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MINIMAX CONTROLLER DESIGN

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When an uncertain dynamic system is to be controlled in an optimal manner, that is when the controller is required to minimize a performance index, the controller design requires a compromise between controller complexity and system performance. At one extreme is the optimal-adaptive controller which is difficult to realize but yields ideal performance, and at the other is any overly simplified controller which yields unacceptable performance. The reasonable controller structures for a given system can often be determined by the designer in terms of a number of free parameters. Then for each structure it is desired to find those parameters which yield the "best" system performance. This thesis develops minimax methods for determining such controller parameters.

The concept of performance-sensitivity is introduced to meet the usual criticism of minimax or "worst-case" design, that it is too pessimistic in concentrating all attention on the worst parameters. Properties of minimax control with a performance-sensitivity as the index are developed. It is shown that the usually desired range of system properties can be achieved by minimaximizing either the system performance index or a performance-sensitivity.

A new algorithm for solving algebraic minimax problems, regardless of the presence of a saddle point, is presented and proved to converge. The rate of convergence, and simplifications which occur when the system is linear or when the index has convexity properties, are discussed. The algorithm is extended to the case of time-varying minimizing parameters, and methods and problems of computation are discussed.

A method of obtaining multi-point optimality with time-varying controllers is presented. This method is of particular value when the computational difficulties of finding minimax time-varying parameters are prohibitive.

A general approach to the design of controllers for uncertain systems is presented. Several examples illustrate the utility and practicality of the methods developed.

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TABLE OF CONTENTS

1. INTRODUCTION

1.1 General Introduction

It is seldom possible to specify accurately all parameters of a physical dynamic system. Uncertainties may arise for many possible reasons. Sometimes, for example, the environment of a system is changeable, as in the case of a jet aircraft which must operate at high and low altitudes. Sometimes the system itself changes, as in the case of the fuel-load of an aircraft or rocket. Often the initial state of a dynamic system is not known precisely. It may even be desirable to allow for the deficiencies of a simplified mathematical model of an actual system, by introducing variable parameters into the model. In all of these cases it is more realistic, when determining the mathematical model of the system, to specify ranges of possible values for some coefficients, rather than to specify fixed nominal values for all coefficients.

If it is desired to control an uncertain dynamic system in an optimal manner, that is if a performance index is to be minimized, the problem of designing the controller can be approached in many ways. When the parameters in question can be measured, and when an optimal control is known for each set of values of the parameters, then at least in theory an optimal-adaptive controller can be constructed. Such a controller measures the parameters and applies the optimal control accordingly. Although this controller yields the best possible system performance it is inherently complex and therefore expensive to realize. Consequently an optimal-adaptive controller can be justified only in a small number of specialized situations. Werner has shown that there are cases in which a controller exists which is equivalent to the optimal-adaptive controller but which does not require direct measurement of the parameters [1]. Unfortunately the control is in terms of an infinite series so the full equivalence cannot be realized in practice. In the past a common approach has been to design as though the system parameters were fixed at nominal values, that is to adopt a nominal-optimal controller. Such a controller is satisfactory when the uncertain parameters have little effect on the system or when they are restricted to small domains. In general, however, the performance with a nominal-optimal controller becomes unsatisfactory as the parameters vary significantly from the nominal values. A variation of this method is to design at nominal values of the parameters but to include sensitivity constraints [2]. This procedure is negative in the sense that making the system insensitive to the parameter variations prevents the possibility of improvement as the parameters become more favorable and may result in a uniformly poor performance at all values of the parameters. Of course the situation is changed if uniformity rather than optimality is desired. In an interesting paper [3] Dorato and Drenick have proposed treating the problem as a game between the controller and nature. There are parallels between the control of an uncertain system and the usual game-theory problem but there is also a fundamental dissimilarity. The game-theory problem requires the determination of rational strategies for opposing players while the control problem requires the determination of a controller only. The game-theory requirement of opposing strategies has led traditionally to the defining of strategies in probabilistic terms but such a step in controller design is usually unnecessary and undesirable.

The final problem which confronts the designer of a controller is the problem of realizing that controller and not simply determining the controls it should generate. Usually there are two basic limits on the system design, the practical or economic limit set on the controller complexity, and the limit on acceptable system performance. Within these limits it is the task of the designer to come to a suitable compromise between the system performance and the controller complexity. A natural question arising in the attempt to reach this compromise is: what is the best possible system performance that can be obtained from a controller of given structure? Schoenberger has considered this question in specific cases in the context of systems with known parameters [4]. When the system includes uncertain or variable parameters, and the control structure is not capable of generating the optimal control at all values of the parameters, the question cannot be answered without first specifying how the performance is to be evaluated. It is necessary to choose a basis for comparing different controllers, or in other words a "supercriterion" [5].

Evidently the supercriterion should be based on the given performance index, and take into account the uncertainty of the variable system parameters. One common supercriterion is the expected value of the performance index. The most desirable controller in this case minimizes the expected value of the index. Another common supercriterion is the maximum value of the performance index. The most desirable controller in this case minimizes the maximum value of the index and is therefore determined by a minimax problem. The choice of an expected value as the supercriterion is most appropriate when the parameters have known probability distributions and when the average performance of a large number of similar systems is of major concern. However, the computational difficulties of minimizing an expected value are often prohibitive. Minimization of a maximum value is a fundamentally simpler problem since for any control the maximum value of an index is usually attained at a small discrete set of parameter values while an expected value cannot be computed without consideration of all values of the parameters.

A common criticism of minimax or "worst-case" design is that it is too pessimistic in concentrating all attention on the worst parameters. An answer to this criticism proposed by Rohrer and Sobral [6] is to use the maximum value of the system "relative sensitivity" (i.e. the fractional deviation of the performance index from optimality) as the supercriterion. This idea has been generalized to the concept of performance-sensitivity in this thesis. In general, a controller which minimizes the maximum value of a performance-sensitivity tends to maintain the system performance near optimality regardless of the values of the variable parameters.

Properties of the two kinds of minimax controller (i.e. with definition of the supercriterion as the maximum value of the performance index or as the maximum value of a performance-sensitivity) are developed in this thesis through several theorems and examples. It is shown that typically desired system properties are achieved with these controllers. A minimax approach to the general problem of designing controllers for uncertain dynamic systems is developed.

Particularly in the case of time-varying controllers it may occur that the system dimensionality prohibits the computation of a minimax controller. In this case the desirable method of determining the controller with the aid of a supercriterion must be abandoned in favor of more heuristic methods. One such method which is developed briefly in this thesis is that of multi-point optimality. Using this method it is sometimes possible to design an excellent controller with a minimum of computation.

1.2 Minimax Problems

It is well recognized that a saddle-point is one form of solution to a minimax problem. This is the type of solution desired in game-theory since it leads to a rational definition of opposing strategies. In the present context the important feature of a saddle-point is that it can be located by variational means [7]. However, in the minimax design of controllers for uncertain systems experience has shown that a saddle point occurs rarely indeed.

The minimax problem without the assumption of a saddle point has received little attention in the literature. In the general case of time-varying minimizing parameters and time-varying maximizing parameters the most prominent available method appears to be dynamic programming. Application of the calculus of variations to this general case is impractical (see Appendix II of [6]). An algorithm has been proposed recently for the case of time-invariant minimizing parameters and a single time-varying maximizing parameter [8]. This algorithm uses a gradient technique and yields a local solution to the minimax problem under certain conditions.

For algebraic minimax problems (i.e. when the minimizing and maximizing parameters are time-invariant) there appear to be three possible methods of solution:

(a) locate a saddle point (if it exists) and show that it represents a global solution

(b) find an <u>analytic</u> solution to the maximization step (if it exists) and then minimize

(c) use a search or iterative procedure.

Since (a) and (b) are seldom relevant to the controller design problem considered here only (c) remains. A direct search procedure is straight-forward but can only be applied in practice if there are few parameters involved.

A new algorithm for numerical solution of algebraic minimax problems is developed and proved to converge in this thesis. The rate of convergence, the methods of computation, and the simplifications which occur in special cases are discussed. The algorithm is also extended to the case of time-varying minimizing parameters and the special methods and difficulties of computation in this case are discussed. Several examples illustrate the utility of these algorithms in controller design.

2. TIME-INVARIANT CONTROLLERS

2.1 Formulation

It is assumed that the dynamic system and the scalar performance index can be expressed in the form:

$$\dot{x}(t) = f[x(t), u(t), v, t], \; x(t_0) = x^0(v)$$

$$J(u,v) = \int_{t_o}^{T} L[x(t), u(t), v, t]dt$$

$$u(t) \in U$$

$$v \in V \tag{2.1}$$

where x(t) is an n-dimensional system state vector, f is an n-dimensional vector-valued function, u(t) is an m-dimensional control vector, v is an s-dimensional vector of unknown parameters and initial conditions, t represents time, J is a performance criterion to be minimized, and L is a non-negative scalar function. V is assumed to be a closed bounded region in E_s. It is assumed that a controller structure has been specified, i.e.

$$u(t) = u[c, x(t)] \tag{2.2}$$

where $c \in W$ is an r-dimensional vector of controller parameters to be determined and W is a closed bounded region in E_r such that $c \in W \Rightarrow u(t) \in U$. In the following c is generally referred to as the control. By substitution of (2.2) into (2.1) J becomes a function of the vectors c and v. For any $v \in V$, the control $u^0(v)$ which realizes $J^0(v)$, the minimum value of J, is the optimal control for that v. The controller capable of measuring v and generating $u^0(v)$ is called the optimal-adaptive controller.

Definition 2.1.1: $S(c,v) = F[J(c,v), J^0(v)]$ is a performance-sensitivity if

(a) F is continuous jointly in its two arguments

(b) $F > 0 \Leftrightarrow J(c,v) > J^0(v)$

(c) $F = 0 \Leftrightarrow J(c,v) = J^0(v)$.

Any continuous function of $[J(c,v) - J^0(v)]$ which is zero only at the origin and contained in the first quadrant, is a performance-sensitivity. Usually the specifications of a system dictate the choice of either one of the relative sensitivity, $S(c,v) = \frac{J(c,v) - J^0(v)}{J^0(v)}$, or (particularly when $J^0(v)$ may be zero) the absolute sensitivity $S(c,v) = J(c,v) - J^0(v)$. However, the definition allows the possibility of weighting the actual value of the performance index, the optimal value of the index, or the deviation of the index from optimality, in any desired manner to achieve special system properties.

Definition 2.1.2: $\hat{c} \in W$ is a minimax performance control if and only if

$$\max_{v \in V} J(\hat{c},v) = \min_{c \in W} \max_{v \in V} J(c,v).$$

Definition 2.1.3: $c^* \in W$ is a minimax performance-sensitivity control if and only if

$$\max_{v \in V} S(c^*,v) = \min_{c \in W} \max_{v \in V} S(c,v).$$

As demonstrated by the examples in this thesis there are many control problems for which one of the two controls $\hat{c}$ or c^* is appropriate.

2.2 Comparison of Controls: An Example

Consider the second-order system

$$\dot{x}_1 = x_2, \qquad x_1(0) = 1$$
$$\dot{x}_2 = -vx_2 + u, \quad x_2(0) = 0$$
$$0 \leq v \leq 2$$
$$J = \int_0^\infty (x_1^2 + x_2^2 + u^2)dt.$$

The optimal control is $u^0(v) = -x_1 - (\sqrt{3+v^2} - v)x_2$ and yields an optimal performance $J^0(v) = \sqrt{3 + v^2}$. An obvious choice for the controller structure is simply $u(t) = c_1\, x_1(t) + c_2\, x_2(t)$.

Consider first a nominal-optimal controller. The choice of $v^{nom} = 1.0$ gives $c^{nom} = (-1.00, -1.00)$. As mentioned in the introduction an extension of this approach is to design at a nominal value of v but to include constraints on the system sensitivity to variation in v [2]. If it is hypothesized that a control c^r exists which is capable of yielding a performance $J(c^r, v)$ completely insensitive to variation in v it follows that

$$J(c^r, v) = \text{const} \geq \max_{v \in V} J(c^0(v), v)$$

since no controller can perform better at any value of v than the optimal-adaptive controller. It is emphasized that c^r is only a hypothetical control, representing the ideal result of an attempt to make the system performance insensitive to variation in v.

For this example the minimax performance control $\hat{c}$ is simply the control which is optimal at $v = 2$, i.e. $\hat{c} = (-1.00, -.65)$.

A suitable choice for the performance-sensitivity is

$$S(c,v) = \frac{J(c,v) - J^0(v)}{J^0(v)} .$$

The minimax performance-sensitivity control c^* (computed by the algorithm in Section 2.3) is $c^* = (-1.20, -1.79)$.

The example in this section was also considered in [6] where c_1 was set equal to -1.00 and the value of c_2 required to realize $\min_{c_2} \max_{v \in V} S(c,v)$ was determined by a graphical procedure. Setting c_1 equal to -1.00, which simplifies the problem, is equivalent to making the constraint on the controller that it must be optimal for some value of v. The resulting control is (-1.00, -1.30) and it yields a minimax value of S which is approximately twice that of the value given by c^*. It is concluded that constraining the controller to be optimal at some value of v is undesirable in this example.

The variation of J with v for the controls c^{nom}, c^r, $\hat{c}$, and c^*, is shown in Figure 1.

Two drawbacks of the nominal-optimal control, c^{nom}, are illustrated by this example. The first is the arbitrary nature of the choice of v^{nom}. The second is that by definition a nominal-optimal controller is constrained to be optimal at some value of v. This is often an unwarranted constraint and in particular it is unwarranted in this example as noted above.

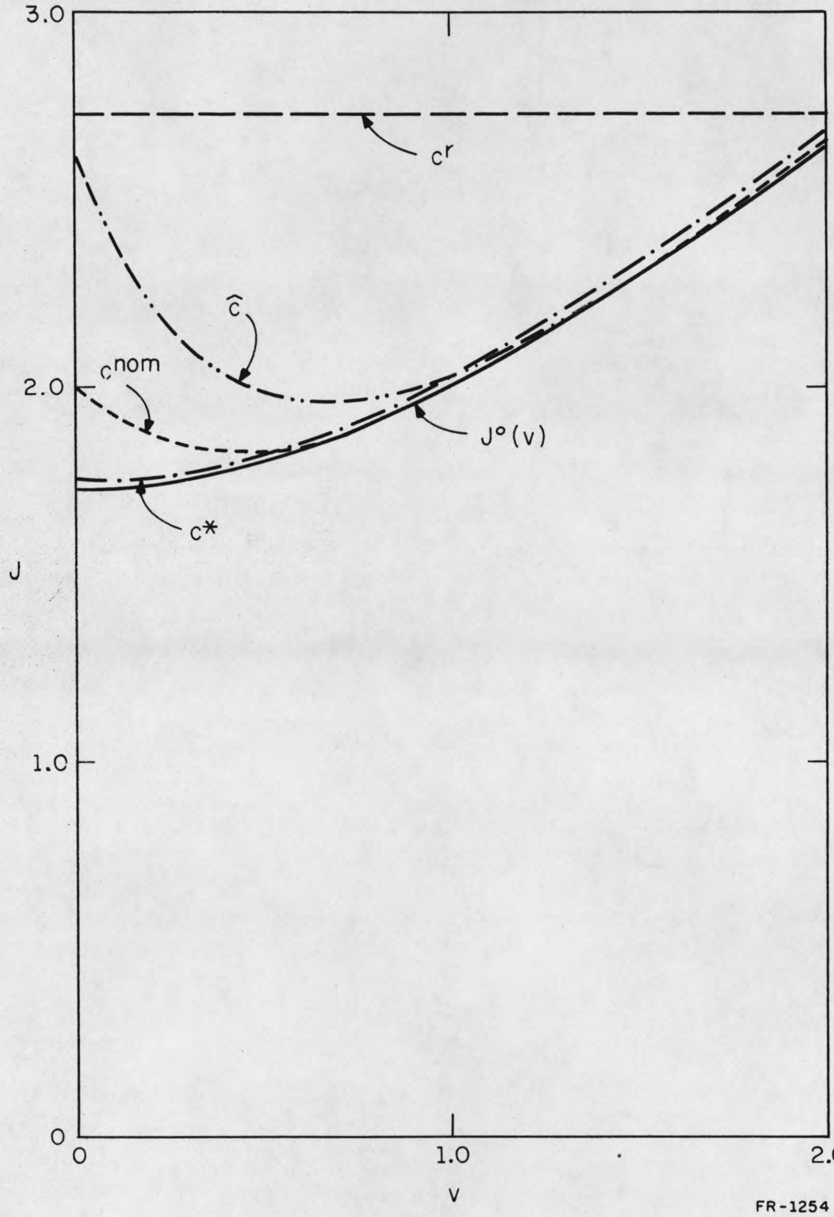

Figure 1. Comparison of controls.

It is clear from the figure that c^r is inferior to all the other controls except in its insensitivity to change as v varies, illustrating the obvious fact that parameter insensitivity is not compatible with near-optimal performance at all parameter values when the optimal performance varies widely.

It is also clear from the diagram that the minimax performance-sensitivity control c* is a very attractive control for this system since J(c*,v) does not exceed the optimal performance $J^0(v)$ by more than 1.9% at any value of v, yet realization of c* does not require the sensing of v nor the variation of any control parameters.

The example also illustrates the range of conservatism possible with minimax controls. The minimax performance control $\hat{c}$ in this example trades good performance at all values of v for a mere 1% improvement at the worst value of v. However, there are situations, such as in the example of Section 2.10, where the worst performance <u>is</u> of prime importance.

Finally, it should be emphasized that c* in this example does not correspond to an optimal control at any value of v. This appears to be a common property of minimax performance-sensitivity controls.

2.3 An Algorithm for Solving Algebraic Minimax Problems

Let S(c,v) be any continuous function of two vectors $c \in W$, $v \in V$, where W and V are closed bounded regions in E_r and E_s respectively. It is desired to determine c* satisfying

$\max_{v \in V} S(c^*,v) = \min_{c \in W} \max_{v \in V} S(c,v)$. The following algorithm replaces this minimax problem by a series of ordinary minimizations and maximizations.

Initialization: The process may be started either by choice of a control c^0 or by choice of an initial vector or vectors v^0. (If $S(c,v)$ is a performance-sensitivity, at least two vectors v^0 should be chosen in view of theorem 2.6.2.) For the purposes of demonstration assume it is convenient to choose v^{10}, v^{20}, $\in V$ $v^{10} \neq v^{20}$. Let $V_0 = \{v^{10}, v^{20}\}$.

First iteration: Minimize wrt* $c \in W$, $\max_{v \in V_0} S(c,v)$. Let c^1 be the minimizing value of c and define $S_1^m = \max_{v \in V_0} S(c^1,v)$.

Maximize wrt $v \in V$, $S(c^1,v)$. Let v^1 be the maximizing value of v and define $S_1^M = S(c^1,v^1)$. (If $S_1^m = S_1^M$ then $c^1 = c^*$ by the corollary to lemma 2.3.1.) Let V_1^a be any set of $v \in V_0$ satisfying $\max_{v \in V_1^a} S(c,v) = \max_{v \in V_0} S(c,v)$, $\forall c \in W$. Then define $V_1 = V_1^a \cup \{v^1\}$.

n^{th} iteration: Minimize wrt $c \in W$, $\max_{v \in V_{n-1}} S(c,v)$. Let c^n be the minimizing value of c and define $S_n^m = \max_{v \in V_{n-1}} S(c^n,v)$.

Maximize wrt $v \in V$, $S(c^n,v)$. Let v^n be the maximizing value of v and define $S_n^M = S(c^m,v^n)$. (If $S_n^m = S_n^M$ then $c^n = c^*$ by the corollary to lemma 2.3.1.) Let V_n^a be any set of $v \in V_{n-1}$ satisfying $\max_{v \in V_n^a} S(c,v) = \max_{v \in V_{n-1}} S(c,v)$, $\forall c \in W$. Then define $V_n = V_n^a \cup \{v^n\}$.

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*with respect to

<u>Termination</u>: If $S_i^m = S_i^M$ for any integer i then the algorithm terminates immediately. Alternatively in view of lemma 2.3.1 below, suitable termination criteria are to stop when $\frac{S_i^M - S_i^m}{S_i^m} \leq \xi_1$ or when $S_i^M - S_i^m \leq \xi_2$ where ξ_1 and ξ_2 are preset positive constants.

If for any c^i there are values of v which yield relative maxima of $S(c^i,v)$ greater than S_i^m, convergence may possibly be speeded by including these v's in V_i in addition to v^i. (The rate of convergence cannot be decreased by such additions to V_i.)

The algorithm is summarized in Figure 2.

The two lemmas and the theorem which follow prove that the algorithm solves the minimax problem. Lemmas 2.3.1 and 2.3.2 show that $\min_{c \in W} \max_{v \in V} S(c,v)$ is bounded below by S_i^m and above by S_i^M for each integer i, and that $\{S_i^m\}$ is a monotonic increasing sequence. Theorem 2.3.1 proves that the limit of the two sequences $\{S_i^m\}$ and $\{S_i^M\}$ is $\min_{c \in W} \max_{v \in V} S(c,v)$. It is a direct result of this theorem that c* can be found to any desired accuracy from the sequence $\{c^i\}$.

<u>Lemma 2.3.1</u>:

$$S_i^m \leq \min_{c \in W} \max_{v \in V} S(c,v) \leq S_i^M \text{ for } i = 1, 2, \ldots$$

<u>Proof</u>:

$$S_i^m = \min_{c \in W} \max_{v \in V_{i-1}} S(c,v) \text{ by definition}$$

$$\leq \min_{c \in W} \max_{v \in V} S(c,v) \qquad \text{since } V_{i-1} \subset V.$$

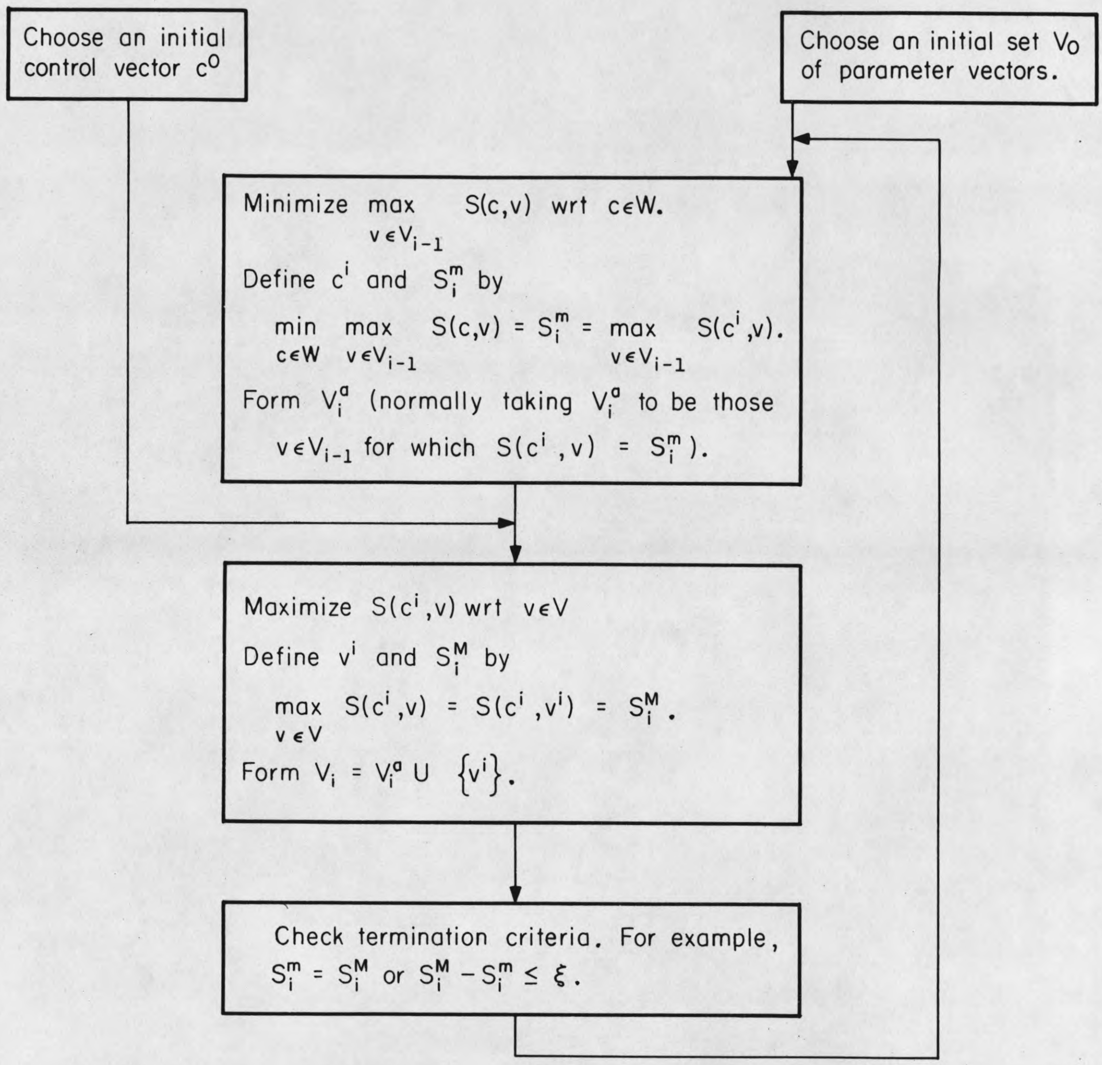

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Figure 2. Summary of the minimax algorithm.

$$S_i^M = \max_{v \in V} S(c^i, v) \quad \text{by definition}$$

$$\geq \min_{c \in W} \max_{v \in V} S(c,v).$$

<u>Corollary</u>: $S_i^m = S_i^M \Rightarrow c^i = c^*$

<u>Proof</u>: $S_i^m = S_i^M \Rightarrow S_i^m = \max_{v \in V} S(c^i, v)$ since $S_i^M = \max_{v \in V} S(c^i, v)$

$$= \min_{c \in W} \max_{v \in V} S(c,v) \quad \text{using lemma 2.3.1}$$

i.e. $c^i = c^*$ by definition of c^*.

<u>Lemma 2.3.2</u>: $S_1^m, S_2^m \ldots,$ is a monotonic increasing sequence.

<u>Proof</u>: $S_i^m = \min_{c \in W} \max_{v \in V_{i-1}} S(c,v)$ by definition of S_i^m

$$\geq \min_{c \in W} \max_{v \in V_{i-1}^a} S(c,v) \quad \text{since } V_{i-1}^a \subset V_{i-1}$$

$$= \min_{c \in W} \max_{v \in V_{i-2}} S(c,v) \quad \text{by definition of } V_{i-1}^a$$

$$= S_{i-1}^m .$$

<u>Theorem 2.3.1</u>: The sequences $\{S_i^m\}$ and $\{S_i^M\}$ both converge to $\min_{c \in W} \max_{v \in V} S(c,v)$.

<u>Proof</u>: If the algorithm terminates in a finite number of iterations the finite sequences $\{S_i^m\}$ and $\{S_i^M\}$ converge to $\min_{c \in W} \max_{v \in V} S(c,v)$ by Lemma 2.3.1. Therefore assume the sequences are infinite.

Since $S(c,v)$ is a continuous function and W and V are compact it follows that $S^* = \min_{c \in W} \max_{v \in V} S(c,v)$ exists. From Lemmas 2.3.1 and 2.3.2 $\{S_i^m\}$ is a monotonic increasing sequence bounded above by S*. It follows that the limit exists. Let $\lim_{i\to\infty} S_i^m = \hat{S}$.

If it can be shown that the sequence $\{S_i^M - S_i^m\}$ converges to zero it follows from Lemma 2.3.1 that

$$\lim_{i\to\infty} S_i^m = \hat{S} = \min_{c\in W} \max_{v\in V} S(c,v) = \lim_{i\to\infty} S_i^M ,$$

which is the desired result.

Suppose $\{S_i^M - S_i^m\}$ does not converge to zero. Then for some $\mathcal{E} > 0$ there is an infinite set $\{\bar{S}_m^M\}$ of elements of $\{S_i^M\}$ such that $\bar{S}_m^M - \hat{S} \geq \mathcal{E}$ for $\forall m$. Let $\{\bar{c}^m\}$ be the sequence of controls corresponding to $\{\bar{S}_m^M\}$. Then since $\{\bar{c}^m\}$ is an infinite sequence on a compact set W there is a convergent subsequence $\{\hat{c}^n\}$. Let the corresponding subsequence of $\{\bar{S}_m^M\}$ be $\{\hat{S}_n^M\}$. Clearly $\hat{S}_n^M - \hat{S} \geq \mathcal{E}$, $\forall n$.

Since S(c,v) is uniformly continuous on W x V $\exists \delta > 0$ such that

$$|S(c^a,v) - S(c^b,v)| \leq \frac{\mathcal{E}}{2} \text{ whenever } |c_i^a - c_i^b| < \delta,\ 1 \leq i \leq r,\ \forall v\in V .$$

Let $\hat{c}^n$, $\hat{c}^m$, $m > n$, be any two elements of $\{\hat{c}^n\}$ where n, m, are sufficiently large that $|\hat{c}_i^n - \hat{c}_i^m| \leq \delta$, $1 \leq i \leq r$. Then

$$|S(\hat{c}^n,v) - S(\hat{c}^m,v)| \leq \frac{\mathcal{E}}{2},\ \forall v\in V . \tag{2.3}$$

Let $\hat{S}_m^m$, $\hat{v}^n$, $\hat{V}_n$, $\hat{V}_{m-1}$, $\hat{V}_n^a$, be the values defined by the algorithm which correspond to $\hat{c}^n$, $\hat{c}^m$. Then

$$\hat{V}_n = \hat{V}_n^a \cup \{\hat{v}^n\}$$

and

$$\max_{v\in V} S(\hat{c}^n,v) = S(\hat{c}^n,\hat{v}^n) = \hat{S}_n^M \geq \hat{S} + \mathcal{E}. \tag{2.4}$$

By definition of V_i it follows that

$$\max_{v \in V_i} S(c,v) \geq \max_{v \in V_k} S(c,v), \ \forall i \geq k, \ \forall c \in W .$$

Therefore since $m > n$

$$\max_{v \in V_{m-1}} S(c,v) \geq \max_{v \in V_n} S(c,v), \ \forall c \in W .$$

$$\therefore \ \hat{S}^m_m = \max_{v \in V_{m-1}} S(\hat{c}^m,v) \geq \max_{v \in V_n} S(\hat{c}^m,v) \geq S(\hat{c}^m,\hat{v}^n) \qquad (2.5)$$

since $\hat{v}^n \in \hat{V}_n$.

Using (2.3),

$$S(\hat{c}^m,\hat{v}^n) \geq S(\hat{c}^n,\hat{v}^n) - \frac{\varepsilon}{2} .$$

Using this result, (2.4) and (2.5),

$$\hat{S}^m_m \geq S(\hat{c}^m,\hat{v}^n) \geq S(\hat{c}^n,\hat{v}^n) - \frac{\varepsilon}{2} \geq \hat{S} + \frac{\varepsilon}{2} > \hat{S} ,$$

which is the desired contradiction since $\hat{S}$ is the limit of the monotonic increasing subsequence $\{\hat{S}^m_i\}$.

The definition of a minimax control c* is that it minimizes with respect to $c \in W$ the quantity $\max_{v \in V} S(c,v)$. Since the sequence of terms $S^M_n = \max_{v \in V} S(c^n,v)$ converges to $S^* = \min_{c \in W} \max_{v \in V} S(c,v)$, it follows that the controls c^n are minimax controls to any desired degree of accuracy in achieving S*, when n is sufficiently large. If c* is a unique control it follows that $\{c^i\}$ converges to c*. If c* is not unique then any limit point $\overline{c}$ of $\{c^i\}$ is a solution for c*.

Choice of the V_i^a: The set V_i^a is defined to be any set of $v \in V_{i-1}$ satisfying

$$\max_{v \in V_i^a} S(c,v) = \max_{v \in V_{i-1}} S(c,v),\ \forall c \in W\ . \tag{2.6}$$

Clearly V_{i-1} is a satisfactory choice according to this criterion. However, it is desirable for V_i^a to be as small a set as possible in order to simplify the succeeding minimization. Thus it is desirable to retain in V_i^a only those values of v which are dominant in the sense of (2.6). Evidently (2.6) requires a strong dominance in that it must hold for all values of $c \in W$. A weaker form of dominance is exhibited by the set V_i^b of all values of $v \in V_{i-1}$ at which S_i^m is attained when $c = c^i$, i.e. the set of all $v \in V_{i-1}$ satisfying $S(c^i,v) = S_i^m$. Namely, there is by continuity of S(c,v), a neighborhood N of c^i, in which

$$\max_{v \in V_i^b} S(c,v) = \max_{v \in V_{i-1}} S(c,v),\ \forall c \in N.$$

With most functions S(c,v), and in particular those which typically arise in control problems, this weaker form of dominance is sufficient for convergence of the algorithm. Furthermore, the information required for the determination of V_i^b is available automatically from the minimization step yielding c^i. For this reason the sets V_i^b should be used in practical application of the algorithm. If for some function S(c,v) the sequence $\{S_i^M - S_i^m\}$ does not converge to zero with use of the sets V_i^b then the sets V_i^a can be employed in order to guarantee convergence.

Maximin problems: The algorithm is equally effective on maximin problems, which can be handled as above simply by using the negative of the pay-off function.

Minimization steps: While minimization wrt $c \in W$ of a function such as $\varphi = \max\{S(c,v^1), S(c,v^2)\}$ is much simpler than the original minimax problem there remains some difficulty because the partial derivatives of φ do not exist at the minimizing c. Figure 3 gives contours of constant φ for the first iteration of the example in Section 2.2. Methods which evaluate gradients by making perturbations of c are not capable of minimizing such a function. The author has used the Rosenbrock rotating coordinate search method as given by Wood [9]. An advantage of this method is that once the search point is on a sharp ridge like the one in Figure 3 and the search direction is along that ridge the minimum is quickly found. If necessary, the direction of the ridge can be determined (in two or n dimensions) by the simple expedient of locating it at two nearby points and using this information to compute the direction cosines.

Bounds on the number of values of v: Suppose the number of values of v at which the minimax value of S is attained is n, i.e. there are n values of v which satisfy

$$\max_{v \in V} S(c^*,v) = \min_{c \in W} \max_{v \in V} S(c,v).$$

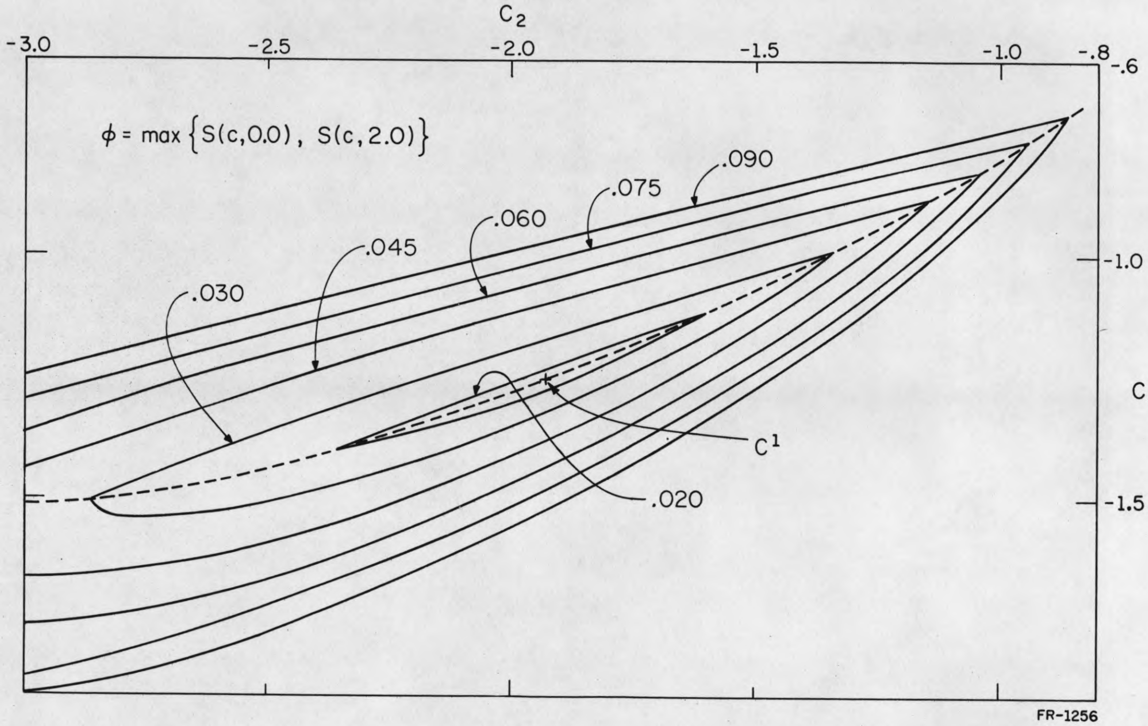

Figure 3. Contours of constant φ.

Then for a large class of systems the largest number of values of v required by the algorithm in the minimization steps is (n + 1). Therefore it is of interest to consider cases in which n can be determined a-priori.

If (1) S(c,v) is continuous jointly in c and v,

(2) v is a scalar

(3) $\frac{\partial S(c,v)}{\partial v}$ exists at all $c \in W$, $v \in V$,

(4) $\frac{\partial S(c,v)}{\partial v}$ has at most m zeroes wrt $v \in V$ for any $c \in W$,

then the value of n is at most $(\frac{m}{2} + 1)$ when m is even, and at most $(\frac{m+1}{2} + 1)$ when m is odd. This result follows from considering the possible configurations of $S(c^*,v)$ as a function of v.

A weaker result is possible when v is of arbitrary dimension. If S(c,v) satisfies (1) and

(5) $\text{grad}_v\, S(c,v)$ exists at all $c \in W$, $v \in V$,

(6) $\text{grad}_v\, S(c,v)$ has no more than m zeroes wrt $v \in V$, for any c,

then there are most m values of v at which the minimax value of S is attained and which are internal points of V.

2.4 Results for Convex Functions

Often the performance index in a control problem is chosen to be convex with respect to the control signal to ensure a unique optimal control. Also it sometimes occurs that the index is convex with respect to the variable parameters. It is therefore of practical importance to consider the effects of convexity on the minimax problem.

The following definitions and properties of convex functions which are required later in this section are presented here for convenience.

<u>Convexity</u>: A scalar function f(x) of a vector x is <u>convex</u> wrt $x \in X$ if for any $x^1, x^2 \in X$,

$$f[\alpha x^1 + (1-\alpha)x^2] \leq \alpha f(x^1) + (1-\alpha)f(x^2)$$

where α is any scalar, $0 \leq \alpha \leq 1$. If

$$f[\alpha x^1 + (1-\alpha)x^2] < \alpha f(x^1) + (1-\alpha)f(x^2)$$

for $0 < \alpha < 1$, then f is <u>strictly convex</u> wrt $x \in X$.

<u>Property (1)</u>: If f(x) is a continuous scalar function of a vector x and is strictly convex wrt $x \in X$ where X is closed then there is precisely one value of x at which f assumes its minimum value.

<u>Proof</u>: Since f is continuous and X is closed there is at least one value of x at which the minimum of f is assumed. Suppose there are two values, x^a, x^b. Then by the strict convexity

$$f\left(\frac{x^a + x^b}{2}\right) < \tfrac{1}{2} f(x^a) + \tfrac{1}{2} f(x^b) = f(x^a) = f(x^b).$$

Contradiction.

<u>Property (2)</u>: If $f_1(x)$, $f_2(x)$, are two scalar functions of a vector x, and are both strictly convex wrt $x \in X$ then $\max\{f_1(x), f_2(x)\}$ is also strictly convex wrt $x \in X$.

<u>Proof</u>: $\max\{f_1[\alpha x^1 + (1-\alpha)x^2],\ f_2[\alpha x^1 + (1-\alpha)x^2]\}$

$$< \max\{\alpha f_1(x^1) + (1-\alpha)f_1(x^2),\ \alpha f_2(x^1) + (1-\alpha)f_2(x^2)\}$$

$$\leq \alpha \max\{f_1(x^1),\ f_2(x^1)\} + (1-\alpha)\max\{f_1(x^2),\ f_2(x^2)\} .$$

Property(3): If $f(x)$ is a scalar function of an n-dimensional vector x and is convex wrt $x \in X$ where X is a convex polyhedron in n-dimensional space then $\max_{x \in X} f(x) = \max_{x \in X_0} f(x)$ where X_0 is the set of vertices of X.

Proof: Suppose a unique maximum is attained at x^a not a vertex of X. Then there exist x^1, x^2, $\in X$, such that $x^a = \alpha x^1 + (1-\alpha)\, x^2$ for $0 < \alpha < 1$. Therefore

$$f(x^a) = f[\alpha x^1 + (1-\alpha)x^2] \leq \alpha f(x^1) + (1-\alpha) f(x^2)$$

$$\leq \max\{f(x^1),\ f(x^2)\}.$$

Contradiction.

The algorithm in Section 2.3 shows that an algebraic minimax problem can always be decoupled into a series of minimizations and maximizations. In a sense the algorithm provides a method for finding the values of v at which minimization will yield the minimax control. The following theorem shows that when the function is strictly convex with respect to c the values of v required are precisely those at which the minimax value is attained when $c = c^*$.

Theorem 2.4.1: Let $S(c,v)$ be a function of an r-dimensional vector $c \in W$ and an s-dimensional vector $v \in V$ with the properties:

(a) $S(c,v)$ is continuous jointly in c and v, $\forall c \in W$, $\forall v \in V$,

(b) $\frac{\partial S(c,v)}{\partial c_i}$ exists and is continuous jointly in c and v, $\forall c \in W$, $\forall v \in V$, $1 \leq i \leq r$,

(c) $S(c,v)$ is strictly convex wrt $c \in W$ for each $v \in V$.

Let c^* be the minimax control and suppose the minimax value of S is attained only at $v^1, \ldots, v^k$ when $c = c^*$. Let $V_0 \equiv \{v^1, \ldots, v^k\}$. Let c^0 be defined by $\max_{v \in V_0} S(c^0,v) = \min_{c \in W} \max_{v \in V_0} S(c,v)$. Then $c^0 = c^*$.

Proof: By an obvious extension of property (2) for convex functions, $\max_{v \in V_0} S(c,v)$ and $\max_{v \in V} S(c,v)$ are both strictly convex wrt $c \in W$. Then by property (1) c^0 and c^* are unique minimizing values of c.

Suppose the theorem is false and $c^0 \neq c^*$. Then by the uniqueness of c^0, and the definition of V_0,

$$\max_{v \in V_0} S(c^*,v) = S(c^*,v^i) > \max_{v \in V_0} S(c^0,v) \geq S(c^0,v^i), \text{ for } 1 \leq i \leq k.$$

Therefore $\exists j > 0$ such that

$$S(c^*,v^i) \geq S(c^0,v^i) + j \text{ for } 1 \leq i \leq k.$$

Let Δc be the vector, $\Delta c \equiv (c^* - c^0)$. Then using (c),

$$S(\alpha c^0 + (1-\alpha)c^*,v^i) < \alpha S(c^0,v^i) + (1-\alpha)S(c^*,v^i) \text{ for } 0 < \alpha < 1.$$

$$\therefore \quad S(c^*-\alpha\Delta c,v^i) - S(c^*,v^i) < \alpha[S(c^0,v^i) - S(c^*,v^i)]$$

i.e., $S(c^*,v^i) - S(c^*-\alpha\Delta c,v^i) > \alpha j$ for $0 < \alpha < 1$.

This is true in particular for α arbitrarily close to zero.

But using (b),

$$S(c^*,v^i) - S(c^*-\alpha\Delta c,v^i) = \langle \text{grad}_c\, S(c,v^i)\big|_{c=c^*}, \alpha\Delta c\rangle$$

when α is arbitrarily small.

$\therefore$ $\langle \text{grad}_c\, S(c^*,v^i), \Delta c\rangle \geq j$ where the notation $\text{grad}_c\, S(c,v^i)|_{c=c^*} = \text{grad}_c\, S(c^*,v^i)$ has been adopted.

Since all partial derivatives wrt c_i are continuous jointly in c and v there is a positive number d_i such that $\langle \text{grad}_c\, S(c,v), \Delta c\rangle > 0$ for all $c \in W$, $v \in V$, satisfying $|c_n - c_n^*| < d_i$, $1 \leq n \leq r$, and $|v_n - v_n^i| \leq d_i$, $1 \leq n \leq s$. Let d be the number $d = \min_{1 \leq i \leq k} d_i$. Let N be the region in c-v space defined by

$$\{N : (c,v)\in N \Leftrightarrow |c_n - c_n^*| \leq d,\ 1 \leq n \leq r, \text{ and } |v_n - v_n^i| \leq d,$$
$$1 \leq n \leq s,\ 1 \leq i \leq k\}.$$

Then $\langle \text{grad}_c\, S(c,v), \Delta c\rangle$ is positive for all $(c,v)\in N$.

$$\therefore \max_{\substack{v\in V\\(c^*,v)\in N}} S(c^*-\beta\Delta c,v) < \max_{\substack{v\in V\\(c^*,v)\in N}} S(c^*,v) \text{ whenever } |\beta\Delta c_n| \leq d \tag{2.7}$$

$$\text{for } 1 \leq n \leq r.$$

Also, by the definition of V_0, $\exists h > 0$, such that

$$\max_{v\in V_0} S(c^*,v) \geq S(c^*,v) + h,\ \forall v\in V,\ (c^*,v)\notin N.$$

By the continuity of $S(c,v)$, $\exists e > 0$, such that for any $v\in V$,

$$|S(c^*-\gamma\Delta c,v) - S(c^*,v)| \leq \frac{h}{2} \text{ whenever } |\gamma\Delta c_n| \leq e,\ 1 \leq n \leq r.$$

$$\therefore S(c^*-\gamma\Delta c,v) \leq S(c^*,v) + \frac{h}{2} \leq \max_{v\in V_0} S(c^*,v) - \frac{h}{2} < \max_{v\in V_0} S(c^*,v),$$

$\forall (c^*,v)\notin N$.

$$\therefore \max_{\substack{v\in V\\(c^*,v)\notin N}} S(c^*-\gamma\Delta c,v) < \max_{v\in V_0} S(c^*,v) \text{ for } |\gamma\Delta c_n| \leq e,\ 1 \leq n \leq r. \tag{2.8}$$

Let $\Delta c^* = \delta\Delta c$ where $|\delta\Delta c_n| \leq d$ and $|\delta\Delta c_n| \leq e$, $1 \leq n \leq r$.

Then combining (2.7) and (2.8),

$$\max_{v \in V} S(c^*-\Delta c^*, v) < \max_{v \in V} S(c^*, v) = \min_{c \in W} \max_{v \in V} S(c, v),$$

which is the desired contradiction.

The minimax problem is also simplified if the function is convex with respect to v as the following lemma shows.

Lemma 2.4.1: Let $S(c,v)$ be a function which is continuous jointly in c and v, $\forall c \in W$, $\forall v \in V$, and convex wrt $v \in V$ for every $c \in W$. Suppose V is a convex polyhedron in s-dimensional Euclidean space. Let V_0 be the set of vertices of V. Then if $\bar{c}$ satisfies

$$\max_{v \in V_0} S(\bar{c}, v) = \min_{c \in W} \max_{v \in V_0} S(c, v)$$

it follows that $\bar{c} = c^*$ the minimax control.

Proof: By property (3) of convex functions

$$\max_{v \in V_0} S(c, v) = \max_{v \in V} S(c, v) \text{ for any } c \in W.$$

$$\therefore \min_{c \in W} \max_{v \in V_0} S(c, v) = \min_{c \in W} \max_{v \in V} S(c, v)$$

i.e. $$\max_{v \in V_0} S(\bar{c}, v) = \max_{v \in V} S(\bar{c}, v) = \min_{c \in W} \max_{v \in V} S(c, v).$$

$\therefore$ $\bar{c} = c^*$ by definition 2.1.3.

The following theorem shows that strict convexity with respect to v also implies a weaker result of the same type as that given by theorem 2.4.1 for strict convexity with respect to c.

Theorem 2.4.2: Let $S(c,v)$ be a function which is continuous jointly in c and v, $\forall c \in W$, $\forall v \in V$. Suppose that for each $c \in W$, $S(c,v)$ is strictly convex wrt $v \in V$. Suppose also that V is a convex polyhedron in s-dimensional Euclidean space and that $\min_{c \in W} \max_{v \in V} S(c,v)$ is attained only at $v^1, \ldots, v^k$, when $c = c^*$. Let $V_0 \equiv \{v^1, \ldots, v^k\}$. Then there is a neighborhood $W_0 \subset W$ surrounding c^* such that

$$\min_{c \in W_0} \max_{v \in V_0} S(c,v) = \max_{v \in V_0} S(c^*,v) = \min_{c \in W} \max_{v \in V} S(c,v). \quad (2.9)$$

Proof: Let V_1 be the set of all vertices of V. By property (3)

$$\max_{v \in V} S(c,v) = \max_{v \in V_1} S(c,v), \ \forall c \in W, \quad (2.10)$$

and therefore $V_0 \subset V_1$.

Assume that there is no neighborhood W_0 of c^* which satisfies (2.9). Then in each neighborhood of c^* there is at least one control which violates (2.9), so it is possible to construct an infinite sequence $c^1, \ldots, c^n, \ldots$ satisfying

$$\lim_{n \to \infty} c^n = c^*$$

and

$$\max_{v \in V_0} S(c^n,v) < \min_{c \in W} \max_{v \in V} S(c,v), \ \forall n. \quad (2.11)$$

Case (a): Assume $V_0 \equiv V_1$. Then $\max_{v \in V_0} S(c^n,v) = \max_{v \in V_1} S(c^n,v)$. Using this, (2.10) and (2.11),

$$\max_{v \in V} S(c^n, v) < \min_{c \in W} \max_{v \in V} S(c,v), \ \forall n,$$

which is the desired contradiction.

Case (b): Assume $V_0 \subset V_1$, $V_0 \neq V_1$. Then there are elements $v^{k+1}, \ldots, v^{k+\ell}$, satisfying $v^{k+i} \in V_1$, $v^{k+i} \notin V_0$. From the definition of V_0,

$$\max_{v \in V} S(c^*, v) = \max_{v \in V_0} S(c^*, v) > S(c^*, v^{k+i}), \text{ for } i = 1, \ldots, \ell.$$

Therefore $\exists j > 0$ such that

$$\max_{v \in V_0} S(c^*, v) \geq S(c^*, v^{k+i}) + j, \text{ for } i = 1, \ldots, \ell. \quad (2.12)$$

By continuity of $S(c,v)$, $\exists \delta > 0$, such that

$$|S(c, v^{k+i}) - S(c^*, v^{k+i})| \leq \frac{j}{2}$$

whenever $|c_k^* - c_k| \leq \delta$, $\forall k$, for $i = 1, \ldots, \ell$. Since $c^n \to c^*$, $\exists m$ sufficiently large that

$$|S(c^m, v^{k+i}) - S(c^*, v^{k+i})| \leq \frac{j}{2}, \text{ for } i = 1, \ldots, \ell. \quad (2.13)$$

Then using (2.13) and (2.12)

$$S(c^m, v^{k+i}) \leq S(c^*, v^{k+i}) + \frac{j}{2} \leq \max_{v \in V_0} S(c^*, v) - \frac{j}{2} < \max_{v \in V_0} S(c^*, v),$$

for $i = 1, \ldots, \ell$. Therefore by definition of V_0,

$$S(c^m, v^{k+i}) < \min_{c \in W} \max_{v \in V} S(c,v) \text{ for } i = 1, \ldots, \ell. \quad (2.14)$$

Also from (2.11)

$$\max_{v \in V_0} S(c^m, v) < \min_{c \in W} \max_{v \in V} S(c,v). \quad (2.15)$$

Combining (2.14), (2.15), and using (2.10),

$$\max_{v\in V_1} S(c^m,v) = \max_{v\in V} S(c^m,v) < \min_{c\in W} \max_{v\in V} S(c,v),$$

which is the desired contradiction.

Lemma 2.4.1 shows that when S(c,v) is convex with respect to v, and V is a convex polyhedron, a single minimization at the vertices of V yields the minimax solution. Therefore whenever V has a small number of vertices this is a particularly easy problem. If, however, V has a large number of vertices it is of advantage to use the algorithm with attention restricted to the vertices of V only. In this connection it is interesting to note that for strict convexity with respect to v the quantity, $\max_{v\in V_0} S(c,v)$, (where V_0 is the set of vertices of V at which the minimax value of S is attained when c = c*) has at least a local minimum at c = c*, as proved in theorem 2.4.2.

2.5 Rate of Convergence of the Algorithm

First consider the following simple case:

(1) S(c,v) is continuous jointly in c and v, $\forall c\in W$, $\forall v\in V$,

(2) v is a scalar, $v^a \leq v \leq v^b$

(3) $\frac{\partial S(c,v)}{\partial v}$ has no more than two zeroes wrt $v\in V$ for any $c\in W$,

(4) $\frac{\partial^2 S(c,v)}{\partial v^2}$ exists $\forall c\in W$, $\forall v\in V$, and $\left|\frac{\partial^2 S(c,v)}{\partial v^2}\right| \leq K$,

(5) $\min_{c \in W} \max_{v \in V_k} S(c,v) = \min_{c \in W} \max_{v \in V} S(c,v)$, where V_k is the set of all v satisfying $S(c^*,v) = \min_{c \in W} \max_{v \in V} S(c,v)$.

A sufficient condition for (5) is given by theorem 2.4.1. From (3) it follows that there are at most two elements in V_k and that if there are two elements at least one is on the boundary of V. Thus the problem becomes a search for the value of $v \in V_k$ which may be an interior point of V. Consider step by step the application of the algorithm in Section 2.3. A crude upper bound on the rate of convergence to zero of $\{S_i^M - S_i^m\}$ can be obtained by assuming at each step that S_i^M is as large as possible. Possible "worst-case" configurations for $S(c^i,v)$ are shown in Figure 4.

Assume that the algorithm is started by choosing $V_0 = \{v^a, v^b\}$. After minimizing, the greatest possible value for S_i^M occurs if $v^1 = \frac{v^a + v^b}{2}$ which implies that $\frac{\partial S}{\partial v}$ is zero at $\frac{v^a + v^b}{2}$. Then $\max_{v \in V} \left|\frac{\partial S}{\partial v}\right| = K \frac{v^b - v^a}{2}$ and $S_1^M < S_1^m + K\frac{(v^b - v^a)^2}{4}$. The set V_1 is then $V_1 = \{v^a, v^b, v^1\}$. After minimizing, the greatest possible value for S_2^M occurs if $v^2 = v^a + \frac{1}{4}(v^b - v^a)$ or if $v^2 = v^a + \frac{3}{4}(v^b - v^a)$. In either case $S_2^M < S_2^m + K\frac{(v^b - v^a)^2}{4^2}$.

In Figure 4 the assumption is made that $v^2 = v^a + \frac{1}{4}(v^b - v^a)$. In this case it is clear that the value of $v \in V_k$ which is internal to V cannot be in the region $[\frac{v^b + v^a}{2}, v^b]$ since minimization at v^a, v^b, and any point in this region would obviously have given a value $\overline{S}_2^m$ at least as small as S_2^m.

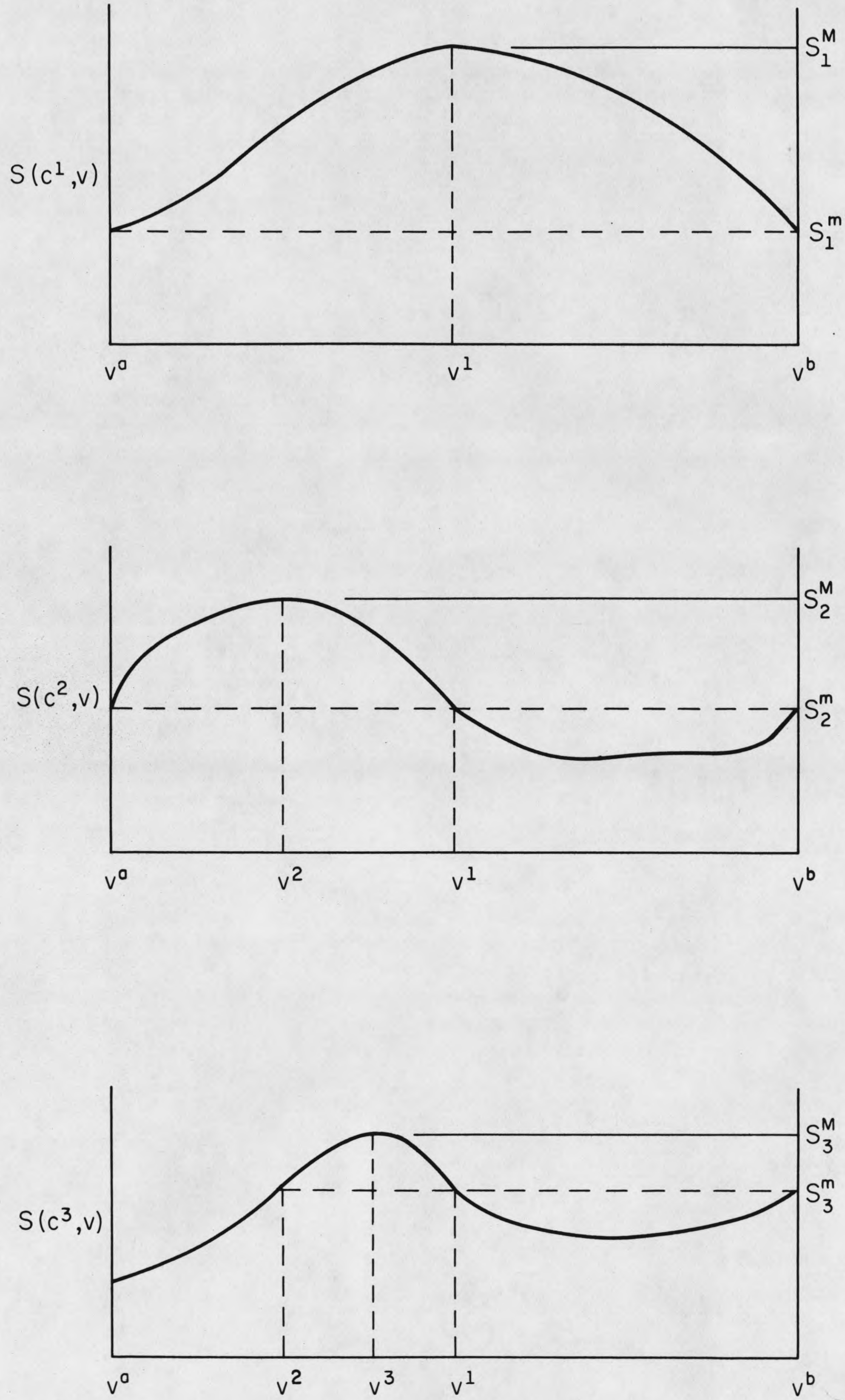

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Figure 4. Worst-case configurations for $S(c^i,v)$.

Continuing with the algorithm, the possible region for the value of $v \in V_k$ which is internal to V is halved at each iteration with the result that $S_n^M - S_n^m < \frac{K(v^b - v^a)^2}{4^n}$. Thus for this case the maximum possible distance between the upper and lower bounds of $\min_{c \in W} \max_{v \in V} S(c,v)$ decreases by a factor of at least four at each succeeding iteration.

Figure 5 summarizes the application of the algorithm to the example in Section 2.2 which is of the type discussed above. The process was started by choosing $V_0 = \{0.0, 2.0\}$. The second minimization, giving c^2, reduced the greatest possible error in $\min_{c \in W} \max_{v \in V} S(c,v)$ to less than 1.8%.

Unfortunately a realistic bound on the rate of convergence when v is of arbitrary dimension is not easily determined, but experience with a number of control problems has shown that rapid convergence is normal. Heuristically, it is reasonable to expect that $S_{i+1}^m \gg S_i^m$ whenever $S_i^M \gg S_i^m$ since the $(i+1)^{st}$ control represents a compromise between the values of v which led to S_i^m and the value of v which yielded S_i^M. On the other hand, when $S_i^m \approx S_i^M$ and $c^i \approx c^*$, rapid convergence is anticipated since the maximizing values of v which are on the boundary of V are insensitive to moderate control changes while the maximizing values of v^m in the interior of V need not be determined precisely because $\text{grad}_v \, S(c,v)\big|_{\substack{c=c^* \\ v=v^m}} = 0$. These statements are illustrated by Figure 5.

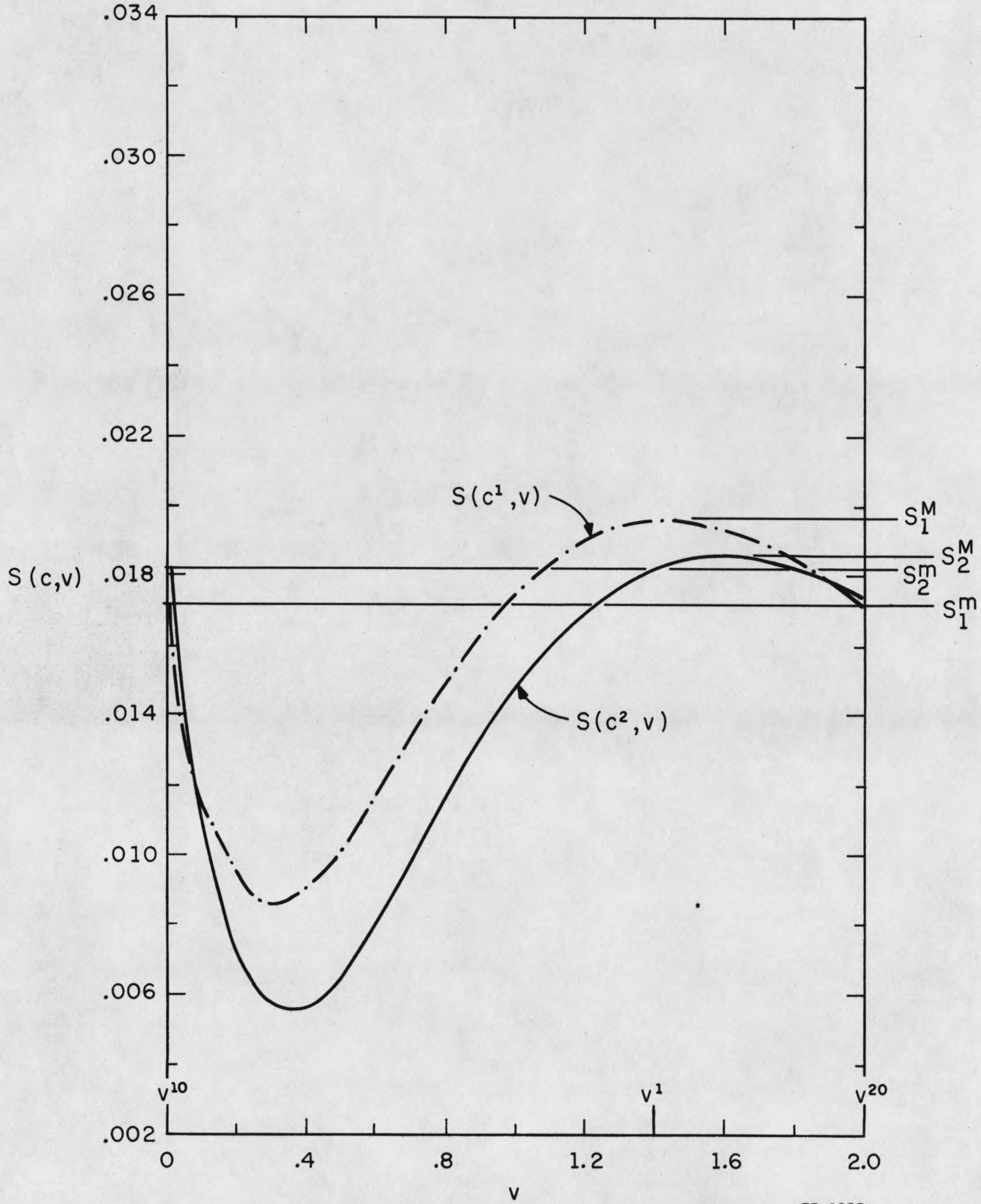

Figure 5. Determination of c* for the example in Section 2.2.

2.6 Properties of Minimax Performance-Sensitivity Control

Some specific properties of minimax performance-sensitivity control, which are relevant to intelligent application of the algorithm in Section 2.3, are developed in this section. Theorem 2.6.1 gives conditions under which a-priori bounds can be placed on the minimax performance-sensitivity control c* when c is a scalar. Theorem 2.6.1 also shows that in this case the minimax value of S(c,v) is attained at more than one value of v when c = c*. Theorem 2.6.2 generalizes the latter result to controls of arbitrary dimension.

Theorem 2.6.1: Let J(c,v) be a performance functional to be minimized, where $c \in W$ is a scalar and $v \in V$ is an s-dimensional vector, with the properties:

(1) J(c,v) is continuous jointly in c and v,

(2) $\frac{\partial}{\partial c} J(c,v)$ exists and is continuous jointly in c and v,

(3) for each $v \in V$ $\exists$ a unique function $c^0(v) \in W$, continuous in v, such that $J(c^0(v), v) = J^0(v)$,

(4) $\frac{\partial}{\partial c} J(c,v)\Big|_{\substack{c=\bar{c} \\ v=\bar{v}}} = 0 \Rightarrow J(\bar{c},\bar{v}) = J^0(\bar{v})$.

Let $S(c,v) = F[J, J^0(v)]$ be any performance-sensitivity which satisfies:

(5) $F[J, J^0(v)]$ is a monotonic strictly increasing function of J for each $v \in V$.

Then if c* is the minimax performance-sensitivity control:

(a) $\exists$ $v^1, v^2, \in V$, $v^1 \neq v^2$, such that $S(c^*,v^1) = S(c^*,v^2) = \min_{c \in W} \max_{v \in V} S(c,v)$.

(b) $\min_{v \in V} c^0(v) \leq c^* \leq \max_{v \in V} c^0(v)$.

(Condition (3) implies that the assumed controller structure $u(t) = u[c,x(t)]$ is capable of generating the optimal performance at any $v \in V$. Condition (4) implies that $J(c,v)$ is stationary wrt c only when c is optimal.)

Proof: Since $c^0(v)$ is continuous and defined for every $v \in V$ it forms an s-dimensional hypersurface Γ which cuts the (s+1)-dimensional c-v space.

Let A be the region, $\{A : (c,v) \in A \Leftrightarrow c \in W, v \in V, c \geq c^0(v)\}$

B be the region, $\{B : (c,v) \in B \Leftrightarrow c \in W, v \in V, c \leq c^0(v)\}$

$$\alpha = \min_{v \in V} c^0(v)$$

$$\beta = \max_{v \in V} c^0(v).$$

In the case that $\alpha = \beta$, $c^0(v)$ is constant, $c^* = c^0(v) = \alpha = \beta$, and $S(\alpha,v) = 0$, $\forall v \in V$, so that the theorem is trivially satisfied.

Therefore assume $\alpha < \beta$.

It is claimed that the function of c, $\max_{\substack{v \in V \\ (c,v) \in A}} S(c,v)$, has the properties:

(i) it is a monotonic strictly increasing function of c, for $\forall c \in W$, $c > \alpha$,

(ii) it is a continuous function of c,

(iii) $\min_{\substack{c \in W \\ c \geq \alpha}} \max_{\substack{v \in V \\ (c,v) \in A}} S(c,v) = 0.$

Proof of (i): From (2), (3) and (4), $J(c,\overline{v})$ is a monotonic strictly increasing function of c for each $\overline{v}\in V$, $(c,\overline{v})\in A$, $c > \alpha$. Then using (5), $S(c,\overline{v})$ is also monotonic strictly increasing for each $\overline{v}\in V$, $(c,\overline{v})\in A$, $c > \alpha$. Since maximization over v does not affect the monotonicity the result follows.

Proof of (ii): Continuity follows directly from (1) and the definition of performance-sensitivity.

Proof of (iii): By definition of α and A, every $(\alpha,v)\in A$ satisfies $(\alpha,v)\in\Gamma$, so that $S(\alpha,v) = 0$ for every $(\alpha,v)\in A$. Therefore $\max_{\substack{v\in V\\(\alpha,v)\in A}} S(\alpha,v) = 0$. Since $S(c,v) \geq 0$, it follows that

$\min_{\substack{c\in W\\c\geq\alpha}} \max_{\substack{v\in V\\(c,v)\in A}} S(c,v) = 0$. Similarly $\max_{\substack{v\in V\\(c,v)\in B}} S(c,v)$ has the properties:

(iv) it is a monotonic strictly decreasing function of c, for $\forall c\in W$, $c < \beta$,

(v) it is a continuous function of c,

(vi) $\min_{\substack{c\in W\\c\leq\beta}} \max_{\substack{v\in V\\(c,v)\in B}} S(c,v) = 0$.

Also since $V = A \cup B$

$$\max_{v\in V} S(c,v) = \max \left\{ \max_{\substack{v\in V\\(c,v)\in A}} S(c,v),\ \max_{\substack{v\in V\\(c,v)\in B}} S(c,v)\right\}. \quad (2.16)$$

From (i) through (vi) it is clear that $\max_{\substack{v\in V\\(c,v)\in A}} S(c,v)$ and $\max_{\substack{v\in V\\(c,v)\in B}} S(c,v)$ are equal at a unique value $\overline{c}$ of c, $\alpha \leq \overline{c} \leq \beta$, and using (2.16) it can be shown that $\bar{c} = c^*$. (Assumption of the contrary leads to the contradiction, $\max_{v \in V} S(c^*,v) > \min_{c \in W} \max_{v \in V} S(c,v)$.)

Therefore

$$\min_{v \in V} c^0(v) \le c^* \le \max_{v \in V} c^0(v)$$

and

$$\max_{\substack{v \in V \\ (c,v) \in A}} S(c^*,v) = \max_{\substack{v \in V \\ (c,v) \in B}} S(c^*,v). \qquad (2.17)$$

Let $\bar{v}$ be a value of v satisfying $\max_{v \in V} S(c^*,v) = S(c^*,\bar{v})$. If $(c^*,\bar{v}) \in \Gamma$ than $\max_{v \in V} S(c^*,v) = 0$ and (a) is trivially satisfied. If $(c^*,\bar{v}) \notin \Gamma$ then using (2.17) there are two values v^1, v^2, of v, such that $(c^*,v^1) \in A$, $(c^*,v^2) \in B$, and $(c^*,v^1) \notin \Gamma$ (i.e. $v^1 \in V$, $v^2 \in V$, $v^1 \neq v^2$) satisfying

$$\max_{v \in V} S(c^*,v) = S(c^*,v^1) = S(c^*,v^2).$$

<u>Theorem 2.6.2</u>: Let J(c,v) be a performance functional to be minimized, ($c \in W$, $v \in V$) with the properties:

(1) J(c,v) is continuous jointly wrt c and v,

(2) $\frac{\partial J}{\partial c_i}$ is continuous jointly wrt c and v, $1 \le i \le r$,

(3) for each $v \in V$ $\exists c^0(v)$ such that $J(c^0(v),v) = J^0(v)$,

(4) $\text{grad}_c J(c,v)\Big|_{\substack{c=\bar{c} \\ v=\bar{v}}} = 0 \Rightarrow J(\bar{c},\bar{v}) = J^0(\bar{v})$.

((3) implies that the assumed structure $u(t) = u(c,x(t))$ is capable of generating the optimal performance at any $v \in V$. (4) implies that J(c,v) is stationary wrt c only when c is optimal.)

Let $S(c,v)$ be any performance-sensitivity such that the minimax performance-sensitivity control c^* is an interior point of W and such that

(5) $\frac{\partial S}{\partial c_i}$ is continuous jointly wrt c and v, $1 \leq i \leq r$.

Then the minimax value of S is attained at two or more distinct values of v, i.e. $\exists v^1, v^2, \in V, v^1 \neq v^2$, such that

$$\min_{c \in W} \max_{v \in V} S(c,v) = S(c^*,v^1) = S(c^*,v^2).$$

Proof: Since $S(c,v) \geq 0$ the statement of the theorem is trivially correct if $\max_{v \in V} S(c^*,v) = 0$. Therefore assume $\max_{v \in V} S(c^*,v) > 0$.

Suppose $\max_{v \in V} S(c^*,v) = S(c^*,v^a)$ where v^a is unique. (2.18)

By assumption $S(c^*,v^a) > 0$ which implies $\text{grad}_c\, S(c,v)\Big|_{\substack{c=c^* \\ v=v^a}} \neq 0$ (using (4) and the definition of S). Therefore $\exists j$ such that $\frac{\partial S}{\partial c_j}\Big|_{\substack{c=c^* \\ v=v^a}} = a \neq 0$. Assume $a > 0$. By continuity of $\frac{\partial S}{\partial c_j}$ (using (2)) $\exists d > 0$ such that $\frac{\partial S(c,v)}{\partial c_j} > 0$ for $c \in W$, $v \in V$, satisfying

$|c_i - c_i^*| \leq d$, $i = 1, 2, \ldots, r$, and $|v_k - v_k^a| \leq d$, $k = 1, 2, \ldots, s$.

Let N be the neighborhood of (c^*,v^a) so defined, so that

$$\frac{\partial S(c,v)}{\partial c_j} > 0, \ \forall (c,v) \in N. \qquad (2.19)$$

Let Δc^j be an r-vector with a single non-zero component Δc in the j^{th} row, $0 < \Delta c \leq d$. Clearly $(c^* - \Delta c^j, v) \in N$ whenever $(c^*,v) \in N$. Then using (2.19)

$$\max_{\substack{v \in V \\ (c^*,v) \in N}} S(c^*-\Delta c^j, v) < \max_{\substack{v \in V \\ (c^*,v) \in N}} S(c^*,v) = S(c^*,v^a). \quad (2.20)$$

Also, using (2.18), $\exists h > 0$, such that

$$S(c^*,v^a) \geq S(c^*,v) + h, \ \forall v \in V, \ (c^*,v) \notin N. \quad (2.21)$$

By continuity of $S(c,v)$, $\exists \delta > 0$ such that for any $v \in V$

$$|S(c^*-\Delta c^j, v) - S(c^*,v)| \leq \frac{h}{2} \text{ whenever } \Delta c \leq \delta. \quad (2.22)$$

Then using (2.22) and (2.21),

$S(c^*-\Delta c^j,v) \leq S(c^*,v) + \frac{h}{2} \leq S(c^*,v^a) - \frac{h}{2} < S(c^*,v^a), \forall v \in V, (c^*,v) \notin N$, and $\Delta c \leq \delta$.

Therefore

$$\max_{\substack{v \in V \\ (c^*,v) \notin N}} S(c^*-\Delta c^j, v) < S(c^*,v^a) \text{ whenever } \Delta c \leq \delta. \quad (2.23)$$

Let Δc^* be Δc^j for any $\Delta c \leq \delta$ and $\Delta c \leq d$. Then combining (2.20) and (2.23)

$$\max_{v \in V} S(c^*-\Delta c^*, v) < S(c^*,v^a).$$

But from the definition of c^* and (2.18), $S(c^*,v^a) = \min_{c \in W} \max_{v \in V} S(c,v)$, which gives the desired contradiction.

If $a < 0$ a similar argument shows that $\max S(c^*+\Delta c^*,v) < S(c^*,v^a)$ which also gives a contradiction.

Since a performance index is normally chosen so that it will yield a unique optimal control and no locally optimal controls, condition (4) of Theorem 2.6.2 is often satisfied in practice. In Section 2.7 it is shown that, in particular, (4) is satisfied by linear systems with quadratic indices and linear state-feedback controllers.

Theorem 2.6.2 shows that there is an interesting difference between minimax performance controls and minimax performance-sensitivity controls when the controller structure is capable of optimality at any single value of v. In the case of the former it is possible for the minimax value of the index to be attained at only one value of v as in the example of Section 2.2. When this occurs the minimax control is the optimal control at that value of v. In the case of the latter, regardless of the particular performance-sensitivity chosen, the minimax value is always attained at more than one value of v (for illustration see the examples in Sections 2.2 and 2.9). This implies that the control c* compromises between different values of v. Such a compromise rarely results in a control which is optimal at any single value of $v \in V$. It is therefore undesirable when minimizing the maximum value of a performance-sensitivity to constrain the controller to be optimal at some value of v.

2.7 Results for the Linear Case

Since the special case of a linear system with a quadratic performance index is important in automatic control theory it is of interest to consider here some results for this case.

Assume that the dynamic system and performance index can be expressed in the form:

$$\dot{x}(t) = A(v)x(t) + B(v)u(t), \qquad x(t_0) = x_0(v),$$

$$J(u,v) = \tfrac{1}{2}\int_0^\infty [x'(t)Qx(t) + u'(t)Ru(t)]dt$$

$$v \in V$$

where $x(t)$ is an n-dimensional state vector

$u(t)$ is an s-dimensional control vector

v is an r-dimensional time-invariant vector containing unknown system parameters

V is a closed bounded region in Er

$A(v)$ is a time-invariant n x n matrix

$B(v)$ is a time-invariant n x s matrix

Q is a constant, positive semi-definite n x n matrix

R is a constant positive definite s x s matrix.

It is assumed that the elements of A, B, and x_0, are continuous wrt v, $\forall v \in V$. The desired form of control is assumed to be constant state feedback. If the system is completely controllable for each $v \in V$ it is well known [10] that there is a unique optimal control for each $v \in V$ given by

$$u^0(v) = -R^{-1}B'Kx(t) \tag{2.24}$$

where $K = K(v)$ is the positive definite solution of

$$KA + A'K - KBR^{-1}B'K + Q = 0 . \tag{2.25}$$

The corresponding optimal value of J is given by

$$J[u^0(v),v] = \tfrac{1}{2}\, x'_0 K x_0. \tag{2.26}$$

It is desired to find minimax state-feedback controllers of form $u = Cx(t)$ where C is an s x n constant matrix. Let $c \in W$ be an (s x n)-dimensional vector containing the elements of C. W is assumed to be a bounded closed set such that $\dot{x} = (A + BC)\,x$ is asymptotically stable, $\forall v \in V$, whenever $c \in W$.

Claim 2.7.1: For any control, $u = Cx(t)$, $c \in W$, the value of the performance index is given by

$$J = \tfrac{1}{2}\, x_0' \, L x_0 \qquad (2.27)$$

where L is positive definite and is the unique solution of

$$(A + BC)'L + L(A + BC) + Q + C'RC = 0. \qquad (2.28)$$

Proof: Let $J(t) = \tfrac{1}{2}\int_t^{\infty} [x'(\sigma)Qx(\sigma) + u'(\sigma)Ru(\sigma)]d\sigma.$

Then

$$\frac{d}{dt} J(t) + \tfrac{1}{2}\, x'(t)Qx(t) + \tfrac{1}{2}\, u'(t)Ru(t) = 0 \qquad (2.29)$$

and

$$\lim_{t\to\infty} J(t) = 0 \qquad (2.30)$$

since $c \in W$ implies asymptotic stability. As a solution to (2.29) and (2.30) try $J(t) = \tfrac{1}{2}\, x'(t)Lx(t)$. Then

$$\frac{d}{dt}[x'(t)Lx(t)] = \frac{d}{dt}\langle x(t), Lx(t)\rangle = \langle \dot{x}(t), Lx(t)\rangle + \langle x(t), L\dot{x}(t)\rangle$$

$$= x'(t)[(A + BC)'L + L(A + BC)]x(t).$$

Substituting into (2.29),

$$x'(t)[(A + BC)'L + L(A + BC) + Q + C'RC]x(t) = 0.$$

Therefore $J(t) = \tfrac{1}{2}\, x'(t)Lx(t)$ satisfies (2.29) for all $x(t)$ if L satisfies (2.28). Also $\lim_{t\to\infty} x'(t)Lx(t) = 0$ since $\lim_{t\to\infty} x(t) = 0$, $\forall c \in W$.

$\therefore J(t) = \tfrac{1}{2}\, x'(t)Lx(t), \ \forall t.$

$\therefore J(0) = \tfrac{1}{2}\int_0^{\infty} [x'(t)Qx(t) + u'(t)Ru(t)]dt = \tfrac{1}{2}\, x(0)'Lx(0) = \tfrac{1}{2}\, x_0'Lx_0.$

Uniqueness of L follows from the known fact that a matrix equation

$GX - XE = D$ has a unique solution for X if G and E have no common eigenvalues. In this case the requirement of asymptotic stability implies that every eigenvalue of $(A + BC)'$ has a negative real part, and every eigenvalue of $-(A + BC)$ has a positive real part. Positive definiteness of L follows from the fact that

$$x_0'Lx_0 = \int_0^\infty [x'(t)Qx(t) + u'(t)Ru(t)]dt > 0, \ \forall x_0 \neq 0.$$

Then using (2.26) and (2.27), the performance-sensitivities $S = \frac{J - J^0(v)}{J^0(v)}$ and $S = J - J^0(v)$ can be expressed respectively as:

$$S(c,v) = \frac{J(c,v) - J^0(v)}{J^0(v)} = \frac{x_0'(L-K)x_0}{x_0'Kx_0}$$

and

$$S(c,v) = J(c,v) - J^0(v) = x_0'(L - K)x_0 \ .$$

In either case the sensitivity $S(c,v)$ is a rational function of the elements of c and v. The matrix L is defined in terms of the $\frac{n(n+1)}{2}$ independent linear equations in the elements of L given by (2.28), and therefore an analytic solution for L is possible (although tedious) for reasonably high dimension n. The matrix K, however, is defined in terms of the $\frac{n(n+1)}{2}$ independent quadratic equations given by (2.25). Isolation of the positive definite solution for K in analytic form is too difficult, in general, for $n > 2$. In this case a solution can be found by determining the steady-state solution of a set of $\frac{n(n+1)}{2}$ first order differential equations [10].

The following claim shows that condition (4) of Theorem 2.6.2 is satisfied by any linear system with a quadratic performance index and a linear state-feedback controller.

Claim 2.7.2: $\text{grad}_c\ J(c,v) = 0 \Leftrightarrow C = -R^{-1} B'K$.

Proof: Let Δc be an arbitrary first order variation in c, and let ΔC, ΔL, and ΔJ, be the corresponding variations in C, L, and J. Then

$$\text{grad}_c\ J(c,v) = 0 \Leftrightarrow \Delta J = \tfrac{1}{2}x_0'\Delta L x_0 = 0 \text{ for arbitrary (first order) } \Delta C. \tag{2.31}$$

Let C become $C + \Delta C$ in (2.28). Then

$$[A+B(C+\Delta C)]'(L+\Delta L) + (L+\Delta L)[A+B(C+\Delta C)] + Q + (C+\Delta C)'R(C+\Delta C) = 0.$$

Expanding, neglecting second order quantities, and subtracting (2.28) gives,

$$(A + BC)'\Delta L + \Delta L(A + BC) + (LB + C'R)\Delta C + \Delta C'(B'L + RC) = 0. \tag{2.32}$$

Case (1): $(B'L + RC) \neq 0$. By comparison with (2.28), ΔL is positive definite (and $\Delta J = \frac{1}{2} x_0'\Delta L x_0 \neq 0$) when $[(LB + C'R)\Delta C + \Delta C'(B'L + RC)]$ is positive definite. But this is the case whenever $\Delta C = \alpha P(LB + C'R)'$ where α is a scalar, $0 < \alpha \ll 1$, and P is any positive definite matrix. Therefore $\Delta J \neq 0$ for arbitrary first order ΔC when $(B'L + RC) \neq 0$.

Case (2): $(B'L + RC) = 0$. Then (2.32) becomes independently of ΔC,

$$(A + BC)'\Delta L + \Delta L(A + BC) = 0\ ,$$

which has the unique solution $\Delta L = 0$, which in turn implies $\Delta J = 0$.

Summarizing cases (1) and (2),

$$(B'L + RC) = 0 \Rightarrow \text{grad}_c\ J(c,v) = 0$$

$$(B'L + RC) \neq 0 \Rightarrow \text{grad}_c\ J(c,v) \neq 0\ .$$

$$\therefore\ \text{grad}_c\ J(c,v) = 0 \Leftrightarrow C = -R^{-1}B'L.$$

Substituting $C = -R^{-1}B'L$ into (2.28) gives,

$$LA + A'L - LBR^{-1}\,B'L + Q = 0,$$

which is identical in form to Eq. (2.25) in K. But (2.25) is known to have a unique positive definite solution for K [11]. Therefore since L is positive definite, L = K, and $\text{grad}_c\ J(c,v) = 0 \Leftrightarrow C = R^{-1}B'K$.

<u>Convexity with respect to initial conditions</u>: It is well known that the optimal controller for a linear system with a quadratic performance index is optimal for all initial conditions. However, this property does not extend to controllers of form $u(t) = Cx(t)$ if there is reduced feedback (i.e. if some states are not available to the controller), and it may not extend to the case of complete feedback if there are variable system parameters. It is therefore necessary at times to consider the range of system initial states as a set of variable parameters included in the vector v. The following claim is of interest in view of the fact that the minimax problem is simplified when the pay-off function is convex with respect to v (see lemma 2.4.1 and theorem 2.4.2).

Claim 2.7.3: For a linear system with a quadratic performance index J, and a linear feedback controller, J(c,v) is strictly convex with respect to that part of v which represents the variable initial states.

Proof: Let $\bar{v}$ be the vector of variable initial states. Then from Claim 2.7.1 $J = \frac{1}{2}\bar{v}' L \bar{v}$ where L is positive definite. Let $\bar{v}^a$, $\bar{v}^b$, be two distinct values of the vector $\bar{v}$. It is necessary to show that

$$\alpha J(\bar{v}^a) + (1-\alpha)J(\bar{v}^b) > J[\alpha\bar{v}^a + (1-\alpha)\bar{v}^b] \text{ for } 0 < \alpha < 1.$$

But this is the case since

$$\alpha J(\bar{v}^a) + (1-\alpha)J(\bar{v}^b) - J[\alpha\bar{v}^a + (1-\alpha)\bar{v}^b]$$

$$= \alpha(\bar{v}^a)'L\bar{v}^a + (1-\alpha)(\bar{v}^b)'L\bar{v}^b - [\alpha\bar{v}^a + (1-\alpha)\bar{v}^b]'L[\alpha\bar{v}^a + (1-\alpha)\bar{v}^b]$$

$$= \alpha(1-\alpha)(\bar{v}^a - \bar{v}^b)'L(\bar{v}^a - \bar{v}^b) > 0,$$

by the positive definiteness of L.

2.8 Minimax Controller Design

In those cases where a controller can reasonably be specified in terms of a time-invariant vector c and where the system complexity is such that the available computational facilities are adequate to solve the minimization and maximization problems required by the algorithm in Section 2.3 the following is a reasonable approach to the design of the controller.

(a) Choose an appropriate performance index or performance-sensitivity based on the system specifications.

(b) Choose a controller structure from physical or other considerations.

(c) Determine the minimax controller parameters.

(d) Evaluate the resulting system performance.

(e) If the performance is unsatisfactory choose a different controller structure (probably more complex). Repeat from (c) until a satisfactory compromise between system performance and controller complexity is attained.

This procedure places emphasis on the physical realization of the controller in contrast to the computation of an optimal control which places full emphasis on the mathematics of the process and none on the physical realization. Thus in comparison with the optimal-adaptive controller the resulting minimax controller will sacrifice some system performance for controller realizability but may sacrifice little performance for a considerable reduction in controller complexity.

While the computational requirements for a minimax design are non-negligible they are often warranted in practical situations since the procedure is only applied in the system design stage and since the method offers the possibility of attaining excellent performance with a controller which is easily realized.

2.9 A Fuel-Time Optimal Example

Consider the position control of a unit mass moving in a straight line under the influence of a limited force.

Let $y(t)$ be the position of the mass,

y_d be the desired position of the mass,

$u(t)$ be the force applied, $|u(t)| \leq 1$.

It is desired to find the control $u(t)$ for which at some time T, $y(T) = y_d$, $\dot{y}(T) = 0$, and for which the fuel-time performance index

$$J = \int_o^T (k + |u(t)|)dt, \; k > 0 ,$$

is minimized. By defining $x_1(t) = y(t) - y_d$, and $x_2(t) = \dot{y}(t)$, the system equations become

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u(t).$$

The actual value of k is chosen according to the relative importance of minimizing time or fuel. In the following k is taken to be 1.0.

It may happen for a number of practical reasons that the effective thrust is variable. For example, the actual mass may be variable, or the quality of the fuel or the motor parameters may be variable. Therefore consider the following system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = v\, u(t), \; .5 \leq v \leq 1.5, \; |u(t)| \leq 1,$$
$$J = \int_o^T (v|u(t)| + k)dt, \; k = 1.0.$$

The performance index includes the term $v|u(t)|$, corresponding to the assumption that the fuel consumption is proportional to this term and not $|u(t)|$ alone.

The optimal adaptive control for this system can be derived by standard methods and is shown in Figure 6. The corresponding optimal value of the performance index for an initial

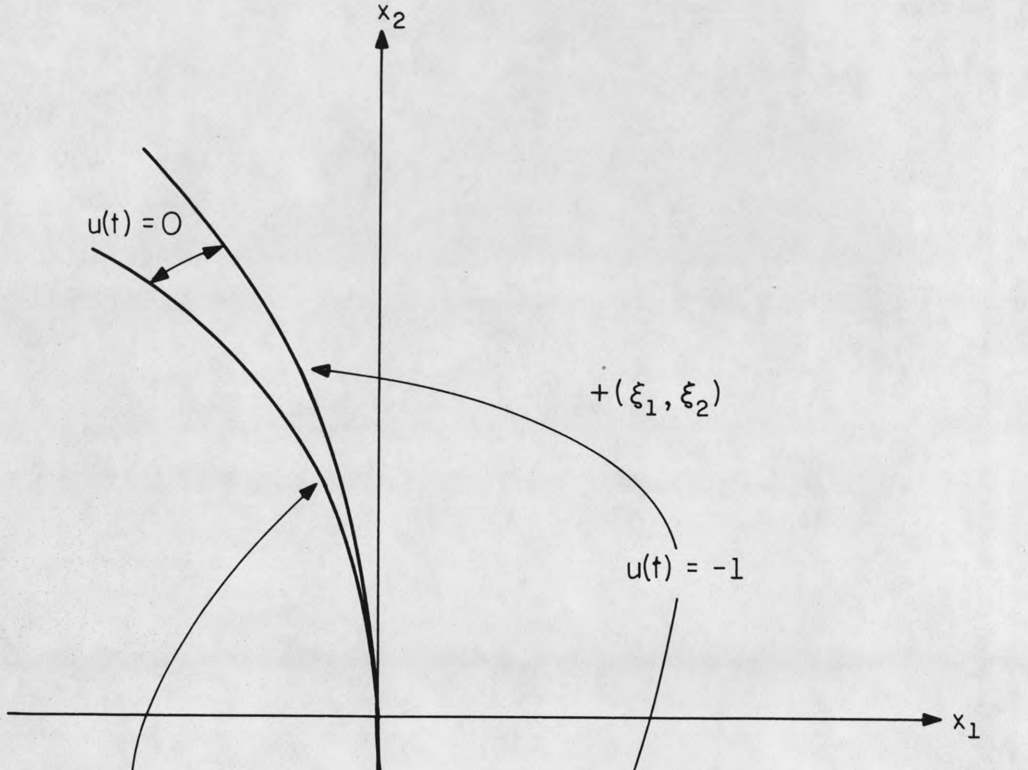

Figure 6. Optimal-adaptive control for the fuel-time example.

condition (ξ_1, ξ_2) as shown in Figure 6 is

$$J^0(v) = (\frac{3}{2} k + 2v + kvg_{kv})t_a - (1 + \frac{k}{2v} + kg_{kv})\xi_2$$

where

$$t_a = \frac{\xi_2}{v} + \frac{1}{v}[(\xi_2^2 + 2v\xi_1)/(1 + 2vg_{kv})]^{\frac{1}{2}}$$

and

$$g_{kv} = \frac{k + 4v}{2kv} .$$

Since the physical realization of this controller is difficult, assume a controller with two constant parabolic switching curves as shown in Figure 7. For an initial condition (ξ_1, ξ_2) there are two types of trajectory, as shown in the figure, according to whether $v \geq \frac{1}{2c_1}$ or $v < \frac{1}{2c_1}$.

For $v \geq \frac{1}{2c_1}$ it can be shown that

$$J(c,v) = (k + v)t_b + (vt_b - \xi_2)(2vc_1 + kc_1 + kc_2)$$

where

$$t_b = \frac{\xi_2}{v} + \frac{1}{v}[(\xi_2^2 + 2v\,\xi_1)/(1 + 2vc_2)]^{\frac{1}{2}} .$$

For $v < \frac{1}{2c_1}$ it can be shown that

$$J(c,v) = (k + v)t_b + [(\xi_2^2 + 2v\,\xi_1)/(1 + 2vc_2)]^{\frac{1}{2}}b$$

where
$$b = \left[\frac{k(c_2-c_1)}{(1-a)} + \frac{(k+v)\ (1+a)}{v(1-a)}\right]$$

and
$$a = [(1 - 2vc_1)/(1 + 2vc_2)]^{\frac{1}{2}} .$$

In deriving these results it has been assumed that the switchings are ideal.

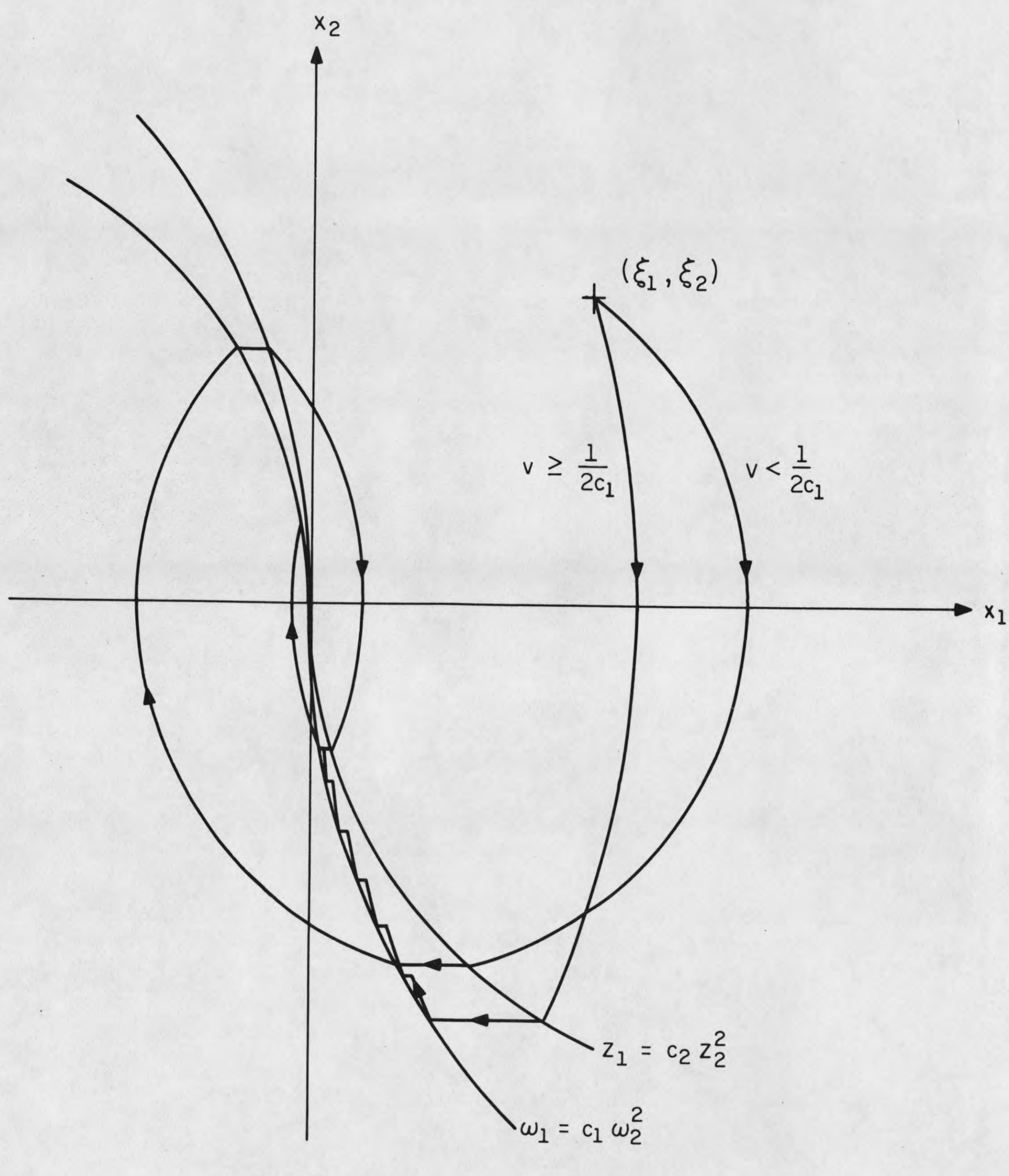

Figure 7. Trajectories with constant switching curves for the fuel-time example.

From the form of $J(c,v)$ and $J^0(v)$ it follows that the performance-sensitivity $S(c,v) = \frac{J(c,v) - J^0(v)}{J^0(v)}$ is independent of the initial state (ξ_1, ξ_2) if $\xi_2 = 0$ or if $\frac{\xi_1^2}{\xi_2^2}$ is a known constant. In the following the special case of $\xi_2 = 0$ is considered, i.e. the initial position is arbitrary but the initial velocity is zero.

The minimax performance-sensitivity control c* for this case can be found by the algorithm in Section 2.3. The choice of $V_0 \equiv \{.5, 1.5\}$ leads to exact convergence with a single iteration, giving c* = (.78, 4.72). The corresponding minimax value of the performance-sensitivity is .29 so that the deviation of the performance from optimality is always less than 30%. Figure 8 shows the variation with v of $S(c^*,v)$, $J(c^*,v)$, and $J^0(v)$.

It is interesting to note that c* is quite dissimilar to an optimal control. In fact, with v in the range .5 to 1.5, the range of values of c_2 for optimality is 2.33 to 3.00, while c_2^* is 4.72.

Several extensions to the approach taken here are possible. For instance, the case of arbitrary but bounded initial states (i.e. position and velocity) is of interest. For this case v would be a three-dimensional vector with two of its components representing $x_1(0)$ and $x_2(0)$. For this case it would also be interesting to choose switching curves of simpler form, such as straight-line segments. Such controllers could also be specified in terms of a time-invariant control vector c.

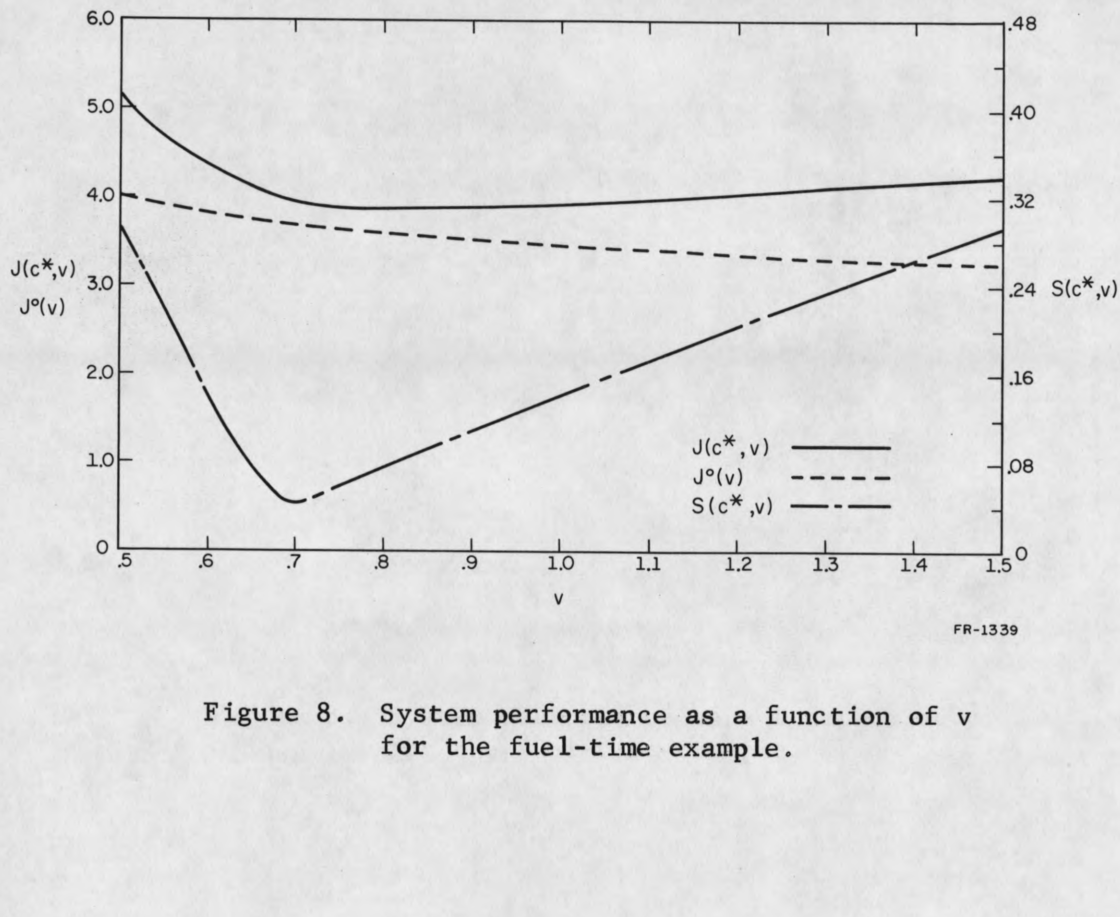

Figure 8. System performance as a function of v for the fuel-time example.

2.10 Position Control of a String of Vehicles

Figure 9 represents a string of three electronically-coupled vehicles. The system equations are

$$\delta\dot{y}_1 = -\frac{\alpha}{m_1}\delta y_1 + \frac{\delta f_1}{m_1}$$

$$\delta\dot{w}_1 = \delta y_1 - \delta y_2$$

$$\delta\dot{y}_2 = -\frac{\alpha}{m_2}\delta y_2 + \frac{\delta f_2}{m_2}$$

$$\delta\dot{w}_2 = \delta y_2 - \delta y_3$$

$$\delta\dot{y}_3 = -\frac{\alpha}{m_3}\delta y_3 + \frac{\delta f_3}{m_3} \qquad (2.33)$$

where δy_i is the velocity deviation of the i^{th} vehicle from the desired string velocity, δw_i is the deviation from the desired spacing, α is a drag coefficient linearized about the nominal string velocity, m_i is the mass of the i^{th} vehicle, and the control signal δf_i is the incremental force applied to the i^{th} vehicle. The performance index is

$$J = \frac{1}{2}\int_0^\infty [10(\delta w_1^2 + \delta w_2^2) + (\delta f_1^2 + \delta f_2^2 + \delta f_3^2)]dt.$$

This system was presented by Levine and Athans in a recent paper [12]. Theoptimal control for $\alpha = m_1 = m_2 = m_3 = 1$ is derived in that paper and it is indicated that the resulting transient performance is satisfactory.

Since the masses of the vehicles would vary in an actual system let $\frac{1}{m_1} = v_1$, $\frac{1}{m_2} = v_2$, and $\frac{1}{m_3} = v_3$ where v_1, v_2, and v_3 are

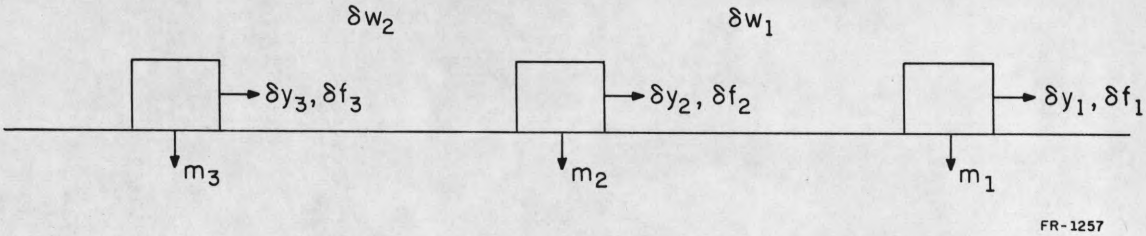

Figure 9. A string of three vehicles.

in the domain [.667, 2.0] (corresponding to vehicles able to transport twice their own weight). Since α is dependent on the string velocity let $\alpha = v_4$ where $v_4 \in [.75, 1.25]$.

Realization of the optimal adaptive control, c^{ideal}, requires the sensing of v_1 through v_4 and also requires that the entire 5-dimensional state vector be fed to each vehicle. It is apparent that an optimal adaptive controller for a string of say fifty vehicles would be unrealistic from an engineering viewpoint. Therefore assume a controller of the form

$$\delta f_1 = c_1 \, \delta y_1 + c_2 \, \delta w_1$$

$$\delta f_2 = c_3 \, \delta w_1 + c_4 \, \delta y_2 + c_5 \, \delta w_2$$

$$\delta f_3 = c_6 \, \delta w_2 + c_7 \, \delta y_3,$$

i.e. the incremental force applied to each vehicle is allowed to depend only on the velocity of that vehicle and the distance between it and the adjacent vehicles. By virtue of the system symmetry it is reasonable to make the restrictions $c_5 = -c_3$, $c_6 = -c_2$ and $c_7 = c_1$.

Since the values of the controller parameters will depend on the initial state let $\delta w_1(0) = v_5$, $\delta w_2(0) = v_6$, $\delta y_1(0) = v_7$, $\delta y_2(0) = v_8$, and $\delta y_3(0) = v_9$. It is assumed that v_5 and v_6 are in the domain [-5, 5] while v_7, v_8, and v_9 are in the domain [-3, 3]. (The fact that the controller parameters are dependent on the initial state is a price that is paid for assuming a simple controller. It can be argued, however, that the optimal system is unnecessarily versatile with respect to initial conditions since for any practical system there are many states that will never arise and are consequently of no interest.)

As formulated above, the system has a four dimensional control vector c and a nine dimensional parameter vector v.

The performance-sensitivity chosen for this example is simply

$$S(c,v) = J(c,v) - J_{ideal}(v)$$

where $J_{ideal}(v)$ is the value of the performance criterion corresponding to c^{ideal} the optimal adaptive control. The performance-sensitivity $S = \frac{J - J_{ideal}(v)}{J_{ideal}(v)}$ is not suitable for this problem because $J_{ideal}(v)$ is zero at some values of v.

<u>Computational aspects of minimaximizing J(c,v) and S(c,v)</u>: As shown by the algorithm in Section 2.3 these minimax problems can be solved by a series of minimizations of form $\min_{c \in W} \varphi(c)$ where $\varphi(c) = \max\{S(c,v^1), \ldots, S(c,v^k)\}$, and a series of maximizations of form $\max_{v \in V} \Psi(v)$ where $\Psi(v) = S(c^i,v)$. Experience with the Rosenbrock rotating coordinate search method [9] has shown that each minimization or maximization requires about (40 x n) computations of φ or Ψ, where n is the dimension of the variable ($n \geq 2$). By using this figure the computation time for each iteration of the algorithm can be estimated if the time required for computing J(c,v) and $J_{ideal}(v)$ is known.

In Section 2.7 it is shown that an analytic solution for J(c,v) is possible in the form $J(c,v) = x_0' L x_0$, where L is a

5 x 5 symmetric matrix, and x_0 is the initial state. An analytic solution for L requires, in the case of this example, the solution of 15 simultaneous linear equations. Since such a solution is excessively tedious the alternate approach can be taken of appending to the system equations (2.33) the equation

$$\dot{y}_4 = 5(\delta w_1^2 + \delta w_2^2) + .5(\delta f_1^2 + \delta f_2^2 + \delta f_3^2),\ y_4(0) = 0,$$

and using $J(c,v) = \lim_{t\to\infty} y_4(t)$. In practice adequate accuracy is obtained by taking $J(c,v) \doteq y_4(4)$. Computation of $J_{ideal}(v)$ is more difficult, requiring the steady-state solution of a set of 15 simultaneous, first-order, nonlinear, differential equations (see Section 2.7). In this case also the steady-state solution is achieved with sufficient accuracy in four seconds of system time.

It is evident that the minimax problem with $J(c,v)$ is easier than the problem with $S(c,v)$ since $J_{ideal}(v)$ is not required in the former case. The minimization steps in the latter case are no more difficult as the values of v are fixed and $J_{ideal}(v)$ can be computed beforehand. However, the maximizations require computation of $J_{ideal}(v)$ at each step of the search. For this reason four values of v were included in the sets V_i in an attempt to speed convergence and reduce the number of maximizations required.

It is noted that a hybrid computer with an analog facility capable of solving for $J(c,v)$ or $J_{ideal}(v)$ in a fraction of a second is ideal for the minimax problems considered here. The overall computation time with such a computer would be a few minutes.

<u>Computation of c^*</u>: The algorithm terminated at the fifth iteration, yielding $(c_1, c_2, c_3, c_4) = (-1.11, -2.27, 2.10, -2.65)$ and $S^* = \min_c \max_{v \in V} S(c,v) = 59$. The minimax value, S^*, was attained at

$$v^{10} = (.667, 2.00, .667, .75, 5.0, -5.0, 3.0, 3.0, 3.0)$$

$$v^{11} = (.667, .667, .667, .75, 5.0, -5.0, -3.0, -3.0, -3.0)$$

$$v^{12} = (.667, 2.0, 2.0, .75, 5.0, 5.0, 3.0, 3.0, 3.0)$$

and other values symmetrical with these.

<u>Computation of $\hat{c}$</u>: The algorithm terminated at the second iteration, yielding $(c_1, c_2, c_3, c_4) = (-2.25, -2.78, 1.51, -2.77)$ and $J^* = \min_c \max_{v \in V} J(c,v) = 445$. The minimax value, J^*, was attained at

$$\hat{v} = (.667, .667, .667, .75, 5.0, -5.0, 3.0, -3.0, 3.0)$$

$$\tilde{v} = (.667, .667, .667, .75, 5.0, 5.0, 3.0, 3.0, -3.0)$$

and other values symmetrical with these.

<u>Nominal optimal control</u>: For comparison, a controller of the same form but optimal for a nominal v was computed. The chosen value of v was

$$v^{nom} = (1.0, 1.0, 1.0, 1.0, 5.0, -5.0, 3.0, -3.0, 3.0)$$

and the resulting control c^{nom} was given by $(c_1, c_2, c_3, c_4) = (-2.1, -1.51, 2.02, -2.65)$.

<u>Evaluation</u>: Table 1 gives the values of J and S for the controllers c^*, $\hat{c}$, c^{nom}, and c^{ideal}, at v^{10}, v^{11}, v^{12}, v^{nom}, $\hat{v}$, and $\tilde{v}$. Figures 10, 11, 12, 13, 14, and 15 illustrate the corresponding transient responses. It should be emphasized that v^{10}, v^{11}, and v^{12}, are

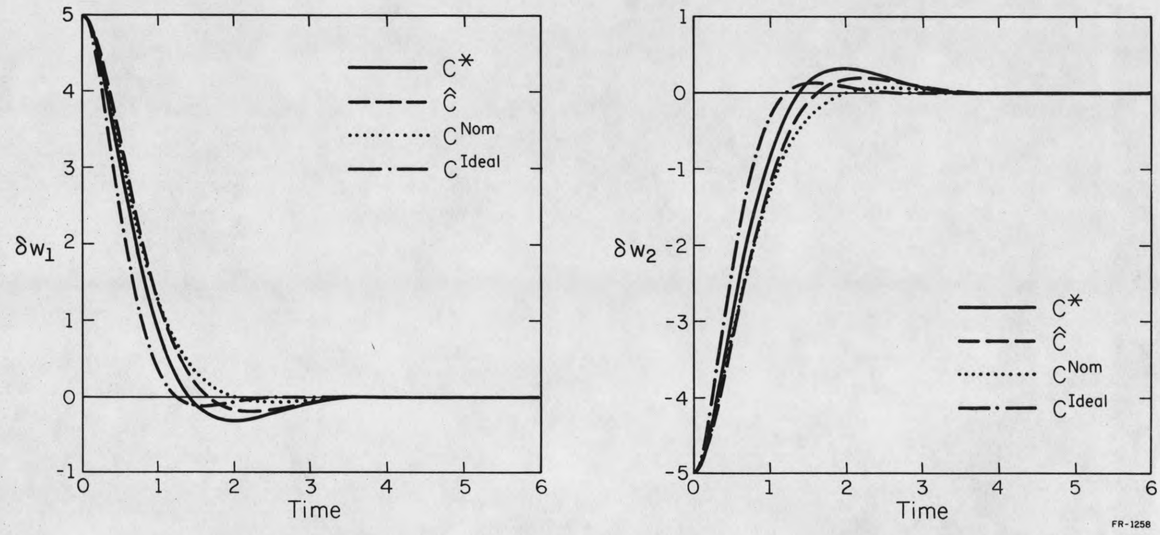

Figure 10. Transient responses at $v = v^{10}$.

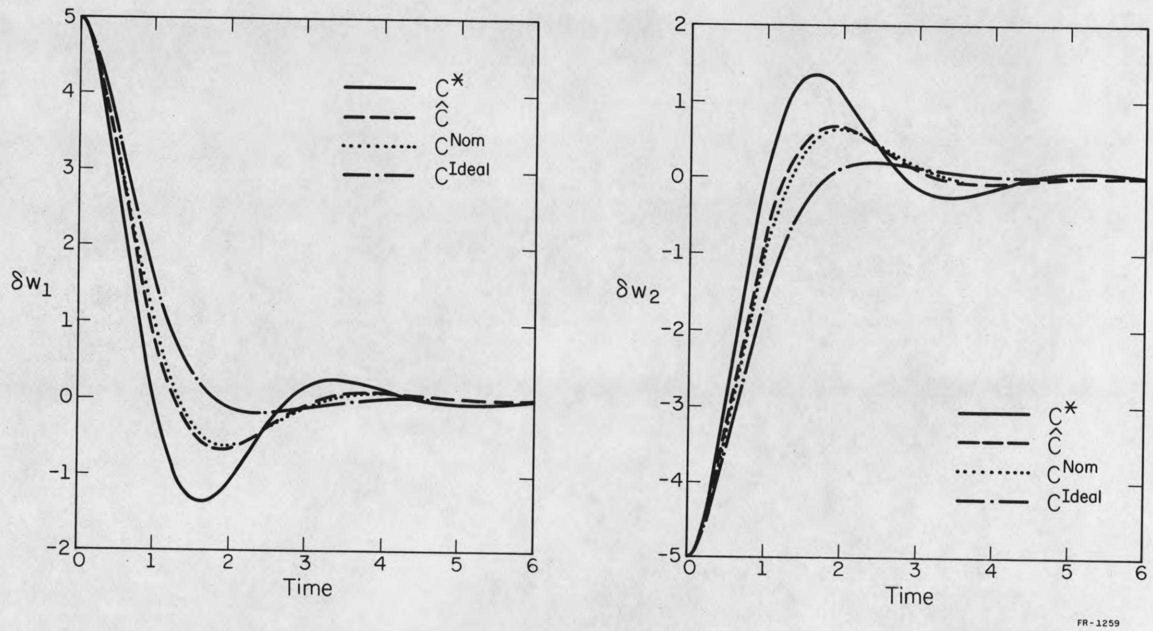

Figure 11. Transient responses at $v = v^{11}$.

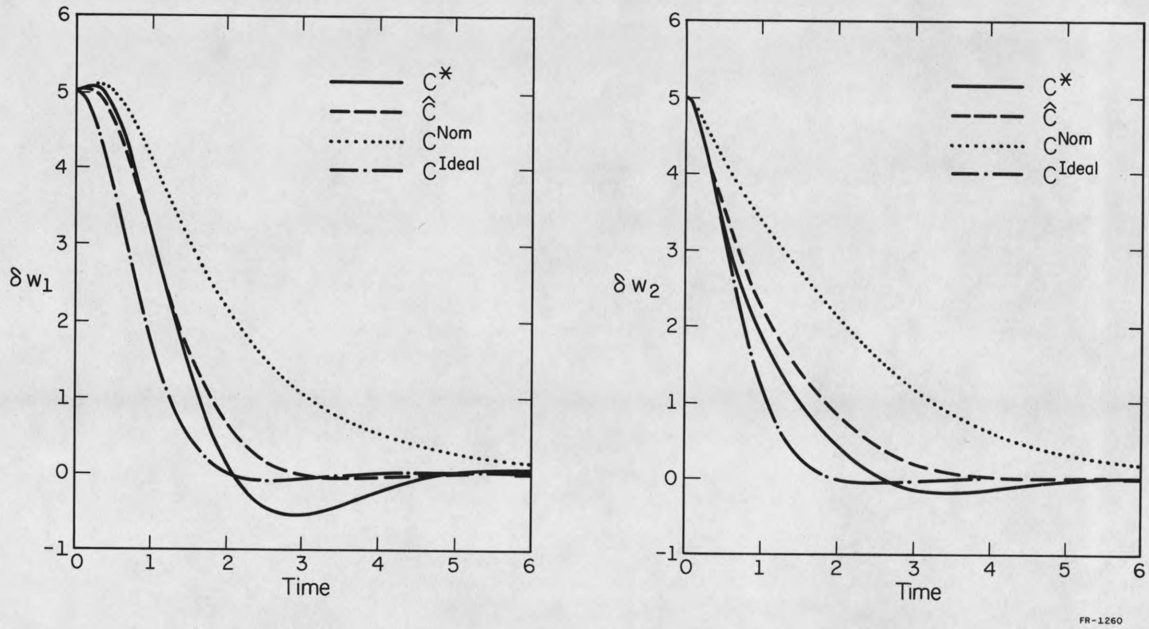

Figure 12. Transient responses at $v = v^{12}$.

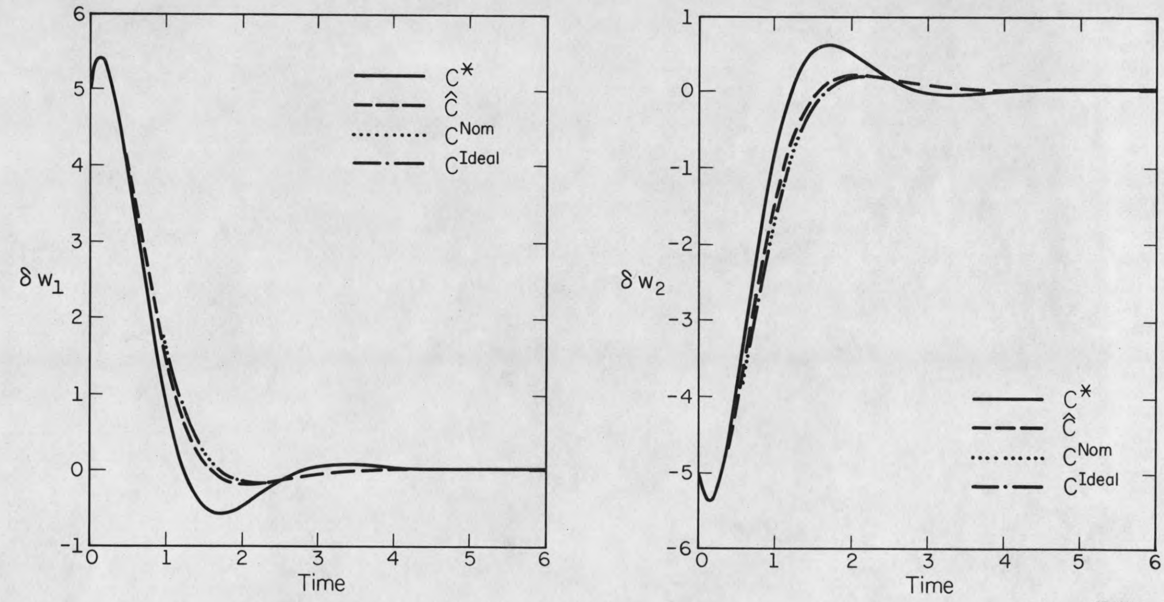

Figure 13. Transient responses at $v = v^{nom}$.

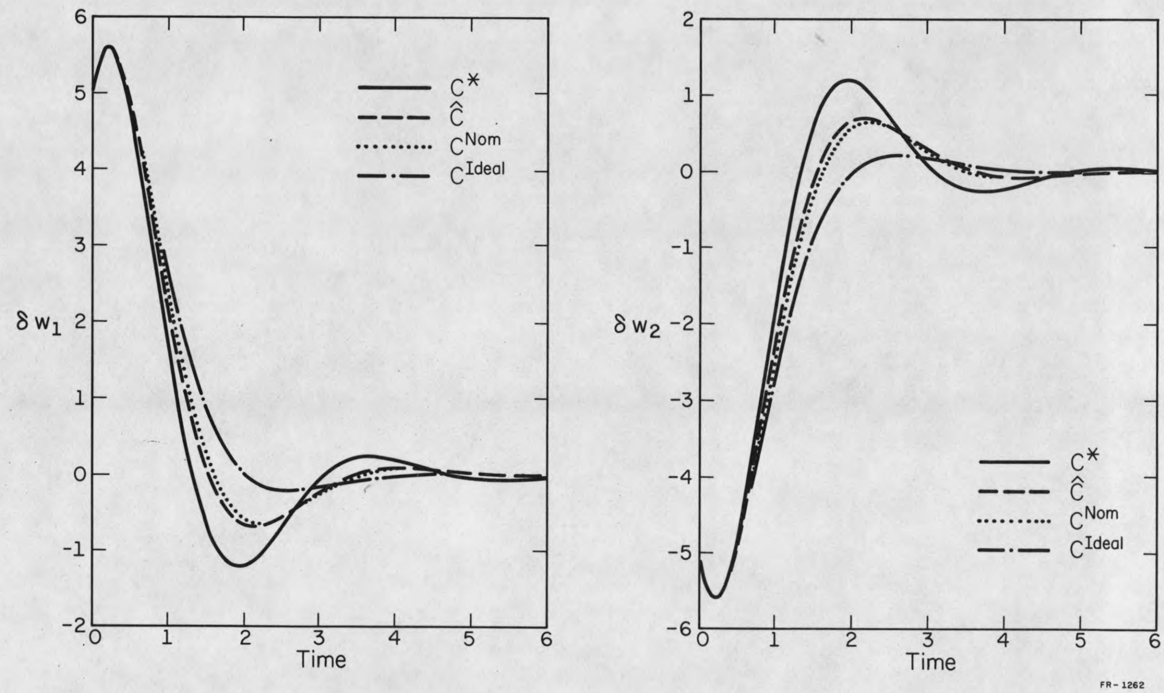

Figure 14. Transient responses at $v = \hat{v}$.

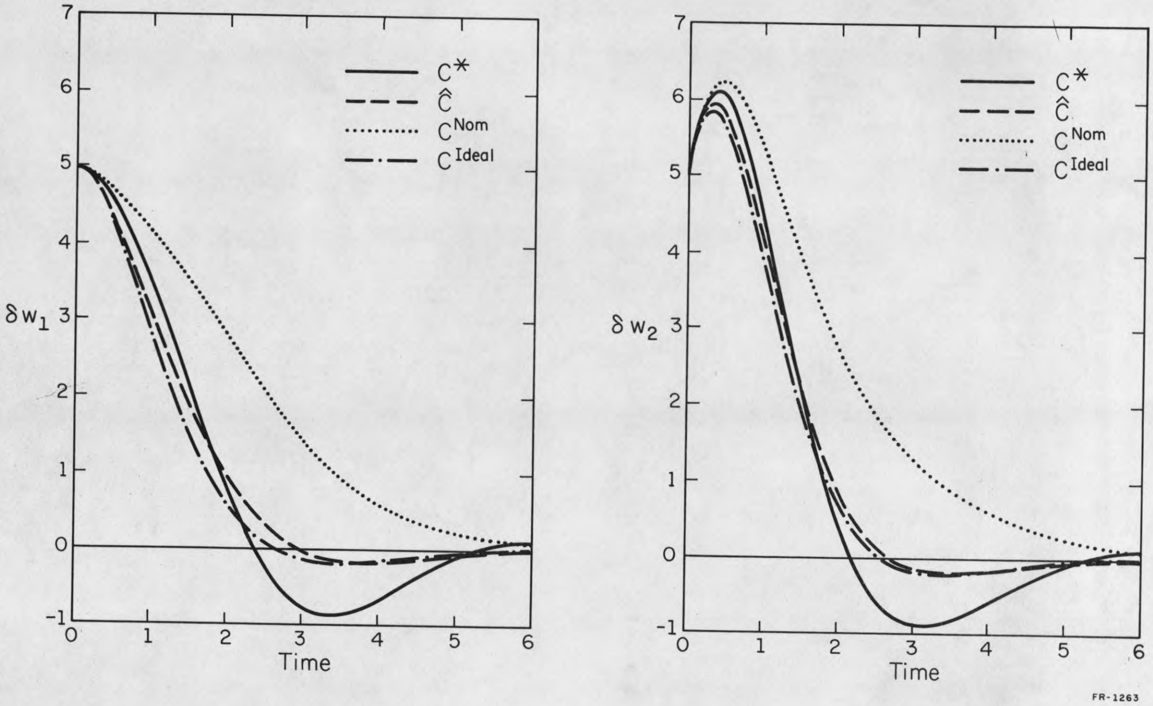

Figure 15. Transient responses at $v = \tilde{v}$.

the values of v which yield the worst performance in terms of S for c*, while $\hat{v}$ and $\tilde{v}$ represent the worst performance in terms of J for $\hat{c}$.

The nominal optimal control c^{nom}, while equivalent to c^{ideal} at v^{nom}, leads to a high value of J and a very slow response at other values of v (Table 1 and Figures 12 and 15). The minimax performance-sensitivity control c* gives a more uniform performance as v varies but sometimes leads to considerable overshoot (Table 1 and Figures 11, 12, and 15). The minimax performance control $\hat{c}$ is superior to c^{nom} and c* since it gives a transient response which is neither very slow nor very oscillatory for any v.

A comparison of $\hat{c}$ with c^{ideal} shows that the former control causes slightly more overshoot at some values of v (Figures 11 and 14) and sometimes leads to a slightly slower response (Figure 12) but these differences are relatively minor. In view of the considerable reduction in controller complexity the minimax performance control $\hat{c}$ is a very attractive choice.

TABLE 1

Values of J and S

	v^{10}	v^{11}	v^{12}	v^{nom}	$\hat{v}$	$\tilde{v}$
c^{ideal}	129, 0	188, 0	196, 0	297, 0	413, 0	440, 0
c*	188, 59	247, 59	255, 59	312, 15	459, 46	471, 31
$\hat{c}$	227, 98	205, 17	268, 72	311, 14	446, 33	446, 6
c^{nom}	175, 46	220, 32	364, 168	297, 0	421, 8	562, 122

It is not unreasonable that the minimax performance control should be suited to this system since the rapid (or ideal) correction of major state errors is vital while non-ideal correction of minor errors is of little concern.

The successful computation of the minimax performance control $\hat{c}$ and its adequate performance suggest several interesting avenues for further research. If it is agreed that the performance with $\hat{c}$ is closer to the ideal than the system requires, then the investigation of an even simpler controller is indicated. For example, the force applied to each vehicle could be allowed to depend only on the velocity of that vehicle, and the distance between it and the preceding vehicle. Another interesting avenue of investigation would be to evaluate the performance of strings of four or more vehicles with the controller parameters $\hat{c}$ obtained for a string of three vehicles. In the case of unsatisfactory performance the next step could be to compute $\hat{c}$ for strings of four and more vehicles. If the controller parameters thus obtained quickly converge to a limit, then the possibility of having a small number of possible gains in each vehicle, to be set according to the number of vehicles in the string, could be considered.

The possibilities discussed above give an illustration of the general design approach presented in Section 2.8.

3. TIME-VARYING CONTROLLERS

3.1 Introduction

It is assumed here as in Section 2.1 that the dynamic system and the performance index can be expressed in the form:

$$\dot{x}(t) = f[x(t), u(t), v, t], \; x(t_0) = x^0(v),$$
$$J(u,v) = \int_o^T L[x(t), u(t), v, t]dt$$
$$u(t) \in U$$
$$v \in V \; . \qquad (3.1)$$

The present chapter is concerned with the class of problems where the controller structure is specified in terms of a time-varying vector, i.e.

$$u(t) = u[c(t), x(t)]$$

where $c(t) \in W$ is an r-dimensional time-varying vector, and W is the allowable domain of c(t) such that $c(t) \in W \Rightarrow u(t) \in U$.

In the following the notation [c(t)] is used to indicate consideration of c(t) over the entire range $t_0 \leq t \leq T$.

Definition 3.1.1: $[\hat{c}(t)] \in W$ is a minimax performance control if and only if

$$\max_{v \in V} J([\hat{c}(t)],v) = \min_{[c(t)] \in W} \max_{v \in V} J([c(t)],v).$$

Definition 3.1.2: $[c^*(t)] \in W$ is a minimax performance-sensitivity control if and only if

$$\max_{v \in V} S([c^*(t)],v) = \min_{[c(t)] \in W} \max_{v \in V} S([c(t)],v).$$

The choice of a controller structure with time-varying parameters greatly increases the difficulty of determining minimax controls since each element of the control vector must be determined at infinitely many instants of time. Therefore in many cases it should be questioned whether the arbitrariness of the proposed structure is necessary. For example, if it occurs that a time-varying minimax parameter can be closely approximated by a small number of orthogonal polynomials (such as Chebyshev polynomials) then it is evident that the minimax problem could have been restated with only the time-invariant polynomial coefficients to be determined. This approach, which reduces the problem to the time-invariant case considered in Chapter 2, may prove to be the most practical method for computation of minimax time-varying controls when $c(t)$ is of moderate or high dimension.

In some cases the enormous freedom remaining after the choice of a time-varying controller structure may be used to determine a good controller by more direct methods than solution of the minimax problem. Section 3.4, titled "Multi-point Optimality," gives some results in this area, and the example in Section 3.5 demonstrates that the resulting control may be so close to optimality at all parameter values that solution of the minimax problem to obtain further improvement is not warranted.

In spite of these interesting possibilities it is desirable to be able to solve the time-varying minimax problem directly without reducing it to a time-invariant problem or by-passing it entirely. This is simply because the minimax controller parameters give the best possible performance for a given structure (at least in the sense of minimizing the maximum value of the performance index or the performance-sensitivity). As noted in the introduction of this thesis the fundamental design compromise is that between the system performance and the controller complexity. Other compromises are justified only in the face of prohibitive computational difficulty or when the control obtained by some method is so close to optimality at all values of the variable parameters that further computation is unwarranted.

The present chapter considers the direct computation of time-varying controls and also presents some preliminary results in the area of time-varying controllers which exhibit multi-point optimality.

3.2 An Algorithm for Time-Varying Minimax Problems

Let $S([c(t)],v)$ be a continuous functional of the r-dimensional vector $c(t)\in W$, $t_0 \leq t \leq T$, and the s-dimensional vector $v\in V$. It is assumed that a norm, $\| \|$, has been defined on W and that W is a compact domain. It is assumed as in the time-invariant case that V is a closed bounded region in E_s. It is desired to determine $[c^*(t)]$ satisfying

$$\max_{v\in V} S([c^*(t)],v) = \min_{[c(t)]\in W} \max_{v\in V} S([c(t)],v).$$

The minimax algorithm for this time-varying case is obtained simply by substituting $[c(t)]$ for c in the algorithm for the time-invariant case presented in Section 2.3. The algorithm defines the sequences $\{S_i^m\}$, $\{S_i^M\}$, $\{[c^n(t)]\}$, $\{v^n\}$, and the sets V_n and V_n^a for n = 1, 2, ...

The two lemmas and the theorem which follow prove that the algorithm solves the minimax problem.

Lemma 3.2.1: $S_i^m \leq \min_{[c(t)]\in W} \max_{v\in V} S([c(t)],v) \leq S_i^M$ for i = 1, 2, ...

Corollary: $S_i^m = S_i^M \Rightarrow [c^i(t)] = [c^*(t)]$.

Lemma 3.2.2: S_1^m, S_2^m, ..., is a monotonic increasing sequence.

The proofs of these lemmas follow exactly the proofs given for Lemmas 2.3.1 and 2.3.2.

Theorem 3.2.1: The sequences $\{S_i^m\}$ and $\{S_i^M\}$ both converge to the limit

$$\min_{[c(t)]\in W} \max_{v\in V} S([c(t)],v) .$$

Proof: Since $S([c(t)],v)$ is a continuous functional of c(t) and is a continuous function of v, and W and V are compact, it follows that $S^* = \min_{[c(t)]\in W} \max_{v\in V} S([c(t)],v)$ exists. Then from Lemmas 3.2.1 and 3.2.2, $\{S_i^m\}$ is a monotonic increasing sequence bounded above by S^*. It follows that the limit exists. Let $\lim_{i\to\infty} S_i^m = \hat{S}$.

If it can be shown that the sequence $\{S_i^M - S_i^m\}$ converges to zero it follows from Lemma 3.2.1 that

$$\lim_{i\to\infty} S_i^m = \hat{S} = \min_{[c(t)]\in W} \max_{v\in V} S([c(t)],v) = \lim_{i\to\infty} S_i^M,$$

which is the desired result.

Suppose $\{S_i^M - S_i^m\}$ does not converge to zero. Then for some $\mathcal{E} > 0$ there is an infinite set $\{\bar{S}_m^M\}$ of elements of $\{S_i^M\}$ such that $\bar{S}_m^M - \hat{S} \geq \mathcal{E}$, for $\forall m$. Let $\{[\bar{c}^m(t)]\}$ be the sequence of controls corresponding to $\{\bar{S}_m^M\}$. Since $\{[\bar{c}^m(t)]\}$ is an infinite sequence on a compact set W there is a convergent subsequence $\{[\hat{c}^n(t)]\}$. Let $[\hat{c}(t)]$ be the limit. Let the corresponding subsequence of $\{\bar{S}_m^M\}$ be $\{\hat{S}_n^M\}$. Then $\hat{S}_n^M - \hat{S} \geq \mathcal{E}$, $\forall n$.

By continuity of $S([c(t)],v)$ there is for each v a $\delta(v) > 0$, such that

$$|S([c(t)],v) - S([\hat{c}(t)],v)| \leq \frac{\mathcal{E}}{4}$$

whenever $\|[c(t)] - [\hat{c}(t)]\| \leq \delta(v)$. Since v is in a compact subset V of E_s it follows that the continuity is uniform wrt $v\in V$, i.e. $\exists\delta > 0$, such that

$$|S([c(t)],v)-S([\hat{c}(t)],v)| \leq \frac{\mathcal{E}}{4} \text{ whenever } \|[c(t)]-[\hat{c}(t)]\| \leq \delta, \ \forall v\in V . \tag{3.1}$$

Let $[\hat{c}^n(t)]$, $[\hat{c}^m(t)]$, $m > n$, be any two elements of $\{[\hat{c}^n(t)]\}$ where n, m, are sufficiently large that

$$\|[\hat{c}^n(t)] - [\hat{c}(t)]\| \leq \delta,$$

and

$$\|[\hat{c}^m(t)] - [\hat{c}(t)]\| \leq \delta.$$

Then using (3.1),

$$|S([\hat{c}^n(t)],v) - S([\hat{c}^m(t)],v)| \leq \frac{\mathcal{E}}{2}, \ \forall v\in V. \tag{3.2}$$

Let $\hat{S}_m^m$, $\hat{v}^n$, $\hat{V}_n$, $\hat{V}_{m-1}$, $\hat{V}_n^a$ corresponding to $[\hat{c}^n(t)]$ and $[\hat{c}^m(t)]$ be as defined by the algorithm. Then

$$\hat{V}_n = \hat{V}_n^a \cup \{\hat{v}^n\}$$

and

$$\max_{v \in V} S([\hat{c}^n(t)],v) = S([\hat{c}^n(t)],\hat{v}^n) = \hat{S}_n^M \geq \hat{S} + \varepsilon. \qquad (3.3)$$

By definition of V_i it follows that

$$\max_{v \in V_i} S([c(t)],v) \geq \max_{v \in V_k} S([c(t)],v), \ \forall i \geq k, \ \forall [c(t)] \in W .$$

Therefore since $m > n$

$$\max_{v \in \hat{V}_{m-1}} S([c(t)],v) \geq \max_{v \in \hat{V}_n} S([c(t)],v), \forall [c(t)] \in W.$$

$$\therefore \ \hat{S}_m^m = \max_{v \in \hat{V}_{m-1}} S([\hat{c}^m(t)],v) \geq \max_{v \in \hat{V}_n} S([\hat{c}^m(t)],v).$$

Therefore since $\hat{v}^n \in \hat{V}_n$,

$$\hat{S}_m^m \geq S([\hat{c}^m(t)],\hat{v}^n). \qquad (3.4)$$

Also, using (3.2)

$$\hat{S}_m^m \geq S([\hat{c}^m(t)],\hat{v}^n) \geq S([\hat{c}^n(t)],\hat{v}^n) - \frac{\varepsilon}{2} .$$

Using this result and (3.3),

$$\hat{S}_m^m \geq S([\hat{c}^n(t)],\hat{v}^n) - \frac{\varepsilon}{2} \geq \hat{S} + \frac{\varepsilon}{2} > \hat{S},$$

which is the desired contradiction since $\hat{S}$ is the limit of the monotonic increasing subsequence $\{\hat{S}_i^m\}$.

As in the time-invariant case it follows that the controls $[c^n(t)]$ are minimax controls to any desired degree of accuracy in achieving $S^* = \min_{[c(t)]\in W} \max_{v\in V} S([c(t)],v)$, when n is sufficiently large. If $[c^*(t)]$ is a unique control the sequence $\{[c^n(t)]\}$ converges to $[c^*(t)]$. If $[c^*(t)]$ is not unique then any limit point of $\{[c^n(t)]\}$ is a solution for $[c^*(t)]$.

In practical application of the algorithm the sets V_i^a should be chosen according to the method outlined for the time-invariant case.

Minimization Steps: While the algorithm has simplified the individual steps in comparison with the original minimax problem there remans some difficulty in the minimization steps. When a minimax control is to be determined the minimization is constrained by the system differential equations and requires the solution of a two point boundary value problem. Consider the minimization with respect to $c(t)\in W$, $t_0 \leq t \leq T$, of

$$\max\{J([c(t)],v^1),\ \ldots,\ J([c(t)],v^k)\}$$

where a performance functional J has been taken for simplicity. The method to be described is also applicable to the performance-sensitivities $S = \frac{J - J^0(v)}{J^0(v)}$, and $S = J - J^0(v)$, with minor modifications. Since the different vectors $v^1, \ldots, v^k$, relate to changes in the system or its initial state the problem may be considered as involving k systems:

$$\dot{x}^1 = f(x^1(t), c(t), v^1, t) = f^1(x^1(t), c(t), t),\ x^1(t_0) = x_0^1\ ,$$

....

$$\dot{x}^k = \qquad\qquad = f^k(x^k(t), c(t), t),\ x^k(t_0) = x_0^k\ ,$$

$$J_1 = \int_{t_0}^{T} L(x^1(t), c(t), v^1, t)dt = \int_{t_0}^{T} L_1(x^1(t), c(t), t)dt,$$

....

$$J_k = \qquad\qquad = \int_{t_0}^{T} L_k(x^k(t), c(t), t)dt, \quad (3.5)$$

where each x^i and f^i is an n-dimensional vector.

It is desired to find $\bar{c}(t)$, $t_0 \leq t \leq T$, to realize

$$\bar{\varphi} = \min_{[c(t)]\in W} \max\{J_1([c(t)]), \ldots, J_k([c(t)])\}.$$

There are two types of solution to a problem of this type. If, for any j, the optimal control $[c^0(t)]_{v=v^j}$ is such that maximization over the J_i with this control yields J_j then it is evident that $[\bar{c}(t)] = [c^0(t)]_{v=v^j}$. More commonly no optimal control satisfies the requirement and $[\bar{c}(t)]$ is in effect a compromise between several values of v^j. In this case it is expected that $\bar{\varphi}$ will be attained at these values of v when $[c(t)] = [\bar{c}(t)]$ and that removal of the J_i corresponding to other values of v would not affect the result.

The most promising approach to problems of this type appears to be by a variation of one of the "gradient in function space" methods. Since the problem is already made unwieldy by the requirement of considering k systems, the simplest of these methods, steepest descent of the Hamiltonian is presented here.

Form the k Hamiltonians and k sets of adjoint equations corresponding to the k systems:

$$H_1 = L_1 + \langle p^1, f^1 \rangle, \quad \dot{p}^1 = -\text{grad}_x H_1, \quad p^1(T) = 0,$$

....

$$H_k = L_k + \langle p^k, f^k \rangle, \quad \dot{p}^k = -\text{grad}_x H_k, \quad p^k(T) = 0. \qquad (3.6)$$

(a) Guess a solution for $c(t)$, $t_0 \leq t \leq T$.

(b) Integrate the system equations (3.5) forwards in time and store $x^1(t), \ldots, x^k(t), J_1, \ldots, J_k$. Set $J_{max} = \max_{1 \leq i \leq k} J_i$.

(c) Integrate the adjoint equations (3.6) backwards in time. At each step improve the estimate of $c(t)$ according to

$$c_i(t)^{new} = c_i(t)^{old} - \mathcal{E} \sum_{j=1}^{k} k_j \frac{\partial H_j}{\partial c_i}, \quad 1 \leq i \leq r,$$

where $\mathcal{E}$ and the k_j are chosen by appropriate rules based on past iterations and discussed below.

If at any step $J_{max}^{new} > J_{max}^{old}$ then $c(t)^{new}$ should be rejected and the value of $\mathcal{E}$ reduced. If $J_{max}^{new} < J_{max}^{old}$ then it may be possible to speed convergence by increasing $\mathcal{E}$. The values of the k_j should be altered at each iteration in such a manner as to increase the emphasis on those Hamiltonians corresponding to the larger performance indices, i.e. the following rules should be followed:

(i) $\sum_{j=1}^{k} k_j = 1$

$$\text{(ii)} \quad J_m > J_n \Rightarrow \left(\frac{k_m}{k_n}\right)^{new} > \left(\frac{k_m}{k_n}\right)^{old}.$$

One realization of the second rule is to use simply

$$\left(\frac{k_m}{k_n}\right)^{new} = \left(\frac{J_m}{J_n}\right)^{\beta}\left(\frac{k_m}{k_n}\right)^{old} \quad \text{for each } J_m > J_n,$$

where β is a small positive number, say $0 < \beta \leq 1$.

Providing the k_j are chosen appropriately the convergence properties of this method are similar to those of the analogous method in computing a single optimal control. Often the method of steepest descent of the Hamiltonian gives good initial convergence but very slow final convergence and in these cases it can be regarded as a method of obtaining an approximate solution. When either the system equations are highly non-linear or the performance index is far from quadratic and the method of steepest descent is not capable of yielding even an approximate solution, a "gradient in function space" method based on second variations should be considered. However, such a method typically requires the integration of $n + \frac{1}{2}n(n+1)$ simultaneous differential equations for each system in the reverse-time steps. When k sets of system equations must be considered it is apparent that quite small numbers k and n will be sufficient to saturate most computational facilities.

3.3 Results for the Linear Case with an Open-Loop Controller

When the system is linear, the performance index is quadratic, and the controller has an open-loop structure, the minimizations required by the minimax algorithm reduce to a simple form of two-point boundary value problem. Since the optimal control can be generated for all initial states by a linear state-feedback controller the choice of an open-loop controller is unusual but it may have practical application where the initial state is within a small region of state space and the state variables are not available to the controller.

For simplicity the method is presented in terms of a performance functional J and two values of v. Given

$$\dot{x} = A(v)x + B(v)u, \ x(t_0) = x_0(v)$$

$$J([u(t)],v) = \tfrac{1}{2}\int_{t_0}^{T} (\langle x,Qx\rangle + \langle u,Ru\rangle)dt$$

and given two values v^1, v^2, of v it is desired to find the control $\bar{u}(t)$, $t^0 \leq t \leq T$, which realizes

$$\bar{J} = \min_{[u(t)]} \max\{J([u(t)],v^1), \ J([u(t)],v^2)\}.$$

By substituting for v^1 and v^2 the equations become

$$\dot{x}^1 = A_1x^1 + B_1u^1, \qquad x_1(t_0) = x_0^1,$$

$$\dot{x}^2 = A_2x^2 + B_2u^2, \qquad x_2(t_0) = x_0^2,$$

$$J_1 = \tfrac{1}{2}\int_{t_0}^{T} (\langle x^1,Qx^1\rangle + \langle u^1,Ru^1\rangle)dt$$

$$J_2 = \tfrac{1}{2}\int_{t_0}^{T} (\langle x^2,Qx^2\rangle + \langle u^2,Ru^2\rangle)dt. \tag{3.7}$$

If the optimal control for $v = v^i$, $[u^0(t)]_{v=v^i}$, is such that $J_i > J_k$ with this control then clearly $[\bar{u}(t)] = [u^0(t)]_{v=v^i}$. Therefore when only two values of v are being considered it is useful to first check the optimal controls. If $[\bar{u}(t)]$ is not an optimal control it follows that $J_1 = J_2$ when $[u(t)] = [\bar{u}(t)]$. (Assumption for instance that $J_1 = J_2 + \mathcal{E}$ when $[u(t)] = [\bar{u}(t)]$ leads to a contradiction in the following way. Since $[\bar{u}(t)]$ is not optimal at $v = v^1$ there exist perturbations $[\delta u(t)]$ which reduce J_1. By continuity, the perturbations can be chosen sufficiently small that the change in J_2 is less than $\mathcal{E}$. Thus $[\bar{u}(t)]$ does not realize $\bar{J}$, which is the desired contradiction.)

Assuming that $[\bar{u}(t)]$ is not an optimal control the problem can be restated as follows. Find $[u^1(t)] = [\bar{u}(t)]$ which minimizes J_1 while satisfying the constraints:

(1) $J_1 = J_2$,

(2) $[u^1(t)] = [u^2(t)]$.

Constraint (1) can be handled by minimizing $J = J_1 + \mu J_2$ where μ is a scalar weighting factor. Let

$$x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \qquad u = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \qquad \hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & \mu Q \end{bmatrix} \qquad \hat{R} = \begin{bmatrix} R & 0 \\ 0 & \mu R \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \qquad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

so that $\dot{x} = Ax + Bu$

and
$$J = \tfrac{1}{2} \int_{t_0}^{T} (\langle x, \hat{Q}x\rangle + \langle u, \hat{R}u\rangle)dt.$$

Define

$$E[x,t] = \min_{t \le \sigma \le T} \int_t^T (\tfrac{1}{2}\langle x, \hat{Q}x\rangle + \tfrac{1}{2}\langle u, \hat{R}u\rangle + \langle \lambda, u^1 - u^2\rangle) d\sigma \quad (3.8)$$

where constraint (2) has been appended to the composite performance index J with the r-dimensional Lagrange multiplier λ. By the continuous form of dynamic programming (3.8) becomes

$$\min_{u(t)} \{\tfrac{1}{2}\langle x,\hat{Q}x\rangle + \tfrac{1}{2}\langle u,\hat{R}u\rangle + \langle \lambda, u^1 - u^2\rangle + \langle \dot{x}, \frac{\partial E}{\partial x}\rangle\} + \frac{\partial E}{\partial t} = 0. \tag{3.9}$$

Taking the gradient with respect to u(t),

$$\hat{R}\,\hat{u} + \begin{bmatrix} \lambda \\ -\lambda \end{bmatrix} + \langle B, \frac{\partial E}{\partial x}\rangle = 0$$

where $\hat{u}$ is the minimizing value of u.

$$\therefore\ \hat{u} = -\hat{R}^{-1}\{\langle B, \frac{\partial E}{\partial x}\rangle + \begin{bmatrix} \lambda \\ -\lambda \end{bmatrix}\}. \tag{3.10}$$

The Lagrange multiplier λ will be chosen so that $u^1 = u^2$ as required by constraint (2). Also since $u^1(t)$ can be expressed in the form $u^1(t) = C_1(t)\, x^1(t) + C_2(t)\, x^2(t)$, $\hat{u}(t)$ can be written

$$\hat{u}(t) = \begin{bmatrix} C_1x^1 + C_2x^2 \\ C_1x^1 + C_2x^2 \end{bmatrix} = \hat{C}x. \tag{3.11}$$

Substituting into (3.9)

$$\tfrac{1}{2}\langle x,\hat{Q}x\rangle + \tfrac{1}{2}\langle \hat{C}x, \hat{R}\,\hat{C}x\rangle + \langle Ax + B\hat{C}x, \frac{\partial E}{\partial x}\rangle + \frac{\partial E}{\partial t} = 0. \tag{3.12}$$

As a solution for E try

$$E[x,t] = \tfrac{1}{2}\langle x,Kx\rangle. \tag{3.13}$$

By substituting (3.13) into (3.12) it is seen that (3.13) is a non-trivial solution if

$$\dot{K} + KB\hat{C} + \hat{C}'BK + KA + A'K + \hat{C}'\hat{R}\hat{C} + \hat{Q} = 0 \tag{3.14}$$

Also, since by definition $E[x,T] = 0$, $K(T) = 0$.

Let $K = \begin{bmatrix} K_1 & K_2 \\ K_2' & K_4 \end{bmatrix}$ where each K_i is an n x n matrix. Then substituting into (3.11),

$$\hat{u} = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = - \begin{bmatrix} R^{-1}(B_1' K_1 x^1 + B_1' K_2 x^2 + \lambda) \\ \frac{R^{-1}}{\mu}(B_2' K_2' x^1 + B_2' K_4 x^2 - \lambda) \end{bmatrix} .$$

Since $u^1 = u^2$,

$$\lambda = \frac{-\mu}{1+\mu} \left[(B_1' K_1 - \frac{B_2' K_2'}{\mu})x^1 + (B_1' K_2 - \frac{B_2' K_4}{\mu})x^2 \right]$$

and $u^1(t) = u^2(t) = C_1(t)\ x^1(t) + C_2(t)\ x^2(t)$

where
$$C_1(t) = \frac{-R^{-1}}{1+\mu}(B_1' K_1 + B_2' K_2')$$

$$C_2(t) = \frac{-R^{-1}}{1+\mu}(B_1' K_2 + B_2' K_4) \ . \qquad (3.15)$$

Thus the procedure for computing $[\overline{u}(t)]$ is:

(a) Choose a value of $\mu > 0$.

(b) Solve (3.14) in reverse time (using (3.11) and (3.15) as needed) and store K(t) as a function of time.

(c) Solve the system equations (3.7) forwards in time and use (3.15) to generate $u^1(t)$.

(d) If $J_1 \neq J_2$ adjust the value of the scalar multiplier μ and repeat from (b).

Generally a small number of iterations should be sufficient to determine μ with adequate accuracy. The method can be extended to minimization at k values of v but the extension may not be practical since in general $\frac{kn(kn+1)}{2}$ simultaneous, first-order, non-linear, differential equations must be integrated in the reverse-time steps.

3.4 Multi-Point Optimality

The following design method, which evolved from a discussion between the author and Professor P. Kokotović, has not to the author's knowledge been previously applied to controller design.

Suppose an open-loop controller is specified for a given system with one input. Then it is possible for the controller to generate the optimal control at any one value of v. Suppose a closed-loop structure with two time-varying parameters is specified. Then (providing the given structure is appropriate) it should be possible for the controller to generate the optimal control at any two values of v. In general (providing always that the structure is appropriate) n arbitrary controller parameters provides the possibility of n points of optimality. If the system has normal continuity properties, and if the number of points of optimality is sufficiently large in comparison with the size of the domain V, the controller will generate nearly-optimal controls at all values of $v \in V$.

Consider the design of a feedback controller for the n-dimensional system:

$$\dot{x}(t) = f(x(t), u(t), v, t), \; x(t_0) = x^0(v),$$

$$J([u(t)],v) = \int_{t_0}^{T} L[x(t), u(t), v, t]dt,$$

$u(t) \in U$, $u(t)$ a scalar,

$v \in V$, v s-dimensional.

It is assumed that the system has a single input for convenience of presentation but this is not an essential requirement. Assume that a linear state-feedback controller is desired, i.e.

$$u(t) = c_0(t) + \sum_{i=1}^{n} c_i(t)\, x_i(t) \; . \qquad (3.17)$$

Let $c(t)$ be the $(n+1)$-dimensional vector with components, $c_0(t)$, ..., $c_n(t)$. In general, it is possible to compute the optimal control $u^0(t, v^i)$ as a function of time, for each value v^i, by standard variational methods. Corresponding to each v^i and $u^0(t, v^i)$ there is a vector of optimal state trajectories $x^0(t, v^i)$. Then using (3.17) the following equations can be written:

$$u^0(t,v^1) = c_0(t) + c_1(t)\, x_1^0(t, v^1) + \ldots + c_n(t)\, x_n^0(t, v^1)$$

$$\vdots$$

$$u^0(t, v^{n+1}) = c_0(t) + c_1(t)\, x_1^0(t, v^{n+1}) + \ldots + c_n(t)\, x_n^0(t, v^{n+1}),$$

i.e. using matrix notation $u^0 = Xc(t)$ where

$$u^0 = \begin{bmatrix} u^0(t, v^1) \\ \vdots \\ u^0(t, v^{n+1}) \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & x_1^0(t, v^1) & \ldots & x_n^0(t, v^1) \\ \vdots & & & \\ 1 & x_1^0(t, v^{n+1}) & \ldots & x_n^o(t, v^{n+1}) \end{bmatrix}.$$

Providing X is non-singular for $t_0 \le t \le T$ there is a unique solution for $c(t)$, $t_0 \le t \le T$, namely

$$c(t) = X^{-1}\, u^0 \qquad (3.18)$$

for which the controller (3.17) generates the optimal control at $v^1, \ldots, v^{n+1}$. It is well known that a matrix X is non-singular when its determinant $|X|$ is non-zero which implies in turn that no two rows or columns of X are linearly dependent. There are a number of possible reasons for rows or columns of X to be linearly dependent and for many of these the difficulty can be overcome. If, for example, the system dimension has been overstated and is in fact (n-1) then two columns will be linearly dependent at all values of time. If the dependence is recognized the number of states fed to the controller and the number of controller parameters can be reduced accordingly. It sometimes occurs that two or more rows of X become linearly dependent for all values of time past a certain point. It is probable in this case that the optimal controls corresponding to the rows involved are also linearly dependent or equal. Therefore a smaller number of independent controller parameters is required and again the dimensionality of X may be reduced until a non-singular matrix results.

It is obviously not essential to the method that all of the system state variables be fed to the controller and it is obviously not essential that the controller structure be restricted to linear feedback. It is, however, essential that the controller structure be appropriate for the given system. Fortunately in this regard there are many control problems where the structure is suggested by physical considerations.

The choice of suitable values v^i at which optimality is to be attained is important to the success of the method. Generally a minimax criterion such as minimization of the maximum deviation from optimality is appropriate. However, it is usually not worthwhile to solve this minimax problem accurately because of the computation involved. The multi-point optimality method is primarily of value in the cases where a good set of controller parameters can be found with a minimum of computation.

3.5 Speed Control of a Rotary Shear

A rotary shear is basically a roller with a cutting blade with the function of cutting a continuous sheet of material into specified lengths. When a change from one length to another is required it is desirable to change the speed of rotation quickly to avoid wastage of material and smoothly to avoid excessive stress of the material. It is assumed in this example that the initial speed has been normalized to unity and the final speed to zero. This problem has been presented by DeRusso, Roy, and Close, in [13].

The system equations are:

$$\dot{x}_1 = x_2, \quad .5 \leq x_1(0) \leq .75,$$
$$\dot{x}_2 = -\,.25\, x_1 - .2\, x_2 + 10u, \quad |u(t)| \leq .2, \quad .0 \leq x_2(0) \leq .2,$$
$$y_1 = 2x_1,$$
$$y_2 = x_2, \quad |y_2(t)| \leq .2,$$

where u is a scalar, x_1, x_2, are the state variables, y_1 is the cutting roller speed and y_2 is the drive motor torque. The constraints on the magnitudes of $u(t)$ and $y_2(t)$ are necessary to avoid saturation of the system. The desired trajectory for $y_1(t)$ is

$$y_1^d(t) = \frac{1}{2}\left(1 + \cos\frac{\pi t}{10}\right)$$

and is shown in several of the figures which follow. It is convenient to define a two-dimensional vector v to represent the unknown initial conditions:

$$v_1 = x_1(0),\ .5 \leq v_1 \leq .75\ ,$$

$$v_2 = x_2(0),\ 0. \leq v_2 \leq .2\ \ .$$

It is assumed that a linear state-feedback controller is desirable.

Quadratic performance index: Since the system equations are linear and the specifications are stated in general terms it is reasonable to first consider a quadratic performance index. The index proposed in [13] is

$$J = \frac{1}{2}\int_0^{10} [.068(y_1(t) - y_1^d(t))^2 + .021(y_2(t) - y_2^d(t))^2 + u^2(t)]dt$$

where $y_2^d(t) = \frac{1}{2}\,\dot{y}_1^d(t)$. With this index the optimal controller of form

$$u(t) = c_0(t) + c_1(t)\ x_1(t) + c_2(t)\ x_2(t)$$

generates the optimal control at all values of v. It can be derived by standard methods [10] and is given by

$$u^0(t) = 10[g_2(t) - k_2(t)\ x_1(t) - k_4(t)\ x_2(t)]$$

where

$$\dot{k}_1 - \frac{k_2}{2} - 100\, k_2^2 + .272 = 0, \qquad k_1(10) = 0,$$

$$\dot{k}_2 + k_1 - \frac{k_2}{5} - \frac{k_4}{4} - 100\, k_2 k_4 = 0, \qquad k_2(10) = 0,$$

$$\dot{k}_4 + 2k_2 - .4k_4 - 100k_4^2 + .021 = 0, \qquad k_4(10) = 0,$$

$$\dot{g}_1 - \frac{g_2}{4} - 10g_2k_2 + .068(1 + \cos \frac{\pi t}{10}) = 0, \qquad g_1(10) = 0,$$

$$\dot{g}_2 + g_1 - \frac{g_2}{5} - 100\, g_2 k_4 - \frac{.021\pi}{40} \sin \frac{\pi t}{10} = 0, \qquad g_2(10) = 0.$$

The response with this control, c^{ideal}, is given for two extreme initial conditions in Figure 16. It is seen that the trajectory $y_1(t)$ follows $y_1^d(t)$ closely and smoothly but unfortunately the constraint on $y_2(t)$ is violated by this control at some values of v. Ignoring this fact for the moment it is of interest to consider a simpler controller of form

$$u(t) = c_0(t) + c_1(t)\, x_1(t)$$

in comparison with the complete state-feedback controller given by c^{ideal}. A suitable performance-sensitivity is $S([c(t)],v) = J([c(t)],v) - J_{ideal}(v)$ where c(t) is now two-dimensional and $J_{ideal}(v)$ is the value of J associated with c^{ideal}. By an extension of Claim 2.7.3 to the case of time-varying linear systems it can be shown that $J([c(t)],v)$ and $J_{ideal}(v)$ can be written in the form

$$J([c(t)],v) = \langle v, L_1\, v\rangle$$

$$J_{ideal}(v) = \langle v, L_2\, v\rangle$$

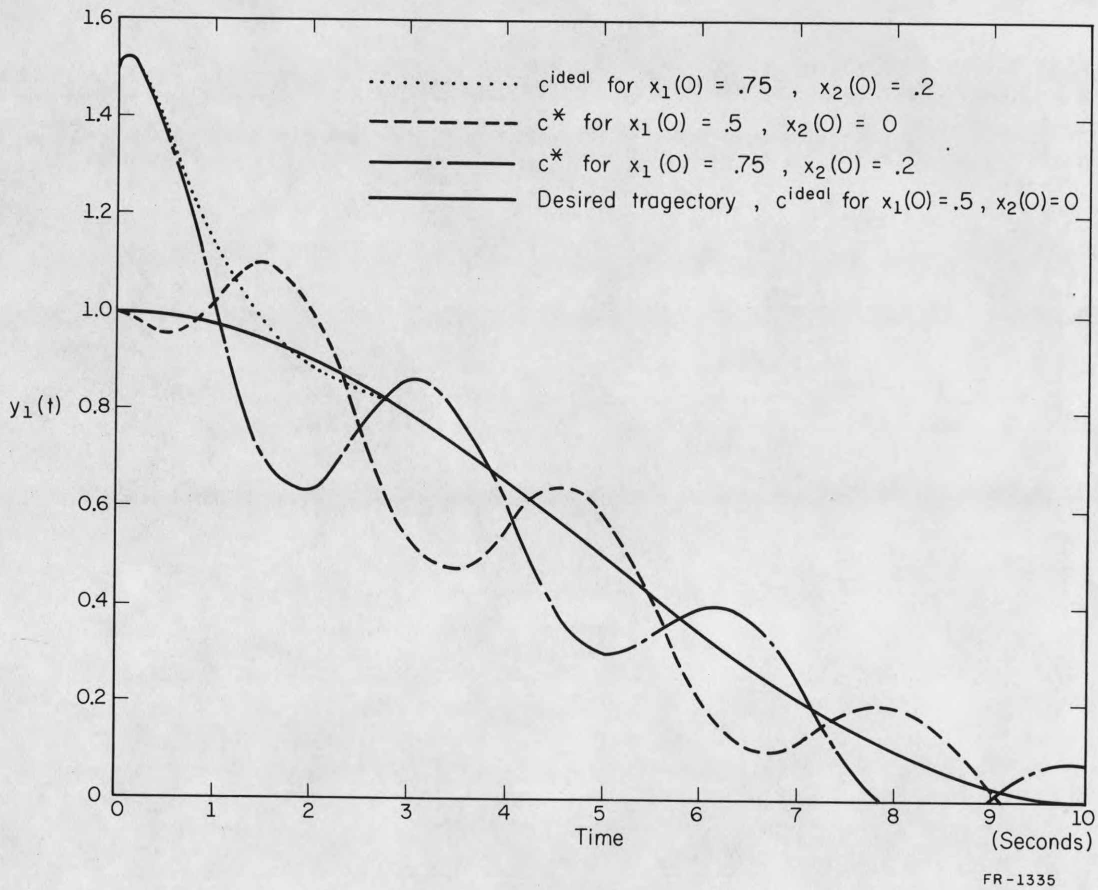

Figure 16. Comparison of controls for the rotary shear example with a quadratic index.

where L_1 and L_2 are positive definite matrices. Since $J([c(t)],v) \geq J_{ideal}(v)$ at all values of v it follows that (L_1-L_2) is a positive semi-definite matrix and $S([c(t)],v)$ is convex with respect to v. By an obvious extension of Lemma 2.4.1 it follows that $[c^*(t)]$ can be obtained with a single minimization of

$$\max\{S([c(t)],v^1),\ S([c(t)],v^2),\ S([c(t)],v^3),\ S([c(t)],v^4)\},$$

where $v^1 = (.5, 0.)$, $v^2 = (.5, .2)$, $v^3 = (.75, 0.)$, $v^4 = (.75, .2)$, are the vertices of V.

The method of steepest descent of the Hamiltonian as described in Section (3.2) was used to solve this minimization problem. When $[c(t)] = [c^*(t)]$ the minimax value of S is attained at v^2 and v^3 only, i.e. these are the values of v at which the greatest deviation from optimality of J occurs when $[c(t)] = [c^*(t)]$. The response of the system with $[c^*(t)]$ is shown in Figure 16. It is clear from the poor response obtained that complete state feedback is essential for satisfactory control of this system.

Non-quadratic performance index: In order to generate a controller which satisfies the constraint on $y_2(t)$ the following performance index is proposed in [13]:

$$J=\tfrac{1}{2}\int_0^{10}[.068(y_1(t)-y_1^d(t))^2+.021(y_2(t)-y_2^d(t))^2+.021\left(\frac{y_2(t)}{.2}\right)^{16}+u^2(t)]dt.$$

The addition of the term $\left(\frac{y_2(t)}{.2}\right)^{16}$ greatly complicates the problem. For instance, to obtain the optimal control in the form u(t) at any given value of v it is necessary to solve the two-point boundary value problem associated with that value of v. Since the index is so far from quadratic the method of steepest descent of the Hamiltonian is not capable of solving for an optimal control, and it is necessary to consider a method based on second variations. In one such method [14] a second-order Taylor expansion of the performance index is made about a nominal trajectory. Then by linearizing any plant non-linearities about the nominal trajectory the approximate index may be minimized by using the theory for linear systems with quadratic indices. By making a number of iterations the optimal control may be generated for any given value of v. The optimal controls generated in this way have the desired form

$$u^0(t) = c_0(t) + c_1(t)\, x_1(t) + c_2(t)\, x_2(t)$$

and are optimal for linear variations of the plant about the nominal-optimal trajectory to the extent that the performance index is approximately quadratic. The approach to the controller design taken in [13] is to adopt such a control c^{nom}. The nominal value of v chosen was $v^{nom} = (.75, .2)$. The response of the system with c^{nom} at $v = (.75, .2)$ is optimal and is shown in Figure 18. However, it is not unexpected to find that with c^{nom} the system performs poorly at other values of v as demonstrated by Figure 17 which gives the response at $v = (.5, 0.)$.

Since c(t) is three-dimensional and a second-variations method is needed to compute an optimal control a considerable amount of computation is required to obtain a minimax control.

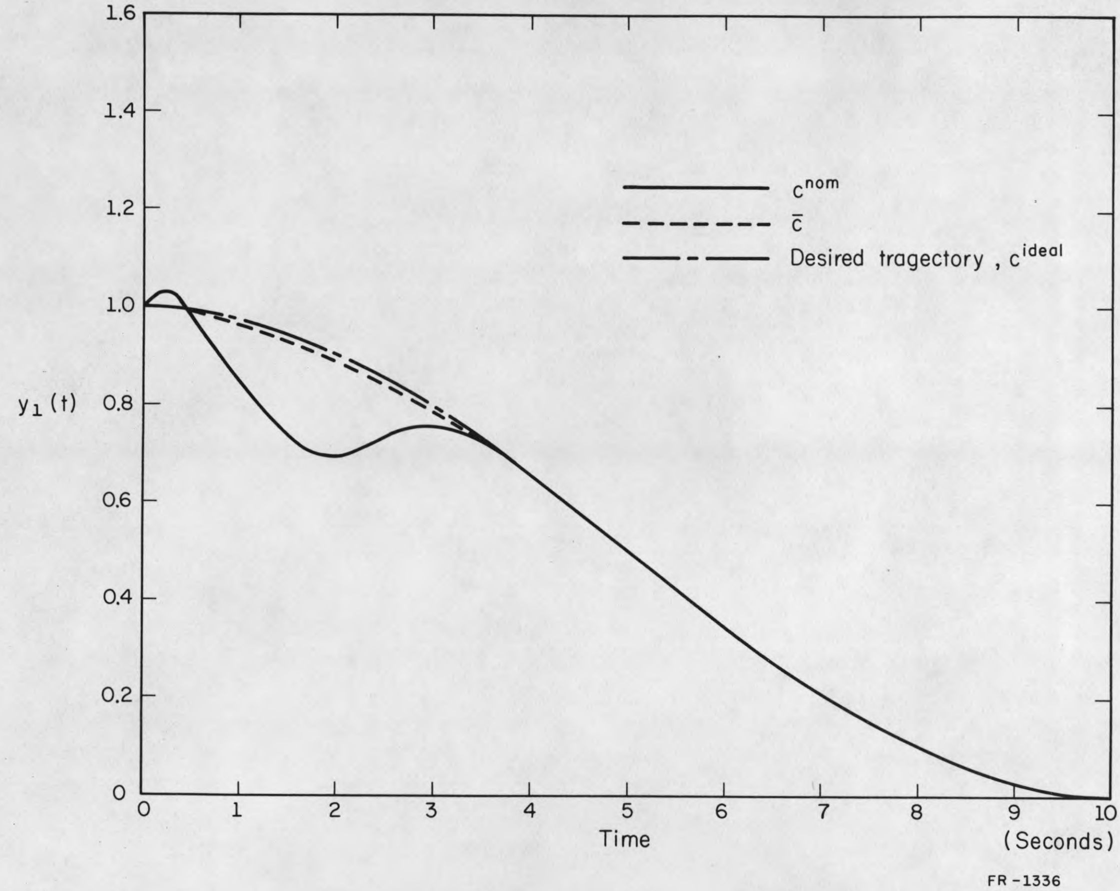

Figure 17. Comparison of controls for the rotary shear example with a non-quadratic index at $x_1(0) = 0.5$, $x_2(0) = 0.0$.

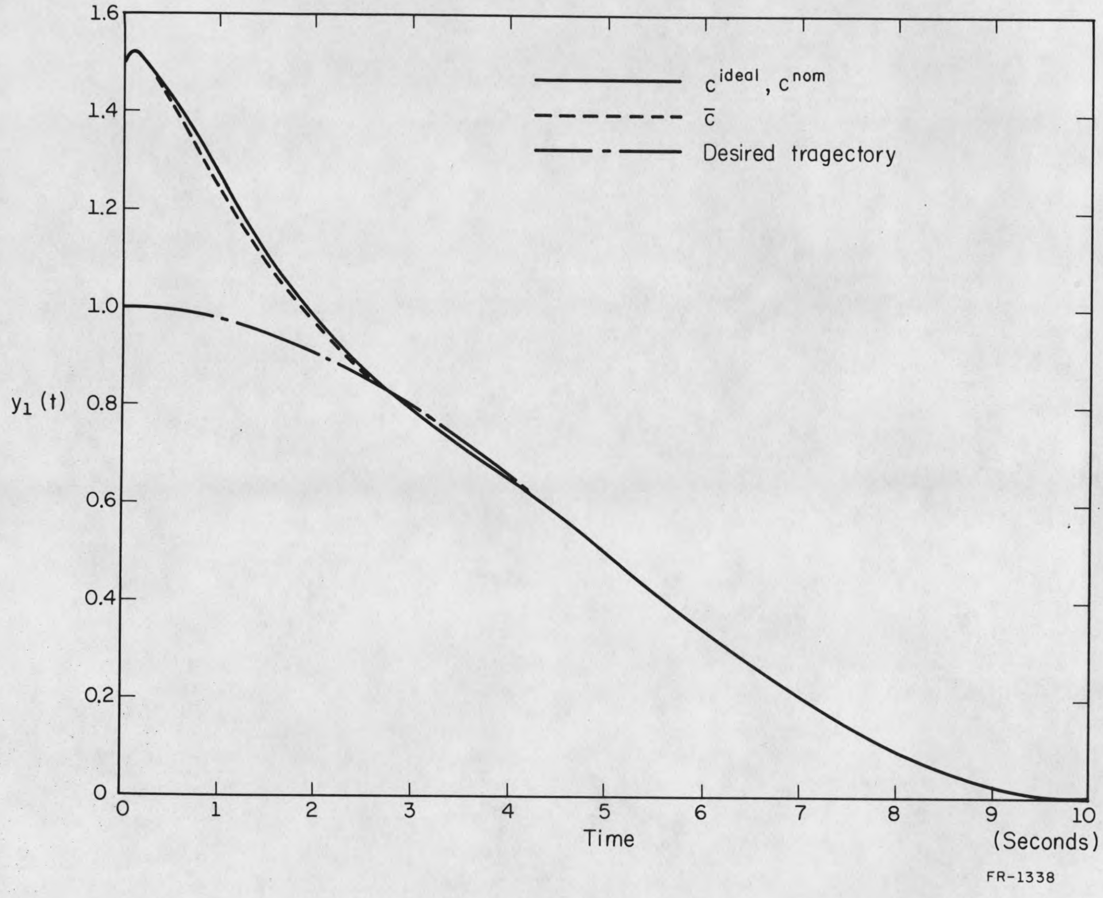

Figure 18. Comparison of controls for the rotary shear example with a non-quadratic index at $x_1(0) = 0.75$, $x_2(0) = 0.2$.

Therefore, the method of multi-point optimality described in Section 3.3 becomes attractive. With the assumed controller structure, optimality at three distinct values of v is possible. Experience has shown that the choice of three well-separated values of v leads to an X matrix with a determinant which is positive at t = 0 and which tends towards zero as time progresses. An inspection of the system responses shows that this is simply because the optimal trajectories are virtually identical from $t \approx 3.5$ to $t = 10$ regardless of the value of v. Therefore, the approach adopted was to use the control parameters obtained from inversion of X for $0 \leq t \leq 3$, to use the average value of the three sets of nominal-optimal control parameters for $3.5 \leq t \leq 10$, and to convert gradually from the former method to the latter in the range $3 \leq t \leq 3.5$.

The choice of the three values of v is not a critical one for this system since any three widely separated values of $v \in V$ yield a good controller. With the values (.5, .1), (.725, 0.), (.725, .2), the resulting control parameters $\bar{c}(t) = (\alpha(t), \beta(t), \gamma(t))$ are as shown in Figures 19 and 20. These time-varying parameters are no more difficult to realize than those required for the nominal-optimal control c^{nom} but the performance with $\bar{c}(t)$ is clearly superior as demonstrated by Table 2 and Figures 17 and 18. In fact the controller represented by $\bar{c}(t)$ generates controls which satisfy all the constraints and are so nearly

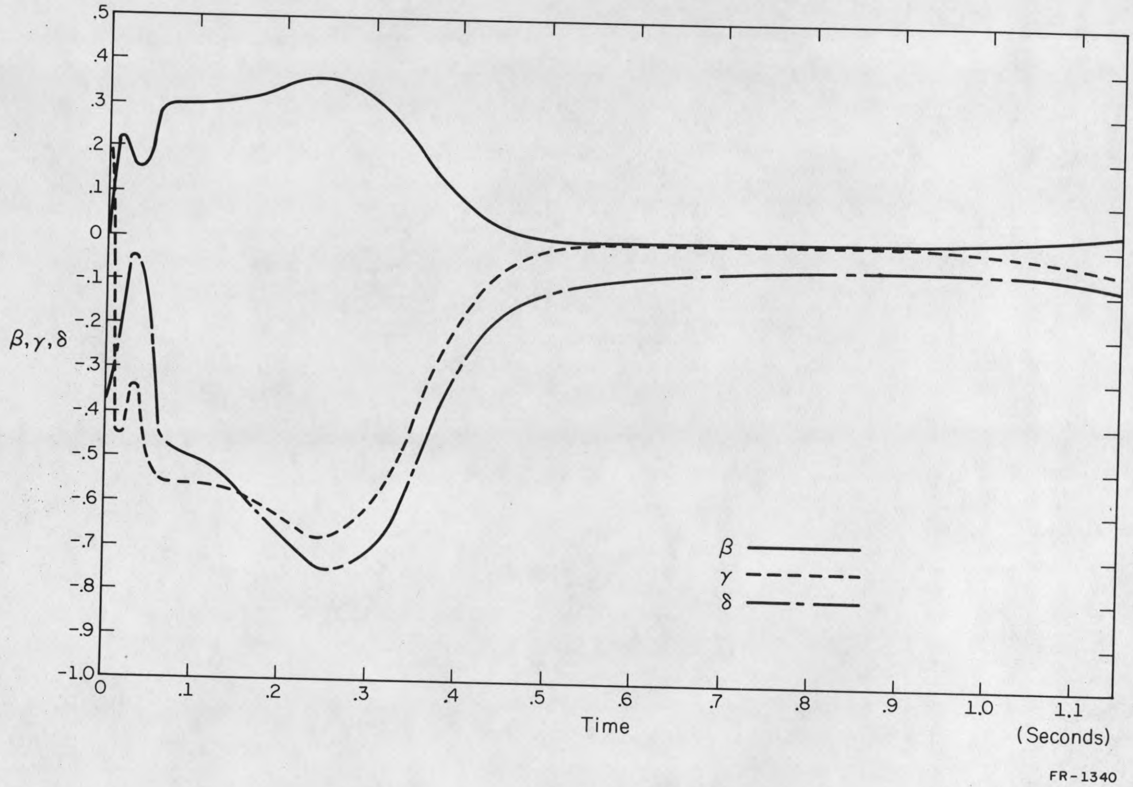

Figure 19. Time functions required to realize $\overline{c}(t)$.

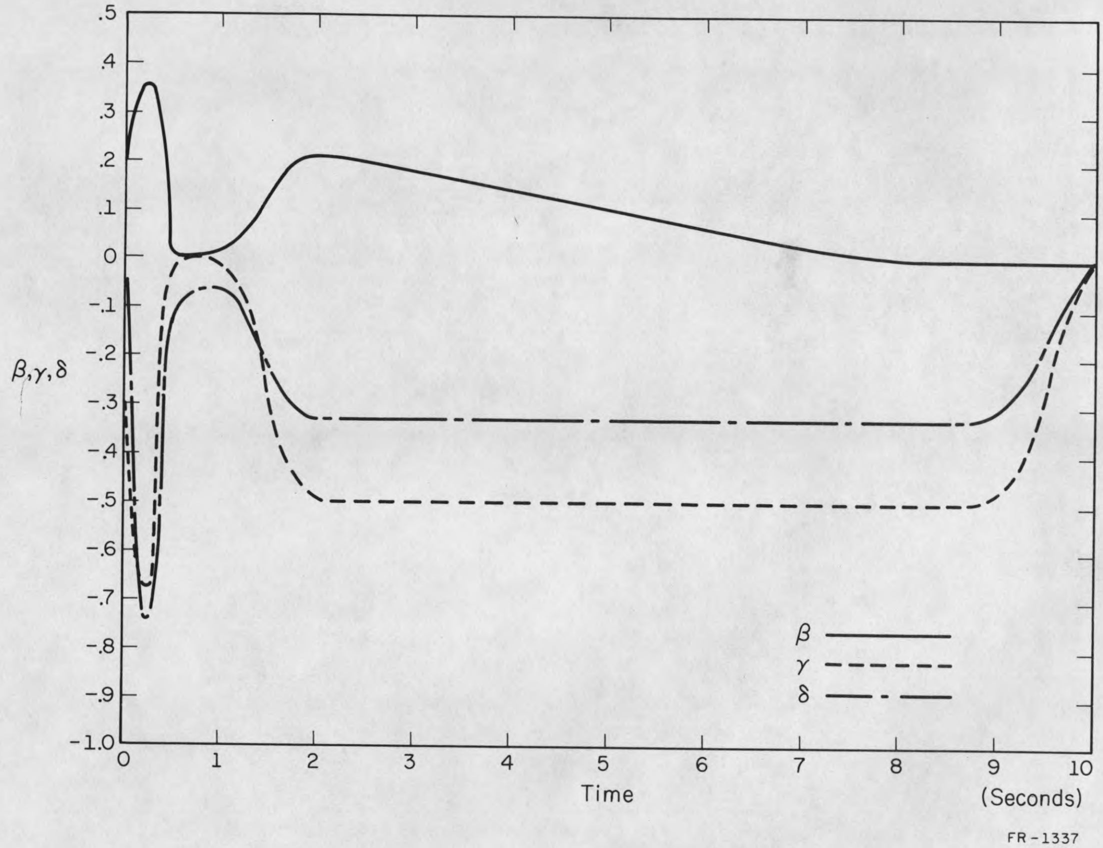

Figure 20. Time functions required to realize $\overline{c}(t)$.

optimal at all values of $v \in V$ that any further computation is not justified.

TABLE 2

Values of J

v	J optimal	$c^{nom}(t)$	c(t)
(.5, 0.)	.00017	.00586	.00028
(.5, .1)	.00024	.00383	.00029
(.5, .2)	.00080	.00371	.00082
(.625, 0.)	.00158	.00460	.00173
(.625, .1)	.00228	.00323	.00245
(.625, .2)	.00344	.00372	.00366
(.75, 0.)	.00695	.00994	.00725
(.75, .1)	.00838	.00925	.00869
(.75, .2)	.01039	.01039	.01075

4. CONCLUSIONS

4.1 Summary

The design of controllers for uncertain dynamic systems has been considered within the framework of minimax methods. Emphasis has been placed on the problem of realization of the controller. With the introduction of performance-sensitivity, minimax methods have been shown to allow the range of system properties which is typically desired. An algorithm capable of solving algebraic minimax problems regardless of the presence of a saddle-point has been developed and applied. The rapid convergence and usefulness of the algorithm in controller design has been illustrated by several examples. In fact, experience shows that convergence in one or two iterations is normal when there is only one variable system parameter. Properties special to minimax performance-sensitivity control have been developed. The simplifications which occur in the minimax problem with convexity assumptions, and also when the system is linear and the performance index quadratic, have been discussed. The minimax design of time-invariant controllers has been shown appropriate for a large class of systems with uncertain parameters.

The methods developed for time-invariant controller design have been extended to time-varying controllers but the problem here is more difficult and the case for minimax control is weakened in practice (but not theory) by the severe computational requirements. The difficulties arise partly because a controller can usually be specified only in terms of several free parameters whereas in contrast an optimal control as a function of time defines only one parameter. Thus computation of the controller parameters is fundamentally more difficult than computation of an optimal control which in itself is usually a difficult problem. However, there are practical systems for which time-varying minimax controllers can be found by the methods presented in this thesis, and as the methods for solving two-point boundary value problems are improved and computational facilities become more powerful the areas of practical application are bound to increase. Two alternate approaches to the design of time-varying controllers have been presented. The first is to simply redefine the problem in terms of time-invariant parameters as discussed briefly in Section 3.1, and the second is to use the method of multi-point optimality presented in Section 3.3. In the latter method advantage is taken of the dimensionality which causes the computational difficulties discussed above.

Several examples have been presented. They illustrate the utility and practicality of the methods developed for design of both time-invariant and time-varying controllers.

4.2 Problems for Further Study

Although this thesis has restricted consideration to system parameters which are unknown but time-invariant (or which vary slowly in comparison with the speed of system operation) it is evident that the philosophy contained herein and some of the methods are applicable to systems with time-varying parameters. The case of a time-invariant controller and time-varying system parameters is of practical interest, for example, in the control of an airplane or radar antenna subject to wind gusts of specified properties. Also the case of a time-varying controller and time-varying system parameters is of interest, for example, in missile intercept problems. There may be problems of this type which can be treated by extensions of the methods in this thesis.

While it is believed that minimax design methods for time-invariant controllers have attained a satisfactory state the chapter on time-varying controllers raises more questions than it answers. There is room for much improvement in the methods of computation of minimax time-varying controllers. It also appears likely that the broad area suggested by the section on multi-point optimality will be a fruitful one for further research.

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13. ABSTRACT

When an uncertain dynamic system is to be controlled in an optimal manner, that is when the controller is required to minimize a performance index, the controller design requires a compromise between controller complexity and system performance. At one extreme is the optimal-adaptive controller which is difficult to realize but yields ideal performance, and at the other is any overly simplified controller which yields unacceptable performance. The reasonable controller structures for a given system can often be determined by the designer in terms of a number of free parameters. Then for each structure it is desired to find those parameters which yield the "best" system performance. This thesis develops minimax methods for determining such controller parameters.

The concept of performance-sensitivity is introduced to meet the usual criticism of minimax or "worst-case" design, that it is too pessimistic in concentrating all attention on the worst parameters. Properties of minimax control with a performance-sensitivity as the index are developed. It is shown that the usually desired range of system properties can be achieved by minimaximizing either the system performance index or a performance-sensitivity.

A new algorithm for solving algebraic minimax problems, regardless of the presence of a saddle point, is presented and proved to converge. The rate of convergence, and simplifications which occur when the system is linear or when

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the index has convexity properties, are discussed. The algorithm is extended to the case of time-varying minimizing parameters, and methods and problems of computation are discussed.

A method of obtaining multi-point optimality with time-varying controllers is presented. This method is of particular value when the computational difficulties of finding minimax time-varying parameters are prohibitive.

A general approach to the design of controllers for uncertain systems is presented. Several examples illustrate the utility and practicality of the methods developed.