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DENNIS F. WILKIE
WILLIAM R. PERKINS

UNIVERSITY OF ILLINOIS – URBANA, ILLINOIS

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Dennis F. Wilkie and William R. Perkins

Coordinated Science Laboratory and
Department of Electrical Engineering
University of Illinois
Urbana, Illinois, U.S.A.

ABSTRACT

A method is developed to generate by one nth order sensitivity model all the sensitivity functions $\left. \frac{\partial x_i}{\partial q_j} \right|_{q^0}$, $i=1, \dots, n$, $j=1, \dots, r$ for a single input, linear, time-invariant, controllable nth order system which depends on r different parameters. This represents an improvement over known methods for generating the sensitivity functions, which generally require a composite dynamic system of order $n(r+1)$. In the development, two useful properties of the sensitivity functions of the states of the companion form of a system (i.e. the system A matrix is in companion form) are used. It is shown that all the sensitivity functions of the states of the companion form system can be generated by linear combinations of the signals on one sensitivity model of the system and the system states. The Leverrier algorithm has been extended so that the method of the paper is readily implemented on a digital or hybrid computer. Finally, an example is given, demonstrating how the technique of the paper can result in considerable time savings in the computer simulation and sensitivity analysis of systems.

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INTRODUCTION

The generation of sensitivity functions (or parameter influence coefficients) of the state of a system with respect to system parameters is an important part of several analysis, synthesis, adaptive, and automatic optimization methods [1-9]. This generation is usually accomplished by using sensitivity models of the system, i.e., dynamic systems which generate the sensitivity functions. These sensitivity models can be obtained by directly differentiating the system equations with respect to the parameters to obtain the sensitivity equations [1], by applying the "sensitivity points method" if the considered system is in a special form [9], or by applying the "structural method" [1]. In the case of linear time-invariant systems described by the state equations

$$\begin{aligned}\dot{\underline{x}} &= A(\underline{q})\underline{x} + B(\underline{q})\underline{u} \\ \underline{y} &= C(\underline{q})\underline{x}\end{aligned}\tag{1}$$

the straightforward application of either "structural methods" or direct differentiation of the system equations to obtain the sensitivity models [1,2,5,9] to generate the sensitivity functions $\left. \frac{\partial x_i}{\partial q_j} \right|_{\underline{q}^0} = \sigma_{ij}(t)$ is well known*. However, this approach will in general lead to r system models of order n in addition to the system itself, where r is the dimension of \underline{q} and n is the order of the system. Thus, the order of the system together with the sensitivity models becomes very large in a high order system containing many parameters. High order models are undesirable when using a digital computer for system simulation, because the

* Since the sensitivity functions are by definition evaluated at some nominal parameter value \underline{q}^0 , the subscript \underline{q}^0 on $\left. \frac{\partial x_i}{\partial q_j} \right|_{\underline{q}^0}$ and any other derivatives is omitted in the remainder of this paper.

most time consuming aspect of the analysis is always the numerical integration of the system equations, and when using analog simulation, the amount of analog equipment necessary for simulating a large system with its sensitivity models may be prohibitive. The "sensitivity points" method does not generate such high order models, but its application is limited to systems which can be represented in a particular form and in which the sensitivity of a scalar output is desired. For these reasons, it is desirable to generate the sensitivity functions of the state of a system with respect to the system parameters by a method which utilizes a sensitivity model of lower order than n as given above and which is applicable to fairly general systems.

It is shown in this paper that for a linear, time-invariant, single input, controllable system, the sensitivity functions $\frac{\partial x_i}{\partial q_j}$ of all states with respect to any number of parameters can be generated by algebraic combination of the signals appearing on one n th order sensitivity model and the system itself. This result follows from two properties of the sensitivity functions for single input linear systems represented in companion form. In order to utilize this result, a new computational algorithm is developed to transform the sensitivity functions of the corresponding companion form of a system back to the original coordinates and parameters.

SPECIAL PROPERTIES OF THE COMPANION FORM

It has been shown [10-14] that for a controllable system described by the state equations

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \quad (2)$$

$$\underline{y} = C\underline{x}$$

where

\underline{x} = n dimensional state vector

u = scalar control

\underline{y} = p dimensional output vector

A = nxn constant matrix

\underline{b} = nx1 constant matrix

C = pxn constant matrix,

a non-singular transformation

$$\underline{x} = T\underline{z} \quad (3)$$

exists such that

$$\dot{\underline{z}} = \hat{A}\underline{z} + \hat{\underline{b}}u$$

where

$$\hat{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} & -\alpha_n \end{bmatrix}$$

and

$$\hat{\underline{b}} = T^{-1}\underline{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \quad (4)$$

The elements $\alpha_1, \alpha_2, \dots, \alpha_n$ in the \hat{A} matrix are the coefficients in the characteristic equation

$$\lambda^n + \alpha_n \lambda^{n-1} + \dots + \alpha_2 \lambda + \alpha_1 = 0.$$

Some interesting properties of the sensitivity functions for a system in companion form have been noted previously in the sensitivity

points method [9]. However, two new and very useful properties for a system in companion form are as follows:

Property I: The Total Symmetry Property

Define a sensitivity vector ξ_i as

$$\xi_i \triangleq \frac{\partial z}{\partial \alpha_i} \quad (5)$$

and a sensitivity matrix $[\xi]$ as

$$[\xi] \triangleq [\xi_1 \xi_2 \dots \xi_n] = [\xi_{ij}(t)]. \quad (6)$$

Then the matrix $[\xi]$ has the following total symmetry property,

$$\xi_{ij}(t) = \xi_{i+1,j-1}(t), \quad \forall i,j=1,\dots,n. \quad (7)$$

Thus, all the elements along the "anti-diagonals" of the system as shown in (8) are equal

$$[\xi] = \begin{bmatrix} \xi_{1,1} & \xi_{1,2} & \xi_{1,3} & \dots & \xi_{1,n} \\ \xi_{1,2} & \xi_{1,3} & \dots & \xi_{1,n} & \xi_{2,n} \\ \xi_{1,3} & & & \vdots & \\ \vdots & & & \xi_{n-1,n} & \\ \xi_{1,n} & \xi_{2,n} & \dots & \xi_{n-1,n} & \xi_{n,n} \end{bmatrix}, \quad (8)$$

and consequently there are only $n+n-1 = 2n-1$ independent sensitivity functions $\frac{\partial z_i}{\partial \alpha_j}$.

Property II: The Complete Simultaneity Property

All of the sensitivity functions $\frac{\partial z_i}{\partial \alpha_j}$, $i,j=1,\dots,n$ for the canonical form of a system (4) can be obtained as algebraic combinations of the signals appearing on one sensitivity model of the system and of the system states.

The proofs of these two properties follow.

For clarity, consider first a third order example as shown in Figure 1. Using the sensitivity points method [9], it follows that the sensitivity functions $\frac{\partial z_1}{\partial \alpha_1}$, $\frac{\partial z_1}{\partial \alpha_2}$ and $\frac{\partial z_1}{\partial \alpha_3}$ are the signals appearing at nodes 1-4 respectively in the sensitivity model. However, since the system is described by the equations

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= -\alpha_1 z_1 - \alpha_2 z_2 - \alpha_3 z_3,\end{aligned}\tag{9}$$

it follows that

$$\frac{\partial}{\partial \alpha_1}(\dot{z}_1) = \frac{d}{dt}\left(\frac{\partial z_1}{\partial \alpha_1}\right) = \dot{\xi}_{11} = \frac{\partial z_2}{\partial \alpha_1}\tag{10}$$

and

$$\frac{\partial}{\partial \alpha_1}(\dot{z}_2) = \frac{d}{dt}\left(\frac{\partial z_2}{\partial \alpha_1}\right) = \dot{\xi}_{21} = \frac{\partial z_3}{\partial \alpha_1},\tag{11}$$

since the conditions for interchanging the order of differentiation in (10) and (11) are met. Now, by inspection of the sensitivity model, the signals $\dot{\xi}_{11}$ and $\dot{\xi}_{21}$ are easily identified as those appearing at nodes 2 and 3. Similarly, it can be shown that $\xi_{22} = \dot{\xi}_{12}$, $\xi_{32} = \dot{\xi}_{22}$, and $\xi_{23} = \dot{\xi}_{13}$. Again, these signals are easily identified on the sensitivity model at nodes 3 and 4. The only sensitivity function not yet obtained is $\xi_{33} \triangleq \frac{\partial z_3}{\partial \alpha_3}$. However, by the procedure used above, $\xi_{33} = \dot{\xi}_{23}$, and ξ_{23} is the signal at node 4 of the sensitivity model. Further inspection shows that the derivative of this signal can be obtained as an algebraic combination of the states and sensitivity function already identified as shown in Figure 1. Thus, both Properties I and II have been verified in

the third order case. For an nth order system, the same ideas as used above in the third order system can be extended. With these ideas and systematic inspection, it follows that the signal at node 1 of the sensitivity model is (as shown in Figure 2)

$$\text{node 1: } \frac{\partial z_1}{\partial \alpha_1},$$

the signal at node 2 is

$$\text{node 2: } \frac{\partial z_1}{\partial \alpha_2} = \frac{\partial z_2}{\partial \alpha_1},$$

and the signals at nodes 3 through n+1 are

$$\text{node 3: } \frac{\partial z_1}{\partial \alpha_3} = \frac{\partial z_2}{\partial \alpha_2} = \frac{\partial z_3}{\partial \alpha_1}$$

\vdots

\vdots

$$\text{node n-1: } \frac{\partial z_1}{\partial \alpha_{n-1}} = \frac{\partial z_2}{\partial \alpha_{n-2}} = \dots = \frac{\partial z_{n-1}}{\partial \alpha_1}$$

$$\text{node n: } \frac{\partial z_1}{\partial \alpha_n} = \frac{\partial z_2}{\partial \alpha_{n-1}} = \frac{\partial z_3}{\partial \alpha_{n-2}} = \dots = \frac{\partial z_{n-1}}{\partial \alpha_2} = \frac{\partial z_n}{\partial \alpha_1}$$

$$\text{node n+1: } \frac{\partial z_2}{\partial \alpha_n} = \frac{\partial z_3}{\partial \alpha_{n-1}} = \frac{\partial z_4}{\partial \alpha_{n-2}} = \dots = \frac{\partial z_n}{\partial \alpha_2}.$$

Now by referring to the sensitivity matrix $[\xi]$ as shown in (8), it is evident that the signals at nodes 1 to n in the sensitivity model represent the elements in the "upper anti-triangle" of the $[\xi]$ matrix along with the elements along the "main anti-diagonal." Also, the symmetry mentioned in Property I is evident for this portion of the $[\xi]$ matrix.

Further inspection of the system equations shows that the elements of the lower anti-triangle of the $[\xi]$ matrix can be obtained by successively differentiating the signal at node $n+1$ of the sensitivity model. Let $(n+1)^p$ denote the signal obtained by differentiating p times (with respect to time) the signal at node $n+1$. Then it follows that the signals $(n+1)^1, \dots, (n+1)^{n-2}$ are:

$$\underline{(n+1)^1}: \quad \frac{\partial z_2}{\partial \alpha_n} = \frac{\partial z_3}{\partial \alpha_{n-1}} = \dots = \frac{\partial z_{n-1}}{\partial \alpha_3} = \frac{\partial z_n}{\partial \alpha_2}$$

$$\underline{(n+1)^2}: \quad \frac{\partial z_3}{\partial \alpha_n} = \frac{\partial z_4}{\partial \alpha_{n-1}} = \dots = \frac{\partial z_n}{\partial \alpha_3}$$

$$\underline{(n+1)^3}: \quad \frac{\partial z_4}{\partial \alpha_n} = \frac{\partial z_5}{\partial \alpha_{n-1}} = \dots = \frac{\partial z_n}{\partial \alpha_4}$$

$$\quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\underline{(n+1)^{n-3}}: \quad \frac{\partial z_{n-1}}{\partial \alpha_n} = \frac{\partial z_n}{\partial \alpha_{n-1}}$$

$$\underline{(n+1)^{n-2}}: \quad \frac{\partial z_n}{\partial \alpha_n}.$$

These are all the lower anti-triangular elements of $[\xi]$. Now it is also true that all of the signals $(n+1)^1, \dots, (n+1)^{n-2}$ can be obtained as linear combinations of the signals at nodes 1 through $n+1$ of the sensitivity model and the signals z_1, \dots, z_{n-1} , rather than by differentiating $n-2$ times the signal at node $n+1$. As an example, consider $\frac{\partial z_n}{\partial \alpha_4}$. By the symmetry exhibited,

$$\frac{\partial z_n}{\partial \alpha_4} = \left(\frac{\partial z_{n-1}}{\partial \alpha_4} \right) = \left(\frac{\partial z_n}{\partial \alpha_3} \right) = \left(\frac{\partial z_{n-1}}{\partial \alpha_3} \right) = \left(\frac{\partial z_n}{\partial \alpha_2} \right). \quad (12)$$

But by inspection,

$$\left(\frac{\partial \dot{z}_n}{\partial \alpha_2} \right) = -z_2 - \alpha_n \frac{\partial z_n}{\partial \alpha_2} - \alpha_{n-1} \frac{\partial z_1}{\partial \alpha_n} - \dots - \alpha_1 \frac{\partial z_1}{\partial \alpha_2}, \quad (13)$$

and hence

$$\frac{\partial z_n}{\partial \alpha_4} = -z_3 - \alpha_n \left(\frac{\partial z_n}{\partial \alpha_2} \right) - \alpha_{n-1} \left(\frac{\partial z_1}{\partial \alpha_n} \right) - \dots - \alpha_1 \left(\frac{\partial z_1}{\partial \alpha_2} \right) \quad (14)$$

which is equivalent to

$$\begin{aligned} \frac{\partial z_n}{\partial \alpha_4} = & -z_3 + \alpha_n z_2 + \alpha_n^2 \frac{\partial z_n}{\partial \alpha_2} + \alpha_n \alpha_{n-1} \frac{\partial z_1}{\partial \alpha_n} + \dots + \alpha_n \alpha_1 \frac{\partial z_1}{\partial \alpha_2} \\ & - \alpha_{n-1} \frac{\partial z_n}{\partial \alpha_2} - \dots - \alpha_1 \frac{\partial z_1}{\partial \alpha_3}. \end{aligned} \quad (15)$$

In order to obtain the signal $(n+1)^P$, the signals z_1, \dots, z_{p+1} are needed in addition to those at nodes 1 through $n+1$. Thus, in order to obtain all the signals $(n+1)^1, \dots, (n+1)^{n-2}$, the signals z_1, \dots, z_{n-1} are needed in addition to those at nodes 1 through $n+1$.

The complexity of proof of the total symmetry and complete simultaneity properties should not obscure the basic simplicity of application of the result. In order to obtain all the elements of the matrix [5], only $2n-1$ signals need be obtained from the system and one sensitivity model. Of these, $n+1$ are obtained directly as the signals at nodes 1 through $n+1$ of the sensitivity model. The remaining $n-2$ functions are obtained as linear combinations of the signals at nodes 1 through $n+1$ of the sensitivity model and z_1, \dots, z_{n-1} as described in the proof of Properties I and II.

APPLICATION TO A GENERAL SYSTEM

In this section, a method of utilizing Properties I and II for systems in companion form in the sensitivity analysis of general single-input, linear, time-invariant controllable systems is developed. Consider the system of (2), with the matrices A , b , and C depending on an r -dimensional parameter vector \underline{q} . Then it is evident that the transformation T in (3) will depend on \underline{q} in general, i.e.,

$$T = T(\underline{q}). \quad (16)$$

Further, the coefficients of the characteristic equation $\alpha_1, \dots, \alpha_n$ will also depend on \underline{q} . For convenience, define the vector $\underline{\alpha}$ as

$$\underline{\alpha}(\underline{q}) \triangleq \begin{bmatrix} \alpha_1(\underline{q}) \\ \alpha_2(\underline{q}) \\ \vdots \\ \alpha_n(\underline{q}) \end{bmatrix} \quad (17)$$

Now with the system (2) in the canonical form (4), it is evident that $\underline{\alpha}$ represents a new parameter vector (n dimensional) in the canonic system, and that no other parameters exist in this system to affect the state \underline{z} . The parameters $\alpha_1, \dots, \alpha_n$ will be referred to as essential parameters.

An important result which follows from the above discussion is:

Assertion 1:

The sensitivity functions of all states x_i in the system (2) with respect to all r parameters ($r \leq n$) can be obtained as linear combinations of the sensitivity functions $\frac{\partial z_i}{\partial \alpha_j}$, $i, j=1, \dots, n$, of the canonic system and the states \underline{x} .

Proof:

This follows directly from (3) and (16). For any parameter q_i ,

$$\frac{\partial \tilde{x}}{\partial q_i} = \frac{\partial T}{\partial q_i} \tilde{z} + T(q) \frac{\partial \tilde{z}}{\partial \alpha} \frac{\partial \alpha}{\partial q_i} = \frac{\partial T}{\partial q_i} T^{-1} \tilde{x} + T(q) \frac{\partial \tilde{z}}{\partial \alpha} \frac{\partial \alpha}{\partial q_i}. \quad (18)$$

At this point, by combining Properties I and II with Assertion 1, the following important result is obtained:

Assertion 2:

Consider a linear, time-invariant, single input controllable system described by the equations

$$\dot{\tilde{x}} = A(q)\tilde{x} + b(q)u, \quad y = C(q)\tilde{x}.$$

The sensitivity functions of all states (or outputs) with respect to all r parameters can be generated by algebraically combining the signals on a single n th order canonical sensitivity model in addition to the considered system. This is accomplished by combining the results of Properties I and II and Assertion 1 as shown in Figure 3.

In order to realize the utility of this result, suppose the sensitivity functions of the states of a linear, single input, time-invariant, controllable twentieth order system with respect to six variable parameters are desired. By straightforward use of currently-available methods, this would require 6 twentieth order sensitivity models in addition to the system itself, i.e., a 140th order dynamic system. However, using the techniques of this paper, only one twentieth

order system would be needed in addition to the given system, i.e., a 40th order system. Indeed this represents a considerable savings.

THE COMPUTATIONAL ALGORITHM

To implement the results of the previous section in a computer simulation, the transformation T of (3) as well as $\frac{\partial T}{\partial q_i}$ and $\frac{\partial \alpha_k}{\partial q_i}$, $k=1, \dots, n$, $i=1, \dots, r$ must be obtained. If it were necessary to find T and the α_k 's as general functions of q to compute these necessary derivatives, the utility of Assertion 2 would be questionable. However, it is now shown that $\frac{\partial T}{\partial q_i}$ and $\frac{\partial \alpha_k}{\partial q_i}$ at any $q \in Q$ can be obtained recursively by an extension of the Leverrier algorithm.

It has been shown that the matrix T can be calculated by the relations [14,15,16]

$$T = [\tilde{t}_1 \dots \tilde{t}_n]$$

where

$$\begin{aligned} \tilde{t}_n &= b \\ \tilde{t}_{n-1} &= A\tilde{t}_n + \alpha_n \tilde{t}_n \\ \tilde{t}_{n-2} &= A\tilde{t}_{n-1} + \alpha_{n-1} \tilde{t}_n \\ &\vdots \\ \tilde{t}_1 &= A\tilde{t}_2 + \alpha_2 \tilde{t}_n. \end{aligned} \tag{19}$$

The coefficients of the characteristic equation which are required to find the transformation can be calculated by using Leverrier's algorithm [15-17]:

$$(sI-A)^{-1} = \frac{\Gamma(s)}{\Delta(s)} = \sum_{i=1}^n \frac{s^{i-1}}{\Delta(s)} S_{i+1} \tag{20}$$

where $\Delta(s) \triangleq \det(sI-A) = \sum_{i=0}^n \alpha_{i+1} s^i$

and where the S_{i+1} and α_{i+1} are determined by the relationships

$$\begin{cases} \alpha_{n+1} = 1 & S_{n+1} = I \\ \alpha_{n-j+1} = -\left(\frac{1}{j}\right) \text{tr}(AS_{n-j+2}), & S_{n-j+1} = \alpha_{(n-j+1)} I + AS_{n-j+2}. \end{cases} \quad (21)$$

A check on the numerical calculations is that $S_1 = 0$ should be obtained.

Thus one can generate the canonical form for any single input, linear, time-invariant controllable system with relative ease on a digital computer by using Equations (19) and (21).

A new algorithm can now be developed from the above, so $\frac{\partial T}{\partial q_i}$ and $\frac{\partial \alpha_k}{\partial q_i}$, $k=1, \dots, n$, $i=1, \dots, \gamma$, can be calculated without knowing the general functional dependence of T and α_k on q . By differentiating (19), we obtain

$$\begin{aligned} \frac{\partial T}{\partial q_i} &= \left[\frac{\partial \tilde{t}_1}{\partial q_i} \dots \frac{\partial \tilde{t}_n}{\partial q_i} \right] \\ \frac{\partial \tilde{t}_n}{\partial q_i} &= \frac{\partial \tilde{b}}{\partial q_i} \\ \frac{\partial \tilde{t}_{n-1}}{\partial q_i} &= \frac{\partial A}{\partial q_i} \tilde{t}_n + A \frac{\partial \tilde{t}_n}{\partial q_i} + \frac{\partial \alpha_n}{\partial q_i} \tilde{b} + \alpha_n \frac{\partial \tilde{b}}{\partial q_i} \\ \frac{\partial \tilde{t}_{n-2}}{\partial q_i} &= \frac{\partial A}{\partial q_i} \tilde{t}_{n-1} + A \frac{\partial \tilde{t}_{n-1}}{\partial q_i} + \frac{\partial \alpha_{n-1}}{\partial q_i} \tilde{b} + \alpha_{n-1} \frac{\partial \tilde{b}}{\partial q_i} \\ &\vdots \\ \frac{\partial \tilde{t}_1}{\partial q_i} &= \frac{\partial A}{\partial q_i} \tilde{t}_2 + A \frac{\partial \tilde{t}_2}{\partial q_i} + \frac{\partial \alpha_2}{\partial q_i} \tilde{b} + \alpha_2 \frac{\partial \tilde{b}}{\partial q_i}. \end{aligned} \quad (22)$$

Thus, if the functional dependence of the system matrices A and \tilde{b} on q is

assumed known (so that $\frac{\partial A}{\partial q_i}$ and $\frac{\partial b}{\partial q_i}$ are known), only $\frac{\partial \alpha_k}{\partial q_i}$, $k=2, \dots, n$, need be found to calculate $\frac{\partial T}{\partial q_i}$ with the same ease that T is calculated. Again, it would not be an easy task to find the α_k 's as general functions of q in order to find $\frac{\partial \alpha_k}{\partial q_i}$ at a point $q^0 \in Q$. However, these derivatives can be calculated recursively by extending Leverrier's algorithm, (21). If (21) is differentiated with respect to q_i , one has the algorithm

$$\begin{aligned} \frac{\partial \alpha_{n+1}}{\partial q_i} &= 0 & \frac{\partial S_{n+1}}{\partial q_i} &= 0 \\ \frac{\partial \alpha_{n-j+1}}{\partial q_i} &= -\left(\frac{1}{j}\right) \text{tr} \left(\frac{\partial A}{\partial q_i} S_{n-j+2} + A \frac{\partial S_{n-j+2}}{\partial q_i} \right) \\ \frac{\partial S_{n-j+1}}{\partial q_i} &= \frac{\partial \alpha_{n-j+1}}{\partial q_i} I + \frac{\partial A}{\partial q_i} S_{n-j+2} + A \frac{\partial S_{n-j+2}}{\partial q_i}. \end{aligned} \quad (24)$$

Thus, by using Equations (5), (22), (23), and (24) $T(q)$, $\frac{\partial T}{\partial q_i}$, and $\frac{\partial \alpha_k}{\partial q_i}$ ($k=1, \dots, n$ $i=1, \dots, r$) at any point $q \in Q$ can be calculated without the need of knowing the functional dependence of T on q . With this result, the utility of Assertion 2 in the sensitivity analysis of systems by computer simulation is greatly enhanced.

EXAMPLE

As an application of the above results, consider the system shown in Figure 4. This is a fourth order system with eight variable parameters q_1, \dots, q_8 . Suppose all the sensitivity functions $\frac{\partial x_i}{\partial q_j}$, $i=1, \dots, 4$, $j=1, \dots, 8$ are desired. Then, by straightforward application of the structural methods for obtaining sensitivity functions [1,2,5,9], eight sensitivity models in addition to the system model are required.

(Shrewd application of the "sensitivity points" method may reduce this number by one or two.) That is, a 36th order dynamic system is needed to generate the sensitivity functions. However, using the methods of this paper, only an 8th order dynamic system is needed to generate the sensitivity functions. Both of these methods were simulated on a digital computer to compare the solution times required. The nominal parameter values q_1^0, \dots, q_8^0 were all 1 and the sensitivity functions were obtained for $t=0$ to $t=25$ sec. Using the eight sensitivity models required 1 minute and 4 sec. to obtain the sensitivity functions, whereas only 28 sec. were required for the solution when essential parameter techniques were used. One might have assumed the solution time ratios should have been of the order 4:1, due to the order of the systems (36/8). However, some time is required to calculate the transformation T , α_n , $\frac{\partial T}{\partial q_i}$, $\frac{\partial \alpha_n}{\partial q_i}$, and z . The above time saving is considerable, however, especially when the sensitivity functions are to be generated iteratively for many parameter values as in an automatic parameter optimization or adaptive problem [6-8]. In 10 iterations, more than five minutes of computer time would be saved. Further, in a hybrid or analog computer application of these methods, the equipment saved when using essential parameter techniques would be considerable.

CONCLUSIONS

Two useful properties of the sensitivity functions for a system in the companion form have been demonstrated. By combining these properties with a known transformation for mapping a single input, linear,

time-invariant, controllable n th order system into companion form, a new method of generating the sensitivity functions of all the state variables with respect to the variable parameters in a system by using only one n th order canonical sensitivity model in addition to the system itself was developed. The Leverrier algorithm has been extended to facilitate use of the techniques presented in digital or hybrid computer sensitivity analysis. An example demonstrated that use of the techniques given in the paper can result in considerable time savings in computer sensitivity analysis, which is especially important in applications where the sensitivity functions must be generated iteratively at many points in the parameter space.

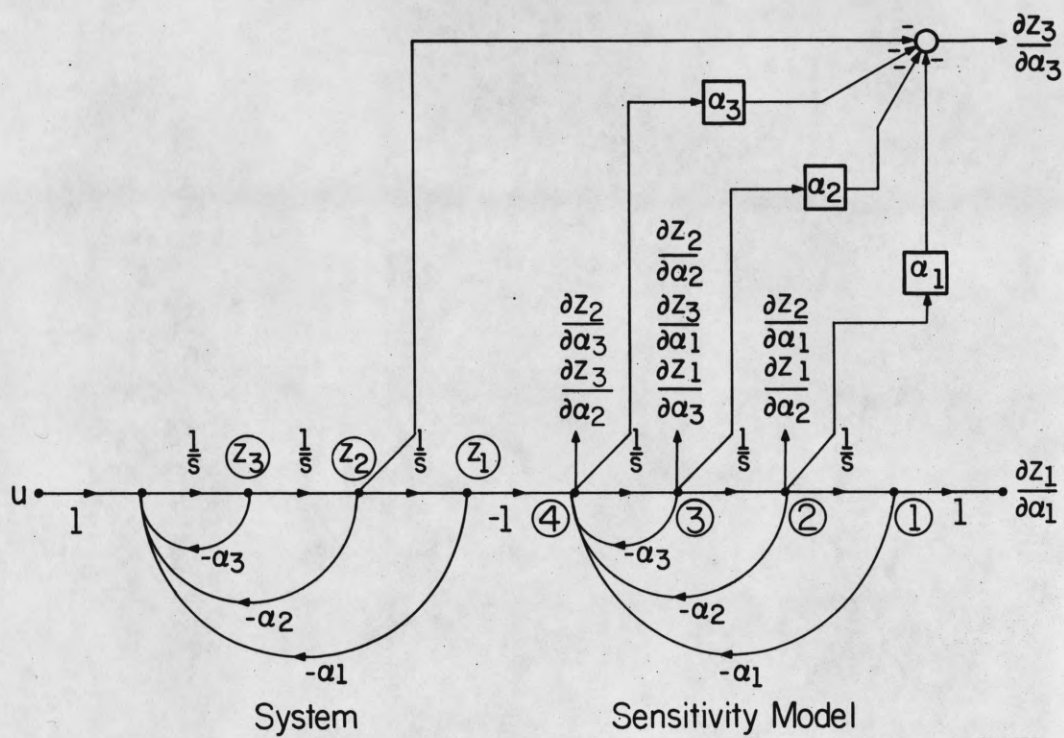
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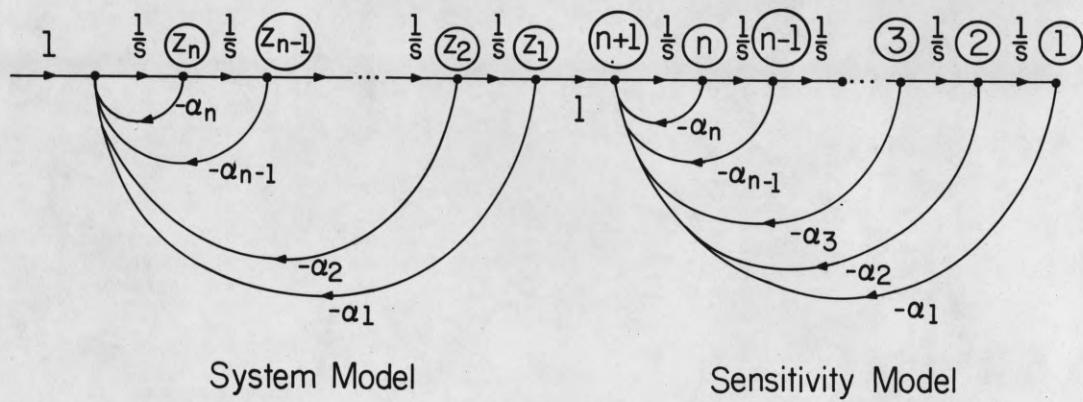
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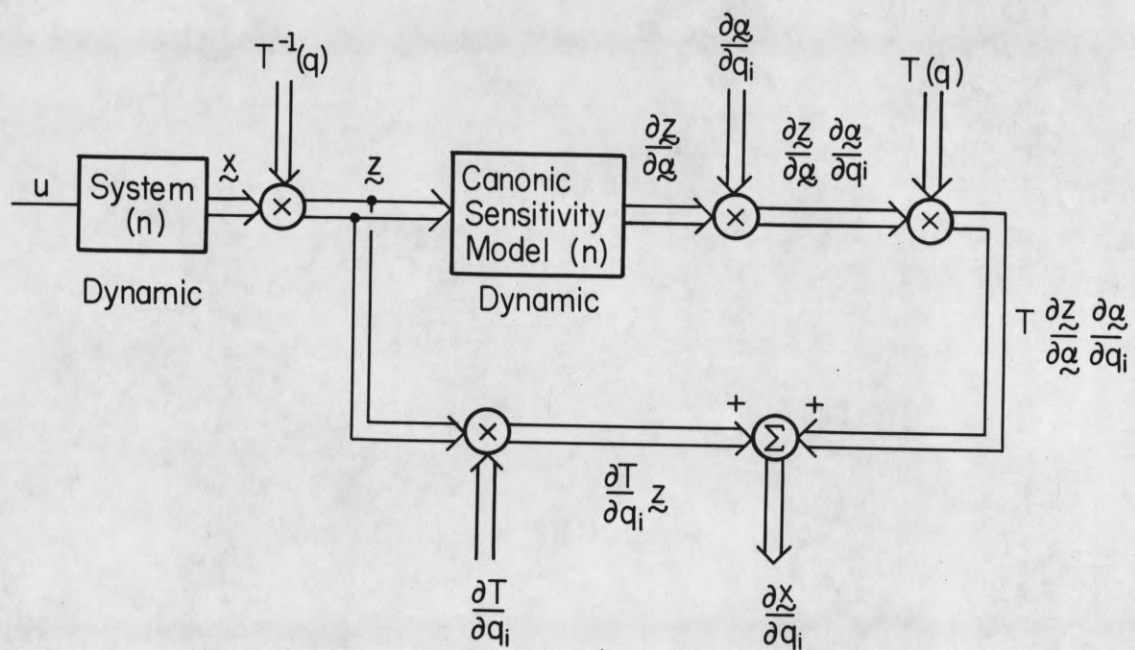
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Figure 1. Third order example



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Figure 2. Nth order companion system and sensitivity model.



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Figure 3. Block diagram for generating the sensitivity functions.

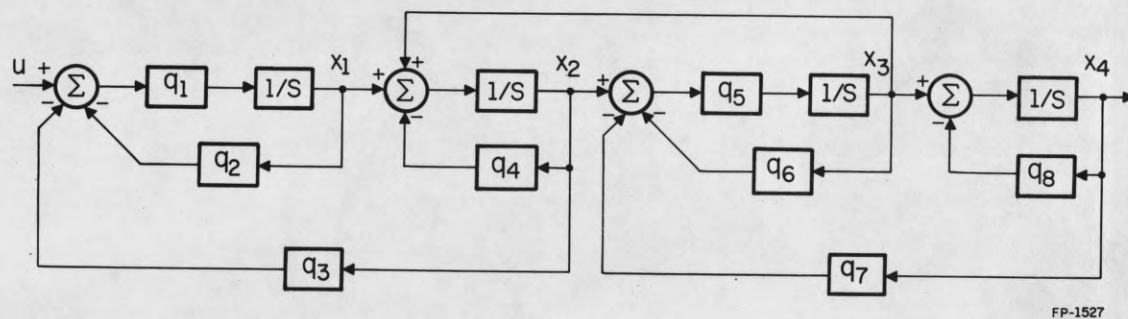


Figure 4. Fourth order example.

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