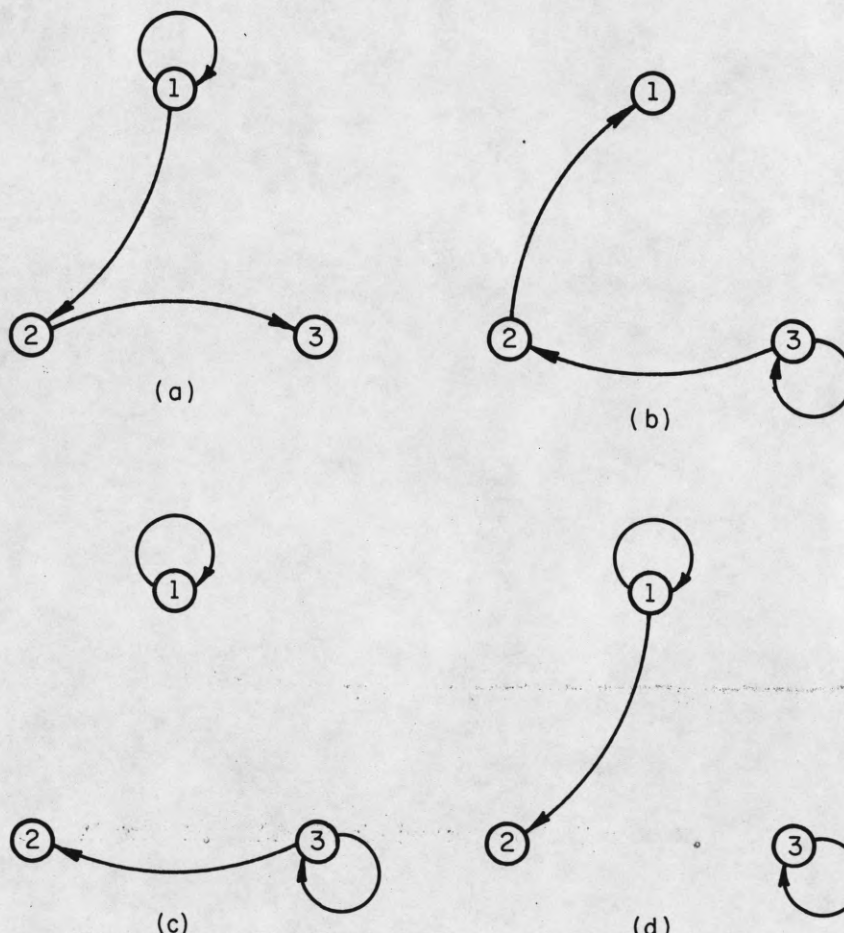


Figure 3. The Semi-Factors of Figure 2.

One immediately observes that the cancellations due to the passive elements do not exist. In general, there may be some cancellations due to the active elements, but in this particular example such terms do not appear. In case the given network is passive, this method does not calculate any superfluous terms as do many others [1-4].

In order that the above theorem may be used effectively, the choice of variables is very important. One way to accomplish this is to draw the corresponding flow graph either from the node-admittance matrix or the loop-impedance matrix of a given network. However, the restriction imposed on the choice of the variables is not serious, and this is the price one has to pay for a more efficient formula.

Definition 2: A subgraph denoted by $R^{j_1 j_2 \dots j_k}$ of a flow graph $G(V, E, f)$ is said to be a k-semi-factor of $G(V, E, f)$ if when the edges (j_1, j_1) , $(j_2, j_2), \dots, (j_k, j_k)$ are added to $R^{j_1 j_2 \dots j_k}$ the resulting graph becomes a semi-factor of $G(V, E \cup (j_1, j_1) \cup (j_2, j_2) \cup \dots \cup (j_k, j_k), f)$ where $E \cup (j_1, j_1) \cup (j_2, j_2) \cup \dots \cup (j_k, j_k)$ is the set union of E and the edges $(j_1, j_1), (j_2, j_2), \dots, (j_k, j_k)$.

Consequently, each component of $R^{j_1 j_2 \dots j_k}$ contains either a self-loop or no self-loop at all. The number of components which do not contain any self-loop is precisely k .

Theorem 3: Suppose $G'(V', E', f')$ is the modified flow graph of a given flow graph $G(V, E, f)$, then the i^{th} row, j^{th} column cofactor Δ_{ij} of $\det G$ is given by

$$\Delta_{ij} = \sum_{R_j^i} (-1)^{q_i - 1} f'(R_j^i)$$

where R_j^i is a 1-semi-factor of $G'(V', E', f')$ where the subscript j indicates that the nodes i and j are contained in the same component of R_j^i ; q_i' is the number of the even components in R_j^i ; and the summation is taken over all $R_j^i \in G'(V', E', f')$.

Proof: Let $G''(V'', E'', f'')$ be the flow graph obtained from $G'(V', E', f')$ in the following way

$$V'' = V'$$

$$E'' = E' - \{(k, i) \mid (k, i) \in E' \text{ and } k \neq i, j\}$$

$$\begin{aligned} f''((t, u)) &= f'((t, u)) \quad \text{for } u \neq i \\ &= 1 \quad \text{for } t = u = i \text{ or} \\ &\quad t = j, u = i. \end{aligned}$$

It follows that

$$\Delta_{ij} = \det G''.$$

By Theorem 2 one obtains

$$\Delta_{ij} = \sum_{S''} (-1)^{q''} f''(S'')$$

where S'' is a semi-factor in $G''(V'', E'', f'')$; q'' is the number of the even components in S'' and the summation is taken over all $S'' \in G''(V'', E'', f'')$.

Since S'' must be one of the following two forms

$$S'' = S^* \vee (k, i) \quad k = j \text{ or } i$$

where S^* is the subgraph obtained from S'' by the removal of the edge (j, i) (or (i, i)), one readily realizes that the corresponding subgraph S' of S^* in $G'(V', E', f')$ forms a 1-semi-factor in $G'(V', E', f')$ such that

$$f''(S'') = f'(S').$$

Next, consider a semi-factor of the form $S^*V(j,i) \in G''(V'', E'', f'')$. There always exists a unique semi-factor $S^*V(i,i) \in G''(V'', E'', f'')$ such that

$$(-1)^{q_1} f''(S^*V(j,i)) = (-1)^{q_1+1} f''(S^*V(i,i)) \quad i \neq j$$

where q_1 is the number of the even components in $S^*V(j,i)$. Conversely, if $S^*V(i,i)$ is a semi-factor of $G''(V'', E'', f'')$ such that nodes i and j are not contained in the same component, then there always exists a unique semi-factor $S^*V(j,i) \in G''(V'', E'', f'')$ with the above property. Therefore, it follows that the only terms which will appear in the final expansion of Δ_{ij} are those semi-factors $S^*V(i,i)$ with nodes i and j belonging to the same component.

Finally, observe the one-to-one correspondence between the uncanceled semi-factors $S^*V(i,i) \in G''(V'', E'', f'')$ and 1-semi-factors $R_j^1 \in G'(V', E', f')$. Observe also that the removal of the edge (i,i) from $S^*V(i,i)$ increases or decreases the number of the even components of $S^*V(i,i)$ by one, so it follows:

$$(-1)^{q_1} f''(S^*V(i,i)) = (-1)^{q_1-1} f'(R_j^1)$$

This completes the proof of this theorem.

Again, consider the example in Figure 2. The 1-semi-factors of the forms R_2^1 and R^1 are shown in Figure 4 and Figure 5, respectively. Therefore, one gets

$$\Delta_{12} = \sum_{R_2^1} (-1)^{q_1-1} f'(R_2^1) = (Z_3+Z_2)(Z_5+Z_6) + (Z_3+Z_2)Z_4 = (Z_3+Z_2)(Z_4+Z_5+Z_6)$$

$$\Delta_{11} = \sum_{R^1} (-1)^{q_1-1} f'(R^1) = (Z_5+Z_6)Z_4 + (Z_3+Z_2)Z_4 + (Z_3+Z_2)(Z_5+Z_6) = (Z_5+Z_6)(Z_2+Z_3+Z_4) + Z_4(Z_3+Z_2).$$

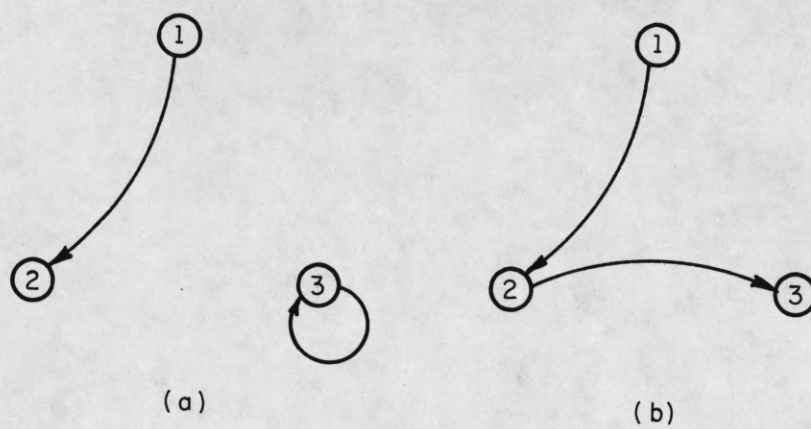


Figure 4. The 1-semi-factors R_2^1 .

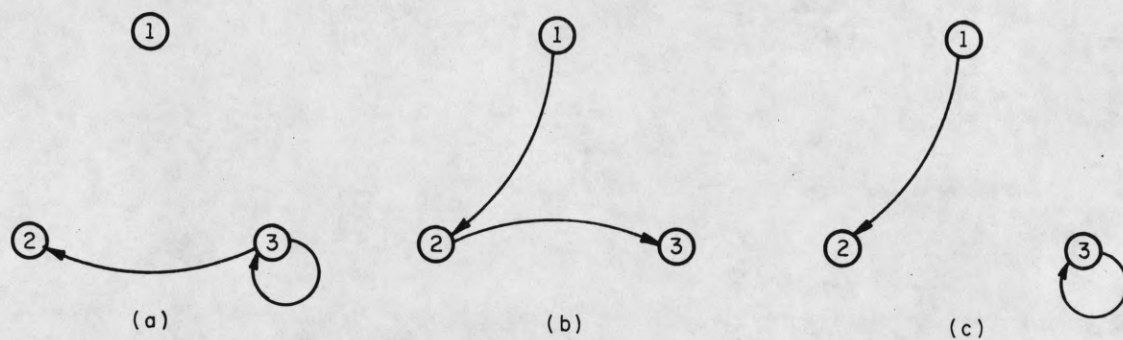


Figure 5. The 1-semi-factors R^1 .

4. The Enumeration of Semi-factors and 1-semi-factors

Let $G'(V', E', f')$ be the modified flow graph of a given flow graph $G(V, E, f)$ and C_1 be the Cartesian product of the subsets S_1 , i.e.,

$$C_1 = \prod_{k=1}^n S'_k$$

where $S'_1 = \{(t, i) \mid (t, i) \in E'; t = 1, 2, \dots, n\}$. It follows that each $c \in C_1$ is either a semi-factor or a subgraph which contains at least one directed circuit of length ≥ 2 . By eliminating the latter from C_1 it can easily be shown that the remaining elements of C_1 will give all possible semi-factors of $G'(V', E', f')$. Similarly, the 1-semi-factors of the form R^j are contained in

$$C_2 = \prod_{\substack{k=1 \\ k \neq j}}^n S'_k.$$

Consider the graph in Figure 2. One gets

$$C_1 = \{(1, 1), (2, 1)\} \times \{(3, 2), (1, 2)\} \times \{(2, 3), (3, 3)\}.$$

The semi-factors are

$$(1, 1)(3, 2)(3, 3); (1, 1)(1, 2)(2, 3); (1, 1)(1, 2)(3, 3); (2, 1)(3, 2)(3, 3).$$

Elements in C_1 which contain at least one directed circuit of length ≥ 2 are

$$(1, 1)(3, 2)(2, 3); (2, 1)(1, 2)(3, 3); \\ (2, 1)(3, 2)(2, 3); (2, 1)(1, 2)(2, 3).$$

Next, consider the product

$$C_2 = \{(1, 2), (3, 2)\} \times \{(2, 3), (3, 3)\}.$$

The 1-semi-factors are

$$(1,2)(2,3); (1,2)(3,3); (3,2)(3,3),$$

and $(3,2)(2,3)$ is the only element in C_2 which contains a directed circuit of length ≥ 2 .

5. Conclusions

It has been shown that the efficient formulas can be obtained for the modified flow graphs. The modifications are simple and the formulas are also very compact. It is true that a "best method" depends upon one's familiarity, and in fact most people are so familiar with the existing formulas of a flow graph, they probably do not like the idea of modifications. Nevertheless, this is a new approach to the problems. For a given system one technique may work better than another. It is always better to know two ways of solving a problem rather than one, for then one can choose a particular approach or combination of approaches, so as to solve the problem at hand in the simplest and most satisfying manner.

The extension of the formulas for the cofactors to minor determinants of any order can be easily obtained. The results (corresponding to k -semi-factors) are only trivially different from those discussed in Section 3. Therefore they will not be repeated here.

APPENDIX I

The following theorem has been shown by Coates [3,5].

Theorem A: Suppose $G(V,E,f)$ is the associated flow graph of a matrix $A_{n \times n}$. Then

$$\det A = (-1)^n \sum_h (-1)^{L_p} P_{f(h)}$$

where h is a 1-factor in $G(V,E,f)$; L_p is the number of directed circuits in h ; and the summation is taken over all $h \in G(V,E,f)$.

There is an alternative way of finding the signs associated with each h , and the result is given below without proof [12].

$$(-1)^n (-1)^{L_p} P_{f(h)} = (-1)^q P_{f(h)}$$

where q is the number of even components in h .

This result is useful in the sense that one can associate each component of h either a plus or a minus sign, according to whether the component is odd or even.

APPENDIX II

Outline of a Proof of Theorem 2

Let $A = [a_{ij}]_{n \times n}$ be either the node-admittance or loop-impedance matrix of a given electrical network, and $G^*(V^*, E^*, f^*)$ be the flow graph constructed as follows:

- (1) Each row (or column) is represented by a node and is labelled by one of the integers from 1 to n such that the node labelled k is associated with k^{th} row (or k^{th} column);
- (2) If $a_{ji} \neq 0$, $i \neq j$, there is an edge (i, j) directed from node i to node j and also a self-loop denoted by $(j, j)_i$ at j such that

$$\begin{aligned} f^*((i, j)) &= a_{ji} \\ f^*((j, j)_i) &= -a_{ji} \end{aligned} \quad i, j = 1, 2, \dots, n \text{ and } i \neq j$$

- (3) If $\sum_{x=1}^n a_{ix} \neq 0$, there is a self-loop $(i, i)_i$ at i such that

$$f^*((i, i)_i) = \sum_{x=1}^n a_{ix} \quad i = 1, 2, \dots, n.$$

In the light of Theorem A in Appendix I, it is easy to show that the following lemma holds.

Lemma 1: $\det A = \det G^* = \sum_{h^*} (-1)^{q^*} f^*(h^*)$

where h^* is a 1-factor of $G^*(V^*, E^*, f^*)$; q^* is the number of the even components in h^* ; the summation is over all $h^* \in G^*(V^*, E^*, f^*)$; and $\det G^*$ is the determinant of the matrix associated with $G^*(V^*, E^*, f^*)$.

In fact, Lemma 1 still holds in case $G^*(V^*, E^*, f^*)$ contains parallel edges.

Lemma 2: Suppose h_1 is a 1-factor of $G^*(V^*, E^*, f^*)$ such that it contains at least one directed circuit of length ≥ 2 , then there exists a 1-factor $h_2 \in G^*(V^*, E^*, f^*)$, $h_2 \neq h_1$, such that $f^*(h_1) = -f^*(h_2)$.

Proof: By Lemma 1 it is always possible to write

$$h_1 = (i_1, i_1)_{k_1} \cdots (i_t, i_t)_{k_t} P_1^{(1)} P_1^{(2)} \cdots P_1^{(s)}$$

where $P_1^{(u)}$, $u = 1, \dots, s$ are directed disjoint circuits of $G^*(V^*, E^*, f^*)$ and such that $P_1^{(1)}$ has the form

$$P_1^{(1)} = (j_1, j_2)(j_2, j_3) \cdots (j_m, j_1)$$

where t, s, m are appropriate integers.

There always exists a 1-factor $h_2 \in G^*(V^*, E^*, f^*)$ such that

$$h_2 = (i_1, i_1)_{k_1} \cdots (i_t, i_t)_{k_t} [(j_2, j_2)_{j_1} (j_3, j_3)_{j_2} \cdots (j_1, j_1)_{j_m}] P_1^{(2)} \cdots P_1^{(s)}.$$

The mapping of h_1 and h_2 with associated signs are

$$f^*(h_1) = (-1)^{q_1} (-1)^{t a_{i_1 k_1} \cdots a_{i_t k_t}} [(-1)^{m-1} a_{j_2 j_1} a_{j_3 j_2} \cdots a_{j_1 j_m}] f^*(P_1^{(2)} \cdots P_1^{(s)})$$

and

$$f^*(h_2) = (-1)^{q_1} (-1)^{t+m a_{i_1 k_1} \cdots a_{i_t k_t} a_{j_2 j_1} a_{j_3 j_2} \cdots a_{j_1 j_m}} f^*(P_1^{(2)} \cdots P_1^{(s)})$$

where q_1 is the number of the even components in $P_1^{(2)} \cdots P_1^{(s)}$. The lemma follows immediately.

Lemma 3: Let A_1 be the set of all 1-factors of $G^*(V^*, E^*, f^*)$ each of which contains at least one directed circuit of length ≥ 2 ; also let A_2 be the set of all 1-factors of $G^*(V^*, E^*, f^*)$ each of which contains only self-loops of the form

$$(i_1, i_1)_{k_1} (i_2, i_2)_{k_2} \cdots (i_j, i_j)_{k_j}$$

such that there exists a set of subindices $k_{t_1} k_{t_2} \cdots k_{t_r}$, $r \geq 2$, of k 's and also a corresponding set of subindices $i_{t_1} i_{t_2} \cdots i_{t_r}$ of i 's with $i_{t_{\rho+1}} = k_{t_\rho}$, $\rho = 1, \dots, r-1$ and $i_{t_1} = k_{t_r}$ where j and r are appropriate integers; then

$$f^*(A_1 \vee A_2) = 0.$$

Proof: Let

$$h_i^u = (j_1, j_1)_{k_1} \cdots (j_t, j_t)_{k_t} p_i^1 p_i^2 \cdots p_i^\alpha$$

where P_i^m , $m = 1, \dots, \alpha$ are directed disjoint circuits of length at least ≥ 2 ; t, α, i, u are appropriate integers, and $h_i^u \in A_1 \vee A_2$ be such that there exists no $h_i^{u'}$ $\in A_1 \vee A_2$ which has the property

$$h_i^{u'} = (j_1, j_1)_{k_1} \cdots (j_{t'}, j_{t'})_{k_{t'}} p_i^1 p_i^2 \cdots p_i^{\alpha+\beta}$$

with $\beta \geq 1$ and $t' \leq t-2$ where β and t' are some positive integers.

For each edge (i, j) , $i \neq j$, $\in G^*(V^*, E^*, f^*)$ there exists a unique corresponding self-loop $(j, j)_i$ also $\in G^*(V^*, E^*, f^*)$. If \tilde{P}_i^m is defined to be a graph obtained from P_i^m by replacing edges in P_i^m with their corresponding unique self-loops, then the elements in $A_1 \vee A_2$ can always be grouped into a set S of subsets such that for each $S_i \in S$

$S_i = \{h_i^k | h_i^k \in A_1 \vee A_2, \text{ and } h_i^k = (j_1, j_1)_{k_1} \cdots (j_t, j_t)_{k_t} \tilde{p}_i^{i_1}$
 $\tilde{p}_i^{i_2} \cdots \tilde{p}_i^{i_x} p_i^{j_1} p_i^{j_2} \cdots p_i^{j_{\alpha-x}}, \text{ where } i_1 i_2 \cdots i_x \text{ and } j_1 j_2 \cdots j_{\alpha-x}$
 are the complementary indices of the integers $1 \ 2 \cdots \alpha$
 in h_i^u ; all other notation is defined the same as in h_i^u }.

Since all the 1-factors in $A_1 \vee A_2$ are distinct it follows that $S_i \wedge S_j = \varnothing$ for $i \neq j$.

It is easy to show that there are 2^α 1-factors in S_i . Half of them contains p_i^1 , say, the other half does not. By replacing p_i^1 by \tilde{p}_i^1 in the former half, one obtains the latter half. Therefore, it is always possible to form $2^{\alpha-1}$ disjoint pairs $(h_i^{k_1}, h_i^{k_2})$ such that (by lemma 2)

$$f^*(h_i^{k_1}) = -f^*(h_i^{k_2}) \quad \text{for all pairs in } S_i$$

or

$$f^*(S_i) = 0.$$

This, in turn, implies $f^*(S) = f^*(A_1 \vee A_2) = 0$.

Lemma 4: Suppose B_1 is the set of all 1-factors of $G^*(V^*, E^*, f^*)$, and B_2 is the set of all semi-factors of $G^i(V^i, E^i, f^i)$, then there exists a one-to-one correspondence between the elements in the set $B_1 - A_1 - A_2$ and the elements in the set B_2 . Furthermore, the mappings of the corresponding elements $h^* \in B_1 - A_1 - A_2$ and $R^i \in B_2$ are such that

$$f^*(h^*) = (-1)^{q^i} f^i(R^i)$$

where q^i is the number of even components in R^i .

Proof: If one associates each edge $(j,j)_i \in E^*$ with an edge $(i,j) \in E'$ for all $i \neq j$, and $(i,i)_j \in E^*$ with $(i,i) \in E'$ for all $i = j$, the first part of the theorem follows immediately. Since

$$\begin{aligned} f*((j,j)_i) &= -f'((i,j)) && \text{for } i \neq j \\ &= f'((i,i)) && \text{for } i = j \end{aligned}$$

by Theorem 1, it is possible to associate each component of R' either a plus or a minus sign, according as the component is odd or even. This completes the proof of this lemma.

At this point it is obvious that Theorem 2 is a direct consequence of Lemma 1 and Lemma 4.

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PART II

FLOW GRAPHS AND BIPARTITE GRAPHS

1. Introduction

In a recent communication, Chow and Roe [1] introduced a new graph called the "matrix graph" in connection with the solution of a system of linear equations. However, this method is not practical in the sense that it is difficult to find all possible "simple subgraphs" even for a system with moderate order. Furthermore, the "matrix graphs" are actually the well-known bipartite graphs in literature [2-6]. The purpose of this note is to develop some relations and properties between flow graphs and bipartite graphs, and use bipartite graphs as an intermediate step in the simplification of the corresponding flow graphs.

2. Definitions and Notation

A directed (linear) graph (denoted by $G(V, E, f)$ or simply G) consists of a node set V and an edge set E of ordered pairs of the form (i, j) , $i, j \in V$; and a mapping function f with domain in E and range in the complex field. i is called the initial node and j the terminal node.

For any $i \in V$ the notation $\rho(i)$ and $\rho^*(i)$ denote the cardinals of the sets of edges of G having i as initial and terminal nodes, respectively. If Q is a subgraph of G and each node of G is a node of Q , then Q is a spanning subgraph of G . A directed graph is regular of degree k if $\rho(i) = \rho^*(i) = k$ for all $i \in V$. An n -factor of G is a regular subgraph of degree n . For an undirected graph H the symbol $p(i)$ will be used to indicate the cardinal of the set of edges of H incident at node i . An undirected graph H is regular of degree k if $p(i) = k$ for

all i in H . A matching subgraph[†] of H is a spanning subgraph of H which is regular of degree 1. For both directed and undirected graphs, the notation $\mu(A)$ denotes the number of nodes contained in the node set A .

3. Bipartite Graphs Associated with Flow Graphs

A bipartite graph [6] is a linear graph $B(V'_1, V'_2, E', f')$ (or simply B) in which the node set decomposes into two disjoint sets V'_1 and V'_2 such that each edge $(i', j') \in E'$ connects a node $i' \in V'_1$ with a node $j' \in V'_2$ where E' is a set of undirected edges and f' is a mapping function. This special type of graph plays an important role in the analysis of the associated flow graphs of the matrices.

For any flow graph $G(V, E, f)$ there is an equivalent representation as a bipartite graph $B(V'_1, V'_2, E', f)$: to the node set V of $G(V, E, f)$, one constructs a replica V'_2 which is in a one-to-one correspondence with $V = V'_1$. For each edge $(i', j') \in E'$ if and only if there exists an edge $(i, j) \in E$. The mapping function f' is defined by

$$f'((i', j')) = f((i, j)) \text{ for all edges } \in E' \text{ and } E$$

Conversely, it is evident that when there is a bipartite graph B with $\mu(V'_1) = \mu(V'_2)$ there is a one-to-one correspondence between edges, it can be represented as a flow graph in V . It is always possible to associate a matrix with a bipartite graph [4].

4. Relations Between Flow Graphs and Bipartite Graphs

Definition 1: An edge of $B(V'_1, V'_2, E', f')$ is said to be essential if and only if it is contained at least in one of the matching subgraphs

[†]Chow calls this a "simple graph." However, the above definition is more commonly used in literature [4,5].

of B. Otherwise it is a non-essential edge.

Theorem 1: Suppose $B(V'_1, V'_2, E', f')$ is the corresponding bipartite graph of a flow graph $G(V, E, f)$, then any subgraph Q of G is an n -factor if and only if the corresponding subgraph C of Q in $B(V'_1, V'_2, E', f')$ is a regular graph of degree n .

Proof: Suppose Q is an n -factor of G , it is obvious that C is a spanning subgraph of B . Suppose C is not regular of degree n , then there exists a node i of degree $> n$ in C . The degree of the corresponding node i in Q is such that either $\rho(i) > n$ or $\rho^*(i) > n$ which contradicts the assumption that Q is an n -factor. Therefore C is a regular subgraph of degree n . Similarly, the converse is true.

Corollary 1: Any subgraph h of G is a 1-factor[†] in G if and only if the corresponding subgraph M of h in the corresponding bipartite graph B is a matching subgraph.

Corollary 2: Any subgraph h of G is a 2-factor in G if and only if the corresponding subgraph M of h in B forms a set of spanning non-touching circuits.

Corollary 3: An edge is essential in B if and only if the corresponding edge in G is contained at least in one of the 1-factors of G .

For any given flow graph $G(V, E, f)$ let

$$\sigma(S) = \mu(S) - \mu(R(S))$$

$$\sigma^*(S) = \mu(S) - \mu(R^*(S))$$

[†]Sometimes this is called a connection [7,8].

where S is a subset of V ; $R(S)$ is a subset of V and contains the set of all terminal nodes of edges having their initial nodes in S ; and $R^*(S)$ is also a subset of V and contains the set of all initial nodes of edges having their terminal nodes in S . $\sigma(S)$ and $\sigma^*(S)$ are usually referred to as the deficiency and the converse deficiency [2-6] of a directed graph, respectively.

A matrix A is said to be reducible if and only if it can be partitioned into the form

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square submatrices. One extends this definition also to include as reducible those matrices which can be transformed into a partitioned matrix with the above property, by interchanging the lines of A . It should be noted that the corresponding rows and columns are not required to permute simultaneously.

Definition 2: The associated flow graph $G(V, E, f)$ of a given matrix A is said to be separable if and only if the matrix A is reducible. Otherwise it is inseparable.

The definition of separability defined here for flow graphs is different from that defined in the graph theory [9].

Theorem 2: The necessary and sufficient condition that the associated flow graph $G(V, E, f)$ of a given matrix A be separable is that there exists at least one proper subset S , $S \neq \emptyset$ (empty set), of V such that either $\sigma(S)$ or $\sigma^*(S)$ is equal to or greater than zero. Furthermore,

if either $\sigma(S)$ or $\sigma^*(S)$ is greater than zero then the determinant of the matrix associated with G (or simply determinant of G) vanishes.

Proof: Suppose there exists a proper subset S , $S \neq \emptyset$, such that $\sigma(S) = \mu(S) - \mu(R(S)) \geq 0$; in terms of the rows and columns of the matrix A , $\mu(R(S))$ is the number of rows in A having at least one non-vanishing element in common with one of the columns corresponding to S . It follows that G is separable. Conversely, if G is separable then there exists at least one proper subset S , $S \neq \emptyset$, such that either $\sigma(S) \geq 0$ or $\sigma^*(S) \geq 0$. Similarly for the other case.

If either $\sigma(S) > 0$ or $\sigma^*(S) > 0$ there exists at least one line containing all zeros in A_{11} or A_{22} in addition to that either A_{12} or A_{21} is zero. This completes the proof of this theorem.

It is obvious that this theorem provides an alternative in determining whether a given matrix can be written as the direct sum of submatrices [10,11].

The deficiency functions defined for flow graphs have a corresponding meaning in bipartite $B(V'_1, V'_2, E', f')$, i.e.,

$$\delta(N'_1) = \mu(N'_1) - \mu(D(N'_1))$$

$$\delta^*(N'_2) = \mu(N'_2) - \mu(D(N'_2))$$

where N'_1 and N'_2 are subsets of V'_1 and V'_2 , respectively; $D(N'_1)$ denotes the set of all nodes in V'_2 which are joined by an edge to at least one node in N'_1 of V'_1 ; $D(N'_2)$ denotes the set of all nodes in V'_1 which are joined by an edge to at least one node in N'_2 of V'_2 .

Theorem 3: If $G(V, E, f)$ is separable and the corresponding determinant is non-zero, then it can be split into at least two unconnected components such that the union of the node sets of the components is V and furthermore the determinant of $G(V, E, f)$ is equal to the product of the determinants of the components.

Proof: Suppose $B(V'_1, V'_2, E', f')$ is the corresponding bipartite graph of $G(V, E, f)$, then by Theorem 2 there exists a proper subset $N'_1 \subset V'_1$, $N'_1 \neq \emptyset$, in $B(V'_1, V'_2, E', f')$ such that $\mu(N'_1) - \mu(D(N'_1)) = 0$. It is evident from Corollary 1 and Corollary 3 that all the edges (i', j') , $i' \in V'_1 - N'_1$, $j' \in D(N'_1)$ are non-essential, and can be removed without changing the value of the determinant of B . Since the interchange of the labels of any two nodes in V'_2 corresponds to the interchange of the corresponding columns in the associated matrix, it follows that it is always possible, by using an even number of interchanges, to rename the nodes in V'_2 in such a way that N'_1 and $D(N'_1)$ have the same designations. Consequently, the corresponding flow graph of this modified bipartite graph contains at least two unconnected components, and the theorem follows immediately.

Corollary 4: If S is a subset of V in $G(V, E, f)$ such that $\mu(S) = \mu(R(S))$ then all the edges of the form (i, j) , $i \in (V - S)$ and $j \in R(S)$, can be removed without changing the value of the determinant of G . Similar result is obtained if $\mu(S) = \mu(R^*(S))$.

5. The Determinant of the Product of Two Matrices

Let

$$A = [a_{ij}]_{n \times m} \quad C = [c_{ij}]_{m \times n} \quad m \geq n$$

be two given matrices, and $B_d(V_1, V_2, E, f)$ (or simply B_d) be a special bipartite graph such that

- (1) $\mu(V_1) = n$ and $\mu(V_2) = m$
- (2) There is a directed edge $(i, j) \in E$, $i \in V_1$ and $j \in V_2$ with $f((i, j)) = a_{ij}$ if and only if $a_{ij} \neq 0$
- (3) There is a directed edge $(i, j) \in E$, $i \in V_2$ and $j \in V_1$, with $f((i, j)) = c_{ij}$ if and only if $c_{ij} \neq 0$.

Theorem 4: $\det AC = (-1)^n \sum (-1)^q f(h)$

where h is a set of non-touching circuits in B_d which contains all the nodes in V_1 ; q is the number of circuits in h ; and the summation is over all h in B_d .

Proof: Let the nodes in V_1 be labelled from 1 to n and the nodes in V_2 from $n+1$ to $n+m$. (The rows and columns of A and C are labelled accordingly; i.e., the rows of A and the columns of C are labelled from 1 to n , and the columns of A and the rows of C from $n+1$ to $n+m$.) If $A(j_1, j_2, \dots, j_n)$ is a major determinant of A formed by the columns j_1, j_2, \dots, j_n and $C(j_1, j_2, \dots, j_n)$ is the corresponding major determinant of $A(j_1, j_2, \dots, j_n)$ in C with $j_1 < j_2 < \dots < j_n$, then

$$A(j_1, j_2, \dots, j_n) C(j_1, j_2, \dots, j_n) = \left(\sum_{k_1 k_2 \dots k_n} \epsilon_{k_1 k_2 \dots k_n}^{j_1 j_2 \dots j_n} a_{1k_1} a_{2k_2} \dots a_{nk_n} \right) \left(\sum_{i_1 i_2 \dots i_n} \epsilon_{i_1 i_2 \dots i_n}^{j_1 j_2 \dots j_n} c_{j_1 i_1} c_{j_2 i_2} \dots c_{j_n i_n} \right)$$

where $k_1 k_2 \dots k_n$ is a permutation of the integers j_1, j_2, \dots, j_n ; and $i_1 i_2 \dots i_n$ is a permutation of the integers $1, 2, \dots, n$. $\epsilon_{k_1 k_2 \dots k_n}^{j_1 j_2 \dots j_n}$ is +1 or -1 according as the permutation $k_1 k_2 \dots k_n$ is even or odd.

Similarly for $\epsilon_{i_1 i_2 \dots i_n}^{j_1 j_2 \dots j_n}$. Since

$$\begin{aligned}
& \epsilon_{i_1 i_2 \dots i_n}^{12 \dots n} \epsilon_{k_1 k_2 \dots k_n}^{j_1 j_2 \dots j_n} a_{1k_1} a_{2k_2} \dots a_{nk_n} c_{j_1 i_1} c_{j_2 i_2} \dots c_{j_n i_n} \\
& = (-1)^n \epsilon_{k_1 k_2 \dots k_n}^{12 \dots n} \epsilon_{i_1 i_2 \dots i_n}^{j_1 j_2 \dots j_n} a_{1k_1} a_{2k_2} \dots a_{nk_n} c_{j_1 i_1} c_{j_2 i_2} \dots c_{j_n i_n}
\end{aligned}$$

and all the row and column indices are distinct, respectively, by following the same argument as in the derivation of the gain of a flow graph [8], one obtains

$$A(j_1, j_2, \dots, j_n) C(j_1, j_2, \dots, j_n) = (-1)^n \sum (-1)^{q'} f(h')$$

where h' is a set of non-touching circuits which contains all the nodes $1, 2, \dots, n, j_1, j_2, \dots, j_n$; q' is the number of circuits in h' ; and the summation is taken over all possible such h' in B_d .

Since $\det AC$ is equal to the sum of the products of all corresponding major determinants, the theorem follows immediately.

Corollary 5: In Theorem 4, if $m = n$ then h becomes a 1-factor in B_d .

6. Illustrative Example:

Consider the flow graph $G(V, E, f)$ shown in Fig. 1. Since there exists non-empty proper subset S of the node set V such that

$$S = \{3, 4\}$$

$$R(S) = \{3, 2\}$$

and $\mu(S) = \mu(R(S))$

by Theorem 2 and Corollary 4, it follows immediately that G is separable and the edges $(5, 3)$, $(2, 2)$, $(1, 2)$ of G are non-essential, so they can be removed without changing the value of the determinant of G . Further-

more, since G is separable it is possible to construct (at least) two unconnected components (by Theorem 3) such that the determinant of G is equal to the product of determinants of the components. (Fig. 2)

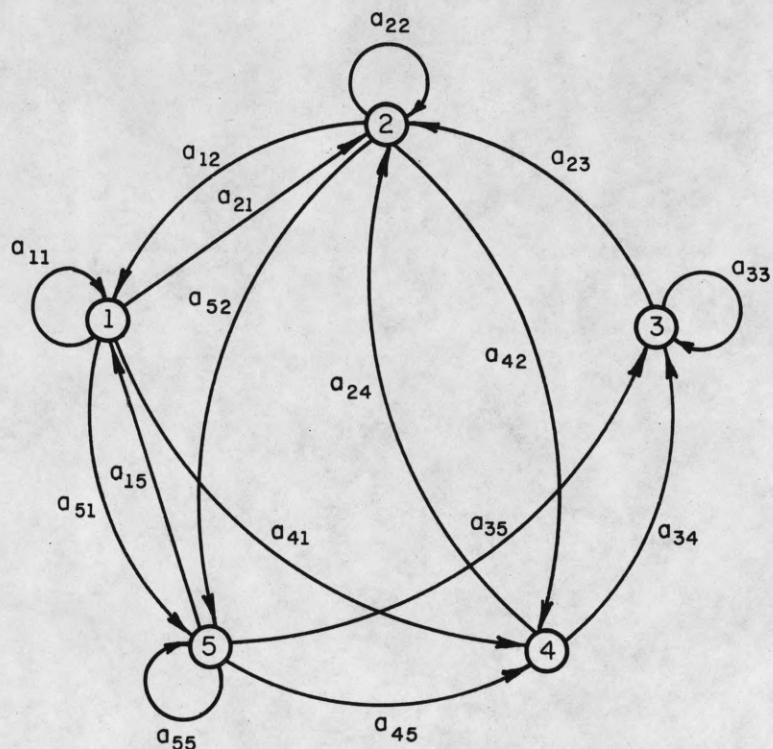


Fig. 1. The Illustrative Example

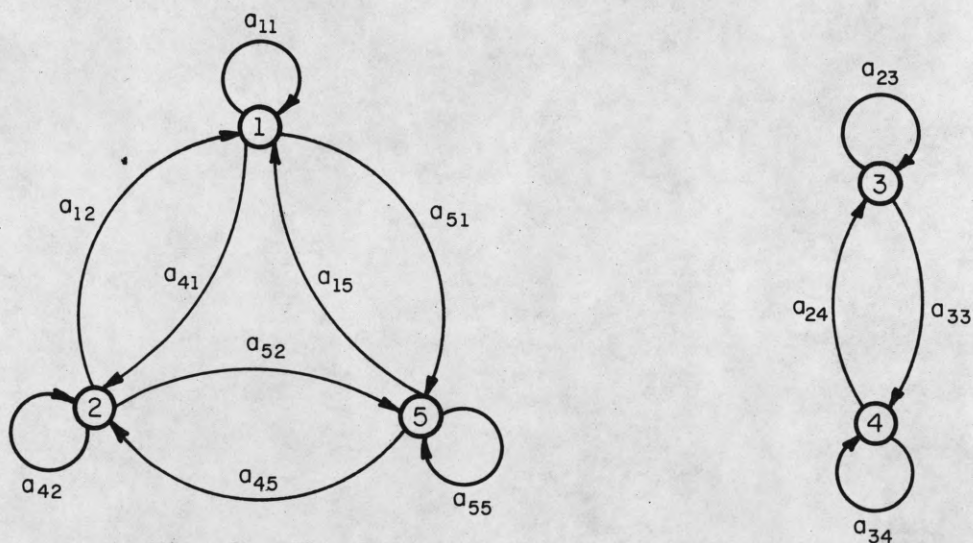


Fig. 2. The Corresponding Two Unconnected Components of Fig. 1.

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PART III

AN EXTENSION OF THE STAR-MESH TRANSFORMATION

The basic problem of analysis of combinational circuits is the determination of the relation between any given connection matrix¹ and the corresponding output matrix. To accomplish this, Hohn [1] showed first how to obtain from a given circuit an equivalent circuit using one less non-terminal node in the formation of the connection matrix. This operation (the star-mesh transformation [1]) is repeated until there is no non-terminal nodes in the accounting. Yoeli [2] generalized it to the process of multiple-node removal in an algebraic form. The purpose of this note is to point out that Yoeli's process can be accomplished topologically.

Let C be the connection matrix (not necessarily symmetrical) of a given circuit G and be partitioned in such way that

$$C = [c_{ij}]_{n \times n} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}_{n \times n}$$

where C_{11} is a square submatrix of order p which corresponds to the terminal nodes of G . For convenience, a mapping function f is defined from the branches (i,j) of G into the Boolean algebra such that

$$f((i,j)) = c_{ij} \quad \text{for all branches in } G$$

Also define

$$f(R) = \prod f((u_1, u_2))$$

where (u_1, u_2) is a branch in R , and R is any non-empty subgraph of G .

Next let

$$C_{t_1 t_2} = [c_{ij}^{t_1 t_2}] \quad t_1, t_2 = 1, 2$$

$$\{C_{22}\} = [m_{ij}]_{(n-p) \times (n-p)}$$

and

$$F = [f_{ij}]_{p \times p}$$

where $\{C_{22}\}$ is the adjoint matrix of C_{22} [2].

Yoeli [2] has shown that if F is the corresponding output matrix of order p , then²

$$F = \{C_{11} + C_{12} \{C_{22}\} C_{21}\}$$

$$\text{Suppose } C_{11} + C_{12} \{C_{22}\} C_{21} = [d_{ij}]$$

then

$$d_{ij} = c_{ij}^{11} + \sum_{k,l}^{n-p} c_{ik}^{12} m_{kl} c_{lj}^{21}$$

Since m_{kl} is the switching function from node k to node l of the circuit corresponding to C_{22} , it follows that

$$m_{kl} = \sum_{P'_{kl}} f(P'_{kl})$$

where P'_{kl} is a directed path from node k to node l in the sectional graph³

¹The definitions and notation used by Hohn [1] and Yoeli [2] will be used here.

$G[A]$ where A is the node set corresponding to the submatrix C_{22} , and the Boolean sum is taken over all possible $P'_{kl} \in G[A]$.

Therefore

$$d_{ij} = \sum_{P_{ij}} f(P_{ij}) \quad \text{for } i \neq j$$

$$= 1 \quad \text{for } i=j$$

where P_{ij} is a directed path from node i to node j in the sectional graph $G[A \cup i \cup j]$ where $A \cup i \cup j$ is the set union of the nodes i, j , and the node set A ; and the Boolean sum is taken over all possible $P_{ij} \in G[A \cup i \cup j]$.

Theorem 1: With the notation used above if G_r is the circuit obtained from G in such a way: (1) G_r only contains p terminal nodes of G ; (2) if $d_{ij} \neq 0$, there is a branch from node i to node j with associated weight d_{ij} ; then G_r and G are equivalent (same output matrix).

Corollary 1: If C_{22} is of order 1, the above theorem reduces to the star-mesh transformation in Hohn's paper [2].

At this point, it is obvious that the topological reduction process not only displays in a very intuitive manner the causal relationships among the variables of the system under study, but also shows that the process is independent of the labelling of the nodes. As a matter of fact, Theorem 1 still holds even if part of the non-terminal nodes is removed.

Example 1: Consider the circuit shown in Fig. 1(a). The dotted part is the sectional graph to be removed. The reduced circuit is shown in Fig. 1(b).

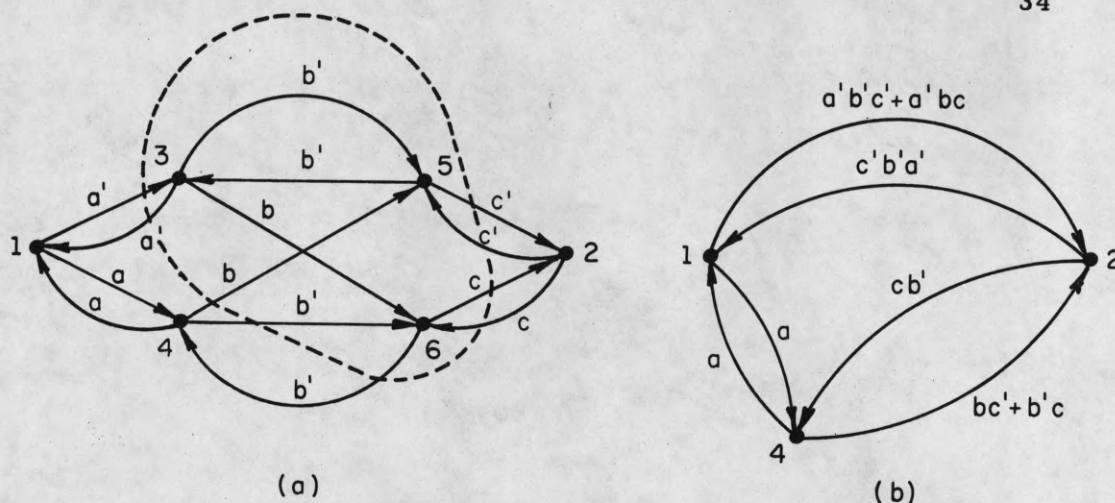


Fig. 1. (a) Example for Multiple-node Removal
(b) The Corresponding Reduced Circuit of (a)

It is interesting to note that this process can be easily applied to sequential machines to give the corresponding multiple "state removal" algorithm with minor modifications.

If $G[A]$ is the sectional graph to be removed from a state diagram G where A is a set of states and if there exist self-loops in $G[A]$, then the process shown in Fig. 2 must be used in order to eliminate all such self-loops (1 is used as identity for multiplication but $1 + b \neq 1$). k is a non-negative integer.

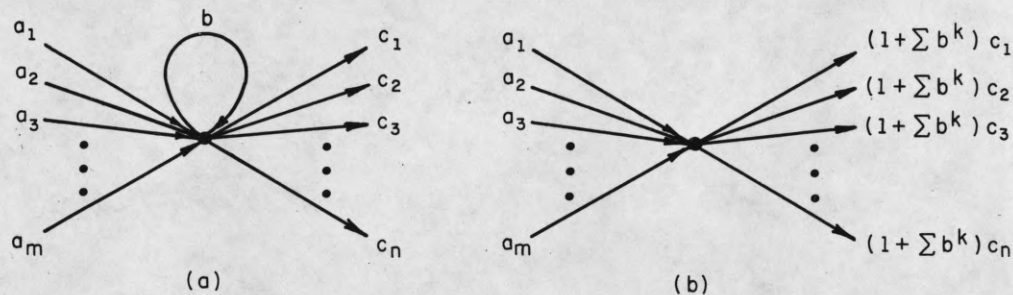


Fig. 2. (a) A Node with a Self-Loop
(b) With the Self-Loop Removed

After all the self-loops having been removed, Theorem 1 now can be applied to obtain the corresponding reduced state diagram.

Example 2: Consider the state diagram of Fig. 3(a). The dotted part is the sectional graph to be removed. Fig. 3(b) is the corresponding reduced state diagram.

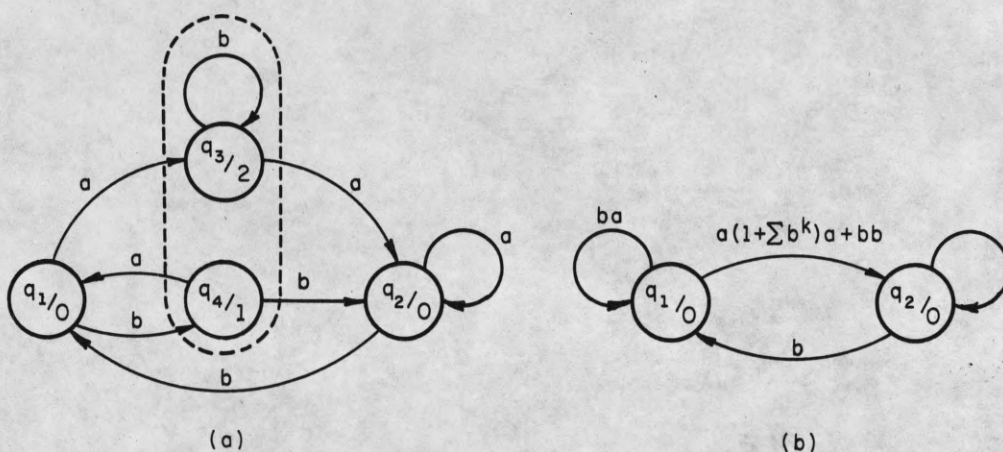


Fig. 3. Example for Multiple State Removal.

Footnotes:

1. The definitions and notation used by Hohn [1] and Yoeli [2] will be used here.
2. This result was first discovered by Shekel and published in [2]. In a recent communication, Brown [4] restated the same result for a more restricted class of elements, i.e., Boolean algebra.
3. A sectional graph (denoted by $G[A]$) of G defined by a node set A is the subgraph whose node set is A and whose branches are all those branches in G which connect two nodes in A .

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A SIMPLIFIED WAY OF SOLVING A SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

1. Introduction

Many methods of solving a system of simultaneous linear equations have been published in literature [1], but no "best method" can be recommended. For a given system, one technique may work better than the others. Therefore, it only depends upon the nature of the problems and one's familiarity with a particular method. In this paper, a new method based on the reduced flow graphs is presented.

2. Definitions and Notation

It is always possible to associate a directed linear graph, called a flow graph [2,3], with a given square matrix $A = [a_{ij}]$ of order n . Each row (or column) is represented by a node and is labelled by one of the integers from 1 to n such that the node labelled k is associated with k^{th} row (or k^{th} column). If $a_{ji} \neq 0$, there is an edge (i,j) directed from i to j with associated weight a_{ji} . For a more compact description of a flow graph, the notation of 3-tuple $G(V,E,f)$ is used where V is a set of nodes; E is a set of directed edges; and f is a mapping function from E to the complex field such that $f((i,j)) = a_{ji}$ for all $i,j \in V$. Also it is convenient to extend the mapping function f from a single edge $(i,j) \in E$ to any subgraph R of $G(V,E,f)$ such that

$$f(R) = \prod f((t,k))$$

where the product is taken over all edges $(t,k) \in R$.

For any $i \in V$ the notation $\rho(i)$ and $\rho^*(i)$ denote the cardinals of the sets of edges of G having i as initial and terminal nodes, respectively. They are called the outgoing and incoming degrees of G at i . A directed graph is regular of degree k if $\rho(i) = \rho^*(i) = k$ for each $i \in V$.

If A is a subset of V , the sectional graph (denoted by $G[A]$) of G defined by A is the subgraph whose node set is A and whose edges are all those edges in G which connect two nodes in A . When $A = V$ the sectional graph is G itself. Two subgraphs are disjoint when they have no edges and nodes in common. If S is a subgraph of G and each node of G is a node of S , then S is a spanning subgraph of G . A component of $G(V,E,f)$ is a maximal connected subgraph of $G(V,E,f)$. A connection of $G(V,E,f)$ is a spanning subgraph of $G(V,E,f)$ which is regular of degree 1. A subgraph (denoted by H_{ij}) of $G(V,E,f)$ is said to be a one-connection from node i to node j if it contains: (1) a directed path from node i to node j ; (2) a set of disjoint circuits which include all nodes of $G(V,E,f)$ except those contained in (1).

3. Main Result

It will be convenient to write the system of simultaneous (consistent) linear equations in matrix form

$$(1) AX = B$$

where $A = [a_{ij}]$ is the coefficient matrix of order n ; and X and B are the column vectors. Frequently, to save space, the column vectors X and B are written in the form $\{x_1, x_2, \dots, x_n\}$ and $\{b_1, b_2, \dots, b_n\}$, respectively.

Let $G(V, E, f)$ be the flow graph associated with the system (1), i.e., associate a flow graph to the coefficient matrix A and then attach a source node (denoted by $n+1$) to it in such a way if $b_k \neq 0$ there is an edge $(n+1, k)$ with $f((n+1, k)) = -b_k$. Also let $G_c(V_c, E_c, f_c)$ be the flow graph obtained from $G(V, E, f)$ such that

$$V_c = V$$

$$\text{and} \quad f_c((i, j)) = 0 \quad \text{for } i \leq j$$

$$f_c((i, j)) = \frac{\sum_{H_{ij}} (-1)^{q_1-1} f(H_{ij})}{\sum_h (-1)^q f(h)} \quad \text{for } i > j$$

where H_{ij} is a one-connection in the sectional graph $G[lv2v \dots vjvi]$ where $lv2v \dots vjvi$ is the node set containing the nodes $1, 2, \dots, j$ and i ; h is a connection in $G[lv2v \dots vj]$ where $lv2v \dots vj$ is the node set containing the nodes $1, 2, \dots, j$; q_1 and q are the numbers of circuits in H_{ij} and h , respectively; and the summations are taken over all H_{ij} and h in $G[lv2v \dots vjvi]$ and $G[lv2v \dots vj]$, respectively. It is also assumed that the determinants of the matrices associated with the sectional graphs $G[lv2v \dots vk]$, $k=1, \dots, n$ are non-zero.

The reduced flow graph $G_c(V_c, E_c, f_c)$ may be defined as the canonical form of $G(V, E, f)$, and usually has the form shown in Fig. 1.

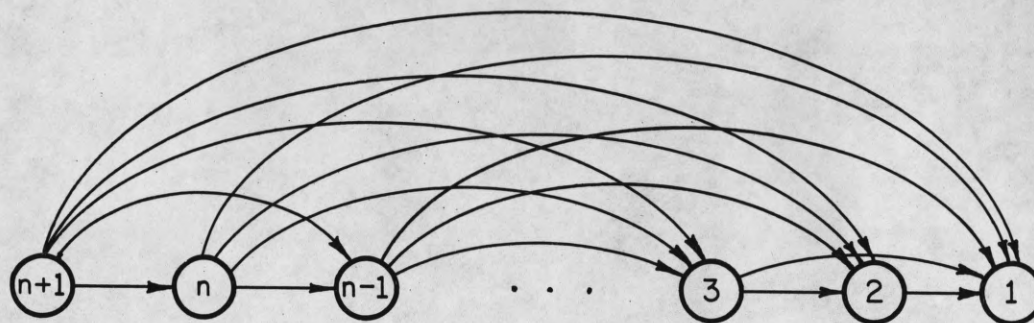


Fig. 1. The Canonical Form of $G(V, E, f)$

Theorem 1: With the same notation used above, the solution of Eq. (1) is given by

$$x_j = \sum_{P_{(n+1)j}} (-1)^{q_p} f_c(P_{(n+1)j}) \quad j=1, \dots, n$$

where $P_{(n+1)j}$ is a directed path from node $n+1$ to node $j \in G_c(V_c, E_c, f_c)$; q_p is the number of edges in $P_{(n+1)j}$; and the summation is taken over all $P_{(n+1)j} \in G_c(V_c, E_c, f_c)$.

Proof: If a row of zeros is attached to the bottom of the matrix $[A, -B]$ where $[A, -B]$ is obtained from A by attaching $-B$ to the right of A , and then an operation similar to the "sweep-out" process [4] for evaluating determinants is applied, the resultant matrix A' turns out to be

$$A' = \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \dots & \frac{a_{1j}}{a_{11}} & \dots & \frac{a_{1(n-1)}}{a_{11}} & \frac{a_{1n}}{a_{11}} & -\frac{a_{1(n+1)}}{a_{11}} \\ 0 & 1 & \dots & \frac{\Delta_2^j}{\Delta_2} & \dots & \frac{\Delta_2^{n-1}}{\Delta_2} & \frac{\Delta_2^n}{\Delta_2} & -\frac{\Delta_2^{n+1}}{\Delta_2} \\ \vdots & \vdots & & & & & & \\ 0 & 0 & \dots & 1 & \dots & \frac{\Delta_j^{n-1}}{\Delta_j} & \frac{\Delta_j^n}{\Delta_j} & -\frac{\Delta_j^{n+1}}{\Delta_j} \\ \vdots & \vdots & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \frac{\Delta_{n-1}^n}{\Delta_{n-1}} & -\frac{\Delta_{n-1}^{n+1}}{\Delta_{n-1}} \\ & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 & -\frac{\Delta_n^{n+1}}{\Delta_n} \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

where

$$\Delta_j = \det \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jj} \end{bmatrix} \quad \text{for } j=1, \dots, n$$

and

$$\Delta_i^k = \det \begin{bmatrix} a_{11} & \dots & a_{1(i-1)} & a_{1k} \\ a_{21} & \dots & a_{2(i-1)} & a_{2k} \\ \vdots & & & \vdots \\ a_{i1} & \dots & a_{i(i-1)} & a_{ik} \end{bmatrix} \quad \begin{array}{l} \text{for } k > i \\ i = 2, \dots, n \\ k = 3, \dots, n+1 \end{array}$$

with

$$a_{m(n+1)} = b_m \quad \text{for } m=1, \dots, n$$

It is obvious that the systems (1) and

$$(2) A'X' = 0$$

where $X' = \{x_1, x_2, \dots, x_n, 1\}$ are equivalent. Therefore, by Cramer's rule one gets

$$x_j = (-1)^{n-j-1} \det A'_{(n+1)j}$$

where $A'_{(n+1)j}$ is the submatrix obtained from A' by striking out the $(n+1)^{th}$ row and j^{th} column of A' . By a theorem due to Coates [2,3] (which is contained in appendix), one readily realizes that the flow graph $G'(V', E', f')$ associated with the system (2) is precisely the reduced flow graph $G_c(V_c, E_c, f_c)$ of $G(V, E, f)$ except that the weights associated with the self-loops (i, i) , $i=1, \dots, n$ are 1's rather than zeros. Therefore

$$x_j = (-1)^{n-j-1} \{ (-1)^{(n+1)+j} [(-1)^{n+1} \sum_{H'_{(n+1)j}} (-1)^{L'_\sigma-1} f'(H'_{(n+1)j})] \}$$

where $H'_{(n+1)j}$ is a one-connection from node $(n+1)$ to node j in $G'(V', E', f')$, L'_σ is the number of circuits in $H'_{(n+1)j}$; and the summation is taken over all $H'_{(n+1)j} \in G'(V', E', f')$. If $P_{(n+1)j}$ is the directed path from node $(n+1)$ to node $j \in H'_{(n+1)j}$, then

$$L'_\sigma = (n+1) - (q_p + 1)$$

where q_p is the number of edges contained in $P_{(n+1)j}$. It follows that

$$x_j = [\sum_{P_{(n+1)j}} (-1)^{q_p} f_c(P_{(n+1)j})]$$

This completes the proof.

In case some of the determinants Δ_j , $j=1,2,\dots,n-1$ are zero, it can be shown that it is always possible either by relabelling or shifting [2] the nodes of $G(V,E,f)$ in such a way that the assumption holds. If $\Delta_j = 0$ for all $j \geq k$, $0 < k \leq n$, the extension is trivial since $(n-k)$ vectors of A can be treated as B .

It should be noted that in the process of the construction of $G_c(V_c, E_c, f_c)$ the most complicated mapping term is $f_c((n+1, n))$ which corresponds to calculate $x_n \in G(V, E, f)$ by coates' method [2]. The remaining terms are obtained from the sectional graphs of $G(V, E, f)$. Therefore, if all x_j , $j=1,\dots,n$ are required, this method certainly will demonstrate its superiority. This is best illustrated by the following example.

Example 1: Consider the system of equations

$$x_1 + x_2 + x_3 + x_4 - 1 = 0$$

$$2x_1 + x_2 + 2x_3 + x_4 - 1 = 0$$

$$2x_1 + 2x_2 - x_3 + 2x_4 - 5 = 0$$

$$x_1 + 2x_2 + 3x_3 + 3x_4 - 3 = 0$$

The flow graph $G(V, E, f)$ and its canonical form $G_c(V_c, E_c, f_c)$ are shown in Fig. 2. Then, by Theorem 1, one has

$$x_4 = 3$$

$$x_3 = -1$$

$$x_2 = -(-1) + (-3)(1) = -2$$

$$\begin{aligned} x_1 &= -(-1) + (-1)(1) + (1)(1) - (-3)(1)(1) + (-3)(1) \\ &= 1 \end{aligned}$$

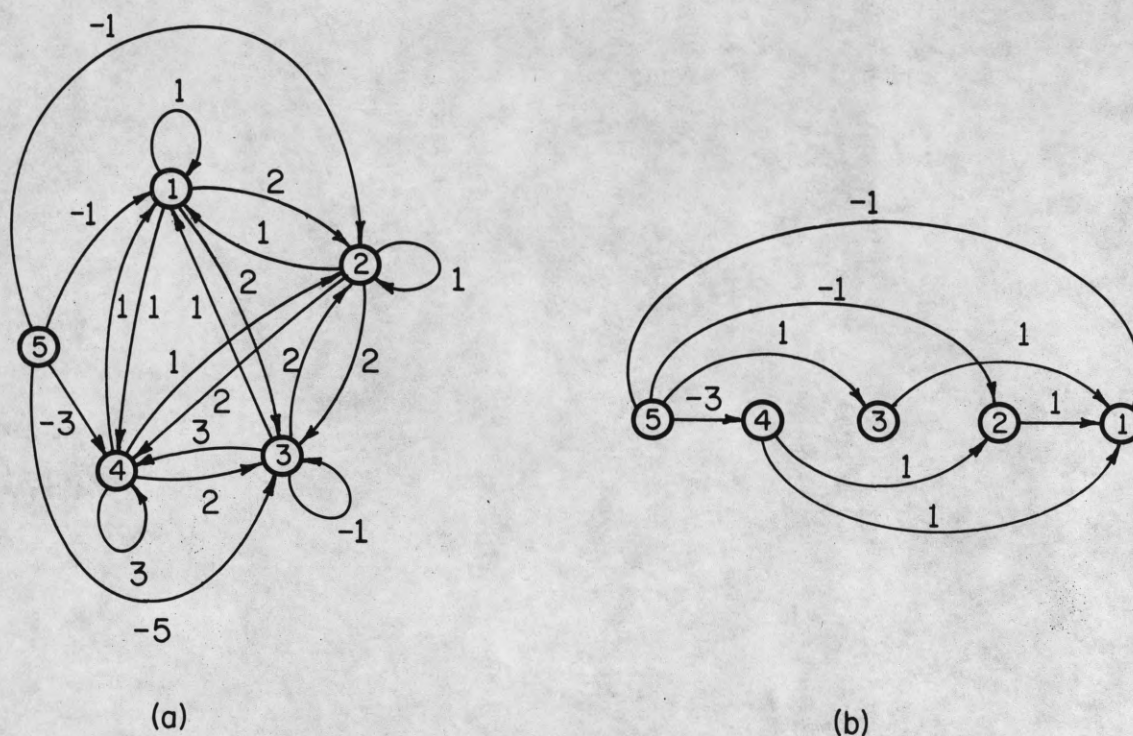


Fig. 2. The Flow Graph $G(V, E, f)$ and its Canonical Form $G_c(V_c, E_c, f_c)$ of Example 1.

4. Conclusions

The topological approach offers an alternative viewpoint which complements and enhances the more familiar classical methods of solving a system of simultaneous linear equations. It is always better to know two ways of solving a problem rather than one, for then one can choose a particular approach or combination of approaches, so as to solve the problem at hand in the simplest and most satisfying manner.

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APPENDIX

The following theorem has been shown by Coates and Desoer [2,3].

Theorem: Suppose $G(V,E,f)$ is the associated flow graph of a matrix $A_{n \times n}$ then

$$\det A = (-1)^n \sum_h (-1)^{L_p} f(h)$$

$$(-1)^{i+j} \det A_{ij} = (-1)^n \sum_{H_{ij}} (-1)^{L_\sigma - 1} f(H_{ij}) \quad \text{for } i \neq j$$

where h is a connection in $G(V,E,f)$; H_{ij} is a one-connection from node i to node j in $G(V,E,f)$; L_p and L_σ are the numbers of circuits in h and H_{ij} , respectively; and the summations are taken over all h and H_{ij} in $G(V,E,f)$. A_{ij} is obtained from A by striking out the i^{th} row and j^{th} column of A .

On Signal-Flow Graphs

Methods of simplification for signal-flow graphs have been treated extensively in the literature [1,2]. The purpose of this note is to generalize a single-node removal algorithm to a multiple-node removal algorithm.

For convenience the notation $f(R)$ will be used to represent the product of the weights associated with the edges in R where R is a subgraph of some signal-flow (or flow) graph G . If A is a subset of the node set of G , the sectional graph (denoted by $G[A]$) of G defined by A is the subgraph whose node set is A and whose edges are all those edges in G which connect two nodes in A . The following theorem is obtained for the flow-graphs [3].

Theorem 1: Suppose V is the node set of a flow graph G , and V_m is a proper subset of V such that $\det G[V_m] \neq 0$, then $\det G = K \det G_r$ with $K = \det G[V_m]$ where G_r is the reduced flow graph obtained from G by the following process:

- (1) Remove $G[V_m]$ from G , i.e., remove all nodes and edges incident to and from any node in V_m ;
- (2) The weight b_{ij} associated with the edge $(i,j) \in G_r$ is given by

$$b_{ij} = \frac{(-1)^{\eta(V_m)}}{K} \sum_{H_{ij}^m} (-1)^{q_m} f(H_{ij}^m) \quad \text{for all } i, j \in (V - V_m)$$

where H_{ij}^m is a one-connection from i to j in $G[V_m \cup i \cup j]$ where $V_m \cup i \cup j$ is the set union of the nodes i, j and the nodes in V_m ; q_m is the number of the loops in H_{ij}^m ; $\eta(V_m)$ represents the number of nodes in V_m ; and the summation is taken over all $H_{ij}^m \in G[V_m \cup i \cup j]$. when $i = j$, $G[V_m \cup i \cup j]$ reduces to $G[V_m \cup i]$ and the one-connections from i to j become the connections in $G[V_m \cup i]$ by definition.

A similar result is obtained for the signal-flow graphs.

Theorem 2: Suppose V' is the node set of a signal-flow graph G' , and V'_m is a proper subset of V' such that $\det G'[V'_m] \neq 0$, then $\det G' = K' \det G'_r$ with $K' = \det G'[V'_m]$ where G'_r is the reduced signal-flow graph obtained from G' by the following process:

- (1) Remove $G'[V'_m]$;
- (2) The weight b'_{ij} associated with the edge $(i,j) \in G'_r$ is given by

$$b'_{ij} = \frac{1}{K'} \sum_k P_k \Delta_k \quad \text{for all } i, j \in (V' - V'_m)$$

where P_k is the gain of the k^{th} forward path from i to j in $G'[V'_m \cup i \cup j]$; Δ_k is the value of $\det G'[V'_m \cup i \cup j]$ for that part of the graph not touching the k^{th} forward path; and the summation is taken over all $P_k \Delta_k \in G'[V'_m \cup i \cup j]$.

It should be noted that the determinants of the graphs used in the above theorem are according to Mason's definition [1-4].

The verbal aspect of the above theorems seems very involved, but the topological structure is rather simple.

The following corollary is seen to be true.

Corollary: The gain between the nodes i and j , $i, j \notin V'_m$, in G' is equal to the gain between the same nodes in G'_r .

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