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DISTRIBUTED RC NETWORKS WITH RATIONAL TRANSFER FUNCTIONS

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ABSTRACT

A distributed RC circuit analogous to a continuously tapped transmission line can be made to have a rational short-circuit transfer admittance and one rational short-circuit driving-point admittance. A subcircuit of the same structure has a rational open circuit transfer impedance and one rational open circuit driving-point impedance. Hence, rational transfer functions may be obtained while considering either generator impedance or load impedance. The functions have poles only along the negative real axis. Although the number of poles is arbitrary, only two may be chosen with complete freedom in a single unit. The residue of a pole in the short-circuit transfer admittance is related to the corresponding residue of the rational driving-point short-circuit admittance. A wide class of transfer functions may be realized by placing one distributed circuit in parallel with the output of another distributed circuit. The loss and total capacity required are often the same order of magnitude as those for lumped circuits. Complex zeroes may readily be obtained with these circuits. In many cases active elements may be used to an advantage in the same manner as with lumped RC networks.

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CONTENTS

ILLUSTRATIONS

1. INTRODUCTION

During the past decade, RC distributed networks have been used primarily for the purpose of phase-shift oscillators, null networks and low pass filters with relatively slow roll-off characteristics. When applicable, the RC distributed network has the advantage over lumped element RC microcircuits in improved reliability due to the reduction of the number of interconnections[1,2]. However, these networks have not found extensive use in system design. This is due largely to the fact that RC distributed networks which have been previously described[3,4,1,5,6] have transfer functions which are irrational functions of frequency while present methods of system design are based on rational functions. It is the purpose of this paper to present a method by which rational functions may be obtained from distributed RC networks.

It has been recognized that with proper control of the morphology and the control of the parameters that a wide range of functions should be possible[7,8]. The problem in general is formidable. The solution of the partial differential equations which describe a given structure is usually obtainable in a series form. It then becomes necessary to satisfy the boundary conditions. Although for a given structure a computer solution may be obtained with reasonable accuracy, an analytic solution has been obtained only for a few simple configurations.

The approach used here limits the morphology to one that can be solved in closed form. The basic structure is analogous to a distributed RC transmission line which is continuously tapped.

2. QUALITATIVE APPROACH

Let us consider briefly a distributed RC transmission line which may be continuously tapped without being unproperly loaded. The general solution for this line is easy to obtain. The propagation constant is

$$\Gamma = \sqrt{rcs + rg} \tag{1}$$

where r is the series resistance per unit length, c is the capacitance per unit length and g is the shunt conductance per unit length. The characteristic impedance is

$$Z_o = \frac{r}{\Gamma} \tag{2}$$

If the line is infinitely long the potential at any point along the line is given by the expression

$$V = V_1 e^{-\Gamma x} \tag{3}$$

Let the continuous tap be a function of position such that the output current due to the potential at point x of width Δx is

$$\Delta I_2 = Y(x) V_1^{-\Gamma x} \Delta x \tag{4}$$

In such a case it is recognized that the total output current is

$$I_2 = \int_0^\infty Y(x) V_1 e^{-\Gamma x} dx \tag{5}$$

This integral is the LaPlace integral and hence may be evaluated by existing LaPlace transform tables. As such the transfer admittance is

$$Y_{21} = \frac{I_2}{V_1} = \mathcal{L}\, Y(x) \tag{6}$$

where Γ is the LaPlace transform variable as well as the propagation constant. Replacing Γ by its equivalent from Equation (1) the transfer admittance Y_{21} may be expressed as a function of s.

A line of this type may be produced physically if the admittance function Y(x) in Equation (5) is a part of the shunt admittance of the transmission line. As such, it is necessary that there be no additional time delay in summing the current components. These requirements are satisfied by the construction indicated in Figure 1. The structure is composed of five different layers. The first and fifth are good conductors, the second and fourth are dielectrics which may be lossy, and the third layer is a resistive film. A given short-circuit transfer admittance function of s may be synthesized if s is replaced by its equivalent value in Γ. The inverse transform of this function with Γ as the LaPlace transform variable gives the desired function Y(x). In order for the synthesized function Y(x) not to have rising exponential components, all the poles of $Y_{21}(s)$ are required to be on the negative real axis.

In order to achieve a physical structure the infinite line must be reduced to one of finite length. The error caused by chopping off the infinite line may be zero in some cases. Consider the synthesis of the function with a single pole on the negative real axis.

$$Y_{21} = \frac{k}{s + \lambda} \tag{7}$$

which, with the substitution of the value of s from Equation (1), allows Y(x) to be expressed as

$$Y(x) = \mathcal{L}^{-1} \frac{krc}{\Gamma^2 - rg + rc\lambda} = \frac{krc}{\sqrt{rc\lambda - rg}} \operatorname{Sin}[(\sqrt{rc\lambda - rg})x] \tag{8}$$

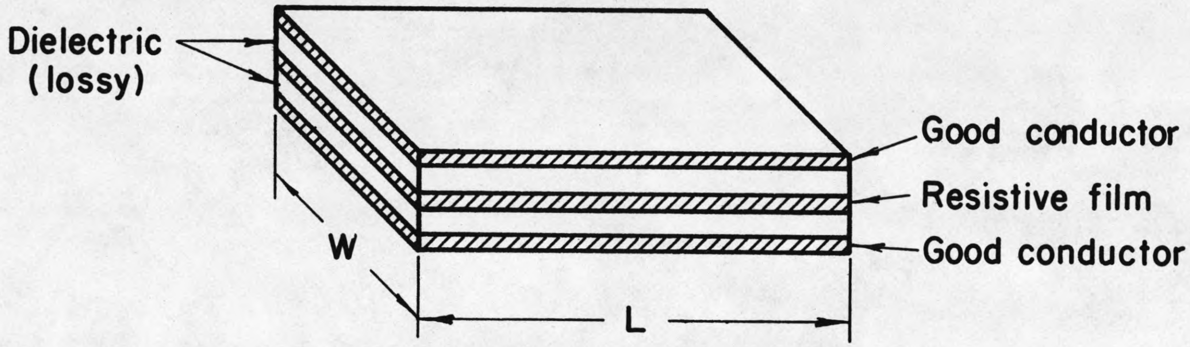


Figure 1. Layer structure

In this case if

$$L = \frac{\pi}{2}\sqrt{rc\lambda - rg} \tag{9}$$

the transfer admittance is the same for the infinite line and the line of length L. The same statement may be made in other cases where Y(x) is a repeating function of x such that

$$Y(x) = (x + 4L) = -Y(x + 2L) = Y(2L - x) \tag{10}$$

This is obvious if one considers the voltage along the line of finite length to be composed of the first incident value plus an infinite number of inner reflected components. The reflection factor at the open end of the line is +1 and the reflection factor at the generator end is -1 since the potential at that point is fixed. Thus the output current I_{2a} due to the incident component is given by

$$I_{2a} = \int_0^L V_1 Y(x) e^{-\Gamma x}\, dx \tag{11}$$

that component due to the first reflection is

$$I_{2b} = \int_0^L V_1 e^{-2\Gamma L} Y(x) e^{\Gamma x}\, dx \tag{12}$$

which upon replacing x by 2L - x becomes

$$\begin{aligned} I_{2b} &= \int_{2L}^{L} V_I e^{-2\Gamma L} Y(2L - x) e^{\Gamma(2L - x)} (-dx) \\ &= \int_{L}^{2L} V_1 Y(x) e^{-\Gamma x}\, dx \end{aligned} \tag{13}$$

This component is reversed in phase when reflected from the generator end and hence the component due to this reflection is

$$I_{2c} = \int_{2L}^{3L} V_1 Y(x) e^{-\Gamma x} \, dx \tag{14}$$

This process is continued indefinitely, and it is seen that the output current is exactly the same for the line of infinite length as the line of length L.

Consider a structure of finite length L which has an admittance function Y(x) that may be expressed as

$$Y(x) = \sum_{n=0}^{\infty} a_n \, \mathrm{Sin} \, \frac{(2n + 1)\pi x}{2L} \tag{15}$$

If this expression has only a finite number of terms, the transfer admittance of the system is rational. The basic type of structure therefore deserves a closer examination as is presented in the next section.

3. QUANTITATIVE APPROACH

3.1 Layer Structure

As a first consideration let the two good conductors of Figure 1 be at the same potential. If the x-y coordinates are chosen so that the resistive film lies in the x-y plane, the potential V between the resistive sheet and the exterior conductors is a function of x, y, and time. Let $\overline{J}_s$ by the surface current density which is a vector in the x-y plane. It is here assumed that all dimensions are small in comparison to a wave length within the dielectric and the good conductors are perfect.

The system is then described by Equations (16) and (17), where

$$\nabla \cdot \overline{J}_s = - C \frac{\partial V}{\partial t} - GV \tag{16}$$

$$\nabla V = -R\overline{J}_s \tag{17}$$

C is the capacitance between the resistive sheet and the exterior conductors per unit area; G is the conductance from the resistive film to the exterior conductors per unit area; R is the resistance per square of the resistive film. R, C, and G are considered to be constants independent of time and position.

Elimination of the current density between Equations (16) and (17) results in Equation (18) in which the potential may be determined as a function of position and time.

$$\nabla \cdot \nabla V = RC \frac{\partial V}{\partial t} + RGV \tag{18}$$

Assuming a product type of solution in which the time dependent component is e^{st}, the general solution of Equation (18) is expressed in Equations (19) and (20).

$$V(x,y,t) = (V_x^+ e^{-\gamma_x x} + V_x^- e^{\gamma_x x})(V_y^+ e^{\gamma_y y} + V_y^- e^{\gamma_y y})e^{st} \qquad (19)$$

$$\gamma_x^2 + \gamma_y^2 = RG + sCR \qquad (20)$$

where the constants V_x^+, V_x^-, V_y^+, V_y^-, γ_x and γ_y are to be determined by the boundary conditions. The component of current in the x direction is then

$$J_x e^{st} = \frac{\gamma_x}{R}(V_y^+ e^{-\gamma_y y} + V_y^- e^{\gamma_y y})(V_x^- e^{-\gamma_x x} - V_x^- e^{\gamma_x x})e^{st} \qquad (21)$$

The component in the y direction is

$$J_y e^{st} = \frac{\gamma_v}{R}(V_x^+ e^{-\gamma_x x} + V_x^- e^{\gamma_x x})(V_y^+ e^{-\gamma_y y} - V_y^- e^{\gamma_y y})e^{st} \qquad (22)$$

Next suppose that the resistive film forms a rectangle in the x-y plane such that $0 \leq x \leq L$, and $0 \leq y \leq W$. Let the potential $V = V_1 e^{st}$ for $x = 0$. Along the remainder of the boundary, let the normal component of the surface current density be zero. Under these conditions, the potential at any point is expressible as

$$V = V_1 e^{st} \frac{\text{Cosh}\, \gamma\, (x - L)}{\text{Cosh}\, \gamma\, L}, \quad \text{where} \quad \gamma = \sqrt{RG + sCR} \qquad (23)$$

This equation satisfies the partial differential equations and the boundary conditions, hence it is the unique solution. The total capacitance between the resistive film and the exterior conductors is a constant independent of position. However, let the capacitance per unit area between the resistive film and the lower conductor be $C_1(x)$, a function of x only and the capacitance between the resistive film and the upper conductor be $C_2(x)$. It is necessary that the sum

$$C_1(x) + C_2(x) = C \tag{24}$$

be a constant. Let the function $C_2(x)$ satisfy the Dirichlet[9] conditions and hence be expressible as

$$C_2(x) = \sum_{n=0}^{\infty} c_n \operatorname{Sin} (2n + 1) \frac{\pi x}{2L} \tag{25}$$

The fact that only odd terms are used does not restrict the type of function since only a quarter of a fundamental length is specified.

The solution is unchanged if the conductivities are defined in a corresponding manner, i.e., $G_1(x)$ is the conductance per unit area from the resistive film to the lower conductor and $G_2(x)$ the conductance per unit area from the resistive film to the upper conductor. It is necessary that the sum

$$G_1(x) + G_2(x) = G(x) \tag{26}$$

be a constant. Let the function $G_2(x)$ be expressed as

$$G_2(x) = \sum_{n=0}^{\infty} b_n \operatorname{Sin} (2n + 1) \frac{\pi x}{2L} \tag{27}$$

3.2 First Two-Port Connection

Consider the layer structure as a two-port device as indicated in Figure 2. When the end on the right is short-circuited, the conditions as described above apply. If we define the total current to the upper conductor as $-I_2e^{st}$ then

$$-I_2 = \int_0^L VW\,[G_2(x) + sC_2(x)]dx \tag{28}$$

This integral with the aid of Equations (23), (25) and (27) may be written as

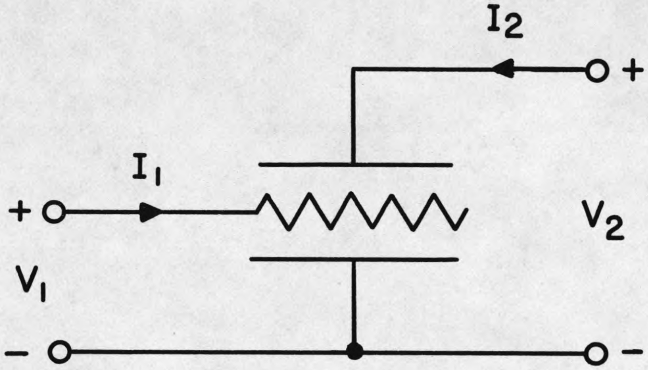


Figure 2. First two-port connection

$$-I_2 = V_1 W \int_0^L \left[\frac{\text{Cosh } \gamma(x - L)}{\text{Cosh } \gamma L} \sum_n^\infty \left[(b_n + sc_n)\text{Sin}(2n + 1)\frac{\pi x}{2L}\right]\right]dx \tag{29}$$

Since each term in this expression is integrable and since the series converges uniformly within this interval, the order of integration and summation may be interchanged. Then each term to be summed is of the form

$$\int_0^L \text{Sin } \frac{(2n + 1)\pi x}{2L} \text{ Cosh } \gamma (x - L)dx = $$

$$\left.\frac{\gamma \text{ Sinh } \gamma (x - L)\text{Sin } \frac{(2n + 1)\pi x}{2L} - \frac{(2n + 1)\pi}{2L} \text{ Cosh } \gamma(x - L)\text{Cos } \frac{(2n + 1)\pi x}{2L}}{\gamma^2 + \frac{(2n + 1)^2\pi^2}{4L^2}}\right|_0^L \tag{30}$$

which, upon substitution of the limits of integration, becomes

$$\frac{(2n + 1)\pi \text{ Cosh } \gamma L}{2L \gamma^2 + \frac{(2n + 1)^2\pi^2}{2L}}$$

Hence the solution to Equation (29) is

$$-I_2 = WV_1 \sum_n^\infty \frac{(2n + 1)\pi(b_n + sc_n)}{2L \gamma^2 + \frac{(2n + 1)^2\pi^2}{2L}} \tag{31}$$

A more useful form is obtained when γ is replaced by its equivalent value from Equation (23).

$$-I_2 = \frac{WV_1}{2LRC} \sum_n^\infty \frac{(2n + 1)\pi(b_n + sc_n)}{s + \frac{G}{C} + \frac{(2n + 1)^2\pi^2}{4L^2RC}} \tag{32}$$

The short-circuit transfer admittance defined as $\frac{+I_2}{V_1}$ is then

$$-Y_{21} = \frac{\pi W}{2LRC} \sum_n^{\infty} \frac{(b_n + sc_n)(2n + 1)}{s + \frac{G}{C} + \frac{(2n + 1)^2 \pi^2}{4L^2 RC}} \tag{33}$$

If the functions $C_2(x)$ and $G_2(x)$ contain only a finite number of non-zero terms, then the short-circuit transfer admittance of Equation (33) may be written in closed form and is rational. These conditions are imposed throughout the remainder of this paper. The current I_1 may be obtained from Equation (21). The short-circuit driving-point admittance for the left end is then

$$Y_{11} = \frac{W}{R} \sqrt{RCs + RG}\ \text{Tanh}\ [L \sqrt{RCs + RG}] \tag{34}$$

The short-circuit driving-point admittance Y_{22} may be obtained after first finding the solution for voltage along the line when a current is injected at a point along the line, or the Green's function[9], when both ends of the two-port are short-circuited. In this case, the boundary conditions become: $J_y = 0$ at $y = 0$ and $y = W$, $J_x = 0$ at $x = L$, $V = 0$ at $x = 0$, and a surface current density is uniformly injected into the resistive film along a line where $x = x_1$. There is still no variation in the y direction, hence Equation (16) and (17) may be expressed as

$$\frac{\partial J_x}{\partial x} = -(sC + G)V + J_{x_1} \delta(x - x_1) \tag{35}$$

$$\frac{\partial V}{\partial x} = - RJ_x \tag{36}$$

where $\delta(x - x_1)$ is the unit impulse or delta function and the time function e^{st} has been divided out of both sides of the equations. Equations (35) and

(36) are functions of x only and may be solved by the LaPlace transform method. The transform of each is given in Equations (37) and (38).

$$pJ_x(p) - J_x(0 +) = - (sC + G)V(p) + J_{x_1} e^{-x_1 p} \tag{37}$$

$$pV(p) = - RJ_x(p) \tag{38}$$

where p is the LaPlace transform variable and it has been recognized that $V(0 +) = 0$. Solving Equations (37) and (38) for V(p) yields

$$V(p) = \frac{-R[J_x(0 +) + J_{x_1} e^{-x_1 p}]}{p^2 - \gamma^2} \tag{39}$$

The inverse transform of Equation (39) is

$$V(x) = \frac{-RJ_x(0 +)}{\gamma} \text{Sinh } \gamma x - RJ_{x_1} U(x - x_1) \frac{\text{Sinh } \gamma(x - x_1)}{\gamma} \tag{40}$$

where $U(x - x_1)$ is the unit step at $x = x_1$. The current density may be found from Equations (36) and (40) or by taking the inverse transform of $J_x(p)$ as determined by Equations (38) and (39).

$$J(x) = J_x(0 +) \text{ Cosh } \gamma x + J_{x_1} U(x - x_1) \text{ Cosh } \gamma(x - x_1) \tag{41}$$

Applying the boundary condition at x = L, it is seen that

$$J_x(0 +) = -J_{x_1} \frac{\text{Cosh } \gamma (L - x_1)}{\text{Cosh } \gamma L} \tag{42}$$

The Green's function for these conditions is then obtained from Equations (40) and (42) as

$$V(x) = \frac{RJ_{x_1}}{\gamma} \left[\frac{\text{Cosh } \gamma(x_1 - L)}{\text{Cosh } \gamma L} \text{Sinh } \gamma x - U(x - x_1) \text{Sinh } \gamma (x - x_1) \right] \quad (43)$$

Now if a voltage V is applied at end two of Figure 2, the equivalent surface current density input at point $x = x_1$ of length Δx_1 may be evaluated by the equivalent circuit of Figure 3.

$$J_{x_1} = V_2 [G_2(x_1) + sC_2(x_1)] \Delta x_1 \quad (44)$$

The potential at any point along the resistive film may then be determined by the relation

$$V = \frac{RV_2}{\gamma} \left[\int_0^L [G_2(x_1) + sC_2(x_1)] \frac{\text{Cosh } \gamma(x_1 - L)}{\text{Cosh } \gamma L} \text{Sinh } \gamma x \, dx_1 + \int_0^x [G_2(x_1) + sC_2(x_1)] \text{Sinh } \gamma(x_1 - x) dx_1 \right] \quad (45)$$

Evaluation of these integrals reveals that

$$V = RV_2 \sum_n \frac{(b_n + sc_n) \text{Sin } \frac{(2n + 1)\pi x}{2L}}{\gamma^2 + \frac{(2n + 1)^2 \pi^2}{4L^2}} \quad (46)$$

It is interesting to use this result to obtain the current at the shorted terminals. The current density at any point along the resistive film may be obtained by taking the gradient of the potential as indicated in Equation (17). Thus using V of Equation (46)

$$J_x = -\frac{1}{R} \frac{\partial V}{\partial x} = \frac{-V_2 \pi}{2L} \sum_n \left[\frac{(2n + 1)(b_n + sc_n) \text{Cos} \frac{(2n + 1)\pi x}{2L}}{\gamma^2 + \frac{(2n + 1)^2 \pi^2}{4L^2}} \right] \quad (47)$$

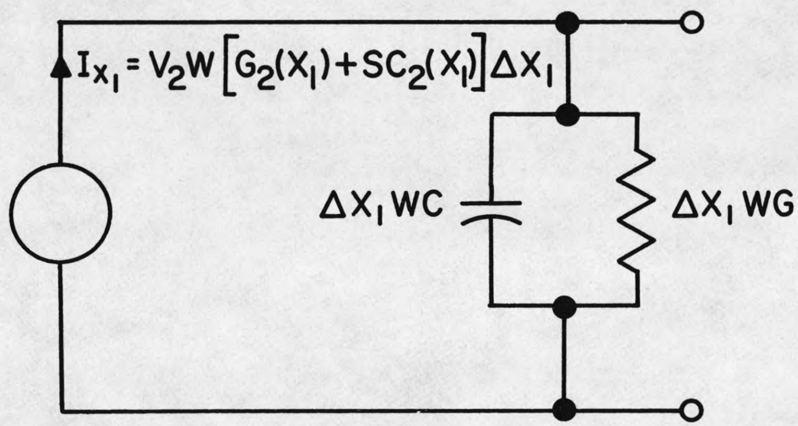


Figure 3. Equivalent current generator

The current density in the y direction is zero. If γ is replaced by its equivalent value as given in Equation (23), the expression multiplied by the width and evaluated at $x = 0$, the current in the shorted terminals is found to be

$$I_1 = \frac{-WV_2\pi}{2LRC} \sum_n \frac{(2n+1)(b_n + sc_n)}{s + \frac{G}{C} + \frac{(2n+1)^2\pi^2}{4L^2RC}} \tag{48}$$

The short-circuit transfer admittance Y_{12} as determined from this relation is in agreement with the value of Y_{21} obtained in Equation (33) by quite a different procedure.

The current I_2 when end one is short-circuited is determined in the following manner:

$$I_2 = W \int_0^L (V_2 - V)[G_2(x) + sC_2(x)]dx \tag{49}$$

Making use of Equation (46), this integral may be evaluated. The short-circuit admittance Y_{22} is then

$$Y_{22} = \frac{2LW}{\pi} \sum_n \left[\frac{b_n + sc_n}{2n+1} - \frac{\pi}{4C} \frac{(b_n + sc_n)^2}{s + \frac{G}{C} + \frac{(2n+1)^2\pi^2}{4L^2RC}} \right] \tag{50}$$

The short-circuit transfer admittance and one short-circuit driving-point admittance are rational functions of s. Hence, rational transfer functions may be obtained.

It should be emphasized here that the rationality of these admittance functions follows in a logical manner, without approximations or assumptions, from the basic Equations (16) and (17), the boundary conditions specified and the partitioning of the admittance parameters as given in Equations (24) - (27).

Equations (16) and (17) are based on the assumption that their dimensions are much smaller than a wave length and the exterior conductors are perfect.

3.3 Second Two-Port Connection

Figure 4 represents an alternate connection of the same layer structure. In this case the resistive film is connected to the lower conductor at the left end. It is seen immediately that the open circuit input impedance Z_{11} for this connection is the reciprocal of the short-circuit admittance Y_{22} of the previous case. Thus

$$Z_{11} = \frac{\pi}{2LW \sum_n \left[\frac{b_n + sc_n}{2n+1} - \frac{\pi}{4C} \frac{(b_n + sc_n)^2}{s + \frac{G}{C} + \frac{(2n+1)^2 \pi^2}{4L^2 RC}} \right]} \tag{51}$$

Next consider end one short-circuited, this places the two good conductors at the same potential so that the original differential equations hold. If a current I_2 is now injected into the right end, the potential along the line is seen to be

$$V = \frac{I_2 R \operatorname{Sinh} \gamma x}{W \gamma \operatorname{Cosh} \gamma L} \tag{52}$$

The current to the upper conductor is now $-I_1$ and may be determined as in Equation (28) with the proper value of V from Equation (52). Then

$$\frac{-I_1}{I_2} = \frac{Z_{12}}{Z_{11}} = \frac{1}{C} \sum_n \frac{[b_n + sc_n](-1)^n}{s + \frac{G}{C} + \frac{(2n+1)^2 \pi^2}{4L^2 RC}} \tag{53}$$

Thus both Z_{11} and Z_{12} are rational functions of the complex frequency variable s. As in the previous case the third open circuit impedance is irrational.

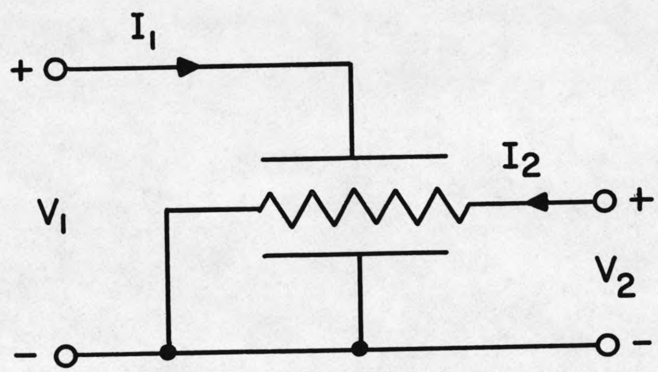


Figure 4. Second two-port connection

It may be determined by the relation

$$Z_{22} = \frac{1}{Y_{22}} + \frac{Z_{12}^2}{Z_{11}} \tag{54}$$

The value of Y_{22} is easily obtained by placing x = L in Equation (52).

$$Y_{22} = \frac{I_2}{V_2} = \frac{W\gamma}{R} \text{ Coth } \gamma L \tag{55}$$

Then,

$$Z_{22} = \frac{R \text{ Tanh } [L\sqrt{RCs - RG}]}{W\sqrt{RCs - RG}} + \frac{Z_{12}^2}{Z_{11}} \tag{56}$$

3.4 Alternate Distribution

In cases where the open circuit voltage ratio or the short-circuit current ratio is to be specified, some functions are more easily realized by choosing the capacitance and conductance functions as

$$C_2(x) = \sum_n a_{cn} \text{ Cos } \frac{n\pi}{L} x \tag{57}$$

and

$$G_2(x) = \sum_n a_{gn} \text{ Cos } \frac{n\pi}{L} x \tag{58}$$

Referring to the two-port connection of Figure 2, the short-circuit transfer admittance may be found to be

$$-Y_{21} = \frac{W\gamma}{RC} [\text{Tanh } \gamma L] \sum_n \frac{a_{gn} + a_{cn} s}{s + \frac{G}{C} + \frac{n^2 \pi^2}{4L^2 RC}} \tag{59}$$

The input short-circuit admittance is of course the same as Equation (35). Neither of these functions is rational, but, their ratio is rational. The short-circuit current transfer ratio, and the open circuit voltage ratio in the reverse direction are rational.

$$- \frac{Y_{21}}{Y_{11}} = \frac{1}{C} \sum_{n} \frac{a_{gn} + a_{cn} s}{s + \frac{G}{C} + \frac{n^2 \pi^2}{4L^2 RC}} \tag{60}$$

The direct derivation of the short-circuit current transfer ratio reveals that Equation (60) is dependent upon the partitioning of the conductance and the capacitance in the same way as the other structure. This relationship is given in the next chapter.

4. CONSTRUCTION

4.1 Alternate Morphology

The construction illustrated in Figure 1 is quite adequate mathematically, but presents a few difficulties to the builder. For ease of construction the structure of Figure 5 is desirable. Particularly for the cases where only the capacitance function needs to be varied, the function $u = C_2(x)$ may be cut or etched onto a rectangular distributed circuit with constant capacitance per unit area. The short-circuit transfer function is readily attainable. However, when the two good conducting surfaces are not at the same potential, additional modes will be excited in the y direction unless the resistance of the resistive film in the y direction is essentially zero. When the resistance in the y direction is essentially zero, it is possible to add the conductance functions $G_1(x)$ and $G_2(x)$ externally from the film to the two good conductors. If the conductivities are added externally, it will be necessary to consider the added conductivity in the x direction or build the conductances with essentially zero conductance in the x direction. Shaping the exterior conductances represents an easy method of control as a function of x.

4.2 Equivalent Distributed Systems

The continuously tapped nonuniform line also has potential. The general problem of the nonuniform transmission line has received some renewed attention recently[10,11]. The primary purpose of these lines in the past has been for the purpose of impedance transformation. These nonuniform lines are generally more difficult to construct than the uniform lines. A class of these lines which are equivalent to uniform transmission lines has been investigated[12]. This idea may be extended since each nonuniform line is usually one of a class

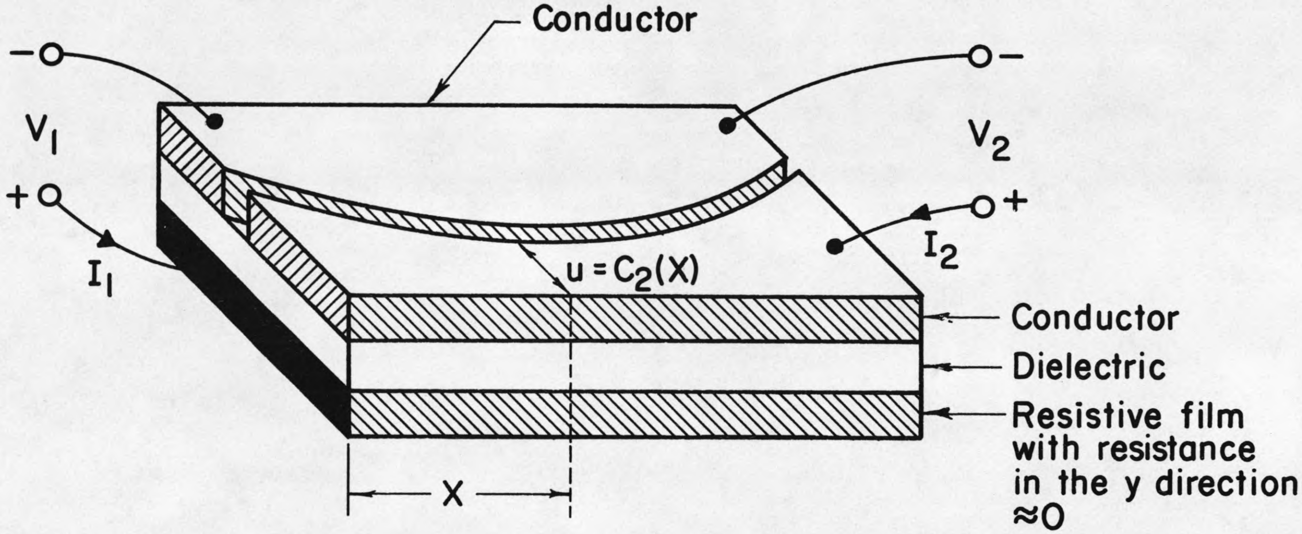


Figure 5. Equivalent construction

of equivalent lines. Some of the equivalent lines are desirable from the standpoint of construction. For instance, it may be desirable to have either the loop impedance or the shunt admittance constant[13]. In most cases this is possible as a special case of the following theorem.

<u>Theorem</u>: Given a transmission line which is characterized by

$$\frac{dV}{dx} = - Zf_1(x)I \tag{61}$$

and

$$\frac{dI}{dx} = - Yf_2(x)V \tag{62}$$

where Z and Y are functions of frequency only and both $f_1(x)$ and $f_2(x)$ are differentiable positive non-zero functions over the interval, and given a differentiable, positive, non-zero function $f_b(x)$, there exists a function $f_a(x)$ such that a transmission line which is described by

$$\frac{dV}{dx} = - Z\, f_a(x)I \tag{63}$$

and

$$\frac{dI}{dx} = - Y\, f_b(x)V \tag{64}$$

behaves in exactly the same manner as the given line at any number of terminal points. A similar theorem holds for the existence of $f_b(x)$ if $f_a(x)$ is given.

<u>Proof</u>: Consider a transmission line described by Equations (61) and (62). Let a new variable be defined by the single valued function

$$u = h(x) \tag{65}$$

where u is zero when x is zero. Then from Equations (63), (64) and (65)

$$\frac{dV}{du} = \frac{dV}{dx}\frac{dx}{du} = - Z\ f_a(x)\ \frac{dx}{du}\ I \tag{66}$$

$$\frac{dI}{du} = \frac{dI}{dx}\frac{dx}{du} = - Y\ f_b(x)\ \frac{dx}{du}\ V \tag{67}$$

The solution of Equations (66) and (67) as a function of u may be made to be the same as the solution of Equations (61) and (62) as a function of x by imposing that

$$f_a(x)\ \frac{dx}{du} = f_1(u) \tag{68}$$

and

$$f_b(x)\ \frac{dx}{du} = f_2(u) \tag{69}$$

Now h(x) may be determined from Equation (68) if $f_a(x)$ is preassigned or from Equation (69) if $f_b(x)$ is preassigned. For a given $f_b(x)$, Equations (65) and (69) reveal that

$$\int_0^u f_2(u)\ du = \int_0^x f_b(x)\ dx \tag{70}$$

which has a solution of the form

$$F_2(u) = F_b(x) \tag{71}$$

The requirement that $f_2(x)$ be positive and non-zero over the interval insures that Equation (71) has only one solution for u within the interval. Thus h(x) may be determined for each value of x.

The dependent function $f_a(x)$ may now be determined from Equations (65),

(68) and (69) as

$$f_a(x) = \frac{f_1[g(x)]}{f_2[g(x)]} f_b(x) \tag{72}$$

A completely analogous procedure may be followed if $f_a(x)$ is specified. Hence the theorem is proved and a procedure established for finding the other equivalent transmission line.

In order to show the general usefulness of the above theorem, a typical nonuniform line has been chosen. The exponentially tapered distributed RC line has series resistance and shunt capacitance that vary with x as

$$r = r_o e^{ax} \tag{73}$$

and

$$c = c_o e^{-ax} \tag{74}$$

where "a" is a constant either positive or negative[3]. Let it be desired to find an equivalent line in which the capacitance per unit length is a constant. Then

$$f_b(x) = m \tag{75}$$

where "m" is a constant which determines the amount of capacitance per unit length. The solution of Equations (65) and (70) determine that

$$h(x) = \frac{1}{a} \ln \frac{1}{(1 - amx)} \tag{76}$$

Hence from (72)

$$f_a(x) = \frac{m}{(1 - amx)^2} \tag{77}$$

The equivalent line thus has

$$r = \frac{r_o m}{(1 - amx)^2} \tag{78}$$

and

$$c = c_o m \tag{79}$$

If the length of the exponential line is L, then the length of the equivalent line L_e is

$$L_e = \frac{1 - e^{-aL}}{am} \tag{80}$$

As a two-port the two lines are completely equivalent. If taps are placed on one line to form an n-port, taps placed on the corresponding points of the other line as determined by h(x) make an equivalent n-port.

When the general theorem above is applied to the layer structure of Figure 1, the problem is somewhat different. The series resistance and the total shunt capacitance of the line are constants in th original form. In the most desirable case

$$G_1(x) = \frac{G}{C} c_1(x) \tag{81}$$

where G may be zero. In this case it is only necessary to control the thickness of the (lossy) dielectric. From the standpoint of construction, things would be much simpler if the lower dielectric were of constant thickness. The price is nonuniform resistance and an exaggerated variation of the upper dielectric thickness.

Let the line for which a solution has been obtained be of length L and represented by: R, C and W which are constant and

$$c_2(x) = 0.5C \text{ Sin } \frac{\pi x}{2L} \tag{82}$$

$$c_1(x) = C(1 - 0.5 \text{ Sin } \frac{\pi x}{2L}) \tag{83}$$

Then

$$f_1(x) = 1, \quad Z = RW \tag{84}$$

and

$$f_2(x) = 1, \quad Y = sCW \tag{85}$$

The new distributed circuit will be of constant width W' and the parameters will be functions of x only. Primed letters will denote corresponding properties, i.e., C', $R'(x)$, etc. The change of variable may be viewed as a change of scale along the x direction. Therefore all functions are transformed in a relation similar to Equations (68) and (69). In order to make c_1 a constant it is necessary that

$$\frac{dx}{du} = \frac{Wc_1(u)}{W'c_1'} = \frac{WC(1 - 0.5 \text{ Sin } \frac{\pi u}{2L})}{W'c_1'} \tag{86}$$

Following the procedure indicated by Equation (70) it is apparent that

$$\frac{x}{L} = \frac{WC}{W'c_1'} \left[\frac{u}{L} - \frac{1}{\pi} + \frac{1}{\pi} \text{ Cos } \frac{\pi u}{2L} \right] \tag{87}$$

Noting that when $u = L$, $x = L'$, it is seen that

$$\frac{L'}{L} = \frac{WC}{W'c_1'}(0.6819) \tag{88}$$

Hence,

$$\frac{x}{L'} = \frac{1}{0.6819} \left[\frac{u}{L} - \frac{1}{\pi} + \frac{1}{\pi} \text{ Cos } \frac{\pi u}{2L} \right] \tag{89}$$

The function $h(x) = u$ is thus transcendental. It may be represented graphically in a striaghtforward manner since for each value of u, x may be determined.

It is not necessary to plot this function. For each value of u the function of the corresponding value of x may be determined from the relation

$$f_a(x) = f_b(x) = \frac{.6819 \frac{L}{L'}}{1 - .5 \sin \frac{\pi u}{2L}} \tag{90}$$

Then the transformations may be performed as follows:

$$\frac{R'(x)}{W'} = \frac{Rf_a(x)}{W} \tag{91}$$

$$W'C'(x) = WCf_b(x) \tag{92}$$

$$W'c_2'(x) = Wc_2(u)\ f_b(x) = WC'(x) - Wc_1' \tag{93}$$

$$W'c_1' = \frac{WL'}{L} C(0.6819) \tag{94}$$

The results of this transformation are given in Table I, and the parameters are plotted in Figure 6 for the special case of $L' = L$ and $W' = W$.

In general the transformations keep the total series resistance, the total capacitance, the total capacitance above the film, and the total capacitance below the film constant. In other words, the following integrals

$$\int_0^L WRdx, \qquad \int_0^L WCdx, \qquad \int_0^L Wc_1(x)dx, \qquad \text{and} \qquad \int_0^L Wc_2(x)dx$$

are invariant. This may be accomplished by choosing

$$f_b(x) = m \tag{95}$$

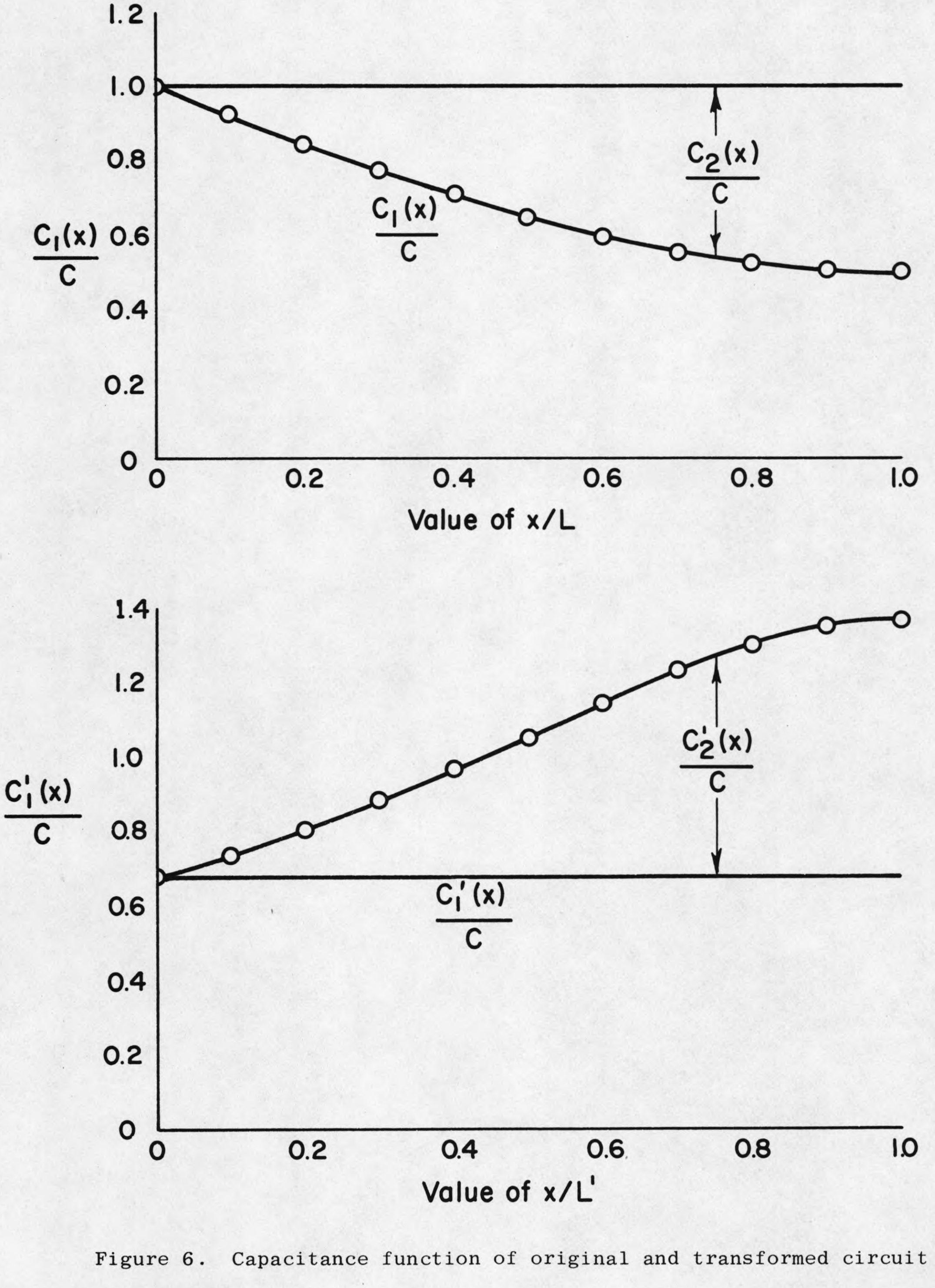


Figure 6. Capacitance function of original and transformed circuit

TABLE I

Calculated Values for the Equivalent Structure

$\frac{u}{L}$	$\frac{x}{L'}$	$\frac{L'}{L} f_b(x)$	$\frac{W'L'}{WLC} c_2'(x)$
0	0	0.6819	0
0.1	0.141	0.739	0.058
0.2	0.271	0.806	0.1245
0.3	0.389	0.881	0.200
0.4	0.498	0.964	0.283
0.5	0.597	1.053	0.372
0.6	0.688	1.142	0.462
0.7	0.772	1.228	0.547
0.8	0 852	1.300	0.619
0.9	0.926	1.345	0.664
1.0	1.000	1.362	0.682

then

$$u = mx \tag{96}$$

$$L = mL' \tag{97}$$

$$\frac{R'}{W'} = \frac{mR}{W} \tag{98}$$

$$\frac{C'}{W'} = \frac{mC}{W} \tag{99}$$

$$\frac{c_1'(x)}{W'} = \frac{mc_1(x)}{W} \tag{100}$$

Hence, it is always possible to scale the dimensions in a linear manner by any amount, provided the total parametric values remain constant.

4.3 Tolerance

The question arises here as to the tolerance which may be required for the partitioning of G(x) and C(x) If the tolerance could easily be specified in terms of additional harmonic content, then the answer would be apparent from the residues of the poles as given in Equation (12). In order that the admittance functions be entirely rational, it is necessary that the functions $C_2(x)$ and $G_2(x)$ as expressed in Equations (8) and (10) contain only a finite number of terms. Errors in the production of either of these functions perhaps produces additional harmonics and, hence, additional poles with comporatively small residues. In order to establish an upper limit to the effect of these unwanted components, let the function $C_2(x)$ be broken down into two components so that

$$C_2(x) = C_{2d} + C_{2e} \tag{101}$$

where $C_{2d}(x)$ is the desired function and $C_{2e}(x)$ is the error component due to manufacturing process, etc. The short-circuit current I_{2e} due to this error in $C_2(x)$ is then given by the expression

$$-I_{2e} = \int_0^L sC_{2e}(x)Vdx \tag{102}$$

The potential as expressed in Equation (6) has a maximum magnitude of V_1 at $x = 0$. Hence,

$$\left|I_{2e}\right| \leq \left|V_1 s\right| \int_0^L \left|C_{2e}(x)\right| dx \tag{103}$$

Thus the magnitude of the error current is no greater than the current through a capacitor with a value of

$$C_e = \int_0^L \left|C_{2e}(x)\right| dx \tag{104}$$

This error may be quite small with reasonable tolerance requirements. It should be noted that at the higher frequencies, due to the attenuation, only the capacitance error near the input is important. The error current is then perhaps much less than the upper limit established.

A similar argument establishes the upper limit of error current due to errors in the production of $G_2(x)$ as the current which would flow through a conductance of the value

$$G_e = \int_0^L \left|G_{2e}(x)\right| dx \tag{105}$$

5. PROPERTIES OF THE CAPACITANCE AND CONDUCTANCE FUNCTIONS

The functions $C_2(x)$ and $G_2(x)$ are normally expressed in the form of a trigonometric series as in Equations (25) and (27). It is often necessary to examine such a function to see if it meets the physical restrictions which are most compactly stated as

$$0 \leqq C_2(x) \leqq C \tag{106}$$

and

$$0 \leqq G_2(x) \leqq G \tag{107}$$

These assure that each of the functions $C_1(x)$, $C_2(x)$, $G_1(x)$ and $G_2(x)$ are positive over the interval $0 \leqq x \leqq L$.

In checking a given function to see if it meets the requirement of Equation (106) or (107) it is usually easier to transform the trigonometric equation into a polynomial. The requirement for the simplest case of three poles is worked out in detail to illustrate a method and to give this useful solution.

The general form of a conductance or capacitance function for three poles with n = 0, 1, 2 is

$$f = a_0 \text{ Sin } \frac{\pi x}{2L} + a_1 \text{ Sin } 3\frac{\pi x}{2L} + a_2 \text{ Sin } 5\frac{\pi x}{2L} \tag{108}$$

A change of variable which amounts to the measurement of an angle from the opposite end of the structure converts the function in Equation (108) to the form

$$f = a_0 \text{ Cos } \nu - a_1 \text{ Cos } 3\nu + a_2 \text{ Cos } 5\nu \tag{109}$$

where

$$\nu = \frac{\pi}{2} - \frac{\pi x}{2L} \tag{110}$$

Equation (109) may be put in the form of a polynomial since it may be represented as the sum of Tchebyscheff polynomials. For this, let

$$\nu = \text{Cos}^{-1} u \tag{111}$$

then

$$\text{Cos } n\nu = T_n(u) \tag{112}$$

Now Equation (109) becomes

$$f = a_0 T_1(u) - a_1 T_3(u) + a_2 T_5(u) \tag{113}$$

or

$$f = a_0 u - a_1(4u^3 - 3u) + a_2(16u^5 - 20u^3 + 5u) \tag{114}$$

Since the relative values of the coefficient are improtant it is convenient to normalize the coefficients so that $a_0 = 1$. This is the most logical choice since in all functions a_0 must be greater than zero. The function to be examined is then

$$F = 16A_2 u^5 - (20A_2 + 4A_1)u^3 + (1 + 3A_1 + 5A_2)u \tag{115}$$

The interval over which Equation (115) must be examined is $0 \leqq x \leqq L$ which corresponds to $0 \leq u \leq 1$. If Equation (115) is to be positive over this interval, then F must not have any roots of odd multiplicity within the interval and F must be positive at some point. Equation (115) may be still further simplified before examination. Since u is positive over the range, then $\frac{F}{u}$ must also be positive over the range. Hence, let $y = u^2$, then

$$\frac{F}{u} = g(y) = 16A_2y^2 - (20A_2 + 4A_1)y + 1 + 3A_1 + 5A_2 \tag{116}$$

Once again the range is $0 \leqq y \leqq 1$, for which it is necessary that there be no roots of odd multiplicity and that the function be positive at some point within the interval of interest. First let us examine Equation (116) for complex roots. Complex roots occur if the discriminant is negative or

$$(20A_2 + 4A_1)^2 - 4(16A_2)(1 + 3A_1 + 5A_2) \leqq 0 \tag{117}$$

Setting the discriminant to zero and performing the indicated operations and reducing yields

$$5A_2^2 - 2A_1A_2 + A_1^2 - 4A_2 = 0 \tag{118}$$

This is the equation of an ellipse in the $A_1 - A_2$ plane. If the axis are rotated by the transformation

$$\xi = -0.2297A_1 + 0.973A_5 \tag{119}$$

$$\eta = 0.973A_1 + 0.2297A_5 \tag{120}$$

the center of the ellipse is at $\xi = 0.373$ and $\eta = 0.600$. The major axis is in the η direction and is 2(1.142). The minor axis is in the ξ direction and is 2(0.437). Points within the ellipse correspond to complex roots of Equation (116) and represent acceptable values. As will be shown later, the function is positive for these values.

Next, let us use a Sturm test to determine the roots within the desired interval. The derivative of Equation (116) is

$$g'(y) = 32A_2y - 20A_2 + 4A_1 \qquad (121)$$

This function will be divided by 4 to form the Sturm function $g_1(y)$. The next function is

$$g_2(y) = \frac{1}{A_2}(A_1^2 + 5A_2^2 - 2A_1A_2 - 4A_2) \qquad (122)$$

The part within the parenthesis in Equation (122) is exactly the same as the discriminant which is given in Equation (118). Hence $g_2(y)$ is negative within the ellipse and in the lower half plane. It is positive elsewhere.

The Sturm's functions given by Equations (116), (121) and (122) may now be checked at $y = 0$ and $y = 1$. With each substitution into Equations (116) and (121) they become straight lines in the $A_1 - A_2$ plane. They are positive on one side of the line and negative on the other. Thus the region in the A plane for which the function f is non-negative over the length of line is determined. This region is indicated in Figure 7.

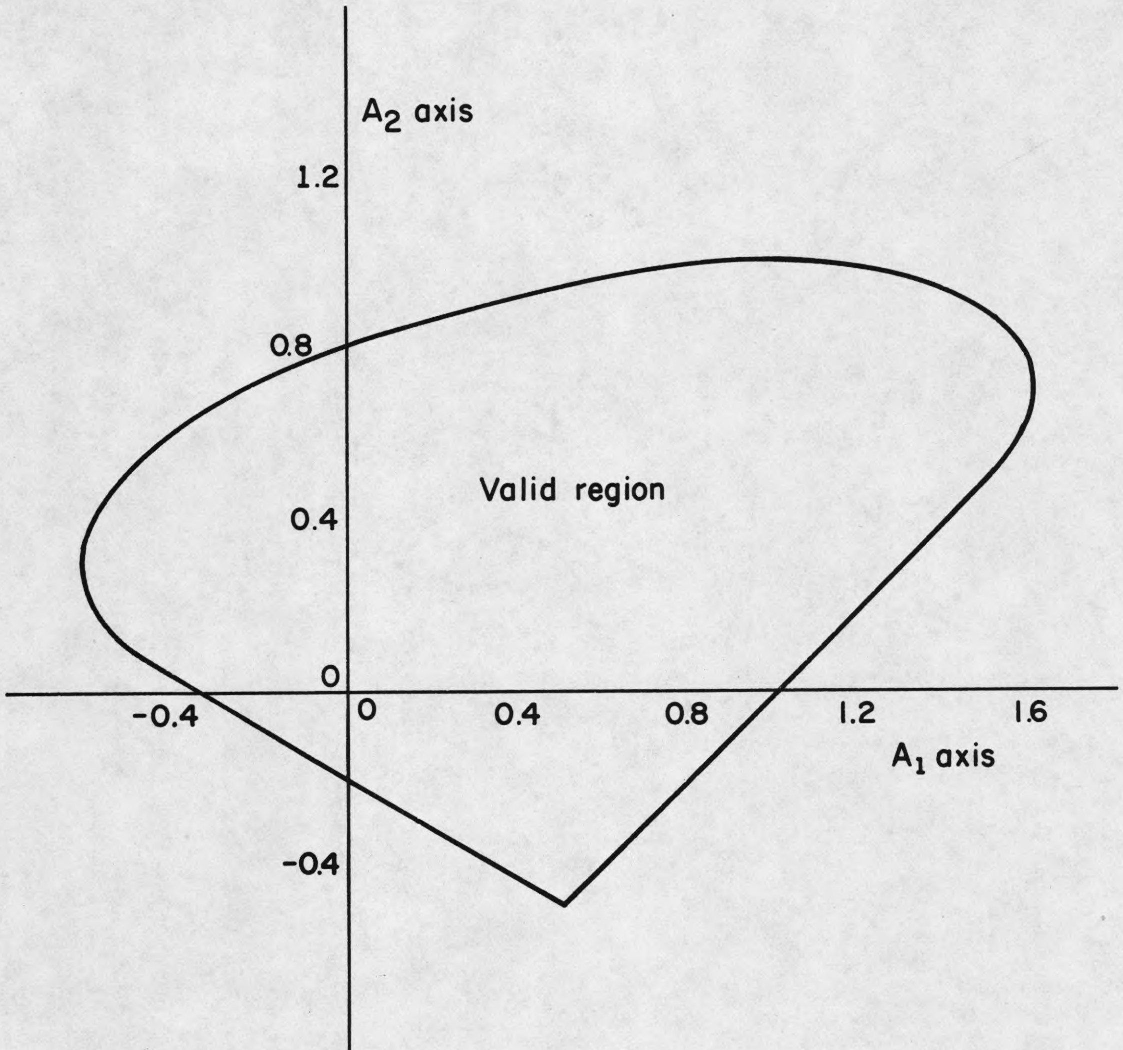


Figure 7. Three pole coefficient region

6. PROPERTIES OF THE FIRST TWO-PORT CONNECTION

6.1 Poles and their Residues

There are limitations to the functions that can be obtained from the layer structure of either Figure 1 or Figure 5. If the admittance parameters for the two-port of Figure 2 are expanded in partial fraction form, Equations (33) and (50) become Equations (123) and (124) respectively.

$$-Y_{21} = \frac{\pi W}{2LRC} \sum_n \left[(2n + 1) \left[c_n + \frac{b_n - c_n \lambda_n}{s + \lambda_n} \right] \right] \tag{123}$$

$$Y_{22} = \frac{W2L}{\pi} \sum_n \left[\frac{bn}{2n + 1} - (2b_n - \lambda_n c_n) c_n \left(\frac{\pi}{4C}\right) - \left(\frac{\pi}{4C}\right) \frac{(b_n - \lambda_n c_n)^2}{s + \lambda_n} + s\left(\frac{c_n}{2n + 1} - \frac{\pi}{4C} c_n^2\right) \right] \tag{124}$$

where

$$\lambda_n = \frac{G}{C} + \frac{(2n + 1)^2 \pi^2}{4L^2 RC} \tag{125}$$

There are two restrictions immediately apparent. First, the residues of the poles of Y_{21} are related to the residues of the poles of Y_{22} in the following manner:

$$\text{residue of the general pole in } Y_{21} = \frac{\pi W (2n + 1)(b_n - \lambda_n c_n)}{2LRC} \tag{126}$$

$$\text{residue of the general pole in } Y_{22} = \frac{WL(b_n - \lambda_n c_n)^2}{2C} \tag{127}$$

Second, there are two degrees of freedom plus the choice of odd integers squared in determining the poles.

If only two poles are to be used, they may be chosen as single poles on the negative real axis without difficulty. However, when three poles are present, the following relations hold.

$$\lambda_0 = \frac{G}{C} + \frac{\pi^2}{4L^2RC} \tag{128}$$

$$\lambda_1 = \frac{G}{C} + \frac{(2n_1 + 1)^2\pi^2}{4L^2RC} \tag{129}$$

$$\lambda_2 = \frac{G}{C} + \frac{(2n_2 + 1)^2\pi^2}{4L^2RC} \tag{130}$$

Then

$$\frac{\lambda_2 - \lambda_0}{\lambda_1 - \lambda_0} = \frac{(2n_2 + 1)^2 - 1}{(2n_1 + 1)^2 - 1} \tag{131}$$

The ratio $\frac{\lambda_2 - \lambda_0}{\lambda_1 - \lambda_0}$ can be made arbitrarily close to any positive value greater than unity. But two physical requirements must be faced: (1) the higher harmonics may require tolerances which can not be achieved, and (2) the value of $\frac{G}{C}$ must be positive and less than λ_o. Ratios which may be obtained with small order harmonics are given in Table II. The possible location of additional poles may be obtained from the same table if subscript "2" is changed to "k". Upon solving Equations (128) and (129) for $\frac{G}{C}$, it is seen that the relation

$$1 < \frac{\lambda_1}{\lambda_0} \leqq (2n_1 + 1)^2 \tag{132}$$

must hold.

TABLE II

Ratio of $\frac{\lambda_2 - \lambda_0}{\lambda_1 - \lambda_0}$

n_1 \ n_2	1	2	3	4	5	6	7
1	1	3	6	10	15	21	28
2		1	2	3.33	5	7	9.33
3			1	1.67	2.5	3.5	4.66
4				1	1.5	2.1	2.8
5					1	1.4	1.87
6						1	1.33
7							1

6.2 Two-Terminal Networks

If degenerate forms of the layer structure are considered, then all RC driving-point admittance functions are obtainable as a circuit formed by the interconnection of a number of distributed circuits. This statement is certainly true since a single resistor and a single capacitor are degenerate forms of the layer structure.

There are limitations on the admittance functions which may be obtained by a single distributed network. The general form of an RC admittance function is

$$Y = k_0 - \left(\frac{k_1}{s + \lambda_1} + \frac{k_2}{s + \lambda_2} + \cdots \frac{k_n}{s + \lambda_n}\right) + k_{n+1}s \tag{133}$$

The necessary and sufficient conditions that Equation (133) be an RC admittance function is that all the k's are positive and the value at s = 0 is nonnegative as expressed by

$$k_0 \geqq \frac{k_1}{\lambda_1} + \frac{k_2}{\lambda_2} + \cdots \frac{k_n}{\lambda_n} \tag{134}$$

These two necessary conditions are sufficient since the expression

$$\frac{k_j}{\lambda_j} - \frac{k_j}{s + \lambda_j} = \frac{\frac{k_j}{\lambda_j} s}{s + \lambda_j} \tag{135}$$

is realizable as a series resistor and capacitor.

The general form of a short-circuit driving-point admittance of Equation (123) should be compared to Equation (133). Here it is seen that k_{n+1} is determined altogether by $C_2(x)$. Although this residue of the pole at infinity is composed of the sum of terms which are both positive and negative, it can not arbitrarily be made zero. If one observes the physical system, it is apparent that as s approaches infinity, the effect of the loss becomes insignificant and the value of $\frac{Y_{22}}{s}$ approaches the value of the integral

$$\int_0^L \left\{ \frac{dx}{\frac{1}{C_1(x)} + \frac{1}{C_2(x)}} \right\}$$

This integral is certainly nonnegative since both $C_1(x)$ and $C_2(x)$ are nonegative. Moreover, it can be zero only if either $C_1(x)$ or $C_2(x)$ is zero for all values of x within the interval. Hence if the system contains a finite number of poles, the residue of the pole at infinity is zero only in case $c_n = 0$ for all n.

The value at s = 0 is completely determined by $G_2(x)$ as may be seen from Equation (50). Again the physical system requires that the value at s = 0 is zero only if $G_2(x) = 0$. Therefore, the RC driving-point admittance functions with a finite number of poles which may be realized with these distributed networks, have the property that there is a zero at the origin if and only if there is a pole at infinity.

If during the synthesis procedure the pole at infinite and the value at the origin are neglected, reasonably good control may be maintained over the relative values of the residues of the internal poles. If such a procedure results in a network function which has a residue of the pole at infinity which is too small and a value at the origin which is too small, then a single resistor and capacitor in parallel with the structure completes the synthesis.

Whether the complete structure is to be "molecular" or deposited film type, one of the limiting factors determining the overall size of a unit is the total capacitance. It is interesting to compare the total capacitance of a distributed network to that of an equivalent lumped element network.

Consider first the case where there is no pole at infinity, then $C_2(x) = 0$. The negative of the residue of each pole may be expressed

$$k_n = \frac{WLb_n^2}{2C} \tag{136}$$

which by the aid of Equation (125) becomes

$$k_n = \frac{WLC}{2}(\lambda_n - \frac{(2n+1)^2\pi^2}{4L^2RC})^2(\frac{b_n}{G})^2 \tag{137}$$

The total capacitance C_f required by a circuit obtained by the Foster admittance pole expansion is

$$C_f = \sum_n \frac{k_n}{\lambda_n^2} = \frac{WLC}{2}\sum_n \left[1 - \frac{(2n+1)^2\pi^2}{4L^2RC\,\lambda_n}\right]^2 (\frac{b_n}{G})^2 \tag{138}$$

Hence the total capacitance C_t required for the distributed circuit is

$$C_t = WLC > C_f \tag{139}$$

The case for zero at the origin is somewhat similar. The negative of a single residue is

$$k_n = \frac{WLC}{2}\lambda_n^2 (\frac{c_n}{C})^2 \tag{140}$$

Hence the total capacitance

$$C_t = WLC = \frac{2k_n}{\lambda_n^2(\frac{c_n}{C})^2} \tag{141}$$

where n is any value. This circuit not only takes care of the poles mentioned, but also provides a pole at infinity. Thus the equivalent Foster admittance pole expansion requires a total capacitance of

$$C_f = \sum_n \frac{k_n}{\lambda_n^2} + \frac{2C_t}{\pi} \sum_n \frac{c_n}{C(2n+1)} - \frac{\pi c_n^2}{4C^2} \tag{142}$$

or

$$C_f = \frac{2C_t}{\pi} \sum_n \frac{c_n}{C(2n+1)} \tag{143}$$

For this case no other canonical form requires less total capacitance than the Foster admittance pole expansion. The situation here is slightly more favorable. As an example, take the distributed network for which G = 0.

Let $c_1 = .333c_0$, $c_2 = 0.200c_0$. Then the total capacitance required by the distributed network is $0.934\ c_0WL$ whereas the total capacitance required for the lumped equivalent is $0.734c_0WL$. Generally speaking, the total capacitance required by the distributed network, whether it is obtained as a single structure or the interconnection of several is of the same order of magnitude as that required by the equivalent lumped circuit. Usually it is within the range obtained by the four canonic forms.

6.3 Transfer Admittance Functions

Not all RC transfer admittance functions may be obtained by a single distributed network. Consider the single pole element first.

$$-Y_{21} = \frac{\pi W}{2LRC} \left[c_o + \frac{b_0 - c_0 \lambda_0}{s + \lambda_0} \right] \tag{144}$$

A pole at infinity is not obtainable without an additional parallel capacitor which may easily be built into the structure of Figure 1 or Figure 5. The zero of Equation (144) is at

$$s = -\frac{b_0}{c_0} \qquad (145)$$

and may be anywhere along the negative real axis.

The two pole case has a short-circuit transfer admittance function

$$-Y_{21} = \frac{\pi W}{2LRC}\left[c_0 + (2n + 1)c_n + \frac{b_0 - c_0\lambda_0}{s + \lambda_0} + \frac{(2n + 1)(b_n - c_n\lambda_n)}{s + \lambda_n}\right] \qquad (146)$$

as determined by Equation (123). Except for the case of $c_0 + (2n + 1)c_n = 0$, there are two zeroes of transmission and they are located at

$$s = -\alpha \pm \sqrt{\alpha^2 - b_0\lambda_n - (2n + 1)\lambda_0 b_n} \qquad (147)$$

where

$$\alpha = \frac{c_0\lambda_n + (2n + 1)c_n\lambda_0 + b_0 + (2n + 1)b_n}{2[c_0 + (2n + 1)c_n]} \qquad (148)$$

If $c_0 + (2n + 1)c_n = 0$ then one of the zeroes is at infinity. If the zeroes are complex, they have a real part which is negative since

$$b_0 + (2n + 1)b_n \geq 0 \qquad (149)$$

and

$$c_0\lambda_n + (2n + 1)c_n\lambda_0 > 0 \qquad (150)$$

When the two roots are real, a single zero may be at the origin.

In any case zeroes are restricted from the positive real axis as this is a three-terminal network. Poles occur singularly on the negative real axis but not at infinity.

The general case of three or more poles is more complex, but zeroes can be obtained on the $j\omega$ axis as well as in the right half plane. This may be illustrated by the following example.

Suppose that it is desirable to produce a transfer admittance of

$$-Y_{21} = \frac{s(s^2 + 1)}{(s + 0.05)(s + 0.45)(s + 1.25)} \tag{151}$$

The poles for this example have been chosen so that a lossless dielectric may be employed. In this case the relation in Equation (33) is easier to employ than Equation (123). When $G = 0$,

$$-Y_{21} = \frac{\pi W}{2LRC}\left[\frac{c_0 s}{s + 0.05} + \frac{3c_1 s}{s + 0.45} + \frac{5c_2 s}{s + 1.25}\right] \tag{152}$$

If the desired function is divided by s, expanded into partial fractions and then multiplied by s,

$$-Y_{21} = \frac{2.09s}{s + 0.05} - \frac{3.755s}{s + 0.45} + \frac{2.67s}{s + 1.25} \tag{153}$$

Hence

$$\frac{c_1}{c_0} = -\ 0.599 \tag{154}$$

and

$$\frac{c_2}{c_0} = 0.2555 \tag{155}$$

When these values are checked by Figure 7, they are seen to be possible although they are not far from the boundary. In this case we see that

$$C_2(x)_{max} = C_2(L) = c_0 - c_1 + c_2 \qquad (156)$$

If we let the maximum value of $C_2(x)$ just equal C, then the condition of Equation (106) is satisfied and the maximum gain may be obtained. Then

$$\frac{C}{c_0} = 1.854 \qquad (157)$$

and from Equations (152) and (153) the total resistance required is

$$\frac{RL}{W} = 0.405 \text{ ohms} \qquad (158)$$

From Equation (33)

$$\frac{\pi^2}{4L^2RC} = 0.05 \qquad (159)$$

Therefore the total capacitance is

$$WLC = 121\ 8 \text{ farads} \qquad (160)$$

The value of the output admittance may be determined by Equation (124) and is

$$Y_{22} = \frac{10.35s^4 + 23.3s^3 + 12.31s^2 + 0.707s}{(s + 0.05)(s + 0.45)(s + 1.25)} \qquad (161)$$

A lumped circuit which has the same value of Y_{22} and a value of $0.1135Y_{12}$ as obtained by the Guillemin method[14] is shown in Figure 8. The total capacitance required by the lumped network is 26 farads. However, the gain is only 0.1135 times that of the distributed circuit.

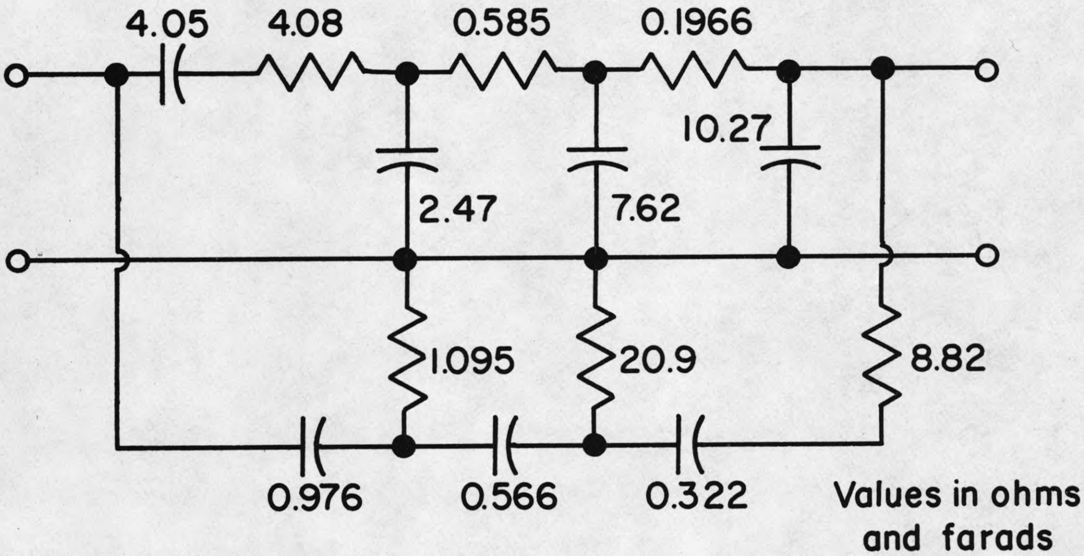


Figure 8. Equivalent circuit by Guillemin's method

It should be observed that the total capacitance required for a given voltage transfer is not a fixed quantity. If the load impedance is real and fixed, the total capacitance may always be reduced with a corresponding increase in loss. This is true since for a network which produces a given transfer function with a given load resistance, a similar network with the impedance level scaled up by a factor of k will produce the same transfer function divided by k if a resistor which is equal to (1 - k) times the load resistance is placed in series with the load and the network. A figure of merit is then defined as a constant times the gain factor divided by the total capacitance. This figure of merit does not change with the type of transformation mentioned above.

If the constant is chosen so the figure of merit for the distributed circuit of the example is 1.0, the corresponding value for the lumped equivalent circuit is 0.557.

When a three terminal network has a load of G_L, the voltage transfer function is

$$\frac{V_2}{V_1} = \frac{Y_{21}}{G_L + Y_{22}} \tag{162}$$

The first two-port connection of the layer structure always produces a zero in this voltage transfer ratio at $s = \infty$. This is due to the fact that Y_{22} has a pole at infinity which is not in Y_{21} if each $c_n \neq 0$ and Y_{21} has a zero at infinity if each $c_n = 0$. The second two-port connection of Figure 4 does not have this restriction.

7. PASSIVE NETWORK SYNTHESIS WITH DISTRIBUTED NETWORKS

7.1 A General Synthesis Procedure

As has been pointed out, not all driving-point admittances nor all short-circuit transfer admittances may be obtained by means of the distributed RC network of the first two-port connection. But, given a Y_{21} and a Y_{22} each of which are realizable in the form of a distributed RC network, then kY_{21} and Y_{22} are realized simultaneously if Y_{22} contains all the poles of Y_{21}.

The proof of the above statement is as follows: given a Y_{21} it may be expanded in partial fraction form, and, if possible, a distributed network may be found which has this short-circuit transfer admittance. The distributed network then has an output short-circuit admittance Y'_{22}. These values may be scaled either up or down so that a network is possible which has the short-circuit parameters kY_{21} and kY'_{22}. If the admittance Y''_{22} is placed in parallel with the output terminals of this network, then the synthesis is complete provided $Y''_{22} = Y_{22} - kY'_{22}$. This is a realizable RC admittance function for k equal to or greater than zero since Y'_{22} contains only poles which are in Y_{22}. Furthermore, Y''_{22} is realizable as a distributed network and k equal to zero only in the case that Y_{22} lies on the boundary of possible functions realizable as RC distributed networks. This is true, since given the poles os an RC admittance function, there is a group of intervals in which the zeroes may lie and be realizable as an RC distributed network. The zeroes of Y''_{22} are continuous functions of k, and hence in all but the limiting cases, k may be greater than zero.

As for lumped element synthesis, there is considerable choice in the values of Y_{21} and Y_{22} required to obtain a given voltage transfer function. The overall gain constant is, of course, dependent upon the choice of these functions. In the first example to follow the poles are chosen arbitrarily and the procedure carried out as explained above. In the next example the poles are chosen for an advantage from the standpoint of configuration.

7.2 Example Illustrating the General Procedure

It is desired to find a distributed network which will provide a voltage transfer function of

$$\frac{V_2}{V_1} = \frac{k(s^2 + 2s + 2)}{(s + 1)(s + 2)(s + 4)} = \frac{-Y_{21}}{1 + Y_{22}} \tag{163}$$

when loaded in a one-ohm load.

The zeroes of Y_{21} are thus given. The restrictions on the poles of Y_{22} are that one must lie within the interval -1 to -2, one within the interval -2 to -4 and one within the interval less than -4. It is further necessary that the corresponding Y_{21} and Y_{22} be realizable as properties of the distributed networks.

Choosing the poles of Y_{22} at -1.5, -3 and infinity, one may expand Y_{21} as

$$-Y_{21} = \frac{k(s^2 + 2s + 2)}{(s + 1.5)(s + 3)} = k\left[1 + \frac{0.833}{s + 1.5} - \frac{3.33}{s + 3}\right] \tag{164}$$

From Equation (123) we see that this may also be expanded as

$$-Y_{21} = \frac{\pi W}{2LRC}\left[c_0 + 3c_1 + \frac{b_0 - 1.5c_0}{s + 1.5} + \frac{3b_1 - 9b_1}{s + 3}\right] \tag{165}$$

Hence we have the following equalities.

$$c_0 + 3c_1 = \frac{2kLRC}{\pi W} \tag{166}$$

$$k_0 - 1.5c_0 = \frac{0.833k(2LRC)}{\pi W} \tag{167}$$

$$3b_1 - 9c_1 = \frac{-3.33k(2LRC)}{\pi W} \tag{168}$$

In these three equations there are effectively five unknowns. It is necessary that the solution be such that the ratios $\frac{c_1}{c_0}$ and $\frac{b_1}{b_0}$ be physically possible. Thus let

$$b_1 = -\frac{b_0}{3} \tag{169}$$

Then

$$c_0 = \frac{0.333k(2LRC)}{\pi W} \tag{170}$$

$$c_1 = \frac{0.2222k(2LRC)}{\pi W} \tag{171}$$

$$b_0 = \frac{1.333k(2LRC)}{\pi W} \tag{172}$$

$$b_1 = \frac{-0.444k(2LRC)}{\pi W} \tag{173}$$

From this selection of poles and Equation (125) it is seen that

$$1.5 = \frac{G}{C} + \frac{\pi^2}{4L^2RC} \tag{174}$$

and

$$s = \frac{G}{C} + \frac{9\pi^2}{4L^2RC} \tag{175}$$

Hence

$$\frac{G}{C} = 1.3125 \tag{176}$$

and

$$\frac{\pi^2}{4L^2RC} = 0.1875 \tag{177}$$

The condition expressed by Equation (107) in this case demands that

$$G \leqq \frac{4}{3}\, b_0 \tag{178}$$

Thus

$$\frac{kLR}{W} \geqq 1.158 \tag{179}$$

The maximum value is chosen here, then

$$\frac{kLR}{W} = 1.158 \tag{180}$$

$$\frac{c_0}{C} = 0.245 \tag{181}$$

$$\frac{c_1}{C} = 0.1636 \tag{182}$$

$$\frac{b_0}{C} = 0.982 \tag{183}$$

$$\frac{b_1}{C} = -0.3275 \tag{184}$$

and

$$k = 0.0880WLC \tag{185}$$

The corresponding output admittance for this distributed network is obtained from Equation (124).

$$Y'_{22} = \frac{WLC}{2}\left[\frac{4}{\pi}\left(\frac{b_0}{C} + \frac{b_1}{3C}\right) - \left(\frac{2b_0}{C} - 1.5\,\frac{c_0}{C}\right)\frac{c_0}{C} - \left(\frac{2b_1}{C} - \frac{3c_1}{C}\right)\left(\frac{c_1}{C}\right)\right. \tag{186}$$

$$\left. - \frac{\left(\frac{b_0}{C} - 1.5\,\frac{c_0}{C}\right)^2}{s + 1.5} - \frac{\left(\frac{b_1}{C} - 3\,\frac{c_1}{C}\right)^2}{s + 3} + s\left(\frac{4}{\pi}\,\frac{c_0}{C} + \frac{4c_1}{3\pi C} - \frac{c_0^2}{C^2} - \frac{c_1^2}{C^2}\right)\right]$$

Then

$$Y'_{22} = WLC\left[0.452 - \frac{0.01892}{s + 1.5} - \frac{0.334}{s + 3} + 0.1475s\right] \tag{187}$$

Now the output admittance required from the voltage transfer function specified is

$$Y_{22} = \frac{(s + 1)(s + 2)(s + 4)}{(s + 1.5)(s + 3)} - 1 = 1.5 - \frac{0.416}{s + 1.5} - \frac{1.333}{s + 3} + s \tag{188}$$

The additional output admittance required is then

$$Y''_{22} = Y_{22} - Y'_{22} \tag{189}$$

which is a realizable RC function if

$$WLC \leqq 2.37 \tag{190}$$

In this case the limiting value is determined by the value at s = 0 rather than by a residue of a pole. Once again the limiting value is chosen to give

$$Y''_{22} = 0.426 - \frac{0.371}{s + 1.5} - \frac{0.651}{s + 3} + 0.650s \tag{191}$$

The chosen value of WLC determines the overall gain constant and completes the specifications for the first distributed network. The total capacitance required for this network is

$$WLC = 2.37 \tag{192}$$

and the overall gain constant is

$$k = 0.208 \tag{193}$$

Therefore Y''_{22} may be realized by a structure having $G_2(x) = 0$. It should be pointed out that if WLC is chosen less than the limit, Y''_{22} will have a non-zero value at s = 0. This excess conductance may be removed and the remaining value of Y''_{22} realized. The excess value of conductance may then be added to the load and the problem scaled so that the load is one ohm. The procedure may be carried out if the limiting value is exceeded, it being only necessary that where the negative conductance so obtained is added to the load, a positive value remains. The optimum gain constant is not necessarily obtained when the limiting value is used. By means of Equations (191) and (124), the following values are needed for Y''_{22}:

$$b''_0 = b''_1 = 0 \tag{194}$$

$$\frac{c_0''}{C''} = 0.271 \tag{195}$$

$$\frac{c_1''}{c_0''} = 0.439 \tag{196}$$

$$\frac{G''}{C''} = 1.3125 \tag{197}$$

and the total capacitance required here is

$$W''L''C'' = 4.49 \text{ farads} \tag{198}$$

The total resistance required for Y_{22}'' is

$$\frac{R''L''}{W''} = 0.550 \text{ ohms} \tag{199}$$

It is interesting to note that the total capacitance required by this solution is 6.84 farads. Figure 9 shows this circuit.

7.3 Example Illustrating A Special Case

Let us again consider the transfer function of Equation (163). Often it is possible to choose the poles in such a manner that the additional shunt admittance Y_{22}'' is zero. This example illustrated such a procedure.

Let the desired voltage ratio be expressed as

$$\frac{V_2}{V_1} = \frac{KP(s)}{M(s)} \tag{200}$$

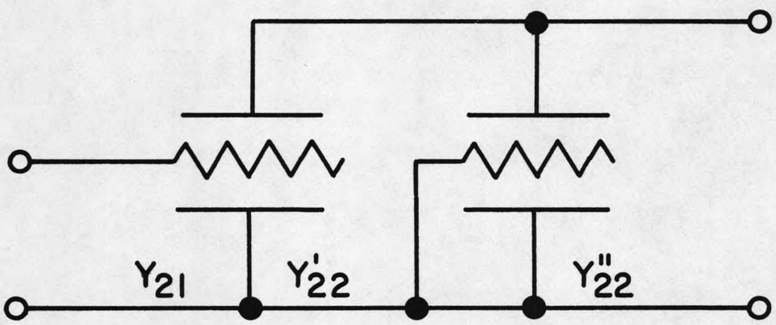


Figure 9. Circuit obtained by the general procedure

Recognizing the relative values of the residues of the poles in Y_{21} and Y_{22} as given in Equations (123) and (124), this voltage transfer ratio may also be written as

$$\frac{V_2}{V_1} = \frac{\frac{\pi}{L^2 R}\left[k_2 + \frac{k_0}{s + \lambda_0} + \frac{3k_1}{s + \lambda_1}\right]}{k_3 - \frac{k_0^2}{s + \lambda_0} - \frac{k_1^2}{s + \lambda_1} + k_4 s} \tag{201}$$

where

$$k_0 = b_0 - c_0 \lambda_0 \tag{202}$$

$$k_1 = b_1 - c_1 \lambda_1 \tag{203}$$

$$k_2 = c_0 + 3c_1 \tag{204}$$

$$k_3 = \frac{2C}{WL} + \frac{4C}{\pi}\left(b_0 + \frac{b_1}{3}\right) - (2b_0 - \lambda_0 c_0)c_0 - (2b_1 - \lambda_1 c_1)c_1 \tag{205}$$

$$k_4 = \frac{4Cc_0}{\pi} + \frac{4Cc_1}{3\pi} - c_0^2 - c_1^2 \tag{206}$$

If the constant in Equation (200) is expressed as

$$K = \frac{\pi k_2}{L^2 R k_4} \tag{207}$$

then the polynomials P(s) and M(s) may be expressed as

$$P(s) = \left[1 + \frac{k_0}{k_2(s + \lambda_0)} + \frac{3k_1}{k_2(s + \lambda_1)}\right] (s + \lambda_0)(s + \lambda_1) \tag{208}$$

and

$$M(s) = \left[\frac{k_3}{k_4} - \frac{k_0^2}{k_4(s + \lambda_0)} \mp \frac{k_1^2}{k_4(s + \lambda_1)} + s\right] (s + \lambda_0)(s + \lambda_1) \tag{209}$$

These polynomials each have a coefficient of one for the term of the highest power of s and are thus known. The following identities are apparent.

$$\frac{k_0}{k_2} = \frac{P(-\lambda_0)}{\lambda_1 - \lambda_0} \tag{210}$$

$$\frac{3k_1}{k_2} = - \frac{P(-\lambda_1)}{\lambda_1 - \lambda_0} \tag{211}$$

$$\frac{k_0^2}{k_4} = - \frac{M(-\lambda_0)}{\lambda_1 - \lambda_0} \tag{212}$$

$$\frac{k_1^2}{k_4} = \frac{M(-\lambda_1)}{\lambda_1 - \lambda_0} \tag{213}$$

$$\frac{k_3}{k_4} = \left[\frac{M(s)}{(s + \lambda_0)(s + \lambda_1)} - s\right]_{s = \infty} \tag{214}$$

Although there are seven unknowns in the last five equations, there are only four independent ratios of constants plus the two pole positions. Note from

Equations (204) and (211)

$$\frac{k_0}{3k_1} = - \frac{P(-\lambda_0)}{P(-\lambda_1)} \tag{215}$$

and from Equations (211) and (214)

$$\frac{k_0^2}{k_1^2} = - \frac{M(-\lambda_0)}{M(-\lambda_1)} \tag{216}$$

Therefore,

$$\left[\frac{3P(-\lambda_0)}{P(-\lambda_1)}\right]^2 = - \frac{M(-\lambda_0)}{M(-\lambda_1)} \tag{217}$$

Hence, if the position of one pole is chosen, the other is fixed by Equation (217). There is only a narrow range for which a complete solution is possible, however. If λ_1 is chosen as 3.500, then λ_0 is found to be 1.134.

The ratios expressed in Equation (210) through Equation (214) are now determined. The five equations beginning with Equation (202) have effectively six unknowns. If c_1 is made to be equal to c_{0_1} then a solution is given by

$$WLC = 98.88 \tag{218}$$

$$C_2(x) = 0.0736C \left[\mathrm{Sin}\,\frac{\pi x}{2L} + \mathrm{Sin}\,\frac{3\pi x}{2L}\right] \tag{219}$$

$$G_2(x) = 0.0210C\,\mathrm{Sin}\,\frac{\pi x}{2L} - 0.0431C\,\mathrm{Sin}\,\frac{3\pi x}{2L} \tag{220}$$

$$\frac{RL}{W} = 0.0844 \tag{221}$$

$$G = 0.838C \tag{222}$$

Then

$$K = 0.972 \tag{223}$$

$$-Y_{21} = \frac{5.48(s^2 + 2s + 2)}{(s + 1.134)(s + 3.50)} \tag{224}$$

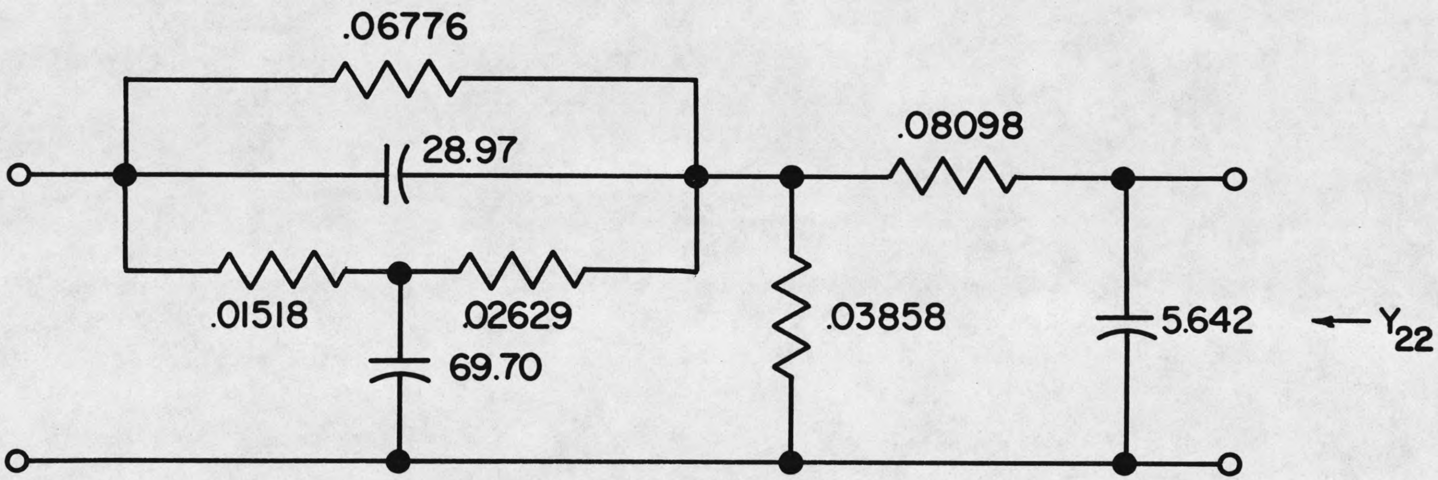
$$Y_{22} = \frac{5.64(s^3 + 6.82s^2 + 13.18s + 7.30)}{(s + 1.134)(s + 3.50)} \tag{225}$$

$$Y_{11} = 34.2\sqrt{s + 0.838} \quad \text{Tanh}\left[2.89\sqrt{s + 0.838}\right] \tag{226}$$

In comparing this result with that of the general procedure, we see that the total capacitance here is approximately fifteen times as large and the gain constant is approximately five times as large. If a reference is chosen so that the circuit of Figure 9 has a figure of merit of 1.0, this circuit has a figure of merit of 0.323. In order to obtain a rough comparison to a lumped circuit, this Y_{22} and Y_{21} to within a multiplying constant were synthesized by a Dasher procedure[4]. The lumped circuit has a short-circuit transfer admittance of

$$-Y_{21} = \frac{12.32(s^2 + 2s + 2)}{(s + 1.134)(s + 3.50)} \tag{227}$$

and an input short-circuit admittance of



Values in ohms and farads

Figure 10. Equivalent dasher circuit

$$Y_{11} = \frac{140.8\ s^2 + 2.165\ s + 0.543}{(s + 1.134)(s + 3.50)} \tag{228}$$

The corresponding figure of merit for this circuit is 0.784.

7.4 Effect of Generator Impedance

The low loss obtained by the two circuits is accompanied by a very low input impedance. Since both procedures assume a generator with zero impedance, it is interesting to note the effect of adding a small generator impedance. This is illustrated in Figure 11. The relative response for a zero impedance generator is shown along with the response of each circuit when the generator has an impedance of one percent of the load impedance. The reference level for each circuit is the zero frequency value when the generator has zero impedance. In this particular case, it is seen that the response is very sensitive with respect to the generator impedance.

7.5 Unit Size

In order to illustrate the approximate size of the circuits which are discussed, the circuit just obtained will be scaled up to a useful frequency and impedance range. If the frequency scale is multiplied by 10^3 and the impedance scale by 10^4, the total capacitance required is 9.888 microfarads. The size required depends upon the type of structure and the type of dielectric. Using a film type[15] with a capacitance of 0.5 microfarads per $cm.^2$, a total area of 19.78 cm^2 is required. This may be obtained with a width of 1.0 inches and a length of 3.07 inches. By comparison the circuit obtained by the general procedure has an input impedance which is much higher and is thereby less sensitive to the generator impedance and requires a total area of only 1.29 cm^2. It was assumed earlier that the dimensions of the system are small in comparison

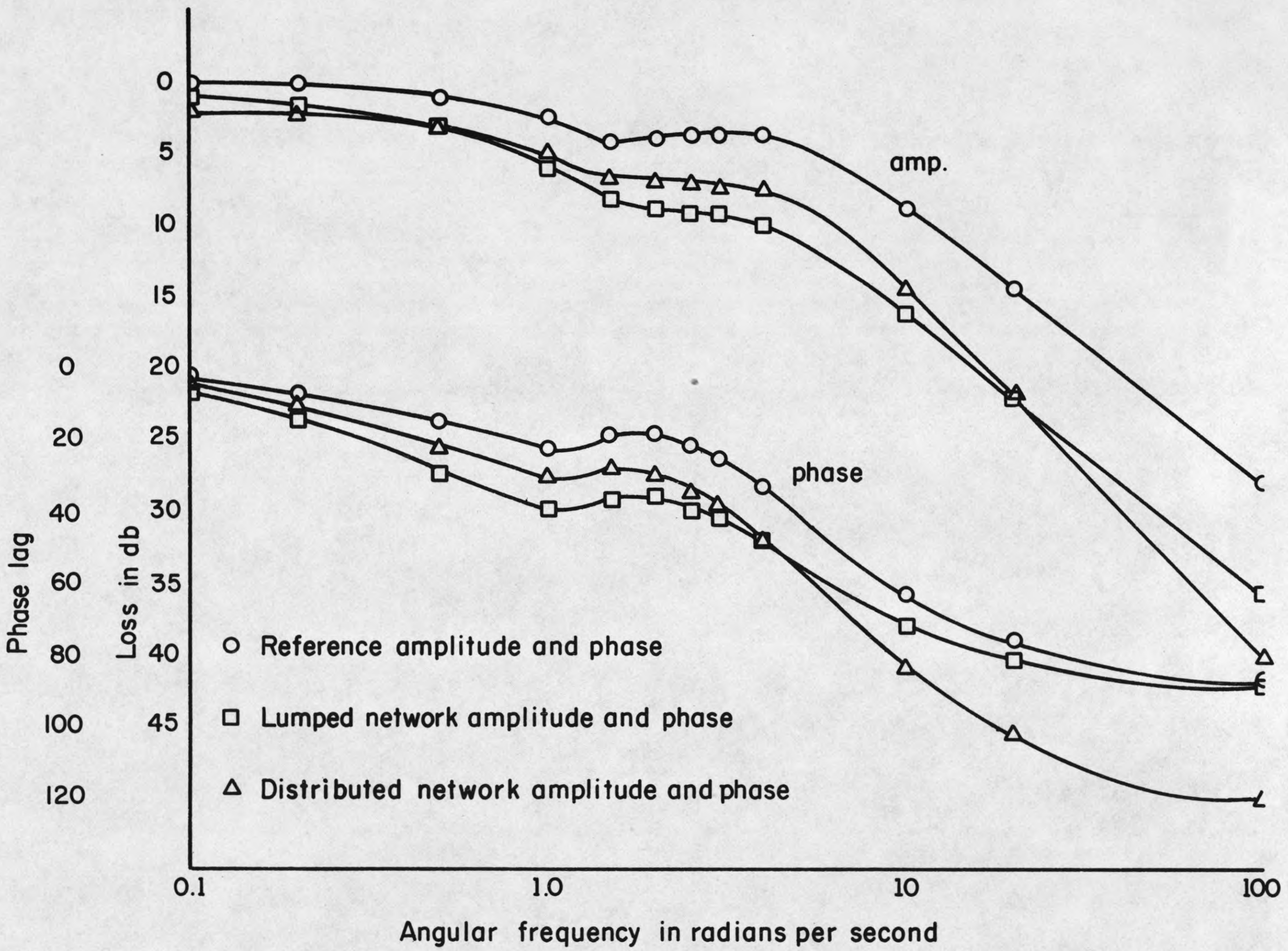


Figure 11. Graph showing the effect of generator impedance

to a wave length within the dielectric. The relative dielectric constant of the tantalum dioxide assumed above is approximately 26. Hence, within the dielectric,

$$\text{the wave length} = \frac{3(10)^{10}}{f\sqrt{26}} \text{ cm.} = \frac{5.88(10)^{9}}{f} \text{ cm.} \tag{229}$$

where f is in c.p.s. The highest frequency shown in Figure 11 when scaled up as in the example is $\frac{10^5}{2\pi}$. Hence the shortest wave length considered there is approximately $3.7(10)^5$ cm. The approximation that the dimensions are small in comparison to a wave length within the dielectric is certainly valid for this example.

8. ACTIVE NETWORKS

Many of the restrictions which have been cited may be removed by means of active elements. Two examples are included here in order to illustrate some of the advantages gained by active circuits.

An old, widely used active network employed the RC twin-T network in the feedback path of an amplifier[16]. The open circuit voltage transfer function of the twin-T RC network is of the form

$$\frac{V_2}{V_1} = K \frac{(s^2 + 1)}{s^2 + as + 1} \tag{230}$$

where $a \geqq 4$. The constant "a" is usually chosen to be equal to 4 as this gives the steepest slope approaching the null point. One of the principal differences between this function and that obtained heretofore with distributed networks is that the phase shift changes from -90 to +90 degrees as the null point is traversed along the $j\omega$ axis. Thus, if the amplifier itself has only 180^o phase shift, the system is stable regardless of the gain of the amplifier where the twin-T is the feedback path.

It has been pointed out that the function of Equation (230) is not possible with the first two-port connection alone since it does not have a zero at $s = \infty$. However, the function is realized in the general form of Figure 12. This general circuit is very useful in removing some of the earlier restrictions. In this circuit a phase inverter is used instead of a voltage inversion type NIC. The reason for this is simply that where one is concerned with voltage ratio only, and where the generator impedance is small in comparison to the driving-point impedance of the network, the two voltages which are needed are

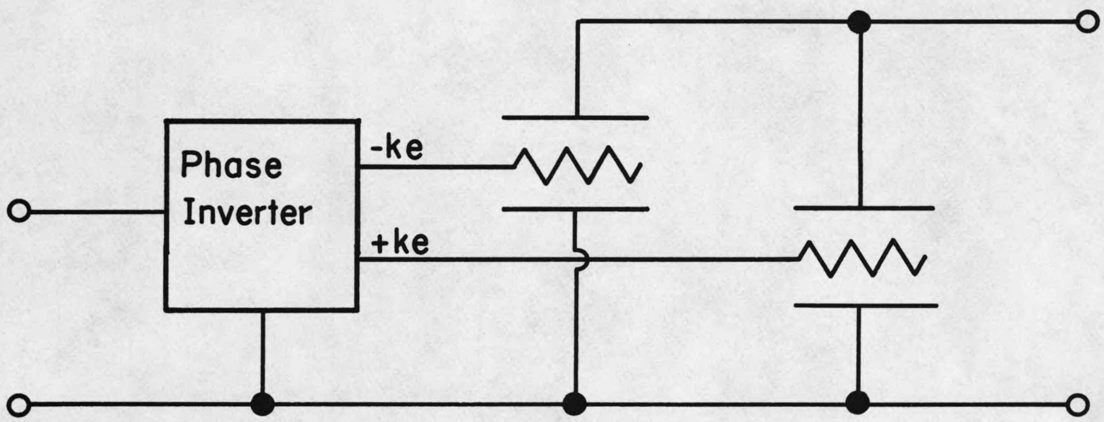


Figure 12. Circuit showing use of phase inverter

often obtainable from the original driving sources. This may be accomplished, for example, by taking one point from the collector, the other from the unbypassed emittor if the driver is a transistor stage.

The circuit of Figure 12 places the restriction on Equation (230) that $a > 2$. This simply requires that the poles be distinct and lie on the negative real axis. This network may produce a slope approaching the null point almost twice that of the twin-T.

At the lower frequencies, when a low impedance generator is coupled to the low input impedance of a transistor through an RC network, it is sometimes advisable to make the general impedance level of the network somewhat higher than either the load or the generator impedance. This is a means of keeping the size of the RC network reasonable. When this is done, the important property of the network is the short-circuit transfer admittance. With this in mind, the function of Equation (230) will be realized as a short-circuit transfer admittance. This may be accomplished in a number of ways. In order that one of the distributed networks of Figure 12 degenerate into a single resistor, and the other have a lossless dielectric, $a = 3\ 1/3$. For this case, the distributed network is defined by

$$\frac{c_0}{C} = \frac{3}{4} \tag{231}$$

$$\frac{c_1}{C} = -\frac{1}{4} \tag{232}$$

$$G = 0 \tag{233}$$

$$\frac{LR}{W} = \frac{7.40}{WLC} \tag{234}$$

The complete circuit is shown in Figure 13 and the overall short-circuit transfer admittance is

$$-Y_{21} = \frac{0.1273AWLC[s^2 + 1]}{s^2 + 3.33\,s + 1} \tag{235}$$

The example just given does not illustrate all of the advantages which may be gained by using the circuit of Figure 12. The fact that the overall transfer admittance of this circuit is the difference between the short-circuit transfer admittances of the individual networks, removes the restrictions on the placement of the zeroes. The poles, however, are still restricted to the negative real axis. This restriction may be removed if the phase inverter is replaced by a NIC and the circuit arranged in a manner analogous to that given by Yanagisawa[17]. The general circuit is shown in Figure 14. The short-circuit parameters for this complete circuit are

$$Y_{11} = Y_{11a} + Y_{11b} \tag{236}$$

$$Y_{21} = Y_{21b} \pm Y_{21a} \tag{237}$$

$$Y_{22} = Y_{22b} - Y_{22a} \tag{238}$$

$$Y_{12} = Y_{12b} \mp Y_{12a} \tag{239}$$

A unity gain NIC is assumed. The + sign in Equation (237) and the - sign in Equation (239) are needed if a voltage inversion type NIC is employed; the opposite sign is used in each case for a current-inversion NIC. Three of the

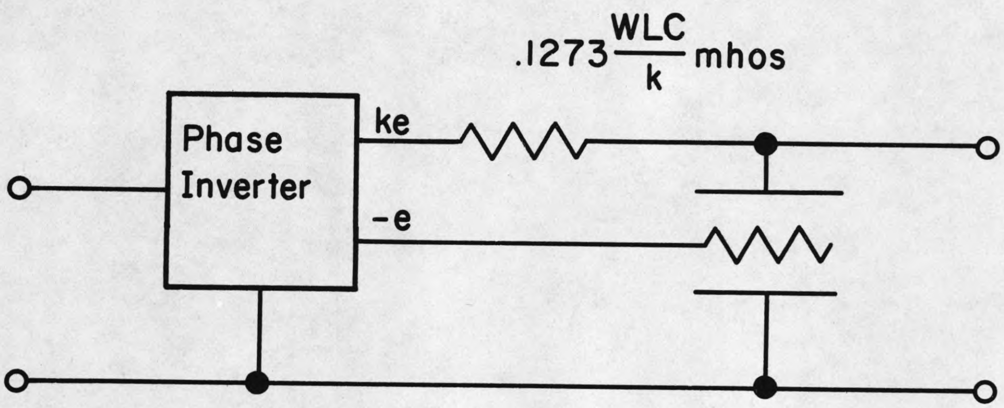


Figure 13. Circuit with active element

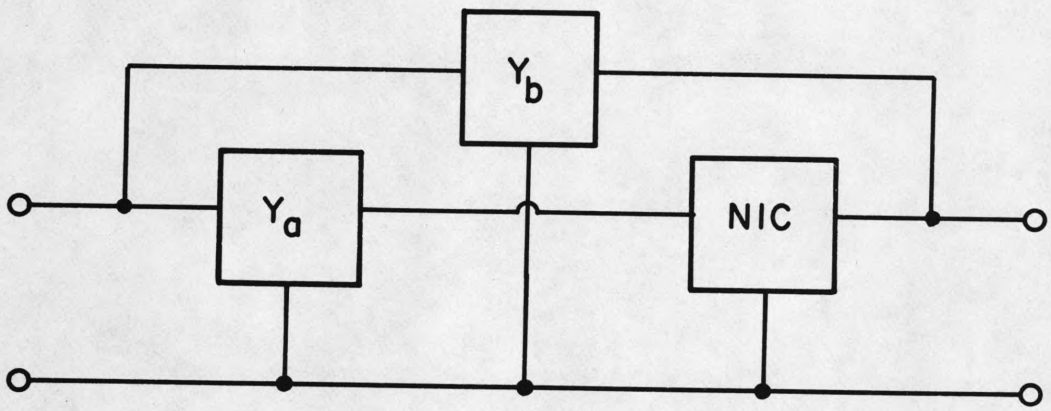


Figure 14. Circuit showing use of NIC

four short-circuit admittance functions may be made to be rational if both the networks "Y_a" and "Y_b" are distributed networks. As in the case of lumped circuits, the fact that Y_{22} is the difference between two RC admittance functions, allows it to have zeroes which are complex.

For an example of the usefulness of the circuit of Figure 14, consider the open circuit voltage transfer ratio of

$$\frac{V_2}{V_1} = \frac{K(s^2 + 5.15)}{(s + 0.695)(s^2 + 0.540\ s + 1.151)} = \frac{KP(s)}{M(s)} \tag{240}$$

This is a low pass filter function which has a 0.5 db. ripple within the pass band defined by $0 \leqq \omega \leqq 1$ and has an attenuation greater than 30 db. for $\omega \geqq 2$. To obtain the function of the form in Equation (240), it is necessary to choose a polynomial Q(s) with all roots distinct along the negative real axis. This polynomial should be of degree equal to, or one less than the highest power of s in P(s) and M(s). Then

$$\frac{KP(s)}{Q(s)} = -(Y_{21b} - Y_{21a})K_1 \tag{241}$$

and

$$\frac{M(s)}{Q(s)} = (Y_{22b} - Y_{22c})K_1 \tag{242}$$

When the ratio $\frac{M(s)}{Q(s)}$ is expanded in partial fraction form, one observes that the residues of the internal poles alternate in sign if M(s) is the same sign when evaluated at any point along the negative real axis. If M(s) changes sign within some interval, then this changes the sign of the residues within that interval.

Hence, if M(s) is examined along the negative real axis, it is often possible to arrange the roots of Q(s) so that the resulting admittance functions have poles which may be obtained with the distributed circuits. In the present problem M(s) changes from positive to negative at s = - 0.695. Hence both residues may be obtained in the same network if one root of Q(s) is made above this value and the other below. These may be chosen so that a lossless dielectric may be employed. Thus let

$$M(s) = (s + 0.2)(s + 1.8) \tag{243}$$

then for Equations (241) and (242)

$$-(Y_{21b} - Y_{21a})K_1 = K\left[1 + \frac{3.24}{s + 0.2} - \frac{5.24}{s + 1.8}\right] \tag{244}$$

$$(Y_{22b} - Y_{22a})K_1 = -\,0.765 + \frac{0.335}{s + 0.2} + \frac{2.36}{s + 1.8} + s \tag{245}$$

Since the residue of each of the internal poles is positive, each of these poles must be in Y_{22a}. The admittance, Y_{22b}, could have one or both of these poles, but the residues connot be as large in magnitude as the corresponding residue of Y_{22a}. Let Y_{22a} be chosen to contain the complete residue of each of the poles and to be composed of the sum of two values $Y'_{22a} + Y''_{22a}$. The network for Y'_{22a} will produce zeroes of Y_{21a}. Y''_{22a} will contain the remaining amount necessary for Y_{22a}. Thus if the network Y'_a is defined by the parameters

$$\frac{c_1}{C} = -\,0.0544 \tag{246}$$

$$\frac{c_0}{C} = 0.909 \tag{247}$$

$$\frac{RL}{W} = \frac{12.32}{WLC} \tag{248}$$

$$G = 0 \tag{249}$$

then

$$-K_1 Y'_{21a} = K \left[13.31 - \frac{3.24}{s + 0.2} + \frac{5.24}{s + 1.8} \right] \tag{250}$$

Where

$$K = 0.00714\ WLCK_1 \tag{251}$$

The required value of Y_{21b} is thus

$$-K_1 Y_{21b} = 14.31K \tag{252}$$

The value of the short-circuit output admittance of the network described by Equations (246) - (249) is

$$Y'_{22a} = \frac{WLC}{2} \left[0.1706 - \frac{0.0331}{s + 0.2} - \frac{0.00957}{s + 1.8} + 0.304s \right] \tag{253}$$

Hence, if

$$WLCK_1 = 20.22 \tag{254}$$

then

$$K_1 Y'_{21a} = 1.726 - \frac{0.335}{s + 0.2} - \frac{0.0969}{s + 1.8} + 3.08s \tag{255}$$

The pole at s = -0.2 is completely taken care of by the network Y_a'. The remaining part is the sum of Equations (246) and (255).

$$K_1(Y_{22a} - Y''_{22b}) = 0.961 + \frac{2.26}{s + 1.8} + 4.08s \tag{256}$$

If we are to continue with networks which have G = 0, then the internal pole of Equation (256) must be in Y''_{22a}. Let

$$c''_0 = C'' \tag{257}$$

then

$$K_1 W'' L'' C'' = 1.395 \tag{258}$$

and

$$\frac{R'' L''}{W''} = 0.981 K_1 \tag{259}$$

Hence

$$K_1 Y''_{22a} = 1.255 - \frac{2.26}{s + 1.8} + 0.1892s \tag{260}$$

The sum of Equations (260) and (256) yields

$$K_1 Y_{22b} = 2.216 + 4.27s \tag{261}$$

Now for Equations (251) and (254)

$$K = 0.1445 \tag{262}$$

Then from Equations (252) and (262)

$$-K_1 Y_{21b} = 2.07 \tag{263}$$

The complete network is shown in Figure 15.

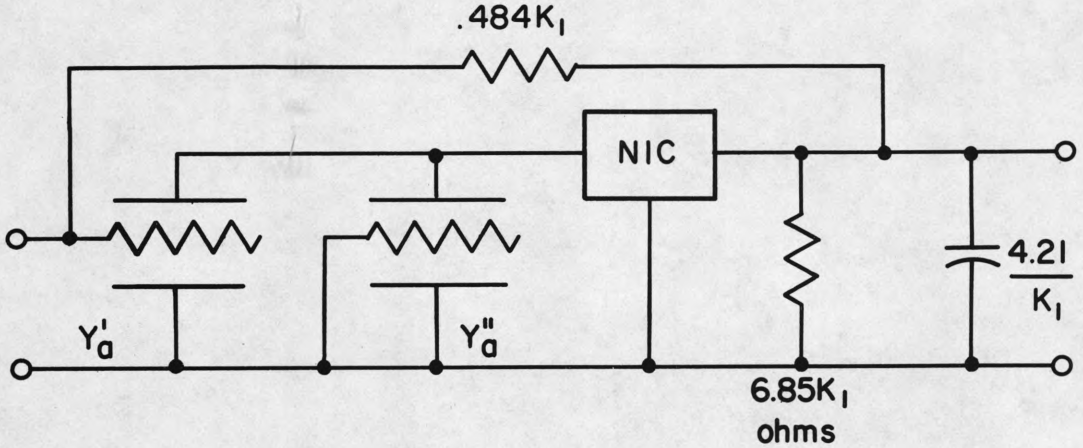


Figure 15. Active circuit of example problem

There are not a great number of components in this circuit, but the function contains only 3 poles. An active circuit which realizes a function very similar to this was obtained by Yanagisawa[17]. The lumped circuit required five capacitors and five resistors in addition to the NIC. This method should be useful for more complex methods where a much greater advantage in component number is realized with the distributed network.

Throughout this example the constant K_1 has been left in the design equations and hence shows how the impedance level of this network can be scaled without effecting the overall gain. There are, of course, practical limits on the impedance level as imposed by the driving circuit and the NIC which is less than ideal. A load may be placed on the circuit without upsetting this result if an equal value is placed in parallel with Y_{22a}. This is evident from Equation (239) as well as a casual observance of the NIC in Figure 15.

The distributed circuit may be used in any other active circuit in place of a lumped RC circuit if it is sufficient that one driving-point admittance and the short-circuit transfer admittance are rational.

9. CONCLUSIONS AND FURTHER PROBLEMS

It has been shown that it is possible to construct a distributed type RC network which has a rational short-circuit transfer admittance and one rational short-circuit driving-point admittance. A subcircuit of this same network has an open-circuit transfer impedance and one open-circuit driving-point impedance which are rational functions of s. This distributed circuit is analogous to a continuously tapped RC transmission line. One approximation made in the derivation is that all dimensions of the structure are small in comparison to a wave length within the dielectric. The name RC distributed circuit normally implies this assumption. An example has justified this assumption for a typical case. The only other assumption is that the good conductors are perfect. It has further been shown that reasonable tolerances may be allowed in the manufacturing without unduly large errors in the admittance functions.

In addition to the principal structure given, a simple transformation was included which allows the generation of any number of equivalent structures.

The properties of the rational admittance functions have been outlined. A method of synthesizing kY_{21} and Y_{22} has been developed for the class of functions that can be expressed as one of the rational admittance functions.

Examples have been included which show how active elements may be used to obtain properties which are not characteristic of these networks.

Some additional problems which need to be solved in this area include determining the properties of the rational impedance functions of the second two-port connection. A method of synthesis using these properties along with the admittance functions may prove to be valuable. The synthesis of a voltage transfer ratio which was presented here did not make provision for the generator impedance; further work seems necessary in this direction.

Following this introduction to distributed networks with rational transfer functions, the search should continue for other forms of morphology which are not obtainable from the simple transformations given, but which possess similar characteristics.

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VITA

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He attended North Texas Agricultural College, Arlington, Texas, John Tarleton Agricultural College, Stephenville, Texas, and received the B.S. and M.S. degrees in Electrical Engineering from Southern Methodist University, Dallas, Texas in 1950 and 1951 respectively. In the summer of 1955 he attended the University of Michigan, Ann Arbor, Michigan and has been a student at the University of Illinois, Urbana, Illinois since 1960.

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Mr. Heizer received the Outstanding Engineering Teacher Award at Southern Methodist University in 1958 and again in 1959. He holds two patents in the field of character recognition. He is a member of Eta Kappa Nu, Sigma Tau, I.R.E. and A.S.E.E.

ERRATA SHEET FOR R-153

p. 26 - Just above eq. 8-1 - change "th" to "the".

p. 31 - 3rd line from bottom - "comparatively"; also period after C(X).

p. 34 - Just below eq. 114 - "important"

p. 35 - After eq. 118 - change "axis" to "axes".

p. 47 - Eq. 160 - needs decimal point

p. 49 - Change k - 1 to 1 - k.

p. 50 - 7 lines from botton - change "os" to "of".