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LINEAR CONTROL LAWS FOR SINGULAR LINEAR SYSTEMS

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ABSTRACT

For a class of quadratic performance indices the optimal control law is a combination of maximum effort (bang-bang) and singular. The singular control law is linear and it is optimal in a hyperplane in the n-dimensional state-space. For practical purposes it is desirable to restrict the class of admissible control laws to be linear. This investigation presents a method of finding a linear control law which is optimal in the sense that it is the singular control law in the singular surface and the best possible linear law elsewhere. Classical calculus of variations and the more sophisticated maximum principle of Pontryagin are the mathematical tools used; the former provides a simple and straight forward method for obtaining the singular solutions while the latter is used to extend the linear singular law to the entire state-space.

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1. INTRODUCTION

1.1 General

In any practical optimization problem the control signal is constrained in some way. In mathematical terminology, the set U of admissible control signals u(t) is defined to be a certain set of functions of the real variable t. In most problems U is defined as the set of all piecewise continuous functions u(t) such that the absolute value $|u(t)|$ is uniformly bounded by a fixed number M. Another way of constraining the control signal is to include it in the performance index. This is done in general by adding to the performance index a term which is the integral of some non-negative definite form in u - for instance λu^2. For obvious physical reasons the performance index is then said to include the "cost" of control. Consider now the following optimization problem. Given a linear plant characterized by the equation (in state-space notation)

$$\dot{\underline{x}} = \underline{A}(t)\ \underline{x} + \underline{B}(t)\ \underline{u} \quad ,$$

where $\underline{x}$ is a n x 1 state vector, $\underline{A}$, $\underline{B}$ are n x n matrices and $\underline{u}$ is the n x 1 control vector, it is desired to find $\underline{u}(t)$ which minimizes the performance index

$$\int_0^T \left(\underline{x}^T(t)\ \underline{Q}\ \underline{x}(t) + \underline{u}^T(t)\ \underline{R}\ \underline{u}(t) \right) dt \quad ,$$

where $\underline{Q}$ and $\underline{R}$ are non-negative definite matrices of constants. It is a well known result of optimal control theory[1,2] that the optimal control law

$\underline{u}(t)$ is a linear combination of the state variable of the system, i.e.

$$\underline{u}(t) = \underline{C}(t)\ \underline{x}(t)\ .$$

Consider now the singular problem, i.e., $\underline{R} \equiv \underline{0}$. If the performance index above does not include the cost of control, it has been shown[3,4] that $\underline{u}(t)$ is linear when $\underline{x}(t)$ lies in a certain hyperplane Γ in the n-dimensional state space. However, outside of Γ the control is made by impulses (assuming no bounds in $|\underline{u}|$). If $\underline{u}$ is constrained in magnitude, the familiar bang-bang control is optimal outside of Γ_r (portion of Γ_r where the linear control law satisfies the constraint on the magnitude of $\underline{u}$). Therefore, in the case $\underline{R} \equiv \underline{0}$, the optimal control is not linear everywhere but only in Γ_r. For practical reasons one may desire to have linear control everywhere. Therefore the following question is posed: which (if any) of the control laws given by $\underline{u} = \underline{C}\ \underline{x}$ is the best approximation for the bang-bang control (i.e., the control which minimizes the performance index) outside of Γ_r? In the next paragraph a concise formulation of the problem is presented; only single input-single output linear time invariant plants are considered.

1.2 Formulation of the Problem

Consider a linear, time invariant plant described by

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{b}\ u(t)\ , \qquad (1)$$

where $\underline{A}$ is an n x n matrix of constants, $\underline{x}$ is the n x 1 state vector, $\underline{b}$ is a n x 1 constant matrix and u(t) is the control signal. Assuming complete controllability for system (1), there is no loss of generality[5] in taking

$\underline{A}$, $\underline{b}$, $\underline{x}$ to have the forms

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} ,$$

$$\underline{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} , \qquad (2)$$

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ \overset{(n-1)}{x} \end{bmatrix} .$$

Consider now the functional

$$S(u) = \int_0^\infty \left[\underline{x}^T(t)\ \underline{Q}\ \underline{x}(t)\right] dt , \qquad (3)$$

where $\underline{Q}$ is non-negative definite, n x n matrix of constants. The set U of admissible controls is defined as being the set of all signals u(t) such that

$$|u(t)| \leq k_A \qquad (4)$$

and

$$u(t) = \underline{c}^T \, \underline{x}(t) \ , \tag{5}$$

where $\underline{x}(t)$ is the solution of system (1) and $\underline{c}$ is a n x 1 constant vector. Since (4) is to be valid at all times, it is obvious (in view of (5)) that the initial conditions must be bounded. Therefore let the set of admissible initial conditions be X, X being the set of all $\underline{x}_o$ such that

$$|| \, \underline{x}_o \, || \leq k_\beta \ , \tag{6}$$

where $||\underline{x}_o||$ = euclidean norm of $\underline{x}_o$. The set U is then the set of all controls given by (5), which satisfy (4) at all times and for any initial condition satisfying (6). The optimization problem is to find u in U which minimizes the functional (3) subject to the restrictions

$$\underline{x}(o) = \underline{x}_o \tag{7a}$$

and

$$\underline{x} \, (\infty) = \underline{o} \tag{7b}$$

Moreover, u must be the singular optimal control on the singular surface. Denoting the integrand in (3) as $F(x, \dot{x}, \ldots \overset{(n-j)}{x})$, two cases will be treated separately: (a) j=1 and (b) j>1. Case (a) is called the case of singularity of first order and is treated in the next section. Case (b) is called the case of singularity of order j and will be treated in the succeeding section.

2. SYSTEMS OF SINGULARITY OF FIRST ORDER

2.1 Singular Linear Control Law

In this section performance indices, in which the order of the highest derivative of x(t) in the integrand is n-1, are considered. First the so-called singular trajectories are obtained in a simple and straight forward manner;[6] then conditions which the desired linear control law must satisfy in order to be the optimal control law on the singular surface are derived. To do so the integrand in the performance index (3) is written as

$$F\left(x,\dot{x},\ddot{x},\ldots,\overset{(n-1)}{x}\right) = \sum_{i,j=1}^{n} q_{i,j} \overset{(i-1)}{x} \overset{(j-1)}{x} \quad . \tag{8}$$

From the calculus of variations a fundamental necessary condition for an extremum is that the integrand (8) satisfies the Euler-Poisson equation[7]

$$\sum_{k=0}^{n-1} (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial F}{\partial \overset{(k)}{x}}\right) = 0 \ . \tag{9}$$

From (8)

$$\frac{\partial F}{\partial \overset{(k)}{x}} = 2 \sum_{i=1}^{n} q_{k+1,i} \overset{(i-1)}{x} \tag{10}$$

therefore the Euler-Poisson equation is

$$2 \sum_{k=o}^{n-1} \sum_{i=o}^{n-1} (-1)^k q_{k+1,\, i+1} \overset{(i+k)}{x} = 0. \tag{11}$$

It is not difficult to see that the above equation is even in the derivatives of x, i.e., it can be written as

$$2 \sum_{p=0}^{n-1} K_p \overset{(2p)}{x} = 0 \ , \qquad (12)$$

where

$$K_p = \sum_{\substack{k,\ell \\ k+\ell=2p}} q_{k+1,\ \ell+1} \ ; \ p = 0,1 \ldots \ n-1 \ . \qquad (13)$$

Taking the characteristic equation associated with (12) we obtain

$$P(s^2) = 2 \sum_{p=o}^{n-1} K_p \, s^{2p} = 0 \qquad (14)$$

$P(s^2)$ is even in s; therefore it can be factored into a unique product of a Hurwitz and an anti-Hurwitz polynomial; since the final state of the system is the stable origin (as given by (7)), the unbounded solutions of (12) must be discarded. Hence, if one writes

$$P(s^2) = H(s)\ H(-s) \ , \qquad (15)$$

the differential equation which the optimal trajectory must satisfy is

$$\sum_{i=0}^{n-1} h_i \overset{(i)}{x} = 0 \ , \quad h_{n-1} = 1 \ . \qquad (16)$$

The trajectories given by (16) are the so-called singular solutions.[3,4]

Since (16) is an equation of order n-1 and the system equation is of order n, the Euler-Poisson equation gives the optimal trajectories only for some initial conditions. These initial conditions form a hyperplane Γ_1 in the n-dimensional state space. Obviously Γ_1 is the set of all $\underline{x}_o$ such that

$$\sum_{i=o}^{n-1} h_i \overset{(i)}{x}_o = 0 . \tag{17}$$

In what follows we derive conditions that the desired linear control law $u(t) = \underline{c}^T \underline{x}(t)$ must satisfy in order that Equation (16) be satisfied by $\underline{x}(t)$ when $\underline{x}_o$ is in Γ_1. To do so the system Equation (1) is written as follows.

$$\sum_{i=o}^{n-1} (a_i - c_i) \overset{(i)}{x} + \overset{(n)}{x} = 0 . \tag{18}$$

The characteristic equation associated with (18) is

$$H_T(s) = \sum_{i=o}^{n-1} (a_i - c_i) s^i + s^n = 0 . \tag{19}$$

If the solution of (18) is to be identical to the solution of (16) when $\underline{x}_o$ is in Γ_1, the system (18) must contain all modes of (16). Hence, we must equate

$$H_T(s) = Z(s) H(s) , \tag{20}$$

where $Z(s) = (s+z_o)$. Equating coefficients of the same power in s, we obtain the following relationships:

$$z_o = \frac{a_o - c_o}{h_o} \tag{21}$$

and

$$c_i = a_i - z_o h_i - h_{i-1}, \; i=1,\ldots n-1 \; . \qquad (22)$$

Equation (22) expresses every component of $\underline{c}$ (but the first) as a function of c_o. Hence, by imposing the condition that the linear control law $u = \underline{c}^T\underline{x}$ be the optimal singular control on the singular surface we have reduced the n-variable problem to a single variable one. Note also that if Equations (22) are satisfied, the control $u = \underline{c}^T\underline{x}$ is independent of c_o when $\underline{x}$ is on the singular surface Γ_1 (see Appendix A). Another immediate result is the condition for stability. In view of (20) and (21) the necessary and sufficient condition for assuring a stable system is simply that

$$a_o - c_o > 0 \quad . \qquad (23)$$

Note that under the stated end point condition (Equation (7b)), (23) is a condition for existence of a solution.

2.2 Extended Linear Control Law

In order to extend the linear control to the entire state space one must find a vector $\underline{c}^*$ which has the following properties: (a) Its components c_i^*, $i=1,\ldots n-1$ must satisfy (22); (b) the control $u^*(t) = (\underline{c}^*)^T \underline{x}(t)$ must be admissible, i.e., u^* must be in U; (c) u^* must minimize the performance index (3), i.e.,

$$\min_{u \in U} S(u) = S(u^*) \; .$$

If Pontryagin's formulation is used, the problem is to find $\underline{c}$ such that the Hamiltonian

$$M\,(\underline{p},\ \underline{x},\ \underline{c}) = p^T\,\dot{\underline{x}} - \underline{x}^T\,\underline{Q}\,\underline{x} \tag{24}$$

is maximized (in $\underline{c}$). The maximization is subject to the constraint

$$u^2 = \underline{x}^T\ \underline{c}\,\underline{c}^T\ \underline{x} \leq k_A^2\ , \tag{25}$$

for every $\underline{x}_o$ such that (6) holds. Moreover, Equations (22) must be satisfied. The auxiliary variable $\underline{p}$ in (24) is defined be its components

$$(\dot{\underline{p}})_i = -\frac{\partial M}{\partial (\overset{i-1}{x})}\ ,\ i=1, \ldots n. \tag{26}$$

Substitution of (1) into (24) gives

$$M(\underline{p},\underline{x},\underline{c}) = p_n\,\underline{c}^T\ \underline{x} + \underline{p}^T\ \underline{A}\ \underline{x} - \underline{x}^T\ \underline{Q}\ \underline{x}\ . \tag{27}$$

Note that if we set $\frac{\partial M}{\partial c_i} = 0$, the condition $p_n(t) = 0$ is obtained. This condition leads to the singular solution already obtained. Since the only term in (27) dependent on $\underline{c}$ is

$$N(\underline{p},\ \underline{x},\ \underline{c}) = p_n\ \underline{c}^T\ \underline{x}\ \ , \tag{28}$$

we have to maximize $N(\underline{p},\underline{x},\underline{c})$ restricted to (25). Furthermore, since (22) is to be satisfied, the only variable in $\underline{c}$ is c_o. Using lagrange multipliers, we set

$$\frac{\partial}{\partial c_o}\left[N(\underline{p},\ \underline{x},\ \underline{c}) + \rho\ \underline{x}^T \underline{c}\ \underline{c}^T\ \underline{x}\right] = 0 \ . \tag{29}$$

Performing the differentiation indicated in (29), we obtain

$$p_n \frac{\partial \underline{c}^T}{\partial c_o}\ \underline{x} + \rho\ \underline{x}^T \left[\frac{\partial \underline{c}}{\partial c_o}\ \underline{c}^T + \underline{c}\ \frac{\partial \underline{c}^T}{\partial c_o}\right] \underline{x} = 0 \ , \tag{30}$$

or

$$p_n \frac{\partial \underline{c}^T}{\partial c_o}\ \underline{x} + 2\ \rho\ \underline{x}^T\ \underline{c}\ \frac{\partial \underline{c}^T}{\partial c_o}\ \underline{x} = 0 \ . \tag{31}$$

As is shown in Appendix B, the function $p_n(t)$ is given by

$$p_n(t) = \sqrt{\frac{q_{11} q_{nn}}{c_o - a_o}} \sum_{i=o}^{n-1} h_i\ x^{(i)}(t) \ . \tag{32}$$

Note that p_n is the n^{th} component of $\underline{p}$; $\underline{p}$ is the solution of

$$\dot{\underline{p}} = -\ \underline{F}^T\ \underline{p} + 2\ \underline{Q}\ \underline{x} \tag{33}$$

(see Appendix B), where

$$\underline{F} = \underline{A} + \underline{b}\ \underline{c}^T \ . \tag{34}$$

Letting

$$\underline{h} = \begin{bmatrix} h_o \\ \vdots \\ h_{n-1} \end{bmatrix} ,$$

we have

$$p_n(t) = \sqrt{\frac{q_{11}q_{nn}}{c_o - a_o}} \quad \underline{x}^T \underline{h} \ . \tag{35}$$

Substituting (35) into (31), we obtain

$$\underline{x}^T \left[\sqrt{\frac{q_{11}q_{nn}}{c_o - a_o}} \ \underline{h} + 2\,\rho\,\underline{c} \right] \frac{\partial \underline{c}^T}{\partial c_o} \ \underline{x} = 0 \ . \tag{36}$$

Letting

$$\underline{D} = \left[\sqrt{\frac{q_{11}q_{nn}}{c_o - a_o}} \ \underline{h} + 2\,\rho\,\underline{c} \right] \left[\frac{\partial \underline{c}^T}{\partial c_o} \right] , \tag{37}$$

it is easy to see that $\underline{D}$ (a dyad) is of rank at most one and its only (possibly) non-zero eigenvalue is

$$\lambda\,(\underline{D}) = \left[\frac{\partial \underline{c}^T}{\partial c_o} \right] \left[\sqrt{\frac{q_{11}q_{nn}}{c_o - a_o}} \ \underline{h} + 2\,\rho\,\underline{c} \right] . \tag{38}$$

As (36) is to be true for any $\underline{x}(t)$, all the eigenvalues of $\underline{D}$ must be zero. Performing the differentiation indicated in (38) and setting $\lambda(\underline{D}) = 0$ we obtain

$$\sqrt{\frac{q_{11}q_{nn}}{2\,\rho}} \ \underline{h}^T\underline{h} = (a_o - c_o)\ \underline{h}^T \ \underline{c} \ . \tag{39}$$

Equation (39) relates the optimal value of c_o to the Lagrange multiplier ρ. The remaining problem is to find ρ in such a way that the constraint in

$u^2(t)$ be satisfied whenever $\underline{x}_o$ is in Γ_1. A numerical iterative procedure is unavoidable in general. However, the following considerations considerably simplify the search for the optimal ρ (or equivalently, c_o). If (22) is substituted into (28), we obtain

$$N(\underline{p}, \underline{x}, \underline{c}) = p_n \sum_{i=0}^{n-1} (a_i - h_{i-1}) \overset{(i)}{x} - p_n z_o \underline{h}^T \underline{x} . \tag{40}$$

The only term in $N(\underline{p}, \underline{x}, \underline{c})$ which depends on c_o is $p_n z_o \underline{h}^T \underline{x}$. Since by (35) and (23) p_n has the sign opposite to the sign of $\underline{h}^T\underline{x}$, in order to maximize $N(\underline{p}, \underline{x}, \underline{c})$ (in $\underline{c}$) one has to maximize z_o. That is, to make c_o as small as the constraint in $|u(t)|$ allows. This result confirms the intuitive notion that the free pole at $-\frac{a_o - c_o}{h_o}$ should be placed as far as possible from the origin. Note that Equation (39) also carries this information since a loose constraint in $|u|$ should result in a "small" value for ρ and hence a "large" value for $(a_o - c_o)$. On the other hand, since (note that $\underline{c}\,\underline{c}^T$ is of rank one)

$$\max_{||\underline{x}_o|| \leq k_B} u^2(0) = \max_{||\underline{x}_o|| \leq k_B} \underline{x}_o^T \underline{c}\,\underline{c}^T \underline{x}_o = \underline{c}^T \underline{c}\; k_B^2 , \tag{41}$$

a necessary condition for $\underline{c}$ to satisfy the constraint in $|u|$ for all $\underline{x}_o$ such that $||\underline{x}_o|| \leq k_B$ is

$$\underline{c}^T \underline{c} \leq \left(\frac{k_A}{k_B} \right)^2 . \tag{42}$$

Condition (42) is by no means sufficient in general. However, if $\underline{x}^T\underline{x}$ is a

Lyapunov function for system (1) Equation (42) is also sufficient (because in this case $||\underline{x}(t)||^2 \leq ||\underline{x}_o||^2$).

2.3 Illustrative Example

Given the system equation (Figure 1)

$$\ddot{x} = u \tag{43}$$

and the performance index

$$\int_o^\infty (x^2 + \dot{x}^2)\, dt \quad , \tag{44}$$

find a linear control law

$$u = c_o x + c_1 \dot{x} \; , \tag{45}$$

such that u minimizes the performance index (45) subject to the following restrictions

$$\underline{x}(o) = \underline{x}_o, \; ||x_o|| \leq 1,$$

$$\underline{x}(\infty) = \underline{o} \; , \; |u(t)|^2 \leq 5, \tag{46}$$

From the performance index, the Euler-Poisson equation is

$$\ddot{x} - x = 0 \; ; \tag{47}$$

the stable part is found to be

$$\dot{x} + x = 0 \; . \tag{48}$$

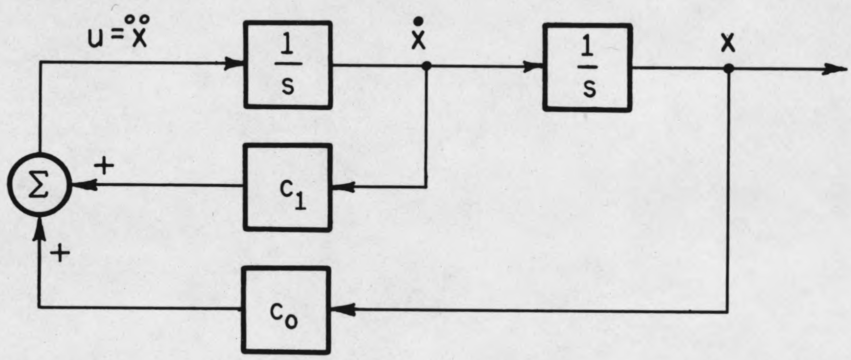

Figure 1. System for Example 2.3.

Therefore,

$$h = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad . \tag{49}$$

From (21) and (22)

$$z_o = - c_o \tag{50}$$

$$c_1 = c_o - 1 \tag{51}$$

Equation (42) gives

$$c_o^2 + (c_o - 1)^2 \leq 5 \ . \tag{52}$$

Then c_o must satisfy the inequalities

$$-1 \leq c_o \leq 2 \ . \tag{53}$$

But $c_o < o$ for stability; hence, the optimal c_o lies in the interval

$$-1 \leq c_o < o \ . \tag{54}$$

On the other hand (39) gives

$$\frac{1}{\rho} = c_o \, (-2 \, c_o + 1) \ , \tag{55}$$

and, therefore, ρ should lie in the interval

$$-\infty < \rho \leq - 1/3 \ . \tag{56}$$

Since the optimal c_o should be the smallest possible, the numerical procedure starts upon our letting $c_o = -1$ (or alternatively , $\rho = -1/3$). In this particular case since the function $V(t) = \underline{x}^T(t)\,\underline{x}(t)$ is non-increasing,

$$\frac{dV}{dt} = -\,4\,\mathring{x}^2(t) \ , \tag{57}$$

the optimal value of c_o is -1. The singular surface Γ_1 for this example is shown in Figure 2. Also in this figure is shown the set X of admissible initial conditions plus the initial condition $\underline{x}_o^T = \frac{1}{5}(-1,-2)$; for this particular initial condition $u^2(o) = 5$. The reader may wonder why Equation (39) was derived since it was stated that c_o should be chosen as the smallest possible value satisfying the constraint in $|u(t)|$. Indeed (39) is not necessary for purposes of calculation in the first order case; however an equation similar to (39) will be necessary in the higher order case, as will be shown in the next section.

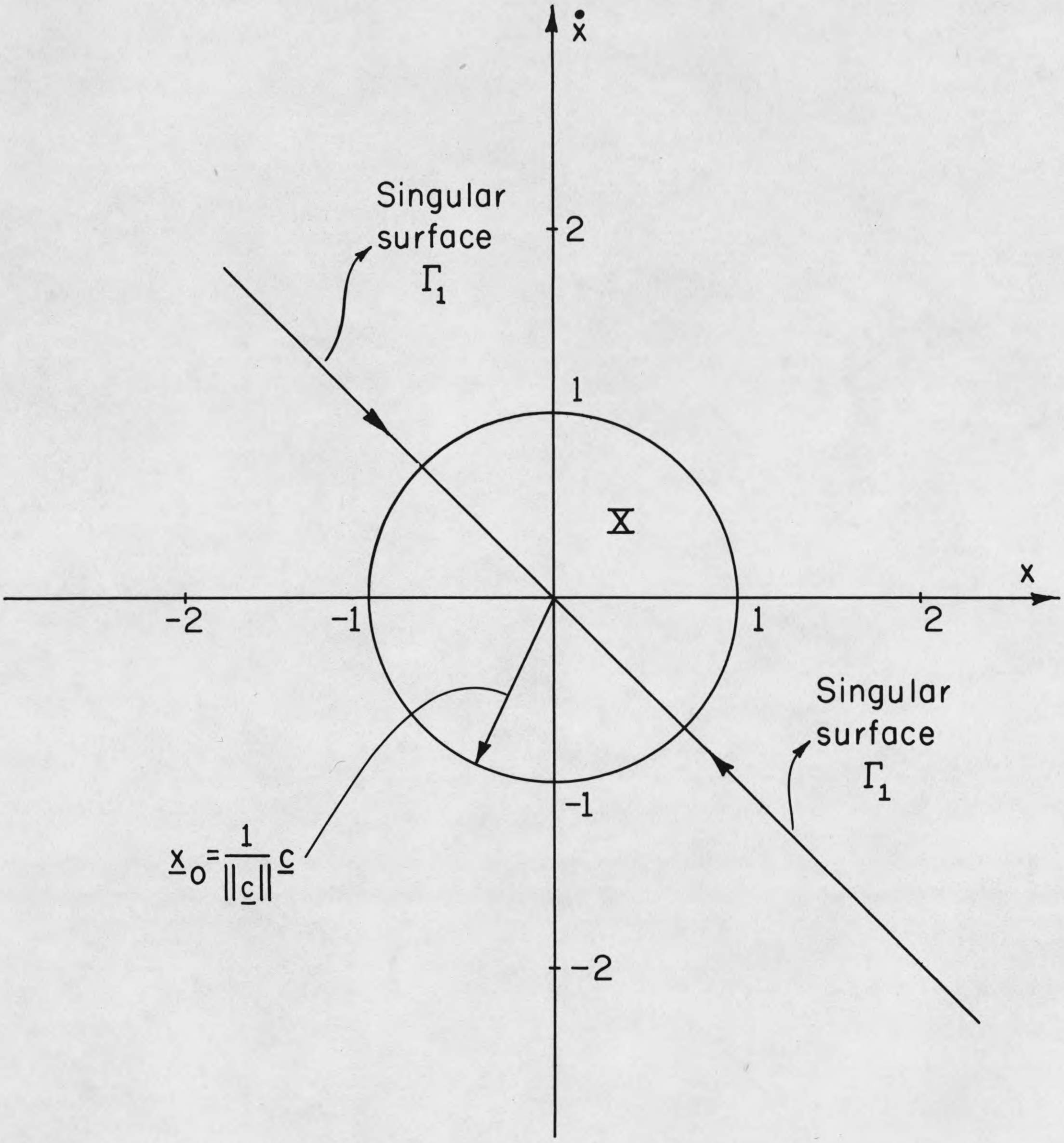

Figure 2. Sets X, Γ_1 and initial condition $\underline{x}_o = \frac{k_A}{||c||} \underline{c}$ for Example 2.3.

3. SYSTEMS OF SINGULARITY OF ORDER j

3.1 Singular Linear Control Law

In this section performance indices in which the integrand is

$$F(x, \overset{\circ}{x}, \ldots \overset{n-j}{x}) = \sum_{i,k=1}^{n-j+1} q_{i,k} \overset{(i-1)}{x} \overset{(k-1)}{x} \qquad (58)$$

are considered. As before, the singular trajectories are obtained; then conditions which the linear control must satisfy in order to be the optimal law on the singular surface are derived. First, in view of (58) the Euler-Poisson equation is of order 2(n-j). Proceeding as in Section 2, we obtain the stable Euler-Poisson equation

$$\sum_{i=o}^{n-j} h_i \overset{(i)}{x}(t) = 0. \qquad (59)$$

The singular surface is then the hyperplane Γ_j, defined as the set of all $\underline{x}_o$ such that

$$\sum_{i=o}^{n-j} h_i \overset{(i+k)}{x_o} = 0, \ k = 0,1 \ldots j-1 . \qquad (60)$$

The conditions on $\underline{c}$ are obtained by equating the polynomial $H_T(s)$ (given by 19) to the product of two polynomials as follows:

$$H_T(s) = Z(s)\, H_j(s) , \qquad (61)$$

where

$$H_j(s) = \sum_{i=o}^{n-j} h_i s^i, \; h_{n-j} = 1 \; . \tag{62}$$

In view of (62), Z(s) is a polynomial of order j in s. Equating terms of the same power in (61), we obtain

$$a_i - c_i = \sum_{k=o}^{j} h_{i-k} z_k \; ; \; i=0, 1 \ldots n-1 \; . \tag{63}$$

If we solve the first j equations of (63) for the z_k's as functions of $c_o, c_1, \ldots c_{j-1}$, the remaining (n-j) c_i's will then be given by

$$c_i = a_i - \sum_{k=o}^{j} h_{i-k} z_k(c_o, c_1 \quad c_{j-1}), \; i=j, \ldots n-1 \; . \tag{64}$$

Therefore, analogous to the special case j=1, the restriction that the linear control law be optimal on Γ_j reduces the n-variable problem to a j-variable problem. Again, stability is assured if the polynomial Z(s) is Hurwitz.

3.2 Extended Linear Control Law

Using Pontryagin's formulation one arrives (as in Section 2) at the problem of maximizing in $\underline{c}$ the function $N(\underline{p}, \underline{x}, \underline{c})$ (given by 28), subject to the constraint relation (25). As before, $\underline{c}$ must also satisfy the singularity conditions, i.e., Equation (64). Using the Lagrange multiplier technique, we set

$$\frac{\partial}{\partial c_m} \left[N(\underline{p}, \underline{x}, \underline{c}) + \rho \, \underline{x}^T \underline{c} \, \underline{c}^T \underline{x} \right] = 0; \quad m=o, \ldots j-1 \; . \tag{65}$$

Performing the differentiation in (65), we arrive at

$$p_n \frac{\partial \underline{c}^T}{\partial c_m} \underline{x} + 2\rho \underline{x}^T \underline{c} \frac{\partial \underline{c}^T}{\partial c_m} \underline{x} = 0, \; m = o, 1 \ldots j-1. \tag{66}$$

As shown in Appendix B, $p_n(t)$ is given by the following:

$$p_n(t) = \sum_{i=o}^{j-1} P_i \frac{d^i}{dt^i} \sum_{\ell=o}^{n-j} h_\ell \, x^{(\ell)} , \tag{67}$$

where the coefficients P_i are functions of $c_o, c_1 \ldots c_{j-1}$. Let $\underline{h}^m$ represent an nx1 vector such that: (a) its first m-1 components are zero; (b) for $i \leq m+n-j$, the $(m+i)^{\underline{th}}$ component is h_i and (c) for $i > m+n-j$, the $(m+i)^{\underline{th}}$ component is zero. With this notation we can write

$$p_n(t) = \sum_{i=o}^{j-1} P_i \, \underline{x}^T \underline{h}^i . \tag{68}$$

Substituting (68) into (66), yields

$$\underline{x}^T \left[\left[\sum_{i=o}^{j-1} P_i \underline{h}^i + 2\rho \underline{c} \right] \frac{\partial \underline{c}^T}{\partial c_m} \right] \underline{x} = 0, \; m = 0, \ldots j-1. \tag{69}$$

Let

$$\underline{D}_m = \left[\sum_{i=o}^{j-1} P_i \underline{h}^i + 2\rho \underline{c} \right] \frac{\partial \underline{c}^T}{\partial c_m} , \quad m = 0, \ldots j-1 . \tag{70}$$

As before $\underline{D}_m$ is recognized to be a matrix of rank one, the only (possibly) non-zero eigenvalue being

$$\lambda(\underline{D}_m) = \frac{\partial \underline{c}^T}{\partial c_m} \left[\sum_{i=o}^{j-1} P_i \underline{h}^i + 2\rho \underline{c} \right] . \tag{71}$$

Hence, in order to make (69) true for all $\underline{x}(t)$ we need

$$\lambda(\underline{D}_m) = 0; \quad m = 0, 1 \ldots j-1 . \tag{72}$$

The j Equations (72) are then solved for the c_i's, $i = o, \ldots j-1$, as functions of ρ. Again the choice of the Lagrange multiplier ρ is made through an iterative procedure. Analogous to the case j=1, Equation (42) gives a necessary condition that the c_i's must satisfy.

3.3 Illustrative Example

Given the system equation

$$\ddot{x} = u , \tag{73}$$

and the performance index

$$\int_o^\infty x^2(t)\, dt , \tag{74}$$

find a linear control law u(t) such that it minimizes the performance index (74) subject to restrictions

$$\underline{x}(o) = \underline{x}_o, \quad ||\underline{x}_o|| \leq 1 \tag{75}$$

and

$$\underline{x}(\infty) = \underline{0} \tag{76}$$

and

$$u^2(t) \leq 2 . \tag{77}$$

The Euler-Poisson equation degenerates to x=0. Hence

$$\underline{h}^{o} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \underline{h}^{1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{78}$$

From (64) we obtain

$$z_o = -c_o \tag{79}$$

and

$$z_1 = -c_1 \tag{80}$$

From Appendix B,

$$P_o = -\frac{1}{c_o} \tag{81}$$

and

$$P_1 = -\frac{1}{c_o c_1} \tag{82}$$

Therefore

$$\frac{\partial \underline{c}^T}{\partial c_o} = [1 , 0] \tag{83}$$

and

$$\frac{\partial \underline{c}^T}{\partial c_1} = [0 , 1] . \tag{84}$$

Equation (72) yields

$$2 \rho c_o + \frac{1}{c_o} = 0 \tag{85a}$$

and

$$2\rho c_1 - \frac{1}{c_o c_1} = 0 \quad . \tag{85b}$$

Hence

$$c_1^2 = - c_o \; . \tag{86}$$

The system equation is then

$$\ddot{x} + \sqrt{-c_o}\;\dot{x} - c_o\, x = 0; \tag{87}$$

that is a damping ratio of 0.5. The value of c_o is restricted by (42) to lie in the interval

$$-1 \leq c_o \leq 2 \; . \tag{88}$$

For stability, $c_o < 0$; hence the optimal c_o lies in the interval

$$-1 \leq c_o < 0 \; . \tag{89}$$

The reader may verify that the solution

$$c_o = - 1 \tag{90}$$

minimizes the performance index; moreover since $V(t) = \underline{x}^T \underline{x}$ is non-decreasing,

$$\frac{dV}{dt} = - 2\, \dot{x}^2 \,(t) \quad , \tag{91}$$

it satisfies the constraint in $u^2(t)$.

4. CONCLUSIONS

An optimization problem with a new kind of constraint has been presented Besides the usual constraint in magnitude, the control signal is restricted to belong to the class of linear controls. Because the performance index (a non-negative definite quadratic form) does not include the so-called "cost of control", singular solutions are present. The linear control law obtained, besides being optimal on the singular surface, is optimal outside of it in the sense that no other linear law gives a smaller value for the given performance index while satisfying the magnitude constraint for all admissible initial conditions. The computational procedure involved in a given problem is straight-forward and easily executed. The extension of these results to multi-input systems is foreseen as a simple theoretical (although perhaps computationally arduous) task.

APPENDIX A : SOME CHARACTERISTICS OF THE LINEAR OPTIMAL CONTROL

In what follows some characteristics of the linear optimal control law are derived. All the derivations are for the general case, i.e., for systems of singularity of order j. As we know the control is given by

$$u(t) = \sum_{i=o}^{n-1} c_i \overset{(i)}{x}(t) \quad . \tag{A.1}$$

Substituting (63) into (A.1) we obtain

$$u = \sum_{i=o}^{n-1} a_i \overset{(i)}{x} - \sum_{i=o}^{n-1} \sum_{k=o}^{n-1} h_{i-k} \; z_k \overset{(i)}{x} \quad . \tag{A.2}$$

Interchanging the order of summation in the second term of (A.2) and then letting $i-k=\ell$ we arrive at

$$u = \sum_{i=o}^{n-1} a_i \overset{(i)}{x} - \sum_{k=o}^{j} z_k \sum_{\ell=o}^{n-1-k} h_\ell \overset{(k+\ell)}{x} \quad . \tag{A.3}$$

Since $h_\ell=o$ for $\ell > n-j$, the second term in (A.3) can be separated into two terms as

$$\sum_{k=o}^{j} z_k \sum_{\ell=o}^{n-1-k} h_\ell \overset{(k+\ell)}{x} = z_j \sum_{\ell=o}^{n-1-j} h_\ell \overset{(j+\ell)}{x} + \sum_{k=o}^{j-1} z_k \sum_{\ell=0}^{n-j} h_\ell \overset{(\ell+k)}{x} \quad . \tag{A.4}$$

Substituting (A.4) into (A.3), and since $z_j = 1$, we obtain

$$u = \sum_{i=o}^{n-1} a_i \overset{(i)}{x} - \sum_{\ell=o}^{n-1-j} h_\ell \overset{(j+\ell)}{x} - \sum_{k=o}^{j-1} z_k \sum_{\ell=o}^{n-j} h_\ell \overset{(\ell+k)}{x} . \quad (A.5)$$

Let

$$u_s = \sum_{i=o}^{n-1} a_i \overset{(i)}{x} - \sum_{i=o}^{n-1-j} h_i \overset{(i+j)}{x} \quad (A.6)$$

and

$$u_{ns} = -\sum_{k=o}^{j-1} z_k \frac{d^k}{dt^k} \sum_{i=o}^{n-j} h_i \overset{(i)}{x} , \quad (A.7)$$

then

$$u(t) = u_s(t) + u_{ns}(t) . \quad (A.8)$$

On Γ_j we have

$$\sum_{i=o}^{n-j} h_i \overset{(i)}{x}(t) = 0 ; \quad (A.9)$$

hence, on Γ_j we have

$$u_{ns}(t) = 0 . \quad (A.10)$$

Therefore the control u(t) can be considered as a sum of two signals $u_{ns}(t)$ and $u_s(t)$; the former is zero on Γ_j and hence, the latter is the only control when $\underline{x}(t)$ is in Γ_j. Obviously $u_s(t)$ is independent of the coefficients $c_o, c_1, \ldots c_{j-1}$.

Consider now the system equation

$$\sum_{i=o}^{n} a_i \overset{(i)}{x}(t) = u_s(t) + u_{ns}(t) . \quad (A.11)$$

Substituting (A.6) into (A.11) and since $a_n=1$, we can write the following,

$$\frac{d^j}{dt^j} \sum_{i=o}^{n-j} h_i \overset{(i)}{x}(t) = u_{ns}(t) = - \sum_{k=o}^{j-1} z_k \frac{d^k}{dt^k} \sum_{i=o}^{n-j} h_i \overset{(i)}{x}(t) \text{ .} \tag{A.12}$$

Letting

$$g(t) = \sum_{i=o}^{n-j} h_i \overset{(i)}{x}(t) \text{ ,} \tag{A.13}$$

we write for (A.12)

$$\frac{d^j}{dt^j} \left(g(t)\right) = \sum_{k=o}^{j-1} z_k \overset{(k)}{g}(t) \text{ .} \tag{A.14}$$

Therefore g(t) is the solution of (A.14) and as so depends only on $c_o, c_1 \ldots c_{j-1}$. In the particular case j=1

$$g(t) = g(o)\, e^{-z_o t} \text{ .} \tag{A.15}$$

APPENDIX B : THE FUNCTION $p_n(t)$.

The auxiliary variable $\underline{p}(t)$ is defined in the text by Equation (26). If we let

$$\underline{F} = \underline{A} + \underline{b}\,\underline{c}^T , \tag{B.1}$$

then the system equation can be written as

$$\dot{\underline{x}} = \underline{F}\,\underline{x} . \tag{B.2}$$

Therefore, the Hamiltonian function is written as follows,

$$M(\underline{p}, \underline{x}, \underline{c}) = \underline{p}^T \underline{F}\,\underline{x} - \underline{x}^T \underline{Q}\,\underline{x} . \tag{B.3}$$

Let $\nabla_{\underline{x}} M$ be a vector whose coordinates are given by

$$(\nabla_{\underline{x}} M)_i = \frac{\partial M}{\partial \, (i_{\underline{x}}^{-1})} . \tag{B.4}$$

Using (26), we obtain

$$\dot{\underline{p}} = - \nabla_{\underline{x}} M = - \underline{F}^T \underline{p} + 2\,\underline{Q}\,\underline{x} . \tag{B.5}$$

In order to find the differential equation that the n^{th} coordinate of $\underline{p}$ satisfies we proceed as follows. First (B.5) is differentiated m times with respect to t, yielding

$$\underline{p}^{(m+1)} + \underline{F}^{T}\,\underline{p}^{(m)} = 2\,\underline{Q}\,\underline{x}^{(m)} \quad . \tag{B.6}$$

As we know

$$\underline{F} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ f_o & f_1 & f_2 & \cdots & f_{n-1} \end{bmatrix} , \tag{B.7}$$

where

$$f_i = c_i - a_i \quad . \tag{B.8}$$

Letting m = n-1 in (B.6) we obtain

$$\overset{(n)}{p}_n + \overset{(n-1)}{p}_{n-1} + f_{n-1}\,\overset{(n-1)}{p}_n = \left[2\,\underline{Q}\,\underline{x}^{(n-1)} \right]_n \quad . \tag{B.9}$$

From (B.6) and for m = n-2, we have

$$\overset{(n-1)}{p}_{n-1} = - \overset{(n-2)}{p}_{n-2} - f_{n-2}\,\overset{(n-2)}{p}_n + \left[2\,\underline{Q}\,\underline{x}^{(n-2)} \right]_{n-1} \quad . \tag{B.10}$$

Substitution of (B.10) in to (B.9) yields

$$\overset{(n)}{p}_n + f_{n-1}\,\overset{(n-1)}{p}_n - f_{(n-2)}\,\overset{(n-2)}{p}_n - \overset{(n-2)}{p}_{n-2} = \left[2\,\underline{Q}\,\underline{x}^{(n-1)} \right]_n - \left[2\,\underline{Q}\,\underline{x}^{n-2} \right]_{n-1} \tag{B.11}$$

It is now clear that p_{n-2} and then p_{n-3}, p_{n-4}, etc., can be eliminated from (B.11) by repeated use of Equation (B.6). If this is done, we finally arrive at the following differential equation.

$$\sum_{i=o}^{n} (-1)^i f_i \overset{(i)}{p}_n = \sum_{i=o}^{n-1} (-1)^i \left[2 \underline{Q} \overset{(i)}{\underline{x}} \right]_{i+1} , \tag{B.12}$$

where

$$f_n \equiv -1 \quad . \tag{B.13}$$

The solution of (B.12) provides the desired $p_n(t)$. Note first the identity

$$\sum_{i=o}^{n-1} (-1)^i \left[2 \underline{Q} \overset{(i)}{\underline{x}} \right]_{i+1} = \sum_{i=o}^{n-1} (-1)^i \frac{d^i}{dt^i} \frac{\partial}{\partial \overset{(i)}{x}} \left[\underline{x}^T \underline{Q} \underline{x} \right] , \tag{B.14}$$

i.e., the right hand side of (B.12) is the left hand side of the Euler-Poisson equation. In view of the factorization expressed by (15) we have

$$\sum_{i=o}^{n-1} (-1)^i \left[2 \underline{Q} \overset{(i)}{\underline{x}} \right]_{i+1} = 2\, q_{n-j+1,n-j+1} \sum_{i=o}^{n-j} (-1)^i h_i \frac{d^i}{dt^i} \sum_{k=o}^{n-j} h_k \overset{(k)}{x} . \tag{B.15}$$

From (63) and (B.8) if follows that

$$f_i = - \sum_{k=o}^{j} h_{i-r} z_k . \tag{B.16}$$

The left hand side of (B.12) is then written as

$$\sum_{i=o}^{n} (-1)^{i} f_{i} \overset{(i)}{p}_{n} = \sum_{i=o}^{n} (-1)^{i+1} \sum_{k=o}^{j} h_{i-k} z_{k} \overset{(i)}{p}_{n} . \qquad (B.17)$$

Expression (B.17) is easily simplified to

$$\sum_{i=o}^{n} (-1)^{i} f_{i} \overset{(i)}{p}_{n} = \sum_{i=o}^{n-j} (-1)^{i} h_{i} \frac{d^{i}}{dt^{i}} \sum_{k=o}^{j} (-1)^{k+1} z_{k} \overset{(k)}{p}_{n} . \qquad (B.18)$$

In view of (B.14), (B.15) and (B.18), Equation (B.12) is now

$$\sum_{i=o}^{n-j} (-1)^{i} h_{i} \frac{d^{i}}{dt^{i}} \left[\sum_{k=o}^{j} (-1)^{k+1} z_{k} \overset{(k)}{p}_{n}(t) - 2\, q_{n-j+1,n-j+1}\, g(t) \right], \qquad (B.19)$$

where g(t) is given by (A.13). Since $\underline{x}(\infty) = \underline{o}$, both $p_{n}(t)$ and g(g) must be bounded (in fact, $p_{n}(\infty) = g(\infty) = o$); hence, if we define y(t) by the expression

$$y(t) = \sum_{k=o}^{j} (-1)^{k+1} z_{k} \overset{(k)}{p}_{n}(t) - 2\, q_{n-j+1,n-j+1}\, g(t) \quad ; \qquad (B.20)$$

y(t) must be bounded, But the only bounded solution of the equation

$$\sum_{i=o}^{n-j} (-1)^{i} h_{i} \overset{(i)}{y}(t) = 0 \qquad (B.21)$$

is y(t) = 0. Hence, the desired $p_{n}(t)$ is a solution of the equation y(t) = 0, i.e., a solution of

$$\sum_{k=o}^{j} (-1)^{k+1} z_k \overset{(k)}{p}_n(t) = 2\, q_{n-j+1,n-j+1}\, g(t) . \tag{B.22}$$

Note that the polynomial $\sum_{k=o}^{j} (-1)^{k+1} z_k s^k$ is a anti-Hurwitz; therefore the initial conditions $p_n(o)$, $\dot{p}_n(o)$, etc., must be such that none of the natural modes of (B.22) are excited. Hence, only the forced solution is different from zero. Since we know that $p_n(t) = o$ whenever $\underline{x}(t)$ is in the singular surface Γ_j, the forced solution should be of the form

$$p_n(t) = \sum_{i=o}^{j-1} P_i \overset{(i)}{g}(t) . \tag{B.23}$$

The coefficients P_i are obtained by substituting (B.23) into (B.22). For the cases j=1,2 and 3 the values of P_i are the following:

(a) $j = 1$

$$P_o = \frac{q_{nn}}{-z_o} ; \tag{B.24}$$

(b) $j = 2$

$$P_o = \frac{q_{n-1,n-1}}{-z_o} , \tag{B.25}$$

$$P_1 = \frac{P_o}{z_1} ; \tag{B.26}$$

(c) $j = 3$

$$P_o = \frac{q_{n-2,n-2}}{-z_o} \quad , \tag{B.27}$$

$$P_1 = \frac{z_2^2 \, q_{n-2,n-2}}{z_o(z_o - z_1 z_2)} \quad , \tag{B.28}$$

$$P_2 = \frac{P_1}{z_2} \quad . \tag{B.29}$$

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