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ON MINIMAL-VARIANCE CONTROL OF LINEAR SYSTEMS WITH QUADRATIC LOSS

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ERRATA

Page 22, Eqn. (3.47) $\int_{t_0}^{T}$ instead of $\int_{0}^{T}$.

Page 23, Eqn. (3.52) same correction as Eqn. (3.47).

Page 32, Eqn. (5.2) $\Phi^T(\tau-\sigma)$ instead of $\Phi^T(t-\sigma)$.

ON MINIMAL-VARIANCE CONTROL OF LINEAR SYSTEMS WITH QUADRATIC LOSS

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ABSTRACT

The problem of minimizing the variance of a quadratic performance index, in the presence of control noise whose properties are known a priori, has been studied for linear, constant systems with open-loop control. The expected value of the performance index is constrained to be a positive number, which cannot be less than the optimal mean index when variance is free.

Solution is by means of the calculus of variations, which is applied to an equivalent noise-free problem. The necessary (Euler) equations are integro-differential and have a kernel matrix derived from output correlation functions. In general, these equations contain a forcing vector which depends upon the third-moment properties of the noise process. For systems in which the state vector can be chosen as the output, the existence of an inverse for the kernel matrix can be related to the total state controllability of an equivalent linear, noise-free plant which incorporates statistical data from the disturbance process.

A maximum estimate for the number of eigenvalues is given, and is refined for the special case of single-input, state-output control. Necessary and sufficient conditions for a unique solution can be found in specific examples.

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1. INTRODUCTION

A typical problem in the theory of optimal control may be defined in the following manner. Select an m-dimensional control vector $\underline{u}(t)$, $t \in [0,T]$, such that the linear, constant system

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \ , \tag{1.1}$$

$$\underline{y} = H\underline{x} \tag{1.2}$$

moves from an initial output $\underline{y}_0$ to a final output $\underline{y}_T$ while minimizing the performance index

$$J = \int_0^T (\underline{y}^T P \underline{y} + \underline{u}^T R \underline{u}) dt. \tag{1.3}$$

The vectors $\underline{x}$ and $\underline{y}$ have dimension n and r; the time-invariant matrices A, B, H, P, and R are $n \times n$, $n \times m$, $r \times n$, $r \times r$, and $m \times m$ respectively; P and R are symmetric and positive-definite.

It often happens that the control input is accompanied by undesirable disturbances. For example, an antenna positioning device may experience fluctuating loads because of wind gusts. One method of representing such effects is to add a term $C\underline{v}$ in the right member of (1.1), where C is an $n \times p$ constant matrix and $\underline{v}$ is a p-vector, continuous parameter random process.

In such cases, it is possible to replace (1.3) with an essentially different criterion, such as the probability of achieving a given task within prescribed tolerances. An example related to this viewpoint has been formulated by Pontryagin, et. al. [1]. However, if the random processes associated with an optimization problem are true disturbances --

that is, not completely dominating system behavior, yet adding uncertainty to any performance -- then the noise-free index (1.3) appears to retain its original significance. It is itself a random variable, assuming values corresponding to sample functions of the $\underline{v}$ process.

From this latter point of view, the statistical moments of J may be used to define extensions of (1.3). If the system is to be used for a large number of identical performance attempts, as for example at a target practice, a criterion of the form $E\{J\}$ tends to give satisfactory results. Thus it is reasonable to believe that a control which optimizes the mean or ensemble average $E\{J\}$ will, in at least one instance, perform very nearly as expected. This minimal-expectation problem has received considerable attention in the literature. Kushner [2] has developed a stochastic maximum principle for these indices; Kipiniak [3] has investigated the class with the calculus of variations; Bellman and others [4,5,6,7] have used dynamic programming.

Consider, however, an application in which but a single performance is possible; only one opportunity is available to achieve the design objectives. Average behavior may then be quite unsatisfactory, especially if the standard deviation is large. With this motivation, a control will be optimal in the sense of this investigation if it brings about a local minimum of index variance, while satisfying a constraint on index expectation. Very little explicit work on this topic has been published. For systems of the type (1.1), with a term $C\underline{v}$ included, problems which seek to minimize the mean value of a quadratic form in final state or output seem close in concept to the work of this paper. But it can be shown, as

Dreyfus [8] has demonstrated, that these reduce to the solution of the same system with no disturbances. Moreover, as observed by Kushner [9], problems which attempt to optimize $E\{J\}$, J being defined as in (1.3), have this property in general. This does not occur in the minimal-variance problem.

As considered herein, the minimal-variance control problem can, of course, be written in final-value form, as in Orford [10]; but the system equations are then nonlinear, so that many analytical advantages are lost. Equations in this paper are integro-differential, but linear nonetheless. Finally, it is also possible to make use of the terminology of minimal-variance, unbiased-estimation theory. Because the functional estimator is quadratic in the unknowns, however, the problem differs from the work of other investigators, for example, Kalman [11].

2. PROBLEM STATEMENT

In this initial investigation, no attempt has been made to formulate the minimal-variance problem in its most general form. A high priority on analytical results which display the structure of solution makes certain simplifying assumptions necessary.

This analysis therefore considers a linear system, additive control noise, a quadratic performance index, a priori knowledge about the statistical nature of the disturbance process, and open-loop control. If the state is known as time evolves, a performance improvement accrues from application of the open-loop, optimal feedback realization [8].

The problem is to select a control vector $\underline{u}(t)$, $t \in [0,T]$, so that the system

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} + C\underline{v} \ , \tag{2.1}$$

$$\underline{y} = H\underline{x} \tag{2.2}$$

moves from an expected initial output

$$E\{\underline{y}(0)\} = \underline{y}_0 \text{ (given)} \tag{2.3}$$

to an expected final output

$$E\{\underline{y}(T)\} = \underline{y}_T \text{ (given)} \tag{2.4}$$

at time T (given), while satisfying the constraint

$$E\{K\} = K_o \text{ (given)}, \tag{2.5}$$

and achieving a local minimum J^o of the functional

$$J = E\{(K - K_o)^2\} \ , \tag{2.6}$$

where

$$K \triangleq \int_0^T (\underline{y}^T P \underline{y} + \underline{u}^T R \underline{u}) dt. \tag{2.7}$$

In expressions (2.1) through (2.7), $\underline{x}$ is an n-vector; $\underline{u}$ is an m-vector; $\underline{y}$ is an r-vector; and $\underline{v}$ is a p-vector random process which characterizes the control disturbance. It follows that $\underline{x}$ and $\underline{y}$ are also random processes. The constant matrices A, B, C, and H have dimension $n \times n$, $n \times m$, $n \times p$, and $r \times n$ respectively; the positive-definite, symmetric matrices P and R have dimension $r \times r$ and $m \times m$. The initial plant state $\underline{x}(0)$ is a random vector independent of the $\underline{v}$ process. In (2.5), K_o is positive and bounded below by the solution of the minimal-expectation problem. The duration T of operation is finite.

The solution replaces the given problem by an equivalent deterministic counterpart, applies the calculus of variations, and obtains necessary (Euler) equations. Since (2.1) and (2.2) are linear, the method is feasible if $\underline{v}$ has at least a stationary zero mean, and if it is permissible to exchange the order of expectation and integration in (2.5) and (2.6).

Although the necessary equations depend only upon the above assumptions, discussion of the solution simplifies if $\underline{v}$ is also gaussian, white, and has mutually uncorrelated components. The assumption that $\underline{v}$ is gaussian corresponds to many physical cases in which the noise process comes about as a result of the combined effects of a large number of minute disturbances. Moreover, Sivan [12] has shown, for linear plants with quadratic indices, that the gaussian case is "worst to control," in the minimal-expectation problem; that is, the index value is larger for gaussian than for other noises. This result indicates an additional incentive for the gaussian assumption. If $\underline{v}$ is not white, it can be represented by the output of a linear system driven by white noise [13].

The mathematical meaning of (2.1) can be related to the concept of a generalized random process [9,11]; or it may be discussed in terms of a differential stochastic process, as for example in Doob [14]. However, the solution can also proceed in the manner of this paper, since only integrals of the $\underline{v}$ process occur. Such an approach, though formal insofar as the definition of (2.1) is concerned, may be put on a rigorous basis by means of its integral equivalent [11,15].

The necessary equations for time-varying systems are straightforward extensions of those for the constant system of this section. They appear in Section 3.3.2.

3. OPTIMIZATION

Because it is assumed that a priori information concerning the $\underline{v}$ process is available, the averaging operations indicated in the previous section can be carried out directly to convert the problem into a disturbance-free counterpart. The calculus of variations is then applicable directly; and linear, integro-differential equations arise as necessary (Euler) conditions for solution. Besides the output controllability conditions implied in (2.3) and (2.4), similar requirements can be stated in controllability terms to establish the nonsingularity of the Euler kernel matrix, if H has an inverse. A constant and a time-varying example reflect the elementary characteristics of the minimal-variance control, and lead naturally to a discussion of relationships with the minimal-expectation problem.

3.1 A Noise-Free Equivalent Problem

Let $\underline{y}$ be the sum of two parts $\underline{y}_u$ and $\underline{y}_v$, which depend upon $\underline{u}$ and $\underline{v}$ respectively. Such a division is possible since (2.1) and (2.2) are linear. Thus

$$\underline{x} = \underline{z} + \underline{w} \ , \tag{3.1}$$

where

$$\dot{\underline{z}} = A\underline{z} + B\underline{u}, \ \underline{z}(0) = E\{\underline{x}(0)\} \ , \tag{3.2}$$

and

$$\underline{w} = \int_0^t \Phi(t-\sigma)C\underline{v}(\sigma)d\sigma \ . \tag{3.3}$$

In (3.3), $\Phi(t)$ is the state transition matrix [16] corresponding to (2.1). It follows immediately that

$$\underline{y}_u = H\underline{z}; \ \underline{y}_v = H\underline{w} \ ; \tag{3.4}$$

and

$$E\{\underline{y}_v\} = \underline{0} \ . \tag{3.5}$$

Interchange of expectation and integration in (2.5) gives

$$K_o = \int_0^T (E\{\underline{y}^T P \underline{y}\} + \underline{u}^T R \underline{u}) dt , \tag{3.6}$$

since $\underline{u}$ is an open-loop control. Moreover, (2.6) and (2.7) lead in a similar manner to

$$J = \int_0^T \int_0^T E\{\Delta_{yp}(t)\ \Delta_{yp}(\tau)\}\ dt\ d\tau , \tag{3.7}$$

where

$$\Delta_{yp} = \underline{y}^T P \underline{y} - E\{\underline{y}^T P \underline{y}\} . \tag{3.8}$$

Now $\underline{y} = \underline{y}_u + \underline{y}_v$, and

$$E\{\underline{y}^T P \underline{y}\} = \underline{y}_u{}^T P \underline{y}_u + E\{\underline{y}_v{}^T P \underline{y}_v\} . \tag{3.9}$$

An equivalent problem, therefore, is to select a control vector $\underline{u}(t)$, $t \in [0,T]$, so that the system

$$\dot{\underline{z}} = A\underline{z} + B\underline{u} \tag{3.10}$$

satisfies the boundary conditions

$$H\underline{z}(0) = \underline{y}_0,\ H\underline{z}(T) = \underline{y}_T , \tag{3.11}$$

obeys the constraint

$$\int_0^T (\underline{z}^T H^T P H \underline{z} + \underline{u}^T R \underline{u} + \sqrt{q_1(t,t)})\ dt = K_o , \tag{3.12}$$

and minimizes the index

$$J = \int_0^T \int_0^T [4\underline{z}^T(\tau) H^T Q(\tau,t) H \underline{z}(t) + 4\underline{z}^T(t) H^T \underline{q}(t,\tau) + q_2(\tau,t) - q_1(\tau,t)] d\tau\ dt . \tag{3.13}$$

It has been convenient to define

$$Q(\tau,t) = PE\{\underline{y}_v(\tau)\underline{y}_v{}^T(t)\}P \ , \tag{3.14}$$

$$\underline{q}(\tau,t) = PE\{\underline{y}_v(\tau)[\underline{y}_v{}^T(t)P\underline{y}_v(t)]\} \ , \tag{3.15}$$

$$q_1(\tau,t) = E\{\underline{y}_v{}^T(\tau)P\underline{y}_v(\tau)\}E\{\underline{y}_v{}^T(t)P\underline{y}_v(t)\} \ , \tag{3.16}$$

$$q_2(\tau,t) = E\{[\underline{y}_v{}^T(\tau)P\underline{y}_v(\tau)][\underline{y}_v{}^T(t)P\underline{y}_v(t)]\} \ . \tag{3.17}$$

It will be seen in the following section that the scalar functions $q_1(t,\tau)$ and $q_2(t,\tau)$ affect K_o and J^o but do not appear in the necessary equations. The matrix $Q(t,\tau)$ and the vector $\underline{q}(t,\tau)$ contribute to the kernel and driving functions respectively in these relations.

3.2 Necessary Conditions and Equations

In the calculus of variations, (3.12) is an isoperimetric constraint. The technique of solution appends (3.10) and (3.12) to (3.13) with Lagrange multipliers $\underline{\lambda}$ and μ to give a new, unconstrained index

$$\begin{aligned} J' = \int_0^T & [\underline{\lambda}^T(\dot{\underline{z}} - A\underline{z} - B\underline{u}) + \mu(\underline{z}^T H^T P H\underline{z} + \underline{u}^T R\underline{u} + \sqrt{q_1(t,t)}) \\ & + \int_0^T (4\underline{z}^T(\tau)H^T Q(\tau,t)H\underline{z}(t) + 4\underline{z}^T(t)H^T\underline{q}(t,\tau) \\ & + q_2(\tau,t) - q_1(\tau,t))\, d\tau]\, dt \ . \end{aligned} \tag{3.18}$$

The first variation is

$$\begin{aligned} \delta J' = \underline{\lambda}^T(T)\ \delta\underline{z}(T) - \underline{\lambda}^T(0)\delta\underline{z}(0) + \int_0^T (2\mu\underline{u}^T R - \underline{\lambda}^T B)\delta\underline{u}\,dt \\ + \int_0^T [2\mu\underline{z}^T H^T P H - \dot{\underline{\lambda}}^T - \underline{\lambda}^T A \\ + 4\int_0^T (2\underline{z}^T(\tau)H^T Q(\tau,t)H + \underline{q}^T(t,\tau)H)d\tau]\delta\underline{z}(t)dt \ . \end{aligned} \tag{3.19}$$

The necessary equations are then

$$2\mu R\underline{u} = B^T\underline{\lambda} \ ; \tag{3.20}$$

$$4H^T \int_0^T (2Q(t,\tau)H\underline{z}(\tau) + \underline{q}(t,\tau))d\tau + 2\mu H^T PH\underline{z} = \dot{\underline{\lambda}} + A^T\underline{\lambda} \ . \tag{3.21}$$

The foregoing procedure uses the property that

$$Q^T(t,\tau) = Q(\tau,t) \ , \tag{3.22}$$

which follows from (3.14).

Consider now the special case in which $\underline{v}$ is white and wide-sense stationary [14]; then

$$E\{\underline{v}(t)\underline{v}^T(\tau)\} = S\ \delta(t-\tau) \ ; \tag{3.23}$$

the covariance matrix S is symmetric and positive semi-definite. Thus there exists a matrix M with the property that

$$S = MM^T \ . \tag{3.24}$$

The kernel matrix $H^TQ(t,\tau)H$ of (3.21) can be written, by means of (3.3), (3.14), and (3.24), as

$$H^TQ(t,\tau)H = H^TPH \int_0^{(t,\tau)} \Phi(t-\sigma)CMM^TC^T\Phi^T(\tau-\sigma)d\sigma\ H^TPH \ , \tag{3.25}$$

where the notation $(\cdot,\cdot)$ selects the smaller of its arguments. If H^{-1} exists (in which event the output relations can be incorporated into the plant equations), the kernel matrix is nonsingular if and only if the Gramian matrix

$$P((t,\tau)) = \int_0^{(t,\tau)} \Phi((t,\tau)-\sigma)CMM^TC^T\Phi^T((t,\tau)-\sigma)d\sigma \tag{3.26}$$

has an inverse for all $(t,\tau) \in [0,T]$. This requirement is equivalent to the condition [17] that the plant (3.10), with B replaced by CM, be totally state controllable on the interval, which fact can be determined by the test

$$\text{rank}\,[CM \mid ACM \mid A^2CM \mid \ldots \mid A^{n-1}CM] = n. \tag{3.27}$$

This interpretation is not possible if H does not have an inverse.

Another necessary condition, valid for any H, is that

$$\text{rank}\,[HB \mid HAB \mid HA^2B \mid \ldots \mid HA^{n-1}B] = r, \tag{3.28}$$

which implies total output controllability of (3.10) with output equations $\underline{y}_u = H\underline{z}$ and assures that (3.11) can be satisfied for at least one control $\underline{u}$.

The vector $\underline{q}(t,\tau)$ derives from the third-moment properties of the $\underline{v}$ process; and so the optimal control must use, in general, more information than that provided by the covariance matrix of the disturbance. However, if $\underline{v}$ is gaussian, white, and has mutually uncorrelated components, then $\underline{q}(t,\tau)$ vanishes; and the equations (3.20) and (3.21) are homogeneous.

Consider also the second variation

$$\begin{aligned} \delta^2 J' = {} & 2\mu \int_0^T \delta\underline{z}^T H^T P H \delta\underline{z}\,dt + 2\mu \int_0^T \delta\underline{u}^T R \delta\underline{u}\,dt \\ & + 8 \int_0^T \int_0^T \delta\underline{z}^T(\tau) H^T Q(\tau,t) H \delta\underline{z}(t)\,d\tau\,dt \;. \end{aligned} \tag{3.29}$$

The matrices P and R are positive-definite; and the third term is non-negative; thus $\mu > 0$ is sufficient to ensure (3.29) positive, and thereby that the solution is a local minimum. Although this technique may not correspond to classical sufficiency conditions [18], it has fundamental implications in the following example.

3.3 First-Order Examples

In order to demonstrate the solution of (3.10), (3.20), and (3.21), it is instructive to examine the first-order (n=1) case. A detailed investigation follows for a constant system; and some corresponding properties are shown for a time-varying example.

3.3.1 A Constant Plant

Let the system equations be

$$\dot{x} = - ax + u + v , \tag{3.30}$$

$$y = x , \tag{3.31}$$

where $a > 0$ and

$$\varphi(t) = e^{-at} . \tag{3.32}$$

If $P = R = 1$, then

$$Q(t,\tau) = \frac{\sigma_v^2}{2a} (e^{-a|t-\tau|} - e^{-a(t+\tau)}) ; \tag{3.33}$$

$$\underline{q}(t,\tau) = \frac{\alpha_v^3 \, e^{-a(t+2\tau)}}{3a} (e^{3a(t,\tau)} - 1) ; \tag{3.34}$$

$$q_1(t,\tau) = \frac{(\sigma_v^2)^2}{4a^2} (e^{-2at}-1)(e^{-2a\tau}-1) ; \tag{3.35}$$

$$\begin{aligned} q_2(t,\tau) &= \frac{\beta_v^4}{4a} (e^{-2a|t-\tau|} - e^{-2a(t+\tau)}) \\ &+ \frac{(\sigma_v^2)^2}{4a^2} (e^{-2at}-1)(e^{-2a\tau}-1) \\ &+ \frac{(\sigma_v^2)^2}{2a^2} (e^{-2a(t+\tau)} - 2e^{-2a[t,\tau]} + e^{-2a|t-\tau|}) , \end{aligned} \tag{3.36}$$

where $[\cdot,\cdot]$ selects the larger of its arguments. Here v is white and stationary, with second, third, and fourth moments σ_v^2, α_v^3, and β_v^4 respectively.

Because there is only one control variable, (3.10), (3.20), and (3.21) combine to give a single integro-differential equation,

$$\frac{2\sigma_v^2}{a}\int_0^T (e^{-a|t-\tau|}-e^{-a(t+\tau)})z(\tau)d\tau + \frac{\alpha_v^3}{a^2}(1-e^{-at}) + \frac{\alpha_v^3}{3a^2}e^{-2aT}(e^{-at}-e^{2at}) + \mu z = \mu(z^{(2)} - a^2 z) . \tag{3.37}$$

The superscript notation (i) indicates the ith time derivative. Equation (3.37) is equivalent to an ordinary differential equation of fourth order. The reader may verify that

$$z^{(4)} - (1+2a^2)z^{(2)} + \left[\frac{4\sigma_v^2}{\mu} + a^2(a^2+1)\right]z = -\frac{\alpha_v^3}{\mu}\left[1+e^{-2a(T-t)}\right] \tag{3.38}$$

is equivalent to (3.37) by a twofold time differentiation and substitution. The solution is

$$z(t) = \sum_{i=1}^{4} c_i e^{p_i t} - \frac{\alpha_v^3}{4\sigma_v^2 + \mu a^2(a^2+1)} - \frac{\alpha_v^3 e^{-2a(T-t)}}{4\sigma_v^2 + 3\mu a^2(3a^2-1)} , \tag{3.39}$$

where the p_i are the (distinct) roots of

$$p_i^4 - (1+2a^2)p_i^2 + \frac{4\sigma_v^2}{\mu} + a^2(a^2+1) = 0 . \tag{3.40}$$

In case $\mu = 16\sigma_v^2$ or $\mu = -4\sigma_v^2/[a^2(a^2+1)]$, the p_i are not distinct; and appropriate changes occur in (3.39).

Two conditions arise from (3.11); and these are

$$\sum_{i=1}^{4} c_i = z(0) + \frac{\alpha_v^3}{4\sigma_v^2 + \mu a^2(a^2+1)} + \frac{\alpha_v^3 e^{-2aT}}{4\sigma_v^2 + 3\mu a^2(3a^2-1)} \quad ; \tag{3.41}$$

$$\sum_{i=1}^{4} c_i e^{p_i T} = z(T) + \frac{\alpha_v^3}{4\sigma_v^2 + \mu a^2(a^2+1)} + \frac{\alpha_v^3}{4\sigma_v^2 + 3\mu a^2(3a^2-1)} \tag{3.42}$$

Two more conditions result from substituting (3.39) into (3.37). Linearly independent functions of the form $e^{p_i t}$ and $e^{\pm at}$ result from this substitution. Since the coefficients of $e^{p_i t}$ terms vanish identically if p_i is a solution of (3.40), a necessary and sufficient condition for (3.39) to satisfy (3.37) is that the coefficients of $e^{\pm at}$ terms also vanish identically. This will occur if the following two conditions are satisfied:

$$\sum_{i=1}^{4} c_i \left[\frac{e^{(p_i-a)T}}{p_i-a} + \frac{2a}{a^2-p_i^2} \right] - \frac{\alpha_v^3}{4\sigma_v^2+\mu a^2(a^2+1)} \left[\frac{2}{a} - \frac{e^{-aT}}{a} \right]$$

$$- \frac{\alpha_v^3}{4\sigma_v^2+3\mu a^2(3a^2-1)} \left[\frac{e^{-aT}}{a} - \frac{2e^{-2aT}}{3a} \right] \tag{3.43}$$

$$+ \frac{\alpha_v^3}{6a\sigma_v^2} (3-e^{-2aT}) = 0 \quad ;$$

$$\sum_{i=1}^{4} c_i \left[\frac{e^{(p_i-a)T}}{p_i-a} \right] + \frac{\alpha_v^3}{4\sigma_v^2 + \mu a^2(a^2+1)} \left[\frac{e^{-aT}}{a} \right]$$

$$- \frac{\alpha_v^3}{4\sigma_v^2 + 3\mu a^2(3a^2-1)} \left[\frac{e^{-aT}}{a} \right] = 0 \ . \tag{3.44}$$

These equations are solvable for the constants $\{c_i\}$, $i = 1,2,3,4$; the control follows from (3.10).

If v is a gaussian process, then $\alpha_v^3 = 0$. Let $\sigma_v^2 = 1$, $z(0) = 1$, $z(T) = 0$, $a = 0.1$, and $T = 1$. Figures 1 and 2 present curves of K_o and J^o versus μ. For a given K_o, it may happen that more than one value of μ can satisfy the constraint (3.12); however, the numerical values of J^o are smallest for positive μ. It is evident that the sensitivity of J^o and K_o to changes in μ is uniformly better for $\mu > 0$.

In general, as $\mu \to 0^+$, the solution corresponds to a minimal-variance problem with no constraints on $E\{K\}$. The control is impulsive, since $\underline{u}$ is not weighted in the J index (3.13). As $\mu \to \infty$, the solution

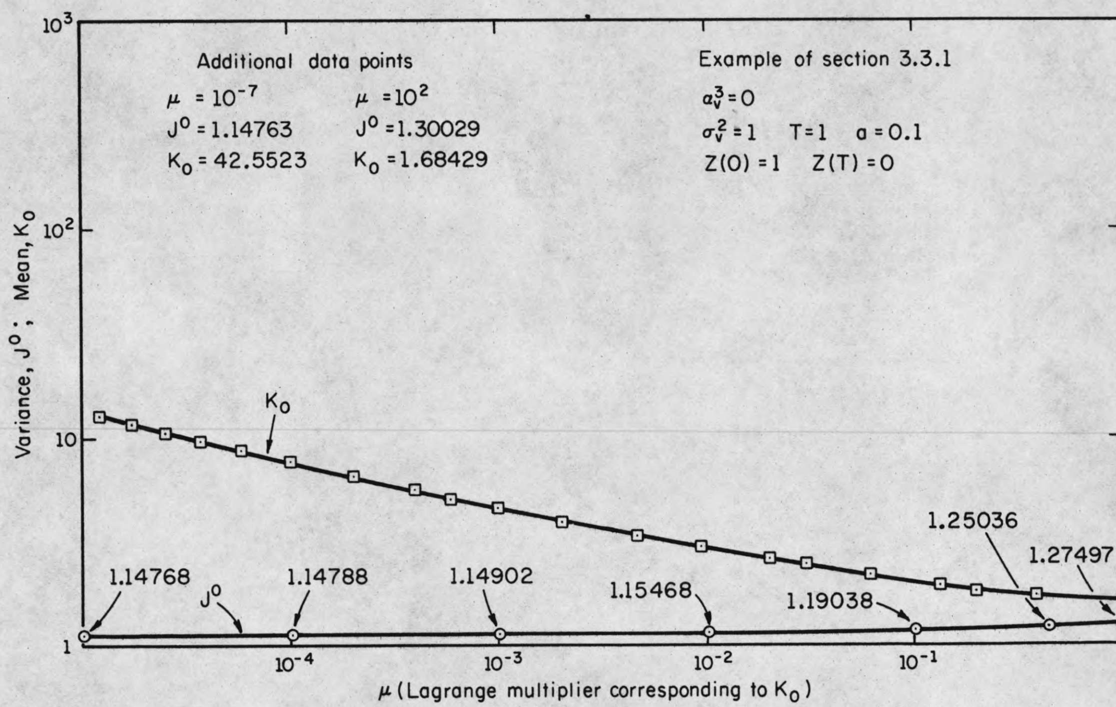

Fig. 1. Variance J^o and Mean K_o as a Function of $\mu(\mu > 0)$.

Fig. 2. Variance J^o and Mean K_o as a Function of $\mu(\mu < 0)$.

minimizes $E\{K\}$, with no constraint on J; Section 3.4 contains a discussion of this case.

By analysis of the c_i equations, (3.41) through (3.44), it is found that the excursions of Figure 2 are such as to satisfy a transcendental equation of the type

$$(a^2 - \alpha_1^2)(a + \beta_3 \cot \beta_3 T) = (a^2 + \beta_3^2)(a + \alpha_1 \coth \alpha_1 T), \quad (3.45)$$

where $p_i = \alpha_i + j\beta_i$. The character of (3.45) indicates an increasing number of excursions as $\mu \to 0^-$.

For positive μ, K_o as a function of J^o appears in Figure 3. The optimal variance J^o changes relatively little as K_o varies, since the contribution to J^o by the process v dominates that of the control u (through z). Increased J^o changes occur if z(0) becomes larger in comparison with σ_v^2. Of course, K_o also grows in this case. Figure 4 substantiates these observations. The singular properties of u, as $\mu \to 0^+$, are also observed in these curves.

The relative size of J^o and K_o, for a fixed μ, depends upon T and a, if z(0) and σ_v^2 are constant, as seen in Figure 5. Asymptotic behavior of K_o and J^o occurs because of steady v process contributions after controlled dynamic responses have decreased, which is in agreement with (3.40), an equation that does not contain T.

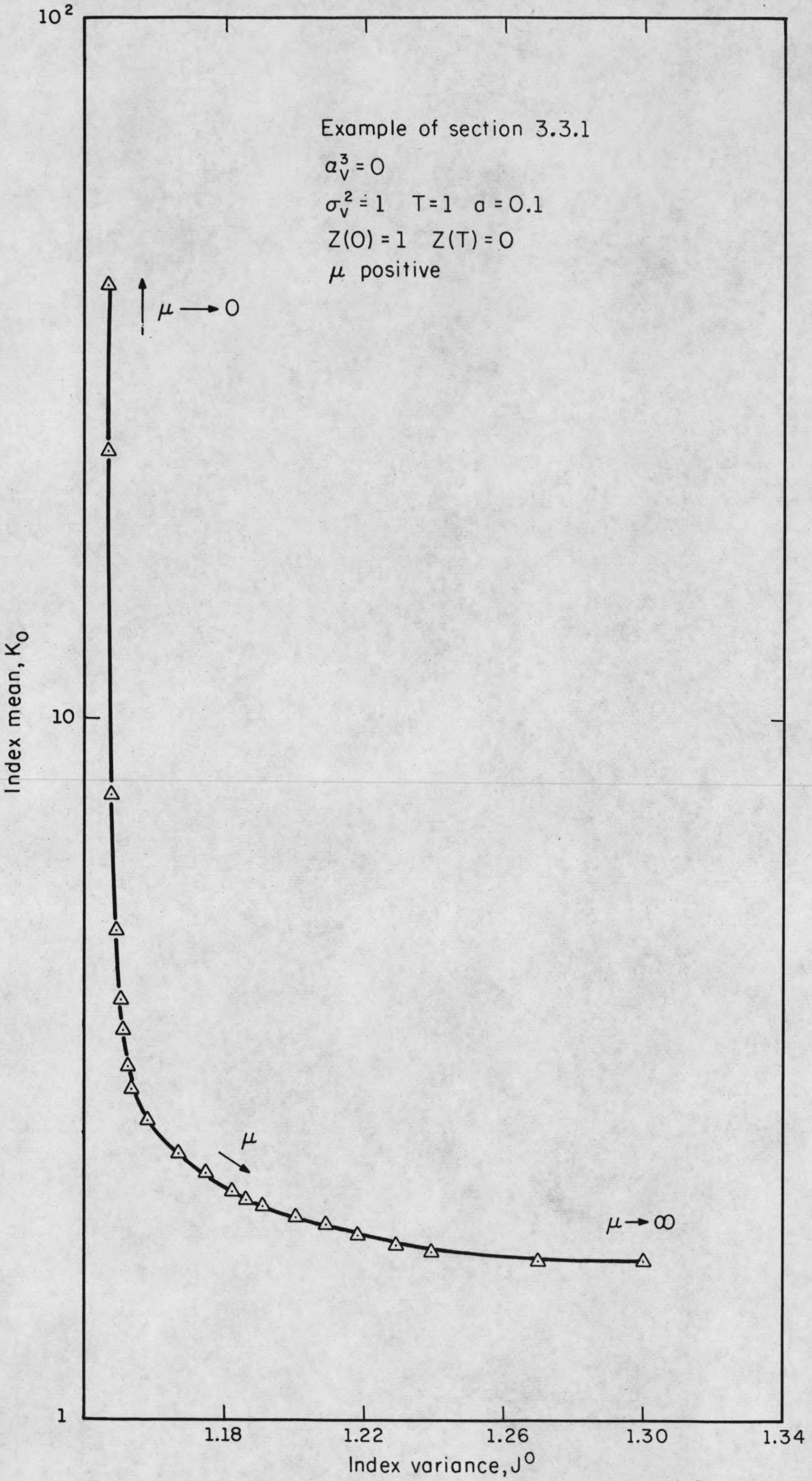

Fig. 3. Mean K_o as a Function of Variance J^o.

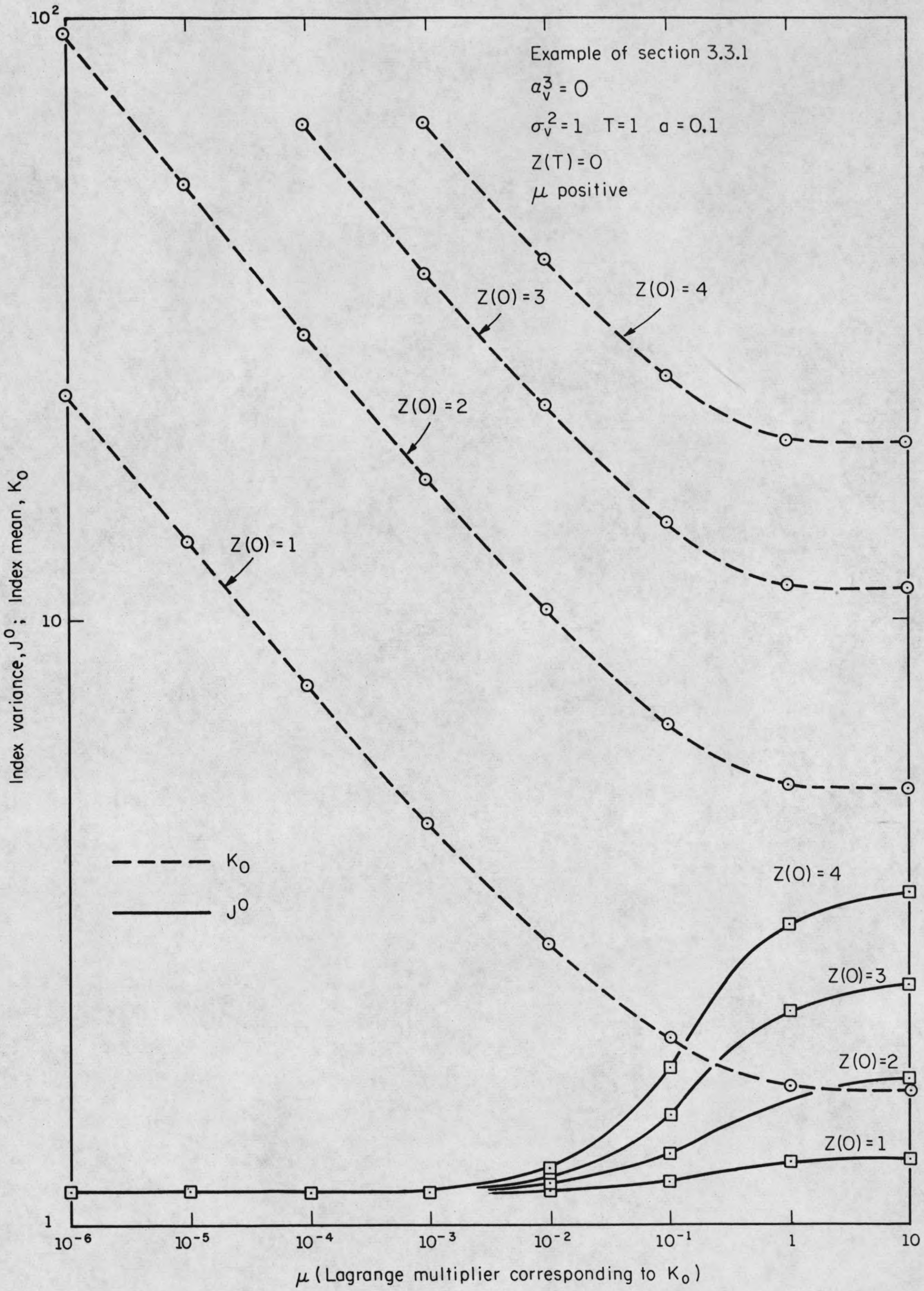

Fig. 4. Variance J^O and Mean K_o as a Function of $\mu(\mu > 0)$ with Parameter z(0).

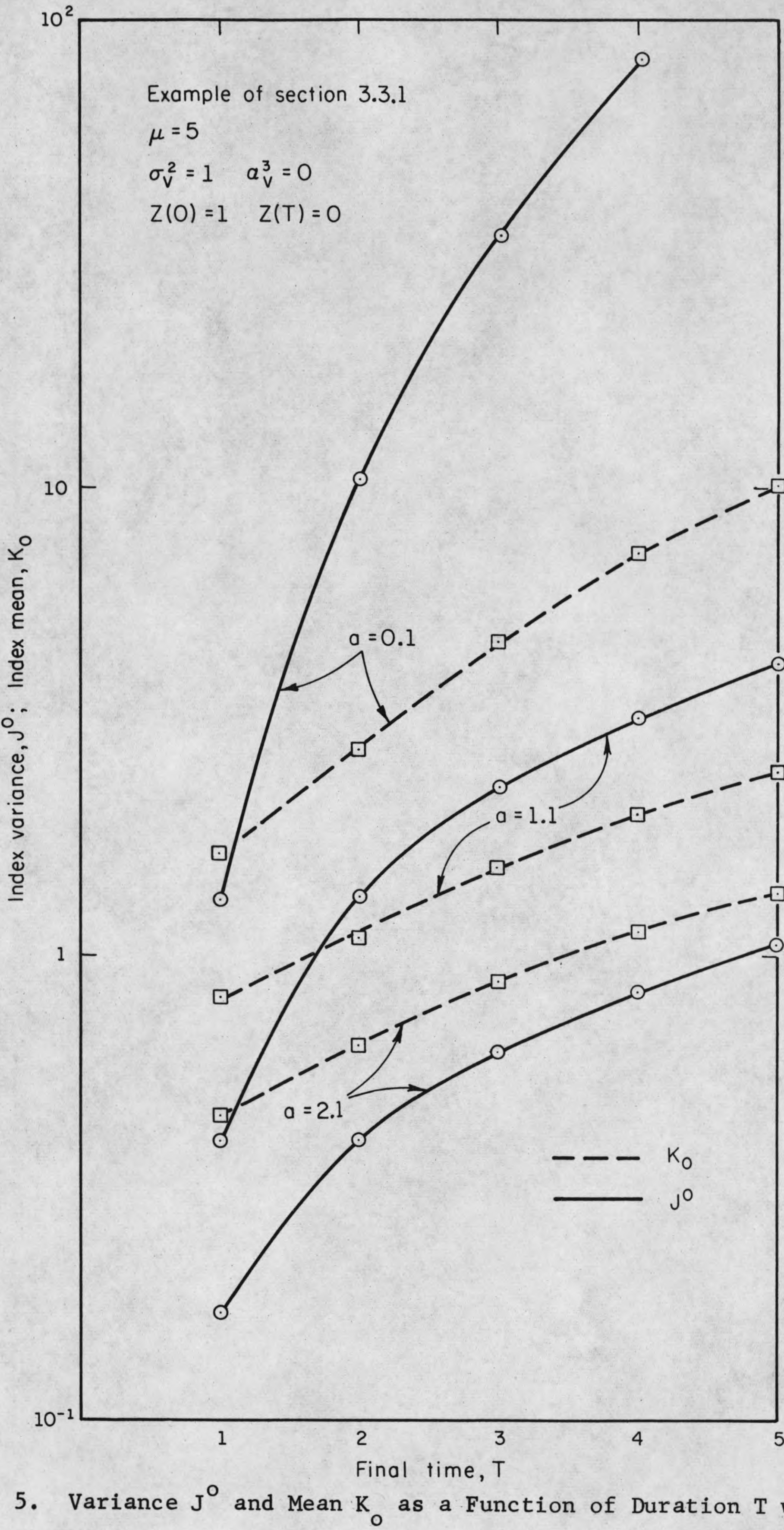

Fig. 5. Variance J^0 and Mean K_0 as a Function of Duration T with Parameter a, for Constant μ.

3.3.2 A Time-Varying Plant

In case the system is time-varying, the optimization proceeds again as in Section 3.2 to give the necessary equations

$$2\mu R(t)\underline{u} = B^T(t)\underline{\lambda} \,, \tag{3.46}$$

$$4H^T(t) \int_0^T [2Q(t,\tau)H(\tau)\underline{z}(\tau) + \underline{q}(t,\tau)]d\tau + 2\mu H^T(t)P(t)H(t)\underline{z} = \dot{\underline{\lambda}} + A^T(t)\underline{\lambda} \,. \tag{3.47}$$

Corresponding modifications must be made in the plant equations, boundary conditions, and expressions for K and J; definitions (3.14) through (3.17) now depend on t_0 through (3.3), which must have a lower limit t_0 and incorporate a transition matrix $\Phi(t,\sigma)$.

Although many of the properties of the preceding example carry over to the time-varying example below, lack of a closed-form solution prevents development of a complete analogy. Let the system be

$$\dot{x} = -\frac{1}{t}x + u + v \,, \tag{3.48}$$

$$y = x \,, \tag{3.49}$$

where

$$\varphi(t,t_0) = t_0/t \,, \tag{3.50}$$

and

$$Q(t,\tau) = \frac{\sigma_v^2 p^2}{3t\tau} \left((t,\tau)^3 - t_0^3\right). \tag{3.51}$$

Here P = p, R = r, and v is white and gaussian. The integro-differential equation corresponding to (3.37) is

$$\frac{4\sigma_v^2 p^2}{3t} \int_0^T \frac{((t,\tau)^3 - t_0^3)}{\tau} z(\tau)d\tau$$

$$+ \mu pz = \mu r(z^{(2)} - \frac{2}{t^2} z) . \tag{3.52}$$

The method of differentiation and substitution is applicable again; and the ordinary differential equation corresponding to (3.38) is

$$z^{(4)} - \left[\frac{4}{t^2} + \frac{p}{r}\right] z^{(2)} + \frac{8}{t^3} z^{(1)}$$

$$+ \left[\frac{2}{r}\left(\frac{p}{t^2} + \frac{2\sigma_v^2 p^2}{\mu}\right) - \frac{8}{t^4}\right] z = 0 . \tag{3.53}$$

Thus the optimal trajectory consists once more of 4n characteristic functions. This property appears general, if $\underline{u}$ has only one component. Section 4 considers this class of problems in more detail for constant systems.

3.4 Relation to Minimal-Expectation Problem

Consider the problem of selecting a control vector $\underline{u}(t)$, $t \in [0,T]$, so that the system (2.1), (2.2) moves from an expected initial output (2.3) to an expected final output (2.4) and minimizes the expectation of (2.7). It has been noted in Section 1 that the control based upon $\underline{v} = \underline{0}$ is the same as the control of this problem. Therefore, by the techniques of Sections 3.1 and 3.2, or by optimization of the noise-free problem,

$$2R\underline{u} = B^T\underline{\lambda} , \tag{3.54}$$

$$2H^T PH\underline{z} = \dot{\underline{\lambda}} + A^T\underline{\lambda} . \tag{3.55}$$

In expressions (3.20) and (3.21), define a new $\underline{\lambda}_1 = \underline{\lambda}/\mu$, and pass to the limit as $\mu \to \infty$, assuming $\underline{\lambda}_1$ to exist. The resulting expressions are the same as (3.54) and (3.55), except for the subscript on $\underline{\lambda}_1$.

Solution to the problem of this section is therefore a limiting case of the problem of this paper, when the limit exists. In particular, the analytical results of Section 4, as $\mu \to \infty$, agree with a direct analysis of (3.54) and (3.55); moreover, the numerical results of Section 3.3.1, for large μ, are in close agreement with those of a corresponding example selected from Section 3.4.

The minimal-variance problem of Section 2 therefore leads to a set of linear, integro-differential, necessary equations. In simple cases, the solution is straightforward; and the numerical results of Section 3.3.1 support (3.29) insofar as the relation of sufficiency conditions and $\mu > 0$ is concerned. Preliminary examples exhibit 4n characteristic functions; this property is generalized in the next section to a larger class of problems having only one control variable. Numerical results indicate a reduction in index variance for an increase in index expectation, and vice-versa. Such data is in accordance with the concept of a density function. Finally, the quantities $Q(t,\tau)$, $\underline{q}(t,\tau)$, $q_1(t,\tau)$, and $q_2(t,\tau)$ relate simply to system parameters in these examples. This does not happen in general; and so a general matrix theory of solution is not straightforward.

4. SINGLE-INPUT CONTROL

Suppose there is a single control variable and a single disturbance process, that both enter the plant equations in like manner ($B = C$), and that the output matrix $H = I$. Further, let the v process be white and stationary. Then the necessary conditions (3.27) and (3.28) are equivalent. Wonham and Johnson [19] have found a canonical form for this case. It is shown in this section that there are at most 4n characteristic functions associated with the optimal trajectory z and each of its first n-1 derivatives, and therefore with the optimal control. It may happen that there are fewer than 4n such functions; an example of such a control appears in Section 4.2. It is convenient to assume that v is gaussian.

4.1 Structure of Solution

For a plant ($H = I$) which satisfies (3.28), therefore, there is no loss of generality in a study of the case below:

$$A = \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ \vdots & & & & & & & \vdots \\ 0 & . & . & . & . & . & 0 & 1 \\ -a_0 & -a_1 & . & . & . & . & . & -a_{n-1} \end{bmatrix} ; \qquad (4.1)$$

$$B = \underline{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} ; \qquad C = \underline{c} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} . \qquad (4.2)$$

The plant has a characteristic equation

$$\sum_{i=0}^{n} a_i s^i = 0, \; a_n = 1 \; ; \qquad (4.3)$$

and the $\underline{z}$ vector has components

$$z_i = z^{(i-1)} \; . \qquad (4.4)$$

The Laplace transform of the state transition matrix is

$$\Phi_{ij}(s) = \sum_{k=j}^{n} a_k s^{k+i-j-1} / \sum_{k=0}^{n} a_k s^k \; , \; j \geq i \; ; \qquad (4.5)$$

$$\Phi_{ij}(s) = - \sum_{k=0}^{j-1} a_k s^{k+i-j-1} / \sum_{k=0}^{n} a_k s^k \; , \; j < i \; . \qquad (4.6)$$

If (4.3) has distinct roots $\{s_i\}$, $i = 1,2,\ldots,n$, then

$$\varphi_{ij}(t) = \sum_{m=1}^{n} \left[\sum_{k=j}^{n} a_k s_m^{k+i-j-1} / \Delta(m) \right] e^{s_m t} \; , \; j \geq i \; ; \qquad (4.7)$$

$$\varphi_{ij}(t) = - \sum_{m=1}^{n} \left[\sum_{k=0}^{j-1} a_k s_m^{k+i-j-1} / \Delta(m) \right] e^{s_m t} \; , \; j < i \; , \qquad (4.8)$$

where

$$\Delta(m) = \sum_{k=1}^{n} k a_k s_m^{k-1} \; . \qquad (4.9)$$

Appropriate changes in (4.7) and (4.8) occur if the s_i are not all distinct. If P is diagonal, having elements P_{ii}, then (3.14), (4.2), and (4.7) give

$$Q_{ij}(t,\tau) = \sigma_v^2 P_{ii} P_{jj} \sum_{m=1}^{n} \sum_{p=1}^{n} \left[\frac{s_p^{j-1} s_m^{i-1}}{\Delta(m)\,\Delta(p)\,(s_m + s_p)} - \frac{e^{-(s_m + s_p)(t,\tau)}}{s_m + s_p} \right] e^{s_m t} e^{s_p \tau} . \tag{4.10}$$

The first necessary equation (3.20) gives

$$2\mu r u = \lambda_n , \tag{4.11}$$

which combines with the plant equations

$$\sum_{i=0}^{n} a_i z^{(i)} = u \tag{4.12}$$

to give

$$\lambda_n = 2\mu r \sum_{i=0}^{n} a_i z^{(i)} . \tag{4.13}$$

Note that the control weighting matrix R is a scalar r. Since v is gaussian, the second necessary equation (3.21) becomes

$$8 \int_0^T Q(t,\tau)\underline{z}(\tau)d\tau + 2\mu P\underline{z} = \dot{\underline{\lambda}} + A^T\underline{\lambda} . \tag{4.14}$$

A systematic elimination of Lagrange variables is possible; and the result is a single integro-differential equation

$$4 \sum_{k=1}^{n} \sum_{j=1}^{n} (-1)^{j+1} \frac{\partial^{j-1}}{\partial t^{j-1}} \int_0^T Q_{jk}(t,\tau) z^{(k-1)}(\tau)d\tau + \mu \sum_{j=1}^{n} (-1)^{j+1} P_{jj} z^{(2j-2)} + \mu r \sum_{j=0}^{n} \sum_{\ell=0}^{n} (-1)^{\ell} a_j a_\ell z^{(j+\ell)} = 0. \tag{4.15}$$

The extension of the Section 3 methods of differentiation and substitution to (4.15) is difficult. However, it is also possible to proceed as follows. Let

$$z = \sum_{i=1}^{4n} c_i e^{p_i t} \quad ; \tag{4.16}$$

in this case (4.15) becomes

$$\begin{aligned}
0 = & - 4\sigma_v^2 \sum_{i=1}^{4n} c_i \sum_{k=1}^{n} p_i^{k-1} P_{kk} \sum_{j=1}^{n} (-1)^{j+1} P_{jj} \sum_{p=1}^{n} \sum_{m=1}^{n} \Bigg\{ \frac{s_m^{j-1}}{\Delta(m)} \cdot \\
& \cdot \frac{s_p^{k-1}}{\Delta(p)} \frac{\partial^{j-1}}{\partial t^{j-1}} \left[\frac{e^{p_i t}}{(p_i - s_m)(p_i + s_p)} \right. \\
& \left. - \frac{e^{s_m t}}{(p_i - s_m)(p_i + s_p)} + \frac{e^{(p_i + s_p)T} (e^{-s_p t} - e^{s_m t})}{(s_m + s_p)(p_i + s_p)} \right] \Bigg\} \\
& + \mu \sum_{i=1}^{4n} c_i \left[\sum_{j=1}^{n} (-1)^{j+1} P_{ii} p_i^{2j-2} \right] e^{p_i t} \\
& + \mu r \sum_{i=1}^{4n} c_i \left[\sum_{j=0}^{n} a_j \sum_{\ell=0}^{n} (-1)^{\ell} a_{\ell} p_i^{j+\ell} \right] e^{p_i t} \quad .
\end{aligned} \tag{4.17}$$

The coefficient of $e^{p_i t}$ must vanish identically for each i; and this occurs if the equation

$$-4\sigma_v^2 \sum_{k=1}^{n} p_i^{k-1} P_{kk} \sum_{j=1}^{n} (-1)^{j+1} P_{jj} p_i^{j-1} \sum_{p=1}^{n} \sum_{m=1}^{n} \left[\frac{s_m^{j-1}}{(p_i - s_m)\Delta(m)} \cdot \frac{s_p^{k-1}}{\Delta(p)(p_i + s_p)} \right] + \mu \sum_{j=1}^{n} (-1)^{j+1} P_{jj} p_i^{2j-2} \tag{4.18}$$

$$+ \mu r \sum_{j=0}^{n} a_j \left[\sum_{\ell=0}^{n} (-1)^{\ell} a_\ell p_i^{j+\ell} \right] = 0$$

is satisfied by (at most) 4n distinct p_i. This expression is, in general, of degree 2n in p_i^2. The coefficients of terms of the form $e^{\pm s_i t}$ in (4.17) must also vanish identically for solution, giving 2n conditions. There are also 2n conditions on the c_i to meet boundary specifications on $\underline{z}(0)$ and $\underline{z}(T)$. The set of 4n equations in the 4n unknown c_i appear below.

$$\sum_{i=1}^{4n} c_i p_i^k e^{p_i T} = z^{(k)}(T), \quad k = 0,1,\ldots,n-1 ; \tag{4.19}$$

$$\sum_{i=1}^{4n} c_i p_i^k = z^{(k)}(0), \quad k = 0,1,\ldots,n-1 ; \tag{4.20}$$

$$\sum_{i=1}^{4n} c_i \sum_{k=1}^{n} p_i^{k-1} P_{kk} \sum_{j=1}^{n} (-1)^{j+1} P_{jj} \sum_{p=1}^{n} \left\{ \frac{s_m^{2j-2} s_p^{k-1}}{\Delta(m)\,\Delta(p)} \cdot \left[\frac{1}{(p_i - s_m)(p_i + s_p)} + \frac{e^{(p_i + s_p)T}}{(s_m + s_p)(p_i + s_p)} \right] \right\} = 0, \tag{4.21}$$

$$m = 1,2,\ldots,n ;$$

$$\sum_{i=1}^{4n} c_i \sum_{k=1}^{n} p_i^{k-1} P_{kk} \sum_{j=1}^{n} (-1)^{j+1} P_{jj} \sum_{m=1}^{n} \left\{ \frac{s_m^{j-1} s_p^{j+k-2}}{\Delta(m)\,\Delta(p)} \cdot \right.$$

$$\left. \cdot \left[\frac{e^{(p_i+s_p)T}}{(s_m+s_p)(p_i+s_p)} \right] \right\} = 0, \quad p = 1,2,\ldots,n . \tag{4.22}$$

If these equations are solvable for the constants c_i, then the vector $\underline{z}$ is given by (4.16) and (4.4); and the open-loop control by (4.12). A necessary and sufficient condition for unique solution is that these equations determine the $\{c_i\}$, $i = 1,2,\ldots,4n$, uniquely.

4.2 A Degenerate Example

As observed at the beginning of Section 4, it may happen that (4.18) is of degree less than 4n; that is, the roots of (4.3) have caused cancellations in (4.18). The problem remains determinate, however, since corresponding conditions in the sets (4.21) and (4.22) also vanish. The following example shows a degeneracy of this type.

Let the plant matrix be

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \tag{4.23}$$

with $s_1 = -1$, $s_2 = -2$. The p_i are roots of

$$p_i^6 - (9 + \frac{1}{r})p_i^4 + [24 + \frac{1}{r}(5 + \frac{4}{\mu})]p_i^2$$

$$- [16 + \frac{4}{r}(1 + \frac{1}{\mu})] = 0. \tag{4.24}$$

There are six c_i to determine. Four conditions satisfy specifications on $z(0)$, $z(T)$, $z^{(1)}(0)$, and $z^{(1)}(T)$. The remaining two conditions guarantee that coefficients of $e^{\pm 2t}$ vanish in (4.17). The reader may verify that coefficients of $e^{\pm t}$ vanish also without further conditions.

For a large class of problems having a single control variable, therefore, the optimal trajectory and control consist in general of at most 4n characteristic functions. The uniqueness of the control depends upon the inversion of an equation

$$N \underline{c} = \underline{f} \qquad (4.25)$$

for the unknown coefficient vector $\underline{c}$. Equation (4.25) represents the aggregate of (4.19) through (4.22). A matrix expression for N is not generally available, because of the difficulty in expressing $Q(t,\tau)$ in simple form, or equivalently in performing the integration in (3.25).

5. MULTI-INPUT SYSTEMS

If the control has more than one component, no systematic elimination of Lagrange variables is possible, so that a reduction to scalar equations offers no simplification. Moreover, a matrix analysis depends upon evaluating

$$Q(t,\tau) = PHW(t,\tau)H^T P \; , \tag{5.1}$$

where

$$W(t,\tau) \triangleq \int_0^{(t,\tau)} \Phi(t-\sigma)CSC^T\Phi^T(t-\sigma)d\sigma. \tag{5.2}$$

Here S is the covariance matrix of the $\underline{v}$ process, which is assumed to be at least wide-sense stationary and white in (5.2). It has not been possible to give a general, algebraic form for (5.2); however, in any specific example the integration is straightforward. It is necessary to discuss the form of $Q(t,\tau)$, rather than its explicit value, in a general formulation.

5.1 A Typical Form for $Q(t,\tau)$

From (5.2), the initial step is to determine the form of the transition matrix. If the eigenvalues of A are distinct, then $\varphi_{ij}(t)$ may be written as

$$\varphi_{ij}(t) = \sum_{k=1}^{n} c_{kij} e^{s_k t} \; ; \tag{5.3}$$

and the equation

$$W_{ij}(t,\tau) = \int_0^{(t,\tau)} \sum_{k=1}^{n} \sum_{\ell=1}^{n} (CSC^T)_{k\ell}\varphi_{ik}(t-\sigma)\varphi_{\ell j}(\tau-\sigma)d\sigma \tag{5.4}$$

becomes

$$W_{ij}(t,\tau) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} \sum_{p=1}^{n} \left(\frac{(CSC^T)_{k\ell} c_{mik} c_{p\ell j}}{(s_m + s_p)} \cdot \right.$$

$$\left. \cdot \left[e^{s_m t} e^{s_p \tau} - \begin{Bmatrix} e^{s_p(\tau - t)} & t < \tau \\ e^{s_m(t-\tau)} & t > \tau \end{Bmatrix} \right] \right). \tag{5.5}$$

If the eigenvalues of A are not distinct, then modifications occur in (5.3) and (5.5); such changes do not affect the procedures which follow, except to substitute other linearly independent functions of the type te^{pt} or $e^{\alpha t} \cos \beta t$ for the simple exponentials of (5.3).

The use for (5.5) is in the evaluation of the integral

$$\int_0^T Q(t,\tau) e^{p_i \tau} d\tau \, , \tag{5.6}$$

where $e^{p_i \tau}$ is a scalar function. From the $W(t,\tau)$ matrix, it follows that

$$\left(\int_0^T W(t,\tau) e^{p_i \tau}) d\tau \right)_{ij} = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} \sum_{p=1}^{n} \left(\frac{(CSC^T)_{k\ell} c_{mik} c_{p\ell j}}{(s_m + s_p)} \cdot \right.$$

$$\cdot \left[\left\{ \frac{1}{p_i + s_p} - \frac{1}{p_i - s_m} \right\} e^{p_i t} - \left\{ \frac{e^{(p_i + s_p)T}}{p_i + s_p} \right\} e^{-s_p t} \right. \tag{5.7}$$

$$\left. \left. + \left\{ \frac{1}{p_i - s_m} + \frac{e^{(s_p + p_i)T} - 1}{p_i + s_p} \right\} e^{s_m t} \right] \right) .$$

It is therefore possible to write

$$\int_0^T Q(t,\tau)e^{p_i\tau}d\tau = PHQ_1(p_i)H^TPe^{p_it}$$

$$+ \sum_{m=1}^{n} PHQ_2(p_i,m)H^TPe^{-s_mt} \qquad (5.8)$$

$$+ \sum_{m=1}^{n} PHQ_3(p_i,m)H^TPe^{s_mt} \quad ,$$

where $Q_1(p_i)$, $Q_2(p_i,m)$, and $Q_3(p_i,m)$ must be found for individual examples. The purpose of illustrating the decomposition (5.8) is to show that the fundamental method of solution introduced in the preceding sections is applicable to multi-input systems.

5.2 A Method of Analysis

Let the components of $\underline{v}$ be mutually uncorrelated, and let $\underline{v}$ be gaussian. Then $\underline{q}(t,\tau) = \underline{0}$, and S is diagonal. It is possible to find the eigenvalues of the integro-differential equations (3.10), (3.20), and (3.21) by assuming solutions of the form

$$\underline{z} = \sum_{i=1}^{d} \underline{c}_{zi}e^{p_it} \quad , \underline{\lambda} = \sum_{i=1}^{d} \underline{c}_{\lambda i}e^{p_it} \quad , \qquad (5.9)$$

where $\underline{c}_{zi}$ and $\underline{c}_{\lambda i}$ are n × 1 constant vectors. Mittra [20] has done related work for integral equations. Then (3.10) and (3.20) combine to give

$$p_i\underline{c}_{zi} = A\underline{c}_{zi} + \frac{1}{2\mu} BR^{-1}B^T\underline{c}_{\lambda i}, \ i = 1,2,\ldots,d \ ; \qquad (5.10)$$

and (3.21) becomes

$$8 \sum_{i=1}^{d} H^T \left(\int_0^T Q(t,\tau) e^{p_i \tau} d\tau \right) H\underline{c}_{zi} + \sum_{i=1}^{d} 2\mu H^T PH\underline{c}_{zi} e^{p_i t}$$

$$= \sum_{i=1}^{d} p_i \underline{c}_{\lambda i} e^{p_i t} + \sum_{i=1}^{d} A^T \underline{c}_{\lambda i} e^{p_i t} \quad . \tag{5.11}$$

From (5.10),

$$\underline{c}_{zi} = \frac{1}{2\mu} (p_i I - A)^{-1} BR^{-1} B^T \underline{c}_{\lambda i}, \; i = 1,2,\ldots,d \quad ; \tag{5.12}$$

and a substitution from (5.8) converts (5.11) to

$$8 \sum_{i=1}^{d} D \sum_{k=1}^{n} [Q_2(p_i,k) e^{-s_k t} + Q_3(p_i,k) e^{s_k t}] D\underline{c}_{zi}$$

$$+ 8 \sum_{i=1}^{d} DQ_1(p_i) D\underline{c}_{zi} e^{p_i t} + 2\mu \sum_{i=1}^{d} D\underline{c}_{zi} e^{p_i t} \tag{5.13}$$

$$= \sum_{i=1}^{d} (p_i I + A^T) \underline{c}_{\lambda i} e^{p_i t} \quad ,$$

where

$$D = H^T PH \; . \tag{5.14}$$

If the substitution (5.9) is valid, then the functions $e^{\pm s_k t}$ and $e^{p_i t}$ are linearly independent; and the conditions for solution are that

$$\sum_{i=1}^{d} DQ_2(p_i,k) D\underline{c}_{zi} = \underline{0}, \; k = 1,2,\ldots,n \; ; \tag{5.15}$$

$$\sum_{i=1}^{d} DQ_3(p_i,k) D\underline{c}_{zi} = \underline{0}, \; k = 1,2,\ldots,n \; ; \tag{5.16}$$

and

$$8DQ_1(p_i)D\underline{c}_{zi} + 2\mu D\underline{c}_{zi} = (p_iI+A^T)\underline{c}_{\lambda i}, \quad i = 1,2,...,d. \tag{5.17}$$

Then (5.12) combines with (5.17) to give an equation of the form

$$T(p_i)\underline{c}_{\lambda i} = \underline{0} , \tag{5.18}$$

where

$$T(p_i) = [D + \frac{4}{\mu} DQ_1(p_i)D](p_iI-A)^{-1}BR^{-1}B^T - (p_iI-A^T) . \tag{5.19}$$

The condition for non-trivial solution is

$$\det T(p_i) = 0 , \tag{5.20}$$

which gives an algebraic equation in the eigenvalues $\{p_i\}$, $i = 1,2,...,d$.

Examination of (5.18), (5.20), and (5.12) indicates that $T(p_i)$ has at most a rank of n-1, $\underline{c}_{\lambda i}$ has at least one component which can be freely chosen for each i, and thus that there are at least d constants to be determined. From the boundary conditions (3.11), and the transversality conditions arising from (3.19), 2n equations in the unknown constants are obtained. Since D, defined in (5.14), has at most rank r, then (5.15) and (5.16) contribute at most 2nr independent conditions. It follows that

$$d_{max} = 2n(r+1) , \tag{5.21}$$

where r is the dimension of $\underline{y}$, the system output. Since only 2n conditions of the form (5.15) and (5.16) occurred in Section 4, d_{max} was simply 4n for that case. It is clear that (5.21) is an equation in d_{max}, because r appears therein as a maximum estimate.

5.3 Second-Order Example

Suppose that the system is described by the following matrices:

$$H = [1 \quad 1] \; ; \; A = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \; ; \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; ;$$

$$P = 1 \; ; \quad C = R = S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; . \tag{5.22}$$

This system satisfies (3.28); and

$$\varphi(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \; ;$$

$$CSC^T = I \; ; \quad D = H^T PH = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \; . \tag{5.23}$$

Under these conditions,

$$W(t,\tau) = \begin{bmatrix} \frac{1}{6}[e^{-3|t-\tau|} - e^{-3(t+\tau)}] & 0 \\ 0 & \frac{1}{4}[e^{-2|t-\tau|} - e^{-2(t+\tau)}] \end{bmatrix} \; ; \tag{5.24}$$

$$Q_1(p_i) = \begin{bmatrix} \dfrac{1}{9-p_i^2} & 0 \\ 0 & \dfrac{1}{4-p_i^2} \end{bmatrix} \; ; \tag{5.25}$$

$$Q_2(p_i,1) = \begin{bmatrix} \dfrac{e^{(p_i-3)T}}{6(p_i-3)} & 0 \\ 0 & 0 \end{bmatrix} \; ; \tag{5.26}$$

$$Q_2(p_i,2) = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{e^{(p_i-2)T}}{4(p_i-2)} \end{bmatrix} ; \qquad (5.27)$$

$$Q_3(p_i,1) = \begin{bmatrix} \frac{1}{6}\left[\dfrac{1-e^{(p_i-3)T}}{p_i-3} - \dfrac{1}{p_i+3}\right] & 0 \\ 0 & 0 \end{bmatrix} ; \qquad (5.28)$$

$$Q_3(p_i,2) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{4}\left[\dfrac{1-e^{(p_i-2)T}}{p_i-2} - \dfrac{1}{p_i+2}\right] \end{bmatrix} ; \qquad (5.29)$$

$$[D + \frac{4}{\mu} DQ_1(p_i)D] = [1 + \frac{4}{\mu}\left(\frac{1}{9-p_i^{\;2}} + \frac{1}{4-p_i^{\;2}}\right)] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} ; \qquad (5.30)$$

$$(p_iI-A)^{-1}BR^{-1}B^T = \begin{bmatrix} \dfrac{1}{p_i+3} & 0 \\ 0 & \dfrac{1}{p_i+2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} . \qquad (5.31)$$

Finally,

$$T(p_i) = \begin{bmatrix} \alpha + 3 - p_i & \alpha \\ \alpha & \alpha + 2 - p_i \end{bmatrix} , \qquad (5.32)$$

where

$$\alpha = \left(\frac{1}{p_i+3} + \frac{1}{p_i+2}\right)\left(1 + \frac{4}{\mu}\left[\frac{1}{9-p_i^{\,2}} + \frac{1}{4-p_i^{\,2}}\right]\right) . \qquad (5.33)$$

By (5.20),

$$p_i^{\,2} - p_i(2\alpha + 5) + (5\alpha + 6) = 0 , \qquad (5.34)$$

which is of eighth degree in p_i, and agrees with (5.21). Moreover, (5.32) does not exist for $p_i = \pm 2, \pm 3$, which validates (5.12). Note also that $\alpha = 0$ is not a possible solution, since (5.18) then demands $p_i = 3,2$, which gives $\alpha = \infty$ in (5.33), a contradiction. Therefore $T(p_i)$ has rank one; and $\underline{c}_{\lambda i}$ has one free component for each i, or a total of 8 free constants.

From (3.11), $z_1(0)$ and $z_1(T)$ relate to $z_2(0)$ and $z_2(T)$, giving two conditions. Two more conditions arise from the requirement that the conjunct of $\delta J'$ vanish (3.19), so that $\lambda_1(0)$ relates to $\lambda_2(0)$, $\lambda_1(T)$ to $\lambda_2(T)$. It remains to examine the conditions (5.15) and (5.16). Inspection of (5.15), for example, shows only one independent equation for each k, when (5.26) or (5.27) is used; a similar statement holds for (5.16). These then are four conditions. So there are $2n(r+1)$ conditions and $2n(r+1)$ undetermined constants.

The method of analysis is therefore possible for general system matrices. The maximum number of eigenvalues is $2n(r+1)$. This estimate should be compared with the maximum number $4n$ which was found for a single-input, state-output control. Because $Q(t,\tau)$ has not been expressed in algebraic form, the necessary and sufficient conditions for a unique solution cannot be given in general, but only in terms of the coefficient matrix N in (4.25). Of course N must be determined for each example.

6. CONCLUSIONS

6.1 Summary

The problem of minimizing the variance of a quadratic performance index, in the presence of control noise the properties of which are known a priori, has been studied for linear, constant systems with open-loop control. The method extends in principle to time-varying cases. The necessary (Euler) equations are integro-differential and have a kernel matrix $H^TQ(t,\tau)H$ derived from output correlation functions. Unless the noise is gaussian, white, and has mutually uncorrelated components, these equations contain a forcing vector $H^T \int_0^T \underline{q}(t,\tau)d\tau$ which depends upon the third-moment properties of the disturbance process. For solution, it is necessary that the noise-free system be totally output-controllable with respect to the control matrix B; if H^{-1} exists, total state controllability with respect to a control matrix CM, where the noise covariance matrix is written as MM^T, guarantees existence of $Q^{-1}(t,\tau)$. Necessary and sufficient conditions for a unique solution may be found for specific examples; a general condition is not possible since an algebraic expression for $Q(t,\tau)$ has not been found.

The expected value of the performance index is constrained to be a positive number K_o, which cannot be less than the optimal mean index when variance is free. This latter problem, and also the impulsive results when K_o is free, are limiting cases of this study. The solution has at most $2n(r+1)$ eigenvalues, where n and r are the dimensions of state and output vectors respectively. If state and output are identical, and if there is a single control variable, this estimate can be refined to 4n eigenvalues.

6.2 Problems for Further Study

An open-loop, optimal feedback implementation of these results is possible. This approach modifies the control vector in accordance with observed future states of the system, and so leads to an improvement in performance. The more general problem of minimal-variance, closed-loop control, when inputs are bounded and disturbance statistics are initially unknown, remains unsolved. For the problem of this paper, the question of necessary and sufficient conditions for a unique solution is as yet unresolved. Related to this question is a method for writing $Q(t,\tau)$ in algebraic form, and a procedure for finding the equivalent differential equations for the integro-differential (Euler) set. Further research on the effects of non-gaussian noises and the solution in terms of the forcing function $\underline{q}(t,\tau)$ will also lead to additional insights.

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