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A TOPOLOGICAL APPROACH TO THE STABILITY OF TIME-VARYING AND NONLINEAR NETWORKS

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A TOPOLOGICAL APPROACH TO THE STABILITY OF TIME-VARYING AND NONLINEAR NETWORKS

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University of Illinois, 1966

ABSTRACT

New techniques are presented which provide stability information for time-varying and nonlinear networks by inspection of the topology and the element values of the network. The derivation of the new techniques is based on Liapunov's second method for time-varying networks, his first method for nonlinear networks and network topology. More specifically, Liapunov's stability theorems of linear time-varying RLC networks are formulated in terms of the tree or chord-set products of the subnetworks derived from the state-space representation of the general networks. The local stability theorems of nonlinear time-invariant RC and RL networks are formulated in terms of the tree or chord-set products of subnetworks. Four examples are given to illustrate the above theorems.

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I. INTRODUCTION

It is well-known that Kirchhoff[14] and Maxwell[20] founded the topological study of electrical networks which permits the evaluation of network functions by inspection without solving network equations. This thesis introduces topological methods which provide stability information for time-varying and nonlinear networks by inspection of the topology and the element values of the network without solving network equations. In other words, this thesis brings together two apparently unrelated topics: Kirchhoff's and Maxwell's work on network topology and Liapunov's work on stability theorems. Thus, a new application of linear graph theory (network topology) is found and new techniques which give stability information are introduced here.

Among recent investigations[8,11,16,23,24] on stability of time-varying networks, Kuh's formulation on Liapunov stability of general linear time-varying networks[16] yields the most convenient form for the topological formulation of stability theorems. He used the state-space representation of general networks by Bashkow[1] and Bryant.[3] In this thesis, Chapter II introduces the resistive multiports and their open- and short-circuit operators determined from the state-space representation which play a central role in the stability analysis. Subsequently, topological formulas for the operator determinants and the coefficients of the charactersitic polynomials of the A-matrices are derived using the topological properties of electrical networks. Chapter IV is devoted to the topological formulation of stability theorems on time-varying networks using the topological formulas developed in the earlier chapter and Kuh's formulation of Liapunov's second method

for time-varying networks. The results, in general, lead to the sufficient conditions for stability and instability of general (not necessarily periodic) time-varying RLC networks.

The topological method is also applied to the stability analysis of some nonlinear networks in Chapter V. The first approximation theorems of Liapunov and the results of the special case in Section 3.2 yield the local stability and instability theorems of nonlinear time-invariant RC and RL networks in terms of the topological quantities.

Throughout the thesis, topological terminology and notation follow closely those in Seshu and Reed.[25] All network variables (voltage, current and network elements) used are in the time-domain and the transformed quantities (frequency-domain) are never used. Mutual inductances are excluded for simplicity of topological formulas.

II. SHORT-CIRCUIT AND OPEN-CIRCUIT OPERATORS

2.1 The State-Space Representation

This section considers the state-space representation of linear time-varying networks. Since there are excellent detailed treatments on this subject,[3,16,17] only a brief discussion is given here. It is desired to represent a linear time-varying network by a first-order matrix differential equation in the form

$$\dot{x}(t) = A(t)\, x(t) \tag{2-1}$$

where $\cdot = \frac{d}{dt}$ and $x(t)$ is a set of the state variables[17,28] which conveys the necessary and sufficient information to characterize any possible future behavior of the network. The voltages across capacitances (except one of the capacitance voltages in each pure capacitance circuit) and the currents through inductances (except one of the inductance currents in each pure inductance cut-set) serve as the state-variables of a linear time-varying network.

First, pick a normal tree* (or proper tree)[1] in the graph of the network which includes the maximum number of capacitances and the minimum number of inductances. Thus, the chords (or links) must contain the maximum number of inductances and the minimum number of capacitances. For convenience, Kuh's notation[16] is employed here. The Kirchhoff current law applied to

---

* Henceforth, a tree means a forest if a network is separated.

each fundamental cut-set states that the tree-branch currents j_C, j_G and j_Γ can be expressed by linear combination with coefficients -1, +1 or 0 of the chord currents i_L, i_R and i_S. Similarly, the Kirchhoff voltage law applied to voltages v_L, v_R and v_S can be expressed by linear combinations with coefficients -1, +1 or 0 of the tree-branch voltages e_C, e_G and e_Γ.

The above two statements are expressed by the following equations using the incidence submatrices α, β, γ, δ, ϵ, and μ.[16]

$$\begin{bmatrix} j_C \\ j_G \\ j_\Gamma \end{bmatrix} = - \begin{bmatrix} \alpha & \beta & \epsilon \\ \gamma & \delta & 0 \\ \mu & 0 & 0 \end{bmatrix} \begin{bmatrix} i_L \\ i_R \\ i_S \end{bmatrix} \tag{2-2a}$$

$$\begin{bmatrix} v_L \\ v_R \\ v_S \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \epsilon \\ \gamma & \delta & 0 \\ \mu & 0 & 0 \end{bmatrix}^t \begin{bmatrix} e_C \\ e_G \\ e_\Gamma \end{bmatrix} \tag{2-2b}$$

where t denotes matrix transpose, the subscripts C, G and Γ denote the tree-branch currents and voltages for the capacitances, conductances and reciprocal inductances, respectively. Similarly, the subscripts L, R and S indicate the chord voltages and currents for the inductances, resistances and elastances, respectively. Then, the voltage-current relations for these elements can be written as

$$\begin{bmatrix} j_C \\ i_S \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} e_C \\ v_S \end{bmatrix} \tag{2-3a}$$

$$\begin{bmatrix} j_G \\ v_R \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} e_G \\ i_R \end{bmatrix} \tag{2-3b}$$

and

$$\begin{bmatrix} e_\Gamma \\ v_L \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} j_\Gamma \\ i_L \end{bmatrix} \tag{2-3c}$$

(recall that mutual inductances are excluded)

where subscripts "1" and "2" designate the tree and chord branches, respectively.

The elimination of non-state-variables j_C, j_G, j_Γ, v_L, v_R, v_S, i_R, i_S, e_G and e_Γ in (2-2) and (2-3) gives the following state-space representation of a linear time-varying RLC network in terms of the state-variables e_C and i_L after a routine but long algebraic process.[16,17]

$$\begin{bmatrix} \mathcal{C}\,\dot{e}_C \\ \mathcal{L}\,\dot{i}_L \end{bmatrix} = -\begin{bmatrix} \mathcal{Y} + \dot{\mathcal{C}} & \mathcal{H} \\ -\mathcal{H}^t & \mathcal{Z} + \dot{\mathcal{L}} \end{bmatrix} \begin{bmatrix} e_C \\ i_L \end{bmatrix} \tag{2-4a}$$

or, in the form of (2-1),

$$\begin{bmatrix} \dot{e}_C \\ \dot{i}_L \end{bmatrix} = -\begin{bmatrix} \mathcal{C}^{-1}(\mathcal{Y} + \dot{\mathcal{C}}) & \mathcal{C}^{-1}\mathcal{H} \\ -\mathcal{L}^{-1}\mathcal{H}^t & \mathcal{L}^{-1}(\mathcal{Z} + \dot{\mathcal{L}}) \end{bmatrix} \begin{bmatrix} e_C \\ i_L \end{bmatrix} \tag{2-4b}$$

where

$$\mathcal{Y} = \beta(R_2 + \delta^t R_1 \delta)^{-1} \beta^t \quad , \quad R_1 = G_1^{-1}$$

$$\mathcal{Z} = \gamma^t (G_1 + \delta\, G_2 \delta^t)^{-1} \gamma \quad , \quad G_2 = R_2^{-1}$$

$$\mathcal{C} = C_1 + \epsilon\, C_2\, \epsilon^t$$

$$\mathcal{L} = L_2 + \mu^t L_1 \mu$$

and

$$\mathcal{H} = \alpha - \beta(R_2 + \delta^t R_1 \delta)^{-1} \delta^t R_1 \gamma$$

2.2 The Decomposition of Time-Varying Reactive Elements

The behavior of time-varying networks is, in general, quite different from that of the familiar time-invariant networks. As shown in the following discussion, however, it is possible to consider that time-varying network elements consist of time-invariant-like elements as far as the voltage-current relationship in the time-domain is concerned. Consider the current i(t) through a time varying capacitance C(t). Since one can write

$$\begin{aligned} i(t) &= \frac{d}{dt}\,[C(t)\ v(t)] \\ &= \dot{C}\, v \ + \ C\, \frac{dv}{dt} \\ &= i_{d1} + i_{d2} \end{aligned} \qquad (2\text{-}5)$$

where v(t) is the voltage across the capacitance, a time-varying capacitance C(t) is looked upon as consisting of two elements in parallel whose voltage-current relations are given, respectively, by

$$i_{d1} = \dot{C}\, v \tag{2-6}$$

and

$$i_{d2} = C \frac{dv}{dt} \tag{2-7}$$

Similarly, the voltage v(t) across an inductance L(t) is

$$\begin{aligned} v(t) &= \frac{d}{dt} [L(t)\ i(t)] \\ &= \dot{L}\, i + L \frac{di}{dt} \\ &= v_{d1} + v_{d2} \end{aligned} \tag{2-8}$$

where i(t) is the current through the inductance. A time-varying inductance L(t) is regarded as consisting of two elements in series whose voltage-current relations are given, respectively, by

$$v_{d1} = \dot{L}\, i \tag{2-9}$$

and

$$v_{d2} = L \frac{di}{dt} \tag{2-10}$$

The voltage-current relations in (2-6) and (2-9) show that the elements $\dot{C}$ and $\dot{L}$ separated from time-varying reactive elements can be treated as a conductance and a resistance, respectively. In fact, the elements $\dot{C}$ and $\dot{L}$ have the same dimensions as those of a conductance and a resistance, respectively. The voltage-current relations in (2-7) and (2-10) show that the other halves of decomposed time-varying reactive elements behave like

time-invariant capacitance and inductance in the time-domain. They will be called the decomposed capacitance and inductance, respectively. The foregoing argument is illustrated in Figure 1.

Note that the voltage across a decomposed capacitance is the same as the voltage across the original capacitance and that the current through a decomposed inductance is the same as the current through the original inductance. Therefore, the state-variables e_C and i_L are invariant under the decomposition of reactive elements and can be looked upon as the voltages across the decomposed capacitances and the currents through the decomposed inductances as well.

2.3 Definitions of the Resistive Multiports and Their Operators

Since the matrices $\mathscr{Y} + \dot{\mathbb{C}}$ and $\mathscr{Z} + \dot{\mathscr{L}}$ in (2-4) play a very important role in the stability analysis of networks as shown later, their physical interpretations are given next.

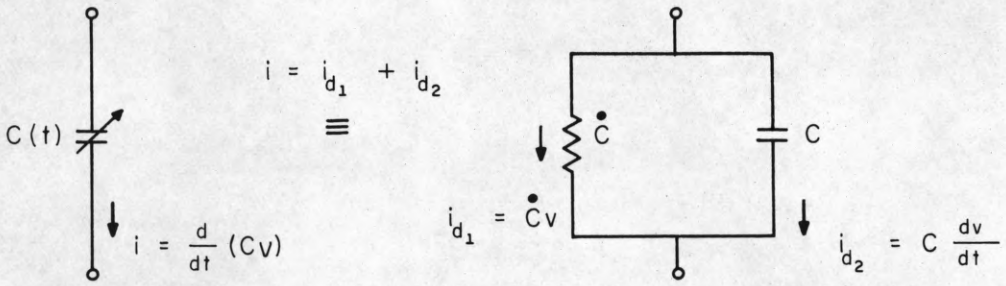
Suppose that all capacitances and all inductances in a given time-varying network are decomposed as described in the previous section. Consider the left side of (2-4a).

$$\mathbb{C}\,\dot{e}_C = (C_1 + \epsilon\, C_2\, \epsilon^t)\,\dot{e}_C$$

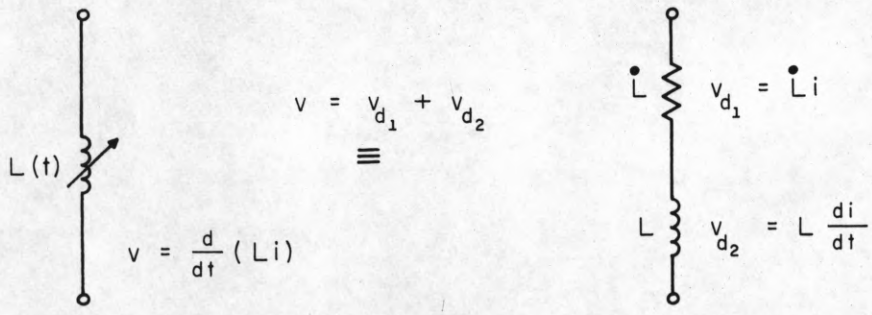
$$= C_1\,\dot{e}_C + \epsilon\, C_2(\epsilon^t\,\dot{e}_C)$$

by the last equation of (2-2b)

$$= C_1\,\dot{e}_C + \epsilon\, C_2\,\dot{v}_S$$



(a)



(b)

Figure 1. The Decomposition of Time-Varying Reactive Elements
(a) Capacitance (b) Inductance

$$= j_{Cd} + \epsilon\, i_{Sd}$$

where j_{Cd} are the currents through the decomposed capacitances belonging to tree-branches and i_{Sd} are the currents through the decomposed capacitances belonging to chords.

Similarly,

$$\mathcal{L}\, \dot{i}_L = (L_2 + \mu^t L_1 \mu)\, \dot{i}_L$$

$$= L_2\, \dot{i}_L + \mu^t\, L_1 (\mu\, \dot{i}_L)$$

by the last equation of (2-2a)

$$= L_2\, \dot{i}_L + \mu^t\, L_1 (-j_{\Gamma})$$

$$= v_{Ld} - \mu^t\, e_{\Gamma d}$$

where v_{Ld} are the voltages across the decomposed inductances belonging to chords and $e_{\Gamma d}$ are the voltages across the decomposed inductances belonging to tree-branches. Upon decomposition of the reactive elements, (2-4a) becomes

$$j_{Cd} + \epsilon\, i_{Sd} = -(\mathcal{Y} + \dot{\mathcal{C}})\, e_{Cd} - \mathcal{H}\, i_{Ld} \qquad (2\text{-}11a)$$

$$v_{Ld} - \mu^t\, e_{\Gamma d} = \mathcal{H}^t\, e_{Cd} - (\mathcal{Z} + \dot{\mathcal{L}})\, i_{Ld} \qquad (2\text{-}11b)$$

where it should be noted that the decomposed capacitance voltages e_{Cd} and

the decomposed inductance currents i_{Ld} are the same as the undecomposed ones, e_C and i_L, respectively. The conditions $i_{Sd} \equiv 0$ and $i_{Ld} \equiv 0$, i.e., the open-circuit of all decomposed reactive elements except for the decomposed tree-branch capacitances,* reduce (2-11a) to the form

$$- j_{Cd} = (\mathcal{Y} + \dot{\mathcal{C}}) e_{Cd} \qquad (2\text{-}12a)$$

Similarly, the conditions $e_{\Gamma d} \equiv 0$ and $e_{Cd} \equiv 0$, i.e., the short-circuits of all decomposed reactive elements except for the decomposed chord inductances* reduce (2-11b) to the form

$$- v_{Ld} = (\mathcal{Z} + \dot{\mathcal{J}}) i_{Ld} \qquad (2\text{-}12b)$$

The above discussion will lead to the following definitions of two sub-networks of a linear time-varying network and their operators.

Definition 1:

The C-multiport denoted by N_C of a linear time-varying network is the resistive multiport network faced by the decomposed tree-branch capacitances when all the remaining decomposed reactive elements are

---

* The decomposed tree-branch inductances (chord capacitances) are automatically open (shorted) since the decomposed tree-branch inductances (chord capacitances) form pure inductance cut-sets (capacitance circuits) with the decomposed chord inductances (tree-branch capacitances).[16,17]

open-circuited.*

Definition 2:

The L-multiport denoted by N_L of a linear time-varying network is the resistive multiport network faced by the decomposed chord-inductances when all the remaining decomposed reactive elements are short-circuited.*

Definition 3:

The short-circuit operator denoted by Y_{SC} of a linear time-varying network is the operator $\mathscr{Y} + \dot{\mathbb{C}}$ defined by (2-12a), which transforms the port voltages into the port currents in the C-multiport.

Definition 4:

The open-circuit operator denoted by Z_{OC} of a linear time-varying network is the operator $\mathscr{Z} + \dot{\mathscr{L}}$ defined by (2-12b), which transforms the port currents into the port voltages in the L-multiport.

It is clear that the operators Y_{SC} and Z_{OC} have the physical meaning of the short-circuit "admittance" and the open-circuit "impedance" matrices in the time-domain of the C- and L-multiports, respectively. The term "operator" is employed here since the terms admittance and impedance are usually defined for the Laplace transformed quantity.

---

* All "dead" branches and nodes resulting from the open- and short-circuits have been removed from the C- and L-multiports. The "dead" means that their removal causes no effect in the port variables.

III. DERIVATION OF SOME TOPOLOGICAL FORMULAS

3.1 Topological Formulas for the Operator Determinants

Topological formulas for the determinants of Y_{SC} and Z_{OC} are the bases for the topological analysis of network stability. Their derivation is given in this section.

Consider the time-varying resistive multiport N which is N_C or N_L, connected with n voltage sources $v_1(t)$, $v_2(t)$, ..., $v_n(t)$ shown in Figure 2. That is, the decomposed tree-branch capacitances for N_C or the decomposed chord inductances for N_L are extracted and replaced by their terminal voltages to leave a time-varying resistive multiport network. It is assumed that there are p independent circuits in the network. Choose a system of the fundamental circuits, 1, 2, ..., n, ..., p, so that each voltage source $v_k(t)$, $k = 1, 2, \ldots, n$, will always be in one of the chords of a tree in N_C or N_L. This choice of circuits will always be possible as long as all reactive elements are time-varying; i.e., $\dot{C}(t) \neq 0$ and $\dot{L}(t) \neq 0$. Because, in this case, there is no cut-set of all decomposed capacitances due to a parallel conductance $\dot{C}$ in each decomposed capacitance and there is no circuit or self-loop of all decomposed inductances due to a series resistance $\dot{L}$ in each decomposed inductance. And obviously, the decomposed tree-branch capacitances cannot form a circuit and the decomposed chord inductances cannot form a cut-set. Time-invariant capacitances may form a cut-set of capacitances only and time-invariant inductances may from a circuit or self-loop of inductances only. These complications may be avoided by considering "practical" reactive elements (a small parallel conductance due to the

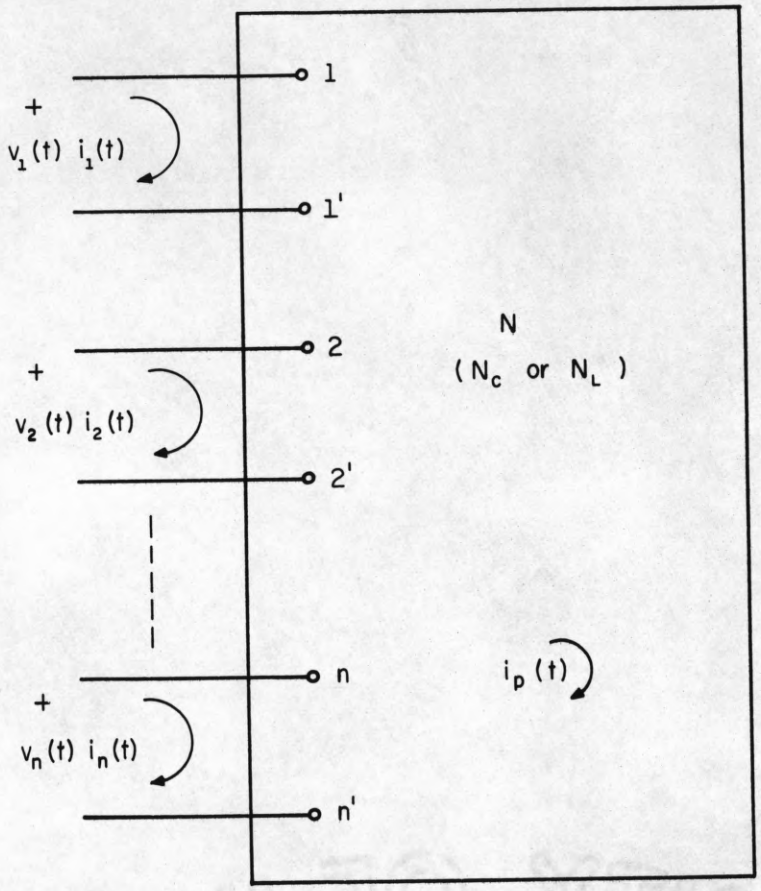


Figure 2. Time-Varying Resistive Multiport Network

leakage loss in each capacitance and a small series resistance due to the copper loss in each inductance) instead of idealized reactive elements as indicated by Darlington[8] and others.[5]

Now the mesh equations of the resistive multiport network in Figure 2 will have the following general form in the time-domain:

$$\begin{bmatrix} r_{11}(t) & r_{12}(t) & \cdots & r_{1n}(t) & \cdots & r_{1p}(t) \\ r_{21}(t) & r_{22}(t) & \cdots & r_{2n}(t) & \cdots & r_{2p}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{n1}(t) & r_{n2}(t) & \cdots & r_{nn}(t) & \cdots & r_{np}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{p1}(t) & r_{p2}(t) & \cdots & r_{pn}(t) & \cdots & r_{pp}(t) \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \\ \cdots \\ i_n(t) \\ \cdots \\ i_p(t) \end{bmatrix} = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \cdots \\ v_n(t) \\ 0 \\ \cdots \\ 0 \end{bmatrix} \qquad (3\text{-}1)$$

(3-1) can be solved uniquely for the mesh currents provided the coefficient matrix is nonsingular.*

---

* Although the choice of fundamental circuits will reduce the possibility of singularity, it may still be singular over certain time intervals since the entries of the matrix are functions of time. This type of singularity is difficult to examine in the absence of explicit functions of network elements.

The solution will be

$$\begin{bmatrix} i_1(t) \\ i_2(t) \\ \cdots \\ i_n(t) \\ \cdots \\ i_p(t) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{n1} & \cdots & \Delta_{p1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{n2} & \cdots & \Delta_{p2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_{1n} & \Delta_{2n} & \cdots & \Delta_{nn} & \cdots & \Delta_{pn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_{1p} & \Delta_{2p} & \cdots & \Delta_{np} & \cdots & \Delta_{pp} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ \cdots \\ v_n(t) \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{n1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_{1n} & \Delta_{2n} & \cdots & \Delta_{nn} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_{1p} & \Delta_{2p} & \cdots & \Delta_{np} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ \cdots \\ v_n(t) \end{bmatrix}$$

where Δ is the determinant of the coefficient matrix of (3-1) and Δ_{ij}'s are the co-factors.

An expression similar to (2-12) may be obtained by solving for the port currents only. The result will be

$$\begin{bmatrix} i_1(t) \\ i_2(t) \\ \cdots \\ i_n(t) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{n1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_{1n} & \Delta_{2n} & \cdots & \Delta_{nn} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ \cdots \\ v_n(t) \end{bmatrix} \qquad (3\text{-}2a)$$

which may be written as the matrix equation

$$i(t) = Y_{SC}(t)\, v(t) \tag{3-2b}$$

Taking the determinant of the coefficient matrix in (3-2)

$$\det Y_{SC}(t) = \frac{1}{\Delta^n} \det \begin{bmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{n1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_{1n} & \Delta_{2n} & \cdots & \Delta_{nn} \end{bmatrix} \tag{3-3}$$

and using Jacobi's Theorem,[13(p. 61)] one obtains

$$\det Y_{SC}(t) = \frac{1}{\Delta^n} \left(\Delta^{n-1} \cdot \Delta_{12 \ldots n,\ 12 \ldots n}\right)$$

$$= \frac{\Delta_{12 \ldots n,\ 12 \ldots n}}{\Delta} \tag{3-4}$$

where $\Delta_{12 \ldots n,\ 12 \ldots n}$ is the co-factor of order b - n of the coefficient matrix in (3-1); that is, the determinant* of the submatrix of order b - n obtained by deleting the first n rows and the first n columns from the coefficient matrix of (3-1).

---

* Note that the sign $(-1)^{n+n} = (-1)^{2n} = 1$ is always plus.

Therefore, for the C-multiport

$$\det Y_{SC} = \det Y_{SC}(t) \Big|_{N_C} = \frac{\Delta_{12 \ldots n,\ 12 \ldots n}}{\Delta} \Big|_{N_C} \tag{3-5}$$

and for the L-multiport, if $\det Y_{SC}(t) \not\equiv 0$ * (in t)

$$\det Z_{OC} = \det Y_{SC}^{-1}(t) \Big|_{N_L} = \frac{\Delta}{\Delta_{12 \ldots n,\ 12 \ldots n}} \Big|_{N_L} \tag{3-6}$$

Using the fundamental circuit matrix B_f [25] and the diagonal matrix of branch resistances $R_b(t)$, one can write the coefficient matrix of (3-1) as

$$B_f\ R_b(t)\ B_f^t \tag{3-7}$$

This is easily derived from Kirchoff's voltage law:

$$B_f\ v_b(t) = 0$$

the relation between branch voltages $v_b(t)$ and branch currents $i_b(t)$:

$$v_b(t) = v_g(t) + R_b(t)\ i_b(t)$$

---

* The elimination of pure inductance circuits makes this less likely singular but it may still be singular over certain time-intervals. This type of singularity is again difficult to study in the absence of explicit functions of network elements.

and the mesh transformation in the time-domain:

$$i_b(t) = B_f^t \, i_m(t)$$

Hence, the mesh determinant in the time-domain will be[25]

$$\begin{aligned}\Delta &= \det B_f \, R_b(t) \, B_f^t \\ &= \Sigma \text{ (Chord-set products of resistances in the time-domain for N+n)} \\ &\triangleq C[V_{N+n}(R)]\end{aligned} \tag{3-8}$$

where N+n denotes the network with the n voltage-source-branches and the co-factor $\Delta_{12 \ldots n,\, 12 \ldots n}$ will be

$$\begin{aligned}\Delta_{12 \ldots n,\, 12 \ldots n} &= \det B_{f\text{-}(12\ldots n)} \, R_b(t) \, B_{f\text{-}(12\ldots n)}^t \\ &= \Sigma \text{ (Chord-set products of resistances in the time-domain for N)} \\ &\triangleq C[V_N(R)]\end{aligned} \tag{3-9}$$

since nonzero majors of $B_{f\text{-}(12\ldots n)}$ which is B_f with rows 1, 2, ..., n deleted, are in one-to-one correspondence with chord-sets of N, where N denotes the network without the n voltage-source-branches.

Since there is the relation[25]

$$(r_1 \cdot r_2 \cdots r_e)(\text{tree product of conductances})$$

$$= (\text{chord-set product of resistances of the same tree})$$

Δ and $\Delta_{12\ldots n,\ 12\ldots n}$ can also be expressed in terms of trees as follows:

$$\Delta = \det R_b(t)\ \Sigma \text{ (tree products of conductances in the time-domain for N+n)}$$

$$\triangleq \det R_b(t)\ V_{N+n}(G) \tag{3-10}$$

and

$$\Delta_{12\ldots n,\ 12\ldots n} = \det R_b(t)\ \Sigma \text{ (tree products of conductances in the time-domain for N)}$$

$$\triangleq \det R_b(t)\ V_N(G) \tag{3-11}$$

where, since voltage sources should be resistanceless, N+n is the network with all the n voltage-source-branches or all the n ports short-circuited and N is the network with all the n ports open-circuited.

Finally, substituting (3-8), (3-9), (3-10) and (3-11) into (3-5) and (3-6), one obtains topological formulas for the determinants of the short-circuit and open-circuit operators, which are summarized in Formulas 1 and 2.

Formula 1:

The operator determinants can be written in terms of the chord-set products as

$$\det Y_{SC} = \frac{C[V_{N_C}(R)]}{C[V_{N_C+n}(R)]} \tag{3-12}$$

and

$$\det Z_{OC} = \frac{C[V_{N_L+n}(R)]}{C[V_{N_L}(R)]} \tag{3-13}$$

where $C[V_N(R)]$ and $C[V_{N+n}(R)]$ are the sums of chord-set products of resistances in the time-domain over all trees in the networks without and with all the n voltage-source branches, respectively (N is N_C or N_L).

Formula 2:

The operator determinants can be written in terms of the tree products as

$$\det Y_{SC} = \frac{V_{N_C}(G)}{V_{N_C+n}(G)} \tag{3-14}$$

and

$$\det Z_{OC} = \frac{V_{N_L+n}(G)}{V_{N_L}(G)} \tag{3-15}$$

where $V_N(G)$ and $V_{N+n}(G)$ are the sums of the tree-products of conductances in the time-domain over all trees in the networks with all the n ports open-circuited and short-circuited, respectively (N is N_C or N_L).

3.2 Topological Formulas for the Coefficients of the Characteristic Polynomials of the A-Matrices

In this section, the topological formulas are obtained for the coefficients of the characteristic polynomials of the A-matrices of RC and RL networks. The derivation is based on the results of the previous section and the following well-known lemma.[13] The special (time-invariant) case of the results in this section will be used in Chapter IV.

Lemma 1:

The coefficient a_r of λ^{n-r} $(r \leqslant n)$ in the characteristic polynomial of a matrix A

$$f(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_r\lambda^{n-r} + \ldots + a_n \tag{3-16}$$

is given by $(-1)^{n-r}$ times the sum of the principal minors of order r of A. In particular, the coefficient a_0 is $(-1)^n$ and the constant term a_n is det A.

Consider the A(t) matrices defined by (2-1) of linear time-varying RC and RL networks. It is easy to see that the A(t) matrix of an RC network is given by

$$A_{RC} = -\mathbb{C}^{-1}(\mathbb{Y} + \dot{\mathbb{C}}) = -\mathbb{C}^{-1}\, Y_{SC} \tag{3-17a}$$

since $\mathcal{L} = 0$ and $\alpha = \gamma = 0$ gives $\mathbb{Z} = 0$ and $\mathbb{H} = 0$ in (2-4) and that the A(t) matrix of an RL network is given by

$$A_{RL} = -\mathcal{L}^{-1}(\mathbb{Z} + \dot{\mathcal{L}}) = -\mathcal{L}^{-1}\, Z_{OC} \tag{3-17b}$$

since $\mathbb{C} = 0$ and $\alpha = \beta = 0$ gives $\mathbb{Y} = 0$ and $\mathbb{H} = 0$ in (2-4).

First, the formulas for RC networks are considered. By Lemma 1,

$$a_r = (-1)^{n-r}\, \Sigma \text{ principal minors of order r of } A_{RC} \tag{3-18}$$

where r = 0, 1, 2, ..., n.

In view of (3-17a), (3-18) becomes

$$a_r = (-1)^n \sum \frac{\text{principal minors of order r of } Y_{S.C}}{\text{principal minors of order r of } \mathfrak{C}} \tag{3-19}$$

With reference to (3-3) and (3-4), the numerator of (3-19) will take the following form:

$$\text{principal minors of order r of } Y_{S.C}$$

$$= \frac{1}{\Delta^r} \det \begin{vmatrix} \Delta_{j_1 j_1} & \Delta_{j_2 j_1} & \cdots & \Delta_{j_r j_1} \\ \Delta_{j_1 j_2} & \Delta_{j_2 j_2} & \cdots & \Delta_{j_r j_2} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_{j_1 j_r} & \Delta_{j_2 j_r} & \cdots & \Delta_{j_r j_r} \end{vmatrix}_{N_C}$$

$$= \left. \frac{\Delta_{j_1 j_2 \cdots j_r,\ j_1 j_2 \cdots j_r}}{\Delta} \right|_{N_C} \tag{3-20}$$

Thus, in a similar way that Formulas 1 and 2 are obtained, (3-20) can be written as

$$\frac{C[V_{N_C+n-r}(R)]}{C[V_{N_C+n}(R)]} \tag{3-21}$$

in terms of the chord-set products, or

$$\frac{V_{N_C+n-r}(G)}{V_{N_C+n}(G)} \tag{3-22}$$

in terms of the tree products.

Next consider the denominator of (3-19). It is clear that

$$\mathbb{C} = \mathbb{Y} + \dot{\mathbb{C}}\Big|_{\text{with } \mathbb{Y} = 0 \text{ and all } \dot{C}_i\text{'s replance by } C_i\text{'s.}}$$

Thus, it is convenient to define the following pure capacitive network.

Definition 3:

The C-network of a time-varying RC network is a pure capacitive multiport network faced by the same ports as those of the C-multiport under the condition that all $\dot{C}_i$'s, including time-invariant ones ($\dot{C}_i = 0$), are replaced by C_i's, all resistive chords are open-circuited and all resistive tree-branches are short-circuited.

Now, with comparison to (3-21) and (3-22), it is easy to obtain the topological expressions for the denominator of (3-19), which will be in terms of the chord-set products

$$\frac{C[V_{C+n-r}(S)]}{C[V_{C+n}(S)]} \quad , \quad (S \triangleq C^{-1}) \tag{3-23}$$

or in terms of the tree products

$$\frac{V_{C+n-r}(C)}{V_{C+n}(C)} \tag{3-24}$$

When there is no pure capacitance circuit, the denominator of (3-19) becomes simply the products of r capacitances. Because in this case the C-network consists of capacitances, each of which is separated or separable, (3-24) will be

$$\frac{V_{C+n-r}(C)}{V_{C+n}(C)} = \frac{C_1\ C_2\ \ldots\ C_r}{1} = C_1\ C_2\ \ldots\ C_r \tag{3-25}$$

Hence, the following formulas are obtained from (3-19), (3-21), (3-22), (3-23), (3-24) and (3-25).

Formula 3:

The coefficients of the characteristic polynomial for a time-varying RC network can be written in terms of the chord-set products as

$$a_r = (-1)^n \Sigma \left(\frac{C[V_{C+n}(S)]}{C[V_{C+n-r}(S)]} \right) \left(\frac{C[V_{N_C+n-r}(R)]}{C[V_{N_C+n}(R)]} \right) \tag{3-26}$$

$$r = 0,\ 1,\ 2,\ \ldots,\ n \quad (S \triangleq C^{-1})$$

where n and n-r added to the subscripts C and N_C denote the numbers of voltage-source-branches present in the C-network, C and the C-multiport, N_C and the summation is taken over all combinations of r voltage-source-branches. In particular, if there is no pure capacitance circuit, (3-26) is simplified as

$$a_r = (-1)^n \Sigma\ \frac{1}{C_1\ C_2\ \ldots\ C_r} \left(\frac{C[V_{N_C+n-r}(R)]}{C[V_{N_C+n}(R)]} \right) \tag{3-27}$$

$$r = 0, 1, 2, \ldots, n \qquad (C_0 \triangleq 1)$$

Formula 4:

The coefficients of the characteristic polynomial for a time-varying RC network can be written in terms of the tree products as

$$a_r = (-1)^n \sum \left(\frac{V_{C+n}(C)}{V_{C+n-r}(C)} \right) \left(\frac{V_{N_C+n-r}(G)}{V_{N_C+n}(G)} \right) \tag{3-28}$$

$$r = 0, 1, 2, \ldots, n$$

where n and n-r added to the subscripts C and N_C denote the number of ports which are to be short-circuited in the C-network and the C-multiport and the summation is taken over all combinations of r ports. In particular, if there is no pure capacitance circuit, (3-28) is simplified as

$$a_r = (-1)^n \sum \frac{1}{C_1 C_2 \cdots C_r} \left(\frac{V_{N_C+n-r}(G)}{V_{N_C+n}(G)} \right) \tag{3-29}$$

$$r = 0, 1, 2, \ldots, n \ .$$

The analogous formulas for a time-varying RL network are obtained in a similar way to the RC case. The analogous definition to Definition 3 is as follows.

Definition 4:

The L-network of a time-varying RL network is a pure inductive multiport network faced by the same ports as those of the L-multiport

under the condition that all $\dot{L}_i$'s, including time-invariant ones ($\dot{L}_i = 0$), are replaced by L_i's, all resistive chords are open-circuited and all resistive tree-branches are short-circuited.

Starting with (3-17b) and making use of Lemma 1 and the results in the previous section, one obtains the following formulas.

Formula 5:

The coefficients of the characteristic polynomial for a time-varying RL network can be written in terms of the chord-set products as

$$a_r = (-1)^n \sum \left(\frac{C[V_{L+n-r}(L)]}{C[V_{L+n}(L)]} \right) \left(\frac{C[V_{N_L+n}(R)]}{C[V_{N_L+n-r}(R)]} \right) \qquad (3\text{-}30)$$

$$r = 0, 1, 2, \ldots, n$$

where n and n-r added to the subscripts L and N_L denote the number of voltage-source-branches present in the L-network, L and the L-multiport, N_L and the summation is taken over all combinations of r voltage-source-branches. In particular, if there is no pure inductance cut-set, (3-30) is simplified as

$$a_r = (-1)^n \sum \frac{1}{L_1 L_2 \cdots L_r} \left(\frac{C[V_{N_L+n}(R)]}{C[V_{N_L+n-r}(R)]} \right) \qquad (3\text{-}31)$$

$$r = 0, 1, 2, \ldots, n \qquad (L_0 \triangleq 1) \ .$$

Formula 6:

The coefficients of the characteristic polynomial for a time-varying RL network can be written in terms of the tree products as

$$a_r = (-1)^n \sum \left(\frac{V_{L+n-r}(\Gamma)}{V_{L+n}(\Gamma)} \right) \left(\frac{V_{N_L+n}(G)}{V_{N_L+n-r}(G)} \right) \qquad (3\text{-}32)$$

$$r = 0, 1, 2, \ldots, n \qquad (\Gamma \triangleq L^{-1})$$

where n and n-r added to the subscripts L and N_L denote the number of ports which are to be short-circuited in the L-network, L and the L-multiport, N_L and the summation is taken over all combinations of r ports. In particular, if there is no pure inductance cut-set, (3-32) is simplified as

$$a_r = (-1)^n \sum \frac{1}{L_1 L_2 \cdots L_r} \left(\frac{V_{N_L+n}(G)}{V_{N_L+n-r}(G)} \right) \qquad (3\text{-}33)$$

$$r = 0, 1, 2, \ldots, n$$

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IV. STABILITY OF LINEAR TIME-VARYING RLC NETWORKS

4.1 Preliminaries

This chapter provides the sufficient conditions for stability and instability of linear time-varying RLC networks in terms of the topological quantities of the modified C- and L-multiports. It is assumed that the A(t) matrix defined by (2-1) be continuous and bounded and all inductances and capacitances be positive for all $t \geqslant 0$.

The derivation of the conditions is based on Liapunov's second method[18] and Kuh's formulation[16] on the Liapunov stability of general linear time-varying networks which states as follows.

Lemma 2:

A general linear time-varying network with positive capacitances and inductances as described by (2-4) is stable in the sense of Liapunov if

$$\mathbb{Y} + \frac{1}{2}\dot{\mathbb{C}} \geqslant 0 \qquad (4\text{-}1a)$$

and

$$\mathbb{Z} + \frac{1}{2}\dot{\mathbb{L}} \geqslant 0 \qquad (4\text{-}1b)$$

It is asymptotically stable if (4-1a) and (4-1b) are satisfied with strict inequality.

To prove Lemma 2, it is sufficient to show that there exists a Liapunov function of the form

$$E(x,\ t) = \frac{1}{2}\, x^t \begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathcal{L} \end{bmatrix} x \qquad (4\text{-}2)$$

where

$$x = \begin{bmatrix} e_C \\ i_L \end{bmatrix}$$

The following form of the time derivative of $E(x,\ t)$ is obtained by making use of (2-4a);

$$\frac{dE}{dt} = -\, e_C^t \left[\mathcal{Y} + \frac{1}{2}\dot{\mathbb{C}}\right] e_C - i_L^t \left[\mathcal{Z} + \frac{1}{2}\dot{\mathcal{L}}\right] i_L$$

$E(x,\ t)$ is indeed a Liapunov function because for all $t \geqslant 0$

i) E is continuous together with its first partial derivatives for all x

ii) for $x = 0$, $E = \frac{dE}{dt} = 0$

and

iii) for $x \neq 0$, $E > 0$, $\frac{dE}{dt} \leqslant 0$ under the hypotheses.

The instability theorem[18] gives the following lemma.

Lemma 3:

The network described in Lemma 2 is unstable if

$$\mathcal{Y} + \frac{1}{2}\dot{\mathbb{C}} < 0$$

and

$$\mathcal{Z} + \frac{1}{2}\dot{\mathcal{L}} < 0$$

4.2 The Modified C- and L-Multiports

According to Lemma 2, positive semidefiniteness of the matrices $\mathbb{Y} + \frac{1}{2}\dot{\mathbb{C}}$ and $\mathbb{Z} + \frac{1}{2}\dot{\mathcal{L}}$ guarantees the stability of a time-varying RLC network. Attention will be paid to these matrices. A simple comparison of these matrices and the operators in (2-12) reveals that the matrix $\mathbb{Y} + \frac{1}{2}\dot{\mathbb{C}}$ is the short-circuit operator with all the entries of $\dot{\mathbb{C}}$ multiplied by the factor 1/2 and that the matrix $\mathbb{Z} + \frac{1}{2}\dot{\mathcal{L}}$ is the open-circuit operator with all the entries of $\dot{\mathcal{L}}$ multiplied by the factor 1/2. Thus, it is convenient to define the following modified resistive multiports.

Definition 5:

The modified C-multiport denoted by $N_{C/2}$ of a linear time-varying network is the C-multiport where all the elements $\dot{C}_i$'s decomposed from capacitances are multiplied by the factor 1/2.

Definition 6:

The modified L-multiport denoted by $N_{L/2}$ of a linear time-varying network is the L-multiport where all the elements $\dot{L}_i$'s decomposed from inductances are multiplied by the factor 1/2.

Now the matrices $\mathbb{Y} + \frac{1}{2}\dot{\mathbb{C}}$ and $\mathbb{Z} + \frac{1}{2}\dot{\mathcal{L}}$ may be interpreted as the short-circuit and open-circuit operators of the modified C- and L-multiports, respectively. They are real symmetric matrices since there are no active elements and no mutual inductances in the networks.

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4.3 Liapunov Stability of Time-Varying Networks in Terms of the Topological Quantities

The topological interpretation of positive semidefiniteness of the matrices $\mathcal{Y} + \frac{1}{2}\dot{\mathcal{C}}$ and $\mathcal{Z} + \frac{1}{2}\dot{\mathcal{L}}$ will lead to the Liapunov stability theorem which can examine the stability of linear time-varying RLC networks directly from the topology of the networks without any knowledge of the solutions. It would, therefore, be better to review here a test for positive semidefiniteness which sometimes tends to be misinterpreted from a test for positive definiteness.

Test for Positive Semidefiniteness*:

A real symmetric matrix A of positive rank $k \leqslant n$ is positive semidefinite if and only if there exists a k by k nonsingular symmetric submatrix B of A such that $\det B = B_k > 0$, $B_{k-1} > 0$, ..., $B_1 > 0$ where B_i, $i = 1, 2, \ldots, k$ are the i^{th} order principal minors of B (the leading principal minors[13] of B suffice them). If $k = n$, the order of A, then it is positive definite.

It has been shown in Chapter III that the principal minor of order r of the matrix $\mathcal{Y} + \dot{\mathcal{C}}$ has the topological form of (3-21) or (3-22). That of the matrix $\mathcal{Y} + \frac{1}{2}\dot{\mathcal{C}}$ can be obtained simply by replacing the C-multiport by the modified C-multiport. Similarly, for the matrix $\mathcal{Z} + \frac{1}{2}\dot{\mathcal{L}}$, its

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* See, for example, p. 257 of Weinberg.[27]

principal minor of order r will have the form of the second multiplier factor in (3-30) or (3-32) except that the L-multiport is replaced by the modified L-multiport. The topological formula for the principal minor of each order is now obtained by setting r = n, n-1, ..., 1. The topological formulas necessary for the stability theorem are summarized below where n and m are the orders of the matrices $\mathcal{Y} + \frac{1}{2}\dot{\mathcal{C}}$ and $\mathcal{Z} + \frac{1}{2}\dot{\mathcal{L}}$, respectively, and r is the order of principal minors.

Formula 7 (In terms of the chord-set products):

A set of principal minors of the matrices $\mathcal{Y} + \frac{1}{2}\dot{\mathcal{C}}$ and $\mathcal{Z} + \frac{1}{2}\dot{\mathcal{L}}$ are given, respectively, by

$$y_r = \frac{C[V_{N_{C/2}+n-r}(R)]}{C[V_{N_{C/2}+n}(R)]} \quad ; \quad r = n, n-1, \ldots, 1 \qquad (4\text{-}4)$$

and

$$z_r = \frac{C[V_{N_{L/2}+m}(R)]}{C[V_{N_{L/2}+m-r}(R)]} \quad ; \quad r = m, m-1, \ldots, 1 \qquad (4\text{-}5)$$

where the numbers n, n-r, m and m-r added to the subscripts $N_{C/2}$ and $N_{L/2}$ denote the numbers of the voltage-source-branches present in the modified C- and L-multiports.

Formula 8 (In terms of the tree products):

A set of principal minors of the matrices $\mathcal{Y} + \frac{1}{2}\dot{\mathcal{C}}$ and $\mathcal{Z} + \frac{1}{2}\dot{\mathcal{L}}$ are given, respectively, by

$$y_r = \frac{V_{N_{C/2}+n-r}(G)}{V_{N_{C/2}+n}(G)} \quad ; \quad r = n, n-1, \ldots, 1 \qquad (4\text{-}6)$$

and

$$z_r = \frac{V_{N_{L/2}+m}(G)}{V_{N_{L/2}+m-r}(G)} \quad ; \quad r = m, m-1, \ldots, 1 \qquad (4\text{-}7)$$

where the numbers n, n-r, m and m-r added to the subscripts $N_{C/2}$ and $N_{L/2}$ denote the numbers of the ports which are to be short-circuited in the modified C- and L-multiports.

Lemmas 2 and 3 and Formula 7 or 8 thus imply the following topological version of Liapunov theorems.

Theorem 1:

A linear time-varying RLC network with positive capacitances and inductances as described by (2-4) is stable in the sense of Liapunov if there exist y_r's and z_r's such that

$$y_{r_p} > 0, \quad y_{r_p-1} > 0, \quad \ldots, \quad y_1 > 0$$

and

$$z_{r_q} > 0, \quad z_{r_q-1} > 0, \quad \ldots, \quad z_1 > 0$$

where r_p and r_q are the first subscripted r's which give nonzero y_r and

z_r in Formula 7 or 8 when r is taken in the descent order of: n, n-1, ..., 1 or m, m-1, ..., 1, respectively. There are two special cases:

i) when $r_p = r_q = 0$; i.e., $y_r = z_r = 0$ for all r, it is still stable, and

ii) when $r_p = n$ and $r_q = m$, it is asymptotically stable.

Theorem 2:

A linear time-varying RLC network with positive capacitances and inductances as described by (2-4) is unstable if $y_r < 0$ and $z_r < 0$ for all r.

Remark:

In Theorem 1, if $r_p < n$, one must search for all possible combinations of r ports until a nonzero y_{r_p} is found; once y_{r_p} is found, it is advisable to number the ports of the multiport and to short-circuit them in the natural order to obtain the numerators for $r = r_p - 1, r_p - 2, \ldots, 1$ in (4-6), if Formula 8 is used. This assures that the sequence $y_{r_p}, y_{r_p - 1}, \ldots, y_1$ represents a set of the r_p principal minors of the r_p^{th} order (maximum order) nonzero submatrix as required in the test for positive semidefiniteness. Similarly, attention should also be paid to the evaluation of the sequence, $z_{r_q}, z_{r_q - 1}, \ldots, z_1$ if $r_q < m$. See examples in the next section.

4.4 Examples

Application of the theorems in the previous section gives the stability conditions of linear time-varying networks in terms of network elements in

the time-domain directly from the topology of the networks without solving or formulating network equations as will be shown in this section. There are two choices of topological quantities: the chord-set products and the tree products. It is advisable to choose one which gives simpler forms of conditions so that it is easier to check the signs of y_r's and z_r's in each case.

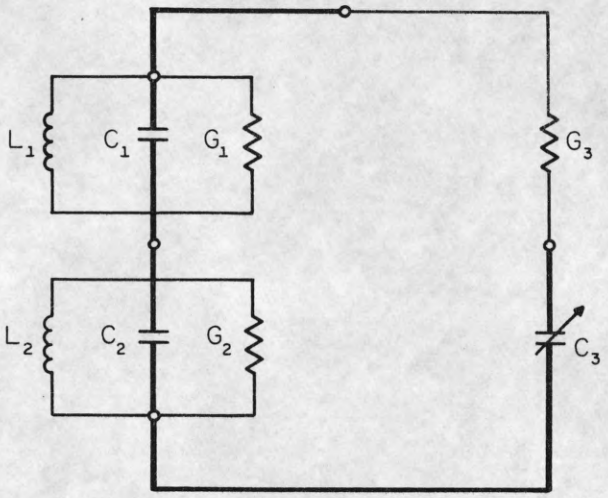
Example 1:

As the first example, consider a shunt version of a negative-resistance parametric amplifier shown in Figure 3(a) which Rohrer[23] and Kuh[16] examined by different methods. Since the C_3 is the only time-varying element and $z_2 = z_1 = 0$, only the modified C-multiport shown in Figure 3(b) needs to be considered. From Formula 8, one obtains

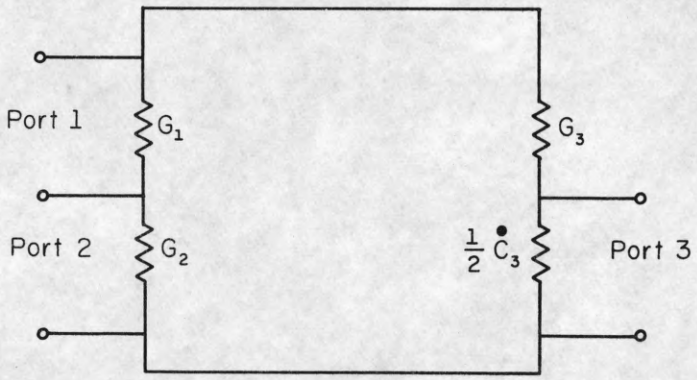
$$y_3 = G_1 G_2 G_3 + \frac{1}{2}\dot{C}_3(G_1 G_2 + G_1 G_3 + G_2 G_3)$$

$$y_2 = \begin{cases} y_2^{(1)} = G_2 G_3 + \frac{1}{2}\dot{C}_3(G_2 + G_3) & \text{(port 1 shorted)} \\ y_2^{(2)} = G_1 G_3 + \frac{1}{2}\dot{C}_3(G_1 + G_3) & \text{(port 2 shorted)} \\ y_2^{(3)} = G_1 G_2 + G_1 G_3 + G_2 G_3 & \text{(port 3 shorted)} \end{cases}$$

$$y_1 = \begin{cases} y_1^{(1)} = G_3 + \frac{1}{2}\dot{C}_3 & \text{(ports 1, 2 shorted)} \\ y_1^{(2)} = G_2 + G_3 & \text{(ports 1, 3 shorted)} \\ y_1^{(3)} = G_1 + G_3 & \text{(ports 2, 3 shorted)} \end{cases}$$



(a)



(b)

Figure 3. Example 1: (a) Given Network
(b) The Modified C-Multiport

Thus, from Theorem 1, the network is asymptotically stable if $y_3 > 0$, $y_2^{(1)} > 0$ and $y_1^{(1)} > 0$ and, even if $y_3 = y_1^{(2)} = y_1^{(3)} = 0$; it is stable if $y_2^{(1)} > 0$ and $y_1^{(1)} > 0$. Although all possible y_r's are listed in the above to illustrate Formula 8, it is not necessary to evaluate all y_r's to examine stability.

Example 2:

For a time-varying RLC network shown in Figure 4(a) where all network elements except C_2 and C_4 are time-varying, the modified C- and L-multiports are obtained as shown in Figures 4(b) and 4(c), respectively, according to Definitions 5 and 6. Then immediately the following conditions for asymptotical stability are obtained using Theorem 1:

$$y_3 = G_4\left(\frac{1}{4}\dot{C}_1\dot{C}_2 + \frac{1}{2}\dot{C}_1 G_3 + \frac{1}{2}\dot{C}_3 G_5 + G_5 G_3\right) > 0$$

$$y_2 = G_4\left(\frac{1}{2}\dot{C}_3 + G_3\right) > 0 \qquad \text{(port 1 shorted)}$$

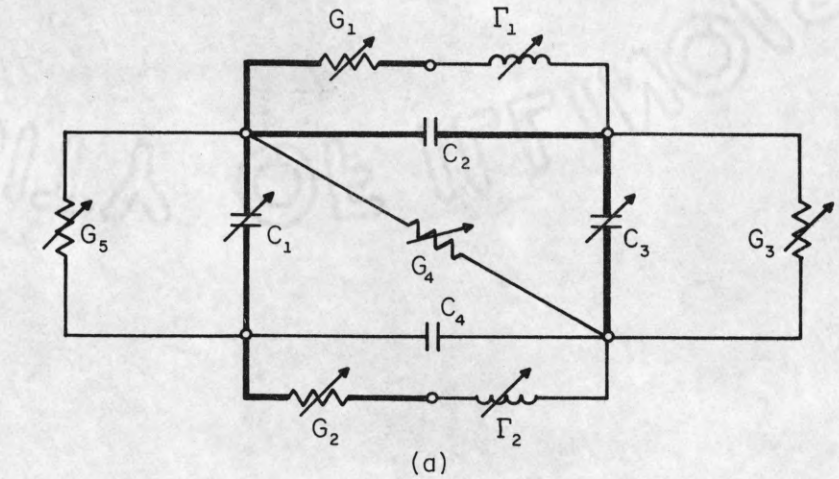
$$y_1 = \frac{1}{2}\dot{C}_3 + G_3 + G_4 > 0 \qquad \text{(ports 1, 2 shorted)}$$

$$z_2 = \frac{4\,\dot{\Gamma}_1\,\dot{\Gamma}_2\,G_1\,G_2}{(G_1 + 2\,\dot{\Gamma}_1)(G_2 + 2\,\dot{\Gamma}_2)} > 0$$

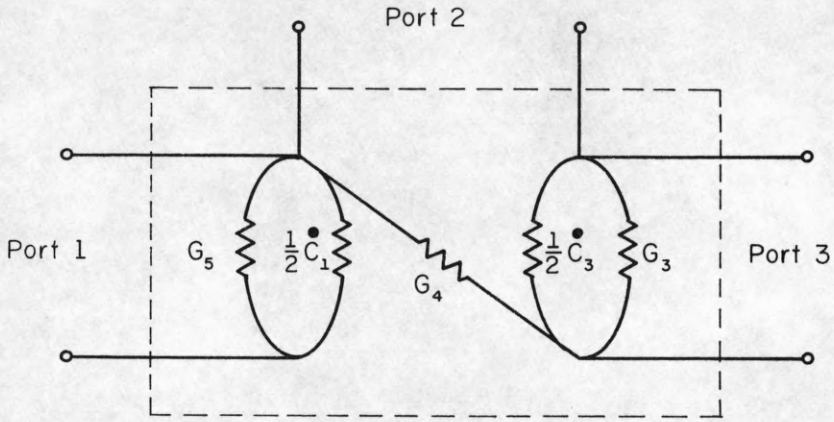
and

$$z_1 = \frac{2\,\dot{\Gamma}_2\,G_2}{G_2 + 2\,\dot{\Gamma}_2} > 0 \qquad \text{(port 1 shorted)}$$

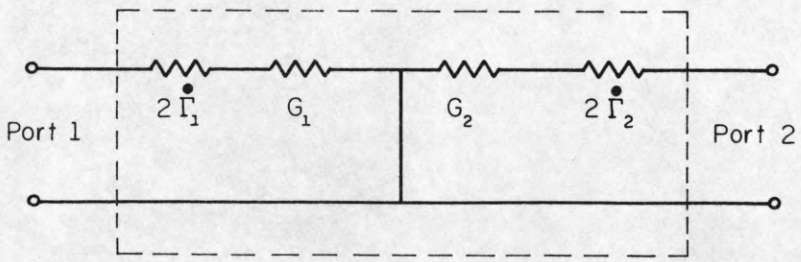
Even if $y_3 = z_2 = 0$, it is stable if $y_2 > 0$, $y_1 > 0$ and $z_1 > 0$.



(a)



(b)



(c)

Figure 4. Example 2: (a) A Time-Varying RLC Network
(b) The Modified C-Multiport
(c) The Modified L-Multiport

V. STABILITY OF NONLINEAR TIME-INVARIANT RC AND RL NETWORKS

5.1 Preliminary Lemmas

In this chapter, the local stability and instability theorems are given in terms of the topological quantities of the linearized networks for certain classes of nonlinear time-invariant RC and RL networks. The derivation is based on Liapunov's first method or the first approximation theorems of Liapunov which may be stated as follows in the special case adapted to this chapter.

Consider time-invariant nonlinear systems of the form

$$\dot{x} = Ax + P(x) \tag{5-1}$$

where $A = [a_{kj}]$ is a square matrix whose elements are constants and each component P_k of P has a development in the second or higher order power series of $x = (x_1, \ldots, x_n)$ which converges absolutely and uniformly for $\| x \| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} < B$ = positive constant. Since $P(0) = 0$, (5-1) has the trivial solution at the origin $x = 0$. Then,

Lemma 4 (Local Stability Theorem):

If all characteristic roots of A in (5-1) have negative real parts, the origin $x = 0$ is asymptotically stable regardless of the higher order terms in $P(x)$.

Lemma 5 (Local Instability Theorem):

If A in (5-1) has at least one characteristic root with a positive

real part, the origin x = 0 is unstable regardless of the higher order terms in P(x).

The proofs of the above lemmas are found elsewhere.[6]

To interpret Lemmas 4 and 5 in terms of the topological quantities, the following Liénard and Chipart test[12,28] will be used. It is more efficient than the Routh-Hurwitz criterion in the sense that fewer Hurwitz determinants need to be tested.

Lemma 6 (Liénard and Chipart):

Let the Hurwitz determinants of polynomial (3-16) be denoted by

$$D_k = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots \\ a_0 & a_2 & a_4 & \cdots \\ 0 & a_1 & a_3 & \cdots \\ 0 & a_0 & a_2 & \cdots \\ \cdots & \cdots & \cdots & a_k \end{vmatrix} \qquad k = 1, 2, \ldots, n \quad (a_k = 0 \text{ for } k > n) \qquad (5\text{-}2)$$

Assume $a_0 > 0$. Then all the roots of polynomial (3-16) with real coefficients have negative real parts if, and only if, any one of the following sets of conditions is satisfied.

(i) $a_n \geq 0,\ a_{n-2} \geq 0,\ \ldots;\ D_1 = a_1 > 0,\ D_3 > 0,\ \ldots,$

(ii) $a_n > 0,\ a_{n-2} > 0,\ \ldots;\ D_2 > 0,\ D_4 > 0,\ \ldots,$

(iii) $a_n > 0;\ a_{n-1} > 0,\ a_{n-3} > 0,\ \ldots;\ D_1 = a_1 > 0,\ D_3 > 0,\ \ldots,$

(iv) $a_n > 0;\ a_{n-1} > 0,\ a_{n-3} > 0,\ \ldots;\ D_2 > 0,\ D_4 > 0,\ \ldots$

5.2 The Power Series Approximation of Nonlinear Network Elements

Before stating the stability theorems in terms of the topological quantities, it will be shown that the constant matrix A in (5-1) comes from only the first approximation term of each network element so that the sufficient conditions for stability and instability of the nonlinear networks can be examined by the linearized networks and so that the topological formulas developed in the earlier chapters are applicable to the stability analysis of the nonlinear systems. As for the formulation and existence problems of the general class of nonlinear networks in the normal form which is more general than (5-1), excellent discussions have been available elsewhere.[2,4,5,10,19,26]

An RC nonlinear network is considered first. It is assumed that nonlinear capacitance C(v) and nonlinear conductance G(v) can be represented by absolutely convergent power series approximations for $|v| < B_0$ with the following terminal voltage-current relations:

$$i = C(v)\frac{dv}{dt} = \left(\sum_{k=0}^{\infty} C_k v^k\right)\frac{dv}{dt} \tag{5-3}$$

in $|v| < B_0 \leqslant \dfrac{1}{\overline{\lim}\ \sqrt[k]{|C_k|}}$ and

$$i = G(v)\,v = \sum_{k=0}^{\infty} g_k v^{k+1} \tag{5-4}$$

in $|v| < B_0 \leqslant \dfrac{1}{\overline{\lim}\ \sqrt[k]{|g_k|}}$.

It is further assumed that the δ in (2-2) is zero which is guaranteed if there is no tree-branch resistance in the RC nonlinear network. Then the corresponding equations to (2-2) and (2-3) for the RC network may be written as

$$j_C = -\beta\, i_R - \epsilon\, i_S \tag{5-5}$$

$$v_R = \beta^t\, e_C \tag{5-6}$$

$$v_S = \epsilon^t\, e_C \tag{5-7}$$

$$\begin{bmatrix} j_C \\ i_S \end{bmatrix} = \begin{bmatrix} C(e_C) & 0 \\ 0 & C_2(v_S) \end{bmatrix} \frac{d}{dt} \begin{bmatrix} e_C \\ v_S \end{bmatrix} \tag{5-8}$$

and

$$i_R = G_2(v_R)\, v_R \tag{5-9}$$

From (5-5), (5-7) and (5-8), one obtains

$$j_C + \epsilon\, i_S = \mathbb{C} \frac{d}{dt} e_C = -\beta\, i_R \tag{5-10}$$

where

$$\mathbb{C} = C_1(e_C) + \epsilon\, C_2(\epsilon^t\, e_C)\, \epsilon^t = \sum_{k=0}^{\infty} \mathbb{C}_k\, e_C^k \tag{5-11}$$

and

$$\mathbb{C}_k = C_{1_k} + \epsilon\, C_{2_k} (\epsilon^t)^{k+1} \quad , \quad k = 0, 1, 2, \ldots \tag{5-12}$$

The substitution of (5-9) and (5-6) into (5-10) yields

$$\mathbb{C} \frac{d}{dt} e_C = - \beta\, G_2(\beta^t e_C)\, \beta^t\, e_C = - \sum_{k=0}^{\infty} \mathbb{Y}_k\, e_C^{k+1} \tag{5-13}$$

where

$$\mathbb{Y}_k = \beta\, G_{2_k} (\beta^t)^{k+1} \quad , \quad k = 0, 1, 2, \ldots \tag{5-14}$$

Thus, if $\mathbb{C}$ and $\mathbb{C}_0$ are nonsingular, (5-13) may be written as

$$\frac{d}{dt} e_C = - \mathbb{C}_0^{-1} (I + \sum_{k=1}^{\infty} \mathbb{C}_0^{-1}\, \mathbb{C}_k\, e_C^k)^{-1} (\mathbb{Y}_0\, e_C + \sum_{k=1}^{\infty} \mathbb{Y}_k\, e_C^{k+1}) \tag{5-15}$$

and for sufficiently small $\|e_C\|$, (5-15) has the form of (5-1)

$$\begin{aligned} \frac{d}{dt} e_C = & - \mathbb{C}_0^{-1}\, \mathbb{Y}_0\, e_C - \sum_{k=1}^{\infty} \mathbb{C}_0^{-1}\, \mathbb{Y}_k\, e_C^{k+1} \\ & + \left[\sum_{k=1}^{\infty} \mathbb{C}_0^{-2}\, \mathbb{C}_k\, e_C^k - \left(\sum_{k=1}^{\infty} \mathbb{C}_0^{-2}\, \mathbb{C}_k\, e_C^k \right)^2 + \ldots \right] \left[\sum_{k=0}^{\infty} \mathbb{Y}_k\, e_C^{k+1} \right] \end{aligned} \tag{5-16}$$

From (5-16), it is clear that the matrix corresponding to the A of (5-1) is

$$A_{RC} = - \mathbb{C}_0^{-1}\, \mathbb{Y}_0 = - (C_{1_0} + \epsilon\, C_{2_0}\, \epsilon^t)^{-1}\, \beta\, G_{2_0}\, \beta^t \tag{5-17}$$

where C_{1_0}, C_{2_0} and G_{2_0} are the diagonal matrices of the first approximation terms of tree-branch capacitances, chord capacitances and chord conductances, respectively. Therefore, the A_{RC} consists entirely of the first approximation terms of nonlinear capacitances and conductances and has the same form as that of (3-17a) with $\dot{\mathcal{C}} = 0$.

The similar argument can be applied to an RL nonlinear network. In this case, it is assumed that nonlinear inductance L(i) and nonlinear resistance R(i) can be represented by absolutely convergent power series approximations for $|i| < B_0$ with the following terminal voltage-current relations:

$$v = L(i)\,\frac{di}{dt} = \left(\sum_{k=0}^{\infty} \ell_k\, i^k\right)\frac{di}{dt} \tag{5-18}$$

in $|i| < B_0 \leqslant \dfrac{1}{\overline{\lim}\sqrt[k]{|\ell_k|}}$, and

$$v = R(i)\, i = \sum_{k=0}^{\infty} r_k\, i^{k+1} \tag{5-19}$$

in $|i| < B_0 \leqslant \dfrac{1}{\overline{\lim}\sqrt[k]{|r_k|}}$.

As in the RC case, assuming $\delta = 0$, which is guaranteed if there is no chord resistance, one obtains the form of (5-1) for the RL nonlinear network when $\mathscr{L}$ and $\mathscr{L}_0$ are nonsingular and for sufficiently small $\left\| i_L \right\|$

$$\frac{d}{dt} i_L = -\mathcal{L}_0^{-1} \mathcal{Z}_0 i_L - \sum_{k=1}^{\infty} \mathcal{L}_0^{-1} \mathcal{Z}_k i_L^{k+1}$$
$$+\left[\sum_{k=1}^{\infty} \mathcal{L}_0^{-2} \mathcal{L}_k i_L^k - \left(\sum_{k=1}^{\infty} \mathcal{L}_0^{-2} \mathcal{L}_k i_L^k\right)^2 + \cdots\right]\left[\sum_{k=0}^{\infty} \mathcal{Z}_k i_L^{k+1}\right] \tag{5-20}$$

where

$$\mathcal{L}_k = L_{2_k} - \mu^t L_{1_k} (-\mu)^{k+1} \quad , \quad k = 0, 1, 2, \ldots \tag{5-21}$$

and

$$\mathcal{Z}_k = -\gamma^t R_{1_k} (-\gamma)^{k+1} \quad , \quad k = 0, 1, 2, \ldots \tag{5-22}$$

As before, it is clear from (5-20) that the matrix corresponding to the A of (5-21) is

$$A_{RL} = -\mathcal{L}_0^{-1} \mathcal{Z}_0 = -(L_{2_0} + \mu^t L_{1_0} \mu) \gamma^t R_{1_0} \gamma \tag{5-23}$$

where L_{1_0}, L_{2_0} and R_{1_0} are the diagonal matrices of the first approximation terms of tree-branch inductances, chord inductances and tree-branch resistances, respectively. Therefore, the A_{RL} consists entirely of the first approximation terms of nonlinear inductances and resistances and has the same form as that of (3-17b) with $\dot{\mathcal{L}} = 0$.

Absolute and uniform convergence of the second and higher order power series in (5-16) and (5-20) certainly holds for sufficiently small $\|e_C\|$

and $\|i_L\|$, respectively, since they are the power series resulting from the sum and products of the absolutely and uniformly convergent power series.[15]

5.3 The First Approximation Theorems of Liapunov in Terms of the Topological Quantities

As pointed out in the previous section, the matrices (5-17) and (5-23) have the same forms as those of (3-17) with $\dot{\mathcal{C}} = 0$ and $\dot{\mathcal{L}} = 0$. Thus, the topological Formulas 4 and 5 in Section 3.2 can be used to state Lemmas 4 and 5 in terms of the topological quantities of the linearized networks of the nonlinear RC and RL networks. In order to apply Lemma 6, the coefficient $a_0 = (-1)^n$ must always be positive. This can be done by multiplying all the coefficients through by $(-1)^n$. Thus, the factor $(-1)^n$ in Formulas 4 and 5 will disappear. Formulas 4 and 5 are used together with (i) of Lemma 6 in the following theorems, where all the network elements are the first approximation terms C_0, G_0, L_0 and R_0 and $\dot{\mathcal{C}} = 0$ in N_C and $\dot{\mathcal{L}} = 0$ in N_L are understood.

Theorem 3:

A nonlinear time-invariant RC network whose elements can be represented by (5-3) and (5-4) is asymptotically stable near the origin $e_C = 0$ if

$$\sum \left(\frac{V_{C+n}(C_0)}{V_{C+n-r}(C_0)} \right) \left(\frac{V_{N_C+n-r}(G_0)}{V_{N_C+n}(G_0)} \right) > 0 \quad \text{for } r = n, n-2, \ldots, 1$$

and D_k with Formula 4 substituted > 0 for $k = 3, 5, \ldots$

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It is unstable if there exists an r such that

$$\sum \left(\frac{V_{C+n}(C_0)}{V_{C+n-r}(C_0)} \right) \left(\frac{V_{N_C+n-r}(G_0)}{V_{N_C+n}(G_0)} \right) \leqslant 0$$

Theorem 4:

A nonlinear time-invariant RL network whose elements can be represented by (5-18) and (5-19) is asymptotically stable near the origin $i_L = 0$ if

$$\sum \left(\frac{C[V_{L+n-r}(L_0)]}{C[V_{L+n}(L_0)]} \right) \left(\frac{C[V_{N_L+n}(R_0)]}{C[V_{N_L+n-r}(R_0)]} \right) > 0 \quad \text{for } r = n, n-2, \ldots, 1$$

and D_k with Formula 5 substituted > 0 for 3, 5, ...

It is unstable if there exists an r such that

$$\sum \left(\frac{C[V_{L+n-r}(L_0)]}{C[V_{L+n}(L_0)]} \right) \left(\frac{C[V_{N_L+n}(R_0)]}{C[V_{N_L+n-r}(R_0)]} \right) < 0$$

The proofs of asymptotical stability in Theorems 3 and 4 simply follow from Lemmas 4 and 6 and Formulas 4 and 5. The instability conditions in Theorems 3 and 4 satisfy sufficiently the hypothesis in Lemma 5 since a necessary condition that all characteristic roots have negative real parts is $a_r > 0$ for all r.

If there is no pure capacitance circuit or no pure inductance cut-set, the conditions in the theorems will become simpler according to (3-29) or

(3-31). Although the first approximation capacitance C_0 and inductance L_0 can be negative, if they are positive then the conditions will be further simplified.

5.4 Examples

In the following examples, the values of all network elements assume to be the first terms of convergent power series approximations to nonlinear elements.

Example 3:

Consider a nonlinear RC network shown in Figure 5(a) which could represent a unit of a computer circuit with tunnel diodes in the place of C_1G_1 and C_2G_2 and a voltage source with a series resistance in the place of G_3 (negative conductance). Its C-multiport and C-network are shown in Figures 5(b) and 5(c), respectively. Since n = 2, $V_{C+n} = 1$ and $V_{N_C+n} = 1$, from Theorem 7 the network is asymptotically stable for small voltages across the capacitances C_1 and C_2 if

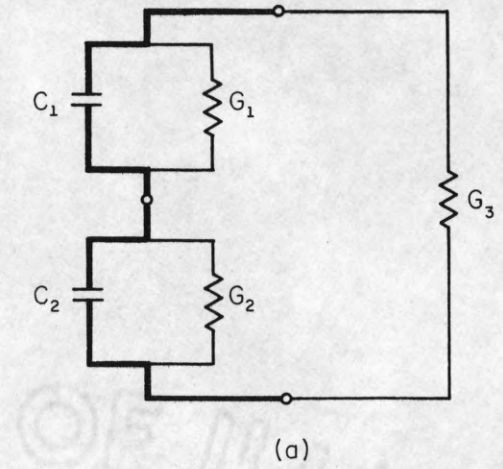
$$\frac{G_1G_2 + G_1G_3 + G_2G_3}{C_1C_2} > 0$$

and

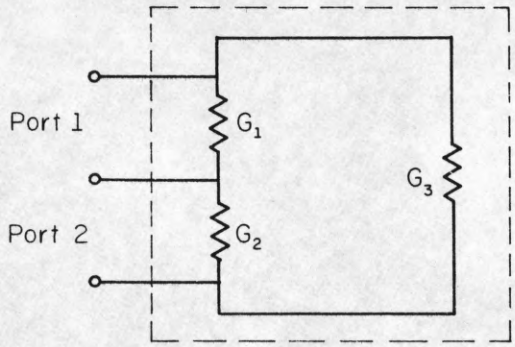
$$\frac{G_1 + G_3}{C_1} + \frac{G_2 + G_3}{C_2} > 0$$

it is unstable if

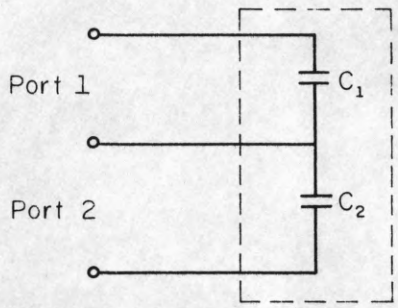
$$\frac{G_1G_2 + G_1G_3 + G_2G_3}{C_1C_2} < 0$$



(a)



(b)



(c)

Figure 5. Example 3: (a) A Nonlinear RC Network
(b) The C-Multiport
(c) The C-Network

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or if

$$\frac{G_1 + G_3}{C_1} + \frac{G_2 + G_3}{C_2} < 0$$

Example 4:

Consider a nonlinear RL network with linearized element values shown in Figure 6(a) where a set of branches, R_1 R_2 R_3 L_3, is chosen as a normal tree. Then its L-multiport, shown in Figure 6(b), is obtained by short-circuiting L_3 and facing by the opened branches L_1 and L_2. The L-network shown in Figure 6(c) is obtained by short-circuiting all resistive tree-branches R_1, R_2 and R_3. From Theorem 4, it is easy to see that the nonlinear RL network is asymptotically stable for small currents through the inductances L_1 and L_2 if

$$\frac{R_1R_2 + R_1R_3 + R_2R_3}{L_1L_2 + L_1L_3 + L_2L_3} > 0$$

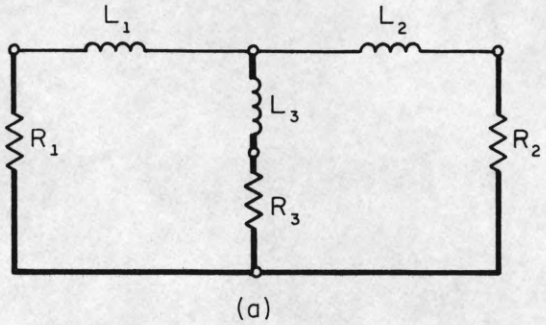
and

$$\frac{L_1 + L_3}{R_1 + R_3} + \frac{L_2 + L_3}{R_2 + R_3} > 0$$

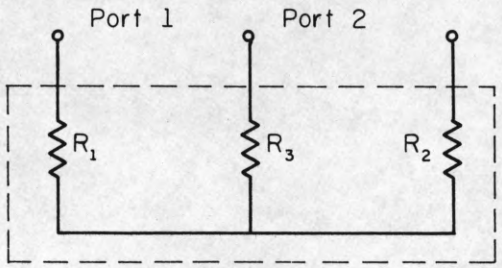
It is unstable if, for example,

$$\frac{R_1R_2 + R_1R_3 + R_2R_3}{L_1L_2 + L_1L_3 + L_2L_3} < 0$$

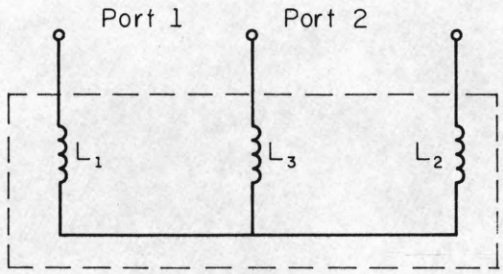
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(a)



(b)



(c)

Figure 6. Example 4: (a) A Nonlinear RL Network
(b) The L-Multiport
(c) The L-Network

VI. CONCLUSIONS AND FURTHER STUDY

6.1 Concluding Remarks

New techniques which provide stability information for time-varying and nonlinear networks simply by inspection of the topology and the element values of the network have been presented. At the same time, a new area of application of linear graph theory (network topology) has been introduced. One significant advantage over other techniques is that stability can be examined directly from the topology and the element values of the network without solving or formulating network equations. As a characteristic of the topological method, it also displays in a very intuitive fashion effects of the topology and the element values of networks to stability. This is very useful in system design in which stability is required. For example, the present work together with an extension to active networks indicated in the next section will be useful in the synthesis of systems utilizing active and nonlinear solid-state devices.

6.2 Further Study and Suggestions

The most important remaining problem which appears to be solvable is an extension of the present work to active networks containing dependent sources and mutual inductances. This requires explicit formulation of active networks in the form of the state equation corresponding to (2-4) using the results specifically stated[9,21,22] or more general forms. Then a theorem corresponding to Lemma 2 should be obtained. Topological formulas corresponding to those in Chapter III will probably be expressed in the

trees common to the voltage graph and the current graph if one employs Mayeda-Coates' method[25] for active networks.

Other suggestions are to obtain topological formulas for the coefficients of the characteristic polynomial of the A-matrix of RLC networks and Hurwitz determinant D_k so that stability analysis of nonlinear RLC networks will become possible and topological formulas of higher-order systems will be simplified. It is also interesting to investigate how the network topology and element values affect stability criteria in the frequency domain.

Although all the topological formulas in this thesis are expressed in terms of trees or chord-sets, they could probably be expressed in terms of multitrees or their complements.

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VITA

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New techniques are presented which provide stability information for time-varying and nonlinear networks by inspection of the topology and the element values of the network. The derivation of the new techniques is based on Liapunov's second method for time-varying networks, his first method for nonlinear networks and network topology. More specifically, Liapunov's stability theorems of linear time-varying RLC networks are formulated in terms of the tree or chord-set products of the subnetworks derived from the state-space representation of the general networks. The local stability theorems of nonlinear time-invariant RC and RL networks are formulated in terms of the tree or chord-set products of subnetworks. Four examples are given to illustrate the above theorems.

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