REPORT R-386 JULY 1968

COORDINATED SCIENCE LABORATORY

USE OF THE COMPANION TRANSFORMATION IN PARAMETER OPTIMIZATION AND ADAPTIVE CONTROL

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This work was supported in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy, and U.S. Air Force) under contract DAAB-07-67-C-0199; and in part by Air Force AFOSR 931-67.

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USE OF THE COMPANION TRANSFORMATION IN PARAMETER OPTIMIZATION AND ADAPTIVE CONTROL

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University of Illinois, 1968

The transformation to companion canonic form for single-input, linear, time-invariant systems is shown to have important applications in the sensitivity analysis of control systems as well as in parameter optimization and model reference adaptive control problems. Two previously unknown properties of a system in companion form are demonstrated. These are the Total Symmetry Property of the sensitivity matrix of a system in companion form and the Complete Simultaneity Property. The latter states that all the sensitivity functions of the states of a companion form system can be generated by linear combinations of the signals on one sensitivity model of the system and the system states. Using these properties, a method is developed to generate by one n^{th} order sensitivity model all the sensitivity functions $\left.\frac{\partial x_i}{\partial v_j}\right|_{\underset{\sim}{v}^o}$, $i = 1,\ldots,n$, $j = 1,\ldots,r$ of the state of any single input, linear, time-invariant, controllable n^{th} order system which depends on r different parameters. This represents an improvement over known methods for generating the sensitivity functions, which generally require a composite dynamic system of order $n(r+1)$. It is demonstrated that use of this result can yield time savings in the sensitivity analysis of systems and in adaptive

control problems. This result is then extended to multi-input normal systems. A study is also made of the effect of performing parameter optimizations in the space of the coefficients of the characteristic equation as opposed to the original parameter space.

Finally, a new approach for the design of model reference adaptive control for single input, linear, time invariant plants is presented. The performance index to be minimized is the norm of the difference between the model and system companion transformations, with the constraint that the system and model eigenvalues are the same. This constrained minimization is converted to an unconstrained algebraic minimization by including state feedback in the controller. Since no iterative solutions of the differential equations are required, the solution time is less than that required for previous approaches to the model reference adaptive problem involving minimization of an integral performance index.

ACKNOWLEDGMENT

The author expresses sincere thanks to Dr. W. R. Perkins, his advisor, for his excellent guidance and continued encouragement during the course of this research as well as for inspiration received in the classroom. He also appreciates helpful discussions with Dr. J. B. Cruz, Jr., and Dr. P. Kokotović. He is indebted to Dr. R. O. Sather, of Wayne State University, for valued advice and counsel during his academic career. He further appreciates the discussions with his associates J. E. Heller, P. Sannuti, and R. G. Stefanek. He expresses heartfelt thanks to his wife Janice whose constant understanding, encouragement, and assistance enabled him to complete his graduate studies. Finally, he acknowledges the financial support of a National Defense Education Act Fellowship and of the Coordinated Science Laboratory of the University of Illinois. Special thanks are also due Mrs. Divona Keel for her expert typing of the manuscript.

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1. INTRODUCTION

A major problem in control system design is minimizing the adverse effects on system performance of uncertainties which may be present in the system. These uncertainties may be in the input, in the system order or structure, or in parameters in the controlled plant. This study is limited to systems in which the uncertainties arise from a set of plant parameters which differ from the nominal values assumed for them in the system design. Such parameter variations may be caused by physical tolerances, aging or damage of components, or conditions of environment which could not be known at the time a system is designed. The parameter values which exhibit these uncertainties will often vary with time. However, in many cases the time variations are of such a nature that the affected parameter values can be considered as piecewise constant time functions. This would be the case for parameter values which vary slowly or experience only abrupt changes at certain times. It is assumed in this study that the variable parameters in a controlled plant can be considered to have piecewise constant values.

One approach to the problem of compensating for uncertainties in a system is to use a feedback controller structure, since it is known that feedback has the potential to reduce the effects of variable parameters on system performance. However, there are cases in which the adverse effects that parameter variations have on a system's response cannot be sufficiently diminished by a feedback controller using only fixed parameters. The types of parameter variations that fall in this latter category cannot be clearly defined, and the system designer must decide by some means whether a fixed parameter controller will produce acceptable system performance in the face of uncertainties in plant parameters. If a fixed controller will not do this, the need for a controller which will adapt itself, i.e. vary its parameters, to adequately compensate for variable plant parameters is apparent.

One means of achieving controller adaptation is by using a model reference adaptive control system [1-3]. The basic block diagram of such a system is shown in Figure 1.1. In this scheme, the reference model is a system chosen to produce a desired output $\underset{\sim}{y}^m(t)$ to a given input u(t). One choice for this model could be the system consisting of the plant with its feedback controller, where all plant and controller parameters are equal to their nominal values. The adaptive mechanism uses information from the plant (with variable parameter vector $\underset{\sim}{p}$) and the error, $\underset{\sim}{e}(t)$, between the system and model responses to adjust the controller parameter vector $\underset{\sim}{v}$. The goal is to make the system and model output vectors $\underset{\sim}{y}$ and $\underset{\sim}{y}^m$ as close as possible at all times. It should be noted that model reference adaptive control problems are a subclass of parameter optimization problems where a parameter vector $\underset{\sim}{v}$ is adjusted in a system to minimize a performance index

$$J = \int_o^T V(\underset{\sim}{x},\underset{\sim}{u},t)dt \ . \qquad (1.1)$$

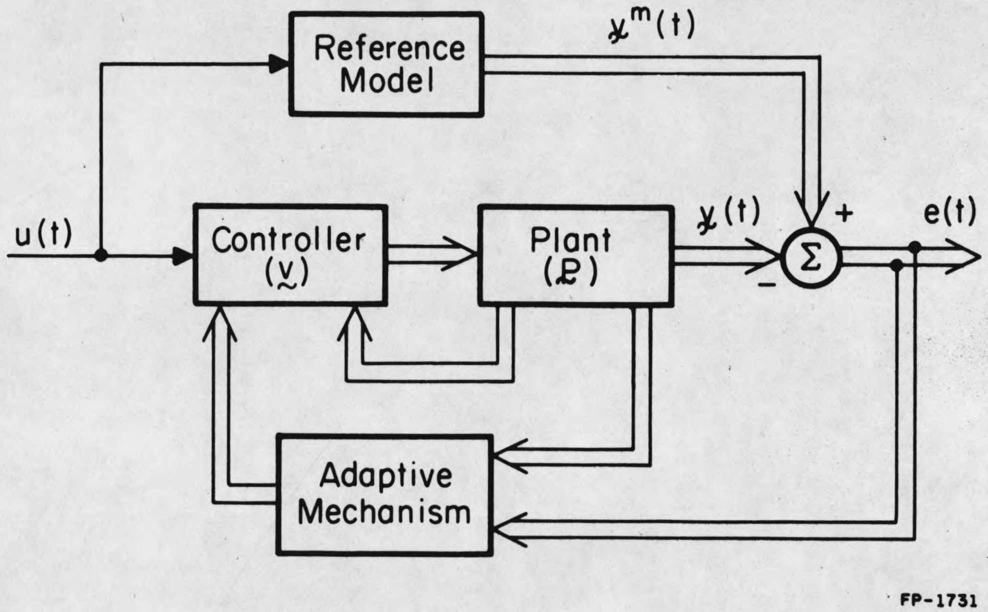

Figure 1.1 Model-Reference Adaptive Control Scheme.

There have been basically two approaches to the model reference adaptive control problem. In the first [1,3], the controller vector $\underset{\sim}{v}$ is adjusted to minimize some performance index of the error, $\underset{\sim}{e}(t)$, and an index often used is

$$J = \int_{o}^{T} \langle (\underset{\sim}{y}-\underset{\sim}{y}^{m}), Q(\underset{\sim}{y}-\underset{\sim}{y}^{m}) \rangle dt \ . \tag{1.2}$$

In the second [2], $\underset{\sim}{v}$ is adjusted to assure $\lim_{t \to \infty} \|\underset{\sim}{e}(t)\|^2 = 0$ by using the second method of Lyapounov. It is known that neither of these approaches is ideal. In the first, the system and model must be iteratively excited by an input so that J and the gradient components of J with respect to the parameters can be generated for use in a numerical minimization procedure. This repeated system excitation may not be possible in a real application. Further, this type of adaptation seems too slow if the controller parameters are adjusted iteratively. However, if the parameters are adjusted continuously, the meaning of the gradient of J with respect to the parameters is not well-defined, and the theory cannot be rigorously justified. In the second approach [2], results have only been given for low order systems (typically second order), often with no more than one controller parameter. For higher order systems with more than one parameter, the choice of a suitable Lyapounov function is not clear, and the method becomes very complicated. Further, nothing is said about the rate of decrease of the error $\underset{\sim}{e}(t)$.

The goal of this study is to improve the techniques of model reference adaptive control and parameter optimization. Since the theory of control system sensitivity is concerned with the changes in system responses caused by parameter variations [4], it plays an important role in adaptive control theory. Thus, the techniques of sensitivity analysis of control systems are used extensively in this work. It is assumed that the plant parameters vary in a piecewise constant manner, so that the adaptation methods discussed are applied during periods when the plant parameters are considered constant. The major results presented are:

(1) Previously unknown properties of a canonic form for a class of systems are used to efficiently generate the sensitivity functions of the system outputs. This simplifies the implementation of the first approach mentioned above to adaptive control and parameter optimization.

(2) A study is made of the choice of adjusted parameters on the adaptation process.

(3) A fundamentally new approach to the adaptive problem for a class of systems is presented. This approach is seen to result in fast adaptation.

A canonic form for single input, linear, time-invariant (SILTIV) control systems, which is known as the companion canonic form [5-9], is used extensively in the thesis. Because of this, the second chapter is devoted to a discussion of the companion canonic form and to a presentation of various important properties (some previously unknown) of this particular type of system. These properties are used in Chapter 3 to develop a method to generate by one n^{th}

order sensitivity model all the sensitivity functions of the state of a system, $\left.\frac{\partial x_i}{\partial v_j}\right|_{\underset{\sim}{v}^o}$, i = 1,...,m, j = 1,...,r, for a SILTIV controllable n^{th} order system which depends on r different parameters. Use of this result in parameter optimization and model reference adaptive control problems is then discussed. In Chapter 4, implications of these results on the choice of parameters to be adjusted in a system are discussed.

In Chapter 5, a new approach is presented for the design of single input, linear, time-invariant model reference adaptive systems. The performance index to be minimized is the norm of the difference between the model and system companion transformations, with the constraint that the system and model eigenvalues are the same. This constrained minimization is converted to an unconstrained algebraic minimization by including state feedback in the controller. Since no iterative solutions of the differential equations are required, the solution time is less than that required for approaches to this problem involving minimization of an integral performance index. The motivation for the new method is discussed, and an example demonstrates how the approach can lead to a fast solution of model reference adaptive control problems.

Throughout the study, it is assumed that the variable plant parameters can be measured. In some practical applications, this assumption may not be valid. In that case, a means of system identification from output measurements [10,11] is required to determine the parameter values.

It will be seen that many of the techniques of the study involve the minimization of performance indices which depend on the variable controller parameters. Since this dependence is implicit and complicated in all but the simplest of problems, it is impractical to obtain an analytical solution for the controller parameter values which will minimize a considered performance index. Thus, some numerical technique of functional minimization is required. The numerical minimization technique used here is Davidon's method [26-28]. It was chosen because it has been shown to converge for initial parameter estimates far from the minimum point and in fewer iterations than are needed for other widely used methods. Further, it is easily implemented. A brief review of Davidon's method is given in Appendix I.

All numerical results given in the thesis were obtained using the CDC 1604 computer of the Coordinated Science Laboratory.

2. SPECIAL PROPERTIES OF THE COMPANION CANONIC FORM

2.1 Introduction

In this chapter, the companion canonic form [5-9] for single input, linear, time-invariant systems is discussed. The algorithm for generating the transformation to companion canonic form is reviewed [5-9], and two properties of this canonic system form are presented which were only recently demonstrated [12]. These are called the Total Symmetry and Complete Simultaneity Properties. In Chapter 3, these properties are shown to have important implications for the sensitivity analysis of systems and consequently for parameter optimization and model reference adaptive control techniques as well. Finally, the invariance of the companion transformation under state feedback, as noticed by Morgan [13], is discussed, and its application to arbitrary eigenvalue assignment in dynamic systems is presented. This invariance property is used in the new approach to model reference adaptive control presented in Chapter 5.

2.2 The Companion Canonic Form

It has been shown [5-9] that for a <u>controllable</u> system described by the state equations

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{b}u$$

$$\underset{\sim}{y} = C\underset{\sim}{x} \qquad (2.1)$$

where

$\underset{\sim}{x}$ = n dimensional state vector

u = scalar control

$\underset{\sim}{y}$ = p dimensional output vector

A = nxn constant matrix

$\underset{\sim}{b}$ = nxl constant matrix

C = pxn constant matrix,

a non-singular transformation

$$\underset{\sim}{x} = T\underset{\sim}{z} \tag{2.2}$$

exists such that

$$\dot{\underset{\sim}{z}} = A_c\underset{\sim}{z} + \underset{\sim}{b}_c\underset{\sim}{u}$$

where

$$A_c = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ & & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_1 & -\alpha_2 & & -\alpha_{n-1} & -\alpha_n \end{bmatrix} \tag{2.3a}$$

and

$$\underset{\sim}{b}_c = T^{-1}\underset{\sim}{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.3b}$$

The elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the A_c matrix are the coefficients in the characteristic equation

$$\lambda^n + \alpha_n\lambda^{n+1} + \ldots + \alpha_2\lambda + \alpha_1 = 0 .$$

The concept of a controllable dynamic system was first presented by Kalman [14]. The implication of controllability which is important in the development of the transformation to companion form is that the matrix

$$Q_c = [\underset{\sim}{b}\ A\underset{\sim}{b}\ A^2\underset{\sim}{b}\ \ldots\ A^{n-1}\underset{\sim}{b}] \tag{2.4}$$

is non-singular for the system (2.1) if it is controllable. Controllability of the system (2.1) is often referred to as controllability of the pair $(A,\underset{\sim}{b})$. In a multi-input linear system

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + B\underset{\sim}{u} \tag{2.5}$$

where $\underset{\sim}{u}$ is an m vector, controllability implies that the matrix

$$Q_c = [B\ AB\ \ldots\ A^{n-1}B] \tag{2.6}$$

has rank n. Again, controllability of the system (2.5) is referred to as controllability of the pair (A,B).

The algorithm for calculating the matrix T [7-9] is as follows:

$$T = [\underset{\sim}{t}_1 \ldots \underset{\sim}{t}_n]$$

where

$$\begin{aligned}
\underset{\sim}{t}_n &= \underset{\sim}{b} \\
\underset{\sim}{t}_{n-1} &= A\underset{\sim}{t}_n + \alpha_n \underset{\sim}{t}_n \\
\underset{\sim}{t}_{n-2} &= A\underset{\sim}{t}_{n-1} + \alpha_{n-1} \underset{\sim}{t}_n \\
&\vdots \\
\underset{\sim}{t}_1 &= A\underset{\sim}{t}_2 + \alpha_2 \underset{\sim}{t}_n \,.
\end{aligned} \tag{2.7}$$

The coefficients of the characteristic equation which are required to find the transformation can be calculated by using Leverrier's algorithm [13,15-17]:

$$(sI-A)^{-1} = \frac{\Gamma(s)}{\Delta(s)} = \sum_{i=1}^{n} \frac{s^{i-1}}{\Delta(s)} S_{i+1} \tag{2.8}$$

where

$$\Delta(s) \triangleq \det(sI-A) = \sum_{i=o}^{n} \alpha_{i+1} s^{i}$$

and where S_{i+1} and α_{i+1} are determined by the relationships

$$\alpha_{n+1} = 1 \ , \ S_{n+1} = I \ ,$$

$$\alpha_{n-j+1} = -\left(\frac{1}{j}\right) \mathrm{tr}\ (AS_{n-j+2}), \ S_{n-j+1} = \alpha_{(n-j+1)} I + AS_{n-j+2}. \tag{2.9}$$

A check on the numerical calculations is that $S_1 = 0$ should be obtained.

Thus one can generate the canonical form for any single input, linear, time-invariant controllable system with relative ease on a digital computer by using (2.7) and (2.9).

2.3 Total Symmetry and Complete Simultaneity Properties of Companion Systems

Some interesting properties of the sensitivity functions for a system in companion form have been noted previously in the sensitivity points method [18]. Recently, it has been demonstrated that the sensitivity functions of a companion form system have even more useful properties [12]. These are

Property 2.1: The Total Symmetry Property

Define a sensitivity vector $\underset{\sim}{\xi}_i$ as

$$\underset{\sim}{\xi}_i \triangleq \left.\frac{\partial \underset{\sim}{z}}{\partial \alpha_i}\right|_{\underline{\alpha}^o} {}^* \tag{2.10}$$

and a sensitivity matrix $[\xi]$ as

$$[\xi] \triangleq [\underset{\sim}{\xi}_1\ \underset{\sim}{\xi}_2\ \cdots\ \underset{\sim}{\xi}_n] = [\xi_{ij}(t)] \ . \tag{2.11}$$

Then the matrix $[\xi]$ has the following <u>total symmetry property</u>,

$$\xi_{ij}(t) = \xi_{i+1,j-1}(t)\ ,\ \forall i,j = 1,\ldots,n \ . \tag{2.12}$$

Thus, all the elements along the "anti-diagonals" of the matrix as shown in (2.13) are equal

$$[\xi] = \begin{bmatrix} \xi_{1,1} & \xi_{1,2} & \xi_{1,3} & \cdots & \xi_{1,n} \\ \xi_{1,2} & \xi_{1,3} & \xi_{1,n} & & \xi_{2,n} \\ \xi_{1,3} & & & & \vdots \\ \vdots & & & & \xi_{n-1,n} \\ \xi_{1,n} & \xi_{2,n} & \cdots & \xi_{n-1,n} & \xi_{n,n} \end{bmatrix} \tag{2.13}$$

and consequently there are at most n+n-1 = 2n-1 independent sensitivity functions $\frac{\partial z_i}{\partial \alpha_j}$.

<u>Property 2.2: The Complete Simultaneity Property</u>

All of the sensitivity functions $\frac{\partial z_i}{\partial \alpha_j}$, i,j = 1,...,n for the canonical form of a system (2.2) can be obtained as algebraic

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*Since the sensitivity functions are by definition evaluated at some nominal parameter value $\underset{\sim}{\alpha}^o$, the subscript $\underset{\sim}{\alpha}^o$ on

$$\left.\frac{\partial \underset{\sim}{z}}{\partial \alpha_i}\right|_{\underset{\sim}{\alpha}^o}$$

and similar subscripts on other derivatives are omitted in the remainder of this report.

combinations of the signals appearing on one sensitivity model of the system and of the system states.

The proofs of these two properties follow.

For clarity, consider first a third order example as shown in Figure 2.1. Using the sensitivity points method [18], it follows that the sensitivity functions $\frac{\partial z_1}{\partial \alpha_1}$, $\frac{\partial z_1}{\partial \alpha_2}$ and $\frac{\partial z_1}{\partial \alpha_3}$ are the signals appearing at nodes 1-3 respectively in the sensitivity model. However, since the system is described by the equations

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= -\alpha_1 z_1 - \alpha_2 z_2 - \alpha_3 z_3 \ , \end{aligned} \tag{2.14}$$

it follows that

$$\frac{\partial}{\partial \alpha_1}(\dot{z}_1) = \frac{d}{dt}\left(\frac{\partial z_1}{\partial \alpha_1}\right) = \dot{\xi}_{11} = \frac{\partial z_2}{\partial \alpha_1} \tag{2.15}$$

and

$$\frac{\partial}{\partial \alpha_1}(\dot{z}_2) = \frac{d}{dt}\left(\frac{\partial z_2}{\partial \alpha_1}\right) = \dot{\xi}_{21} = \frac{\partial z_3}{\partial \alpha_1} \ , \tag{2.16}$$

since the conditions for interchanging the order of differentiation in (2.15) and (2.16) are met. Now, by inspection of the sensitivity model, the signals $\dot{\xi}_{11}$ and $\dot{\xi}_{21}$ are easily identified as those appearing at nodes 2 and 3. Similarly, it can be shown that $\xi_{22} = \dot{\xi}_{12}$, $\xi_{32} = \dot{\xi}_{22}$, and $\xi_{23} = \dot{\xi}_{13}$. Again, these signals are easily identified on the sensitivity model at nodes 3 and 4. The only sensitivity function not yet obtained is $\xi_{33} \triangleq \frac{\partial z_3}{\partial \alpha_3}$. However, by the procedure used above, $\xi_{33} = \dot{\xi}_{23}$, and ξ_{23} is the signal at

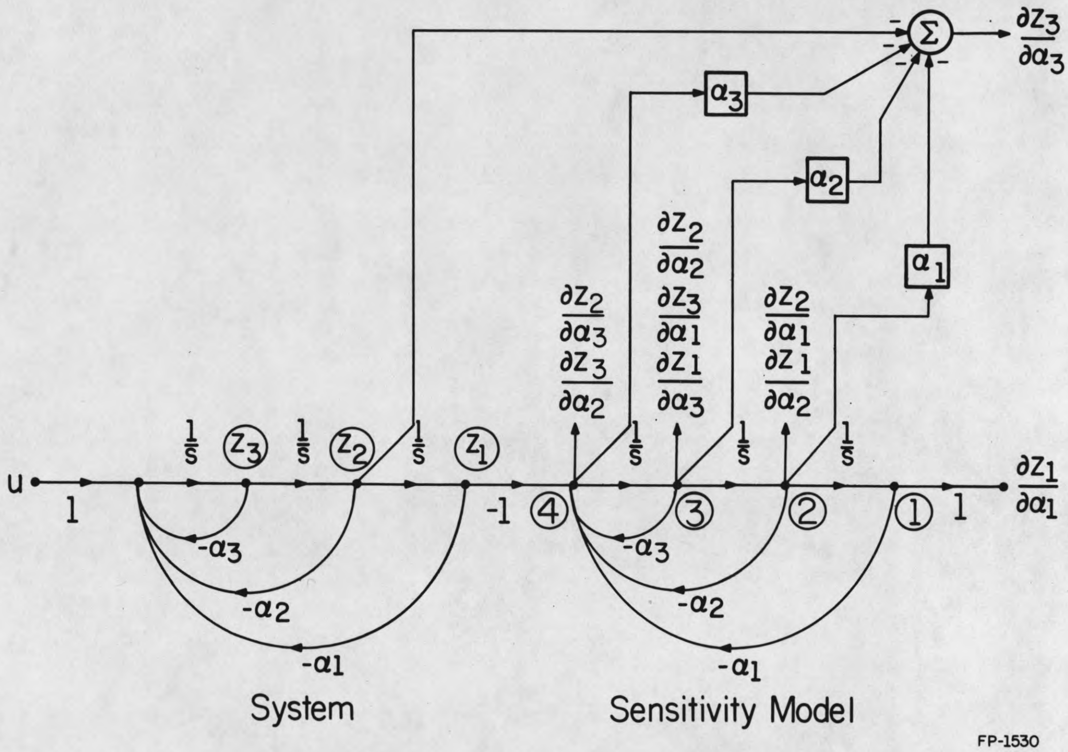

Figure 2.1 Third Order Example.

node 4 of the sensitivity model. Further inspection shows that the derivative of this signal can be obtained as an algebraic combination of the states and sensitivity functions already identified as shown in Figure 2.1. Thus, both Properties 2.1 and 2.2 have been verified in the third order case. For an n^{th} order system, the same ideas as used above in the third order system can be extended. With these ideas and systematic inspection, it follows that the signal at node 1 of the sensitivity model is (as shown in Figure 2.2)

node 1: $$\frac{\partial z_1}{\partial \alpha_1},$$

the signal at node 2 is

node 2: $$\frac{\partial z_1}{\partial \alpha_2} = \frac{\partial z_2}{\partial \alpha_1},$$

and the signals at nodes 3 through n+1 are

node 3: $$\frac{\partial z_1}{\partial \alpha_3} = \frac{\partial z_2}{\partial \alpha_2} = \frac{\partial z_3}{\partial \alpha_1}$$

$\vdots \qquad \vdots$

node n-1: $$\frac{\partial z_1}{\partial \alpha_{n-1}} = \frac{\partial z_2}{\partial \alpha_{n-2}} = \dots = \frac{\partial z_{n-1}}{\partial \alpha_1}$$

node n: $$\frac{\partial z_1}{\partial \alpha_n} = \frac{\partial z_2}{\partial \alpha_{n-1}} = \frac{\partial z_3}{\partial \alpha_{n-2}} = \dots = \frac{\partial z_{n-1}}{\partial \alpha_2} = \frac{\partial z_n}{\partial \alpha_1}$$

node n+1: $$\frac{\partial z_2}{\partial \alpha_n} = \frac{\partial z_3}{\partial \alpha_{n-1}} = \frac{\partial z_4}{\partial \alpha_{n-2}} = \dots = \frac{\partial z_n}{\partial \alpha_2}.$$

Now by referring to the sensitivity matrix $[\xi]$ as shown in (2.13), it is evident that the signals at nodes 1 to n in the sensitivity

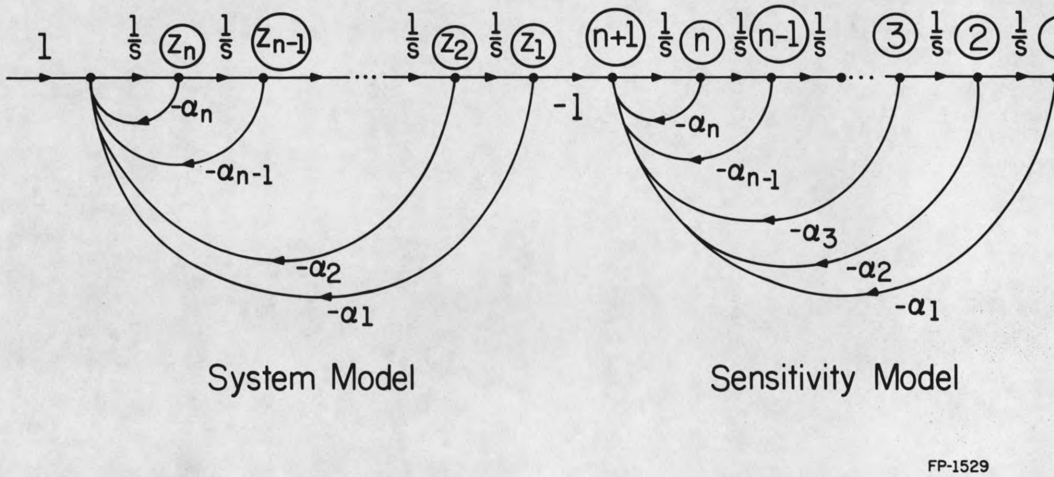

Figure 2.2 N^{th} Order Companion System and Sensitivity Model.

model represent the elements in the "upper anti-triangle" of the $[\xi]$ matrix along with the elements along the "main anti-diagonal." Also, the symmetry mentioned in Property 2.1 is evident for this portion of the $[\xi]$ matrix.

Further inspection of the system equations shows that the elements of the lower anti-triangle of the $[\xi]$ matrix can be obtained by successively differentiating the signal at node n+1 of the sensitivity model. Let $(n+1)^p$ denote the signal obtained by differentiating p times (with respect to time) the signal at node n+1. Then it follows that the signals $(n+1)^o, \ldots (n+1)^{n-2}$ are:

$\underline{(n+1)^0}$: $$\frac{\partial z_2}{\partial \alpha_n} = \frac{\partial z_3}{\partial \alpha_{n-1}} = \ldots = \frac{\partial z_{n-1}}{\partial \alpha_3} = \frac{\partial z_n}{\partial \alpha_2}$$

$\underline{(n+1)^1}$: $$\frac{\partial z_3}{\partial \alpha_n} = \frac{\partial z_4}{\partial \alpha_{n-1}} = \ldots = \frac{\partial z_n}{\partial \alpha_3}$$

$\underline{(n+1)^2}$: $$\frac{\partial z_4}{\partial \alpha_n} = \frac{\partial z_5}{\partial \alpha_{n-1}} = \ldots = \frac{\partial z_n}{\partial \alpha_4}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$\underline{(n+1)^{n-3}}$: $$\frac{\partial z_{n-1}}{\partial \alpha_n} = \frac{\partial z_n}{\partial \alpha_{n-1}}$$

$\underline{(n+1)^{n-2}}$: $$\frac{\partial z_n}{\partial \alpha_n} \quad .$$

These are all the lower anti-triangular elements of $[\xi]$. Now it is also true that all of the signals $(n+1)^0, \ldots, (n+1)^{n-2}$ can be obtained as linear combinations of the signals at nodes 1 through n of the sensitivity model and the signals $z_1,\ldots,z_{n-1}$, rather than by differentiating n-2 times the signal at node n+1. As an example, consider $\frac{\partial z_n}{\partial \alpha_4}$. By the symmetry exhibited,

$$\frac{\partial z_n}{\partial \alpha_4} = \overset{\cdot}{\left(\frac{\partial z_{n-1}}{\partial \alpha_4}\right)} = \overset{\cdot}{\left(\frac{\partial z_n}{\partial \alpha_3}\right)} = \overset{\cdot\cdot}{\left(\frac{\partial z_{n-1}}{\partial \alpha_3}\right)} = \overset{\cdot\cdot}{\left(\frac{\partial z_n}{\partial \alpha_2}\right)} . \tag{2.17}$$

But by inspection,

$$\overset{\cdot}{\left(\frac{\partial z_n}{\partial \alpha_2}\right)} = -z_2 - \alpha_n \frac{\partial z_n}{\partial \alpha_2} - \alpha_{n-1} \frac{\partial z_1}{\partial \alpha_n} - \cdots - \alpha_1 \frac{\partial z_1}{\partial \alpha_2} , \tag{2.18}$$

and hence

$$\frac{\partial z_n}{\partial \alpha_4} = -z_3 - \alpha_n \overset{\cdot}{\left(\frac{\partial z_n}{\partial \alpha_2}\right)} - \alpha_{n-1} \overset{\cdot}{\left(\frac{\partial z_1}{\partial \alpha_n}\right)} - \cdots - \alpha_1 \overset{\cdot}{\left(\frac{\partial z_1}{\partial \alpha_2}\right)} \tag{2.19}$$

which is equivalent to

$$\frac{\partial z_n}{\partial \alpha_4} = -z_3 + \alpha_n z_2 + \alpha_n^2 \frac{\partial z_n}{\partial \alpha_n} + \alpha_n \alpha_{n-1} \frac{\partial z_1}{\partial \alpha_n} + \cdots +$$

$$\alpha_n \alpha_1 \frac{\partial z_1}{\partial \alpha_2} - \alpha_{n-1} \frac{\partial z_n}{\partial \alpha_2} - \cdots - \alpha_1 \frac{\partial z_1}{\partial \alpha_3} . \tag{2.20}$$

In order to obtain the signal $(n+1)^p$, the signals $z_1,\ldots,z_{p+1}$ are needed in addition to those at nodes 1 through n. Thus, in order to obtain all the signals $(n+1)^0,\ldots,(n+1)^{n-2}$, the signals $z_1,\ldots,z_{n-1}$ are needed in addition to those at nodes 1 through n.

The complexity of proof of the total symmetry and complete simultaneity properties should not obscure the basic simplicity of application of the result. In order to obtain all the elements of the matrix $[\xi]$, only 2n-1 signals need be obtained from the system and one sensitivity model. Of these, n+1 are obtained directly as the signals at nodes 1 through n+1 of the sensitivity model. The remaining n-2 functions are obtained as linear combinations of the signals at nodes 1 through n+1 of the sensitivity model and $z_1, \ldots, z_{n-1}$ as described in the proof of Properties 2.1 and 2.2.

2.4 Invariance of the Companion Transformation

An important invariance property of the companion transformation discussed in Section 2.2 is the following:

Property 2.3:

Consider the system described by

$$\dot{\hat{\underset{\sim}{x}}} = (A + \underset{\sim}{b}\underset{\sim}{k}')\hat{\underset{\sim}{x}} + \underset{\sim}{b}u = \hat{A}(\underset{\sim}{k})\hat{\underset{\sim}{x}} + \underset{\sim}{b}u \ , \tag{2.21}$$

where $\underset{\sim}{k}$ is an n dimensional state feedback parameter vector

$$\underset{\sim}{k}' = (k_1, \ldots, k_n) \ .$$

Let the companion transformation for the system (2.21) be denoted by $\hat{T}(\underset{\sim}{k})$. Consider also the system described by

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{b}u \ , \tag{2.22}$$

with corresponding companion transformation T. Then $\hat{T}(\underset{\sim}{k}) \equiv T$, which means the companion transformation is invariant for the class of systems as given in (2.21) and (2.22).

Proof: Using the transformation T in (2.2), let

$$\hat{\underset{\sim}{x}} = T\,\hat{\underset{\sim}{z}}. \tag{2.23}$$

Then (2.21) becomes

$$\dot{\hat{\underset{\sim}{z}}} = [T^{-1} A T + T^{-1} \underset{\sim\sim}{bk}' T]\hat{\underset{\sim}{z}} + T^{-1} \underset{\sim}{bu} \tag{2.24}$$

but using (2.3) this becomes

$$\dot{\hat{\underset{\sim}{z}}} = \left[A_c + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} k'T \right] \hat{\underset{\sim}{z}} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \equiv \tilde{A} \hat{\underset{\sim}{z}} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \ . \tag{2.25}$$

Define

$$\hat{\underset{\sim}{k}}' = \underset{\sim}{k}'T = (\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_n) \ . \tag{2.26}$$

Then, it follows from (2.3), (2.25) and (2.26) that

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & & & & \\ -\alpha_1+\hat{k}_1 & -\alpha_2+\hat{k}_2 & \cdots & & -\alpha_n+\hat{k}_n \end{bmatrix} \tag{2.27}$$

and thus T transforms system (2.21) to companion form, <u>independent</u> of $\underset{\sim}{k}$, i.e. $T \equiv \hat{T}(\underset{\sim}{k})$.

Now, following Morgan [13], it can be shown that any desired eigenvalues (with corresponding $\alpha_1^d,\ldots,\alpha_n^d$) can be achieved in a system of the form (2.21). From (2.26) and (2.27), it is evident that the following choice for $\underset{\sim}{k}$,

$$\underset{\sim}{k} = (T')^{-1}(\underline{\alpha}-\underline{\alpha}^d) \tag{2.28}$$

will ensure that the system (2.21) has eigenvalues corresponding uniquely to $\underline{\alpha}^d = (\alpha_1^d,\ldots,\alpha_n^d)'$.

Recently, Wonham [19] has shown that for a multi-input linear system

$$\dot{\underset{\sim}{x}} = (A+BK)\underset{\sim}{x} + B\underset{\sim}{u} \ , \qquad (2.29)$$

where u is an m vector, arbitrary eigenvalues can be attained by proper choice of the state feedback matrix K, if and only if the pair (A,B) is controllable. Because of the arbitrary nature of the linear system he has chosen, Wonham's proof of this result is quite complicated and the computation of the necessary transformation to obtain the feedback gains is difficult. However, Morgan's approach [13] to the single input case discussed above can be easily generalized to a class of multi-input systems. Suppose that at least one of the inputs u_i for the multi-input system (2.29) can completely control the system. That is, the system

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{b}^i u_i \qquad (2.30)$$

is controllable for some i ϵ (1,...,m). Then for this system, a transformation T^i exists (as given in Section 2.2) such that if

$$\underset{\sim}{x} = T^i \underset{\sim}{z} \qquad (2.31)$$

the system

$$\dot{\underset{\sim}{z}} = A_c \underset{\sim}{z} + \underset{\sim}{b}_c u_i \qquad (2.32)$$

is in companion canonical form. Further, T^i also transforms the system

$$\dot{\underset{\sim}{x}} = (A + \underset{\sim}{b}^i \underset{\sim}{k}')\underset{\sim}{x} + \underset{\sim}{b}^i u_i \qquad (2.33)$$

to the form of (2.25), (2.26), and (2.27). Finally, if the transformation (2.31) is applied to the multi-input system (2.29), then the transformed system is described by

$$\dot{\underset{\sim}{z}} = \tilde{A}\underset{\sim}{z} + \left[\underset{\sim}{b}^{1} \ldots \underset{\sim}{b}^{i-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \underset{\sim}{b}^{i+1} \ldots \underset{\sim}{b}^{m}\right]\underset{\sim}{u} \tag{2.34}$$

where $\tilde{A}$ is the same as in (2.27). Thus, arbitrary poles can be achieved in the multi-input system in a manner similar to the single input case. The advantage of this approach in the case of multi-input systems over that of Wonham [19] are the ease with which T^{i} can be calculated, the result is unique, and only n feedback parameters (as opposed to mn) are required. Of course, as stated earlier, this approach applies only to the class of systems (2.29) where at least one of the pairs $(A,\underset{\sim}{b}^{i})$, $i = 1,\ldots,m$ is controllable.

3. ESSENTIAL PARAMETERS IN SENSITIVITY ANALYSIS

3.1 Introduction

The generation of sensitivity functions (or parameter influence coefficients) of the state of a system with respect to system parameters is an important part of several analysis, synthesis, adaptive, and automatic optimization methods [3,4,18,20-25]. This generation is usually accomplished by using sensitivity models of the system, i.e. dynamic systems which generate the sensitivity functions. These sensitivity models can be obtained by directly differentiating the system equations with respect to the parameters to obtain the sensitivity equations [4], by applying the "sensitivity points method" if the considered system is in a special form [18], or by applying the "structural method" [4]. In the case of linear time-invariant systems described by the state equations

$$\dot{\underset{\sim}{x}} = A(\underset{\sim}{v})\underset{\sim}{x} + B(\underset{\sim}{v})\underset{\sim}{u}$$

$$\underset{\sim}{y} = C(\underset{\sim}{v})\underset{\sim}{x} \qquad (3.1)$$

the straightforward application of either "structural methods" or direct differentiation of the system equations to obtain the sensitivity models [4,18,20,23] to generate the sensitivity functions $\frac{\partial x_i}{\partial v_j} = \sigma_{ij}(t)$ is well known. However, this approach will in general lead to r system models of order n in addition to the system itself, where r is the dimension of $\underset{\sim}{v}$ and n is the order of the system. Thus, the order of the system together with the sensitivity models becomes very large in a high order system containing many parameters. High order models are undesirable when using a digital computer for system simulation, because the most time consuming aspect of the analysis is always the numerical integration of the system equations, and when using analog simulation, the amount of analog equipment necessary for simulating a large system with its sensitivity models may be prohibitive. The "sensitivity points" method does not generate such high order models, but its application is limited to systems which can be represented in a particular form and in which the sensitivity of a scalar output is desired. For these reasons, it is desirable to generate the sensitivity functions of the state of a system by a method which utilizes a sensitivity model of lower order than rn and which is applicable to a broad class of systems.

It has recently been shown [12] that for a linear, time-invariant, single input, controllable system, the sensitivity functions $\frac{\partial x_i}{\partial v_j}$ of all states with respect to any number of parameters can be generated by algebraic combination of the signals appearing on one n^{th} order sensitivity model and the system itself. This result is given in this chapter. It follows from the two properties of the sensitivity functions for single input linear systems represented in companion form which were presented in Section 2.3. In order to utilize this result, a new computational algorithm is developed to enable pointwise transformation of the sensitivity functions of the corresponding companion form of a system back to the original coordinates and parameters. Finally, the major result of the chapter (Section 3.2) is extended to a class of multi-input systems to show that at most 2m-1 dynamic models are needed to generate the sensitivity functions $\frac{\partial x_i}{\partial v_j}$, where m is the dimension of the control vector.

3.2 Low Order Generation of Sensitivity Functions

Consider the system (3.1), with the matrices A, $\underset{\sim}{b}$ and C depending on an r dimensional parameter vector $\underset{\sim}{v}$. Then, in general, the companion transformation (Section 2.2) corresponding to (3.1) will depend on $\underset{\sim}{v}$, i.e.,

$$T = T(\underset{\sim}{v}) \ . \tag{3.2}$$

Further, the coefficients of the characteristic equation of (3.1) $\alpha_1, \alpha_2, \ldots, \alpha_n$ will also depend on $\underset{\sim}{v}$. For convenience, define the vector $\underset{\sim}{\alpha}(\underset{\sim}{v})$ as

$$\alpha(\underset{\sim}{v}) \triangleq \begin{bmatrix} \alpha_1(\underset{\sim}{v}) \\ \alpha_2(\underset{\sim}{v}) \\ \vdots \\ \alpha_n(\underset{\sim}{v}) \end{bmatrix} \ . \tag{3.3}$$

Now with the system (3.1) in the companion canonical form

$$\dot{\underset{\sim}{z}} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 \ldots & 0 \\ \vdots & & & & \vdots \\ -\alpha_1 & -\alpha_2 & \ldots & & -\alpha_n \end{bmatrix} \underset{\sim}{z} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \ , \tag{3.4}$$

it is evident that $\underset{\sim}{\alpha}$ represents a new parameter vector (n dimensional) in the canonic system, and that no other parameters exist in this system to affect the state $\underset{\sim}{z}$. The parameters $\alpha_1, \ldots, \alpha_n$ will be referred to as the essential parameters. An important result which follows from the above discussion is:

<u>Assertion 3.1</u>:

The sensitivity functions of all states x_i in the system (3.1) with respect to all r parameter ($r \lesseqgtr n$) can be obtained as linear combinations of the sensitivity functions $\frac{\partial z_i}{\partial \alpha_j}$, i,j = 1,...,n, of the canonic system and the states $\underset{\sim}{x}$.

<u>Proof</u>: This follows directly since $\underset{\sim}{x} = T(\underset{\sim}{v})\underset{\sim}{z}$. For any parameter v_i,

$$\frac{\partial \underset{\sim}{x}}{\partial v_i} = \frac{\partial T}{\partial v_i} \underset{\sim}{z} + T(\underset{\sim}{v}) \frac{\partial \underset{\sim}{z}}{\partial \underset{\sim}{\alpha}} \frac{\partial \underset{\sim}{\alpha}}{\partial v_i} = \frac{\partial T}{\partial v_i} T^{-1} \underset{\sim}{x} + T(\underset{\sim}{v}) \frac{\partial \underset{\sim}{z}}{\partial \underset{\sim}{\alpha}} \frac{\partial \underset{\sim}{\alpha}}{\partial v_i} \,. \tag{3.5}$$

<u>Assertion 3.2</u>:

Consider a linear, time-invariant, single input controllable system described by the equations

$$\dot{\underset{\sim}{x}} = A(\underset{\sim}{v})\underset{\sim}{x} + \underset{\sim}{b}(\underset{\sim}{v})u$$

$$\underset{\sim}{y} = C(\underset{\sim}{v})\underset{\sim}{x} \,.$$

The sensitivity functions of all states (or outputs) with respect to all r parameters can be generated by algebraically combining the signals on a single n^{th} order canonical sensitivity model in addition to the considered system. This is accomplished by combining the results of Properties 2.1 and 2.2 and Assertion 3.1 as shown in Figure 3.1.

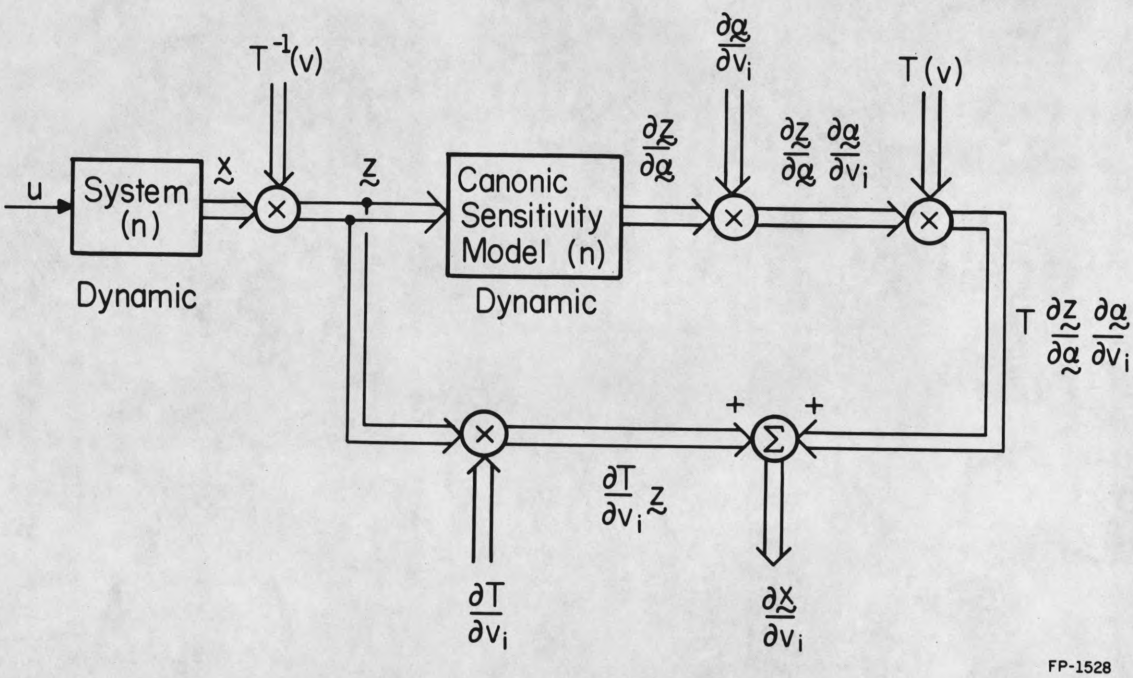

Figure 3.1 Block Diagram for Generating the Sensitivity Functions.

In order to realize the utility of this result, suppose the sensitivity functions of the states of a linear, single input, time-invariant, controllable twentieth order system with respect to six variable parameters are desired. By straightforward use of currently-available methods, this would require 6 twentieth order sensitivity models in addition to the system itself, i.e. a 140th order dynamic system. However, using the techniques of this chapter, only one twentieth order system would be needed in addition to the given system, i.e. a 40th order system. Indeed this represents a considerable savings.

3.3 The Computational Algorithm

To implement the results of the previous section in a computer simulation, the transformation T of (3.2) as well as $\frac{\partial T}{\partial v_i}$ and $\frac{\partial \alpha_k}{\partial v_i}$, $k = 1,\ldots,n$, $i = 1,\ldots,r$ must be obtained. If it were necessary to find T and the α_k's as general functions or $\underset{\sim}{v}$ to compute these necessary derivatives, the utility of Assertion 3.2 would be questionable. However, it is now shown that $\frac{\partial T}{\partial v_i}$ and $\frac{\partial \alpha_k}{\partial v_i}$ at any $\underset{\sim}{v} \in V$ can be obtained recursively by an extension of the Leverrier algorithm.

The algorithm for generating $T(\underset{\sim}{v})$ for a system (3.1) as well as the Leverrier algorithm for generating $\underset{\sim}{\alpha}(\underset{\sim}{v})$ were given previously in Section 2.2. Using these, $T(\underset{\sim}{v})$ and $\underset{\sim}{\alpha}(\underset{\sim}{v})$ can be pointwise generated for various $\underset{\sim}{v}$, as long as the functional dependence of A and $\underset{\sim}{b}$ on $\underset{\sim}{v}$ are known. By differentiating (2.7), the following algorithm is obtained for generating $\frac{\partial T}{\partial v_i}$, $i = 1,\ldots,r$:

$$\frac{\partial T}{\partial v_i} = \left[\frac{\partial \underset{\sim}{t}_1}{\partial v_i} \cdots \frac{\partial \underset{\sim}{t}_n}{\partial v_i}\right]$$

$$\frac{\partial \underset{\sim}{t}_n}{\partial v_i} = \frac{\partial \underset{\sim}{b}}{\partial v_i}$$

$$\frac{\partial \underset{\sim}{t}_{n-1}}{\partial v_i} = \frac{\partial A}{\partial v_i} \underset{\sim}{t}_n + A \frac{\partial \underset{\sim}{t}_n}{\partial v_i} + \frac{\partial \alpha_n}{\partial v_i} \underset{\sim}{b} + \alpha_n \frac{\partial \underset{\sim}{b}}{\partial v_i}$$

$$\frac{\partial \underset{\sim}{t}_{n-2}}{\partial v_i} = \frac{\partial A}{\partial v_i} \underset{\sim}{t}_{n-1} + A \frac{\partial \underset{\sim}{t}_{n-1}}{\partial v_i} + \frac{\partial \alpha_{n-1}}{\partial v_i} \underset{\sim}{b} + \alpha_{n-1} \frac{\partial \underset{\sim}{b}}{\partial v_i}$$

$$\vdots$$

$$\frac{\partial \underset{\sim}{t}_1}{\partial v_i} = \frac{\partial A}{\partial v_i} \underset{\sim}{t}_2 + A \frac{\partial \underset{\sim}{t}_2}{\partial v_i} + \frac{\partial \alpha_2}{\partial v_i} \underset{\sim}{b} + \alpha_2 \frac{\partial \underset{\sim}{b}}{\partial v_i} \; . \qquad (3.6)$$

Thus, if the functional dependence of the system matrices A and $\underset{\sim}{b}$ on $\underset{\sim}{v}$ is assumed known (so that $\frac{\partial A}{\partial v_i}$ and $\frac{\partial \underset{\sim}{b}}{\partial v_i}$ are known), only $\frac{\partial \alpha_k}{\partial v_i}$, $k = 2,\ldots,n$, need be found to calculate $\frac{\partial T}{\partial v_i}$ with the same ease that T is calculated. Again, it would not be an easy task to find the α_k's as general <u>functions</u> of $\underset{\sim}{v}$ in order to find $\frac{\partial \alpha_k}{\partial v_i}$ at a point $\underset{\sim}{v}^o \in V$. However, these derivatives can be calculated recursively by extending Leverrier's algorithm, (2.9). If (2.9) is differentiated with respect to v_i, one has the algorithm

$$\frac{\partial \alpha_{n+1}}{\partial v_i} = 0 \quad \frac{\partial S_{n+1}}{\partial v_i} = 0$$

$$\frac{\partial \alpha_{n-j+1}}{\partial v_i} = -\left(\frac{1}{j}\right) \operatorname{tr} \left(\frac{\partial A}{\partial v_i} S_{n-j+2} + A \frac{\partial S_{n-j+2}}{\partial v_i}\right)$$

$$\frac{\partial S_{n-j+1}}{\partial v_i} = \frac{\partial \alpha_{n-j+1}}{\partial v_i} I + \frac{\partial A}{\partial v_i} S_{n-j+2} + A \frac{\partial S_{n-j+2}}{\partial v_i} \; . \qquad (3.7)$$

Thus, by using (2.7), (2.9), (3.6) and (3.7), $T(\underset{\sim}{v})$, $\frac{\partial T}{\partial v_i}$ and $\frac{\partial \underset{\sim}{\alpha}}{\partial v_i}$ $(i = 1,\ldots,r)$ at any point $\underset{\sim}{v} \in V$ can be calculated without the need of knowing the functional dependence of T on $\underset{\sim}{v}$. With this result, the utility of Assertion 3.2 in the sensitivity analysis of systems by computer simulation is greatly enhanced.

3.4 Examples

In this section, the results of Section 3.2 and 3.3 are applied to two problems. The first involves the sensitivity analysis of a system in which it is desired to generate the sensitivity functions of the state with respect to a variable parameter vector. In the second, the sensitivity functions are to be used to generate the gradient components for a parameter optimization (or adaptation) scheme.

Example 3.1:

Consider the system shown in Figure 3.2. This is a fourth order system with eight variable parameters, $v_1,\ldots,v_8$. Suppose all the sensitivity functions $\frac{\partial x_i}{\partial v_j}$, $i = 1,\ldots,4$, $j = 1,\ldots,8$ are desired. Then, by straightforward application of the structural methods for obtaining sensitivity functions [3,4,18,20], eight sensitivity models in addition to the system model are required. (Shrewd application of the "sensitivity points" method may reduce this number by one or two.) That is, a 36th order dynamic system is needed to generate the sensitivity functions. However, using the methods of this paper, only an 8th order dynamic system is needed to generate the sensitivity functions. Both of these methods were

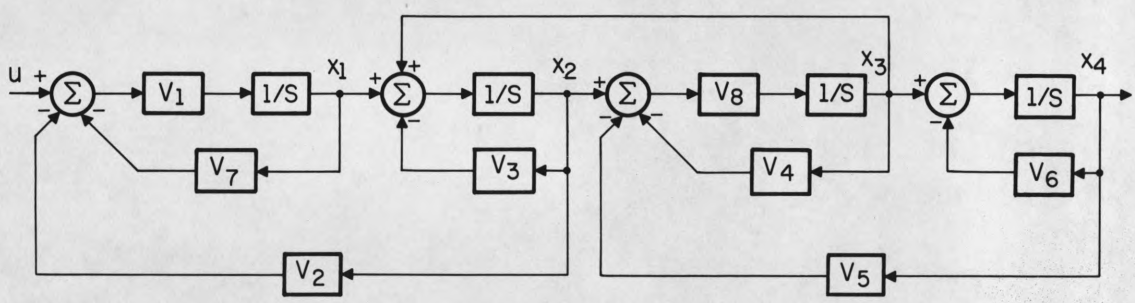

Figure 3.2 Fourth Order Example.

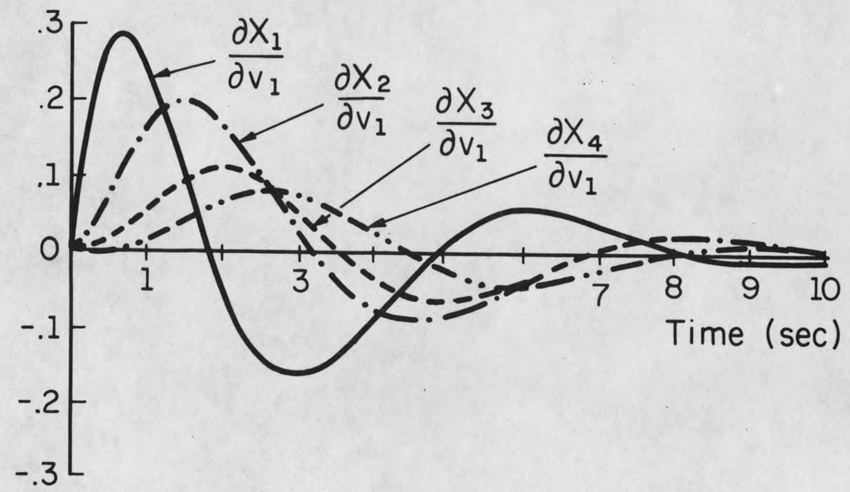

Figure 3.3 Sensitivity Functions $\partial \underset{\sim}{x}/\partial v_1$, for Example 3.1.

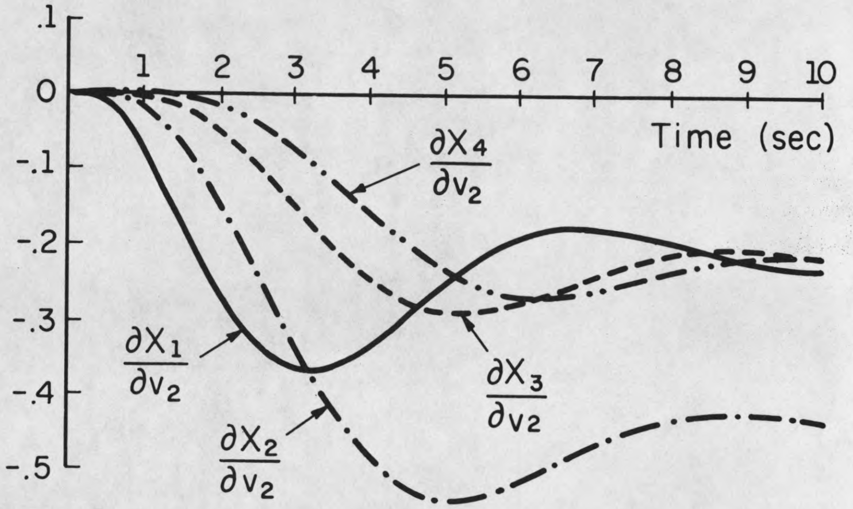

Figure 3.4 Sensitivity Functions $\partial \underset{\sim}{x}/\partial v_2$, for Example 3.1.

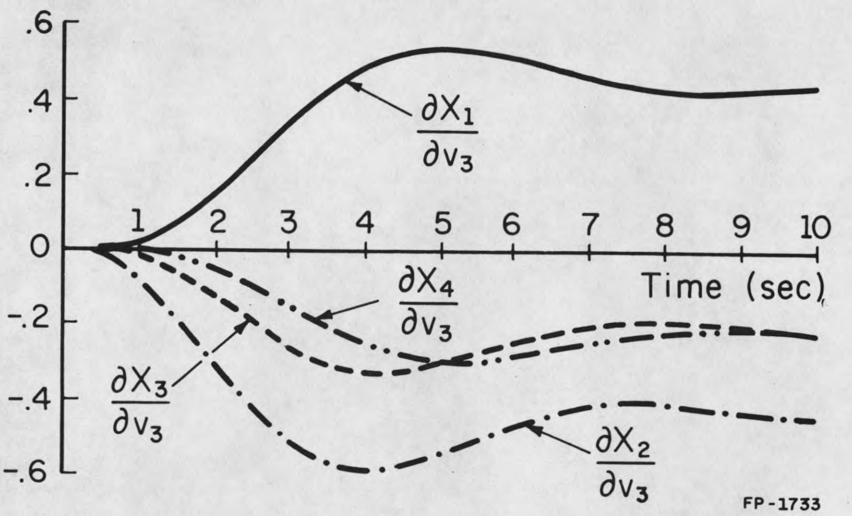

Figure 3.5 Sensitivity Functions $\partial \underset{\sim}{x}/\partial v_3$, for Example 3.1.

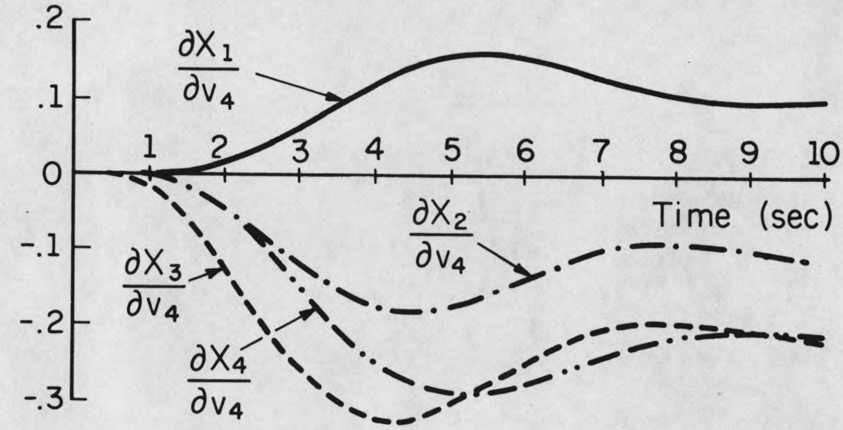

Figure 3.6 Sensitivity Functions $\partial \underset{\sim}{x}/\partial v_4$, for Example 3.1.

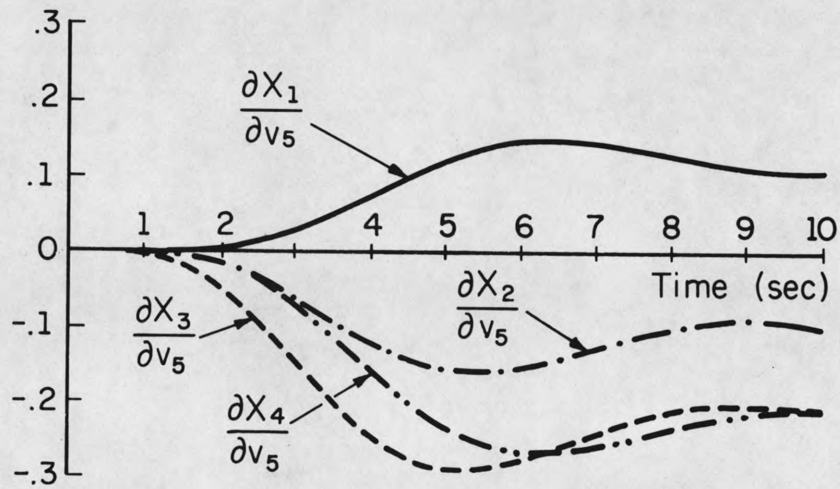

Figure 3.7 Sensitivity Functions $\partial \underset{\sim}{x}/\partial v_5$, for Example 3.1.

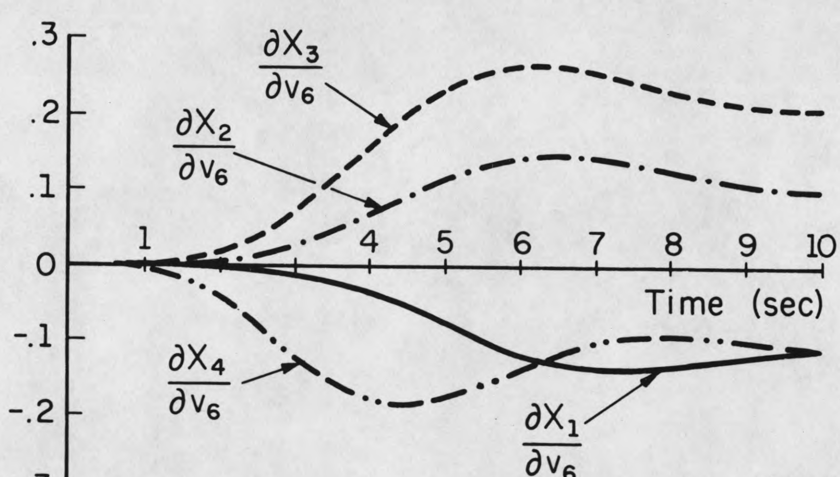

Figure 3.8 Sensitivity Functions $\partial \underset{\sim}{x}/\partial v_6$, for Example 3.1.

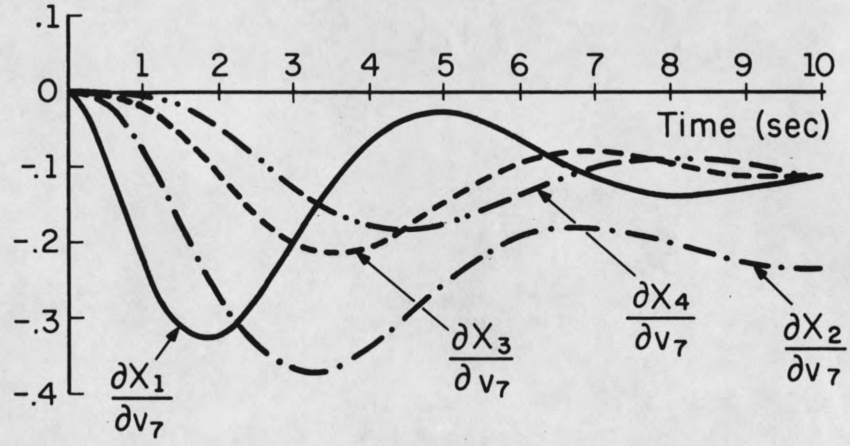

Figure 3.9 Sensitivity Functions $\partial \underset{\sim}{x}/\partial v_7$, for Example 3.1.

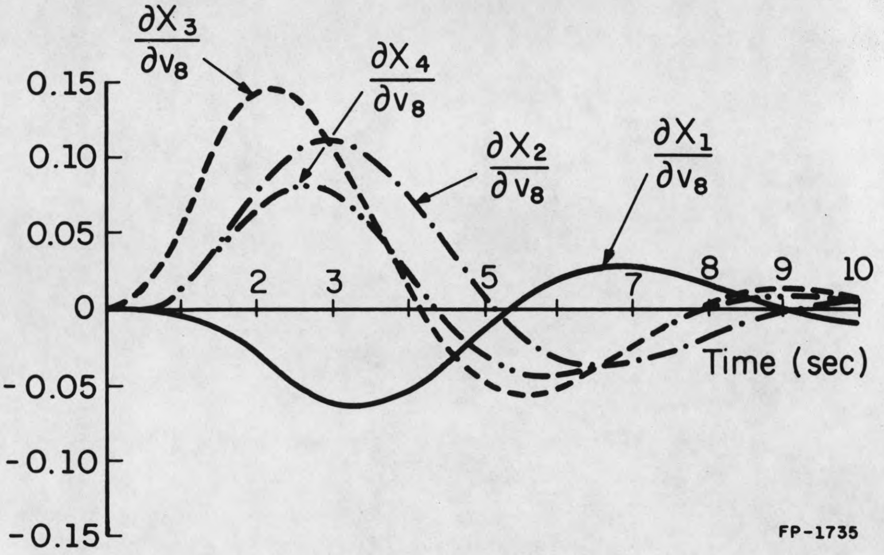

Figure 3.10 Sensitivity Functions $\partial \underset{\sim}{x}/\partial v_8$, for Example 3.1.

simulated on a digital computer to compare the solution times required. The nominal parameter values $v_1^o, \ldots, v_8^o$ were all 1 and the sensitivity functions were obtained for $t = 0$ to $t = 25$ sec.

Using the eight sensitivity models required 1 minute and 4 sec. to obtain the sensitivity functions, whereas only 28 sec. were required for the solution when essential parameter techniques were used. One might have assumed the solution time ratios should have been of the order 4:1, due to the order of the systems (36/8). However, some time is required to calculate the transformation T, α_n, $\frac{\partial T}{\partial v_i}$, $\frac{\partial \alpha_n}{\partial v_i}$ and $\underset{\sim}{z}$. The above time saving is considerable, however, especially when the sensitivity functions are to be generated for many parameter values. If the sensitivity functions were generated at 10 points in parameter space, five minutes of computer time would be saved. Further, in a hybrid or analog computer application of these methods, the equipment saved when using essential parameter techniques would be considerable.

Example 3.2:

Suppose now that the system of Figure 3.2 is to operate in a model reference adaptive control scheme, where $v_1, \ldots, v_6$ are to be adjusted to compensate for changes in v_7 and v_8. The reference model chosen is the same system with the parameters $v_1, \ldots, v_8$ all unity. The performance index to be minimized is

$$J = \frac{1}{2} \int_o^T \langle (\underset{\sim}{x} - \underset{\sim}{x}^m), Q(\underset{\sim}{x} - \underset{\sim}{x}^m) \rangle dt \;. \tag{3.8}$$

In this example, the weighting matrix Q was chosen as the identity matrix, and the parameters v_7 and v_8 in the system were 1.05 and .95

respectively. The parameters were then iteratively adjusted to minimize (3.8) using Davidon's method (Appendix I). The components of the gradient for (3.8) are given by

$$\frac{\partial J}{\partial v_i} = \int_o^T \langle (\underset{\sim}{x}-\underset{\sim}{x}^m), Q \frac{\partial \underset{\sim}{x}}{\partial v_i} \rangle dt, \quad i = 1,\ldots,6 . \tag{3.9}$$

The sensitivity functions needed to generate the gradient components were obtained using presently available techniques (i.e. six sensitivity models) as well as by the method discussed in Sections 3.2 and 3.3. The latter approach required the integration of 19 simultaneous first order differential equations and the former required solution of a 39^{th} order system. When essential parameter techniques are used to generate the sensitivity functions which are integrated to find the gradient components in (3.9), much of the algebra of multiplying T, $\frac{\partial T}{\partial v_i}$, $\frac{\partial \underset{\sim}{z}}{\partial \underset{\sim}{\alpha}}$, etc. must be done "inside" of the differential equation subroutine. This is more time consuming than when the algebra can be done external to the subroutine as in Example 3.1. Despite this fact, the time required for the solution (5 iterations) using essential parameters techniques was 9 minutes 49 seconds as opposed to the 11 minutes 36 seconds required using standard techniques. This represents a 20% time savings for this example, and the savings would be a greater percentage of the total time for a higher order system. Also, the equipment saving that would result from using essential parameters techniques when solving the problem using analog or hybrid computation would be substantial.

The system and model responses for this example are shown in Figures 3.11 through 3.14.

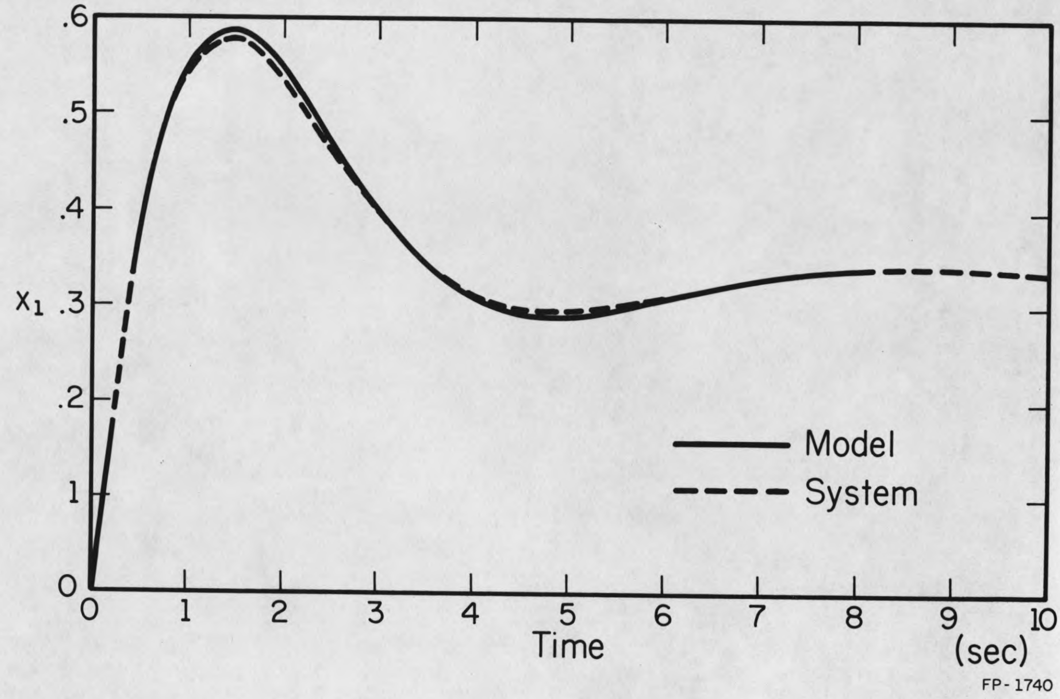

Figure 3.11 System and Model Response of State x_1 for Example 3.2.

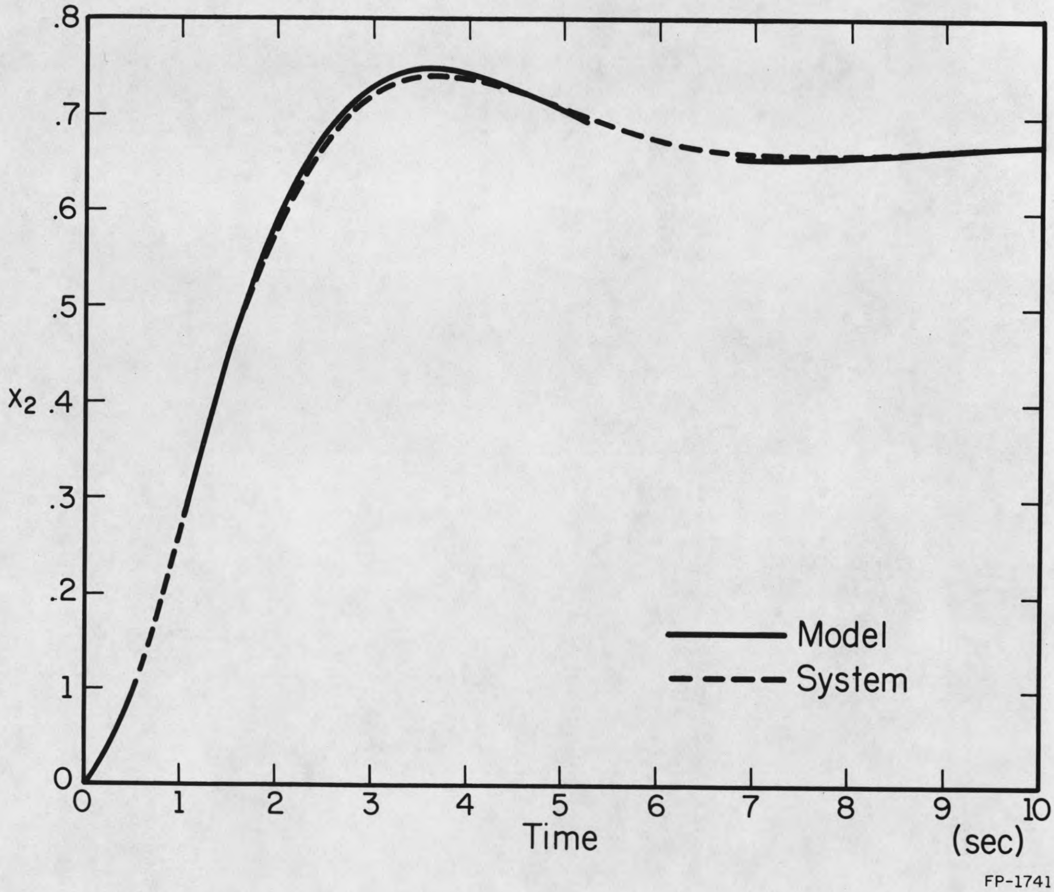

Figure 3.12 System and Model Respone of State x_2 for Example 3.2.

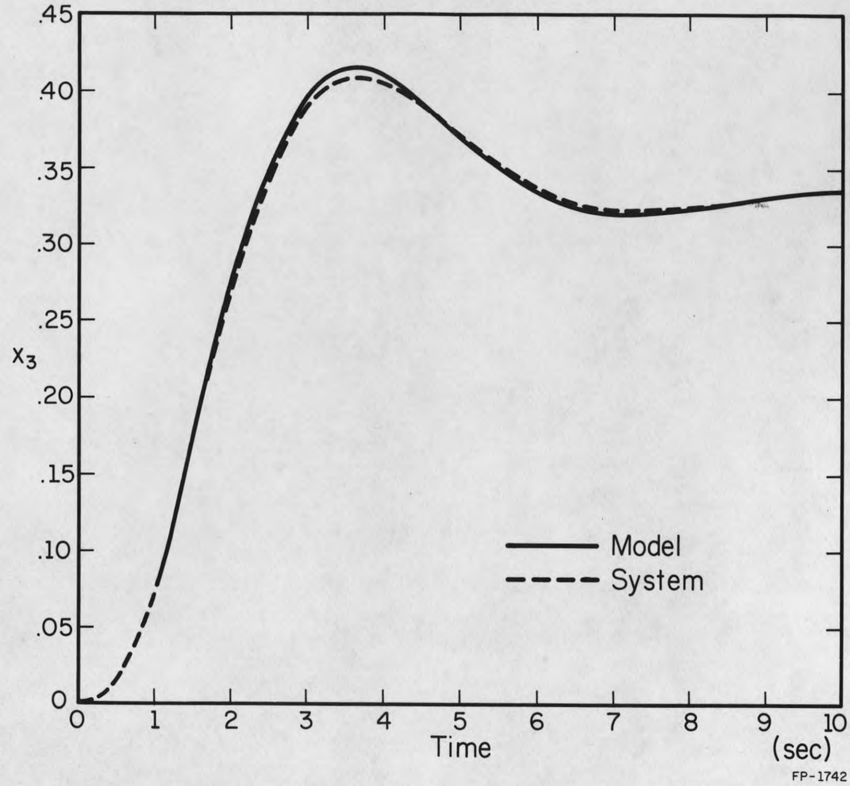

Figure 3.13 System and Model Response of State x_3 for Example 3.2.

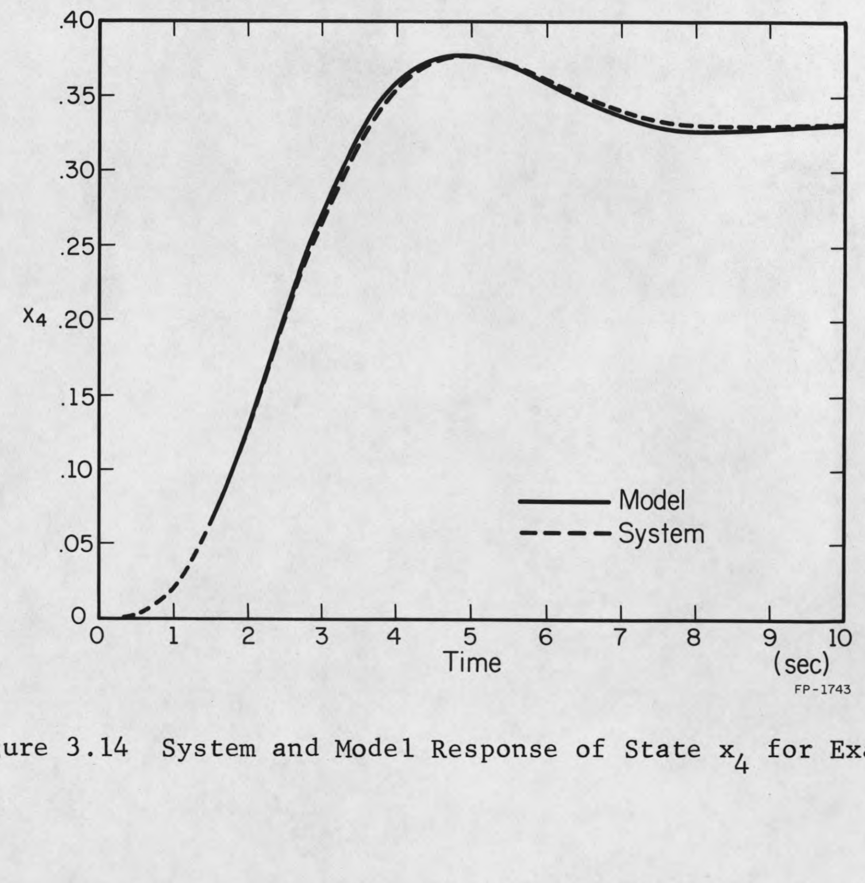

Figure 3.14 System and Model Response of State x_4 for Example 3.2.

3.5 Extension to Multi-Input Systems

The results of the previous sections can be extended to a class of linear, time-invariant, multi-input controllable systems. In this case, the system is described by the state equations

$$\dot{\underset{\sim}{x}} = A(\underset{\sim}{v})\underset{\sim}{x} + B(\underset{\sim}{v})\underset{\sim}{u} ,$$

$$\underset{\sim}{y} = C(\underset{\sim}{v})\underset{\sim}{x} , \qquad (3.10)$$

where

$\underset{\sim}{x}$ = n dimensional state vector

$\underset{\sim}{v}$ = r dimensional parameter vector

$\underset{\sim}{u}$ = m dimensional input vector

$\underset{\sim}{y}$ = p dimensional output vector

and $A(\underset{\sim}{v})$, $B(\underset{\sim}{v})$ and $C(\underset{\sim}{v})$ are nxn, nxm, and pxn matrices, respectively, which are functions of the parameter vector $\underset{\sim}{v}$. Let the following assumptions be made:

(i) Initial conditions in the system are zero.

(ii) The system is normal, i.e. completely controllable from any one input acting alone. This means the pairs $(A,\underset{\sim}{b}_i)$, i = 1,...,m are all controllable.

The first of these assumptions must also be made in the previous results for single input systems. However, in the multi-input case, this restriction is less severe since arbitrary initial conditions can be achieved with an additional input to the system, whose corresponding $\underset{\sim}{b}$ vector will have components depending on initial conditions. In addition, this means other system parameters may depend on initial conditions, which are considered as n more parameters in this case.

For a linear system with zero initial conditions, it follows from the principle of superposition that

$$\underset{\sim}{x} = \underset{\sim}{x}^1 + \underset{\sim}{x}^2 + \ldots + \underset{\sim}{x}^m \tag{3.11}$$

where $\underset{\sim}{x}^1$ is the component of the response due to u_1 acting alone, $\underset{\sim}{x}^2$ is that due to u_2, etc. By assumption (ii), transformations $T_1,\ldots,T_m$ exist which would transform the corresponding single input systems (with inputs $u_1,\ldots,u_m$) into companion form. Now since

$$\frac{\partial \underset{\sim}{x}}{\partial v_i} = \frac{\partial \underset{\sim}{x}^1}{\partial v_i} + \frac{\partial \underset{\sim}{x}^2}{\partial v_i} + \ldots + \frac{\partial \underset{\sim}{x}^m}{\partial v_i}\ , \ i = 1,\ldots,r \tag{3.12}$$

it is evident that the desired sensitivity functions are simply obtained if $\frac{\partial \underset{\sim}{x}^1}{\partial v_i}$, ..., $\frac{\partial \underset{\sim}{x}^m}{\partial v_i}$ are known. But from the results of Sections 3.2 and 3.3, it follows that only one canonic sensitivity model is required for each $\underset{\sim}{x}^k$ to generate $\frac{\partial \underset{\sim}{x}^k}{\partial v_i}$. Thus, m companion sensitivity models are needed to generate the sensitivity functions $\frac{\partial \underset{\sim}{x}}{\partial v_i}$, assuming that $\underset{\sim}{z}^1,\ldots,\underset{\sim}{z}^m$ can be calculated. This is equivalent to determining $\underset{\sim}{x}^1,\ldots,\underset{\sim}{x}^m$ since

$$\underset{\sim}{z}^1 = (T_1)^{-1}\underset{\sim}{x}^1, \ldots \ \underset{\sim}{z}^m = (T_m)^{-1}\underset{\sim}{x}^m \ . \tag{3.13}$$

The only way of obtaining $\underset{\sim}{x}^1,\ldots,\underset{\sim}{x}^m$ simultaneously is by using m-1 system models with scalar inputs $u_1,\ldots,u_m$ in addition to the system itself. Since m-1 system models must be used, and only $\underset{\sim}{z}_1,\ldots,\underset{\sim}{z}_m$ are required (as opposed to $\underset{\sim}{x}^1,\ldots,\underset{\sim}{x}^m$), it is best to use m-1 companion system models. Further, since the only parameters in the companion models are $\alpha_1,\ldots,\alpha_n$ (the coefficients of the characteristic equation), it is evident that all the companion system models are identical. Thus, for a normal system of order n, all the sensitivity functions $\frac{\partial x_i}{\partial v_j}$, $i = 1,\ldots,n$, $j = 1,\ldots,r$, can be generated using at most 2m-1 companion models of order n in addition to the system itself, as shown in Figure 3.15.

Important points to note in comparing this result with the single input results obtained previously are the following:

(1) All of the companion system and sensitivity models are decoupled, have a simple structure, and are identical. This facilitates the analog or digital simulation of the composite system.

(2) Since all of the $\alpha_1,\ldots,\alpha_n$ are the same for every model, they need be calculated only once as in the single input case. This also is true for $\frac{\partial \tilde{\alpha}}{\partial v_i}$, $i = 1,\ldots,r$.

(3) By using companion models of the system, $\tilde{z}^2,\ldots,\tilde{z}^m$ are generated directly and hence $T_2,\ldots,T_m$ need not be inverted. Thus the number of matrix inversions is the same as in the single input case.

(4) The only additional algebraic calculations for the multi-input case are those of calculating $T_2,\ldots,T_m$ and $\frac{\partial T_2}{\partial v_i},\ldots,\frac{\partial T_m}{\partial v_i}$, $i = 1,\ldots,r$.

5. The number of required models (2m-1) is independent of the number of parameters, as is the structure of the models.

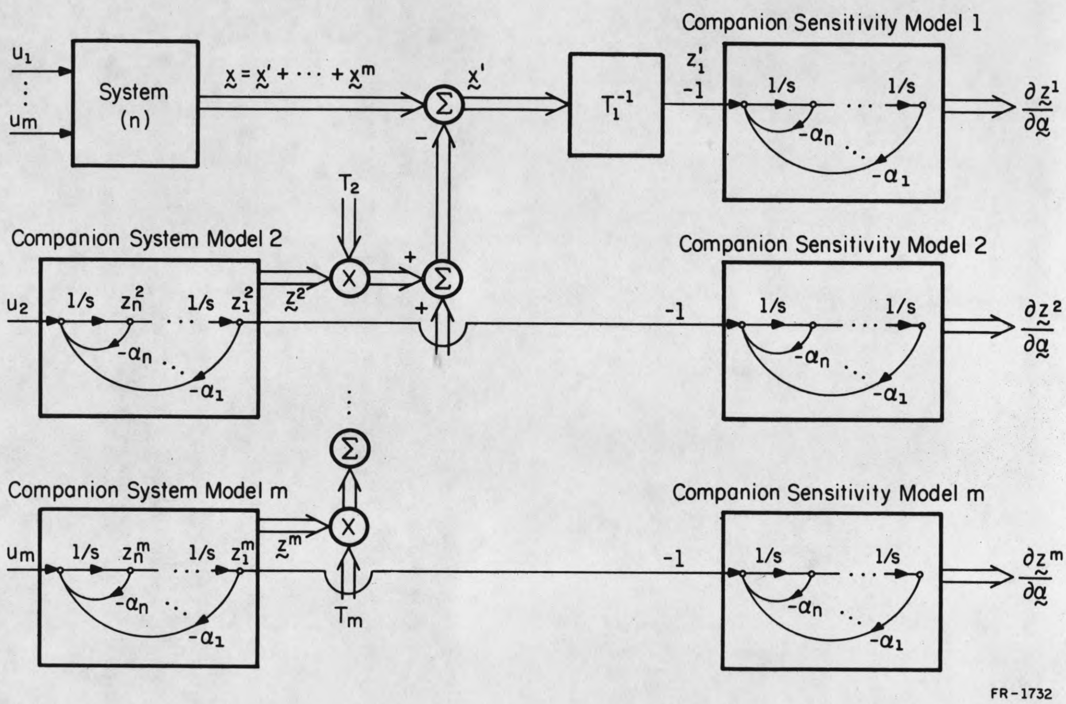

Figure 3.15 Generation of Sensitivity Functions for Normal System with m Inputs.

Now assuming that the number of parameters to be adjusted in an n^{th} order system would be at least n, and that generally $2m-1 < n$, the approach of this section could result in a considerable savings of simulation time (or in the analog case an equipment savings). This would be especially true for high order systems with only a few inputs. These results could be further generalized to non-normal systems which have real eigenvalues and are diagonalizable, but the procedure becomes quite complicated in that case and would probably not result in substantial savings.

4. THE EFFECTS OF PARAMETER CHOICE ON ADAPTATION

4.1 Introduction

From the results given in Chapters 2 and 3, it is apparent that the sensitivity functions of the companion state of a SILTIV system with respect to the coefficients of the system characteristic equation play an important role in sensitivity analysis. It follows from (2.13) that in general at least one of the sensitivity functions $\frac{\partial z_i}{\partial \alpha_j}$, i = 1,...,n, for each α_j, j = 1,...,n is needed to determine the matrix $[\xi_{ij}] = [\frac{\partial z_i}{\partial \alpha_j}]$ and subsequently the sensitivity matrix $\frac{\partial \tilde{x}}{\partial \tilde{v}}$ for a system. The question thus arises as to how these results can best be used in parameter optimization and adaptive control. One possible application of them would be to solve parameter optimization or adaptive control problems using the companion canonic form of a system, rather than some originally chosen system configuration. If this is done, the coefficients of the characteristic equation of the system (i.e. $\tilde{\alpha}$) would be adjusted to minimize a given performance index. After the solution is obtained in terms of $\tilde{\alpha}$, it would be necessary to adjust the parameter in the original system description to achieve the desired $\tilde{\alpha}$ if possible. The possibilities and effects of optimization using the companion canonic form are discussed in this chapter. (This type of optimization is hereafter referred to as an alpha space optimization). It is found that there is no particular advantage to such an approach. However, in Chapter 5 it is seen that the results of the previous chapters can be used advantageously in a new approach to parameter optimization and adaptive control.

4.2 Alpha Space Optimization Using Feedback Parameters

Consider the system described by

$$\dot{\tilde{x}} = A(\tilde{v})\tilde{x} + \tilde{b}(\tilde{v})u \ ,$$

$$\tilde{y} = C\tilde{x} \ , \tag{4.1}$$

where $\tilde{v}$ is an r dimensional parameter vector and all other quantities are as defined in (2.1). Suppose that the performance index

$$J(\tilde{v}) = \frac{1}{2}\int_{o}^{T} \langle(\tilde{y}-\tilde{y}^{m}),\ Q(\tilde{y}-\tilde{y}^{m})\rangle dt \ , \tag{4.2}$$

where $\tilde{y}^m$ is the desired system output, is to be minimized by suitable adjustment of $\tilde{v}$. In order to investigate alpha space optimization, the performance index (4.2) is written in terms of the companion canonic form (Section 2.2) state variables as

$$\hat{J}(\tilde{v},\tilde{\alpha}) = \frac{1}{2}\int_{o}^{T} \langle(CT(\tilde{v})\tilde{z}(\tilde{\alpha})-\tilde{y}^{m}),\ Q(CT(\tilde{v})\tilde{z}(\tilde{\alpha})-\tilde{y}^{m})\rangle dt \ , \tag{4.3}$$

It is desired to minimize the performance index $\hat{J}(\tilde{v},\tilde{\alpha})$ by adjusting $\tilde{\alpha}$, and subsequently by determining $\tilde{v}$ such that the coefficients of the characteristic equation of (4.1) are precisely $\tilde{\alpha}^*$, the $\tilde{\alpha}$ which minimizes (4.3). The gradient components of $\hat{J}(\tilde{v},\tilde{\alpha})$ with respect to α_i, i = 1,...,n, needed for any gradient type numerical minimization procedure are given by

$$\frac{\partial \hat{J}(\underset{\sim}{v},\underset{\sim}{\alpha})}{\partial \alpha_i} = \int_o^T \langle (CT(\underset{\sim}{v}) z(\underset{\sim}{\alpha}) - \underset{\sim}{y}^m) Q (\sum_{j=1}^{r} C \frac{\partial T}{\partial v_j} \frac{\partial v_j}{\partial \alpha_i} z(\underset{\sim}{\alpha}) + CT(\underset{\sim}{v}) \frac{\partial \underset{\sim}{z}}{\partial \alpha_i}) \rangle dt. \quad (4.4)$$

Thus, as $\underset{\sim}{\alpha}$ is adjusted in an iterative procedure, the necessary values for $\underset{\sim}{v}$ to obtain each $\underset{\sim}{\alpha}^k$ must be determined at each point of the procedure so that $T(\underset{\sim}{v})$, $\frac{\partial T}{\partial v_j}$, and $\frac{\partial v_j}{\partial \alpha_i}$ can be obtained. Since in general

$$\underset{\sim}{\alpha}^k = \underset{\sim}{f}(\underset{\sim}{v}^k) \quad (4.5)$$

at the k^{th} iteration, the equation (4.5) must be solved for $\underset{\sim}{v}^k$ given $\underset{\sim}{\alpha}^k$. Obtaining this solution presents substantial difficulties. First, the functional relationship indicated in (4.5) is highly non-linear. Second, if the dimension of $\underset{\sim}{v}$ is less than n there may be no solution, and if it is greater than n the solution is not unique. Investigations of procedures to solve (4.5) for $\underset{\sim}{v}^k$ indicate that the time required to obtain such a solution (if it exists) at each point $\underset{\sim}{\alpha}^k$ in a numerical minimization of $\hat{J}(\underset{\sim}{v},\underset{\sim}{\alpha})$ is prohibitive. However, there is a special choice of parameters $\underset{\sim}{v}$ for which equation (4.5) has a unique and easily obtained solution. Namely, if $\underset{\sim}{v}$ is the feedback parameter vector $\underset{\sim}{k}$, $(\underset{\sim}{v} = \underset{\sim}{k})$ such that (4.1) becomes

$$\dot{\underset{\sim}{x}} = (A + \underset{\sim}{b}\underset{\sim}{k}')\underset{\sim}{x} + \underset{\sim}{b}u ,$$
$$\underset{\sim}{y} = C\underset{\sim}{x} , \quad (4.6)$$

where A, $\underset{\sim}{b}$ and C are independent of $\underset{\sim}{k}$, then the companion form transformation of (4.6) is independent of $\underset{\sim}{k}$ as discussed in Section 2.4. Then (4.4) becomes

$$\frac{\partial \hat{J}(\underset{\sim}{k},\underset{\sim}{\alpha})}{\partial \alpha_i} = \int_o^T \langle (CTz(\underset{\sim}{\alpha}) - \underset{\sim}{y}^m),\ QCT \frac{\partial \underset{\sim}{z}}{\partial \alpha_i} \rangle dt\ . \qquad (4.7)$$

Further, if $\underset{\sim}{\alpha}^A$ corresponds to the coefficients of the characteristic equation of A in (4.6), then the $\underset{\sim}{k}$ required to obtain any $\underset{\sim}{\alpha}^k$ in the numerical minimization of $\hat{J}(\underset{\sim}{k},\underset{\sim}{\alpha})$ is given by

$$\underset{\sim}{k} = (T')^{-1}\ (\underset{\sim}{\alpha}^A - \underset{\sim}{\alpha}^k)\ . \qquad (4.8)$$

Thus an alpha space minimization of (4.2) for the system (4.1) can be achieved with relative ease when the system parameters are the state feedback parameters. The question to be considered now is whether there is any particular advantage to performing the minimization (4.2) as an alpha space minimization (4.3). In general, a conclusive statement cannot be made regarding this question. However, the following example demonstrates that an alpha space optimization may not be more efficient with respect to number of iterations or time required despite the important role of the companion form of a system in sensitivity analysis.

4.3 Example

The example considered is that shown in Figure 5.3. The parameters $k_1,\ldots,k_4$ are to be adjusted to minimize

$$J(\underset{\sim}{k}) = \frac{1}{2}\int_o^T \langle (\underset{\sim}{x} - \underset{\sim}{x}^m),\ Q(\underset{\sim}{x} - \underset{\sim}{x}^m) \rangle dt\ , \qquad (4.9)$$

where $\underset{\sim}{x}^m$ is generated by the system with $v_1,\ldots,v_8$ all unity and $k_1,\ldots,k_4$ all zero. This minimization was achieved by iteratively adjusting $\underset{\sim}{k}$ according to Davidon's method as well as by the alpha space minimization discussed in the previous section. The results for various values of $v_1,\ldots,v_8$ differing from the nominal values of unity are given in Table 4.1. The results show that in this example, the adaptation procedure took longer when $\underset{\sim}{\alpha}$ was adjusted than when $\underset{\sim}{k}$ was adjusted. This indicates that an alpha space optimization is less efficient (with respect to iterations and time required) than a feedback parameter optimization, but it cannot be stated conclusively whether this result would be true for a general set of system parameters.

Table 4.1

Comparison of Adaptation Times for $\underset{\sim}{k}$ and $\underset{\sim}{\alpha}$ Adaptation

System Parameters	Iterations	Time	Performance Index
$v_1=v_2=...=v_6=1.0$	($\underset{\sim}{k}$) 4	5 min. 25 sec.	3.8775×10^{-4}
$v_7=1.1$, $v_8=.9$	($\underset{\sim}{\alpha}$) 7	5 min. 42 sec.	11.436×10^{-4}
$v_1=...=v_6=1.0$	($\underset{\sim}{k}$) 6	6 min. 11 sec.	7.1546×10^{-3}
$v_7=.5$, $v_8=1.75$	($\underset{\sim}{\alpha}$) 17	10 min. 47 sec.	7.6076×10^{-3}
$v_1=...-v_6=1.0$	($\underset{\sim}{k}$) 11	13 min. 45 sec.	1.377×10^{-2}
$v_7=6.$, $v_8=3$	($\underset{\sim}{\alpha}$) 33	23 min. 14 sec.	1.6588×10^{-2}
$v_1=...=v_6=1.0$	($\underset{\sim}{k}$) 15	14 min. 7 sec.	1.9802×10^{-2}
$v_7=.5$, $v_8=5.$	($\underset{\sim}{\alpha}$) 26	16 min. 38 sec.	2.0930×10^{-2}
$v_1=...=v_6=1.0$	($\underset{\sim}{k}$) 3	3 min. 51 sec.	.76169
$v_7=v_8=10.0$	($\underset{\sim}{\alpha}$) 14	10 min. 26 sec.	.83261
$v_1=.5, v_2=1.25, v_3=1.6, v_4=2.$	($\underset{\sim}{k}$) 10	11 min. 36 sec.	1.1628
$v_5=1., v_6=1., v_7=.5, v_8=3.$	($\underset{\sim}{\alpha}$) 26	17 min. 51 sec.	1.1447*

* at 11 minutes 23 seconds this was 1.1627.

5. A NEW APPROACH TO MODEL REFERENCE ADAPTIVE CONTROL

5.1 Introduction

In the previous chapters, various approaches to model reference adaptive control and parameter optimization problems have been discussed, and new results utilizing the companion transformation for SILTIV systems were developed to facilitate the implementation of these approaches. In this chapter, a new approach to the model reference adaptive control problem for single-input, linear, time-invariant systems is presented. The actual system is forced to have dynamic characteristics similar to the model by causing the eigenvalues of the system matrix A to be exactly the same as the eigenvalues of the model. This guarantees that the state vectors of the companion canonical form of the system and model are identical. The norm of the difference between the transformations relating the actual system output and model output to their companion states is selected as an index of the difference between the system and model responses. By minimizing this performance index, the system and model outputs are then forced to be close in norm. This approach is shown to involve only algebraic manipulations of the system A and $\underset{\sim}{b}$ matrices and minimization of an algebraic function. The advantages of this technique over those requiring minimization of an integral square error between the system and model outputs [1,3] are that no system simulations are required, no averaging time for the integration is required, the result is independent of the system input as a time function, and the system and model need not be repeatedly excited in real time optimization as in some model reference adaptive schemes.

Motivation is given for the new approach to the model following problem in the Section 5.2. A discussion of the numerical implementation of the technique is given and an example is considered.

5.2 Motivation of the Approach

In the approach to be presented, it is assumed that the system and model are of the same order. This is certainly the case in model reference adaptive control when the model chosen is the system with all parameters set to their nominal values. In previous approaches [1,3], it has often been true that a model of lower order than the system is chosen. This has sometimes been desirable, for example when the model is simulated by analog means and a lower order model implies less equipment. However, it will be seen that no system simulations are required in this approach, and hence it is not a disadvantage to define a model of the same order as the plant.

The linear, time-invariant system to be controlled is described by the state equations

$$\dot{\underset{\sim}{x}} = A(\underset{\sim}{v})\underset{\sim}{x} + \underset{\sim}{b}(\underset{\sim}{v})u$$

$$\underset{\sim}{y} = C\underset{\sim}{x} \ , \qquad (5.1)$$

and the model is described by

$$\dot{\underset{\sim}{x}}^m = A_m \underset{\sim}{x}^m + \underset{\sim}{b}_m u_o$$

$$\underset{\sim}{y}^m = C_m \underset{\sim}{x}^m \qquad (5.2)$$

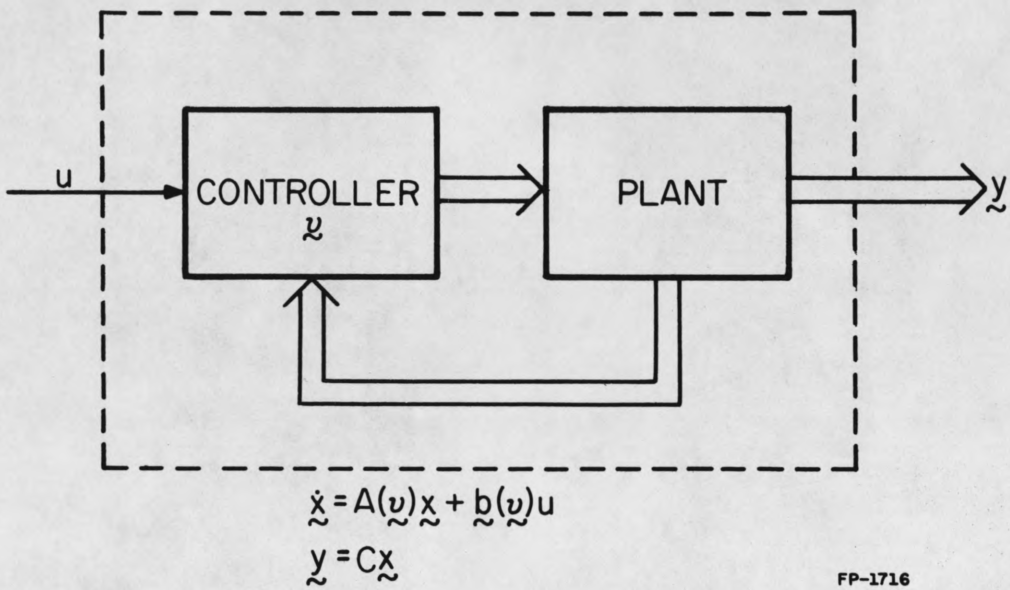

Figure 5.1 System Consisting of Controller and Plant.

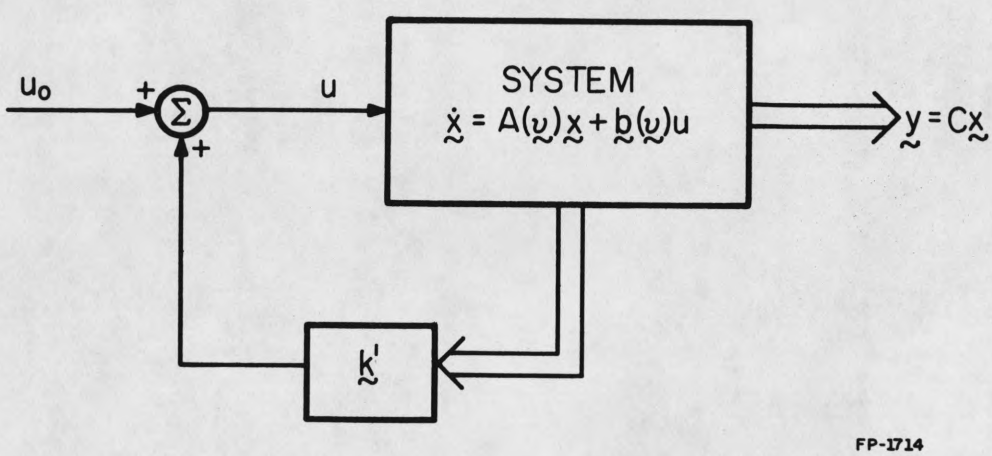

Figure 5.2 System with State Feedback Added.

where

$\underset{\sim}{x}, \underset{\sim}{x}^m$ are n dimensional state vectors

$\underset{\sim}{y}, \underset{\sim}{y}^m$ are p dimensional output vectors

$\underset{\sim}{v}$ is an r dimensional parameter vector

u, u_o are scalar inputs.

The form of the controlled system is shown in Figure 5.1. It is desired to adjust $\underset{\sim}{v}$ to make $\underset{\sim}{y}$ and $\underset{\sim}{y}^m$ as close as possible in some sense. The method of achieving this will be to minimize a measure of the norm $\|\underset{\sim}{y}-\underset{\sim}{y}^m\|$. One way of doing this is to actually minimize a chosen norm by an iterative procedure. This has been done previously [1,3] for the norm

$$\|\underset{\sim}{y}-\underset{\sim}{y}^m\| = \frac{1}{2}\int_o^T \langle(\underset{\sim}{y}-\underset{\sim}{y}^m), Q(\underset{\sim}{y}-\underset{\sim}{y}^m)\rangle dt \ .$$

However, such a minimization requires repeated simulations of the dynamic system and sensitivity equations which is time consuming. The approach here is to minimize a bound on the norm $\|\underset{\sim}{y}-\underset{\sim}{y}^m\|$, and it is shown to result in an optimization involving only algebraic equations (as opposed to differential equations). As shown in the considered examples, this approach can lead to a fast numerical minimization procedure, and the results are very close to those obtained by actually minimizing $\|\underset{\sim}{y}-\underset{\sim}{y}^m\|$.

Since experience has shown that a special choice of state variables may lead to an easier numerical minimization procedure, consider any two nonsingular, time-invariant, linear transformations T and T_m of the system and model state vectors, and denote the transformed states by $\underset{\sim}{z}$ and $\underset{\sim}{z}^m$, so that

$$\underset{\sim}{x} = T(\underset{\sim}{v})\underset{\sim}{z}, \quad \underset{\sim}{x}^m = T_m \underset{\sim}{z}^m . \tag{5.3}$$

Note that in general T is allowed to depend on $\underset{\sim}{v}$.

In terms of the transformed state variables, the problem of causing $\underset{\sim}{y}$ to follow $\underset{\sim}{y}^m$ by minimizing the norm of their difference becomes

$$\min_{\underset{\sim}{v}} \|C\underset{\sim}{x} - C_m \underset{\sim}{x}^m\| = \min_{\underset{\sim}{v}} \|CT(\underset{\sim}{v})\underset{\sim}{z} - C_m T_m \underset{\sim}{z}^m\| . \tag{5.4}$$

It would be desirable if the eigenvalues of the system could be forced to be the same as the model, since these characterize the unforced response of the system. The system and model may then be called "dynamically similar." Further, since the eigenvalues of a system are invariant under any nonsingular linear transformation, the system would be "dynamically similar" to the model independent of the choice of state coordinates.

Pursuing this approach, it will become apparent that a particularly suitable transformation to use in (5.3) and (5.4) is the transformation to companion form, previously discussed in Chapter 2. The companion canonical form of the system (5.1) is then

$$\dot{\underset{\sim}{z}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ -\alpha_1 & -\alpha_2 & \cdots & & & -\alpha_n \end{bmatrix} \underset{\sim}{z} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u . \tag{5.5}$$

Since $\alpha_1,\ldots,\alpha_n$ are the only parameters in state description of the companion system, it is evident that they completely determinze $\underset{\sim}{z}$. Further, $\alpha_1,\ldots,\alpha_n$ are uniquely determined by the system eigenvalues, and vice-versa. Thus, if the system eigenvalues are forced to be the same as those of the model by requiring

$$\underset{\sim}{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \underset{\sim}{\alpha}^m = \begin{bmatrix} \alpha_1^m \\ \vdots \\ \alpha_n^m \end{bmatrix} , \tag{5.6}$$

it follows that $\underset{\sim}{z} = \underset{\sim}{z}^m$. Thus, the problem in (5.4) becomes a minimization with the equality constraint $\underset{\sim}{\alpha} = \underset{\sim}{\alpha}^m$, and then

$$\min_{\underset{\sim}{v}} \|\underset{\sim}{y}-\underset{\sim}{y}^m\| = \min_{\underset{\sim}{v}} \|(CT(\underset{\sim}{v}) - C_mT_m)\underset{\sim}{z}^m\| . \tag{5.7}$$

From this, it follows that

$$\min_{\underset{\sim}{v}} \|\underset{\sim}{y}-\underset{\sim}{y}^m\| \leq \min_{\underset{\sim}{v}} \{\|CT(\underset{\sim}{v}) - C_mT_m\|\} \|\underset{\sim}{z}^m\| , \tag{5.8}$$

where the operator norm indicated in (5.8) is some matrix norm. Suppose now that the upper bound on $\min_{\underset{\sim}{v}} \|\underset{\sim}{y}-\underset{\sim}{y}^m\|$ indicated in (5.8) is minimized. Then the minimization problem in (5.7) becomes the constrained <u>algebraic</u> minimization problem,

$$\min_{\underset{\sim}{v}} \|CT(\underset{\sim}{v}) - C_mT_m\| , \tag{5.9}$$

with the equality constraint $\underset{\sim}{\alpha} = \underset{\sim}{\alpha}^m$. Thus, by choosing to minimize the upper bound indicated in (5.8) rather than $\|\underset{\sim}{y}-\underset{\sim}{y}^m\|$, the problem is an <u>algebraic</u> optimization. This minimization indicated in (5.9) is logical, since if $\underset{\sim}{z} = \underset{\sim}{z}^m$, then in order for $\underset{\sim}{y}$ to be close to $\underset{\sim}{y}^m$ the difference of the transformations relating $\underset{\sim}{y}$ and $\underset{\sim}{y}^m$ to $\underset{\sim}{z}$ and $\underset{\sim}{z}^m$ should be made small. This is exactly what is indicated in (5.9). Further, the considered examples show that minimization of this bound can produce good results in making $\underset{\sim}{y}$ close to $\underset{\sim}{y}^m$. A suitable norm in the space of considered matrices has been found to be

$$\frac{1}{2} \sum_i \sum_j (Ct_{ij}(\underset{\sim}{v}) - C_m t^m_{ij})^2 , \tag{5.10}$$

although other choices for the norm may yield good results in particular cases.

The question arises as to whether <u>any</u> desired $\underset{\sim}{\alpha}^m$ can be achieved by the adjusted system. However, it was shown in Section 2.4 that any desired eigenvalues (with corresponding $\alpha^m_1, \ldots, \alpha^m_n$) can be achieved in a system by using state feedback, and the feedback vector in the single input case is given by

$$\underset{\sim}{k} = (T'(\underset{\sim}{v}))^{-1} (\underset{\sim}{\alpha} - \underset{\sim}{\alpha}^m) \tag{5.11}$$

where $\underset{\sim}{\alpha}$ and $T'(\underset{\sim}{v})$ are the coefficients of the characteristic equation and companion transformation of the system (5.1), respectively. Assume now that state feedback is added to the controller of system (5.1), so that it is as shown in Figure 5.2. The system is then described by

$$\dot{\hat{\underset{\sim}{x}}} = (A(\underset{\sim}{v}) + \underset{\sim}{b}(\underset{\sim}{v})\underset{\sim}{k}')\hat{\underset{\sim}{x}} + \underset{\sim}{b}(\underset{\sim}{v})u_o = \hat{A}(\underset{\sim}{v},\underset{\sim}{k})\hat{\underset{\sim}{x}} + \underset{\sim}{b}(\underset{\sim}{v})u_o . \tag{5.12}$$

For a system in this form, the constrained minimization (5.8) becomes

$$\min_{\underset{\sim}{v},\underset{\sim}{k}} \|T(\underset{\sim}{v},\underset{\sim}{k}) - T_m\| \tag{5.13}$$

subject to the equality constraint

$$\underset{\sim}{\alpha}(\underset{\sim}{v},\underset{\sim}{k}) = \begin{bmatrix} \alpha_1(\underset{\sim}{v},\underset{\sim}{k}) \\ \vdots \\ \alpha_n(\underset{\sim}{v},\underset{\sim}{k}) \end{bmatrix} = \underset{\sim}{\alpha}^m . \tag{5.14}$$

Now due to the invariance of the companion transformation under state feedback (Property 2.3), it is evident that the minimization in (5.13) is independent of $\underset{\sim}{k}$. The problem then is simply a <u>non-constrained minimization</u> with respect to $\underset{\sim}{v}$ of $\|T(\underset{\sim}{v}) - T_m\|$, and the condition (5.14) is satisfied after the optimal $\underset{\sim}{v}^*$ is found by solving (5.11) for the feedback vector $\underset{\sim}{k}$. Thus, by choosing the controller structure of Figure 5.2, the designer can greatly facilitate the computational solution.

5.3 Numerical Implementation of the Method

The algebraic optimization of (5.13) and (5.14) involves the minimization

$$\min_{\underset{\sim}{v}} J(\underset{\sim}{v}) = \min_{\underset{\sim}{v}} \|CT(\underset{\sim}{v}) - C_m T_m\| = \min_{\underset{\sim}{v}} \frac{1}{2} \sum_{ij}\sum (Ct_{ij}(\underset{\sim}{v}) - C_m t^m_{ij})^2 . \tag{5.15}$$

The required minimization can be carried out by a variety of optimization techniques. As was mentioned previously, the method used here is Davidon's method [Appendix I]. This requires generating the gradient of the performance index. For (5.15), the gradient components are given by

$$(\text{grad}_{\underset{\sim}{v}} J(\underset{\sim}{v}))_k = (\nabla_{\underset{\sim}{v}} J(\underset{\sim}{v}))_k = \frac{\partial J}{\partial v_k} = \sum_{ij}\sum (Ct_{ij}(\underset{\sim}{v}) - C_m t^m_{ij}) \frac{\partial t_{ij}}{\partial v_k} . \tag{5.16}$$

A method of generating $\frac{\partial t_{ij}}{\partial v_k}$, i,j = 1,...,n, k = 1,...r, without knowing the functional dependence of T on $\underset{\sim}{v}$ was given previously in Section 3.3. Using these algorithms, and assuming the functional dependences of the system matrices $A(\underset{\sim}{v})$ and $\underset{\sim}{b}(\underset{\sim}{v})$ are known, one can pointwise generate $T(\underset{\sim}{v})$ and $\frac{\partial T(\underset{\sim}{v})}{\partial v_k}$, k = 1,...,r with relative ease on a digital computer. These results were implemented in the example given in the next section.

5.4 Example

Consider the system of Figure 5.3. Without the state feedback parameters (shown dotted), this system is a realization of an arbitrary fourth order transfer function (from u to x_1) if suitable values for $v_1,\dots,v_8$ are chosen. The state equations for the system are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -v_1v_7 & -v_1v_2 & 0 & 0 \\ 1 & -v_3 & 1 & 0 \\ 0 & v_8 & -v_8v_4 & -v_8v_5 \\ 0 & 0 & 1 & -v_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} [k_1k_2k_3k_4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \, . \tag{5.17}$$

Suppose now that this system is to be controlled in a model reference adaptive scheme, where $v_1,\dots,v_6$ are to be adjusted to compensate for changes in v_7 and v_8. The model used in this case

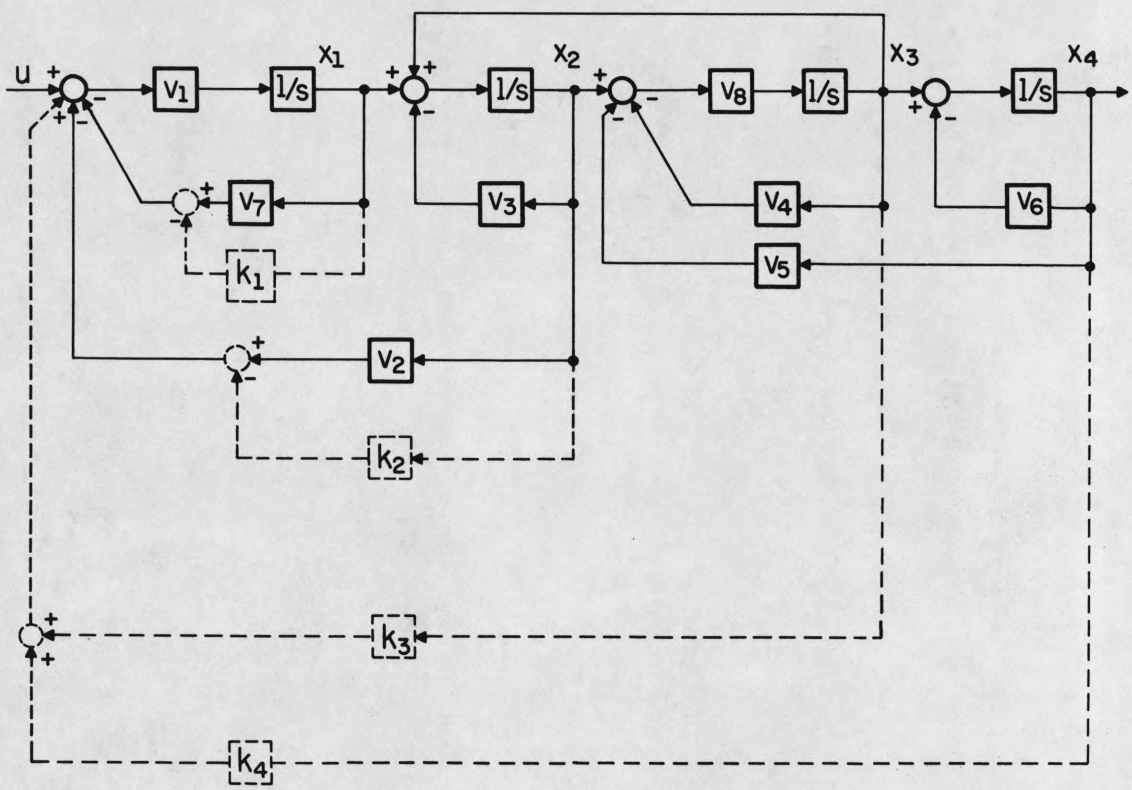

Figure 5.3 Fourth Order System of Example 5.1.

is the system with the nominal parameter values $v_1^o,\dots,v_8^o$ all unity. The state equations of the model are thus

$$\begin{bmatrix} \dot{x}_1^m \\ \dot{x}_2^m \\ \dot{x}_3^m \\ \dot{x}_4^m \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1^m \\ x_2^m \\ x_3^m \\ x_4^m \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u , \qquad (5.18)$$

and the companion transformation and coefficients of the characteristic equation of the model are

$$T_m = \begin{bmatrix} 1.0 & 3.0 & 3.0 & 1.0 \\ 2.0 & 2.0 & 1.0 & 0.0 \\ 1.0 & 1.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}, \quad \underset{\sim}{\alpha}^m = \begin{bmatrix} 3.0 \\ 6.0 \\ 7.0 \\ 4.0 \end{bmatrix} . \qquad (5.19)$$

Consider first the case where the output vector is exactly the state vector, $\underset{\sim}{y} = \underset{\sim}{x}$. For 5% changes in v_7 and v_8 from the nominal, $v_7 = 1.05$ and $v_8 = 0.95$, the computation of $v_1,\dots,v_6$ and $k_1,\dots,k_4$ according to the method of the paper required 6 iterations which took 31 seconds. The computed values for $\underset{\sim}{v}$ and $\underset{\sim}{k}$ were

$$\underset{\sim}{v} = \begin{bmatrix} 1.024 \\ 1.000 \\ 0.992 \\ 0.971 \\ 1.070 \\ 1.020 \end{bmatrix}, \quad \underset{\sim}{k} = \begin{bmatrix} 0.010 \\ 0.081 \\ -0.080 \\ -0.048 \end{bmatrix} , \qquad (5.20)$$

and the companion transformation for the system (5.17) corresponding to the values in (5.20) is

$$T = \begin{bmatrix} 0.99322 & 3.0037 & 3.0061 & 1.0243 \\ 2.00300 & 1.9903 & 1.0243 & 0.0000 \\ 0.99321 & 0.9731 & 0.0000 & 0.0000 \\ 0.97039 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \tag{5.21}$$

The performance index $\|CT-C_m T_m\|$ corresponding to (5.21) and (5.19) is $J = 1.44 \times 10^{-3}$. It is evident that the system and model companion transformation are very close. Note that k_2 and v_2 together constitute only one parameter adjustment. The step responses $x_1,\ldots,x_4$ along with the corresponding model states are shown in Figures 5.4 through 5.7.

As a matter of interest, these results are compared with those of Example 3.2, in which the same system was considered and in which $\underset{\sim}{v}$ was adjusted to minimize the integral square error criterion

$$J(\underset{\sim}{v}) = \frac{1}{2}\int_{o}^{T} \langle (\underset{\sim}{x}-\underset{\sim}{x}^{m}),\ Q(\underset{\sim}{x}-\underset{\sim}{x}^{m})\rangle dt \ . \tag{5.22}$$

In that case, using essential parameters techniques, the solution for the optimal $\underset{\sim}{v}^*$ required 5 iterations and 9 min. 49 sec. of computer time. The excessive solution time for that approach was due to the repeated solution of the system and model state equations. The integral-square-error for the system in Example 3.2 was found to be 3.1597×10^{-4}, while the value was 6.2875×10^{-4} for the parameter values given in (5.20). The important point to note is that the solution in this example was obtained in 31 seconds, while <u>each</u> iteration in the integral-square-error minimization required approximately <u>two minutes</u>. Thus, it is apparent that one of the

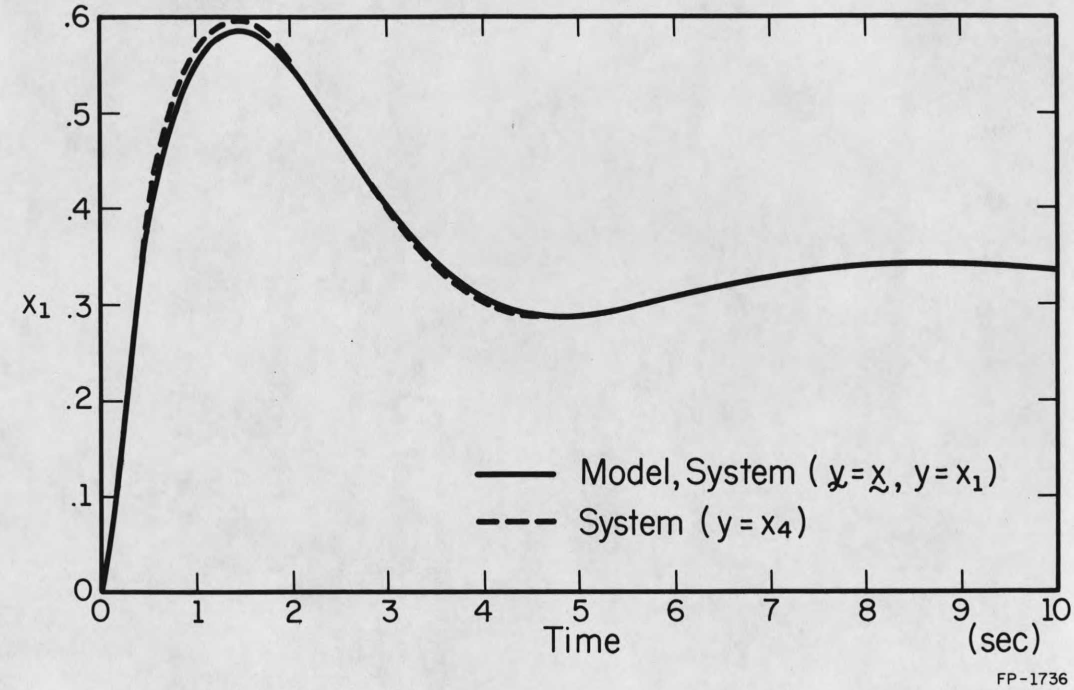

Figure 5.4 System and Model Reponses of State x_1 for Example.

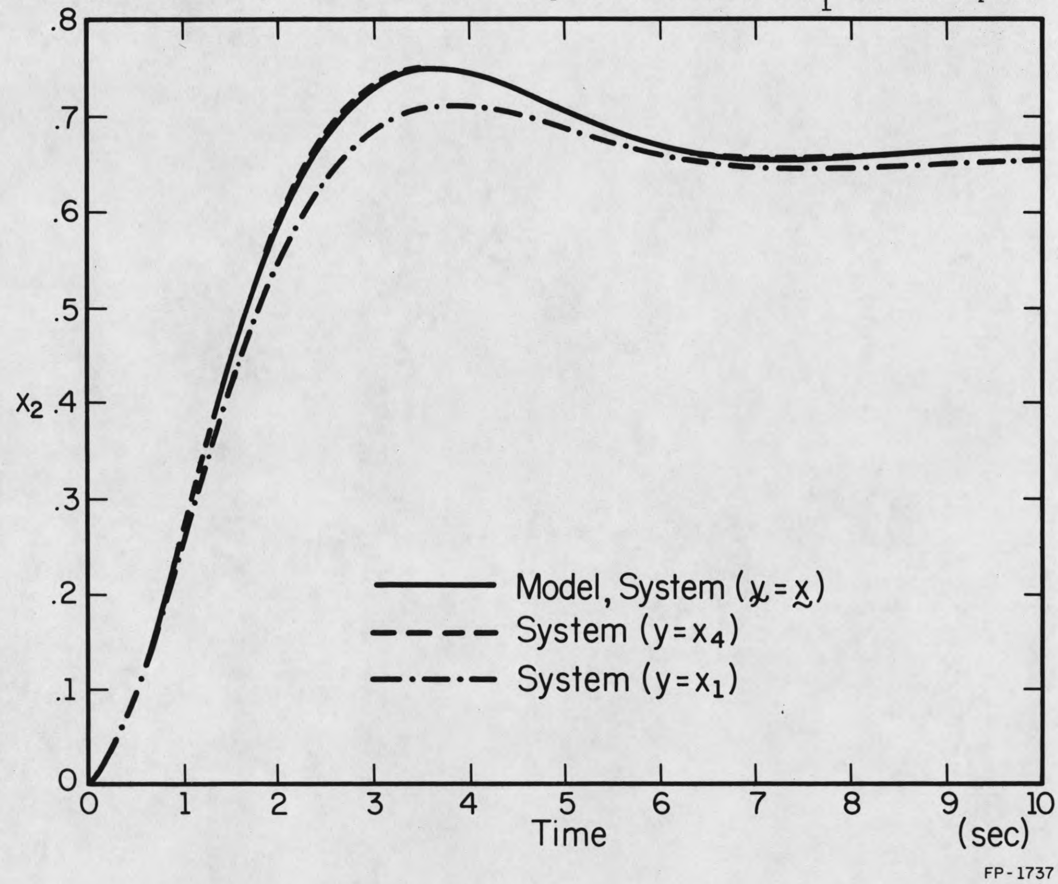

Figure 5.5 System and Model Responses of State x_2 for Example.

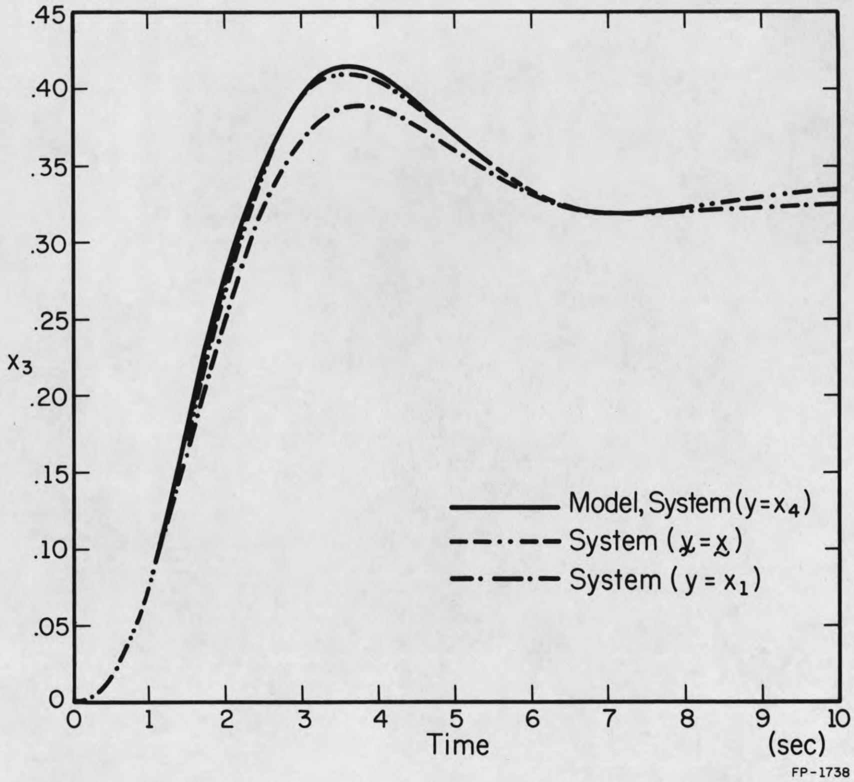

Figure 5.6 System and Model Responses of State x_3 for Example.

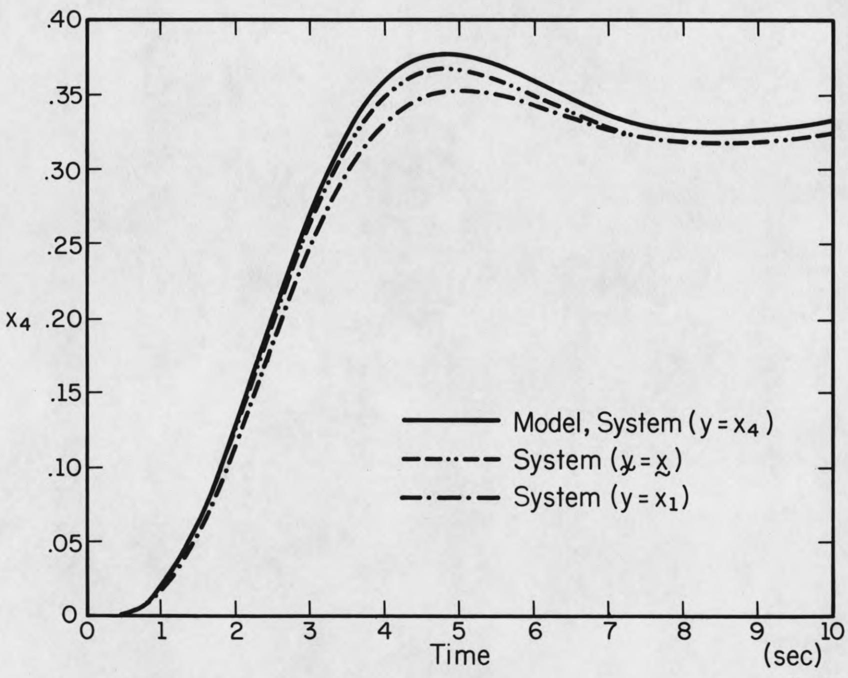

Figure 5.7 System and Model Responses of State x_4 for Example.

advantages of the approach to the model following problem presented in this chapter is the short time required for obtaining a solution.

Consider now the case where $y = x_4$, a scalar output. Then $\underset{\sim}{c}' = (0,0,0,1)$ and only the differences $t_{ij}-t_{ij}^m$ for $j = 4$ are considered in the performance index to be minimized. The computation of $v_1,\ldots,v_6$ and $k_1,\ldots,k_4$ in this case required 1 iteration (8 seconds). The computed values for $\underset{\sim}{v}$ and $\underset{\sim}{k}$ are

$$\underset{\sim}{v} = \begin{bmatrix} 1.05 \\ 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}, \quad \underset{\sim}{k} = \begin{bmatrix} 0.053 \\ 0.048 \\ -0.050 \\ -0.048 \end{bmatrix}, \tag{5.23}$$

and the companion transformation for the system (5.17) corresponding to the values (5.23) is

$$T = \begin{bmatrix} 1.0 & 3.0526 & 3.1053 & 1.0526 \\ 2.0 & 2.0526 & 1.0526 & 0.0000 \\ 1.0 & 1.0000 & 0.0000 & 0.0000 \\ 1.0 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \tag{5.24}$$

The performance index $\|CT-C_mT_m\|$ corresponding to (5.24) and (5.19) is 3.797×10^{-7}, and it is seen that the last two rows of T are essentially identical to those of T_m in (5.19). Since the system and model companion states $\underset{\sim}{z}$ and $\underset{\sim}{z}^m$ are identical, this implies the system and model outputs x_4 and x_4^m are identical also. Again as a matter of interest, the integral square error of (5.22) was found to be 2.846×10^{-4}. This is a <u>better</u> result than that obtained by the integral square error minimization, and the solution time was only 8 seconds!

It is interesting to note that only one parameter, v_1, is changed in this case. The closer correspondence between x_4 and x_4^m as well as the shorter solution time are perhaps to be expected, since all the effect of adjusting $\underset{\sim}{v}$ is concentrated toward making only four terms in $T(\underset{\sim}{v})$ close to those of T_m, rather than on making all 16 terms of the transformations the same. In fact, since 3 of the elements of the last row of T and T_m are identically zero, only the term t_{41} is affected. It is also interesting to note however, that in this problem the step responses of all the states $x_1, \ldots, x_4$ for the case $y = x_4$ were closer to the corresponding model responses than in the previous case when $\underset{\sim}{y} = \underset{\sim}{x}$. These responses are shown in Figures 5.4 through 5.7.

Because of the interesting results obtained in the case $y = x_4$, where only one term of the transformation to companion form is involved in the minimization, it is of further interest to consider the case when $y = x_1$ (i.e $\underset{\sim}{c}' = (1,0,0,0)$). Then all four non-zero terms in the first row of the transformation will be forced to be close to the corresponding terms of T_m. In this case, the computation of $v_1, \ldots, v_6$ and $k_1, \ldots, k_4$ required 26 seconds. This is to be expected because more terms are involved in the minimization than in the second case. The computed values for $\underset{\sim}{v}$ and $\underset{\sim}{k}$ were

$$\underset{\sim}{v} = \begin{bmatrix} 1.0001 \\ 1.0000 \\ 1.0172 \\ 1.0164 \\ 1.0010 \\ 1.0172 \end{bmatrix}, \quad \underset{\sim}{k} = \begin{bmatrix} 0.0500 \\ 0.292 \times 10^{-5} \\ -0.0181 \\ -0.0520 \end{bmatrix} \qquad (5.25)$$

and the companion transformation for the system (5.17) corresponding to the values (5.25) is

$$T = \begin{bmatrix} 1.000 & 3.000 & 3.000 & 1.000 \\ 1.933 & 1.983 & 1.000 & 0.000 \\ 0.966 & 0.950 & 0.000 & 0.000 \\ 0.950 & 0.000 & 0.000 & 0.000 \end{bmatrix} \quad (5.26)$$

The performance index $\|CT-C_mT_m\|$ corresponding to (5.26) and (5.19) is 2.71×10^{-11}. The first rows of T and T_m are essentially identical, and this implies since $\underset{\sim}{z} = \underset{\sim}{z}^m$ that $x_1 = x_1^m$. Again this is a good result. Considering all the states, however, the integral-squared-error of (5.22) in this case was 7.0325×10^{-3}. This is considerably higher than in any of the previous cases. However, the problem here was to make x_1 close to x_1^m ignoring the responses of the other states. Indeed this was achieved, and the higher integral-square-error cannot be considered to indicate a poor result. Again, the system responses are shown in Figures 5.4 through 5.7.

From this example, it is apparent that good results can be obtained when forcing a system to follow a model using the new approach to model reference adaptive control presented in this chapter. The particular advantages of the approach are that the optimum parameter values are obtained quickly and no system or model simulations are required. A short solution time is certainly desirable in adaptive control. Further, if this approach was used in a model reference control scheme, the model would only have to be defined, not built. Finally, no repeated excitation of the plant and model would be required as is necessary in the integral-square-error type of optimization [3]. This could be a definite advantage in an on line adaptive control problem.

6. CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY

The thesis has demonstrated how the companion canonic form for single-input, linear, time-invariant control systems can be used to advantage in parameter optimization and model reference adaptive control problems. The advantages of using the companion canonic form are derived from three useful properties of the companion form and transformation which were presented in Chapter 2. These properties are the Total Symmetry and Complete Simultaneity Properties of the sensitivity functions for companion form systems and the invariance of the companion transformation under state feedback.

In Chapter 3, the Total Symmetry and Complete Simultaneity Properties were used to show how the sensitivity functions of the state of a SILTIV n^{th} order system with respect to any number of parameters can be obtained using only <u>one</u> n^{th} order canonic sensitivity model of the system in addition to the system itself. The Leverrier algorithm was extended to facilitate use of the results of the chapter in digital or hybrid computer sensitivity analysis and in parameter optimization. This result can yield important time and equipment savings when applied in parameter optimization and model reference adaptive control problems. The result was then extended to multi-input normal systems.

In Chapter 4, a study of the effect of solving a parameter optimization (or adaptive control) problem in the parameter space of the coefficients of the characteristic equation of a system was made. It was seen that this alpha space optimization cannot be achieved for an arbitrary set of adjustable parameters in a system. However, in the case when the system parameters are the state feedback parameters, such an alpha space optimization can be achieved. In the example considered, it was found that there was no apparent advantage to performing an alpha space optimization as opposed to direct adjustment of the feedback parameters.

In Chapter 5, a new approach to solving the model following problem for single input, linear, time-invariant control system design was given. The method consists of forcing the system to be "dynamically equivalent" to the model by causing the system eigenvalues to be the same as the model. The index of the difference between the system and model responses chosen to be minimized is the norm of the difference between their companion transformations. This is then an algebraic minimization with an equality side constraint. By utilizing the invariance of the companion transformation under state feedback, it was shown that the constrained algebraic minimization problem could be reduced to an unconstrained minimization, if a controller including state feedback was chosen. Finally, an example demonstrated how the approach of Chapter 5 can lead to a fast solution of the model reference adaptive control problem.

There are several suggestions for further study that follow from the results of this thesis. The first is to investigate further the possibility of extending the results of Chapter 3 to multi-input systems. Perhaps there is a canonic form for multi-input systems which has properties comparable to those demonstrated in Chapter 2 for the companion canonic form for single-input systems but which requires less than 2m-1 dynamic models to generate the sensitivity functions for an m input system. The author investigated this possibility with no noteworthy results. Another important question to consider is whether the results of Chapters 2 and 3 could be extended in some sense to non-linear systems.

Finally, further study and application of the new approach to the model reference adaptive control problem presented in Chapter 5 could be fruitful. In particular, it would be interesting to investigate the possibilities of a minimax type of design utilizing these ideas as well as their application to systems with stochastic parameter variations. Perhaps results could be obtained which would eliminate the necessity for an adaptive controller in a wide range of problems.

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APPENDIX I

Davidon's Method

Davidon's method is a numerical technique of minimizing a scalar function of a vector $J(\underset{\sim}{v})$, i.e. a method of determining $\underset{\sim}{v}^*$ such that

$$J(\underset{\sim}{v}^*) = \min_{\underset{\sim}{v}} J(\underset{\sim}{v}) \ . \qquad \text{(A.1)}$$

The basic procedure begins with an initial guess, $\underset{\sim}{v}^o$, for the minimizing $\underset{\sim}{v}^*$ and iteratively determines what the next value chosen for $\underset{\sim}{v}$ should be to be nearer the minimum. The iteration procedure at the $i+1^{st}$ step is

$$\underset{\sim}{v}^{i+1} = \underset{\sim}{v}^i - \alpha^i H^i \operatorname{grad}_{\underset{\sim}{v}} J \Big|_{\underset{\sim}{v}^i} , \qquad \text{(A.2)}$$

where

α^i = positive scalar constant step size, determined at each iteration.

H^i = positive definite symmetric matrix which is updated at each step of the minimization.

The steps to be implemented in a minimization procedure using Davidon's method are as follows:

(1) Choose $H^o = I$ (rxr identity matrix).

(2) Having $\underset{\sim}{v}^i$ and H^i and $(\nabla_{\underset{\sim}{v}} J)_i$, find α^i such that

$$J(\underset{\sim}{v}^i - \alpha^i H^i (\nabla_{\underset{\sim}{v}} J)_i) = \min_{\lambda} \{ J(\underset{\sim}{v}^i - \lambda H^i (\nabla_{\underset{\sim}{v}} J)_i) \} \ .$$

Then form $\underset{\sim}{v}^{i+1}$ according to (A.2).

(3) Find $(\nabla_{\underset{\sim}{v}} J)_{i+1}$ and define

$$\underset{\sim}{y}^i \triangleq (\nabla_{\underset{\sim}{v}} J)_{i+1} - (\nabla_{\underset{\sim}{v}} J)_i \ .$$

(4) Form the matrix H^{i+1} by

$$H^{i+1} = H^i + A^i + B^i$$

where

$$A^i = \frac{(\alpha^i H^i \nabla_{\underset{\sim}{v}} J|_i)(\alpha^i H^i \nabla_{\underset{\sim}{v}} J|_i)^T}{(-\alpha^i H^i \nabla_{\underset{\sim}{v}} J/_i)^T \ \underset{\sim}{y}^i}$$

and

$$B^i = \frac{(H^i \underset{\sim}{y}^i)(H^i \underset{\sim}{y}^i)^T}{(\underset{\sim}{y}^i)^T \ H^i \underset{\sim}{y}^i} \ .$$

Return to step (2) and repeat the process, continuing until some predetermined stopping criteria (indicating the minimum is obtained) are satisfied.

Proofs of covergence as well as justification of the updating procedure for H^i are extensively discussed by Fletcher and Powell [27] and are thus not given here.

VITA

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* University of Illinois Coordinated Science Laboratory Urbana, Illinois 61801	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

USE OF THE COMPANION TRANSFORMATION IN PARAMETER OPTIMIZATION AND ADAPTIVE CONTROL

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5. AUTHOR(S) *(First name, middle initial, last name)*

WILKIE, Dennis Frank

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July 1968	76	28

8a. CONTRACT OR GRANT NO. DAAB-07-67-C-0199; also in part AFSOR 931-67. b. PROJECT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) R-386
c. d.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

Distribution of this report is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Joint Services Electronics Program thru U.S. Army Electronics Command Ft. Monmouth, New Jersey 07703

13. ABSTRACT

NONE

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Control Systems						
Sensitivity Analysis						
Essential Parameters						
Sensitivity Functions						
Leverrier Algorithm						
Parameter Optimization						
Adaptive Control						
Companion Canonic Form						
Companion Transformation						