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PLANE-WAVE DIFFRACTION BY A WEDGE-A SPECTRAL DOMAIN APPROACH

A. CIARKOWSKI
R. MITTRA

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Plane-Wave Diffraction by a Wedge--A Spectral Domain Approach		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER R-967; UILU-ENG 82-2233
7. AUTHOR(s) A. Ciarkowski and R. Mittra		8. CONTRACT OR GRANT NUMBER(s) N000-1479C-0424
9. PERFORMING ORGANIZATION NAME AND ADDRESS Coordinated Science Laboratory University of Illinois Urbana, IL 61801		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Joint Services Electronics Program		12. REPORT DATE November 1981
		13. NUMBER OF PAGES 23
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Electromagnetic Scattering Wedge diffraction Spectral domain approach Uniform asymptotic expansion		
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A. Ciarkowski and R. Mittra

ABSTRACT

In this paper we investigate the canonical problem of plane wave diffraction by a wedge in the context of the spectral domain approach which exploits the relationship between the induced current on a scatterer and its far field. We show how the Sommerfeld solution to the wedge diffraction problem can be manipulated in a form which enables one to interpret the far scattered field as the Fourier transform of the physical optics current on the two faces of the wedge, augmented by the fringe current near the tip of the wedge. We also show that the uniform asymptotic expansion derived by Lee and Deschamps on the basis of the Lewis, Ahluwalia and Boersma ansatz can be rigorously obtained using the approach presented in this paper.

Keywords: electromagnetic scattering, wedge diffraction, spectral domain approach, uniform asymptotic expansion.

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I. INTRODUCTION

The Spectral Theory of Diffraction (STD), which was introduced recently by Mittra, Rahmat-Samii and Ko [1], has been found to be a useful tool for solving high-frequency diffraction problems. The concepts of STD together with its scope, advantages and disadvantages can be found in a number of publications [1], [2] which have appeared recently in the literature. This theory is founded on the concept that the scattered field is related to the Fourier transform of the induced current on the scatterer and that the ray description of the scattered field is derivable from the spectral representation via an asymptotic evaluation of the integral form of this representation.

The canonical geometries of the semi-infinite half-plane and the wedge play important roles in the development of the various theories for high-frequency scattering. The Sommerfeld half-plane solution has been extensively studied [2] in the context of STD, and the STD formalism for the half-plane diffraction has been applied to planar and curved strips. In this paper, we extend the STD concepts to the geometry of the wedge. We show how the contour integral à la Sommerfeld for the wedge problem can be manipulated to yield a transform representation for the scattered field in terms of the currents flowing on the two faces of the wedge. We also show that it is possible to derive a uniform asymptotic solution for the total field, one which is valid for all observation angles. The uniform solution so derived is found to be identical to that conjectured by Lee and Deschamps [6].

II. SOMMERFELD SOLUTION TO THE WEDGE PROBLEM

The starting point of our analysis is the Sommerfeld solution to the problem of plane-wave diffraction by a wedge. Let a perfectly conducting wedge, with the faces defined by the angles $\phi = 0$ and $\phi = \Phi$, be illuminated by a plane wave*

$$u^i = e^{-ik\rho\cos(\phi-\phi')} \quad (1)$$

incident from the ϕ' direction. (See Fig. 1.)

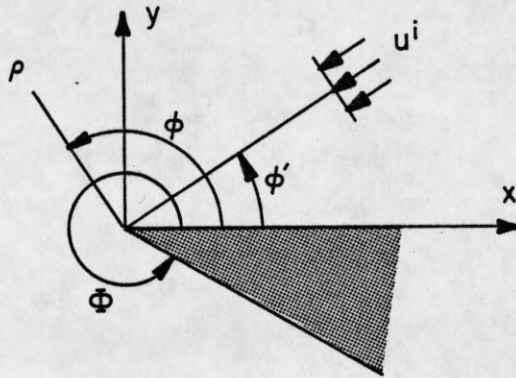


Figure 1. Geometry of the wedge-diffraction problem.

The angles involved have the following ranges:

$$\frac{3}{2}\pi < \Phi < 2\pi; \quad 0 \leq \phi \leq \Phi; \quad 0 < \phi' < \pi.$$

The total field u is subject to the boundary condition

$$u = 0 \text{ on the wedge surface, for } u = E_z \text{ (E-wave), or} \quad (2a)$$

$$\frac{\partial u}{\partial n} = 0 \text{ on the wedge surface, for } u = H_z \text{ (H-wave).} \quad (2b)$$

* $e^{-i\omega t}$ time convention is implicit throughout.

This field, as derived by Sommerfeld, can be represented in the form [3]

$$u(\rho, \phi) = w(\rho, \phi; \phi') + \bar{w}(\rho, \phi; -\phi'), \quad (3)$$

where

$$w(\rho, \phi; \phi) = \frac{\Omega}{2\pi} \int_A e^{-ik\rho \cos(\alpha-\phi)} \frac{d\alpha}{1-e^{i\Omega(\alpha-\psi)}} \quad (4)$$

is the Sommerfeld integral defined on the contour A in the complex α plane (Fig. 2), and $\Omega = \frac{\pi}{\phi}$. The upper (lower) sign in (3) applies to the case of the E-wave (H-wave).

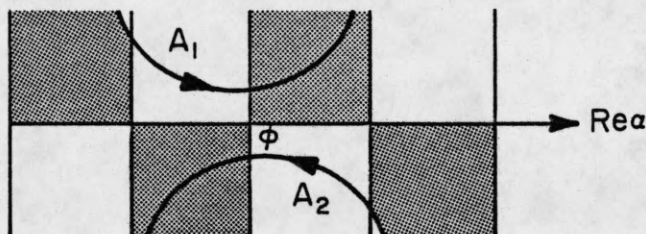


Figure 2. Contour $A = A_1 + A_2$ in the complex α plane. The shadowed areas indicate the convergence regions for the Sommerfeld integral $w(\rho, \phi; \phi)$.

III. ALTERNATE FORM OF THE SOMMERFELD INTEGRAL w

In this section we derive an alternate form of the Sommerfeld integral w , one which would allow us to express the scattered field in terms of the transform of the induced current on the two faces of the wedge.

We begin with the integrand for w given in (4) and note that the poles are located on the real axis in the α plane at the points

$$\alpha_p = \psi + 2p\phi, \quad p = 0, \pm 1, \pm 2, \dots$$

Since $\phi > \pi$, only one of these poles can appear in the region $(-\pi + \phi, \pi + \phi)$. This implies that only the incident and one reflected wave can reach the given observation point (ρ, ϕ) . The next step is to replace the contour A by a sum of the contours K and L ($= L_1 + L_2$), as shown in Fig. 3.

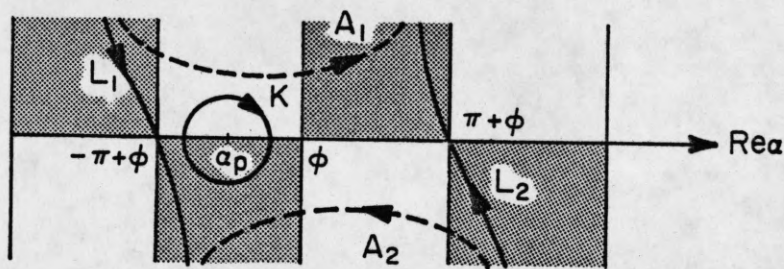


Figure 3. Contours K and L ($= L_1 + L_2$) in the α plane.

Because the contours L_1 and L_2 are separated by 2π , and traverse in opposite directions, we can write

$$\int_{L_2} f(\alpha) d\alpha = - \int_{L_1} f(\alpha + 2\pi) d\alpha.$$

Consequently, the integral along L can be replaced by one along either L_1 or L_2 . Next, let us turn to the integral along the contour K . The contribution to w from the integral along K may be found by evaluating the residue at the pole α_p . Incorporating these manipulations in the expression for w given in (4), we get

$$w(\rho, \phi; \psi) = \sum_p \eta(\pi - [\alpha_p - \phi]) e^{-ik\rho \cos(\alpha_p - \phi)} + I(\rho, \phi; \psi), \quad (5)$$

where

$$I(\rho, \phi; \psi) = \begin{cases} -\frac{i}{2\pi} \int_{L_1} e^{-ik\rho \cos(\alpha - \phi)} \frac{\Omega \sin \Omega \pi \, d\alpha}{\cos \Omega(\alpha - \psi + \pi) - \cos \Omega \pi}, & \text{or} \\ \frac{i}{2\pi} \int_{L_2} e^{-ik\rho \cos(\alpha - \phi)} \frac{\Omega \sin \Omega \pi \, d\alpha}{\cos \Omega(\alpha - \psi - \pi) \cos \Omega \pi}, & \end{cases} \quad (6a)$$

$$(6b)$$

and $\eta(x)$ is the Heaviside step function. The sum in (5) is extended over all p 's for which the inequality $\pi - [\alpha_p - \phi] > 0$ holds at least for some value of ϕ lying in the range $0 < \phi < \Phi$.

Simple asymptotic considerations reveal that the sum over p in the r.h.s. in (5) is of the order of $(k\rho)^0$ as $k\rho \rightarrow \infty$, whereas the integral I is of the order of $(k\rho)^{-1/2}$. Hence the representation (5) can be useful for high-frequency asymptotic analysis. Also, it will be seen from the following that the integral I can be easily converted into Fourier integral form. Both of these features will be taken advantage of in subsequent sections.

IV. RAY-OPTICAL REPRESENTATION OF U

When ϕ' and ϕ lie in the range $0 < \phi' < \pi$, $0 \leq \phi \leq \Phi$, only one pole $\alpha = \alpha_0^i$ of $w(\rho, \phi; \phi')$ and two poles $\alpha = \alpha_0^r$, and $\alpha = \alpha_1^r$ of $w(\rho, \phi; -\phi')$ can satisfy the inequality $-\pi + \phi < \alpha < \pi + \phi$. These are

$$\alpha_0^i = \phi', \text{ due to the incident wave,}$$

$$\alpha_0^r = -\phi', \text{ due to the wave reflected from the wall } \phi = 0, \text{ and}$$

$$\alpha_1^r = 2\Phi - \phi', \text{ due to the wave reflected from the wall } \phi = \Phi.$$

Thus, we find from (5)

$$w(\rho, \phi; \phi') = \eta(\pi + \phi' - \phi) e^{-ik\rho \cos(\phi' - \phi)} + I(\rho, \phi; \phi'), \text{ and} \quad (7)$$

$$w(\rho, \phi; -\phi') = \eta[\pi - (\phi' + \phi)] e^{-ik\rho \cos(\phi' + \phi)} + \eta[2\phi - (\phi' + \phi) - \pi] e^{-ik\rho \cos(2\phi - \phi' - \phi)} \\ + I(\rho, \phi; -\phi'). \quad (8)$$

Substituting (7) and (8) into (3), we obtain the representation for u which is amenable to ray-optical interpretation. Indeed, the first term in (7) and first two terms in (8) are of the order of $(k\rho)^0$ as $k\rho \rightarrow \infty$, and represent the GO approximation to the total field. The remainders in (7) and (8), represented by $I(\phi')$ and $I(-\phi')$, respectively, are of the order of $(k\rho)^{-1/2}$ and represent the field diffracted at the edge of the wedge (for details see [4], [5]).

V. CONVENTIONAL REPRESENTATION OF U

The representation for the total field u , given in (3), (7) and (8), contains terms which are discontinuous functions of the observation angle ϕ though the total field is continuous. It is useful to seek an alternate representation in the context of STD, which consists of continuous functions only. This is accomplished by deforming the contour L such that the poles α_0^i , α_0^r and α_1^r do not cross the new contour as ϕ and ϕ' are varied. We choose this deformed contour to be $L^<$ for $\phi < \pi$ and $L^>$ for $\phi > \pi$. Both of these contours are shown in Fig. 4.

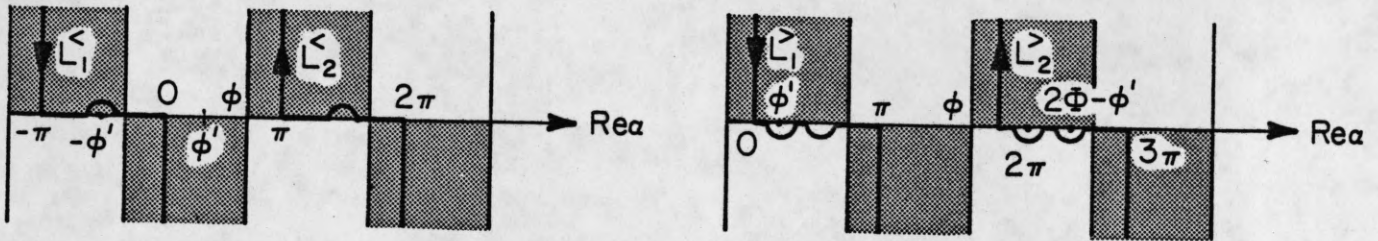


Figure 4. Contours $L^<$ ($\phi < \pi$) and $L^>$ ($\phi > \pi$) in the α plane.

With such a choice for the contours, the pole α_0^i is always located inside the region bounded either by $L_1^<$ and $L_2^<$, or by $L_1^>$ and $L_2^>$, and the poles α_0^r and α_1^r always appear outside of the regions.

The total field u can now be expressed as

$$u = u^i + u^s, \quad (9)$$

where u^i and u^s are, respectively, the incident and scattered fields. The scattered field is defined by the formula

$$u^s = \frac{i\lambda}{2\pi} \int_{P_\lambda} e^{-ik\rho \cos(\alpha - \phi)} W(\alpha) d\alpha, \quad (10)$$

where

$$\lambda = \text{sgn } y, \quad P_\lambda = \begin{cases} L_1^< & \lambda = 1 \\ L_2^> & \lambda = -1 \end{cases}, \quad (10a)$$

and $W(\alpha)$ is defined as

$$W(\alpha) = \Omega \sin \Omega \pi \{ [\cos \Omega(\alpha - \psi + \lambda \pi) - \cos \Omega \pi]^{-1} \mp [\cos \Omega(\alpha + \psi + \lambda \pi) - \cos \Omega \pi]^{-1} \}; \quad (11)$$

the upper (lower) sign corresponds to the case of the E-wave (H-wave).

Our choice of $L_1^<$ and $L_2^>$ for the contour P_λ is arbitrary in that we are free to use either $L_1^<$ or $L_2^<$ for P_- and $L_1^>$ or $L_2^>$ for P_+ , with appropriate choice for the corresponding integrals.

In the literature, the expression given in (9) is referred to as the conventional representation. (See reference [1].) We proceed next to find the Fourier transform representation of the scattered field. This is described in the following section.

VI. FOURIER TRANSFORM REPRESENTATION OF THE SCATTERED FIELD u^s

Let us introduce the following change of variables:

$$\begin{aligned} w &= k \cos \alpha; & v &= \sqrt{k^2 - w^2} = -\lambda k \sin \alpha; & x &= \rho \cos \phi; & y &= \rho \sin \phi; \\ k_x &= -k \cos \phi'; & k_y &= -k \sin \phi'. \end{aligned} \quad (12)$$

For v , the branch of the square root is chosen such that $\text{Im } v > 0$. Since the transformation $w \rightarrow \alpha$ is a multiple-valued function, the two contours $L_1^<$ and $L_2^>$ in the α -plane transform into a single contour C in the w -plane. The resulting contour in the w -plane is shown in Fig. 5.

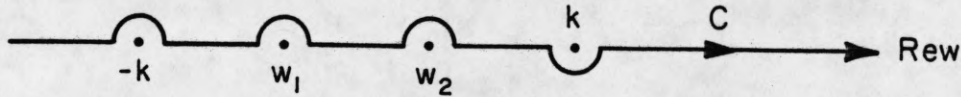


Figure 5. Contour C in the complex w -plane (image of $L_1^<$ and $L_2^>$ in α -plane).

$$w_1 = -k_x, \quad w_2 = -(\cos 2\phi k_x + \sin 2\phi k_y)$$

Thus the scattered field u^s , as defined by (10) and (11), becomes

$$u^s = \frac{1}{2\pi} \int_C dw e^{-iwx} \frac{e^{iv|y|}}{2iv} 2V(w) \quad (13)$$

where

$$V(w) = W(\alpha). \quad (14)$$

If we define the direct and inverse Fourier transforms for the functions $u(x)$ and $U(w)$ as

$$U(w) = \int_{-\infty}^{\infty} e^{iwx} u(x) dx; \quad u(x) = \frac{1}{2\pi} \int_{-\infty+i\Delta}^{\infty+i\Delta} e^{-iwx} U(w) dw, \quad \Delta > 0, \quad (15)$$

then a comparison of (13) and (15) reveals that u^s in (13) has the form of an inverse Fourier transform for the function $\frac{e^{iv|y|}}{2iv} 2V$. We observe that $\frac{e^{iv|y|}}{2iv}$ is the Fourier transform of the free-space Green's function $\frac{i}{4} H_0^{(1)}(kR)$ and that the scattered field has been expressed in the form of a convolution integral involving the Green's function g and the surface current j . These integrals take the form

$$u^s(\underline{r}) = -i\omega\mu \int_S g(\underline{r}-\underline{r}') j(\underline{r}') dr' \quad \text{for E-wave, and} \quad (16a)$$

$$u^s(\underline{r}) = \frac{\partial}{\partial n} \int_S g(\underline{r}-\underline{r}') j(\underline{r}') dr' \quad \text{for H-wave.} \quad (16b)$$

Returning now to (13) we deduce that the function $V(w)$ is proportional to the Fourier transform of the current flowing on the surface of the wedge. This conjecture will now be verified.

VII. ALTERNATE FORMS FOR THE SCATTERED FIELD

In this section we derive alternate forms for the scattered field which can be interpreted in the context of the spectral domain. We consider the E-wave and H-wave cases separately.

7.1. E-Wave

From (16) it follows that the scattered field u^s can be represented as a summation of two terms, viz.,

$$u^s = u_0^s + u_1^s \quad (17)$$

where

$$u_0^s(\underline{r}) = -i\omega\mu \int_0^\infty dx \frac{i}{4} H_0^{(1)}(k|\underline{r}-\underline{\hat{x}}x|) j_0(x), \quad (18a)$$

and

$$u_1^s(\underline{r}) = -i\omega\mu \int_0^\infty dx_1 \frac{i}{4} H_0^{(1)}(k|\underline{r}-\underline{\hat{x}}_1 x_1|) j_1(x_1). \quad (18b)$$

The two terms in the r.h.s. of (17) may be associated with the two currents,

$$j_0(x) = -\frac{1}{i\omega\mu} \frac{\partial u(x, y=0)}{\partial y}, \quad (19a)$$

and

$$j_1(x_1) = \frac{1}{i\omega\mu} \frac{\partial u(x_1, y_1=0)}{\partial y_1}, \quad (19b)$$

which flow on the two surfaces of the wedge defined by $\phi = 0$ and $\phi = \Phi$, respectively (Fig. 6).

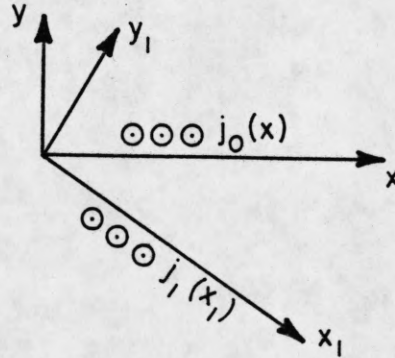


Figure 6. The currents $j_0(x)$ and $j_1(x_1)$ flowing on the wedge surfaces defined by $\phi = 0$ and $\phi = \Phi$, respectively. Both currents flow along the z-axis.

Replacing the Hankel function in (18) by its integral representation [4]

$$\frac{1}{4} H_0^{(1)}(kR) = -\frac{i}{4\pi} \int_C \frac{e^{-i w(x-x') + i v |y-y'|}}{v} dw; \quad v = k^2 - w^2; \quad \text{Im } v > 0;$$

$$R = \sqrt{(x-x')^2 + (y-y')^2}$$

and changing the order of integration leads to the following expressions for u_0^s and u_1^s :

$$u_0^s(\underline{r}) = -\frac{i\omega\mu}{2\pi} \int_C dw e^{-iwx} \frac{e^{iv|y|}}{2iv} J_0(w), \text{ and} \quad (20a)$$

$$u_1^s(\underline{r}) = -\frac{i\omega\mu}{2\pi} \int_C dw_1 e^{-i w_1 x_1} \frac{e^{i v_1 |y_1|}}{2i v_1} J_1(w_1), \text{ where} \quad (20b)$$

$$J_0(w) = \int_0^\infty dx e^{iwx} j_0(x), \text{ and} \quad (21a)$$

$$J_1(w_1) = \int_0^\infty dx_1 e^{i w_1 x_1} j_1(x_1). \quad (21b)$$

We observe from (20) that the component fields u_0^s and u_1^s comprising the scattered field u^s have the form of inverse Fourier transforms of functions related to the Fourier transforms of currents j_0 and j_1 flowing along the faces of the wedge. This is one of the results we had been seeking.

Next, we show that the scattered field u can be written as a Fourier transform of a single quantity which is related to a suitable combination of the two currents. We begin by introducing an additional change of variables, defined by

$$w_1 = k \cos(\alpha - \Phi), \quad -\lambda_1 v_1 = k \sin(\alpha - \Phi), \text{ where } \lambda_1 = \text{sgn } y_1. \quad (22)$$

Such a change of variables is equivalent to the enforcement of the condition

$$-w_1 x_1 + v_1 |y_1| = -wx + v|y|.$$

This may be verified by using the relationship

$$x_1 = x \cos \Phi + y \sin \Phi, \quad y_1 = -x \sin \Phi + y \cos \Phi.$$

When these changes of variables given in (22) are inserted into (20a) we get

$$u_0^s = - \frac{\lambda \omega \mu}{4\pi} \int_{P_\lambda} d\alpha e^{-ik\rho \cos(\alpha-\Phi)} J_0(k \cos \alpha), \quad (23)$$

where P_λ is defined in (10a), and

$$u_1^s = - \frac{\lambda_1 \omega \mu}{4\pi} \int_{L_\mu} d\alpha e^{-ik\rho \cos(\alpha-\Phi)} J_1[k \cos(\alpha-\Phi)], \quad \text{where } L_\mu = \begin{cases} L_3 & \lambda_1=1, \lambda=1, \\ L_4 & \lambda_1=-1, \lambda=1, \\ L_5 & \lambda_1=-1, \lambda=-1, \end{cases}$$

The contours L_3 , L_4 and L_5 are shown in Fig. 7.

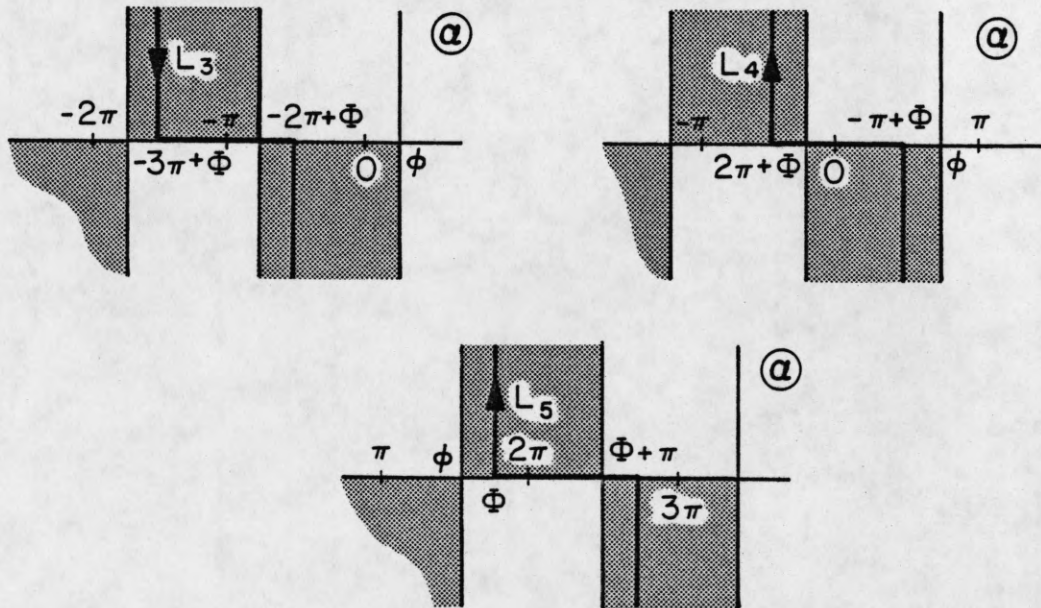


Figure 7. Contours L_3 , L_4 and L_5 in the α plane.

The contour L_3 can be deformed into the contour $L_1^<$ (see Fig. 4), the contour L_4 into the contour which coincides with $L_1^<$ but is orientated along the opposite direction, and the contour L_5 into the contour $L_2^>$. With these changes, u_1^s can be rewritten as

$$u_1^s = - \frac{\lambda \omega \mu}{2\pi} \int_P d\alpha e^{-ik\rho \cos(\alpha-\phi)} J_1[k \cos(\alpha-\phi)]. \quad (24)$$

Using (14) and (23) in (24), we finally obtain

$$u^s = - \frac{i\lambda}{2\pi} \int_{P_\lambda} d\alpha e^{-ik\rho \cos(\alpha-\phi)} \left\{ - \frac{i\omega \mu}{2} [J_0(k \cos \alpha) + J_1[k \cos(\alpha-\phi)]] \right\}. \quad (25)$$

Comparison of this result and (10) gives

$$W(\alpha) = - \frac{i\omega \mu}{2} \{ J(k \cos \alpha) + J_1[k \cos(\alpha-\phi)] \}. \quad (26)$$

This formula is useful for providing a physical interpretation of the function $W(\alpha)$ in that the scattered field u^s is a Fourier transform of $W(\alpha)$ which is related to the transforms of J_0 and J_1 as shown in (26). Note that equating the integrands of (10) and (25) is permissible because the two integral representations for u^s are valid for any contour resulting from the deformation of P_λ in Fig. 5.

Next we proceed to separate the physical optics (PO) and the fringe currents in the transform domain. We begin by introducing the function $W^f(\alpha)$ which is to be interpreted later as the transform of the fringe current:

$$W^f(\alpha) = W(\alpha) - W^{PO}(\alpha), \quad (27)$$

where $W^{PO}(\alpha)$ is obtained from (26) by replacing J_0 and J_1 with J_0^{PO} and J_1^{PO} , respectively, the latter being the transforms of the physical optics

components of the currents j_0 and j_1 . The physical optics currents j_0^{PO} and j_1^{PO} are obtained in the usual way from the knowledge of the incident H-field and are given by

$$j_0^{PO}(x) = \frac{2k\sin\phi'}{\omega\mu} e^{-ikx\cos\phi'} \quad \text{and} \quad j_1^{PO}(x_1) = -\frac{2k\sin(\phi'-\Phi)}{\omega\mu} e^{-ikx_1\cos(\phi'-\Phi)}.$$

Transforming these using (21), we get

$$J_0^{PO}(k\cos\alpha) = -\frac{2}{i\omega\mu} \frac{\sin\phi'}{\cos\alpha - \cos\phi'}, \quad \text{Im}(\cos\alpha - \cos\phi') > 0, \quad \text{and} \quad (28a)$$

$$J_1^{PO}[k\cos(\alpha-\Phi)] = \chi \frac{2}{i\omega\mu} \frac{\sin(\phi'-\Phi)}{\cos(\alpha-\Phi) - \cos(\phi'-\Phi)}, \quad \text{Im}[\cos(\alpha-\Phi) - \cos(\phi'-\Phi)] > 0, \quad (28b)$$

where

$$\chi = \begin{cases} 1 & \text{if the incident wave illuminates the wedge face } \phi = \Phi, \\ 0 & \text{if the wedge face } \phi = \Phi \text{ lies in the shadow region} \\ & \text{of the incident wave.} \end{cases}$$

Note that both $W(\alpha)$ given in (11) and $W^{PO}(\alpha)$ found from (26) and (28) have simple poles at $\alpha = \pm\phi'$ and $\alpha = 2\Phi - \phi'$. However, by considering the limits

$$(\alpha - \phi') \frac{\sin\phi'}{\cos\alpha - \cos\phi'} \rightarrow -1 \quad \text{as } \alpha \rightarrow \phi',$$

$$(\alpha - \phi') \frac{\Omega \sin\Omega\pi}{\cos\Omega(\alpha - \phi' + \pi) - \cos\Omega\pi} \rightarrow -1 \quad \text{as } \alpha \rightarrow \phi',$$

$$(\alpha + \phi') \frac{\sin\phi'}{\cos\alpha - \cos\phi'} \rightarrow 1 \quad \text{as } \alpha \rightarrow -\phi',$$

$$(\alpha + \phi') \frac{-\Omega \sin\Omega\pi}{\cos\Omega(\alpha + \phi' + \pi) - \cos\Omega\pi} \rightarrow 1 \quad \text{as } \alpha \rightarrow -\phi',$$

$$(\alpha - 2\Phi + \phi') \frac{-\sin(\phi' - \Phi)}{\cos(\alpha - \Phi) - \cos(\phi' - \Phi)} \rightarrow -1 \quad \text{as } \alpha \rightarrow 2\Phi - \phi', \quad \text{and}$$

$$(\alpha - 2\Phi + \phi') \frac{-\Omega \sin\Omega\pi}{\cos\Omega(\alpha + \phi' - \pi) - \cos\Omega\pi} \rightarrow -1 \quad \text{as } \alpha \rightarrow 2\Phi - \phi'$$

one can show that the singularities in (27) cancel each other out and, consequently, $W^f(\alpha)$, the transform of the fringe current, is a regular function. If we replace $W(\alpha)$ in (10) with $W^f(\alpha)$ and apply the method of stationary phase to the resulting integral, we obtain a field of the order of $(k\rho)^{-1/2}$ which is regular away from the wedge, and includes the shadow boundaries of the incident and both reflected waves. This field can be attributed to the fringe current, which is the excess over the PO current on the surface of the wedge.

7.2 H-wave.

The scattered field for the H-polarization case can be again expressed by (17), but the function u_0^s and u_1^s now becomes

$$u_0^s(\underline{r}) = \frac{\partial}{\partial y} \int_0^\infty dx \frac{1}{4} H_0^{(1)}(k|\underline{r}-\underline{x}|) j_0(x)$$

and

$$u_1^s(\underline{r}) = -\frac{\partial}{\partial y_1} \int_0^\infty dx_1 \frac{1}{4} H_0^{(1)}(k|\underline{r}-\underline{x}_1|) j_1(x_1)$$

where

$$j_0(x) = u(x, y = 0) \quad \text{and} \quad j_1(x_1) = u(x_1, y_1 = 0).$$

The current $j_0(x)$ flows along the x-axis and the current $j_1(x_1)$ along the negative x_1 - axis. (Fig. 8)

By applying a procedure similar to the one for the E-wave case, we find

$$u^s = -\frac{i\lambda}{2\pi} \int_{P_\lambda} d\alpha e^{-ik\rho\cos(\alpha-\phi)} \left\{ \frac{k}{2i} [\sin\alpha J_0(k\cos\alpha) - \sin(\alpha-\phi) J_1[k\cos(\alpha-\phi)]] \right\}. \quad (29)$$

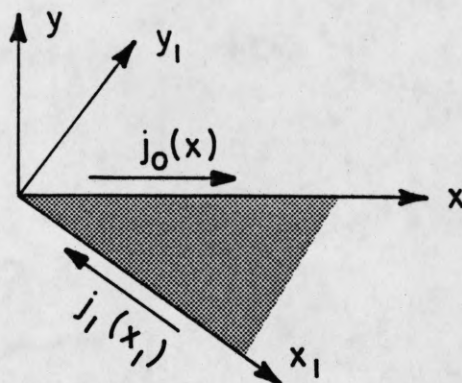


Figure 8. The currents $j_0(x)$ and $j_1(x_1)$ flowing on the surfaces $\phi = 0$ and $\phi = \Phi$.

Then from (10) and (29) we obtain

$$W(\alpha) = \frac{k}{2i} \{ \sin \alpha J_0(k \cos \alpha) - \sin(\alpha - \Phi) J_1[k \cos(\alpha - \Phi)] \}.$$

The function $W(\alpha)$ can again be interpreted as a combination of terms related to the Fourier transform of the currents on the faces of the wedge. All further observations regarding $W^f(\alpha)$ continue to remain valid, except that the expressions for J_0^{PO} and J_1^{PO} are now given by

$$J_0^{PO}(k \cos \alpha) = \frac{2i}{k} \frac{1}{\cos \alpha - \cos \phi'} , \quad \text{Im}(\cos \alpha - \cos \phi') > 0 , \text{ and}$$

$$J_1^{PO}[k \cos(\alpha - \Phi)] = \frac{2i}{k} \frac{1}{\cos(\alpha - \Phi) - \cos(\phi' - \Phi)} , \quad \text{Im}[\cos(\alpha - \Phi) - \cos(\phi' - \Phi)] > 0.$$

The main result of this section is that we represented the scattered field u^s in terms of Fourier transforms of currents flowing along the faces of the wedge. Even though the transforms are discontinuous functions in the spectral domain (compare (26) and (11)), the integral representation for the scattered field (25) remains a uniformly continuous function of the observa-

tion angle ϕ in its range of variation. Of course, the total field is also a continuous function of this angle.

In the next section we construct an approximate, uniformly continuous, asymptotic representation for the total field - one which is expressed in terms of known functions.

VIII. UNIFORM ASYMPTOTIC EXPRESSION FOR THE TOTAL FIELD U

As a first step toward constructing a uniform asymptotic expression (UAE), we return to the representation given in (3) for the total field $u(\rho, \phi)$. Next, we manipulate the expressions for the functions $w(\rho, \phi; \phi')$ and $w(\rho, \phi, -\phi')$ to derive a representation which remains valid for all observation angles. We begin by considering the function $w(\rho, \phi; \phi')$. For the purpose of later manipulation, it is convenient to represent this function in the form

$$w(\rho, \phi; \phi') = \eta(\pi + \phi' - \phi) e^{-ik\rho \cos(\phi' - \phi)} + I_1(\rho, \phi; \phi') + I_2(\rho, \phi; \phi'), \text{ where} \quad (30)$$

$$I_n(\rho, \phi; \phi') = \frac{\Omega}{2\pi} \int_{L_n} \frac{e^{-ik\rho \cos(\alpha - \phi)}}{1 - e^{i\Omega(\alpha - \phi')}} d\alpha.$$

In order to find its UAE we will employ a technique, described in [4], which is suitable for an asymptotic analysis of integrals which contain a simple pole in the neighborhood of the saddle point. According to this technique, the integral

$$I(\lambda) = \int_{SDP} f(\alpha) e^{\lambda g(\alpha)} d\alpha, \quad ,$$

with a simple pole at $\alpha = \alpha_0$ and nearby saddle point at $\alpha = \alpha_s$, has the asymptotic expression:

$$I(\lambda) \sim e^{\lambda g(\alpha_s)} \{ \pm i 2a \sqrt{\pi} e^{-\lambda b^2} Q(\pm i b \sqrt{\lambda}) + \sqrt{\frac{\pi}{\lambda}} T(0) \}, \quad \text{Im} b > 0, \quad \text{as } \lambda \rightarrow \infty, \quad \text{where}$$

$$a = \lim_{\alpha \rightarrow \alpha_0} [(\alpha - \alpha_0) f(\alpha)]; \quad b = \sqrt{g(\alpha_s) - g(\alpha_0)}; \quad T(0) = h f(\alpha_s) + \frac{a}{b};$$

$$h = \sqrt{\frac{-2}{g''(\alpha_s)}}; \quad Q(y) = \int_y^\infty e^{-x^2} dx,$$

and the sign of H is defined such that $\arg h = (\arg d\alpha)_{\alpha_s}$, where $d\alpha$ is an element along the steepest descent path SDP, while $\arg b$ is defined such that

$$b \rightarrow \frac{\alpha_0 - \alpha_s}{h}, \quad \text{as } \alpha_0 \rightarrow \alpha_s.$$

In our case we have

$$g(\alpha) = -i \cos(\alpha - \phi); \quad f(\alpha) = \frac{\Omega}{2\pi} \frac{1}{1 - e^{i\Omega(\alpha - \psi)}}; \quad \alpha_0 = \phi', \quad \text{and on } L_1:$$

$$\alpha_s = -\pi + \phi; \quad a = -\frac{1}{i2\pi}; \quad h = e^{-i\frac{\pi}{4}\sqrt{2}}; \quad b = \sqrt{2} e^{i\frac{\pi}{4}} \cos \frac{\phi' - \phi}{2};$$

$$T(0) = e^{-i\frac{\pi}{4}\sqrt{2}} \frac{\Omega}{2\pi} \frac{1}{1 - e^{i\Omega(-\pi + \phi - \phi')}} - \frac{1}{i2\pi} \frac{1}{\sqrt{2} e^{i\frac{\pi}{4}} \cos \frac{\phi' - \phi}{2}}.$$

Now $w(\rho, \phi; \phi')$ can be rewritten in the form

$$w(\rho, \phi; \phi') = \eta(\text{Im} b) e^{k\rho g(\alpha_0)} + I_1(\rho, \phi; \phi') + I_2(\rho, \phi; \phi').$$

Let us assume that the pole $\alpha = \phi'$ lies near the contour L_1 . We observe that

$$Q(y) = \sqrt{\pi} F(e^{i\frac{\pi}{4}} y), \text{ where } F(z) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{\pi}} \int_z^\infty e^{i\mu^2} d\mu \text{ is the Fresnel integral, and}$$

that $F(z) = 1 - F(-z)$. Consequently,

$$\eta(\text{Im} b) e^{k\rho g(\alpha_0)} \pm i 2a\sqrt{\pi} e^{k\rho g(\alpha_0)} Q(\pm i b\sqrt{k\rho}) = e^{k\rho g(\alpha_0)} F\left(\sqrt{\frac{k\rho}{i}} b\right),$$

and the sum of the first two terms in (30) have the following UAE:

$$\begin{aligned} \eta(\pi + \phi' - \phi) e^{-ik\rho \cos(\phi' - \phi)} + I_1(\rho, \phi; \phi') \sim e^{ik\rho} \{ e^{-ik\rho [1 + \cos(\phi' - \phi)]} F(-\sqrt{2k\rho} \cos \frac{\phi' - \phi}{2} + \\ \frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi k\rho} \cos \frac{\phi' - \phi}{2}} + \frac{\Omega e^{-i\frac{\pi}{4}}}{\sqrt{2\pi k\rho}} \frac{1}{1 - e^{i\Omega(\phi - \pi - \phi')}}} \}. \end{aligned} \quad (31)$$

Since the pole $\alpha = \alpha_0$ is away from the contour L_2 , the contribution to $w(\rho, \phi; \phi')$ from $I_2(\rho, \phi; \phi')$ can be calculated by applying the standard saddle-point method, with the result

$$I_2(\rho, \phi; \phi') \sim - \frac{e^{ik\rho - i\frac{\pi}{4}}}{\sqrt{2\pi k\rho}} \frac{\Omega}{1 - e^{i\Omega(\phi + \pi - \phi')}}. \quad (32)$$

Upon adding (31) and (32) together, we obtain the needed UAE for the function

$w(\rho, \phi; \phi')$:

$$\begin{aligned} w(\rho, \phi; \phi') \sim e^{ik\rho} \{ e^{-ik\rho [1 + \cos(\phi' - \phi)]} F(-\sqrt{2k\rho} \cos \frac{\phi' - \phi}{2}) + \frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi k\rho}} \frac{1}{\cos \frac{\phi' - \phi}{2}} - \\ \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi k\rho}} \frac{\Omega \sin \Omega \pi}{\cos \Omega(\phi - \phi') - \cos \Omega \pi} \}. \end{aligned}$$

If the pole α_0 lies near the contour L_2 , then the above reasoning can be repeated with the contours L_1 and L_2 interchanged. It can be shown that the final result is the same.

To find the UAE for the function $w(\rho, \phi; -\phi')$, let us represent it in the form:

$$w(\rho, \phi; -\phi') = \eta(\pi - \phi' - \phi) e^{-ik\rho \cos(\phi' + \phi)} + \eta(\pi - 2\psi + \phi' + \phi) e^{-ik\rho \cos(2\psi - \phi' - \phi)} +$$

$$I_1(\rho, \phi; -\phi') + I_2(\rho, \phi; -\phi') \quad (33)$$

$$\text{where } I_n(\rho, \phi; -\phi') = \frac{\Omega}{2\pi} \int_{L_n} \frac{e^{-ik\rho \cos(\alpha - \phi)}}{1 - e^{i\Omega(\alpha + \phi')}} d\alpha.$$

In this case the pole $\alpha = -\phi'$ appears near the contour L_1 and another pole $\alpha = 2\psi - \phi'$ lies near the contour L_2 . Therefore, now we employ the same technique to the first and the third and to the second and the fourth terms in (33), respectively, as was used to find (32). The result is

$$w(\rho, \phi; -\phi') \sim e^{ik\rho} \{ e^{-ik\rho[1 + \cos(\phi + \phi')]} F(-\sqrt{2k\rho} \cos \frac{\phi' + \phi}{2}) + \frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi k\rho}} \frac{1}{\cos \frac{\phi' + \phi}{2}} +$$

$$\frac{\Omega e^{-i\frac{\pi}{4}}}{\sqrt{2\pi k\rho}} \frac{1}{1 - e^{i\Omega(\phi - \pi + \phi')}} + e^{-ik\rho[1 + \cos(2\psi - \phi' - \phi)]} F(-\sqrt{2k\rho} \cos \frac{2\psi - \phi' - \phi}{2}) + \quad (34)$$

$$\frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi k\rho}} \frac{1}{\cos \frac{2\psi - \phi' - \phi}{2}} - \frac{\Omega e^{-i\frac{\pi}{4}}}{\sqrt{2\pi k\rho}} \frac{1}{1 - e^{i\Omega(\pi + \phi - \phi')}} \}.$$

The sum of (33) and (34) yields the UAE for the total field u . It will be rewritten in a compact form below. Let us first introduce the notations

$$s = \rho; \quad s^i = -\rho \cos(\phi - \phi'); \quad s^{r1} = -\rho \cos(\phi + \phi'); \quad s^{r2} = -\rho \cos(2\psi - \phi' - \phi).$$

In the context of ray optics, these quantities are the eiconals of the diffracted, incident, and the two reflected waves, respectively. We also introduce the functions ϵ^i , ϵ_1^r and ϵ_2^r , which are equal to -1 in the lit region and +1 in the shadow region of the respective fields. Next we use the notation

$$\tilde{F}(x) = \frac{e^{i(x^2 + \frac{\pi}{4})}}{2\sqrt{\pi} x}$$

which is the first term of the asymptotic expansion of $F(x)$, according to

$$F(x) = \eta(-x) + \tilde{F}(x) + O(x^{-3}).$$

Then the UAE for the total field u takes the form:

$$u \sim u^i [F(\epsilon^i \sqrt{k(s-s^i)}) - \tilde{F}(\epsilon^i \sqrt{k(s-s^i)})] + \{i \rightarrow r_1\} + \{i \rightarrow r_2\} + u^d, \quad (35)$$

where

$$u^i = \exp(is^i), \quad u^{r1} = \exp(is^{r1}), \quad u^{r2} = \exp(is^{r2}) \quad (36a)$$

$$u^d = -\frac{e^{ik\rho + i\frac{\pi}{4}}}{\sqrt{2\pi k\rho}} \Omega \sin \Omega \pi \{ [\cos \Omega(\phi - \phi') - \cos \Omega \pi]^{-1} + [\cos \Omega(\phi + \phi') - \cos \Omega \pi]^{-1} \}. \quad (36b)$$

We note that the expression for the total field u contains u^d , given in (36) which is the diffracted field à la Keller. This diffracted field is non-uniform in nature as it is singular both at the shadow boundary and in the neighborhood of $\rho = 0$. However, the total field given by (35) is uniform at the shadow boundaries because the singularities in \tilde{F} 's appearing in the

first three terms exactly cancel out the singularity in u^d . However, such a cancellation does not occur at $\rho = 0$. Hence the expression for u in (35) is still non-uniform at the tip of the wedge; nonetheless, the uniform nature is maintained at asymptotic distances away from the tip, i.e., for $k\rho$ large.

Finally, we point out that the expression in (35) is identical to the one based on the Lee and Deschamps ansatz [6] which is a generalization of the uniform formula for the half-plane derived by Ahluwalia, Boersma and Lewis [7].

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