3900 CSC

300 RM

No patents

Coordinated Science Laboratory

UNIVERSITY OF ILLINOIS - URBANA, ILLINOIS

ON NETWORKS AND BI-COMPLETE GRAPHS

James A. Resh

REPORT R-174 JULY, 1963

COORDINATED SCIENCE LABORATORY
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

Contract DA-36-039-TR US AMC 02208(E)
DA Project 3A-99-25-004

The research reported in this document was made possible by support extended to the University of Illinois, Coordinated Science Laboratory, jointly by the Department of the Army, Department of the Navy (Office of Naval Research), and the Department of the Air Force (Office of Scientific Research) under Department of Army Contract DA-36-039-TR US AMC 02208(E).

ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor Wataru Mayeda for his interest and assistance in the research for this paper.

TABLE OF CONTENTS

1. EXORDIUM

1.1 Prolegomena

Mathematically, a communication network is a network (by which, in this paper, we mean a bi-complete† linear graph whose edges are weighted by real numbers) whose edge weights are restricted to be nonnegative. Considerable space in the recent (since the mid 50's) applied mathematics literature has been devoted to their study. In the physical interpretations which generated this interest (typically, the edges are identified as channels for some type of flow and the edge weights assume the character of upper bounds on the rates of flow which can be supported by the corresponding channels) the property which occupies the center of attention is the capacity of the network for supporting a flow from one terminal to another. It is this preoccupation with "generalized plumbing" that distinguishes the theory of communication networks from other mathematical systems dealing with weighted graphs (e.g. the theories of resistor networks and automata).

The analysis problem (determine the terminal capacities when the network is given) for both nonoriented and oriented networks has been successfully treated (4,5,6), as has the synthesis problem for the nonoriented case (7). The synthesis problem for oriented communication networks (i.e. for the case when among the specified terminal capacities are some which, between the same pair of terminals, are different in one direction from the other) remains unsolved. It seems that a solution when the physically generated restriction to nonnegative weights is lifted should be a major step in the direction of a solution to the more restricted problem; in any event, networks

† An oriented graph is "bi-complete" if for each pair (v_i, v_j) of distinct vertices it contains both of the edges $\tilde{e}_{ij}$ and $\tilde{e}_{ji}$. For treatises on linear graph theory see (1,2,3).

are interesting in themselves, and for this reason constitute the primary object of this thesis.

1.2 Notation

The vertices (terminals) of an n-terminal network are labeled $v_1, v_2, \ldots, v_n$ and the edge (channel) directed from v_i to v_j is denoted by $\tilde{e}_{ij}$. A particular network N is then completely specified by its <u>edge weight matrix</u> $E_N = [e_{ij}]$, whose row i and column j correspond to vertices v_i and v_j respectively, and where entry $(E_N)_{ij} = e_{ij}$ for $1 \leq i \neq j \leq n$ is the weight in N of edge $\tilde{e}_{ij}$, the entries on the main diagonal remaining undefined (no significance is attached to the main diagonal entries of any matrix considered in the paper -- for our purposes it's almost a shame that square arrays must have main diagonals).

If V is any proper subset of the vertices of a network then the <u>semi-cut</u> $\tilde{c}(V)$ is defined† to be the set of all edges $\tilde{e}_{ij}$ such that $v_i \in V$ and $v_j \in \bar{V}$, where $\bar{V}$ is the complement of V. The sum of the weights in N of the edges of $\tilde{c}(V)$ is called the <u>value in N</u> of $\tilde{c}(V)$ and is denoted by $c_N(V)$. Semi-cut $\tilde{c}(V)$ is said to separate vertex v_i from vertex v_j if $\tilde{e}_{ij} \in \tilde{c}(V)$; thus if $\tilde{c}(V)$ separates v_i from v_j then the removal from the network of all edges in $\tilde{c}(V)$ would break all directed paths from v_i to v_j.

Then the terminal capacity t_{ij} from node v_i to node v_j is the minumum of the values in N of the semi-cuts which separate v_i from v_j (here we take this as a definition but in the theory of communication networks, where the concept of "flow" is a primitive, our defining property is a result

---

† If $V = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ and $\bar{V} = \{v_{j_1}, v_{j_2}, \ldots, v_{j_{n-r}}\}$ then we will sometimes use the notation $\tilde{c}(V) = \tilde{c}(i_1, i_2, \ldots, i_r)(j_1, j_2, \ldots, j_{n-r})$ which, although redundant, has heuristic value when one wants to enumerate the edges of $\tilde{c}(V)$.

originally found by Ford and Fulkerson (5)). The set of terminal capacities of N is conveniently displayed in its terminal capacity matrix $T_N = [t_{ij}]$, whose row i and column j correspond to nodes v_i and v_j respectively, and where entry $(T_N)_{ij} = t_{ij}$ $(1 \leq i \neq j \leq n)$ is defined above.

1.3 Properties of Terminal Capacity Matrices

Tang and Chien (8) found that the terminal capacity matrix T of any communication network has the property that after a suitable rearrangement of its rows and columns (i.e. after suitably reordering the vertices of the network) it can be partitioned as:

$$T = \left[\begin{array}{c|c} T_A & T_O \\ \hline T_{BA} & T_B \end{array}\right]$$

where the resulting principal submatrices T_A and T_B are both square, and where the entries of T_O are all equal to the smallest entry of T; further, the resulting principal submatrices, and all ensuing ones, if of order at least 2, are partitionable in the same manner. For brevity, we shall say that any real square matrix having this property is partitionable.

Mayeda (9) found that the terminal capacity matrix T for any communication network has the property that for each entry t_{ij} there exists at least one proper subset V of the vertices of the network, which contains v_i but not v_j, such that the submatrix T(V) of T whose rows correspond to the vertices in V and whose columns correspond to the vertices in $\bar{V}$ has t_{ij} as its largest entry. This will be referred to as the S-submatrix condition, any submatrix of T which fulfills the condition for a particular t_{ij} being called an S-submatrix for t_{ij}.

The proof of the S-submatrix condition is brief: If semi-cut $\tilde{c}(V)$ separates v_i from v_j and has value t_{ij} then, for any vertex pair $(v_{i'}, v_{j'})$ such that $v_{i'} \in V$ and $v_{j'} \in \bar{V}$, the corresponding terminal capacity is $t_{i'j'} \leq c(V) = t_{ij}$; and since the submatrix T(V) has just these terminal capacities for its entries, it is an S-submatrix for t_{ij} and the set of S-submatrices for t_{ij} is nonempty, Q.E.D.

By noticing that the above proof makes no use of the fact that the edge weights of a communication network are nonnegative we see that, in fact, the terminal capacity matrix of an arbitrary network has the same property.

Let M be any real square matrix which satisfies the S-submatrix condition, let m_o be the value of its smallest entry, and let $M(V_o)$ be any S-submatrix of M for any entry of M whose value is m_o. Since m_o is the largest entry of $M(V_o)$ and the smallest of M it follows that all entries of $M(V_o)$ have value m_o; thus, if we rearrange the rows (and, correspondingly, the columns) of M so that those whose index is in V_o all precede those whose index is in $\bar{V}_o$, it can be partitioned as:

$$M = \begin{array}{c} \\ V_o \\ \bar{V}_o \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\qquad}^{V_o} & \overbrace{\qquad}^{\bar{V}_o} \end{array} \\ \left[\begin{array}{c|c} M_1 & M(V_o) \\ \hline M(\bar{V}_o) & M_2 \end{array} \right] \end{array}$$

Next, let m_1 be the value of the smallest entry of the principal submatrix M_1, and let $M(V_1)$ be any S-submatrix of M for any entry of M_1 whose value is m_1. It is clear that $M_1(V_o \cap V_1)$ is an S-submatrix of M_1 for the same entry so every entry of $M_1(V_o \cap V_1)$ has value m_1; and since the rows and columns of M whose indices appear in V_o can be rearranged at will without destroying the above partitioning, we may rearrange them so that those whose indices are in $V_o \cap V_1$ all precede those whose indices are in $V_o \cap \bar{V}_1$, after which M can be partitioned as:

$$
M = \begin{array}{c|cc|c}
 & V_o \cap V_1 & V_o \cap \bar{V}_1 & \bar{V}_o \\ \hline
V_o \cap V_1 & M_1 & M_1(V_o \cap V_1) & \\
V_o \cap \bar{V}_1 & M_1(V_o \cap \bar{V}_1) & M_2 & M(V_o) \\ \hline
\bar{V}_o & M(\bar{V}_o) & & M_2
\end{array}
$$

It is evident that this process, based entirely on the S-submatrix condition, leads to a proof of Tang and Chien's partitioning condition. Thus, the terminal capacity matrix of an arbitrary network is partitionable.

Another consequence of Mayeda's condition is the

<u>Theorem 1.1</u>: Let $T = [t_{ij}]$ be a real square matrix which satisfies the S-submatrix contition. Then network N has T for its terminal capacity matrix if and only if for each pair (v_i, v_j) of distinct vertices:

$$\min \left\{ c_N(V) \;\middle|\; T(V) \text{ is an S-submatrix for } t_{ij} \right\} = t_{ij} \; .$$

Proof: To show that the condition is sufficient we need only show if N satisfies it and $\tilde{c}(V)$ is a semi-cut separating v_i from v_j but T(V) is not an S-submatrix for t_{ij} then $c_N(V) \geq t_{ij}$. Since T(V) is not an S-submatrix for t_{ij} we know that its largest entry, call it $t_{i'j'}$, is strictly greater than t_{ij}; thus, since T(V) is an S-submatrix for this largest entry, we have by hypothesis that $c_N(V) \geq t_{i'j'} > t_{ij}$.

To see the necessity of the condition suppose that N has terminal capacity matrix T. Then, for each pair (v_i, v_j) of distinct vertices, each semi-cut $\tilde{c}(V)$ separating v_i from v_j has value $c_N(V) \geq t_{ij}$, so in particular each semi-cut corresponding to an S-submatrix for t_{ij} has value at least t_{ij}. Also, since N realizes T, the value in N of at least one semi-cut, say $\tilde{c}(V_o)$, is $c_N(V_o) = t_{ij}$; as we saw in the proof of the S-submatrix condition, $T(V_o)$ is an S-submatrix for t_{ij}. Thus $\min \left\{ c_N(V) \,\middle|\, T(V) \text{ is an S-submatrix for } t_{ij} \right\} = t_{ij}$, Q.E.D.

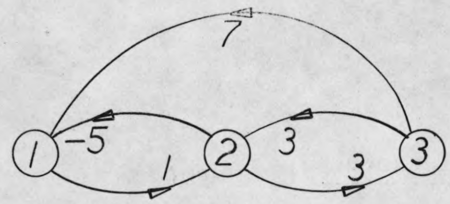
Notice that, given a real square matrix T with S-submatrices, a network N, and a single pair (v_i, v_j) of distinct vertices of N, it doesn't follow from the theorem that $(T_N)_{ij} = t_{ij}$ if

$$\min \left\{ c_N(V) \,\middle|\, T(V) \text{ is an S-submatrix for } t_{ij} \right\} = t_{ij} \ ;$$

the theorem requires that the relation hold for all pairs of distinct vertices. An example to illustrate this point is

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} x & 1 & 1 \\ 2 & x & 3 \\ 2 & 4 & x \end{bmatrix} \end{array}$$

N:



Here the only S-submatrix of T for its entry t_{21} is

$$T(2,3)(1) = \begin{array}{c} \\ 2 \\ 3 \end{array}\begin{array}{c} 1 \\ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{array}$$

and the value in N of the corresponding semi-cut $\tilde{c}\,(2,3)(1)$ is $c_N(2,3)(1) = 2 = t_{21}$, yet the terminal capacity in N from vertex v_2 to vertex v_1 is $(T_N)_{21} = -2 \neq t_{21}$.

2. SEMI-CUTS

2.1 Introduction

In this chapter we shall develop some algebraic properties of the set of semi-cuts associated with the n-vertex bi-complete graph. Convenient use can be made of the Euclidean vector space $\mathcal{E}_{(n-1)n}$ of n x n real matrices if we identify edge $\tilde{e}_{ij}$ of the graph with a matrix whose only non-zero entry is a 1 in the (i, j) position, and a nonoriented subgraph by the sum of its edges. Thus for the semi-cut $\tilde{c}(V)$ we have the representation

$$\tilde{c}(V) = \sum_{\tilde{e}_{ij} \in \tilde{c}(V)} \tilde{e}_{ij} \left(= \sum_{\substack{v_i \in V \\ v_j \in \bar{V}}} \tilde{e}_{ij} \right)$$

in which the (i,j) entry of the matrix $\tilde{c}(V)$ is 1 if edge $\tilde{e}_{ij}$ is in the semi-cut $\tilde{c}(V)$ and 0 otherwise†.

More common in linear graph theory than the semi-cut is the directed cut set. If we denote by $\tilde{\tilde{c}}$ (V) the set (of all edges which have one vertex in V and the other in $\bar{V}$) oriented from V to $\bar{V}$ then a useful representation is

$$\tilde{\tilde{c}}(V) = \sum_{\substack{v_i \in V \\ v_j \in \bar{V}}} (\tilde{e}_{ij} - \tilde{e}_{ji}) \quad (= \tilde{c}(V) - \tilde{c}(\bar{V}))$$

† The double duty served by the symbols $\tilde{c}$ and $\tilde{e}$ should cause no difficulty.

in which the (i,j) entry is + 1/-1 if $\tilde{e}_{ij}$ is in $\tilde{\tilde{c}}(V)$ and agrees/disagrees with it in orientation, 0 otherwise.

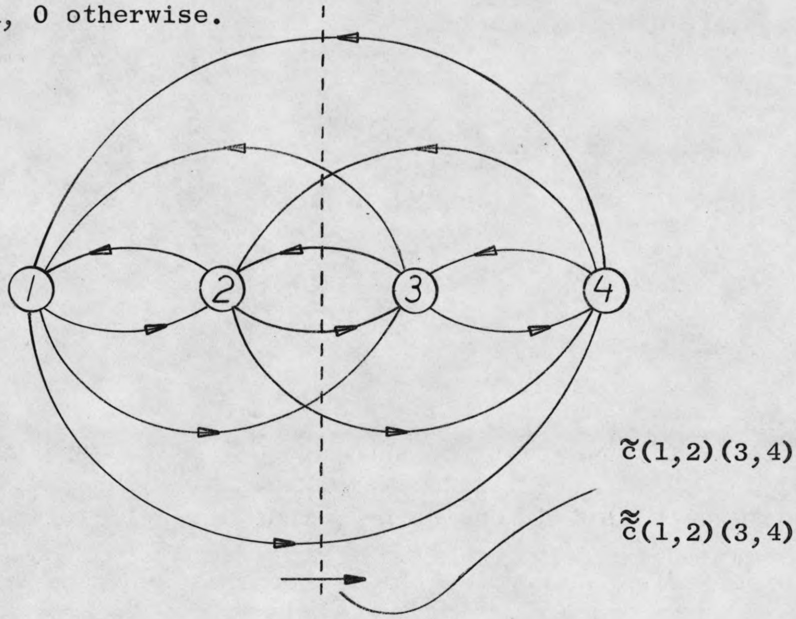


Figure 2.1

As an illustration of the notation we have for the 4-vertex graph (c.f. Figure 2.1):

$$\tilde{c}(1,2)(3,4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 1 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix} \end{array} + \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix} \end{array}$$

$$+ \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix} \end{array} + \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 1 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 1 & 1 \\ 0 & x & 1 & 1 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix} \end{array}$$

and, similarly,

$$\tilde{\tilde{c}}(1,2)(3.4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 1 & 1 \\ 0 & x & 1 & 1 \\ -1 & -1 & x & 0 \\ -1 & -1 & 0 & x \end{bmatrix} \end{array}$$

It is well known that the subspace $\tilde{\tilde{C}} \subseteq \mathcal{E}_{(n-1)n}$ spanned by the set of directed cut sets is of dimension n-1; thus any n-1 linearly independent directed cut sets constitute a basis for this subspace and in particular, for a given basis $\tilde{\tilde{c}}_1, \tilde{\tilde{c}}_2, \ldots, \tilde{\tilde{c}}_{n-1}$ of directed cut sets, any proper subset V of the vertices of the graph uniquely determines a set $\{\alpha_1(V), \alpha_2(V), \ldots, \alpha_{n-1}(V)\}$ of real numbers such that

$$\tilde{\tilde{c}}(V) = \sum_{\nu=1}^{n-1} \alpha_\nu(V)\, \tilde{\tilde{c}}_\nu$$

An often useful class of bases is associated with the trees of the graph, in the following manner. Directed cut set $\tilde{\tilde{c}}$ is called fundamental with respect to tree τ if $\tilde{\tilde{c}}$ contains exactly one edge of τ and this edge agrees with $\tilde{\tilde{c}}$ in orientation; and if $\tilde{\tilde{c}}_1, \tilde{\tilde{c}}_2, \ldots, \tilde{\tilde{c}}_{n-1}$ are the fundamental directed cut sets w.r.t. any tree then they constitute a basis of $\tilde{\tilde{C}}$.

2.2 Semi-cuts

We now transfer our attention to the subspace $\tilde{C}$ of $\mathcal{E}_{(n-1)n}$ spanned by the set of semi-cuts of the graph. After treating the question of the dimensionality of $\tilde{C}$ and establishing the relation between bases of $\tilde{\tilde{C}}$ (which is a subspace of $\tilde{C}$) and bases of $\tilde{C}$, we turn to a detailed consideration of certain special bases which are particularly suited to our needs.

Theorem 2.1: $\tilde{C}$ is of dimension $\frac{(n-1)(n+2)}{2}$.

Before proving this theorem it is convenient to establish the following pair of lemmas. The pairs $\tilde{e}_{ij}$, $\tilde{e}_{ji}$ of edges incident at the same two vertices, or zweiecks†, are important enough in the theory of semi-cuts to warrant assigning a special symbol to the vector $\tilde{e}_{ij} + \tilde{e}_{ji}$ -- we shall use $\tilde{e}^*_{ij}$ ($= \tilde{e}^*_{ji}$) for this purpose.

Lemma 2.1: $\tilde{C}$ contains all the zweiecks of the graph.

Proof: If v_{i_1} and v_{i_2} are any two distinct vertices of the graph then consider the semi-cuts $\tilde{c}_1 = \tilde{c}\,(1, 2, \ldots, i_1-1, i_1+1, \ldots, n)\,(i_1)$ and $\tilde{c}_2 = \tilde{c}\,(1, 2, \ldots, i_2-1, i_2+1, \ldots, n)\,(i_2)$, indicated in Figure 2.2.

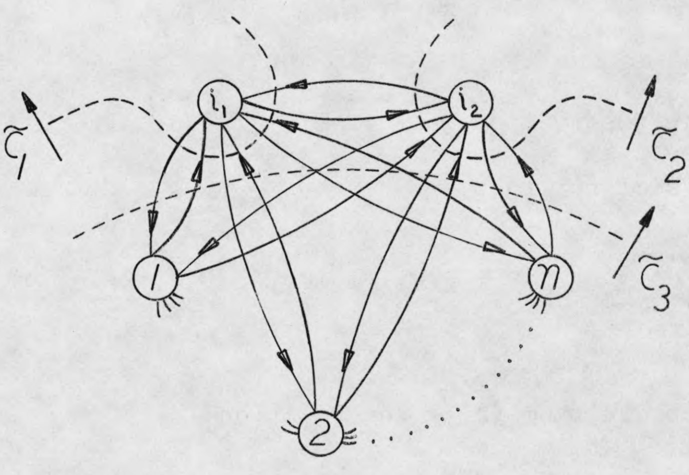


Figure 2.2

We see that

$$\tilde{c}_1 + \tilde{c}_2 = \sum_{\substack{i=1 \\ i \neq i_1}}^{n} \tilde{e}_{ii_1} + \sum_{\substack{i=1 \\ i \neq i_2}}^{n} \tilde{e}_{ii_2}$$

† In English we have triangles and quadrangles, but when it comes to other polygons (witness) we look for a different way of saying it. The Germans, on the other hand, don't mind prefixing "Eck" with any cardinal number — hence, in particular, "die Zweiecke".

$$= \sum_{\substack{i=1\\ i\neq i_1, i_2}}^{n} (\tilde{e}_{ii_1} + \tilde{e}_{ii_2}) + \tilde{e}_{i_1 i_2} + \tilde{e}_{i_2 i_1} = \tilde{c}_3 + \tilde{e}^*_{i_1 i_2}$$

where $\tilde{c}_3 = \tilde{c}(1, 2, \ldots, i_1 - 1, i_1 + 1, \ldots, i_2 - 1, i_2 + 1, \ldots, n)(i_1, i_2)$. Thus we have $\tilde{e}^*_{i_1 i_2} = \tilde{c}_1 + \tilde{c}_2 - \tilde{c}_3 \in \tilde{C}$ which completes the proof since v_{i_1} and v_{i_2} were arbitrary.

<u>Theorem 2.2</u>: If $\left\{\tilde{\tilde{c}}_\nu = \tilde{\tilde{c}}(V_\nu)\right\}_{\nu=1,2,\ldots,n-1}$ is a basis for $\tilde{\tilde{C}}$ then $\left\{\tilde{c}_\nu = \tilde{c}(V_\nu)\right\}_{\nu=1,2,\ldots,n-1} \bigcup \left\{\tilde{e}^*_{ij}\right\}_{1 \leq i < j \leq n}$ is a basis for $\tilde{C}$.

<u>Proof</u>: Lemma 2.1 establishes the fact that $\left\{\tilde{c}_\nu\right\}_\nu \bigcup \left\{\tilde{e}^*_{ij}\right\}_{i,j}$ is a subset of $\tilde{C}$; it will suffice to show that any semi-cut $\tilde{c}(V_o)$ has a unique representation as a linear combination of elements of this subset.

Since $\left\{\tilde{\tilde{c}}_\nu\right\}$ is a basis for $\tilde{\tilde{C}}$ it follows that we can write

$$\tilde{\tilde{c}}(V_o) = \sum_{\nu=1}^{n-1} \alpha_\nu(V_o)\, \tilde{\tilde{c}}_\nu \quad . \tag{2.1}$$

Then the simple observation that if $\tilde{\tilde{c}}(V)$ is any directed cut set and $\tilde{c}(V)$ the corresponding semi-cut then

$$\tilde{\tilde{c}}(V) = 2\tilde{c}(V) - \sum_{\tilde{e}_{ij} \in \tilde{c}(V)} \tilde{e}^*_{ij}$$

allows us to obtain from (2.1) the relation

$$2\tilde{c}(V_o) - \sum_{\tilde{e}_{ij} \in \tilde{c}(V_o)} \tilde{e}^*_{ij} = \sum_{\nu=1}^{n-1} \alpha_\nu (V_o) \left(2\tilde{c}_\nu - \sum_{\tilde{e}_{ij} \in \tilde{c}_\nu} \tilde{e}^*_{ij}\right),$$

which readily reduces to the desired form:

$$\tilde{c}(V_o) = \sum_{\nu=1}^{n-1} \alpha_\nu(V_o)\, \tilde{c}_\nu + \sum_{\substack{i,j=1\\ i<j}}^{n} \beta_{ij} \tilde{e}^*_{ij} \quad . \tag{2.2}$$

Suppose that this representation isn't unique, i.e. there exist constants α'_ν and β'_{ij}, which aren't all equal to the corresponding constants $\alpha_\nu\,(V_o)$ and β_{ij}, such that

$$\tilde{c}(V_o) = \sum_{\nu=1}^{n-1} \alpha'_\nu \; \tilde{c}_\nu + \sum_{\substack{i,j=1 \\ i<j}}^{n} \beta'_{ij} \; \tilde{e}^*_{ij} \; .$$

From these two representations of $\tilde{c}(V_o)$ we obtain the relation

$$\sum_{\nu=1}^{n-1} [\alpha_\nu(V_o) - \alpha_\nu] \; \tilde{c}_\nu + \sum_{\substack{i,j=1 \\ i<j}}^{n} [\beta_{ij} - \beta_{ij}] \; \tilde{e}^*_{ij} = 0$$

since for each ν we have $\tilde{c}_\nu = \frac{1}{2}\left[\tilde{\tilde{c}}_\nu + \sum_{\tilde{e}_{ij} \in \tilde{c}_\nu} \tilde{e}^*_{ij}\right]$, this last relation implies the following relation between the $\tilde{\tilde{c}}_\nu$ and the $\tilde{e}^*_{ij}$:

$$\sum_{\nu=1}^{n-1} [\alpha_\nu(V_o) - \alpha_\nu] \; \tilde{\tilde{c}}_\nu = - \sum_{\nu=1}^{n-1} \left[[\alpha_\nu \; (V_o) - \alpha_\nu] \sum_{\tilde{e}_{ij} \in \tilde{c}_\nu} \tilde{e}^*_{ij} \right]$$

$$-2 \sum_{\substack{i,j=1 \\ l<j}}^{n} [\beta_{ij} - \beta_{ij}] \; \tilde{e}^*_{ij} \quad ; \qquad (2.3)$$

but on the left side of this equation the kl - entry is the negative of the lk - entry while on the right side it is evident that the kl- and lk- entries are equal; this requires that both sides equal the zero matrix, $\tilde{0}$; but

$\sum_{\nu=1}^{n-1} [\alpha_\nu(V_o) - \alpha_\nu] \; \tilde{\tilde{c}}_\nu = \tilde{0}$ implies that $\alpha_\nu(V_o) = \alpha'_\nu$ for each ν, since the $\tilde{\tilde{c}}_\nu$ are linearly independent. Thus equation (2.3) reduces to

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} [\beta_{ij} - \beta'_{ij}] \; \tilde{e}^*_{ij} = \tilde{0}$$

and the linear independence of the members of $\left\{\tilde{e}^*_{ij}\right\}_{1\leq i<j\leq n}$ requires that $\beta_{ij} = \beta'_{ij}$ for each (i,j). This completes the proof of the uniqueness of the representation (2.2), hence of the theorem.

All that remains for a proof of Theorem 2.1 is to count the number of elements of the set $\left\{\tilde{c}_\nu\right\}_\nu \cup \left\{\tilde{e}^*_{ij}\right\}_{i,j}$, which, by Theorem 2.2, is a basis for $\tilde{C}$: but $\left\{\tilde{c}_\nu\right\}_\nu$ has cardinality $n-1$, $\left\{\tilde{e}^*_{ij}\right\}_{i,j}$ has cardinality $(n-1)n/2$, and $\left\{\tilde{c}_\nu\right\}_\nu \cap \left\{\tilde{e}^*_{ij}\right\}_{i,j} = \phi$, so this number is

$$(n-1) + (n-1)n/2 = (n-1)(n+2)/2, \text{ Q.E.D.}$$

2.3 Pseudo bases

Let us call a set $\left\{\tilde{c}_\nu\right\}_{\nu=1,2,\ldots,b}$ of semi-cuts a pseudo basis for $\tilde{C}$ if $\left\{\tilde{c}_\nu\right\}_{\nu=1,2,\ldots,b} \cup \left\{\tilde{e}^*_{ij}\right\}_{1\leq i<j\leq n}$ is a basis for $\tilde{C}$. (Then for a given pseudo basis $\left\{\tilde{c}_\nu\right\}_\nu$, each proper subset $\overline{V}$ of the vertices of the graph uniquely determines real numbers $\alpha_\nu(V)$ and $\beta_{ij}(V)$ such that

$$\tilde{c}(V) = \sum_{\nu=1}^{b} \alpha_\nu(V) \; \tilde{c}_\nu + \sum_{\substack{i,j=1 \\ i<j}}^{n} \beta_{ij}(V) \; \tilde{e}^*_{ij} \Big) .$$

Theorem 2.2 guarantees the existence of pseudo bases, and Theorem 2.1 requires that b = n-1 for any pseudo basis. It is natural to ask whether or not there exist pseudo bases besides those characterized by Theorem 2.2; this question is answered in the negative by

Theorem 2.3: $\left\{\tilde{c}_\nu\right\}_{\nu=1,2,\ldots,n-1}$ is a pseudo basis for $\tilde{C}$ if and only if $\left\{\tilde{\tilde{c}}_\nu\right\}_{\nu=1,2,\ldots,n-1}$ is a basis for $\tilde{\tilde{C}}$.

Proof: Theorem 2.2 proves that the condition is sufficient. To see that it is also necessary, let $\left\{\tilde{c}_\nu\right\}_\nu$ be a pseudo basis and $\tilde{\tilde{c}}(V_o)$ be any directed cut set. Then we have

$$\tilde{c}(V_o) = \sum_\nu \alpha_\nu(V_o)\,\tilde{c}_\nu + \sum_{i<j} \beta_{ij}(V_o)\,\tilde{e}^*_{ij} \tag{2.4}$$

$$\tilde{c}(\bar{V}_o) = \sum_\nu \alpha_\nu(\bar{V}_o)\,\tilde{c}_\nu + \sum_{i<j} \beta_{ij}(\bar{V}_o)\,\tilde{e}^*_{ij} \tag{2.5}$$

by observing that $\tilde{c}(V_o) + \tilde{c}(\bar{V}_o) = \sum_{\tilde{e}_{ij} \in \tilde{c}(V_o)} \tilde{e}^*_{ij}$ we see that $\alpha_\nu(\bar{V}_o) \equiv -\alpha_\nu(V_o)$ since otherwise equation (2.4) and (2.5) would yield a non-trivial relation among the members of the basis $\left\{\tilde{c}_\nu\right\}_\nu \bigcup \left\{\tilde{e}^*_{ij}\right\}_{i<j}$. Thus from (2.4) and (2.5) we have

$$\begin{aligned}\tilde{\tilde{c}}(V_o) &= \tilde{c}(V_o) - \tilde{c}(\bar{V}_o)\\ &= \sum_\nu 2\,\alpha_\nu(V_o)\,\tilde{c}_\nu + \sum_{i<j} [\beta_{ij}(V_o) - \beta_{ij}(\bar{V}_o)]\,\tilde{e}^*_{ij}\ ;\end{aligned}$$

with this, the relations $\tilde{c}_\nu = 1/2 \left(\tilde{\tilde{c}}_\nu + \sum_{\tilde{e}_{ij} \in \tilde{c}_\nu} \tilde{e}^*_{ij} \right)$ yield

$$\underbrace{\tilde{\tilde{c}}(V_o)}_{\substack{(i,j)\text{-entry} \triangleq a_{ij} \\ (j,i)\text{-entry} \triangleq -a_{ij}}} = \underbrace{\sum_\nu \alpha_\nu(V_o)\, \tilde{\tilde{c}}_\nu}_{\substack{(i,j)\text{-entry} \triangleq b_{ij} \\ (j,i)\text{-entry} \triangleq -b_{ij}}} + \underbrace{\sum_\nu \alpha_\nu(V_o) \sum_{\tilde{e}_{ij} \in \tilde{c}_\nu} \tilde{e}^*_{ij} + \sum_{i<j} [\beta_{ij}(V_o) - \beta_{ij}(\bar{V}_o)]\tilde{e}^*_{ij}}_{\substack{(i,j)\text{-entry} \triangleq c_{ij} \\ (j,i)\text{-entry} \triangleq c_{ij}}}$$

With the indicated definitions we see that $a_{ij} = b_{ij} + c_{ij}$ and $-a_{ij} = -b_{ij} + c_{ij}$; thus $c_{ij} = 0$ and we have

$$\tilde{\tilde{c}}(V_o) = \sum_\nu \alpha_\nu(V_o)\, \tilde{\tilde{c}}_\nu \quad ;$$

since $\tilde{\tilde{c}}(V_o)$ was an arbitrary directed cut set we thus see that $\tilde{\tilde{c}}_\nu$ ν at least contains a basis for $\tilde{\tilde{C}}$; and since the dimension of $\tilde{\tilde{C}}$ is exactly n-1 we have proven that $\left\{\tilde{\tilde{c}}_\nu\right\}_\nu$ is in fact a basis, Q.E.D.

Later, when we begin the synthesis study which is this paper's raison d'être, we will be interested in pseudo bases with the properties (1) $\beta_{ij}(V) \geq 0$ for all i,j, and V; and (2) $\beta_{ij}(V) \neq 0$ implies either $\tilde{e}_{ij} \in \tilde{c}(V)$ or $\tilde{e}_{ji} \in \tilde{c}(V)$. That not all pseudo bases have these properties is shown by the pseudo basis

$$\left\{ \tilde{c}_1 = \tilde{c}(1,3,4)(2),\ \tilde{c}_2 = \tilde{c}(1,2,4)(3),\ \tilde{c}_3 = \tilde{c}(1,3,2)(4) \right\}$$

for the 4-vertex graph (since, for example, $\mathfrak{C}(1,2)(3,4) = \mathfrak{C}_2 + \mathfrak{C}_3 - \mathfrak{E}^*_{34}$, so that $\beta_{34}(\{v_1, v_2\}) < 0$ and neither $\mathfrak{E}_{34}$ nor $\mathfrak{E}_{43}$ is an edge of $\mathfrak{C}(\{v_1, v_2\})$).

The following pair of theorems shows that not only are the properties (1) and (2) compatible, they are equivalent.

Theorem 2.4: $\{c_\nu\}_{\nu=1,2,\ldots,n-1}$ is a pseudo basis with the property that $\beta_{ij}(V) \geq 0$ for all i,j and V if and only if for each pair (i,j) either $\mathfrak{E}_{ij}$ is in no $\mathfrak{C}_\nu$ or else $\mathfrak{E}_{ji}$ is in no $\mathfrak{C}_\nu$.

Proof: If for each pair (i,j) either $\mathfrak{E}_{ij}$ is in no $\mathfrak{C}_\nu$ or $\mathfrak{E}_{ji}$ is in no $\mathfrak{C}_\nu$ then in any linear combination $\sum_\nu \alpha_\nu(V)\,\mathfrak{C}_\nu$ either the (i,j)-entry is zero or else the (j,i)-entry is zero; suppose that for some proper subset of the vertices, say V_o, and some pair of integers, say (i_o, j_o) with $1 \leq i_o < j_o \leq n$, it were the case that $\beta_{i_o j_o}(V_o) < 0$; then either the (i_o, j_o) - or the (j_o, i_o) - entry of $\sum_\nu \alpha_\nu(V_o)\,\mathfrak{C}_\nu + \sum_{i<j} \beta_{ij}(V_o)\,\mathfrak{E}^*_{ij}$ would be negative (depending upon whether it is $\mathfrak{E}_{i_o j_o}$ or $\mathfrak{E}_{j_o i_o}$ which appears in no member of $\{c_\nu\}_\nu$); either case is impossible since both the (i_o, j_o) - and (j_o, i_o) - entries of $\mathfrak{C}(V_o)$ are nonnegative (all entries of every semi-cut vector are nonnegative).

Suppose, on the other hand, that the pseudo basis $\{\mathfrak{C}_\nu\}_\nu$ yields $\beta_{ij}(V) \geq 0$ for all i,j and V but that one of the semi-cuts in the pseudo basis, say $\mathfrak{C}_{\nu_1}$, contains edge $\mathfrak{E}_{i_o j_o}$ and another, say $\mathfrak{C}_{\nu_2}$, contains $\mathfrak{E}_{j_o i_o}$. Then, as indicated in Figure 2.3, these

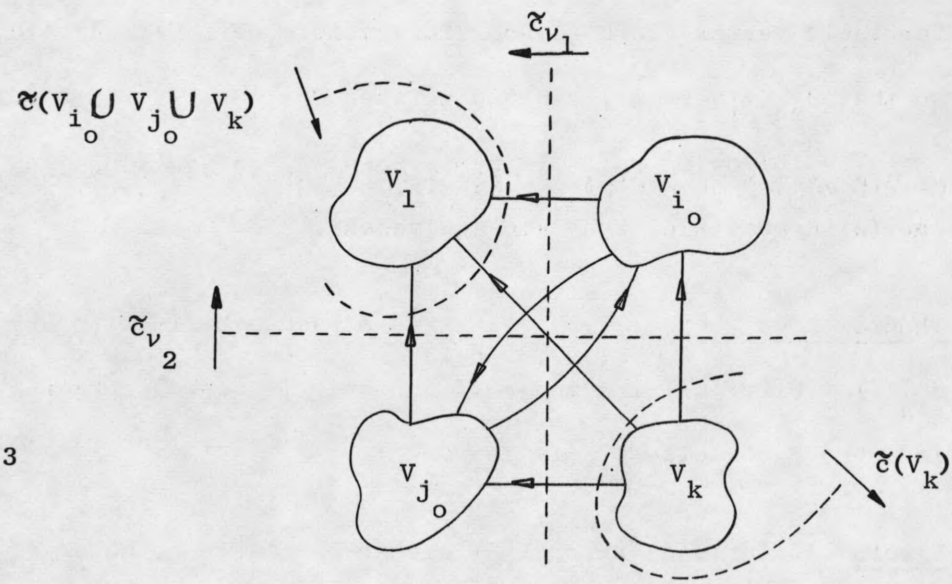


Figure 2.3

two semi-cuts partition the vertices into four subsets: V_{i_o}, V_{j_o}, V_k, and V_1, where $v_{i_o} \in V_{i_o}$ and $v_{j_o} \in V_{j_o}$ (if $\tau_{\nu_1} = \tau(V_{\nu_1})$ and $\tau_{\nu_2} = \tau(V_{\nu_2})$ then $V_{i_o} = V_{\nu_o} \cap \bar{V}_{\nu_2}$, $V_{j_o} = \bar{V}_{\nu_1} \cap V_{\nu_2}$, $V_k = V_{\nu_1} \cap V_{\nu_2}$, and $V_1 = \bar{V}_{\nu_1} \cap V_{\nu_2}$). It is readily verified that

$$\tau_{\nu_1} + \tau_{\nu_2} - \sum_{\substack{v_\sigma \in V_{i_o} \\ v_\tau \in V_{j_o}}} \tau^*_{\sigma\tau} - \tau(V_k) = \tau(V_{i_o} \cup V_{j_o} \cup V_k) \quad ;$$

if we substitute into this the representation of $\tau(V_k)$ (as a linear combination of the elements of the basis) then we obtain the (unique) representation of $\tau(V_{i_o} \cup V_{j_o} \cup V_k)$:

$$\tau(V_{i_o} \cup V_{j_o} \cup V_k) = \tau_{\nu_1} + \tau_{\nu_2} - \sum_{\substack{v_\sigma \in V_{i_o} \\ v_\tau \in V_{j_o}}} \tau^*_{\sigma\tau} - \sum \alpha_\nu(V_k)\tau_\nu - \sum_{i<j} \beta_{ij}(V_k)\tau^*_{ij}$$

from which we see that $\beta_{\sigma\tau}(V_{i_o} \cup V_{j_o} \cup V_k) = -1 - \beta_{\sigma\tau}(V_k)$ if $v_\sigma \in V_{i_o}$ and $v_\tau \in V_{i_o}$; but this means that for each such pair (σ, τ) either $\beta_{\sigma\tau}(V_k)$ or $\beta_{\sigma\tau}(V_{i_o} \cup V_{j_o} \cup V_k)$ must be negative, in contradiction to our hypothesis. This contradition completes the proof of the theorem.

<u>Theorem 2.5</u>: $\left\{\tilde{c}_\nu\right\}_\nu$ is a pseudo basis with the property that $\beta_{ij}(V) \geq 0$ for all i, j, and V if and only if one of $\tilde{e}_{ij}$, $\tilde{e}_{ji}$ is in $\tilde{c}(V)$ whenever $\beta_{ij}(V) \neq 0$.

<u>Proof</u>: Suppose that $\left\{\tilde{c}_\nu\right\}_\nu$ yields $\beta_{ij}(V) \geq 0$ for all i,j, and V and that $\beta_{i_o j_o}(V_o) \neq 0$ but neither $\tilde{e}_{i_o j_o}$ nor $\tilde{e}_{j_o i_o}$ is in $\tilde{c}(V_o)$. This means that the (i_o, j_o) - and (j_o, i_o) - entries of $\sum_\nu \alpha_\nu(V_o)\, \tilde{c}_\nu$ are both equal to $-\beta_{i_o j_o}(V_o)$, hence neither is zero; but this in turn implies that both $\tilde{e}_{i_o j_o}$ and $\tilde{e}_{j_o i_o}$ are edges of semi-cuts in the pseudo basis, in contradiction to Theorem 2.4.

To see that the condition is sufficient, suppose that one of $\tilde{e}_{ij}$, $\tilde{e}_{ji}$ is in $\tilde{c}(V)$ whenever $\beta_{ij}(V) \neq 0$ but that $\beta_{ij}(V) < 0$ for some i, j and V. Then by Theorem 2.4 we know that for some pair (i_o, j_o) the edge $\tilde{e}_{i_o j_o}$ is an edge of a semi-cut, say $\tilde{c}_{\nu_1}$, of the pseudo basis and the edge $\tilde{e}_{j_o i_o}$ is an edge of another pseudo basis semi-cut, say $\tilde{c}_{\nu_2}$. Then, in the notation of Figure 2.3, for any $v_\sigma \in V_{i_o}$ and any $v_\tau \in V_{j_o}$ neither $\tilde{e}_{\sigma\tau}$ nor $\tilde{e}_{\tau\sigma}$ is in either $\tilde{c}(V_k)$ or $\tilde{c}(V_{i_o} \cup V_{j_o} \cup V_k)$ but one of $\beta_{\sigma\tau}(V_k)$, $\beta_{\sigma\tau}(V_{i_o} \cup V_{i_o} \cup V_k)$ is nonzero (since, as we showed in the proof of Theorem 2.4, $\beta_{\sigma\tau}(V_k) + \beta_{\sigma\tau}(V_{i_o} \cup V_{j_o} \cup V_k) = -1$). This contradiction establishes the theorem.

A useful graph-theoretical characterization of the special pseudo bases we have been discussing can be conveniently expressed if we define a semi-cut $\tilde{c}(V)$ to be <u>fundamental</u> with respect to a tree if the corresponding directed cut set $\tilde{\tilde{c}}(V)$ is fundamental with respect to the same tree. It isn't difficult to see that associated with a particular tree is a unique set of n-1 fundamental semi-cuts, and we have the

Theorem 2.6: $\left\{\tilde{c}_\nu\right\}_\nu$ is a pseudo basis with the property that one of $\tilde{e}_{ij}$, $\tilde{e}_{ji}$ is in $\tilde{c}(V)$ whenever $\beta_{ij}(V) \neq 0$ if and only if $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{n-1}$ are the fundamental semi-cuts associated with a tree which is a directed path.

Proof: Let $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{n-1}$ be the fundamental semi-cuts with respect to a tree whose edges are $\tilde{e}_{i_1 i_2}, \tilde{e}_{i_2 i_3}, \ldots, \tilde{e}_{i_{n-1} i_n}$ (Figure 2.4 illustrates this for n = 4); then it is easily seen that $\tilde{c}_\nu = \tilde{c}(i_1, i_2, \ldots, i_\nu)(i_{\nu+1}, i_{\nu+2}, \ldots, i_n)$

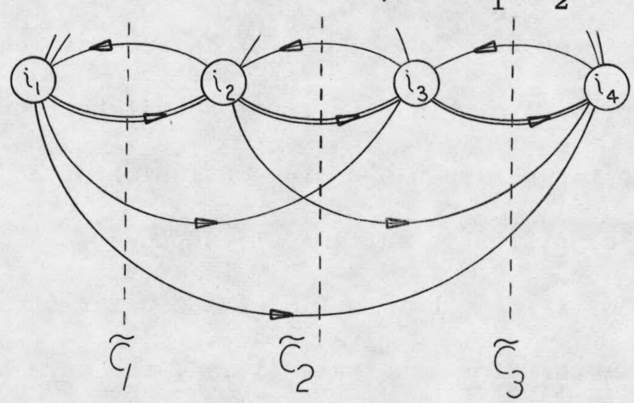


Figure 2.4

for each $\nu = 1, 2, \ldots, n-1$ so that for any pair (i_σ, i_τ) either $\tilde{e}_{i_\sigma i_\tau}$ is in no $\tilde{c}_\nu$ or $\tilde{e}_{i_\tau i_\sigma}$ is in no $\tilde{c}_\nu$ (the former if $\sigma > \tau$ and the latter if $\sigma < \tau$). Thus the sufficiency of the condition is established by Theorems 2.4 and 2.5.

On the other hand, suppose that one of $\tilde{e}_{ij}$, $\tilde{e}_{ji}$ is in $\tilde{c}(V)$ if $\beta_{ij}(V) \neq 0$. Then by Theorems 2.5 and 2.4 we know that for each pair (i,j) either $\tilde{e}_{ij}$ or $\tilde{e}_{ji}$ is in no $\tilde{c}_\nu$. Define the set V_ν by the relation $\tilde{c}_\nu = \tilde{c}(V_\nu)$, and arrange the members of $\left\{V_\nu\right\}_\nu$ in order of increasing cardinality as $V_{\nu_1}, V_{\nu_2}, \ldots, V_{\nu_{n-1}}$. Then $V_{\nu_k} \subseteq V_{\nu_{k+1}}$ for each $k = 1, 2, \ldots, n-2$: assume the contrary; then for some k_o there is a vertex $v_{i_o} \in V_{\nu_{k_o}}$ such that $v_{i_o} \in \bar{V}_{\nu_{k_o+1}}$; and since $V_{\nu_{k_o+1}}$ contains at least as many vertices as $V_{\nu_{k_o}}$ it follows that some other vertex v_{j_o} is in $V_{\nu_{k_o+1}}$ but not in $V_{\nu_{k_o}}$; thus $\tilde{e}_{i_o j_o} \in \tilde{c}_{\nu_{k_o}}$ and $\tilde{e}_{j_o i_o} \in \tilde{c}_{\nu_{k_o+1}}$; but by our hypothesis this violates Theorem 2.4 via Theorem 2.5, a contradiction which establishes the assertion that $V_{\nu_k} \subseteq V_{\nu_{k+1}}$. And since $V_{\nu_k} = V_{\nu_{k+1}}$ would mean that $\tilde{c}_{\nu_k} = \tilde{c}_{\nu_{k+1}}$ (in violation of the definition of pseudo bases) we know that in fact V_{ν_k} is a proper subset of $V_{\nu_{k+1}}$. Thus, since there are only n vertices in the graph, we must conclude that they can be ordered as v_{i_1}, $v_{i_2}, \ldots, v_{i_n}$ where $V_{\nu_1} = \left\{v_{i_1}\right\}$, $V_{\nu_2} = \left\{v_{i_1}, v_{i_2}\right\}, \ldots, V_{\nu_{n-1}} = \left\{v_{i_1}, v_{i_2}, \ldots, v_{i_{n-1}}\right\}$. It follows immediately that the semi-cuts $\tilde{c}_{\nu_1}, \tilde{c}_{\nu_2}, \ldots, \tilde{c}_{\nu_{n-1}}$ are the fundamental semi-cuts associated with the tree $\tilde{e}_{i_1 i_2}, \tilde{e}_{i_2 i_3}, \ldots, \tilde{e}_{i_{n-1} i_n}$ (a directed path), Q.E.D.

2.4 μ-matrices

Consider the subset of $\mathcal{E}_{(n-1)n}$ consisting of the matrices with the properties: (1) Each of the integers 1, 2, ..., n-1 appears as an entry above the main diagonal; (2) each of the integers n, n+1, ..., (n-1)(n+2)/2 appears as an entry below the main diagonal; (3) satisfies Mayeda's S-submatrix condition. A matrix of this subset will be called a <u>μ-matrix</u>. For a given μ-matrix M, let us denote by $S_\nu(M)$ the set of S-submatrices for the entry ν; then the μ-matrices afford the following interesting combinatorial characterization of a class of bases for $\tilde{C}$, each member of which contains only semi-cuts (as contrasted with the class of bases associated with the pseudo bases of $\tilde{C}$):

<u>Theorem 2.7</u>: Given a μ-matrix M, let us select (in any fashion) from each set $S_\nu(M)$ a single submatrix $M(V_\nu)$; then $\left\{\tilde{c}_\nu = \tilde{c}(V_\nu)\right\}_{\nu=1,2,\ldots,(n-1)(n+2)/2}$ is a basis for $\tilde{C}$.

<u>Proof</u>: The semi-cut $\tilde{c}_\nu$ contains one edge which is in none of the semi-cuts $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{\nu-1}$ (the corresponding S-submatrix $M(V_\nu)$ contains the entry ν which is in none of the S-submatrices $M(V_1), M(V_2), \ldots, M(V_{\nu-1})$, whose largest entries are $1, 2, \ldots, \nu-1$ respectively). Thus $\left\{\tilde{c}_\nu\right\}_\nu$ constitutes a linearly independent subset of $\tilde{C}$ whose cardinality is the dimension of $\tilde{C}$, Q.E.D.

Finally, a class of bases for $\tilde{C}$ which contains members of both of the classes discussed previously is established by

<u>Theorem 2.8</u>: Let M be a μ-matrix and suppose that it is the (i_ν, j_ν)-entry of M which equals ν; if we select (in an arbitrary manner) from each set $S_\nu(M)$ a single submatrix $M(V_\nu)$ for $\nu = 1, 2, \ldots, \nu_o - 1$ (where $\nu_o \geq n$) then the subset $\left\{\gamma_\nu\right\}_\nu$ of $\tilde{C}$ is a basis for $\tilde{C}$, where

$$\tilde{\gamma}_\nu = \begin{cases} \tilde{c}(V_\nu) & \text{for } \nu = 1, 2, \ldots, \nu_o - 1 \\ \tilde{c}^*_{i_\nu j_\nu} & \text{for } \nu = \nu_o, \nu_o + 1, \ldots, (n-1)(n+2)/2 \end{cases}$$

The proof of Theorem 2.8 is so similar to that of Theorem 2.7 that it seems unnecessary to state it explicitly. If $\left\{\tilde{\gamma}_\nu\right\}_\nu$ is one of the bases described in Theorem 2.8 then let $\alpha_\nu(V)$ be the coefficient of $\tilde{\gamma}_\nu$ in the representation

$$\tilde{c}(V) = \sum_\nu \alpha_\nu(V) \; \tilde{\gamma}_\nu$$

of the semi-cut $\mathfrak{C}(V)$; then an interesting (and obvious) property of the basis is that, for $\nu \geq n$ and $\alpha_\nu(V) \neq 0$, $M(V) \in S_\nu(M)$ and $\alpha_\nu(V) = 1$ and $\alpha_{\nu+p}(V) = 0$ for $p = 1,2,\ldots$ (this is a generalization of a property, suggested by the theorems of 2.3, of the bases associated with those special pseudo bases for which $\beta_{ij}(V) \neq 0$ implies either $\mathfrak{S}_{ij}$ or $\mathfrak{S}_{ji} \in \mathfrak{C}(V)$).

3. BI-CIRCUIT TRANSFORMATION OF NETWORKS

3.1 Introduction

In this chapter we take a brief look at the orthogonal complement of $\tilde{C}$ in $\mathcal{E}_{(n-1)n}$, then pass on to some corollary results in the theory of networks. In particular, we obtain a group of transformations, from the set of all networks onto itself, which preserve terminal capacity matrices; the group is seen to be generated by a finite set of more elementary transformations (called "bi-circuit transformations") which have a simple graph theoretical characterization.

3.2 Bi-circuits

A directed circuit $\tilde{K}$ is a set of edges $\tilde{e}_{i_1 i_2}, \tilde{e}_{i_2 i_3}, \ldots, \tilde{e}_{i_s i_1}$ where the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$ are distinct. If we take $\tilde{k} = \sum_{\tilde{e}_{ij} \in \tilde{k}} \tilde{e}_{ij}$ as the vector representation of the directed circuit then it is well known in linear graph theory that $\tilde{K}$ (the subspace of $\mathcal{E}_{(n-1)n}$ spanned by the directed circuits of the n-vertex bi-complete graph) and $\tilde{\tilde{C}}$ are orthogonal complements in $\mathcal{E}_{(n-1)n}$, i.e. if $\tilde{\tilde{\gamma}} \in \tilde{\tilde{C}}$ and $\tilde{\kappa} \in \tilde{K}$ then the inner product $(\tilde{\gamma}, \tilde{\kappa})$ is zero, and, taken together, $\tilde{\tilde{C}}$ and $\tilde{K}$ span the whole space $\mathcal{E}_{(n-1)n}$.

Let us orient the zweieck $\tilde{e}^*_{ij}$ by ordering its vertices as (v_i, v_j), and assign to the resulting <u>directed zweieck</u> $\tilde{\tilde{e}}^*_{ij}$ the vector representation $\tilde{e}_{ij} - \tilde{e}_{ji}$. Then a <u>bi-circuit</u> $\tilde{\tilde{K}}$ is a set of directed zweiecks $\tilde{\tilde{e}}^*_{i_1 i_2}, \tilde{\tilde{e}}^*_{i_2 i_3}, \ldots, \tilde{\tilde{e}}^*_{i_s i_1}$, where $s > 2$ and the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$ are distinct; if

$\tilde{\tilde{e}}^*_{ij} \in \tilde{\tilde{K}}$ then edge $\tilde{e}_{ij}/\tilde{e}_{ji}$ is in $\tilde{\tilde{k}}$ and agrees/disagrees with $\tilde{\tilde{k}}$ in orientation.

If we take $\sum_{\tilde{\tilde{e}}^*_{ij} \in \tilde{\tilde{k}}} \tilde{\tilde{e}}^*_{ij}$ as the vector associated with the bi-circuit $\tilde{\tilde{K}}$ and denoted by $\tilde{\tilde{K}}$ the subspace of $\mathcal{E}_{(n-1)n}$ spanned by the bi-circuits of the bi-complete graph then the following two theorems establish the important fact that $\tilde{C}$ and $\tilde{\tilde{K}}$ are orthogonal complements in $\mathcal{E}_{(n-1)n}$.

Theorem 3.1: If $\tilde{\gamma} \in \tilde{C}$ and $\tilde{\tilde{k}} \in \tilde{\tilde{K}}$ then $(\tilde{\gamma}, \tilde{\tilde{k}}) = 0$.

Proof: It will suffice to show that if $\tilde{c}(V)$ is any semi-cut and $\tilde{\tilde{k}}$ any bi-circuit then $(\tilde{c}(V), \tilde{\tilde{k}}) = 0$. Elementary topological considerations show that the number of directed zweiecks $\tilde{\tilde{e}}^*_{ij}$ of $\tilde{\tilde{k}}$ with $v_i \in V$ and $v_j \in \bar{V}$ equals the number which have $v_i \in \bar{V}$ and $v_j \in V$; this means that of the edges which are in both $\tilde{c}(V)$ and $\tilde{\tilde{k}}$, as many disagree as agree with $\tilde{\tilde{k}}$ in orientation. It follows that $(\tilde{c}(V), \tilde{\tilde{k}}) = 0$ since this inner product is just a sum $\sum_{i \neq j} \delta_{ij}$ of real numbers where $\delta_{ij} = 1/-1$ if $\tilde{e}_{ij} \in \tilde{c}(V)$ $\tilde{\tilde{k}}$ and agrees/disagrees with $\tilde{\tilde{k}}$ in orientation, $\delta_{ij} = 0$ otherwise, Q.E.D.

Corollary: The dimension of $\tilde{\tilde{K}}$ is at most $(n-1)(n-2)/2$.

Proof: $\tilde{\tilde{K}}$ is orthogonal to $\tilde{C}$ by the theorem; since the dimension of $\mathcal{E}_{(n-1)n}$ is $(n-1)n$ and that of $\tilde{C}$ is $(n-1)(n+2)/2$ it follows that the dimension of $\tilde{\tilde{K}}$ is $\leq (n-1)n - (n-1)(n+2)/2$, Q.E.D.

Theorem 3.2: The dimension of $\tilde{\tilde{K}}$ is $(n-1)(n-2)/2$.

Proof: By the corollary to Theorem 3.1, it will suffice to show that the dimension of $\tilde{\tilde{K}}$ is $\geq (n-1)(n-2)/2$. Consider the tree $\tau = \left\{\tilde{e}_{1,2}, \tilde{e}_{2,3}, \ldots, \tilde{e}_{n-1,n}\right\}$ of the graph; then each directed zweieck both of whose edges are chords of

T forms with certain of the directed zweiecks in $\left\{\tilde{\tilde{e}}^*_{1,2}, \tilde{\tilde{e}}^*_{2,3}, \ldots, \tilde{\tilde{e}}^*_{n-1,n}\right\}$ a bi-circuit; the set of all such bi-circuits is linearly independent (since each contains one directed zweieck, the one both of whose edges are chords of T, which is in no other) and contains $(n-1)n/2 - (n-1) = (n-1)(n-2)/2$ bi-circuits ($(n-1)n/2$ is the number of directed zweiecks of the graph and $(n-1)$ is the number of directed zweiecks which contain an edge of T), Q.E.D.

3.3 Equivalent networks

Two n-vertex networks will be said to be equivalent if they have the same terminal capacity matrix; they will be called K-equivalent if each semi-cut of $\tilde{G}_n$ has the same value in the one as in the other (both of these are obviously equivalence relations on the set of all networks), and we have

Theorem 3.3: Two networks are equivalent if they are K-equivalent.

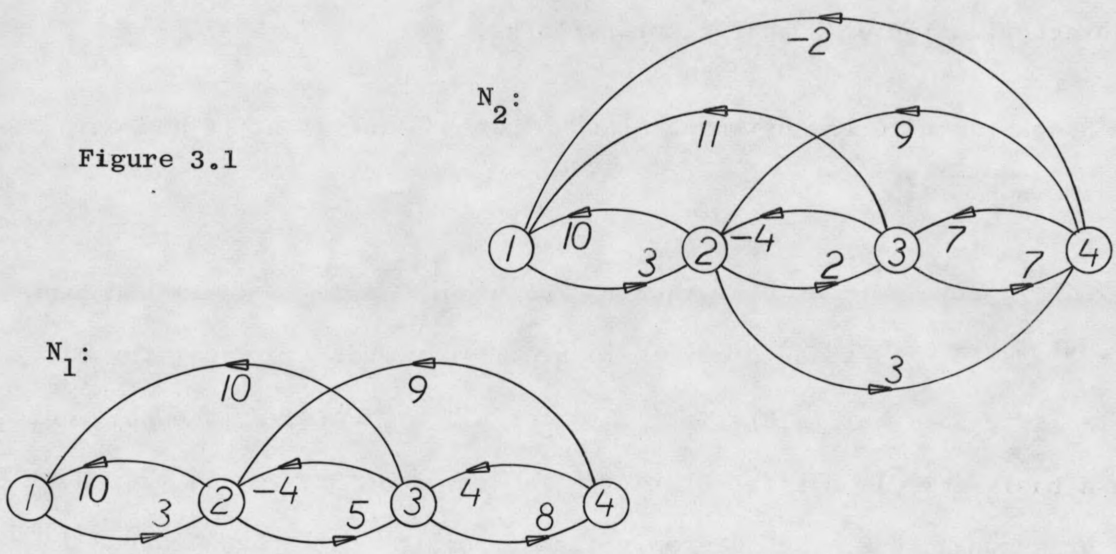
Proof: The terminal capacity matrix of a network is determined solely by the values in the network of the various semi-cuts in $\tilde{G}_n$, so the theorem follows.

Although K-equivalence is thus seen to be sufficient for equivalence, the two networks of Figure 3.1 demonstrate the unfortunate fact that K-equivalence is actually a stronger condition than equivalence (N_1 and N_2 have the same terminal capacity matrix

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 3 & 3 & 3 \\ 15 & x & 5 & 5 \\ 14 & 7 & x & 7 \\ 13 & 8 & 9 & x \end{bmatrix} \end{array}$$

but, e.g. semi-cut $\tilde{c}(3,4)(1,2)$ has the values 15 and 14 in N_1 and N_2, respectively). In fact, two networks can be equivalent and yet a semi-cut

Figure 3.1



can be minimum in one without being minimum in the other; for instance, network N of Figure 3.2 is equivalent to the network of Figure 3.1 but $\tilde{c}(1,2,3)(4)$ is minimum (for t_{34}) in N and non-minimum in N_1 and N_2.

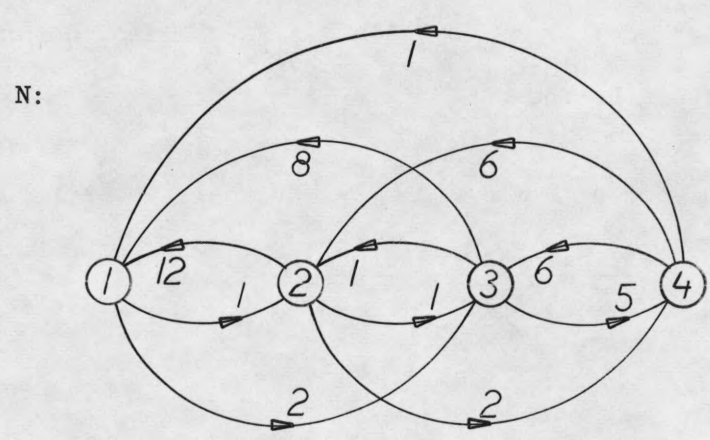


Figure 3.2

Despite the comments of the previous paragraph, the concept of K-equivalence is a useful one and will be explored in the following pages. If E and E' denote the edge weight matrices of N and N', respectively, then the following theorem is a characterization of K-equivalent networks.

Theorem 3.4: The n-vertex networks N and N' are K-equivalent if and only if E-E' is in $\tilde{\tilde{K}}$.

Proof: If $\tilde{\gamma}$ is any semi-cut of $\tilde{G}_n$ then the value of $\tilde{\gamma}$ in N is $(\tilde{\gamma}, E)$ and its value in N' is $(\tilde{\gamma}, E')$; thus to say that these two values are equal for all semi-cuts is to say that $(\tilde{\gamma}, E)-(\tilde{\gamma}, E') = 0$ for all $\tilde{\gamma} \in \tilde{C}$ (since the semi-cuts contain a basis for $\tilde{C}$); but $(\tilde{\gamma}, E) - (\tilde{\gamma}, E') = (\tilde{\gamma}, E-E')$ so we see that E-E' must be orthogonal to $\tilde{C}$; the theorem follows from the fact that $\tilde{C}$ and $\tilde{\tilde{K}}$ are orthogonal complements in $\mathcal{E}_{(n-1)n}$.

Corollary: The weight in N of each zweieck of $\tilde{G}_n$ is the same as its weight in N' if N and N' are K-equivalent.

Proof: Since each zweieck is in $\tilde{C}$ (Th.2.1) and $\tilde{\tilde{K}}$ is orthogonal to $\tilde{C}$(Th.3.1) the result follows from the theorem.

If $\tilde{\tilde{k}}$ is a bi-circuit of $\tilde{G}_n$ and ϵ is any real number then the pair $(\tilde{\tilde{k}}, \epsilon)$ determines a bi-circuit transformation of network N into network N' where

$$E' = E + \epsilon \, \tilde{\tilde{k}}$$

and we have the

Theorem 3.5: Two networks are K-equivalent if and only if either can be obtained from the other by a sequence of bi-circuit transformations.

Proof: Suppose that N' can be obtained from N by the sequence of bi-circuit transformations determined by the pairs $(\tilde{\tilde{k}}_1, \epsilon_1), (\tilde{\tilde{k}}_2, \epsilon_2), \ldots, (\tilde{\tilde{k}}_s, \epsilon_s)$; then $\left(E'-E = \sum_{\sigma=1}^{s} \epsilon_\sigma \tilde{\tilde{k}}_\sigma\right) \epsilon\ \tilde{\tilde{K}}$ and N is K-equivalent to N' by Theorem 3.4. Suppose, on the other hand, that N and N' are K-equivalent; then by Theorem 3.4 we know that $E'-E \in \tilde{\tilde{K}}$ so, if $\left\{\tilde{\tilde{k}}_\mu\right\}_\mu$ is any set of bi-circuits which is a basis for $\tilde{\tilde{K}}$, there exists a set of real numbers $\left\{\epsilon_\mu\right\}_\mu$ such that $E'-E = \sum_\mu \epsilon_\mu \tilde{\tilde{k}}_\mu$; thus N' can be obtained from N by the sequence of bi-circuit transformations determined by the pairs in the set $\left\{(\tilde{\tilde{k}}_\mu, \epsilon_\mu)\right\}_\mu$. Since N' can be obtained from N by a sequence of bi-circuit transformations $(\tilde{\tilde{k}}_1, \epsilon_1)$, $(\tilde{\tilde{k}}_2, \epsilon_2), \ldots$ if and only if N can be obtained from N' by the sequence $(\tilde{\tilde{k}}_1, -\epsilon_1), (\tilde{\tilde{k}}_2, -\epsilon_2), \ldots,$ the theorem follows.

Let us refer to a subgraph of $\tilde{G}_n$ composed entirely of zweiecks as a z-tree, z-cotree, etc. if the subgraph of G_n obtained by considering each zweieck as a single (nonoriented) edge is a tree, cotree, etc. Then the concept of K-equivalence allows us to prove that the weight of one component of each zweieck in any z-cotree can be specified arbitrarily, i.e.

Theorem 3.6: Any network N is K-equivalent to a network N' where the weight in N' of one component of each zweieck in any z-cotree of $\tilde{G}_n$ is arbitrary.

Proof: It will suffice to show that if τ is any z-tree of $\tilde{G}_n$ and $\tilde{e}_{i_o j_o}$ is a component of any zweieck in the z-cotree of τ then there is a bi-circuit transformation relating N to a network N' in which $e_{i_o j_o}$ is an arbitrary real number r and the weights in N' of the edges of all zweiecks (except $\tilde{e}^*_{i_o j_o}$)

in the z-cotree of τ are the same as in N. But $\tilde{e}^*_{i_o j_o}$ forms with certain of the zweiecks of τ a z-circuit whose zweiecks can be oriented to form a bi-circuit $\tilde{\tilde{k}}$ with which $\tilde{e}_{i_o j_o}$ agrees in orientation ($\tilde{\tilde{k}}$ is unique, but this fact is immaterial); then the bi-circuit transformation determined by the pair ($\tilde{\tilde{k}}$, $r-e_{i_o j_o}$) is readily seen to yield a network N' with the desired property, Q.E.D.

If we agree to call an edge of $\tilde{G}_n$ a "true edge" of a network only if its weight in N is nonzero then we have the

<u>Corollary</u>: No terminal capacity matrix requires for its realization a network with more than (n-1)(n+2)/2 true edges.

<u>Proof</u>: By the theorem, any realization N of the terminal capacity matrix is equivalent to a network N' in which the weight of one component of each zweieck of any z-cotree of $\tilde{G}_n$ is 0; such an N' has at most a z-tree (with 2(n-1) edges) and half of a z-cotree (with (n-1)(n-2)/2 edges) of true edges, Q.E.D.

The following weaker theorem can be proven when we restrict ourselves to communication networks:

<u>Theorem 3.7</u>: Any communication network N is K-equivalent to a communication network N' which has no z-circuit of true zweiecks.

<u>Proof</u>: It will suffice to show that if N has at least one z-circuit of true zweiecks then it is K-equivalent to a communication network N' with at least one fewer z-circuits of true zweiecks. But suppose that $\left\{\tilde{e}^*_{i_\sigma j_\sigma}\right\}_{\sigma=1,2,\ldots,s}$

is a true z-circuit of N and that $\tilde{e}_{i_1 j_1}$ is the edge of this z-circuit which has the smallest edge weight; then the zweiecks in $\left\{\tilde{e}^*_{i_\sigma i_\sigma}\right\}_\sigma$ can be oriented to produce a bi-circuit $\approx{k}$ which agrees with $\tilde{e}_{i_1 j_1}$ in orientation, and we see that the bi-circuit transformation determined by $(\approx{k}, - e_{i_1 j_1})$ yields a network N' with the desired property.

Corollary: If a terminal capacity matrix has any realization by a communication network then it has one with no more than (n-1)(n+2)/2 true edges.

Proof: If communication network N realizes the terminal capacity matrix then the communication network N' of the theorem is also a realization and, since it has no z-circuit of true zweiecks, contains at most a z-tree of true zweiecks and the corollary follows.

The final theorem of this chapter expresses an important relationship between networks and communication networks.

Theorem 3.8: A network N is K-equivalent to a communication network N' if and only if each terminal capacity and the weight of each zweieck of N is nonnegative.

Proof: Since bi-circuit transformations preserve the values of terminal capacities and the weights of zweiecks, the "only if" part of the theorem is immediate. Thus, assume the conditions of the converse; it will suffice to show that if N contains a negatively weighted edge then it is K-equivalent to a network with at least one fewer such edges. To do this we will construct a finite sequence of networks $N_o, N_1, \ldots, N_s$ where N_s is the desired network.

Suppose that edge $\tilde{e}_{i_o j_o}$ of $\tilde{G}_n$ has negative weight in N. Take $N_o = N$ and suppose that the networks $N_1, N_2, \ldots, N_{\sigma-1}$ have been constructed and that the weight in $N_{\sigma-1}$ of $\tilde{e}_{i_o j_o}$ is $e^{(\sigma-1)}_{i_o j_o} < 0$, $N_{\sigma-1}$ is K-equivalent to N, the negatively weighted edges in $N_{\sigma-1}$ form a subset of the negatively weighted edges in N, and let $\tilde{\pi}_\sigma = \left\{\tilde{e}_{i_o i_1}, \tilde{e}_{i_1 i_2}, \ldots, \tilde{e}_{i_{k-1} i_k}, \tilde{e}_{i_k j_o}\right\}$ be any directed path from vertex v_{i_o} to vertex v_{j_o} with the properties: (1) the edges of $\tilde{\pi}_\sigma$ are all true edges of $N_{\sigma-1}$ and (2) at least one edge of $N_{\sigma-1}$ has positive weight in $N_{\sigma-1}$. Such a $\tilde{\pi}_\sigma$ exists: for assume the contrary, that all edges of all directed paths of $\tilde{G}_n$ from v_{i_o} to v_{j_o} have nonpositive weights in $N_{\sigma-1}$; since any semi-cut $\tilde{c}$ of $\tilde{G}_n$ which separates v_{i_o} from v_{j_o} is composed entirely of edges of these paths it would follow that the value in $N_{\sigma-1}$ of any such semi-cut would be $c \leq e^{(\sigma-1)}_{i_o j_o} < 0$; thus the terminal capacity from v_{i_o} to v_{j_o} in $N_{\sigma-1}$ would be $t^{(\sigma-1)}_{i_o j_o} \leq c < 0$; but since $N_{\sigma-1}$ is K-equivalent to N by our induction hypothesis we know that $t^{(\sigma-1)}_{i_o j_o} = t_{i_o j_o} > 0$; this contradiction establishes the existence of $\tilde{\pi}_\sigma$. If $\tilde{\tilde{k}}_\sigma$ is the bi-circuit $\left\{\tilde{e}^*_{j_o i_o}, \tilde{e}^*_{i_o i_1}, \tilde{e}^*_{i_1 i_2}, \ldots, \tilde{e}^*_{i_{n-1} i_k}, \tilde{e}^*_{i_k j_o}\right\}$ and $\epsilon_\sigma = \min\left\{e_{ij} \;\middle|\; e_{ij} > 0 \text{ and } \tilde{e}_{ij} \in \tilde{\pi}_\sigma\right\}$ then we now distinguish two cases: (a) $-e_{i_o j_o} \leq \epsilon_\sigma$ and (b) $-e_{i_o j_o} > \epsilon_\sigma$. In case (a) let N_s be the network obtained from $N_{\sigma-1}$ by the bi-circuit transformation $(\tilde{\tilde{k}}_\sigma, e_{i_o j_o})$; N_s is K-equivalent to $N_{\sigma-1}$ by Theorem 3.5, hence to N by our induction hypothesis and the transitivity of equivalence; the weight in N_s of $\tilde{e}_{i_o j_o}$ is zero and from this, the K-equivalence of N_s and N, the fact that the weight of a zweieck is the same in two K-equivalent networks, and the hypothesis that the weight of each zweieck of $\tilde{G}_n$ is nonnegative in N, we see that the weight in N_s of $\tilde{e}_{j_o i_o}$ is nonnegative; if the weight in $N_{\sigma-1}$ of an edge of $\tilde{\pi}_\sigma$ was positive then, since it is less in N_s by only the amount $-e^{(\sigma-1)}_{i_o j_o} < \epsilon_\sigma$, it is still positive in N_s; if an edge of the path $\left\{\tilde{e}_{j_o i_k}, \tilde{e}_{i_k i_{k-1}}, \ldots, \tilde{e}_{i_2 i_1}, \tilde{e}_{i_1 i_o}\right\}$ had nonnegative weight in $N_{\sigma-1}$ then it has positive weight in N_s since its weight in N_s is larger by the positive amount $-\, e^{(\sigma-1)}_{i_o j_o}$; from the foregoing we can conclude that the negatively weighted edges of N_s are a subset of the negatively weighted edges of $N_{\sigma-1}$ and hence of N by our induction hypothesis — in fact, a proper subset since edge $\tilde{e}_{i_o j_o}$ has negative weight in N and zero weight in N_s. In case (b) we let N_σ be the network obtained from $N_{\sigma-1}$ with the bi-circuit transformation determined by $(\tilde{\tilde{k}}_\sigma, -\epsilon_\sigma)$; then arguments similar to those used in case (a) show that N_σ is K-equivalent to N and the negatively weighted edges of N_σ are a subset of the negatively weighted edges of N; and since the weight in N_σ of $\tilde{e}_{i_o j_o}$ is $e^{(\sigma)}_{i_o j_o} = e^{(\sigma-1)}_{i_o j_o} + \epsilon_\sigma < 0$ we have successfully completed the σth induction step. One further relationship between N_σ and $N_{\sigma-1}$ is needed in order that we may guarantee the termination of this procedure after a finite number of steps; such a relationship is: the number of directed paths of $\tilde{G}_n$ from v_{i_o} to v_{j_o} which are composed to true edges is one fewer in N_σ than in $N_{\sigma-1}$ (this is readily verified); since there are only a finite number of such paths in N, the process must terminate after a finite number of steps, Q.E.D.

4. REALIZATION OF NETWORKS WITH SPECIFIED TERMINAL CAPACITY MATRICES

4.1 Introduction

In this chapter we shall show that any real, square matrix which satisfies Mayeda's S-submatrix condition is the terminal capacity matrix of some network. For simplicity of presentation we will treat the problem in three cases of increasing complexity: in 4.1 we assume that the matrix T to be realized has the maximum number of distinct entries; in 4.2 we assume only that T has unique S-submatrices for its above-diagonal† entries $t_{\nu,\nu+1}$ ($\nu=1,2,\ldots,n-1$); and in 4.3 we retain only the necessary condition which was assumed throughout.

4.2 Maximum number of distinct entries

Throughout this section we will be considering a real, square matrix T which

A. satisfies the S-submatrix condition; and

B. has $(n-1)(n+2)/2$ distinct entries.

Under these conditions we have

Theorem 4.1: The semi-cuts in $\left\{\tilde{c}_\nu = \tilde{c}(1,2,\ldots,\nu)(\nu+1,\nu+2,\ldots,n)\right\}_\nu$ are the semi-cuts determining $t_{\nu,\nu+1}$ in any network whose terminal capacity matrix is T.

Proof: Since only the n-1 smallest entries of T appear above the main diagonal it is clear that no S-submatrix for $t_{\nu,\nu+1}$ contains an entry t_{ij} below the main diagonal (hypothesis B requires that all such entries be strictly greater than $t_{\nu,\nu+1}$; but of all the submatrices T(V) which contain $t_{\nu,\nu+1}$ only $T(1,2,\ldots,\nu)(\nu+1,\nu+2,\ldots,n)$ contains no entry t_{ij} with $i>j$; since by hypothesis

† Throughout this chapter we assume that the rows and columns of T have been arranged so that without rearrangment T can be partitioned as in Tang and Chien's theorem.

A we know that $t_{\nu,\nu+1}$ has an S-submatrix it follows from the above that it has exactly one, namely $T(1,2,\ldots,\nu)(\nu+1,\nu+2,\ldots,n)$. Thus, by Theorem 1.1 we see that it is semi-cut $\tilde{c}(1,2,\ldots,\nu)(\nu+1,\nu+2,\ldots,n)$ which determines $t_{\nu,\nu+1}$ in any network realizing T, Q.E.D.

Since the semi-cuts $\tilde{c}_1, \tilde{c}_2,\ldots,\tilde{c}_{n-1}$ of Theorem 4.1 are the fundamental semi-cuts with respect to the path-tree whose edges are $\tilde{e}_{12}, \tilde{e}_{23},\ldots,\tilde{e}_{n-1,n}$ it follows from Theorem 2.6 that they constitute a pseudo-basis for $\tilde{C}$ with properties:

(1) $\beta_{ij}(V)$ equals either 0 or 1 for any semi-cut $\tilde{c}(V)$ and any integer pair (i,j) with $1 \leq i < j \leq n$;

(2) $\beta_{ij}(V) = 1$ (and $i<j$) if and only if edge $\tilde{e}_{ji}$ is in semi-cut $\tilde{c}(V)$.

<u>Theorem 4.2</u>: Let $t_{i_o j_o}$ be any entry below the main diagonal of T, and T(V) be any S-submatrix for $t_{i_o j_o}$. Then the value in any network realizing T of semi-cut $\tilde{c}(V)$ is completely determined by the values of $\tilde{c}_1, \tilde{c}_2,\ldots,\tilde{c}_{n-1}$ and the weights of the zweiecks $\tilde{e}^*_{ij}$ $(i<j)$ such that $t_{ji} \leq t_{i_o j_o}$.

<u>Proof</u>: By property (1) of the pseudo basis $\{\tilde{c}_\nu\}$ we know that $\beta_{ij}(V) \neq 0$ implies that $\beta_{ij}(V) = 1$; and by property (2) that $\beta_{ij}(V) = 1$ (and $i<j$) implies that edge $\tilde{e}_{ji}$ is in $\tilde{c}(V)$; but $\tilde{e}_{ji} \in \tilde{c}(V)$ implies $t_{ji} \in T(V)$; and since T(V) is an S-submatrix for $t_{i_o j_o}$ it follows that $t_{ji} \in T(V)$ implies $t_{ji} \leq t_{i_o j_o}$. Thus, in the representation

$$\tilde{c}(V) = \sum_{\nu=1}^{n-1} \alpha_\nu(V)\tilde{c}_\nu + \sum_{\substack{i,j=1 \\ i<j}}^{n} \beta_{ij}(V)\, \tilde{e}^*_{ij}$$

of $\tilde{c}(V)$, we have shown that $\beta_{ij}(V) \neq 0$ (and $i < j$) implies $t_{ji} \leq t_{i_o j_o}$. The theorem follows immediately.

We are now in a position to realize T. Let's adopt Michelangelo's synthesis philosophy and suppose that we already have a network N which realizes T and all we have to do is discover its edge weights. First of all, by Theorem 4.1 we know that the semi-cuts $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{n-1}$ in N have the values $t_{12}, t_{23}, \ldots, t_{n-1,n}$ respectively. Next, since by property B of T the below-diagonal entries of T are all distinct, they can be ordered as $t_{i_1 j_1}, t_{i_2 j_2}, \ldots, t_{i_s j_s}$ (where $s = (n-1)n/2$) such that

$$t_{i_1 j_1} < t_{i_2 j_2} < \ldots < t_{i_s j_s} .$$

Then, since $\tilde{e}^*_{j_1 i_1}$ is the only zweieck $\tilde{e}^*_{ji}$ $(j < i)$ such that $t_{ij} \leq t_{i_1 j_1}$, Theorem 4.2 and Theorem 1.1 show that we have just the information we need in order to determine its weight — for $e^*_{j_1 i_1}$ is simply the (unique) solution of the system of inequalities

$$\left\{ e^*_{i_1 j_1} \geq t_{i_1 j_1} - \sum_{\nu=1}^{n-1} \alpha_\nu(V)\, c_\nu \;\middle|\; T(V) \text{ is an S-submatrix for } t_{i_1 j_1} \right\}$$

which satisfies at least one of the inequalities as an equation. Similarly, if $e^*_{i_1 j_1}, e^*_{i_2 j_2}, \ldots, e^*_{i_{\sigma-1} j_{\sigma-1}}$ have been determined, and $\mathcal{S}_{i_\sigma j_\sigma}$ denotes the set of S-submatrices for $t_{i_\sigma j_\sigma}$, then $e^*_{i_\sigma j_\sigma}$ is the unique solution of the system of inequalities

$$\left\{ e^*_{i_\sigma j_\sigma} \geq t_{i_\sigma j_\sigma} - \sum_{\nu=1}^{n-1} \alpha_\nu(V)\, t_{\nu,\nu+1} - \sum_{\substack{i<j \\ t_{ij} < t_{i_\sigma j_\sigma}}} \beta_{ij}(V) e^*_{ij} \;\middle|\; T(V) \in \mathcal{S}_{i_\sigma i_\sigma} \right\}$$

which satisfies at least one of the inequalities as an equation; notice that, once again, the only unknown quantity in the system is $e^*_{i_\sigma j_\sigma}$.

<u>Theorem 4.3</u>: Any real, square matrix T satisfying conditions A and B is realizable as the terminal capacity matrix of a network; and the realizing network is unique up to an arbitrary K-transformation.

<u>Proof</u>: A realization of T is obtained by determining the Zweieck weights as in the foregoing discussion and setting

$$e_{ij}=t_{ij} \text{ and } e_{ji}=e^*_{ij}-t_{ij} \text{ if } i=j-1$$

$$e_{ij}=0 \text{ and } e_{ji}=e^*_{ij} \text{ if } 1<j-1 .$$

We have seen that if N_1 and N_2 are any two realizations of T then the values of $\tilde{c}_\nu$ ($\nu=1,2,\ldots,n-1$) and the weights of $\tilde{e}^*_{ij}$ ($1 \leq i < j \leq n$) must be the same in N_2 as in N_1, <u>i.e.</u> the difference $E_{N_1}-E_{N_2}$ between the two edge weight matrices is orthogonal to each of the vectors $\tilde{c}_\nu$ and $\tilde{e}^*_{ij}$. Since, by Theorem 2.2, these vectors constitute a basis for $\tilde{C}$ it follows that $E-E' \in \tilde{\tilde{K}}$ (by Theorems 3.1 and 3.2), Q.E.D.

<u>4.3 Unique S-submatrices</u>

In this section we impose the following restrictions on the matrix T to be realized:

A. it has S-submatrices for each entry; and

C. it has unique S-submatrices for the entries $t_{\nu,\nu+1}$ ($\nu=1,2,\ldots,n-1$).

That these conditions are weaker than those of 4.1 is seen by the matrix

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ 1 & x & 3 & 3 & 3 \\ 2 & 6 & x & 5 & 3 \\ 3 & 6 & 6 & x & 3 \\ 4 & 7 & 8 & 9 & x \end{array}$$

which has less than (4-1)(4+2)/2 = 9 distinct entries.

Theorem 4.4: The semi-cuts in $\left\{\tilde{c}_\nu = \tilde{c}(1,2,\ldots,\nu)(\nu+1,\nu+2,\ldots,n)\right\}_{\nu=1,2,\ldots,n-1}$ are the semi-cuts which determine $t_{\nu,\nu+1}$ ($\nu=1,2,\ldots,n-1$) in any network realizing a T which satisfies conditions A and C of this section.

Proof: The matrix $T(1,2,\ldots,\nu)(\nu+1,\nu+2,\ldots,n)$ is always an S-submatrix for $t_{\nu,\nu+1}$; since hypothesis C requires that there be but one such S-submatrix, this is the one, and the theorem follows from Theorem 1.1, Q.E.D.

The discussion following Theorem 4.1, and Theorem 4.2 itself, carry over directly to the less restricted case under consideration in this section. In the present case, however, we may find that the inequalities for a particular zweieck weight e^*_{ij} may also involve other zweieck weights $e^*_{i'j'}$ which have not yet been determined (this can happen if $t_{i'j'}=t_{ij}$ and an S-submatrix for t_{ij} contains $t_{i'j'}$); thus, instead of a system of inequalities for a single variable we may have a system of simultaneous inequalities involving a number of variables; and any solution of this system such that each variable zweieck weight appears in an equation will lead to a network realizing T. The networks realizing T are not necessarily all K-equivalent, then. Example 4.1 illustrates these points.

Example 4.1:

Let the matrix T to be realized be

$$T = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ 1 & x & -4 & -4 & -4 \\ 2 & 7 & x & -2 & -4 \\ 3 & 3 & 0 & x & -4 \\ 4 & 3 & 3 & 6 & x \end{array}$$

As in section 4.2 we must have c(1)(2,3,4) = -4, c(1,2)(3,4) = -2, and c(1,2,3)(4) = -4, so, still as before, we can take as our "starting network" the partial realization indicated in Figure 4.1:

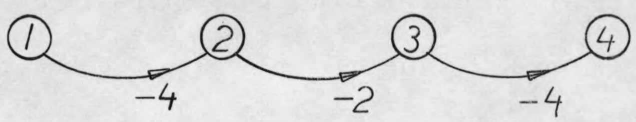


Figure 4.1

The smallest below-diagonal entry of T is $t_{32} = 0$, and its sole S-submatrix is T(1,3)(2,4); from Figure 4.1 we see that the "present value" of c(1,3)(2,4) is -8, which must be increased to t_{32} by altering the weight of edge $\tilde{e}_{32}$; thus $e_{32} = 8$ is needed, and the partial realization to be used in the next step is that of Figure 4.2:

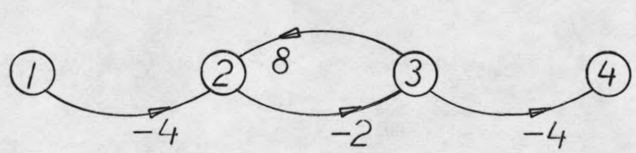


Figure 4.2

One representative of the second smallest below-diagonal entries of T is $t_{31} = 3$; its S-submatrices are T(3)(1,2,4) and T(3,4)(1,2); from Figure 4.2 we see that the present values of the corresponding semi-cuts are c(3)(1,2,4) = 4 and c(3,4)(1,2) = 8. But notice that edge $\tilde{e}_{31}$ is not the only edge in $\tilde{c}$(3,4)(1,2) whose weight has not yet been determined: $\tilde{e}_{41}$ and $\tilde{e}_{42}$ also fall in this category. Thus, instead of fixing the weight of $\tilde{e}_{31}$ immediately, we use the known present values of the semi-cuts corresponding to S-submatrices for t_{31} to write the following two inequalities which Theorem 1.1 demands be satisfied if our network is eventually to be a realization of T:

$$e_{31} + 4 \geq t_{31} = 3$$

$$e_{31} + e_{41} + e_{42} + 8 \geq t_{31} = 3$$

We must next determine the additional conditions imposed on e_{41} and e_{42} by submatrices for t_{41} and t_{42} which aren't also S-submatrices for t_{31}; the only S-submatrix for t_{41} is T(3,4)(1,2), which has already been considered; the S-submatrices for t_{42} are T(3,4)(1,2) and T(1,3,4)(2), the former of which has already been considered, while the latter corresponds to semi-cut $\tilde{c}$(1,3,4)(2) whose present value is 4; since $\tilde{e}_{42}$ is the only edge appearing in this semi-cut whose weight hasn't yet been determined, Theorem 1.1 yields the condition:

$$e_{42} + 4 \geq t_{42} = 3$$

Thus, the system of inequalities to be satisfied by e_{41}, e_{42}, and e_{31} is:

$$e_{31} \geq -1$$

$$e_{31} + e_{41} + e_{42} \geq -5$$

$$e_{42} \geq -1$$

Two examples of acceptable solutions of the above system (by "acceptable" we mean solutions in which each variable appears in at least one inequality which is satisfied as an equality) are:

(a) $e_{31} = -1$, $e_{41} = -3$, $e_{42} = -1$

and:

(b) $e_{31} = 0$, $e_{41} = -5$, $e_{42} = 0$

Since the remaining two entries (t_{43} and t_{21}) are distinct, there are no more anomalies and the realizations of T which correspond to the above two solutions are easily seen to be those indicated by Figure 4.3(a) and (b) respectively:

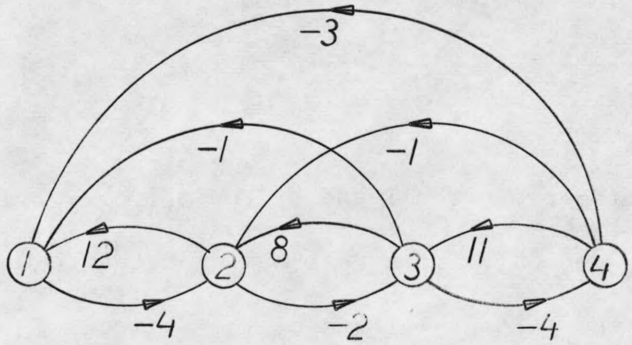


Figure 4.3(a)

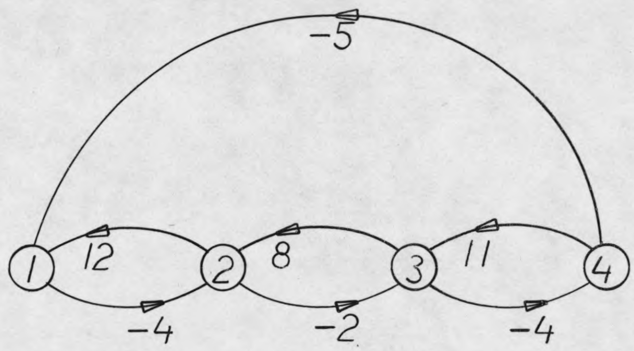


Figure 4.3(b)

Notice that these networks, while equivalent (they both realize T), are not K-equivalent, e.g. the semi-cut $\tilde{c}(4)(1,2,3)$ has value 7 in the network of Figure 4.3(a) but is the determining semi-cut for $t_{43} = 6$ in that of Figure 4.3(b).

In general, suppose that the weights of the zweiecks $\tilde{e}^*_{ij}$ (i<j) such that $t_{ji} < t_{i_1 j_1}$ have been determined, and that $\left\{ t_{i_\sigma j_\sigma} \right\}_{\sigma = 1,2,\ldots,s}$ is the set of below-diagonal entries which equal $t_{i_1 j_1}$. Then each S-submatrix T(V) for each $t_{i_\sigma j_\sigma}$ leads to inequality of the form:

$$e^*_{i_{\sigma_1} j_{\sigma_1}} + e^*_{i_{\sigma_2} j_{\sigma_2}} + \ldots + e^*_{i_{\sigma_k} j_{\sigma_k}} \geq a(V)$$

where k is an integer between 1 and s, the $\sigma_1, \sigma_2, \ldots, \sigma_k$ are distinct integers between 1 and s, and a(V) is a real number determined by the values of $\tilde{c}_1$, $\tilde{c}_2, \ldots, \tilde{c}_{n-1}$ and the (known) weights of the zweiecks $\tilde{e}^*_{ij}$ (i<j) such that $t_{ji} < t_{i_1 j_1}$. Any network whose zweieck weights are acceptable solutions of this system of inequalities (and whose v_i to v_j terminal capacity is larger than $t_{i_1 j_1}$ if $t_{ij} > t_{i_1 j_1}$) will have the required terminal capacities

$t_{i_1j_1}, t_{i_2j_2}, \ldots, t_{i_sj_s}$ (where by an "acceptable" solution we mean a solution in which each of the zweieck weights $e^*_{i_\sigma j_\sigma}$ appears in at least one equation). Since all of the variables in this system have nonnegative coefficients it is clear that the system is consistent; and since each variable appears in at least one inequality (with nonzero coefficient) it follows that an acceptable solution exists.

We have thus shown that the only difficulty encountered in this section which didn't appear in section 4.2 (the zweieck weights being determined in sets of more than one) doesn't prevent us from realizing T in essentially the same manner as in section 4.2 — there is always a realization, but the relation between various realizations is something more general than the K-transformations.

4.4 The general case

In this section our only condition on T is

A. it satisfies the S-submatrix condition.

If the S-submatrices for each above-diagonal entry are unique then this case reduces to that of section 4.3. Thus, assume that some above-diagonal entry has at least two S-submatrices. In this event, if we were to simply set $c_\nu = t_{\nu,\nu+1}$ for each ν and proceed to determine the zweieck weights as before then the network obtained would have a terminal capacity matrix which not only could differ above the main diagonal from T, but below as well. How could this happen? Well, first of all, suppose that one of the above - diagonal entries t_{pq} of T has an S-submatrix all of whose below - diagonal entries are less than t_{pq}; then we have no guarantee that the corresponding semi-cut of the network has value at least as large as t_{pq} so it is possible that the network's terminal capacity from v_p to v_q is less than t_{pq}; in this way the network's terminal capacity matrix could differ from T above the diagonal. This being the case, suppose that submatrix T(V) of T fails to be an S-submatrix for a below-diagonal entry t_{ij} of T only because some of its above-diagonal entries are larger than t_{ij}; then, since the above-diagonal terminal capacities of the network may be smaller than those of T, it could happen that the corresponding submatrix of the networks terminal capacity matrix <u>is</u> an S-submatrix for the network's v_i to v_j terminal capacity; since the method of section 4.3 wouldn't have guaranteed that the corresponding semi-cut has value as large as t_{ij} (it only makes this guarantee for semi-cuts corresponding to S-submatrices of T for t_{ij}) it could very well happen that the network's v_i to v_j terminal capacity is less than t_{ij}. That these possibilities aren't vacuous is shown by the example below, where T satisfies condition A and N is one of the possible networks resulting from the method of section 4.3:

<u>Example 4.2</u>:

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 1 & 1 & 1 \\ 2 & x & 4 & 3 \\ 2 & 5 & x & 3 \\ 2 & 7 & 6 & x \end{bmatrix} \end{array}$$

N:

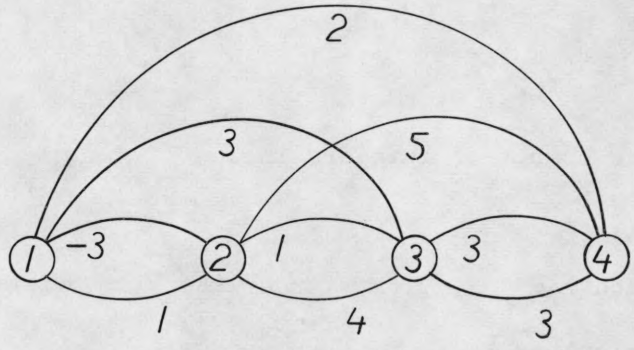


Figure 4.4

$$T_N = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & x & 1 & 1 & 1 \\ 2 & 1 & x & 1 & 1 \\ 3 & 2 & 5 & x & 3 \\ 4 & 2 & 7 & 6 & x \end{array}$$

In addition to condition A we can assume without loss of generality that T satisfies

(D_1) T has n-1 distinct entries above the main diagonal, and

(D_2) no below-diagonal entry of T equals an above-diagonal entry.

That these conditions entail no loss of generality can be seen as follows: if T doesn't satisfy D_1 and D_2 then we perturb it into a new matrix $T(\epsilon)$ which, as long as the nonnegative real variable ϵ is small enough, satisfies A, D_1, and D_2, and reduces to T as $\epsilon \to 0$; any realization $N_{T(\epsilon)}$ of $T(\epsilon)$ will then reduce to a realization of T as $\epsilon \to 0$. $T(\epsilon) = [t_{ij}(\epsilon)]$ is obtained from $T = [t_{ij}]$ in the following way.

(a) Partition $\{1, 2, \ldots, n\}$ into subsets $I_1, I_2, \ldots, I_k$ such that

$$i_1 \in I_{k_1},\ i_2 \in I_{k_2} \Longrightarrow \begin{cases} t_{i_1, i_1+1} = t_{i_2, i_2+1} & \text{if } 1 \le k_1 = k_2 \le k \\ t_{i_1, i_1+1} < t_{i_2, i_2+1} & \text{if } 1 \le k_1 < k_2 \le k \end{cases} ;$$

let n_k denote the number of integers in I_k.

(b) For each $k = 1,2,\ldots,K$: order the elements of I_k (using any criterion) as $\nu_1, \nu_2, \ldots, \nu_{n_k}$ and set

$$t_{\nu_1, \nu_1+1}(\epsilon) = t_{\nu_1, \nu_1+1} + \epsilon^2$$

$$t_{\nu_2, \nu_2+1}(\epsilon) = t_{\nu_2, \nu_2+1} + \epsilon^3$$

$$t_{\nu_{n_k}, \nu_{n_k}+1}(\epsilon) = t_{\nu_{n_k}, \nu_{n_k}+1} + \epsilon^{(n_k+1)} ;$$

(c) Select values for the remaining above-diagonal entries of $T(\epsilon)$ which make $T(\epsilon)$ partitionable without rearrangement.

(d) For each integer pair (i,j) with $1 \le j < i \le n$ set

$$t_{ij}(\epsilon) = t_{ij} + \epsilon .$$

It isn't difficult to see that $T(\epsilon)$ has the cited properties if we restrict ϵ to the range

$$0 < \epsilon < \min \left\{ 1, \min \left\{ t_{ij} - t_{i'j'} \;\middle|\; t_{ij} - t_{i'j'} > 0 \right\} \right\} .$$

For the remainder of this chapter we assume T to satisfy conditions D_1 and D_2.

Consider a network N about which only the following two things are known a priori:

(i) its foward edge weights are $e_{\nu,\nu+1} = t_{\nu,\nu+1}$ $(\nu=1,2,\ldots,n-1)$ and $e_{pq} = 0$ $(1 \leq p < q-1 \leq n)$, and

(ii) the below-diagonal entries of its terminal capacity matrix equal their counterparts in T, i.e. $(T_N)_{ij} = t_{ij}$ $(1 \leq j < i \leq n)$.

In order to discuss the S-submatrices of T_N for its below-diagonal entries we need the estimate (iii) of the above-diagonal entries of T_N. To obtain it, we define for each integer pair (p,q) with $p < q$ the number t'_{pq} which equals

$$\min\left\{\max\left\{t_{ij}\;\middle|\; t_{ij} \in T(V),\ i > j\right\}\right\}$$

where the minimum is over all S-submatrices T(V) of T for t_{pq} which contain a below-diagonal entry, unless there aren't any such S-submatrices in which case t'_{pq} equals t_{pq}. Then from the known properties of N we can conclude that

(iii) $$t'_{pq} \leq (T_N)_{pq} \leq t_{pq}$$

(from property (i) we know that $(T_N)_{pq} \leq t_{pq}$ since vertices v_p and v_q are separated by a semi-cut whose value equals that entry $t_{\nu,\nu+1}$ —unique by D_1 — of T which equals t_{pq}; suppose that $(T_N)_{pq} < t'_{pq}$; then the terminal capacity $(T_N)_{pq}$ is determined by a semi-cut which corresponds to one of the S-submatrices for $(T_N)_{pq}$, say $T_N(V_o)$, which contains a below-diagonal entry of T_N; let $(T_N)_{i_oj_o}$ denote the largest below-diagonal entry of $T_N(V_o)$; then, by property (ii), $t_{i_o j_o} = (T_N)_{i_o j_o}$ and $t_{i_o j_o}$ is the largest below-diagonal entry of $T(V_o)$, so from the definition of t'_{pq} we have $t'_{pq} \leq (T_N)_{i_o j_o}$; but, since $(T_N)_{i_o j_o}$ is in an S-submatrix for $(T_N)_{pq}$, we have $(T_N)_{i_o j_o} \leq (T_N)_{pq}$; this leads to the relation $t'_{pq} \leq (T_N)_{pq}$, in contradiction to our supposition that $(T_N)_{pq} < t'_{pq}$).

Now we fix our attention on any below-diagonal entry $(T_N)_{ij}$ of T_N and inquire about which submatrices $T_N(V)$ that contain $(T_N)_{ij}$ are S-submatrices of T_N for $(T_N)_{ij}$. First,

(iv) if $T(V)$ is an S-submatrix of T for t_{ij} then $T_N(V)$ is an S-submatrix of T_N for $(T_N)_{ij}$

(since properties (ii) and (iii) establish that no entry of $T_N(V)$ is larger than the corresponding entry of $T(V)$ and that $(T_N)_{ij} = t_{ij}$). Next, suppose that $T(V)$ contains the below-diagonal entry t_{ij} but is not among its S-submatrices; then either (a) $T(V)$ contains a below-diagonal entry $t_{i'j'}$ greater than t_{ij}, or (b) all of the entries of $T(V)$ which are larger than t_{ij} lie above the main diagonal of T. In case (a) it follows from property (ii) that $T_N(V)$ is not an S-submatrix for $(T_N)_{ij}$. Case (b) has two subcases: (b_1) $t'_{pq} > t_{ij}$ where t_{pq} is some above-diagonal entry of $T(V)$, and (b_2) $t'_{pq} \leq t_{ij}$ for all above-diagonal entries t_{pq} of $T(V)$. In case (b_1) it follows from property (iii) that $T_N(V)$ is not an S-submatrix of T_N for $(T_N)_{ij}$. All that remains is case (b_2), about which we can only say that $T_N(V)$ can be, but doesn't have to be, an S-submatrix for $(T_N)_{ij}$ (there are examples of each possibility). Thus we have

(v) if $T(V)$ is not an S-submatrix for t_{ij} then $T_N(V)$ is not an S-submatrix for $(T_N)_{ij}$ unless $T(V)$ contains t_{ij} and the only entries larger than t_{ij} are above-diagonal entries t_{pq} with $t'_{pq} \leq t_{ij}$; in this exceptional case $T_N(V)$ may or may not be on S-submatrix for $(T_N)_{ij}$.

With properties (iv) and (v) in mind we can see how to obtain from T a network N with the properties (i) and (ii). The procedure is essentially that of section 4.3, the single difference being occasioned by the exceptional case in property (v): when writing the inequalities of the zweieck weight e^*_{ij} $(l > j)$ we must include not only the inequalities obtained from S-submatrices for t_{ij} but also the inequalities obtained from those non-S-submatrices comprising the exceptional case; one way of accomplishing this is to write the inequalities for e^*_{ij} $(i > j)$ which the procedure of section 4.3 would require if we were realizing T'' instead of T, where

$$(T'')_{ij} = \begin{cases} t'_{ij} & \text{if } i > j \\ t_{ij} & \text{if } i > j \end{cases}$$

<u>Example 4.3:</u>

Consider again the matrix of Example 4.2,

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 1 & 1 & 1 \\ 2 & x & 4 & 3 \\ 2 & 5 & x & 3 \\ 2 & 7 & 6 & x \end{bmatrix} \end{array}.$$

Each of the entries t_{24}, t_{34}, and t_{23} have S-submatrices which contain below-diagonal entries of T: T(2,3)(1,4) is common to t_{24} and t_{34} and its largest below-diagonal entry has value 2, so $t'_{24} = t'_{34} = 2$; T(2)(1,3,4) is the only S-submatrix for t_{23} and its largest entry also has value 2, so $t'_{23} = 2$. Thus:

$$T'' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 1 & 1 & 1 \\ 2 & x & 2 & 2 \\ 2 & 5 & x & 2 \\ 2 & 7 & 6 & x \end{bmatrix} \end{array}$$

We see that when writing the inequalities for t_{21}, for instance, we must include one not only for T(2,3,4)(1) but also for T(2)(1,3,4) and T(2,3)(1,4). If we do this then one of the networks which could result is that of Figure 4.5:

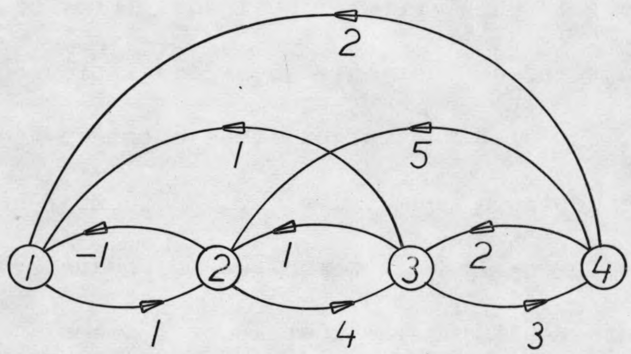


Figure 4.5

which realizes, not T, but

$$T_N = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} x & 1 & 1 & 1 \\ 2 & x & 3 & 3 \\ 2 & 5 & x & 3 \\ 2 & 7 & 6 & x \end{array}\right] \end{array}$$

Thus, let N be any network with the properties

(i') $(T_N)_{ij} \leq t_{ij}$ if $i < j$

(ii') $(T_N)_{ij} = t_{ij}$ if $i > j$

(since properties (i) and (ii) imply (i') and (ii') we know that such an N exists). If $T_N = T$ then N is a realization of T; if for some $i < j$, however, $(T_N)_{ij} < t_{ij}$ then we shall show how to modify N to obtain a network N' which in addition to properties (i') and (ii') has the properties

(iii') $(T_{N'}) = t_{ij}$ if $(T_N) = t_{ij}$

(iv') for at least one vertex pair (v_i, v_j) such that $(T_N)_{ij} < t_{ij}$,

$(T_{N'})_{ij} = t_{ij}$.

Thus, the process can be repeated, starting with N', etc. until eventually a realization of T is obtained.

Choose any vertex pair (v_p, v_q) such that $(T_N)_{pq} < t_{pq}$. We are going to obtain an n x n matrix $\Delta(\epsilon) = [\delta_{ij}(\epsilon)]$ (each of whose entries can be a continuous, piecewise-linear function on some closed interval $0 \leq \epsilon \leq \epsilon_{max}$ of the real line, where the upper limit $\epsilon_{max} > 0$ will be discussed in a moment) such that the network $N(\epsilon)$ whose edge weight matrix is $E_{N(\epsilon)} = E_N + \Delta(\epsilon)$ satisfies

1. $(T_{N(\epsilon)})_{pq}$ increases linearly with ϵ over some upper subinterval of $0 \leq \epsilon \leq \epsilon_{max}$; and
2. $(T_N)_{ij} \leq (T_{N(\epsilon)})_{ij} \leq t_{ij}$ over the whole interval $0 \leq \epsilon \leq \epsilon_{max}$ and for all vertex pairs (v_i, v_j), $1 \leq i \neq j \leq n$.

(Notice that the second property ensures $(T_{N(\epsilon)})_{ij} = t_{ij}$ if $(T_N)_{ij} = t_{ij}$).

The upper limit ϵ_{max} is to be determined as the smallest value of ϵ such that either

(a) $(T_{N(\epsilon)})_{ij} = t_{ij}$ for some $1 \leq i < j \leq n$ for which $(T_N)_{ij} < t_{ij}$; or

(b) $T_N(V)$ is not an S-submatrix for $(T_N)_{pq}$ but $T_{N(\epsilon)}(V)$ is an S-submatrix for $(T_{N(\epsilon)})_{pq}$.

If ϵ_{max} is determined by (a) then we can identify $N(\epsilon_{max})$ as the network N' we want. If case (b) occurs then we let $N(\epsilon_{max}) = N_2$ and start over again, obtaining a matrix $\Delta_2(\epsilon)$ such that $N_2(\epsilon)$ has properties 1,2, and 3 (with N replaced by N_2, $N(\epsilon)$ replaced by $N_2(\epsilon)$, etc.). We continue, in this way obtaining a sequence $N, N_2, N_3, \ldots$ which eventually terminates with case (a) — this must happen after a finite number of steps since eventually we will obtain an N_k with $(T_{N_k})_{pq}$ so large that if T(V) is an S-submatrix for t_{pq} then $T_{N_k}(V)$ is an S-submatrix for $(T_{N_k})_{pq}$.

In preparation for the detailed construction of $\Delta(\epsilon)$ we partition the set $\left\{(T_N)_{ij} \,\middle|\, (T_N)_{ij} \geq (T_N)_{pq}\right\}$ into equivalence classes $\tilde{\mathcal{J}}_1, \tilde{\mathcal{J}}_2, \ldots, \tilde{\mathcal{J}}_R$ such that if $1 \leq r_1 \leq r_2 \leq R$ then $(T_N)_{i_1 j_1} \in \tilde{\mathcal{J}}_{r_1}$ and $(T_N)_{i_2 j_2} \in \tilde{\mathcal{J}}_{r_2}$ implies

$$\begin{cases} (T_N)_{i_1 j_1} = (T_N)_{i_2 j_2} & \text{if } r_1 = r_2 \\ (T_N)_{i_1 j_1} < (T_N)_{i_2 j_2} & \text{if } r_1 < r_2 \, . \end{cases}$$

We also define:

$\mathcal{S}_{ij}(T_N)$ to be the set of S-submatrices of T_N for its entry $(T_N)_{ij}$, and

$N_{\Delta(\epsilon)}$ to be the network whose edge weight matrix is $\Delta(\epsilon)$.

This takes care of the preliminaries.

We take $\delta_{ij}(\epsilon) = 0$ $(0 \leq \epsilon \leq \epsilon_{max})$ if $(T_N)_{ij} < (T_N)_{pq}$; and $\delta_{pq}(\epsilon) = \epsilon$ $(0 \leq \epsilon \leq \epsilon_{max})$, and suppose that $\delta_{ij}(\epsilon)$ has already been determined if $(T_N)_{ij}$ is a member of one of the classes $\tilde{\mathcal{J}}_1, \tilde{\mathcal{J}}_2, \ldots, \tilde{\mathcal{J}}_{r-1}$ (in a moment we will give special consideration to the case $r = 1$). Consider the system of linear inequalities:

$$\left\{ c_{N_{\Delta(\epsilon)}}(V) \geq (T_N)_{i*j*} - c_N(V) \,\middle|\, (T_N)_{i*j*} \in \tilde{\mathcal{J}}_r \text{ and } T_N(V) \in \mathcal{S}_{i*j*}(T_N) \right\}$$

First we observe that the only as yet undetermined quantities appearing in this system are those functions $\delta_{ij}(\epsilon)$ such that $(T_N)_{ij} \in \tilde{\mathcal{J}}_r$. Except in the case $r = 1$ the only condition imposed on these $\delta_{ij}(\epsilon)$ is that they be piecewise linear functions which satisfy the inequalities in such a way that:

If $(T_N)_{ij} = t_{ij}$ then, for each ϵ_o in the interval $[0, \epsilon_{max}]$, $\delta_{ij}(\epsilon_o)$ must appear in (at least) one inequality which for that value of ϵ_o is satisfied as an equality.

53

As in section 4.3, where we discussed a similar system, such a set of functions can always be found.

It is evident that the above discussion ensures that N(ϵ) has the desired property 2. As the following example shows, we have not yet said enough to ensure property 1.

Example 4.4:

$$T = \begin{bmatrix} x & 5 & 5 & 4 \\ 6 & x & 30 & 4 \\ 6 & 50 & x & 4 \\ 8 & 7 & 7 & x \end{bmatrix}$$

A network with properties (i) and (ii) (hence (i') and (ii'), found by a method discussed earlier, is

N:

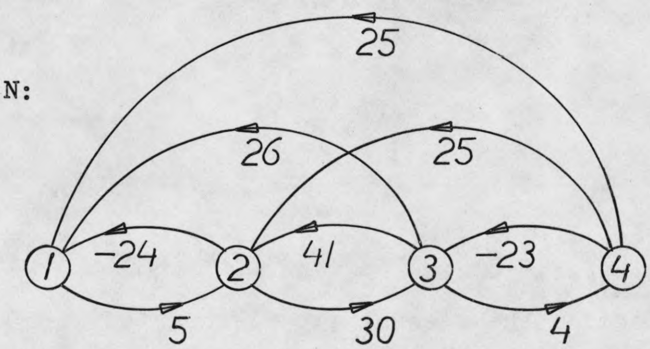


Figure 4.6

which realizes, not T, but

$$T_N = \begin{bmatrix} x & 5 & 5 & 4 \\ 6 & x & 6 & 4 \\ 6 & 50 & x & 4 \\ 8 & 7 & 7 & x \end{bmatrix}$$

$(T_N)_{pq} = (T_N)_{23}$ is the only entry of T_N which is less than t_{pq}. The equivalence classes of $(T_N)_{ij}$ $(T_N)_{ij} \geq (T_N)_{23} = 6$ are

$$\tilde{\mathfrak{J}}_1 = \left\{(T_N)_{23}, (T_N)_{21}, (T_N)_{31}\right\}$$

$$\tilde{\mathfrak{J}}_2 = \left\{(T_N)_{42}, (T_N)_{43}\right\}$$

$$\tilde{\mathfrak{J}}_3 = \left\{(T_N)_{41}\right\}$$

$$\tilde{\mathfrak{J}}_4 = \left\{(T_N)_{32}\right\}$$

We set $\delta_{12}(\epsilon) = \delta_{13}(\epsilon) = \delta_{14}(\epsilon) = \delta_{24}(\epsilon) = \delta_{34}(\epsilon) = 0$ and $\delta_{23}(\epsilon) = \epsilon$, and proceed to determine the inequalities associated with $\tilde{\mathfrak{J}}_1$. Since $\mathcal{S}_{21}(T_N) = \left\{T_N(2)(1,3,4),\ T_N(2,3)(1,4)\right\}$ we have

$$\epsilon + \delta_{21} \geq 6 - 6 = 0$$

$$\delta_{21} + \delta_{31} \geq 6 - 6 = 0$$

and from $\mathcal{S}_{31}(T_N) = \left\{ T_N(2,3)(1,4) \right\}$ we have no new inequalities.

Now, although $\delta_{21}(\epsilon) = \delta_{31}(\epsilon) = 0$ is an acceptable solution (the second inequality, in which both functions appear, is satisfied as an equation for all values of ϵ), we have as yet given no grounds for choosing this solution over such solutions as

$$\begin{cases} \delta_{21}(\epsilon) = -\epsilon \quad \forall \epsilon \\ \delta_{31}(\epsilon) = +\epsilon \quad \forall \epsilon \end{cases}$$

or

$$\begin{cases} \delta_{21}(\epsilon) = \begin{cases} 0 \text{ if } 0 \leq \epsilon \leq 1/2 \\ 1/2-\epsilon \text{ if } 1/2 \leq \epsilon \end{cases} \\ \delta_{31}(\epsilon) = \begin{cases} 0 \text{ if } 0 \leq \epsilon \leq 1/2 \\ -1/2 + \epsilon \text{ if } 1/2 \leq \epsilon \end{cases} \end{cases} .$$

Suppose we selected the latter of these; then (regardless of what solutions are selected for the $\tilde{\mathcal{J}}_2$-, $\tilde{\mathcal{J}}_3$ -, and $\tilde{\mathcal{J}}_4$- systems) the value in $N(\epsilon)$ of $\tau(2)(1,3,4)$ is

$$c_{N(\epsilon)}(2)(1,3,4) = \begin{cases} 6 + \epsilon \text{ if } 0 \leq \epsilon \leq 1/2 \\ 6\ 1/2 \text{ if } 1/2 \leq \epsilon \end{cases} ;$$

and, since $\tau(2)(1,3,4)$ separates v_2 from v_3, $N(\epsilon)$ manifestly fails to have property 1.

Let's examine, in general, the $\tilde{\mathcal{J}}_1$-system of inequalities. In this system the only previously determined nonzero entry of $\Delta(\epsilon)$ is $\delta_{pq}(\epsilon) = \epsilon$. Let's assume momentarily that $\delta_{ij}(\epsilon) = 0$ for all $(T_N)_{ij} \in \tilde{\mathcal{J}}_1$ and see why that doesn't constitute an acceptable solution of the system.

Under this assumption we know that for each $(T_N)_{ij} \in \tilde{\mathcal{J}}_1$ the inequality

$$c_{N_{\Delta(\epsilon)}}(V) \geq (T_N)_{ij} - c_N(V)$$

is satisfied for each S-submatrix $T_N(V)$ of $(T_N)_{ij}$ (because the left-hand side equals either ϵ or 0 depending upon whether or not $\mathfrak{C}(V)$ contains the sole edge $\mathfrak{e}_{pq}$ which has nonzero weight in $N_{\Delta(\epsilon)}$, while the right hand side is either 0 or negative according as $\mathfrak{C}(V)$ is or is not a determining semi-cut in N for $(T_N)_{ij}$). Thus, if "$\delta_{ij}(\epsilon) = 0$ for all $(T_N)_{ij} \in \tilde{\mathcal{J}}_1$" isn't an acceptable solution it can only be because <u>every</u> inequality in which, say, $\delta_{i*j*}(\epsilon)$ appears is thereby satisfied as

$$c_{N_{\Delta(\epsilon)}}(V) > (T_N)_{i*j*} - c_N(V) \quad ;$$

this, in turn, requires that the edge $\mathfrak{e}_{pq}$ appears in every semi-cut which is a determining semi-cut in N for $(T_N)_{i*j*}$; and it also requires that for every S-submatrix $T(V')$ of T for t_{i*j*} (none of which contains t_{pq}, of course) we have

$$c_N(V') > t_{i*j*}$$

(since if $c_N(V') < t_{i*j*}$ then $(T_N)_{i*j*} < t_{i*j*}$ so for an acceptable solution we aren't required to have $\delta_{i*j*}(\epsilon)$ appear in any inequalities which are solved as an equality, while if $c_N(V') = t_{i*j*}$ then $\delta_{i*j*}(\epsilon)$ appears in the equality

$$(0 =)\ c_{N_{\triangle(\epsilon)}}(V') \leqq (T_N)_{i*j*} - c_N(V')$$

$$= 0 \quad ;$$

in neither case, contrary to our choice of i*, j*, is "$\delta_{ij}(\epsilon) = 0$ for all $(T_N)_{ij} \in \tilde{\mathcal{J}}_1$" prevented from being an acceptable solution by virtue of misbehavior with respect to $\delta_{i*j*}(\epsilon)$).

In the discussion to follow, let me retain the symbol $\delta_{i*j*}(\epsilon)$ as the generic name for those problematical entries of $\triangle(\epsilon)$ characterized in the previous paragraph. What I am going to describe is a method of solving the $\tilde{\mathcal{J}}_1$ system of inequalities which will guarantee that $N(\epsilon)$ has property 1.

First, any $\delta_{ij}(\epsilon)$ which is not of the $\delta_{i*j*}(\epsilon)$ type is set identically equal to zero for all values of ϵ.

Next, select any $\delta_{i*j*}(\epsilon)$ as $\delta_{i^*_1 j^*_1}(\epsilon)$ and for small ϵ set $\delta_{i^*_1 j^*_1}(\epsilon) = - \epsilon$; also (for the same range of values of ϵ) set to zero any $\delta_{i*j*}(\epsilon)$ which appears in any homogeneous inequality with $\delta_{i^*_1 j^*_1}(\epsilon)$. The interval over which these assignments are to hold will be discussed in a moment.

Next, select as $\delta_{i^*_2 j^*_2}$ any δ_{i*j*} which appears in no homogeneous inequality with $\delta_{i^*_1 j^*_1}$ and for some small range set $\delta_{i^*_2 j^*_2}(\epsilon) = -\epsilon$ and set to zero any other $\delta_{i*j*}(\epsilon)$ which appears in any homogeneous inequality with $\delta_{i^*_2 j^*_2}$.

Continue in the above manner, selecting $\delta_{i^*_3 j^*_3}$, $\delta_{i^*_4 j^*_4}, \ldots,$ $\delta_{i^*_\omega j^*_\omega}$, where $\delta_{i^*_\omega j^*_\omega}$ appears in no homogeneous inequality with any of $\delta_{i^*_1 j^*_1}$, $\delta_{i^*_2 j^*_2}, \ldots,$ $\delta_{i^*_{\omega-1} j^*_{\omega-1}}$, until for some $\omega = \Omega$ all $\delta_{i^*j^*}$'s have been assigned values over some small range of values of ϵ.

The following three observations should place in evidence the fact that for small ϵ the above assignments constitute an acceptable solution of the $\tilde{\mathfrak{J}}_1$ system of inequalities:

(O_1) Each δ_{ij} (whether a $\delta_{i^*j^*}$ of not) appears in some homogeneous inequality (which means simply that each entry of T_N has value determined by some semi-cut of N which corresponds to an S-submatrix of T_N for that entry);

(O_2) Each δ_{ij} which is not a $\delta_{i^*j^*}$ appears in some homogeneous inequality without δ_{pq} (else it would be a $\delta_{i^*j^*}$);

(O_3) No $\delta_{i^*j^*}$ appears in a homogeneous inequality without δ_{pq} (else it would not be a $\delta_{i^*j^*}$).

In order to discuss the range of validity of the above assignments we need a fourth observation:

(O_4) Each $\delta_{i^*j^*}$ appears in some nonhomogeneous inequality without δ_{pq} (there is an S-submatrix T(V) of T for each $t_{i^*j^*}$; by a trivial extension of property (vi) of networks which have properties (i) and (ii) to networks which have properties (i') and (ii'), $T_N(V)$ is an S-submatrix of T_N for $(T_N)_{i^*j^*}$; and, since $t_{i^*j^*} = (T_N)_{i^*j^*} = (T_N)_{pq} < t_{pq}$, no such S-submatrix can contain the entry $(T_N)_{pq}$; finally, $c_N(V) > (T_N)_{i^*j^*}$, else $\delta_{i^*j^*}$ would not be a $\delta_{i^*j^*}$; thus the inequality $c_{N_{\triangle(\epsilon)}}(V) \geq (T_N)_{i^*j^*} - c_N(V)$ is an inhomogeneous inequality containing δ_{i*j*} but not $\delta_{p\delta}$).

Thus, as ϵ increases from zero, the left-hand sides of certain inhomogeneous inequalities decrease (at least of those inhomogeneous inequalities which contain some $\delta_{i^*_\omega j^*_\omega}$ without δ_{pq}) so there exist values of ϵ for which these are equalities; let ϵ_1 be the smallest such value. Let $\delta_{i^*_{\omega_\sigma} j^*_{\omega_\sigma}}$ $(\sigma = 1,2,\ldots,s)$ be all the $\delta_{i^*_\omega j^*_\omega}$, and $\delta_{i^*_\tau j^*_\tau}$ $(\tau = 1,2,\ldots,t)$ be all the δ_{i*j*}'s which aren't $\delta_{i^*_\omega j^*_\omega}$'s, which for $\epsilon = \epsilon_1$ appear in an inhomogeneous inequality which is satisfied as an equality. If we require

$$\delta_{i^*_{\omega_\sigma} j^*_{\omega_\sigma}}(\epsilon) = \begin{cases} -\epsilon & \text{if } 0 \leq \epsilon \leq \epsilon_1 \\ -\epsilon_1 & \text{if } \epsilon_1 \leq \epsilon \leq \epsilon_{max} \end{cases}$$

and

$$\delta_{i^*_\tau j^*_\tau}(\epsilon) = 0 \text{ if } 0 \leq \epsilon \leq \epsilon_{max}$$

(for each $\sigma = 1,2,\ldots,s$ and $\tau = 1,2,\ldots,t$) then for all ϵ in the interval $[0, \epsilon_{max}]$ each $\delta_{i^*_{\omega_\sigma} j^*_{\omega_\sigma}}$ and $\delta_{i^*_\tau j^*_\tau}$ will appear in an inequality satisfied as an equality (and of course the inequalities of the $\mathfrak{J}_1$-system will all be satisfied).

Note that there may be some δ_{i*j*}'s which aren't among the $\delta_{i^*_{\omega_\sigma} j^*_{\omega_\sigma}}$'s or $\delta_{i^*_\tau j^*_\tau}$'s and which won't appear in equalities when $\epsilon > \epsilon_1$ unless something is done about them. The set of all such δ_{i*j*}'s must be treated at $\epsilon = \epsilon_1$ in a manner similar to our treatment of the set of all δ_{i*j*}'s at $\epsilon = 0$ (it doesn't seem necessary to describe the process in detail).

The result of all this must also be pretty clear by now. The salient feature of the solution just described is that the only $\delta_{ij}(\epsilon)$ (such that $(T_N)_{ij} \in \mathfrak{J}_1$) which isn't a decreasing function of ϵ on the whole interval $[0, \epsilon_{max}]$

is $\delta_{pq}(\epsilon)$. This has guaranteed that, when ϵ is large enough, the determining semi-cuts for each $(T_{N(\epsilon)})_{ij}$ will all have shifted position so that none of them contains edge $\tilde{e}_{pq}$; when this happens $(T_{N(\epsilon)})_{pq}$ is free to increase with ϵ, so $N(\epsilon)$ has property 1, Q.E.D.

5. APERÇU

As mentioned in the exordium, the problem of synthesizing an oriented communication network to realize a given complete set of terminal capacities has been extant for several years. Although I had originally intended this problem to be the subject of this thesis it soon became clear (via the bi-circuit transformations) that the solution of the generalized form would be, if not a prerequisite to, at least a step towards the solution of the communication network problem.

At the time the research reported here began, the known general necessary conditions for realizability of a terminal capacity matrix (by a communication network) were Tang and Chien's partitioning condition and a stronger condition given by Gomory and Hu(10):

$$t_{ij} \geq \min \left\{ t_{ik}, t_{kj} \right\} \forall k \neq i,j \ (i \neq j)$$

which is logically equivalent to (and more concise than) Mayeda's S-submatrix condition, but (for me, at least) heuristically inferior to the latter. Shortly thereafter, Mayeda's paper (in which he introduced the S-submatrix concept for the sole purpose of demonstrating that for each $n > 0$ there is a communication network with $(n-1)(n+2)/2$ distinct terminal capacities — this number being previously known only as an obvious upper bound) appeared.

In chapter 1 of this paper it is shown that the S-submatrix condition (hence the other two, as well) is necessary not only for communication networks but also for arbitrary networks. The greater part of the remainder of the thesis is devoted to the demonstration that this condition is also sufficient when arbitrary networks are allowed.

Using the last theorem of chapter 3 and the results of section 4.2 it is easy to see that the S-submatrices are not sufficient if the realizing network is required to be a communication network: If I is an arbitrary μ-matrix of order n then there exist nth order matrices T such that

$$(I)_{ij} = (I)_{i'j'} \text{ implies } (T)_{ij} = (T)_{i'j'}$$

and

$$(I)_{ij} < (I)_{i'j'} \text{ implies } (T)_{ij} < (T)_{i'j'}$$

which are realizable by communication networks and others which aren't.

Theorem 3.8 and the results of section 4.2 do yield a fairly simple necessary and sufficient condition for realizability of matrices T which are related in the above manner to μ-matrices. The extension of this condition to the classes of matrices discussed in sections 4.3 and 4.4 awaits a transformation (more general, of course, than the bi-circuit transformation) relating their network realizations.

A cursory study of the relationship between members of different K-equivalence subclasses of the set of realizations of a matrix suggests that (contrary to the view I held when writing chapters 2 and 3) the properties of networks depend upon the topology of graphs more general than the bi-complete graphs. This, in turn, suggests that it would be useful to extend the results of chapters 2 and 3 to directed graphs in general.

REFERENCES

1. Berge, C., *Théorie des Graphes et ses Applications.* Paris: Dunod, 1958.

2. Seshu, S., and Reed, M., *Linear Graphs and Electrical Networks.* Reading, Mass.: Addison-Wesley, 1961.

3. König, D., *Theorie der Endlichen und Unendlichen Graphen.* New York: Chelsea, 1950.

4. Boldyreff, A. W., Determination of the Maximal Steady Flow of Traffic Through a Railroad Network, *J. Oper. Res. Soc. of America*, vol. 3, 443-465 (1955).

5. Ford, L. R., and Fulkerson, D. R., Maximal Flow Through a Network, *Can. J. Math.* 8, 399-404 (1956).

6. Elias, P., Feinstein, A., and Shannon, C. E., A Note on the Maximum Flow Through a Network, *IRE Trans. on Info. Theory*, vol. IT-2, 117-119 (Dec. 1956).

7. Mayeda, W., Terminal and Branch Capacity Matrices of a Communication Net, *IRE Trans. on Circuit Theory*, vol. CT-7, 261-269 (Sept. 1960).

8. Tang, D. T., and Chien, R. T., Analysis and Synthesis Techniques of Oriented Communication Nets, *IRE Trans. on Circuit Theory*, vol. CT-8, 39-43 (Sept. 1961).

9. Mayeda, W., On Oriented Communication Nets, *IRE Trans. on Circuit Theory*, vol. CT-9, 261-267 (Sept. 1962).

10. Gomory, R., and Hu, T. C., Multi-Terminal Network Flows, IBM, Yorktown Heights, N.Y., IBM Rept. No. RC-318 (Sept. 1960).

DISTRIBUTION LIST AS OF JULY 12, 1963

1 Director
Air University Library
Maxwell Air Force Base, Alabama
Attn: CR-4803a

1 Redstone Scientific Information Center
U.S. Army Missile Command
Redstone Arsenal, Alabama

1 Electronics Research Laboratory
University of California
Berkeley 4, California

2 Hughes Aircraft Company
Florence and Teale
Culver City, California
Attn: N. E. Devereux
Technical Document Center

3 Autonetics
9150 East Imperial Highway
Downey, California
Attn: Tech. Library, 3041-11

1 Dr. Arnold T. Nordsieck
General Motors Corporation
Defense Research Laboratories
6767 Hollister Avenue
Goleta, California

1 University of California
Lawrence Radiation Laboratory
P. O. Box 808
Livermore, California

1 Mr. Thomas L. Hartwick
Aerospace Corporation
P. O. Box 95085
Los Angeles 45, California

1 Lt. Colonel Willard Levin
Aerospace Corporation
P. O. Box 95085
Los Angeles 45, California

1 Professor Zorab Kaprelian
University of Southern California
University Park
Los Angeles 7, California

1 Sylvania Electronic Systems - West
Electronic Defense Laboratories
P. O. Box 205
Mountain View, California
Attn: Documents Center

1 Varian Associates
611 Hansen Way
Palo Alto, California
Attn: Dr. Ira Weissman

1 Huston Denslow
Library Supervisor
Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

1 Professor Nicholas George
California Institute of Technology
Electrical Engineering Department
Pasadena, California

1 Space Technology Labs., Inc.
One Space Park
Redondo Beach, California
Attn: Acquisitions Group
STL Technical Library

2 Commanding Officer and Director
U.S. Naval Electronics Laboratory
San Diego 52, California
Attn: Code 2800, C. S. Manning

1 Commanding Officer and Director
U.S. Navy Electronics Laboratory
San Diego 52, California
Attn: Library

1 Office of Naval Research Branch Office
1000 Geary Street
San Francisco, California

1 The RAND Corporation
1700 Main Street
Santa Monica, California
Attn: Library

1 Stanford Electronics Laboratories
Stanford University
Stanford, California
Attn: SEL Documents Librarian

1 Dr. L. F. Carter
Chief Scientist Air Force
Room 4E-324, Pentagon
Washington 25, D. C.

1 Mr. Robert L. Feik
Associate Director for Research
Research and Technology Division
AFSC
Bolling Air Force Base 25, D. C.

1 Captain Paul Johnson (USN-Ret)
National Aeronautics and Space
Administration
1520 H Street, N. W.
Washington 25, D. C.

1 Major Edwin M. Myers
Headquarters USAF (AFRDR)
Washington 25, D. C.

1 Dr. James Ward
Office of Deputy Director
(Research and Info)
Department of Defense
Washington 25, D. C.

1 Dr. Alan T. Waterman
Director, National Science Foundation
Washington 25, D. C.

1 Mr. G. D. Watson
Defense Research Member
Canadian Joint Staff
2450 Massachusetts Ave., N. W.
Washington 8, D. C.

1 Mr. Arthur G. Wimer
Chief Scientist
Air Force Systems Command
Andrews Air Force Base
Washington 25, D. C.

1 Director, Advanced Research
Projects Agency
Washington 25, D. C.

1 Air Force Office of Scientific Branch
Directorate of Engineering Sciences
Washington 25, D. C.
Attn: Electronics Division

1 Director of Science and Technology
Headquarters, USAF
Washington 25, D. C.
Attn: AFRST-EL/GU

1 Director of Science and Technology
AFRST - SC
Headquarters, USAF
Washington 25, D. C.

1 Headquarters, R & T Division
Bolling Air Force Base
Washington 25, D. C.
Attn: RTHR

1 Headquarters, U.S. Army Material Command
Research Division, R & D Directorate
Washington 25, D. C.
Attn: Physics & Electronics Branch
Electronics Section

1 Commanding Officer
Diamond Ordnance Fuze Laboratories
Washington 25, D. C.
Attn: Librarian, Room 211, Bldg. 92

1 Operations Evaluation Group
Office of the CNO (Op03EG)
Navy Department
Washington 25, D. C.

1 Chief of Naval Operations
Tech. Analysis & Advisory Group (OP-07T)
Pentagon
Washington 25, D. C.

1 Commanding Officer
U.S. Army Personnel Research Office
Washington 25, D. C.

1 Commanding Officer & Director
David W. Taylor Model Basin
Navy Department
Washington 7, D. C.
Attn: Code 142, Library

1 Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Code 686

1 Bureau of Ships
Navy Department
Washington 25, D. C.
Attn: Code 732

1 Technical Library, DLI-3
Bureau of Naval Weapons
Department of the Navy
Washington 25, D. C.

1 Director
Naval Research Laboratory
Washington 25, D. C.
Attn: Code 5140

1 Department of the Navy
Office of Naval Research
Washington 25, D. C.
Attn: Code 437

1 Dr. H. Wallace Sinaiko
Institute for Defense Analyses
Research & Engineering Support Division
1666 Connecticut Ave., N.W.
Washington 9, D. C.

1 Data Processing Systems Division
National Bureau of Standards
Conn. at Van Ness
Room 239, Bldg. 10
Washington 25, D. C.
Attn: A. K. Smilow

1 Exchange and Gift Division
The Library of Congress
Washington 25, D. C.

1 NASA Headquarters
Office of Applications
400 Maryland Avenue, S.W.
Washington 25, D. C.
Attn: Mr. A. M. Greg Andrus
Code FC

1 AOGC (PGAPI)
Eglin Air Force Base
Florida

1 Commanding Officer
Office of Naval Research, Chicago Branch
6th Floor, 230 North Michigan
Chicago 1, Illinois

1 Laboratories for Applied Sciences
University of Chicago
6220 South Drexel
Chicago 37, Illinois

1 Librarian
School of Electrical Engineering
Purdue University
Lafayette, Indiana

1 Commanding Officer
U.S. Army Medical Research Laboratory
Fort Knox, Kentucky

2 Keats A. Pullen, Jr.
Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland

1 Commander
Air Force Cambridge Research Laboratories
Laurence G. Hanscom Field
Bedford, Massachusetts
Attn: CRXL

1 Director
U.S. Army Human Engineering Laboratories
Aberdeen Proving Ground, Maryland

5 Scientific & Technical Information Facility
P. O. Box 5700
Bethesda, Maryland
Attn: NASA Representative (S-AK/DL)

1 Mr. James Tippett
National Security Agency
Fort Meade, Maryland

1 Dr. Lloyd Hollingsworth
Director, ERD
AFCRL
L. G. Hanscom Field
Bedford, Massachusetts

1 Major William Harris
Electronics Systems Division
L. G. Hanscom Field
Bedford, Massachusetts

1 Instrumentation Laboratory
Massachusetts Institute of Technology
68 Albany Street
Cambridge 39, Massachusetts
Attn: Library W1-109

1 Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge 39, Massachusetts
Attn: Document Room 26-327

1 Dr. Robert Kingston
Lincoln Laboratories
Lexington, Massachusetts

1 Lincoln Laboratory
Massachusetts Institute of Technology
P. O. Box 73
Lexington 73, Massachusetts
Attn: Library, A-082

1 Sylvania Electric Products Inc.
Electronic Systems
Waltham Labs. Library
100 First Avenue
Waltham 54, Massachusetts

1 Minneapolis-Honeywell Regulator Co.
Aeronautical Division
2600 Ridgeway Road
Minneapolis 13, Minnesota
Attn: Mr. D. F. Elwell
Main Station: 625

1 Inspector of Naval Material
Bureau of Ships Technical Representative
1902 West Minnehaha Avenue
St. Paul 4, Minnesota

20 Activity Supply Officer, USAELRDL
Building 2504, Charles Wood Area
Fort Monmouth, New Jersey
For: Accountable Property Officer
Marked: For Inst. for Exploratory Research
Inspect at Destination
Order No. 5776-PM-63-91

1 Commanding General
U.S. Army Electronic Command
Fort Monmouth, New Jersey
Attn: AMSEL-RE

1 Mr. A. A. Lundstrom
Bell Telephone Laboratories
Room 2E-127
Whippany Road
Whippany, New Jersey

1 AFMDC (MDSGP/Capt. Wright)
Holloman Air Force Base
New Mexico

1 Commanding General
White Sands Missile Range
New Mexico

1 Microwave Research Institute
Polytechnic Institute of Brooklyn
55 John Street
Brooklyn 1, New York

1 Cornell Aeronautical Laboratory, Inc.
4455 Genesee Street
Buffalo 21, New York
Attn: J. P. Desmond, Librarian

1 Sperry Gyroscope Company
Marine Division Library
155 Glen Cove Road
Carle Place, L.I., New York
Attn: Mrs. Barbara Judd

1 Rome Air Development Center
Griffiss Air Force Base
New York
Atten: Documents Library
RAALD

1 Library
Light Military Electronics Department
General Electric Company
Armament & Control Products Section
Johnson City, New York

1 Columbia Radiation Laboratory
Columbia University
538 West 120th Street
New York 57, New York

1 Mr. Alan Barnum
Rome Air Development Center
Griffiss Air Force Base
Rome, New York

1 Dr. E. Howard Holt
Director
Plasma Research Laboratory
Rensselaer Polytechnic Institute
Troy, New York

3 Commanding Officer
U.S. Army Research Office (Durham)
Box CM, Duke Station
Durham, North Carolina
Attn: CRD-AA-IP, Mr. Ulsh

1 Battelle-DEFENDER
Battelle Memorial Institute
505 King Avenue
Columbus 1, Ohio

1 Aeronautical Systems Division
Navigation and Guidance Laboratory
Wright-Patterson Air Force Base
Ohio

1 Aeronautical Systems Division
Directorate of Systems Dynamic Analysis
Wright-Patterson Air Force Base
Ohio

1 Commanding Officer (AD-5)
U.S. Naval Air Development Center
Johnsville, Pennsylvania
Attn: NADC Library

2 Commanding Officer
Frankford Arsenal
Philadelphia 37, Pennsylvania
Attn: SMUFA-1300

1 H. E. Cochran
Oak Ridge National Laboratory
P. O. Box X
Oak Ridge, Tennessee

1 U.S. Atomic Energy Commission
Office of Technical Information Extension
P. O. Box 62
Oak Ridge, Tennessee

1 President
U.S. Army Air Defense Board
Fort Bliss, Texas

1 U.S. Air Force Security Service
San Antonio, Texas
Attn: ODC-R

1 Director
Human Resources Research Office
The George Washington University
300 North Washington Street
Alexandria, Virginia

20 ASTIA Technical Library AFL 2824
Arlington Hall Station
Arlington 12, Virginia
Attn: TISLL

1 Commander
U.S. Army Research Office
Highland Building
3045 Columbia Pike
Arlington 4, Virginia

1 U.S. Naval Weapons Laboratory
Computation and Analysis Laboratory
Dahlgren, Virginia
Attn: Mr. Ralph A. Niemann

2 Army Material Command
Research Division
R & D Directorate
Bldg. T-7
Gravelley Point, Virginia