Localized Turbulent Flows on Scouring Granular Beds

G. Gioia
Department of Theoretical and Applied Mechanics, University of Illinois, Urbana, Illinois 61801, USA

Fabián A. Bombardelli
Department of Civil and Environmental Engineering, University of California, Davis, California 95616, USA
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In many applications a sustained, localized turbulent flow scours a cohesionless granular bed to form a pothole. Here we use similarity methods to derive a theoretical formula for the equilibrium depth of the pothole. Whereas the empirical formulas customarily used in applications contain numerous free exponents, the theoretical formula contains a single one, which we show can be determined via the phenomenological theory of turbulence. Our derivation affords insight into how a state of dynamic equilibrium is attained between a granular bed and a localized turbulent flow.

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When a water jet plunges into the free surface of a body of water of uniform depth, a turbulent cauldron is established in the body of water under the point of entrance of the jet. If the body of water lies on a granular bed, the turbulent cauldron starts to scour the bed to form a pothole (Fig. 1). Under a sustained action of the jet, the pothole deepens until a state of dynamic equilibrium is attained between the granular bed and the turbulent cauldron. This scenario is relevant to many applications in hydrology, geomorphology, and hydraulic engineering. For example, in the bed of a stream below an overflowing gate, a pothole forms and may compromise the stability of the gate [1]. This application poses the direct problem: given the power of the jet, determine the depth of the pothole under equilibrium conditions. Other applications may pose the inverse problem. For example, the depth of potholes and other relics of scouring found on the beds of Martian outflow channels could be used to gauge the colossal floods that carved these channels in pre-Amazonian times [2]. Seeking to describe the equilibrium conditions of turbulent cauldrons on scouring beds, researchers have proposed a number of widely used empirical formulas [3]. These formulas have been predicated on dimensional analysis and heuristic arguments, and contain numerous free exponents that have been determined by fitting experimental results. It is hardly surprising, therefore, if, when applied to the same problem, different formulas give badly disparate predictions (see, e.g., [1], p. 658ff.). Yet, for lack of better means, we continue to use empirical formulas to deal with many common applications of turbulent flows. In this Letter, we use dimensional analysis and similarity methods [4] to derive a theoretical formula containing a single free exponent—a similarity exponent. This formula subsumes the empirical formulas proposed so far and is valid asymptotically under conditions that are amply met in applications. Then we show that the same theoretical formula as well as the value of the similarity exponent can be derived using the phenomenological theory of turbulence [5]. To that end, we build on recent work [6] indicating that the phenomenological theory may be applied to turbulence that is both anisotropic and inhomogeneous (as is the case in the turbulent cauldron). Our method of analysis may also be useful in developing a theoretical understanding of mine burial, bridge pier-induced erosion, and other applications in which a turbulent flow interacts with a granular bed.

An exhaustive compilation of empirical formulas for the equilibrium depth of the pothole was undertaken by Mason and Arumugam [3]. From that compilation, it is apparent that all empirical formulas proposed so far are but special cases of the following generalized empirical formula:

$$R = K q^{e_q} h^{e_h} g^{e_g} d^{e_d} \left(\frac{\rho}{\rho_s - \rho} \right)^{e_\rho}, \tag{1}$$

where R is the sum of the depth of the pothole and the depth of the body of water, $R = \Delta + D$, and coincides with the size of the turbulent cauldron (Fig. 1); K is a free multiplicative constant whose value must be determined empirically; q is the volume flux of the jet per unit thickness (measured in the direction perpendicular to the plane in Fig. 1); h is the head of the jet; g is the gravitational acceleration; d is the diameter of the grains of the bed; ρ is

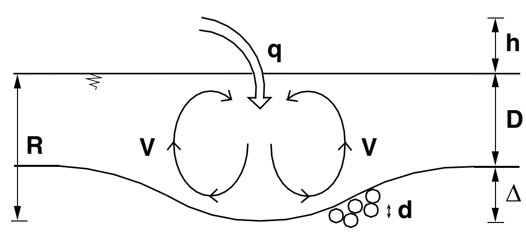

FIG. 1. Geometry and notation. The jet of volume flux q plunges from a height h (called the *head*). The turbulent cauldron is spanned by its largest eddies (of characteristic velocity V). The granular bed is composed of cohesionless grains of diameter d. Note that the geometry is two dimensional: the jet, the turbulent cauldron, and the pothole extend to infinity in the direction perpendicular to the plane of the figure (or thickness).

the density of water; ρ_s is the density of the grains of the bed; and e_q, e_h, e_g, e_d, and e_ρ are free exponents whose values must be determined empirically. The rows of Table I list the values of the exponents determined empirically (or set to zero from the start) by different researchers. (In the case of Bormann and Julien, the values of the exponents were determined semiempirically [7].) Each researcher (or group of researchers) determined the values of several free parameters (including a number of exponents and the multiplicative constant K) by fitting experimental results. As might have been surmised from the number of free parameters and the diversity of experimental results, and Table I confirms, in some cases different researchers obtained widely dissimilar values of a given exponent.

We now ascertain to what extent a theoretical formula may be predicated on dimensional analysis and similarity methods. We start by choosing a suitable set of variables. Because the turbulence is fully developed in applications, the viscosity need not be included in our set of variables. After evaluating a number of alternatives, we decide for the following set of six variables: R, ρ, g, ρ_s, d, and P. P is the power of the jet per unit thickness, $P = q\rho g h$, and therefore the power that sustains the turbulent cauldron. The choice of P places the focus of our analysis on the *energetics* of the turbulent cauldron, and constitutes the key to our results. The dimensional equations $[P] = [\rho][g]^{3/2} \times [R]^{5/2}$, $[\rho_s] = [\rho]$, and $[d] = [R]$ show that the dimensions of three of the variables (P, ρ_s, and d) can be expressed as products of powers of the dimensions of the other variables; it follows from Buckingham's Π theorem [4] that we can reduce the functional relation among P, R, ρ, g, ρ_s, and d to an equivalent functional relation among three dimensionless variables. With the sensible choice of dimensionless variables $\Pi_1 \equiv P/\rho g^{3/2} R^{5/2}$, $\Pi_2 \equiv \rho_s/\rho$ (the relative density of the bed), and $\Pi_3 \equiv d/R$ (the relative roughness of the bed), we may write $\Pi_1 = \mathcal{F}[\Pi_2, \Pi_3]$ or, equivalently,

$$P = \rho g^{3/2} R^{5/2} \mathcal{F}\left[\frac{d}{R}, \frac{\rho_s}{\rho}\right], \tag{2}$$

where $\mathcal{F}$ is a dimensionless function of the relative density and of the relative roughness of the bed. To make further progress, we note that in applications $d/R \ll 1$ and seek to formulate an asymptotic similarity law for $d/R \to 0$. There are two possible similarities: complete and incomplete [4]. In the case of complete similarity in d/R, $\mathcal{F}[d/R, \rho_s/\rho]$ becomes independent of d/R as $d/R \to 0$. If this were the case, R would be independent of d for $d/R \ll 1$, which is incompatible with the empirical values of the exponent e_d in Table I. In the case of incomplete similarity in d/R, Eq. (2) admits the following power-law asymptotic expression [4], $\mathcal{F}[d/R, \rho_s/\rho] = (d/R)^\alpha \mathcal{G}[\rho_s/\rho] + o[(d/R)^\alpha]$, where α is a similarity exponent, which cannot be determined by dimensional analysis, and $\mathcal{G}$ is a dimensionless function of the relative density of the bed, ρ_s/ρ. By substituting the leading term of this asymptotic expression in (2) and rearranging, we obtain the following formula for R:

$$R = K q^{e_q} h^{e_h} g^{e_g} d^{e_d} \mathcal{H}\left[\frac{\rho_s}{\rho}\right], \tag{3}$$

where $e_q = e_h = 2/(5 - 2\alpha)$, $e_g = -1/(5 - 2\alpha)$, $e_d = -2\alpha/(5 - 2\alpha)$, and we have defined $\mathcal{H}[\Pi_2] \equiv 1/K(\mathcal{G}[\Pi_2])^{2/(5-2\alpha)}$, where K is a dimensionless constant. The theoretical formula of Eq. (3) has the same form as the generalized empirical formula of Eq. (1) (provided that $\mathcal{H}[\Pi_2] = 1/(\Pi_2 - 1)^{e_\rho}$). Nevertheless, the exponents that appear in both formulas (with the exception of e_ρ) are now revealed to be functions of a single free parameter, the similarity exponent. Because these functions were unknown, researchers developing empirical formulas treated the exponents of (1) as free parameters whose values had to be determined empirically (Table I). A much improved way of determining these exponents suggests itself now, via the empirical determination of the similarity exponent. Yet we do not pursue this way of determining the exponents. Instead, we show presently that Eq. (3) as well as the function $\mathcal{H}[\rho_s/\rho]$ and the value of the similarity exponent can be derived in a completely independent way using the phenomenological theory of turbulence.

TABLE I. Sets of values of the exponents of Eq. (1) empirically determined (or set to zero) by different researchers. Adapted from [3,7]. Also shown are the theoretical values of the exponents determined here.

Researcher(s) (year)	e_q	e_h	e_g	e_d	e_ρ
Schoklitsch (1932)	0.57	0.2	0	-0.32	0
Veronese (1937)	0.54	0.225	0	-0.42	0
Eggenberger and Müller (1944)	0.6	0.5	-0.3	-0.4	4/9
Hartung (1959)	0.64	0.36	0	-0.32	0
Franke (1960)	0.67	0.5	0	-0.5	0
Kotoulas (1967)	0.7	0.35	-0.35	-0.4	0
Chee and Padiyar (1969)	0.67	0.18	0	-0.063	0
Chee and Kung (1974)	0.6	0.2	0	-0.1	0
Machado (1980)	0.5	0.3145	0	-0.0645	0
Bormann and Julien (1991)	0.6	0.5	-0.3	-0.4	0.8
Theory	2/3	2/3	$-1/3$	$-2/3$	1

The phenomenological theory [5] is based on two tenets pertaining to the steady production of turbulent (kinetic) energy: (1) The production occurs at the length scale of the largest turbulent eddies in the flow, and (2) the rate of production is independent of the viscosity. From these tenets, it is possible to obtain a scaling expression for the rate of production of turbulent energy per unit mass of cauldron (which we denote by ε) in terms of the velocity of the largest eddies (which we have denoted by V) and of the size of the largest eddies (which scales with R) [5]. The result is Taylor's scaling, $\varepsilon \sim V^3/R$ [8], where the symbol "$\sim$" stands for "scales with." Further, it is possible to show that the velocity of the turbulent eddies of size l, u_l, scales in the form $u_l \sim (\varepsilon l)^{1/3}$, valid for $l \gg \eta$, where $\eta = \nu^{3/4}\varepsilon^{-1/4}$ is the Kolmogorov (viscous) length scale and ν the kinematic viscosity [5,6]. By combining $\varepsilon \sim V^3/R$ and $u_l \sim (\varepsilon l)^{1/3}$ we obtain the Kolmogorov scaling, $u_l \sim V(l/R)^{1/3}$ (valid for $l/\eta \gg 1$). We recall these results later on.

Now we consider the energetics of the turbulent cauldron and seek to obtain a scaling expression for V, the velocity of the largest eddies. The production of turbulent energy is driven by the jet, whose power per unit thickness is $P = q\rho gh$. Therefore, P must equal the rate of production of turbulent energy per unit thickness of cauldron (note that P is independent of the viscosity, in accord with the second tenet of the phenomenological theory stated above), and we can write $P = \varepsilon M$, where ε is the rate of production of turbulent energy per unit mass of cauldron, and $M \sim \rho R^2$ is the mass per unit thickness of cauldron. It follows that $\varepsilon \sim qgh/R^2$ and, from a comparison with $\varepsilon \sim V^3/R$, that

$$V \sim \left(qg\frac{h}{R}\right)^{1/3}, \tag{4}$$

which is the sought expression for the velocity of the largest eddies in the cauldron.

Next we consider the surface of the pothole and seek to obtain a scaling expression for the shear stress exerted by the flow on that surface. Let us call S a wetted surface tangent to the peaks of the grains at the surface of the pothole (Fig. 2). Under conditions of fully developed turbulence, the shear stress acting on S is the Reynolds stress, $\tau = \rho|\overline{v_n v_t}|$, where v_n and v_t are the fluctuating velocities normal and tangent to S, respectively, and an overbar denotes the time average. We study v_n first, and start by making a crucial observation: if the relative roughness is small ($d/R \ll 1$), eddies of sizes larger than, say, $2d$ can make only a negligible contribution to v_n (this is entirely a matter of geometry; see Fig. 2). On the other hand, eddies smaller than d fit in the space between successive grains on the bed, so that these eddies can make a sizable contribution to v_n. Nevertheless, where these eddies are smaller than, say, $d/2$, their velocities are negligible compared with the velocity of the eddies of size d. [Recall that, according to the Kolmogorov scaling, $u_l \sim V(l/R)^{1/3}$ (valid for $l/\eta \gg 1$); therefore, the smaller the size of an eddy, l, the smaller its velocity, u_l.] Thus, assuming that $d/\eta \gg 1$, v_n is dominated by u_d, the velocity of the eddies of size d. In other words, $v_n \sim u_d$. We now turn to v_t. Eddies of all sizes can provide a velocity tangent to S. Thus, v_t is dominated by V, the velocity of the largest eddies, and $v_t \sim V$. We conclude that $|\overline{v_n v_t}| \sim u_d V$, and therefore $\tau \sim \rho u_d V$ [9]. We may now substitute (4) and $u_d \sim V(d/R)^{1/3}$ in $\tau \sim \rho u_d V$ to obtain

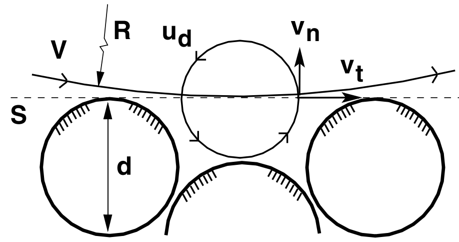

FIG. 2. Three grains of diameter d lying at the surface of the pothole. The dashed line is the trace of a wetted surface S tangent to the peaks of the grains at the surface of the pothole.

$$\tau \sim \rho\frac{(qhg)^{2/3}d^{1/3}}{R}, \tag{5}$$

which is the sought expression for the shear stress exerted by the turbulent cauldron on the surface of the pothole, valid for $\eta \ll d \ll R$. To discuss Eq. (5), it is convenient to rewrite it in terms of the power of the jet per unit thickness, $P = q\rho gh$, with the result $\tau \sim P^{2/3}(\rho d)^{1/3}/R$. Consider now the instant when a jet of power P plunges into the surface of a body of water of uniform depth D. Then, the pothole starts to form, and, as the depth Δ of the pothole increases, the size $R = \Delta + D$ of the cauldron increases accordingly, leading to a decrease in the shear stress on the surface of the pothole. Eventually, the shear stress decreases to a critical value τ_c, and the scouring ceases. Thus the condition of equilibrium between the turbulent cauldron and the granular bed is $\tau = \tau_c$ [10].

To obtain a scaling expression for the critical stress τ_c, we follow Shields [11] in recognizing that the grains at the surface of a granular bed are subjected to a Reynolds stress $\tau \sim \rho u_d V$ (exerted by the turbulent flow), a gravitational stress $\tau_g \sim (\rho_s - \rho)gd$, and a viscous stress $\tau_\nu \sim \rho\nu V/d$. Then, if the equilibrium condition is satisfied, we can perform a straightforward dimensional analysis using three variables: $\tau = \tau_c$, τ_g, and τ_ν. The result is $\tau_c \sim \tau_g \mathcal{I}[\mathrm{Re}_d]$, where $\mathcal{I}$ is a dimensionless function of a Reynolds number $\mathrm{Re}_d \equiv \tau/\tau_\nu = u_d d/\nu$. By recalling that $\varepsilon \sim u_d^3/d$, $\eta = \nu^{3/4}\varepsilon^{-1/4}$, and $d/\eta \gg 1$, we conclude that $\mathrm{Re}_d \sim (d/\eta)^{4/3} \gg 1$, and seek to formulate a similarity law for $\mathrm{Re}_d \to \infty$. If we assume complete similarity in Re_d, then $\mathcal{I}[\mathrm{Re}_d]$ tends to a constant as $\mathrm{Re}_d \to \infty$ (as indicated by experimental results on the incipient motion of granular beds [11]), and therefore $\tau_c \sim (\rho_s - \rho)gd$, which is the sought expression for the critical stress.

We are now ready to impose the equilibrium condition. By substituting (5) and $\tau_c \sim (\rho_s - \rho)gd$ into $\tau = \tau_c$ and rearranging, we obtain the following scaling expression for R:

$$R \sim q^{2/3} h^{2/3} g^{-1/3} d^{-2/3} \left(\frac{\rho}{\rho_s - \rho} \right). \tag{6}$$

This expression gives (up to a multiplicative constant) a complete theoretical formula to compute the equilibrium depth of the pothole as $\Delta = R - D$, where D is the depth of the body of water (Fig. 1). A comparison of (6) with (3) indicates that $e_q = e_h = 2/3$, $e_g = -1/3$, and $e_d = -2/3$, in accord with a similarity exponent of value $\alpha = 1$. Thus, the theory gives values of e_q, e_h, e_g, and e_d that relate to one another in the form necessitated by the independent analysis that yielded (3). Further, a comparison of (6) with (3) indicates that $\mathcal{H}[\Pi_2] = 1/(\Pi_2 - 1)$, which corresponds to the generalized empirical formula of Eq. (1) with $e_\rho = 1$. The theoretical values of the exponents appear in Table I, where they may be compared with the corresponding empirical values determined by various researchers. This comparison affords a considerable degree of experimental support to our theoretical results, even though for each exponent the alternative empirical values vary over a sizable range, as might have been expected where many difficult experiments were involved.

In this Letter, we have focused on the energetics of the turbulent cauldron to derive (up to a multiplicative constant) a theoretical formula for the depth of a pothole in equilibrium with a turbulent cauldron driven by a jet of constant power. The formula represents the power-law asymptotic behavior of a fully developed turbulent flow of incomplete similarity in the relative roughness of the cohesionless granular bed. In deriving the formula based on the phenomenological theory, we have gained insights into the form of interaction between the cauldron and the granular bed. These insights are of obvious theoretical import, but they also suggest improved ways of dealing with applications. Thus, for example, on the bed of an overflowing dam a pothole is frequently excavated in advance in order to confine the dissipation of excess hydraulic power [12]. Our discussion of the equilibrium condition between the bed and the cauldron, $\tau = \tau_c$, suggests that to incorporate a *physically meaningful* safety factor in the design of such a pothole we might impose the *design* condition $\tau = \tau_c/f$, where $f > 1$ is the safety factor. The design value of the depth of the pothole would then be $\Delta_{\text{design}} = f\Delta + (f - 1)D$, where Δ is the equilibrium value of the depth. In conclusion, our results indicate that despite current practice, theory may be advantageously used instead of empirical formulas in the analysis and design of overflowing gates, weirs, dams, and natural obstructions.

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[1] See, for example, W. H. Graf, *Fluvial Hydraulics* (John Wiley & Sons, Chichester, U.K., 1998).

[2] We have discussed this form of paleohydrological inference with J. W. Head III, Brown University (private communication). Potholes hundreds of meters deep have been documented on the beds of Martian outflow channels; see, for example, M. S. Bobinson and K. L. Tanaka, Geology **18**, 902 (1990). On Earth, the potholes that form in the backswamps of a river where a levee is overtopped by a high flood become long-lasting geomorphological relics known as *crevasse lakes* (or *blue holes* to the inhabitants of the lower Mississippi valley).

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[8] D. Lohse, Phys. Rev. Lett. **73**, 3223 (1994). The existence of an upper bound on ε that is independent of the viscosity has been proved *mathematically*; see C. R. Doering and P. Constantin, Phys. Rev. Lett. **69**, 1648 (1992).

[9] G. Gioia and F. A. Bombardelli, Phys. Rev. Lett. **88**, 014501 (2002).

[10] There is experimental evidence that grains from the bed may become entrained in the turbulent cauldron while the pothole deepens. Nevertheless, the grains return to the bed as soon as the scouring ceases; see, for example, the photographs of Fig. 3 in V. D'Agostino and V. Ferro, J. Hydraul. Eng. **130**, 24 (2004). Thus, the condition of equilibrium between the turbulent cauldron and the granular bed is not affected by the entrainment of grains. Now after the grains return to the bed, there may be a change in the size of the coves between grains on the bed (i.e., the roughness of the bed may change). Unless the grains are very small, and therefore highly cohesive, the size of these coves must still scale with the diameter of the grains. If the grains are cohesive, the cohesive energy introduces a new length scale; when the grains return to the bed, this length scale may become manifest in the form of a very porous aggregate—a rougher bed. Therefore, we have assumed the grains to be cohesionless. This is not a very restrictive assumption. In air, the cohesion becomes important if the grains are smaller than about 0.1 mm (that is to say, for silts; sands are cohesionless). Nevertheless, in water the cohesion is largely neutralized, even for silts, and becomes noticeable only if the grains are clayish.

[11] See, for example, A. J. Raudkivi, *Loose Boundary Hydraulics* (Balkema, Rotterdam, 1998).

[12] Such a pothole is known to hydraulic engineers as a *plunge pool.* A well-designed plunge pool prevents the downstream advection of turbulent energy and the attendant environmental damage. Incidentally, as many an angler knows, in small dams the plunge pool also affords fish an attractive hiding place.