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MODEL SIMPLIFICATION AND OPTIMAL CONTROL OF STOCHASTIC SINGULARLY PERTURBED SYSTEMS UNDER EXPONENTIATED QUADRATIC COST

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Model Simplification and Optimal Control of Stochastic Singularly Perturbed Systems under Exponentiated Quadratic Cost*

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Abstract

We study the optimal control of a general class of stochastic singularly perturbed linear systems with perfect and noisy state measurements under positively and negatively exponentiated quadratic cost. Both finite- and infinite-horizon cases are treated, where in the latter case we take as the cost function the long term time average of the logarithm of the expected value of the exponentiated quadratic cost. In each case, we identify appropriate "slow" and "fast" subproblems, obtain their optimum solutions (compatible with the corresponding measurement structure), and subsequently study the performances they achieve on the full-order system as the singular perturbation parameter ϵ becomes sufficiently small, with the expressions given in all cases being exact to within $O(\sqrt{\epsilon})$. It is shown that the composite controller (obtained by appropriately combining the optimum slow and fast controllers) achieves a performance level close to the optimal one whenever the full-order problem has a solution. The slow controller, on the other hand, achieves only a finite performance level (but not necessarily optimal), provided that the fast subsystem is open-loop stable. If the intensity of the noise in the system dynamics decreases to zero, however, the slow controller also achieves a performance level close to the optimal one.

The paper also presents a more direct derivation (than heretofore available) of the solution to the LEQG problem under noisy state measurements, which allows for a general quadratic cost (with cross terms) in the exponent and correlation between system and measurement noises. Such a general LEQG problem is encountered in the slow-fast decomposition of the full-order problem, even if the original problem does not feature correlated noises. In this general context, the paper also establishes the complete equivalence between the LEQG problem and the H^∞ -optimal control problem with measurement feedback, though this equivalence does not extend to the slow and fast subproblems arrived at after time-scale separation.

Key Words—Linear exponential quadratic Gaussian optimal control, generalized Riccati differential equation, generalized algebraic Riccati equation, singular perturbations, H^∞ -optimal control.

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1 Introduction

The problem of optimal control of stochastic linear systems under exponentiated quadratic loss (so called LEQG problem) has been studied extensively in the literature, with new interest arousing on the topic due to the recently established relationship with the H^∞ -optimal control of similar systems (but with deterministic disturbances) under quadratic loss. Perhaps the first formulation of the LEQG problem was given by Jacobson ([1]), in both discrete and continuous time, and using perfect state measurements, motivated by the fact that the exponentiated quadratic cost captures risk seeking or risk averse behavior, not obtainable using the LQG formulation (which is risk neutral). Indeed it was discovered in [1] that the LEQG formulation with a positive exponent is equivalent (as far as the optimal solution goes) to a deterministic zero-sum LQ differential game, which we now know ([2]) is equivalent to an H^∞ -optimal control problem, thus completing the link. The counterparts of the results of [1] in the imperfect state measurement case for discrete and continuous time were later obtained in [3], [4] and [5], with the relationship with the H^∞ -optimal control problem established in a series of subsequent publications, such as [6], [7], [8]; see also the book by Whittle ([9]). Similar relationships (between exponentiated cost stochastic control and worst case designs) exist also for nonlinear problems, as established for some subclass of such problems in [10]. A more recent paper [11] completely establishes this equivalence in the discrete-time case.

Our objective in this paper is to study, under both perfect and noisy state measurements, the robustness properties of the optimal solution of the LEQG problem with respect to unmodeled fast dynamics. This study is conducted in the framework of singularly-perturbed models, with a small positive parameter ϵ quantifying the extent of coupling between the slow and fast dynamics. We seek ϵ -independent controllers that provide good (in a sense to be made precise later) approximation to the optimal controller of the full-order problem in a neighborhood of $\epsilon = 0$.

As mentioned earlier, at the full-order level there is an equivalence between the positively exponentiated subclass and a class of LQ H^∞ -optimal control problems with singularly perturbed dynamics, with this latter class of problems extensively studied recently from the point of view of robustness and model reduction ([12], [13], [14], [15]). This equivalence, however, does not readily carry over to the "model-reduction" stage, and as to be seen here the end results in the two cases are considerably different. One of the reasons for this is that (as it has been studied earlier in [16]) in stochastic problems the parameter ϵ has to enter the system dynamics and the measurement equation in a certain way for the problem to be well-defined as $\epsilon \rightarrow 0$. The exact problem formulation provided in Section 2 shows that indeed in the stochastic case a time-scale separation of the full-order system becomes much more involved. Nevertheless, we still find occasion to use some of our earlier results from [12] and [13] in the present development, to simplify some of the proofs. Furthermore, in the derivation of the optimal solution to the stochastic control problem associated with the slow subsystem, we are faced with the need to obtain a clean and complete solution to the general LEQG problem with general cost structure and correlation between system and measurement noises. This motivates us into the investigation that leads to the results of Section 3, which generalize the earlier results of [5].

The paper is organized as follows. In the next section (Section 2) we formulate the LEQG problem with perfect state measurements for singularly perturbed systems and present the solution to the full-order problem. In Section 3, we present a clean derivation (under least stringent conditions) of a complete solution to the general LEQG problem under noisy state measurements with general

cost structure and correlation between system and measurement noises. In Section 4, we study the singularly perturbed stochastic control problem under perfect state measurements, where we decompose the problem into slow and fast ϵ -free subproblems, obtain optimal controllers for these subproblems, and study the optimality of the composite controller as well as that of the slow controller in terms of the attainable performance for the full-order problem. In Section 5, we study the problem under noisy state measurements, where we identify the slow and fast subproblems to the full-order problem, obtain optimal controllers for these subproblems, construct the composite controller from these controllers, and study the optimality of these suboptimal controllers in terms of the attainable performance for the full-order problem. Three numerical examples are presented in Section 6 to illustrate the theory. The paper ends with the concluding remarks of Section 7, and two Appendices, which contain detailed derivations of some of the results in the main body of the paper.

2 Problem Formulation

The system under consideration, with slow and fast dynamics, is described in the "singularly perturbed" form by¹

$$\begin{cases} dx_1 &= (A_{11}(t)x_1 + A_{12}(t)x_2 + B_1(t)u_t) dt + G_1(t) dw_t; & x_1(t_0) = x_{10} \\ \epsilon dx_2 &= (\epsilon^\alpha A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u_t) dt + \epsilon^{1/2} G_2(t) dw_t; & x_2(t_0) = x_{20} \\ dy_1 &= (C_{11}(t)x_1 + C_{12}(t)x_2) dt + E_1(t) dw_t; & y_1(t_0) = 0 \\ dy_2 &= (\epsilon^\alpha C_{21}(t)x_1 + C_{22}(t)x_2) dt + \epsilon^\beta E_2(t) dw_t; & y_2(t_0) = 0 \end{cases} \quad (2.1)$$

where $x' := (x'_1, x'_2)$ is the n -dimensional state vector, with x_1 of dimension n_1 and x_2 of dimension $n_2 := n - n_1$; $y' := (y'_1, y'_2)'$ is the m -dimensional measurement process, with y_1 of dimension m_1 and y_2 of dimension $m_2 := m - m_1$; u_t is the p -dimensional control input, and w_t is a r -dimensional vector valued standard Wiener process starting at t_0 , which is independent of the initial condition; the small positive scalar ϵ is the singular perturbation parameter. The underlying probability space is the triplet (Ω, \mathcal{F}, P) .

Associated with this system, we introduce the cost function:

$$J_\theta(\mu) = \frac{2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(x'_{t_f} Q_f x_{t_f} + \int_{t_0}^{t_f} (x' Q(t) x + u' u) dt \right) \right] \right\} \right\} \quad (2.2)$$

where the terminal cost weighting matrix Q_f shows dependence on $\epsilon > 0$, as to be specified later. The scalar $\theta \neq 0$ is the parameter in terms of which we are going to parametrize our solution.

In the perfect state measurements case, the initial state is assumed to be known perfectly, and the control input u_t is generated by a closed-loop control policy μ , according to

$$u(t) = \mu(t, x_{[t_0, t]}) \quad (2.3)$$

where $\mu \in \mathcal{M}$ is an admissible control. Furthermore, we let $\alpha = 0$, which makes the system dynamics well-defined as $\epsilon \rightarrow 0$, as shown in [16]. Then, we seek an optimal controller with respect to cost (2.2), i.e., a $u^* = \mu^*(t, x_{[t_0, t]})$ such that

$$J_\theta(\mu^*) = \min_{\mu \in \mathcal{M}} J_\theta(\mu) := J_\theta^* \quad (2.4)$$

¹The well-posedness of the LQG stochastic optimal control problem for this singularly perturbed system has been studied in [16]. The appropriateness of system dynamics (2.1) has been established there.

In the noisy state measurements case, the initial state is taken to be a Gaussian random vector with mean \bar{x}_0 and covariance Σ_0 (where Σ_0 is assumed to be positive definite, and will show dependence on ϵ , as to be specified later). In this case, the control input u is generated by a control policy μ_I , according to

$$u(t) = \mu_I(t, y_{[t_0, t]}) \quad (2.5)$$

where $\mu_I : [t_0, t_f] \times \mathcal{H}_y \rightarrow \mathcal{H}_u$ is piecewise continuous in t and Lipschitz continuous in y , further satisfying the given causality condition. Let us denote the class of all admissible controllers by \mathcal{M}_I . (See the next section for a precise description of the class of admissible controllers.) For this case, we take $\alpha = 1/2$ and $\beta = 1/2$, which again leads to well-defined systems dynamics as $\epsilon \rightarrow 0$, as shown in [16].

Denoting the cost function (2.2) for the imperfect measurements case by $J_{I\theta}(\mu_I)$, we again seek an optimal controller with respect to $J_{I\theta}(\mu_I)$, that is a $u^* = \mu_I^*(t, y_{[t_0, t]})$ such that

$$J_{I\theta}(\mu_I^*) = \min_{\mu_I \in \mathcal{M}_I} J_{I\theta}(\mu_I) := J_{I\theta}^* \quad (2.6)$$

We first make three basic assumptions:

A1 Q_f , Σ_0 and $Q(\cdot)$ in (2.2) are partitioned as

$$Q_f = \begin{bmatrix} Q_{f11} & \epsilon Q_{f12} \\ \epsilon Q_{f21} & \epsilon Q_{f22} \end{bmatrix}; \quad \Sigma_0 = \begin{bmatrix} \Sigma_{011} & \sqrt{\epsilon} \Sigma_{012} \\ \sqrt{\epsilon} \Sigma_{021} & \Sigma_{022} \end{bmatrix}; \quad Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix}$$

where in each case the 11-block is of dimensions $n_1 \times n_1$, and the 22-block is of dimensions $n_2 \times n_2$.

A2 The matrix functions $A_{ij}(t)$, $Q_{ij}(t)$, $B_i(t)$, $G_i(t)$, C_{ij} , E_i ($i = 1, 2$ $j = 1, 2$) are continuously differentiable in $t \geq 0$.

A3 The matrices $A_{22}(t)$, $Q_{22}(t)$, $G_2(t)G_2'(t)$ and $N(t) := E(t)E(t)'$ are invertible for all $t \in [0, t_f]$, where $E' = [E_1' E_2']$.² The system noise and the measurement noise are uncorrelated, i.e., $G_1 E' = 0$ and $G_2 E' = 0$.

Let us further introduce the following notation:

$$\begin{aligned} A_\epsilon(t) &:= \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ \epsilon^{\alpha-1} A_{21}(t) & \frac{1}{\epsilon} A_{22}(t) \end{bmatrix}; & B(t) &:= \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} \\ B_\epsilon(t) &:= \begin{bmatrix} B_1(t) \\ \frac{1}{\epsilon} B_2(t) \end{bmatrix}; & G(t) &:= \begin{bmatrix} G_1(t) \\ 0 \end{bmatrix}; & G_\epsilon(t) &:= \begin{bmatrix} G_1(t) \\ \frac{1}{\sqrt{\epsilon}} G_2(t) \end{bmatrix} \\ C_1(t) &:= \begin{bmatrix} C_{11}(t) \\ C_{21}(t) \end{bmatrix}; & C_2(t) &:= \begin{bmatrix} 0 \\ C_{22}(t) \end{bmatrix}; & C_{2\epsilon}(t) &:= \begin{bmatrix} C_{12}(t) \\ \frac{1}{\sqrt{\epsilon}} C_{22}(t) \end{bmatrix} \\ C_\epsilon(t) &:= \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ \epsilon^\alpha C_{21}(t) & C_{22}(t) \end{bmatrix}; & E_\epsilon(t) &:= \begin{bmatrix} E_1(t) \\ \epsilon^\beta E_2(t) \end{bmatrix} \end{aligned}$$

²The conditions of invertibility of Q_{22} and $G_2 G_2'$ can be further relaxed to the conditions of observability of the pairs (A_{22}, Q_{22}) and (A_{22}, G_2) . To obtain the same results under the relaxed conditions, we perturb the matrices Q_{22} and $G_2 G_2'$ by λI , for some scalar $\lambda > 0$; then all the derivation are correct for the perturbed problem and they converge (to finite values) as $\lambda \rightarrow 0$ (see [17] for details).

and define $S(t; \theta) := B(t)B'(t) - \theta G(t)G'(t)$, $S_\epsilon(t; \theta) := B_\epsilon(t)B'_\epsilon(t) - \theta G_\epsilon(t)G'_\epsilon(t)$, $R_\epsilon(t; \theta) := C'_\epsilon N_\epsilon^{-1} C_\epsilon - \theta Q$ with the ij -th block of $S(t; \theta)$ denoted henceforth by $S_{ij}(t; \theta)$, $i, j = 1, 2$. We also define

$$R_\epsilon(t; \theta) := \begin{bmatrix} R_{\epsilon 11} & \frac{1}{\sqrt{\epsilon}} R_{\epsilon 12} \\ \frac{1}{\sqrt{\epsilon}} R_{\epsilon 21} & \frac{1}{\epsilon} R_{\epsilon 22} \end{bmatrix}; \quad R_{12}(t; \theta) := C'_1 N^{-1} C_2; \quad R_{21}(t; \theta) := R_{12}(t; \theta)';$$

$$R_{22}(t; \theta) := C_2 N^{-1} C_2; \quad R_{11}(t; \theta) := C'_1 N^{-1} C_1 - \theta Q_{11}; \quad R(t; \theta) := \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

For each fixed $\epsilon > 0$, the problem formulated above has been solved in the literature for both the perfect state [1] and noisy state measurements [5] cases. But the computation of the optimal or suboptimal controllers for small values of $\epsilon > 0$ present serious difficulties, due to numerical stiffness. To remedy this, we pose in this paper the question of whether optimal controllers can be determined, for small values of $\epsilon > 0$, by solving well-behaved ϵ -independent smaller-order problems, as in the case of the singularly perturbed linear-quadratic Gaussian regulator problem ([18]).

Under perfect state measurements, for each $\epsilon > 0$ (see [1]), if the GRDE:

$$\dot{\tilde{Z}} + A'_\epsilon \tilde{Z} + \tilde{Z} A_\epsilon - \tilde{Z} S_\epsilon \tilde{Z} + Q = 0; \quad \tilde{Z}(t_f) = Q_f \quad (2.7)$$

admits a nonnegative definite solution $\tilde{Z}(t; \epsilon)$ on $[t_0, t_f]$, then, the optimal controller for the full-order problem is

$$u^*(t) = \mu^*(t, x(t)) = -B'_\epsilon \tilde{Z}(t; \epsilon) x(t), \quad t \geq t_0 \quad (2.8)$$

Let us introduce the following quantity:

$$\theta^*(\epsilon) := \sup\{\theta \in \mathbf{R} : \text{the GRDE (2.7) admits a nonnegative definite solution on } [t_0, t_f].\} \quad (2.9)$$

Then, for $\theta < \theta^*(\epsilon)$, the LEQG problem admits an optimal controller given by (2.8), with the optimal cost being

$$J_\theta^*(\epsilon) = x'_0 \tilde{Z}(0; \epsilon) x_0 + \int_0^T \text{Tr}(G_\epsilon G'_\epsilon \tilde{Z}(t; \epsilon)) dt \quad (2.10)$$

Under noisy state measurements, for each $\epsilon > 0$, the problem above admits an optimal controller³ if the GRDE (2.7) admits a nonnegative definite solution $\tilde{Z}(t; \epsilon)$ on $[t_0, t_f]$, and in addition the following GRDE:

$$\dot{\tilde{\Sigma}} = A_\epsilon \tilde{\Sigma} + \tilde{\Sigma} A'_\epsilon - \tilde{\Sigma} R_\epsilon \tilde{\Sigma} + G_\epsilon G'_\epsilon; \quad \tilde{\Sigma}(t_0) = \Sigma_0 \quad (2.11)$$

admits a positive definite solution $\tilde{\Sigma}(t; \epsilon)$ on $[t_0, t_f]$, and further the following spectral radius condition is satisfied:

$$I - \theta \tilde{\Sigma}(t) \tilde{Z}(t) \quad \text{has only positive eigenvalues} \quad \forall t \in [t_0, t_f]. \quad (2.12)$$

As the counterpart of (2.9), let us introduce:

$$\theta_I^*(\epsilon) := \sup\{\theta \in \mathbf{R} : \text{the GRDEs (2.7) and (2.11) admit nonnegative definite solutions on } [t_0, t_f] \text{ such that (2.12) is satisfied.}\} \quad (2.13)$$

³See [5] and Theorem 1 of next section.

For $\theta < \theta_I^*(\epsilon)$, the optimal controller is given by ⁴

$$u^*(t) = \mu_I^*(t, y_{[t_0, t]}) = -B'_\epsilon \tilde{Z}(t; \epsilon) \hat{x}_t, \quad t \geq t_0 \quad (2.14)$$

$$d\hat{x}_t = (A_\epsilon - S_\epsilon \tilde{Z}) \hat{x}_t dt + (I - \theta \tilde{\Sigma} \tilde{Z})^{-1} \Sigma C'_\epsilon N_\epsilon^{-1} (dy_t - C_\epsilon \hat{x}_t dt); \quad (2.15)$$

$$\hat{x}(t_0) = (I - \theta \tilde{\Sigma}(t_0) \tilde{Z}(t_0))^{-1} \bar{x}_0$$

with the optimal cost being

$$J_{I\theta}^*(\epsilon) = \bar{x}_0' \tilde{Z}(t_0) (I - \theta \tilde{\Sigma}(t_0) \tilde{Z}(t_0))^{-1} \bar{x}_0 + \int_0^T \text{Tr}(\tilde{\Sigma} Q + \tilde{\Sigma} C'_\epsilon N_\epsilon^{-1} C_\epsilon \tilde{\Sigma} \tilde{Z} (I - \theta \tilde{\Sigma} \tilde{Z})^{-1}) dt - \frac{1}{\theta} \ln(\det(I - \theta \tilde{\Sigma}(t_f) \tilde{Z}(t_f))) \quad (2.16)$$

To study the infinite-horizon case, we take A, B, G, Q, C, E to be time-invariant, and $Q_f = 0, t_0 = 0$. We take the cost function to be:

$$J_{\theta\infty}(\mu) = \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (x' Q x + u' u) dt \right) \right] \right\} \right\} \quad (2.17)$$

In the perfect state measurements case, we take (A_ϵ, B_ϵ) to be controllable, and (A_ϵ, Q) to be observable for every $\epsilon > 0$. Then, by Theorem 7 in Appendix A, for each $\epsilon > 0$, if the generalized algebraic Riccati equation (GARE)

$$A'_\epsilon \tilde{Z}_\infty + \tilde{Z}_\infty A_\epsilon - \tilde{Z}_\infty S_\epsilon \tilde{Z}_\infty + Q = 0 \quad (2.18)$$

admits a minimal positive definite solution \tilde{Z}_∞ , the problem admits an optimal solution

$$u^*(t) = \mu^\infty(x) = -B'_\epsilon \tilde{Z}_\infty(\epsilon) x(t), \quad t \geq 0. \quad (2.19)$$

Introducing the quantity:

$$\theta_\infty^*(\epsilon) := \sup \{ \theta \in \mathbb{R} : \text{the GARE (2.18) admits a positive definite solution} \}, \quad (2.20)$$

we have that for $\theta < \theta_\infty^*(\epsilon)$, the infinite-horizon LEQG problem admits an optimal controller given by (2.19), which achieves an optimal cost

$$J_{\theta\infty}^*(\epsilon) = \text{Tr}(G_\epsilon G'_\epsilon \tilde{Z}_\infty(t; \epsilon)) \quad (2.21)$$

For the imperfect state measurements case, we denote the cost function (2.17) by $J_{I\theta\infty}(\mu_{I\infty})$, and assume that (A_ϵ, B_ϵ) and (A_ϵ, G_ϵ) are controllable, and (A_ϵ, Q) and (A_ϵ, C_ϵ) are observable for every $\epsilon > 0$. Then, by Theorem 2 of next section, for each $\epsilon > 0$, if the GARE (2.18) admits a minimal positive definite solution, and the following GARE:

$$A_\epsilon \tilde{\Sigma}_\infty + \tilde{\Sigma}_\infty A'_\epsilon - \tilde{\Sigma}_\infty R_\epsilon \tilde{\Sigma}_\infty + G_\epsilon G'_\epsilon = 0 \quad (2.22)$$

admits a minimal positive definite solution $\tilde{\Sigma}_\infty$, and further satisfies the spectral radius condition:

$$I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty \quad \text{has only positive eigenvalues,} \quad (2.23)$$

⁴It is proven in [2] that \hat{x}_t of (4.2) in [5] is exactly $(I - \theta \tilde{\Sigma}(t) \tilde{Z}(t))^{-1} \hat{x}_t$, where \hat{x}_t is generated by (2.15).

then the problem admits an optimal controller. In particular, defining the quantity:

$$\theta_{I\infty}^*(\epsilon) := \sup\{\theta \in \mathbb{R} : \text{the GAREs (2.18) and (2.22) admit minimal positive definite solutions such that (2.23) is satisfied}\}, \quad (2.24)$$

for $\theta < \theta_{I\infty}^*(\epsilon)$ the optimal controller is given by

$$u_{I\infty}^*(t) = \mu_{I\infty}^*(t, y_{(-\infty, t]}) = -B_\epsilon' \tilde{Z}_\infty \hat{x}_t \quad (2.25)$$

$$d\hat{x}_t = (A_\epsilon - S_\epsilon \tilde{Z}_\infty) \hat{x}_t dt + (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} \tilde{\Sigma}_\infty C_\epsilon' N_\epsilon^{-1} (dy_t - C_\epsilon \hat{x}_t dt) \quad (2.26)$$

with the optimal cost being

$$J_{I\theta\infty}^*(\epsilon) = \text{Tr}(\tilde{\Sigma}_\infty Q + \tilde{\Sigma}_\infty C_\epsilon' N_\epsilon^{-1} C_\epsilon \tilde{\Sigma}_\infty \tilde{Z}_\infty (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1}) \quad (2.27)$$

Thus completing the description of the direct solution to the full-order problem for $\epsilon > 0$, we turn, in the Sections 4 and 5, to the derivation of the approximate solution based on a time-scale decomposition. Before we derive the slow and fast subsystems, however, we present, in next section, a complete solution to the general LEQG problem under imperfect state measurements, with a general cost structure and correlated system and measurement noise. This solution, to be presented in the next section, generalizes the results of [5] by removing some of the technical assumptions made there. This generality will be needed in the derivation of the solution to the slow subsystem in Section 5.

3 Solution to the General LEQG Problem Under Imperfect State Measurements

In this section, we consider the general LEQG problem under imperfect state measurements, with a general exponentiated quadratic cost structure. We let $\epsilon > 0$ be fixed, and suppress it in the system and measurement dynamics, which are simply written as:

$$\begin{cases} dx = (A(t)x + B(t)u_t) dt + G(t) dw_t; & x(t_0) = x_0 \\ dy = C(t)x dt + E(t) dw_t; & y(t_0) = 0 \end{cases} \quad (3.1)$$

We first consider the finite and then the infinite-horizon case.

3.1 The Finite-Horizon Problem

For the finite-horizon case, the cost function associated with the system (3.1) is taken a generalized version of (2.2):

$$J_{I\theta}(\mu_I) = \frac{2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(x_{t_f}' Q_f x_{t_f} + \int_{t_0}^{t_f} (x' Q(t) x + 2x' P(t) u + u' R(t) u) dt \right) \right] \right\} \right\} \quad (3.2)$$

where $R(t) > 0$ and $Q - PR^{-1}P \geq 0$ for all $t \in [t_0, t_f]$.

Let Assumption A2 hold, the matrix Σ_0 be positive definite and the matrix $N(t) := E(t)E(t)'$ be positive definite for all $t \in [t_0, t_f]$. Introduce the notation,

$$\begin{aligned} \bar{A} &:= A - BR^{-1}P'; & \bar{Q} &:= Q - PR^{-1}P'; & \bar{S} &:= BR^{-1}B' - \theta GG'; \\ L &:= GE'; & \bar{A} &:= A - LN^{-1}C; & \bar{M} &:= GG' - LN^{-1}L'; & \bar{R} &:= C'N^{-1}C - \theta Q \end{aligned}$$

and in terms of this the backward GRDE:

$$\dot{\tilde{Z}} + \tilde{A}'\tilde{Z} + \tilde{Z}\tilde{A} - \tilde{Z}\tilde{S}\tilde{Z} + \tilde{Q} = 0; \quad \tilde{Z}(t_f) = Q_f \quad (3.3)$$

and the forward GRDE:

$$\dot{\tilde{\Sigma}} = \tilde{\Sigma}\tilde{A}' + \tilde{A}\tilde{\Sigma} - \tilde{\Sigma}\tilde{R}\tilde{\Sigma} + \tilde{M}; \quad \tilde{\Sigma}(t_0) = \Sigma_0. \quad (3.4)$$

Define the quantity:

$$\theta_I^* := \sup\{\theta \in \mathbf{R} : \text{the GRDEs (3.3) and (3.4) admit nonnegative definite solutions } \tilde{Z} \text{ and } \tilde{\Sigma}, \text{ respectively, on } [t_0, t_f] \text{ such that the matrix } I - \theta\tilde{\Sigma}\tilde{Z} \text{ has only positive eigenvalues, for each } t \in [t_0, t_f] \}, \quad (3.5)$$

and for $\theta < \theta_I^*$, introduce the filter:

$$d\tilde{x} = (A + \theta\tilde{\Sigma}Q)\tilde{x} dt + (B + \theta\tilde{\Sigma}P)u dt + (\tilde{\Sigma}C' + L)N^{-1}(dy - C\tilde{x} dt) \quad (3.6)$$

with initial state $\tilde{x}(t_0) = \bar{x}_0$. Letting $\hat{x} := (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{x}$, it can be shown (see Appendix B) that \hat{x} is generated by the following dynamics:

$$d\hat{x} = (\bar{A} - \bar{S}\tilde{Z})\hat{x} dt + (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(B + \theta\tilde{\Sigma}P)\bar{u} dt + (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(\tilde{\Sigma}C' + L)N^{-1}(dy - (C + \theta L'\tilde{Z})\hat{x} dt) \quad (3.7)$$

where $\bar{u} = u + R^{-1}(B'\tilde{Z} + P')\hat{x}$.

Let $\varepsilon := x - \tilde{x}$ and $e := x - \hat{x}$. We will now restrict our attention to the class of controllers such that the following process $\zeta(t)$ defines a martingale on $[t_0, t_f]$:

$$\zeta(t) = \exp\left\{\int_{t_0}^t (\varepsilon'\tilde{\Sigma}^{-1}G - e'\tilde{\Sigma}^{-1}(\tilde{\Sigma}C' + L)N^{-1}Edw_t - \frac{1}{2}\int_{t_0}^t |G'\tilde{\Sigma}^{-1}\varepsilon - E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Sigma}^{-1}e|^2 dt)\right\} \quad (3.8)$$

Hence, we define the set of admissible controllers \mathcal{M}_I to be all mappings $\mu_I : [t_0, t_f] \times \mathcal{H}_y \rightarrow \mathcal{H}_u$ that are piecewise continuous in t and Lipschitz continuous in y , and further satisfying the given causality condition such that $\zeta(t)$ is a martingale on $[t_0, t_f]$.

The above condition will be needed in the application of Girsanov Theorem [19] for a change of probability measures. A sufficient condition for this condition to be satisfied is the existence of positive constants δ and κ such that

$$E\{\exp\{\delta|G'\tilde{\Sigma}^{-1}\varepsilon - E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Sigma}^{-1}e|^2\}\} \leq \kappa, \quad \forall t \in [t_0, t_f].$$

It is obvious that any linear control law renders this condition. We refer the readers to the recent book [19] for a thorough coverage of this topic.

Now, we prove the following result:

Theorem 1 Consider the general finite-horizon LEQG problem described by (3.1), (3.2). Let Assumption A2 hold, and assume that $\Sigma_0 > 0$, $R(t) > 0$, $Q(t) - P(t)R(t)^{-1}P(t)' \geq 0$ and $N(t) > 0$. For each $\theta < \theta_I^*$, the optimal controller is given by

$$u_I^* = \mu_I^*(t, y_{[t_0, t]}) = -R^{-1}(B'\tilde{Z} + P')(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{x} = -R^{-1}(B'\tilde{Z} + P')\hat{x} \quad (3.9)$$

where \tilde{x} is generated by the filter (3.6), or equivalently, \hat{x} is generated by the filter (3.7). The optimal cost can be written as:

$$J_{I\theta}^* = \inf_{\mu_I \in \mathcal{M}_I} J_{I\theta}(\mu_I) = \bar{x}_0' \tilde{Z}(t_0) (I - \theta \Sigma_0 \tilde{Z}(t_0))^{-1} \bar{x}_0 - \frac{1}{\theta} \ln \{ \det(I - \theta \tilde{\Sigma}(t_f) Q_f) \} \\ + \int_{t_0}^{t_f} \text{Tr}(\tilde{\Sigma} Q + (\tilde{\Sigma} C' + L) N^{-1} (C \tilde{\Sigma} + L') \tilde{Z} (I - \theta \tilde{\Sigma} \tilde{Z})^{-1}) dt \quad (3.10)$$

Furthermore, the above controller is also conditionally optimal.⁵

Proof The differential equation for ε is easily obtained to be:

$$d\varepsilon = (\bar{A} - \tilde{\Sigma} C' N^{-1} C) \varepsilon dt - \theta \tilde{\Sigma} Q \tilde{x} dt - \theta \tilde{\Sigma} P u dt + (G - (\tilde{\Sigma} C' + L) N^{-1} E) dw_t$$

Let $\tilde{\Psi} := \tilde{Z} (I - \theta \tilde{\Sigma} \tilde{Z})^{-1}$, and define

$$\Upsilon(t, \varepsilon, \tilde{x}) := |\varepsilon|_{\frac{1}{\theta} \tilde{\Sigma}^{-1}}^2 + |\tilde{x}|_{\tilde{\Psi}}^2 =: \Upsilon_1(t, \varepsilon) + \Upsilon_2(t, \tilde{x})$$

To derive the differential for Υ , we first obtain the differentials for Υ_1 and Υ_2 :

$$d\Upsilon_1 = \varepsilon' \frac{d}{dt} \left(\frac{1}{\theta} \tilde{\Sigma}^{-1} \right) \varepsilon dt + 2\varepsilon' \frac{1}{\theta} \tilde{\Sigma}^{-1} ((\bar{A} - \tilde{\Sigma} C' N^{-1} C) \varepsilon dt - \theta \tilde{\Sigma} Q \tilde{x} dt - \theta \tilde{\Sigma} P u dt \\ + (G - (\tilde{\Sigma} C' + L) N^{-1} E) dw_t) + \text{Tr}(G' - E' N^{-1} (C \tilde{\Sigma} + L')) \frac{1}{\theta} \tilde{\Sigma}^{-1} (G - (\tilde{\Sigma} C' + L) N^{-1} E) dt$$

Note that $\frac{1}{\theta} \tilde{\Sigma}^{-1}$ satisfies the following GRDE:

$$\frac{d}{dt} \left(\frac{1}{\theta} \tilde{\Sigma}^{-1} \right) + \frac{1}{\theta} \tilde{\Sigma}^{-1} (\bar{A} - \tilde{\Sigma} C' N^{-1} C) + (\bar{A} - \tilde{\Sigma} C' N^{-1} C)' \frac{1}{\theta} \tilde{\Sigma}^{-1} \\ + \frac{1}{\theta} \tilde{\Sigma}^{-1} \theta \tilde{M} \frac{1}{\theta} \tilde{\Sigma}^{-1} + Q + \frac{1}{\theta} C' N^{-1} C = 0$$

Hence,

$$d\Upsilon_1 = -\varepsilon' \left(\frac{1}{\theta} \tilde{\Sigma}^{-1} \tilde{M} \tilde{\Sigma}^{-1} + Q + \frac{1}{\theta} C' N^{-1} C \right) \varepsilon dt - 2\varepsilon' Q \tilde{x} dt - 2\varepsilon' P u dt + 2\varepsilon' \frac{1}{\theta} \tilde{\Sigma}^{-1} (G \\ - (\tilde{\Sigma} C' + L) N^{-1} E) dw_t + \text{Tr}(G' - E' N^{-1} (C \tilde{\Sigma} + L')) \frac{1}{\theta} \tilde{\Sigma}^{-1} (G - (\tilde{\Sigma} C' + L) N^{-1} E) dt$$

It is proven in Appendix B that the matrix $\tilde{\Psi}$ satisfies the following GRDE:

$$\dot{\tilde{\Psi}} + \tilde{\Psi} (A + \theta \tilde{\Sigma} Q) + (A + \theta \tilde{\Sigma} Q)' \tilde{\Psi} + Q + \theta \tilde{\Psi} (\tilde{\Sigma} C' + L) N^{-1} (C \tilde{\Sigma} + L') \tilde{\Psi} \\ - ((I - \theta \tilde{Z} \tilde{\Sigma})^{-1} P + \tilde{\Psi} B) R^{-1} (P' (I - \theta \tilde{\Sigma} \tilde{Z})^{-1} + B' \tilde{\Psi}) = 0 \quad (3.11)$$

This leads to the following expression for the differential of Υ_2 :

$$d\Upsilon_2 = -\tilde{x}' (Q + \theta \tilde{\Psi} (\tilde{\Sigma} C' + L) N^{-1} (C \tilde{\Sigma} + L') \tilde{\Psi} - ((I - \theta \tilde{Z} \tilde{\Sigma})^{-1} P + \tilde{\Psi} B) R^{-1} (P' (I \\ - \theta \tilde{\Sigma} \tilde{Z})^{-1} + B' \tilde{\Psi})) \tilde{x} dt + 2\tilde{x}' \tilde{\Psi} (B + \theta \tilde{\Sigma} P) u dt + 2\tilde{x}' \tilde{\Psi} (\tilde{\Sigma} C' + L) N^{-1} C \varepsilon dt \\ + 2\tilde{x}' \tilde{\Psi} (\tilde{\Sigma} C' + L) N^{-1} E dw_t + \text{Tr}(E' N^{-1} (C \tilde{\Sigma} + L') \tilde{\Psi} (\tilde{\Sigma} C' + L) N^{-1} E) dt$$

Thus, the differential for Υ is, see Appendix B for details of the derivation:

$$d\Upsilon = \text{Tr}(E' N^{-1} (C \tilde{\Sigma} + L') \tilde{\Psi} (\tilde{\Sigma} C' + L) N^{-1} E + (G' - E' N^{-1} (C \tilde{\Sigma} + L')) \frac{1}{\theta} \tilde{\Sigma}^{-1} (G \\ - (\tilde{\Sigma} C' + L) N^{-1} E)) dt - (x' Q x + 2x' P u + u' R u) dt + |\tilde{u}|_R^2 dt + \frac{2}{\theta} (\varepsilon' \tilde{\Sigma}^{-1} G \\ - \varepsilon' \tilde{\Sigma}^{-1} (\tilde{\Sigma} C' + L) N^{-1} E dw_t - \frac{1}{\theta} |G' \tilde{\Sigma}^{-1} \varepsilon - E' N^{-1} (C \tilde{\Sigma} + L') \tilde{\Sigma}^{-1} e|^2 dt \quad (3.12)$$

⁵For a precise definition see [5]. This property is also referred to as *strong time consistency* [20].

Adding the identically zero quantity $(2/\theta) \int_{t_0}^{t_f} d\Upsilon - (2/\theta)(\Upsilon(t_f, \varepsilon(t_f), \dot{x}(t_f)) + \Upsilon(t_0, \varepsilon(t_0), \dot{x}_0))$ to the exponent of $J_{I\theta}$, yields after some re-arrangement:

$$J_{I\theta} = |\bar{x}_0|_{\bar{\Psi}}^2 + \int_{t_0}^{t_f} \text{Tr}(\bar{\Psi}(\bar{\Sigma}C' + L)N^{-1}(C\bar{\Sigma} + L') + \frac{1}{\theta}\bar{\Sigma}^{-1}(\bar{M} + \bar{\Sigma}C'N^{-1}C\bar{\Sigma})) dt + \frac{2}{\theta} \ln\{E\{\exp\{\frac{1}{2}|\varepsilon(t_0)|_{\Sigma_0^{-1}}^2 - \frac{1}{2}|e(t_f)|_{\Sigma(t_f)^{-1}-\theta Q_f}^2 + \frac{\theta}{2} \int_{t_0}^{t_f} |\bar{u}|_R^2 dt\} \zeta(t_f)\}\} \quad (3.13)$$

Introduce a change of probability [19]:

$$\frac{d\tilde{P}}{dP} = \zeta(t_f) \quad (3.14)$$

The measure \tilde{P} is a probability measure for all $\mu_I \in \mathcal{M}_I$, since $\zeta(t)$ is a martingale on $[t_0, t_f]$ by the definition of \mathcal{M}_I .

Under the new probability measure \tilde{P} , the process v_t , defined by:

$$v_t := w_t - \int_{t_0}^{t_f} (G'\bar{\Sigma}^{-1}\varepsilon - E'N^{-1}(C\bar{\Sigma} + L')\bar{\Sigma}^{-1}e) dt$$

is a standard Wiener process starting at t_0 , and it is independent of x_0 .

It is straightforward to derive the following expression for the stochastic differential equation satisfied by y , under the new measure \tilde{P} :

$$dy = (C + \theta L'\bar{Z})\hat{x} dt + E dv_t$$

Hence, we conclude that

$$Y_{t_0}^t := \sigma\{y_s : t_0 \leq s \leq t\} = \sigma\{Ev_s : t_0 \leq s \leq t\}$$

and $Y_{t_0}^t$ is independent to x_0 , for each $t \in [t_0, t_f]$.

Let the expectation with respect to the probability measure \tilde{P} be denoted by \tilde{E} . Then,

$$\begin{aligned} \bar{J}_{I\theta} &:= \frac{2}{\theta} \ln\{E\{\exp\{\frac{1}{2}|\varepsilon(t_0)|_{\Sigma_0^{-1}}^2 - \frac{1}{2}|e(t_f)|_{\Sigma(t_f)^{-1}-\theta Q_f}^2 + \frac{\theta}{2} \int_{t_0}^{t_f} |\bar{u}|_R^2 dt\} \zeta(t_f)\}\} \\ &= \frac{2}{\theta} \ln\{\tilde{E}\{\exp\{\frac{1}{2}|\varepsilon(t_0)|_{\Sigma_0^{-1}}^2 - \frac{1}{2}|e(t_f)|_{\Sigma(t_f)^{-1}-\theta Q_f}^2 + \frac{\theta}{2} \int_{t_0}^{t_f} |\bar{u}|_R^2 dt\}\}\} \\ &= \frac{2}{\theta} \ln\{\tilde{E}\{\tilde{E}\{\exp\{\frac{1}{2}|\varepsilon(t_0)|_{\Sigma_0^{-1}}^2 - \frac{1}{2}|e(t_f)|_{\Sigma(t_f)^{-1}-\theta Q_f}^2 + \frac{\theta}{2} \int_{t_0}^{t_f} |\bar{u}|_R^2 dt\} | Y_{t_0}^{t_f}\}\}\} \\ &= \frac{2}{\theta} \ln\{\tilde{E}\{\exp\{\frac{\theta}{2} \int_{t_0}^{t_f} |\bar{u}|_R^2 dt\} \tilde{E}\{\exp\{\frac{1}{2}|\varepsilon(t_0)|_{\Sigma_0^{-1}}^2 - \frac{1}{2}|e(t_f)|_{\Sigma(t_f)^{-1}-\theta Q_f}^2\} | Y_{t_0}^{t_f}\}\}\} \end{aligned}$$

We will first obtain an expression for the quantity:

$$J_b := \tilde{E}\{\exp\{\frac{1}{2}|\varepsilon(t_0)|_{\Sigma_0^{-1}}^2 - \frac{1}{2}|e(t_f)|_{\Sigma(t_f)^{-1}-\theta Q_f}^2\} | Y_{t_0}^{t_f}\}$$

Toward that end, we first derive a differential equation for e in terms of v_t :

$$de = (\bar{A} + \bar{M}\bar{\Sigma}^{-1})e dt + (B - (I - \theta\bar{\Sigma}\bar{Z})^{-1}(B + \theta\bar{\Sigma}P))\bar{u} dt + (L - (I - \theta\bar{\Sigma}\bar{Z})^{-1}(\bar{\Sigma}C' + L))N^{-1}E dv_t + (G - LN^{-1}E) dv_t$$

Note that the processes $\{Ev_t\}_{t_0 \leq t \leq t_f}$ and $\{(G - LN^{-1}E)v_t\}_{t_0 \leq t \leq t_f}$ are independent, since $(G - LN^{-1}E)E' = 0$. Then, we can decompose e into $e = \hat{e} + \bar{e}$, where

$$\begin{aligned} d\hat{e} &= (\bar{A} + \bar{M}\bar{\Sigma}^{-1})\hat{e} dt + (B - (I - \theta\bar{\Sigma}\bar{Z})^{-1}(B + \theta\bar{\Sigma}P))\bar{u} dt + (L - (I - \theta\bar{\Sigma}\bar{Z})^{-1}(\bar{\Sigma}C' + L))N^{-1}E dv_t \\ d\bar{e} &= (\bar{A} + \bar{M}\bar{\Sigma}^{-1})\bar{e} dt + (G - LN^{-1}E) dv_t; \quad \bar{e}(t_0) = \varepsilon(t_0) \end{aligned}$$

The process \hat{e} belongs to $Y_{t_0}^{t_f}$ and the process \tilde{e} is independent of $Y_{t_0}^{t_f}$. Therefore, the process \hat{e} is the conditional expectation of the process e given $Y_{t_0}^{t_f}$. The conditional distribution of the vector $[\tilde{e}(t_0)' e(t_f)']'$, given the measurement sigma-field $Y_{t_0}^{t_f}$, is Gaussian with mean and covariance,

$$\begin{bmatrix} 0 \\ \hat{e}(t_f) \end{bmatrix}; \begin{bmatrix} \Sigma_0 & D(t_f)' \\ D(t_f) & \Phi(t_f) \end{bmatrix} =: \Lambda, \quad \text{respectively,}$$

where $D(t)$ satisfies:

$$\dot{D} = (\tilde{A} + \tilde{M}\tilde{\Sigma}^{-1})D; \quad D(t_0) = \Sigma_0$$

and $\Phi(t)$ is the solution to the following differential Lyapunov equation:

$$\dot{\Phi} = (\tilde{A} + \tilde{M}\tilde{\Sigma}^{-1})\Phi + \Phi(\tilde{A} + \tilde{M}\tilde{\Sigma}^{-1})' + \tilde{M}; \quad \Phi(t_0) = \Sigma_0.$$

Hence, J_b can be evaluated as:

$$\begin{aligned} J_b &= \int_{R^{2n}} \frac{1}{(2\pi)^n (\det(\Lambda))^{1/2}} \exp\left\{\frac{1}{2}|\tilde{e}(t_0)|_{\Sigma_0^{-1}}^2 - \frac{1}{2}|\tilde{e}(t_f) + \hat{e}(t_f)|_{\tilde{\Sigma}(t_f)^{-1} - \theta Q_f}^2\right. \\ &\quad \left. - \frac{1}{2} \left\| \begin{bmatrix} \tilde{e}(t_0) \\ \tilde{e}(t_f) \end{bmatrix} \right\|_{\Lambda^{-1}}^2\right\} d\tilde{e}(t_0) d\tilde{e}(t_f) = \int_{R^{2n}} \frac{1}{(2\pi)^n (\det(\Lambda))^{1/2}} \exp\left\{-\frac{1}{2} \left\| \begin{bmatrix} \tilde{e}(t_0) \\ \tilde{e}(t_f) \end{bmatrix} \right\|_{\tilde{\Lambda}^{-1}}^2\right\} \\ &\quad + \tilde{\Lambda} \begin{bmatrix} 0 \\ (\tilde{\Sigma}^{-1} - \theta Q_f)\hat{e}(t_f) \end{bmatrix} \Big|_{\tilde{\Lambda}^{-1}}^2\right\} d\tilde{e}(t_0) d\tilde{e}(t_f) = \sqrt{\frac{\det(\tilde{\Lambda})}{\det(\Lambda)}} \end{aligned}$$

where

$$\begin{aligned} \tilde{\Lambda} &= \begin{bmatrix} \Sigma_0^{-1} D' \tilde{\Phi}^{-1} D \Sigma_0^{-1} & -\Sigma_0^{-1} D' \tilde{\Phi}^{-1} \\ -\tilde{\Phi}^{-1} D \Sigma_0^{-1} & \tilde{\Phi}^{-1} + \tilde{\Sigma}^{-1} - \theta Q_f \end{bmatrix}^{-1} \Big|_{t=t_f} \\ \tilde{\Phi} &= \Phi - D \Sigma_0^{-1} D' \end{aligned}$$

Thus, the cost function can be written as follows:

$$\begin{aligned} J_{I\theta} &= |\bar{x}_0|_{\tilde{\Psi}}^2 + \int_{t_0}^{t_f} \text{Tr}(\tilde{\Psi}(\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L') + \frac{1}{\theta}\tilde{\Sigma}^{-1}(\tilde{M} + \tilde{\Sigma}C'N^{-1}C\tilde{\Sigma})) dt \\ &\quad + \frac{1}{\theta} \ln(\det(\tilde{\Lambda})) - \frac{1}{\theta} \ln(\det(\Lambda)) + \frac{2}{\theta} \ln\{E\{\exp\{\frac{\theta}{2} \int_{t_0}^{t_f} |\tilde{u}|_R^2 dt\}\}\} \\ &\geq |\bar{x}_0|_{\tilde{\Psi}}^2 + \int_{t_0}^{t_f} \text{Tr}(\tilde{\Psi}(\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L') + \frac{1}{\theta}\tilde{\Sigma}^{-1}(\tilde{M} + \tilde{\Sigma}C'N^{-1}C\tilde{\Sigma})) dt \\ &\quad + \frac{2}{\theta} \ln(J_b) \end{aligned} \tag{3.15}$$

The controller (3.9) achieves the lower bound above, and hence is optimal. It is easy to see that the controller (3.9) is also conditionally optimal.

To show that the lower bound in (3.15) is indeed the same as (3.10), we first note that

$$\det(\Lambda) = \det(\Sigma_0) \det(\tilde{\Phi}(t_f))$$

and

$$\det(\tilde{\Lambda}) = \frac{1}{\det(\Sigma_0^{-1} D' \tilde{\Phi}^{-1} D \Sigma_0^{-1}) \det(\tilde{\Sigma}^{-1} - \theta Q_f)}$$

Then,

$$J_b = \sqrt{\frac{\det(\Sigma_0) \det(\tilde{\Sigma}(t_f))}{\det(I - \theta \tilde{\Sigma}(t_f) Q_f) \det(D(t_f))}} \frac{1}{\det(D(t_f))}$$

From the differential equations for $D(t)$ and $\tilde{\Sigma}$, we obtain the following differential equations for $\det(D(t))$ and $\det(\tilde{\Sigma})$:

$$\begin{aligned} \frac{d}{dt} \det(D) &= \text{Tr}(\tilde{A} + \tilde{M} \tilde{\Sigma}^{-1}) \det(D) \\ \frac{d}{dt} \det(\tilde{\Sigma}) &= -\text{Tr}(2\tilde{A} + \tilde{M} \tilde{\Sigma}^{-1} - \tilde{\Sigma} \tilde{R}) \det(\tilde{\Sigma}) \end{aligned}$$

Using these in the expression for J_b , and further substituting the resulting expression into (3.15), the desired result (3.10) follows.

This completes the proof. \square

Remark 1 We observe that the optimal controller obtained for the LEQG problem is precisely the central controller for the H^∞ -optimal control problem [2]. The preceding Theorem also subsumes the main result of [5] as a special case, and obtains it under less restrictive conditions. \square

Remark 2 Theorem 1 also holds when $\Sigma_0 \geq 0$, if we restrict the set of admissible controllers to be the set of linear controllers. This generalization can be proved via a standard perturbation analysis (by first replacing Σ_0 by $\Sigma_0 + \rho I$, $\rho > 0$, and then letting $\rho \downarrow 0$). \square

3.2 The Infinite-Horizon Problem

To study the infinite-horizon case, we take A, B, G, Q, P, R, C, E to be time-invariant, and $Q_f = 0, t_0 = 0$. Consider the time-average cost function:

$$J_{I\infty}(\mu_{I\infty}) = \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (x' Q x + u' u) dt \right) \right] \right\} \right\} \quad (3.16)$$

Introduce two GAREs:

$$\tilde{A}' \tilde{Z}_\infty + \tilde{Z}_\infty \tilde{A} - \tilde{Z}_\infty \tilde{S} \tilde{Z}_\infty + \tilde{Q} = 0 \quad (3.17)$$

and

$$\tilde{\Sigma}_\infty \tilde{A}' + \tilde{A} \tilde{\Sigma}_\infty - \tilde{\Sigma}_\infty \tilde{R} \tilde{\Sigma}_\infty + \tilde{M} = 0 \quad (3.18)$$

Define the quantity:

$$\theta_{I\infty}^* := \sup \{ \theta \in \mathbb{R} : \text{the GRDEs (3.17) and (3.18) admit minimal positive definite solutions } \tilde{Z}_\infty \text{ and } \tilde{\Sigma}_\infty, \text{ respectively, such that the matrix } I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty \text{ has only positive eigenvalues.} \} \quad (3.19)$$

For $\theta < \theta_{I\infty}^*$, we introduce the filter:

$$d\tilde{x} = (A + \theta \tilde{\Sigma}_\infty Q) \tilde{x} dt + (B + \theta \tilde{\Sigma}_\infty P) u dt + (\tilde{\Sigma}_\infty C' + L) N^{-1} (dy - C \tilde{x} dt) \quad (3.20)$$

with the initial state $\tilde{x}(t_0) = \tilde{x}_0$. Let $\hat{x} := (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} \tilde{x}$; then we can easily show, as in the finite horizon case, that \hat{x} is generated by the following differential equation:

$$\begin{aligned} d\hat{x} &= (\tilde{A} - \tilde{S} \tilde{Z}_\infty) \hat{x} dt + (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} (B + \theta \tilde{\Sigma}_\infty P) \tilde{u} dt + (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} (\tilde{\Sigma}_\infty C' \\ &\quad + L) N^{-1} (dy - (C + \theta L' \tilde{Z}_\infty) \hat{x} dt) \end{aligned} \quad (3.21)$$

where $\tilde{u} = u + R^{-1}(B'\tilde{Z}_\infty + P')\hat{x}$.

Suppose $\Sigma_0 > 0$, but $\Sigma_0 \leq \tilde{\Sigma}_\infty$; then, the solution to GRDE (3.4) converges to $\tilde{\Sigma}_\infty$ exponentially as $t \rightarrow \infty$.

We will define the set of admissible controllers \mathcal{M}_I to be all mappings $\mu_I : [0, \infty) \times \mathcal{H}_y \rightarrow \mathcal{H}_u$ that are admissible for every finite-horizon problem with initial time 0 and final time $t_f > 0$, for all $t_f \in \mathbb{R}^+$.

Then, we have the following result:

Theorem 2 Consider the general LEQG problem described by (3.1), (3.16), with the matrices A , B , G , Q , P , R , C , E being time-invariant. Let $\Sigma_0 > 0$, $R > 0$, $Q - PR^{-1}P' \geq 0$ and $N > 0$, and assume that the pairs (A, B) and (A, G) are controllable, and the pairs (A, C) and (A, Q) are observable. For each $\theta < \theta_{I\infty}^*$, if $\Sigma_0 \leq \tilde{\Sigma}_\infty$, then the optimal controller is given by

$$u_{I\infty}^* = \mu_{I\infty}^*(t, y_{[t_0, t]}) = -R^{-1}(B'\tilde{Z}_\infty + P')(I - \theta\tilde{\Sigma}_\infty\tilde{Z}_\infty)^{-1}\tilde{x} = -R^{-1}(B'\tilde{Z}_\infty + P')\hat{x} \quad (3.22)$$

where \tilde{x} is generated by the filter (3.20), or equivalently, \hat{x} is generated by filter (3.21). The optimal cost can be written as:

$$\begin{aligned} J_{I\infty}^* &= \inf_{\mu_{I\infty} \in \mathcal{M}_I} J_{I\infty}(\mu_{I\infty}) \\ &= \text{Tr}(\tilde{\Sigma}_\infty Q + (\tilde{\Sigma}_\infty C' + L)N^{-1}(C\tilde{\Sigma}_\infty + L')\tilde{Z}_\infty(I - \theta\tilde{\Sigma}_\infty\tilde{Z}_\infty)^{-1}) \end{aligned} \quad (3.23)$$

Proof By Theorem 1, we have the following relationships, for any admissible controller:

$$\begin{aligned} \frac{2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (x'Qx + u'u) dt \right) \right] \right\} \right\} &\geq \frac{1}{t_f} (\bar{x}_0' \tilde{Z}^{t_f}(0) (I - \theta \Sigma_0 \tilde{Z}^{t_f}(0))^{-1} \bar{x}_0 \\ &- \frac{1}{\theta} \ln \{ \det(I - \theta \tilde{\Sigma}^{t_f}(t_f) Q_f) \} + \int_0^{t_f} \text{Tr}(\tilde{\Sigma}^{t_f} Q + (\tilde{\Sigma}^{t_f} C' + L)N^{-1}(C\tilde{\Sigma}^{t_f} + L')\tilde{Z}^{t_f} \\ &\quad (I - \theta \tilde{\Sigma}^{t_f} \tilde{Z}^{t_f})^{-1}) dt) \end{aligned}$$

where \tilde{Z}^{t_f} and $\tilde{\Sigma}^{t_f}$ are the solutions to GRDE (3.3) and (3.4), respectively, on the time interval $[0, t_f]$. Hence,

$$J_{I\infty}(\mu_{I\infty}) \geq \text{Tr}(\tilde{\Sigma}_\infty Q + (\tilde{\Sigma}_\infty C' + L)N^{-1}(C\tilde{\Sigma}_\infty + L')\tilde{Z}_\infty(I - \theta\tilde{\Sigma}_\infty\tilde{Z}_\infty)^{-1}) = J_{I\infty}^*$$

since, \tilde{Z}^{t_f} converges to \tilde{Z}_∞ as $t_f \rightarrow \infty$ and $\tilde{\Sigma}^{t_f}$ converges to $\tilde{\Sigma}_\infty$ as $t \rightarrow \infty$ exponentially.

To prove the theorem, it is sufficient to show that the controller defined by (3.22) and (3.20), or equivalently, the one given by (3.22) and (3.21), achieves a performance level that is equal to $J_{I\infty}^*$ given by (3.23).

Let

$$\begin{aligned} \bar{J}_{I\theta}^{t_f}(\mu_{I\infty}) &:= \frac{2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (x'Qx + u'u) dt \right) \right] \right\} \right\} \\ \bar{J}_{I\theta}^{t_f}(\mu_{I\infty}) &:= \frac{2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (x'Qx + u'u) dt + x(t_f)' \tilde{Z}_\infty x(t_f) \right) \right] \right\} \right\} \end{aligned}$$

Then, clearly $\bar{J}_{I\theta}^{t_f}(\mu_{I\infty}) \geq J_{I\theta}^{t_f}(\mu_{I\infty})$, and

$$J_{I\infty}(\mu_{I\infty}) = \lim_{t_f \rightarrow \infty} \frac{1}{t_f} J_{I\theta}^{t_f}(\mu_{I\infty})$$

Hence, we have

$$J_{I\theta\infty}(\mu_{I\infty}) \leq \lim_{t_f \rightarrow \infty} \frac{1}{t_f} \bar{J}_{I\theta}^{t_f}(\mu_{I\infty})$$

By the proof of Theorem 1, and in particular by (3.13), we have the identity:

$$\begin{aligned} \bar{J}_{I\theta}^{t_f}(\mu_{I\infty}) &= |\bar{x}_0|_{\bar{\Psi}_\infty}^2 + \int_0^{t_f} \text{Tr}(\bar{\Psi}_\infty(\bar{\Sigma}_\infty C' + L)N^{-1}(C\bar{\Sigma}_\infty + L') + \frac{1}{\theta}\bar{\Sigma}_\infty^{-1}(\bar{M} + \bar{\Sigma}_\infty C' \\ &\quad \cdot N^{-1}C\bar{\Sigma}_\infty) dt + \frac{2}{\theta} \ln\{E\{\exp\{\frac{1}{2}|\varepsilon(t_0)|_{\bar{\Sigma}_\infty^{-1}}^2 - \frac{1}{2}|e(t_f)|_{\bar{\Sigma}_\infty^{-1} - \theta\bar{Z}_\infty}^2} \\ &\quad + \frac{\theta}{2} \int_0^{t_f} |\bar{u}|_R^2 dt\} \zeta(t_f)\}) \end{aligned} \quad (3.24)$$

where $\varepsilon := x - \bar{x}$, $e := x - \hat{x}$, $\bar{u} := u + R^{-1}(B'\bar{Z}_\infty + P')\hat{x}$, $\bar{\Psi}_\infty := \bar{Z}_\infty(I - \theta\bar{\Sigma}_\infty\bar{Z}_\infty)^{-1}$ and $\zeta(t)$ is defined by:

$$\begin{aligned} \zeta(t) &= \exp\left\{\int_0^t (\varepsilon'\bar{\Sigma}_\infty^{-1}G - e'\bar{\Sigma}_\infty^{-1}(\bar{\Sigma}_\infty C' + L)N^{-1}Edw_t \right. \\ &\quad \left. - \frac{1}{2} \int_0^t |G'\bar{\Sigma}_\infty^{-1}\varepsilon - E'N^{-1}(C\bar{\Sigma}_\infty + L')\bar{\Sigma}_\infty^{-1}e|^2 dt\right\} \end{aligned}$$

It is clear that, under the controller $\mu_{I\infty}^*$, as defined by (3.22) and (3.20) (as well as (3.21)), the process $\zeta(t)$ is a martingale on $[0, t_f]$. Introduce a change of probability:

$$\frac{d\bar{P}}{dP} = \zeta(t_f)$$

Then, under the new measure,

$$\begin{aligned} \bar{J}_{I\theta}^{t_f}(\mu_{I\infty}^*) &= |\bar{x}_0|_{\bar{\Psi}_\infty}^2 + \int_0^{t_f} \text{Tr}(\bar{\Psi}_\infty(\bar{\Sigma}_\infty C' + L)N^{-1}(C\bar{\Sigma}_\infty + L') + \frac{1}{\theta}\bar{\Sigma}_\infty^{-1}(\bar{M} + \bar{\Sigma}_\infty C' \\ &\quad N^{-1}C\bar{\Sigma}_\infty) dt + \frac{2}{\theta} \ln\{\bar{E}\{\exp\{\frac{1}{2}|\varepsilon(t_0)|_{\bar{\Sigma}_\infty^{-1}}^2 - \frac{1}{2}|e(t_f)|_{\bar{\Sigma}_\infty^{-1} - \theta\bar{Z}_\infty}^2\}\}\} \end{aligned}$$

This leads to the inequality:

$$J_{I\theta\infty}(\mu_{I\infty}^*) \leq \lim_{t_f \rightarrow \infty} \frac{1}{t_f} \bar{J}_{I\theta}^{t_f}(\mu_{I\infty}^*) \leq J_{I\theta\infty}^*$$

from which the theorem follows. \square

Remark 3 The Theorem can be generalized to the case when $\Sigma_0 \geq 0$, if we restrict the set of admissible controllers to linear controllers, in view of Remark 2 and the independence of the solution on the initial condition.

4 Model Simplification under Perfect State Measurements

We now return to the original goal of this paper, which is the derivation of the optimal solution as $\epsilon \rightarrow 0$, via model simplification. We first consider the perfect state measurements case, where we take $\alpha = 0$. Toward the end of obtaining ϵ -free solutions, we first decompose the system into slow and fast modes as in [12].

4.1 Time-Scale Decomposition

Slow subsystem

The slow subsystem is obtained by letting $\epsilon = 0$ and solving for x_2 (to be denoted \bar{x}_2) in terms of $x_1 =: x_s$, $u =: u_s$, and under the working assumption A3:

$$\bar{x}_2 = -A_{22}^{-1}(A_{21}x_s + B_2u_s). \quad (4.1)$$

Using this in the first equation of (2.1), we obtain the reduced-order (slow) dynamics:

$$dx_{st} = (A_0x_{st} + B_0u_{st}) dt + G_1 dw_t \quad (4.2)$$

where $A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_0 := B_1 - A_{12}A_{22}^{-1}B_2$. Using (4.1) also in the cost function (2.2) leads to the reduced (slow) cost (with $x_1 = x_s$):

$$J_{s\theta}(\mu_s) = \frac{2}{\theta} \ln \{ E \{ \exp \left[\frac{\theta}{2} (|x_s(t_f)|_{Q_{f11}}^2 + \int_{t_0}^{t_f} (|x_s|_{Q_{11}}^2 + x'_s Q_{12} \bar{x}_2 + \bar{x}'_2 Q_{21} x_s + |\bar{x}_2|_{Q_{22}}^2 + |u_s|^2) dt) \right] \} \} \} \quad (4.3)$$

We introduce the following transformation to cancel out cross terms between x_s and u_s :

$$\begin{aligned} \tilde{u}_s = & (I + B'_2 A'_{22}{}^{-1} Q_{22} A_{22}^{-1} B_2)^{1/2} [u_s + (I + B'_2 A'_{22}{}^{-1} Q_{22} A_{22}^{-1} B_2)^{-1} \\ & \cdot B'_2 A'_{22}{}^{-1} Q_{22} A_{22}^{-1} ((A_{21} - A_{22} Q_{22}^{-1} Q_{21}) x_s)] \end{aligned} \quad (4.4)$$

Then, we arrive at the following standard LEQG problem, which has no cross terms between state and control in the cost (see [12] for detailed derivations):

$$dx_{st} = (\tilde{A}_0 x_{st} + \tilde{B}_0 \tilde{u}_{st}) dt + G_1 dw_t; \quad x_s(t_0) = x_{10} \quad (4.5)$$

$$J_{s\theta}(\mu_s) = \frac{2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(|x_s(t_f)|_{Q_{f11}}^2 + \int_{t_0}^{t_f} (|x_s|_{\tilde{Q}}^2 + |\tilde{u}_s|^2) dt \right) \right] \right\} \right\} \quad (4.6)$$

The coefficient matrices above are explicit functions of the parameter θ , and are written as:

$$\begin{aligned} \tilde{A}_0(\theta) = & A_{11} - A_{12} Q_{22}^{-1} Q_{21} - (S_{12} + A_{12} Q_{22}^{-1} A'_{22})(S_{22} + A_{22} Q_{22}^{-1} A'_{22})^{-1} \\ & \cdot (A_{21} - A_{22} Q_{22}^{-1} Q_{21}) \end{aligned} \quad (4.7)$$

$$\tilde{B}_0 = B_0 (I + B'_2 A'_{22}{}^{-1} Q_{22} A_{22}^{-1} B_2)^{-1/2} \quad (4.8)$$

$$\begin{aligned} \tilde{Q} = & Q_{11} - Q_{12} Q_{22}^{-1} Q_{21} + (A'_{21} - Q_{12} Q_{22}^{-1} A'_{22})(S_{22} + A_{22} Q_{22}^{-1} A'_{22})^{-1} \\ & \cdot (A_{21} - A_{22} Q_{22}^{-1} Q_{21}) \end{aligned} \quad (4.9)$$

The above LEQG problem admits an optimal solution if the following GRDE

$$\dot{\tilde{Z}}_s + \tilde{A}'_0 \tilde{Z}_s + \tilde{Z}_s \tilde{A}_0 - \tilde{Z}_s (\tilde{B}_0 \tilde{B}'_0 - \theta G_1 G'_1) \tilde{Z}_s + \tilde{Q} = 0; \quad \tilde{Z}_s(t_f) = Q_{f11}. \quad (4.10)$$

admits a nonnegative definite solution on $[t_0, t_f]$. Let us introduce $S_0 := \tilde{B}_0 \tilde{B}'_0 - \theta G_1 G'_1$, which can be rewritten in terms of the original system matrices as follows (see [12] for details):

$$\begin{aligned} S_0 = & S_{11} + A_{12} Q_{22}^{-1} A'_{12} - (S_{12} + A_{12} Q_{22}^{-1} A'_{22})(S_{22} + A_{22} Q_{22}^{-1} A'_{22})^{-1} \\ & \cdot (S_{21} + A_{22} Q_{22}^{-1} A'_{12}) \end{aligned} \quad (4.11)$$

In view of this, let us define

$$\theta_s := \sup\{\theta \in \mathbb{R} : \text{the GRDE (4.10) admits a nonnegative definite solution on } [t_0, t_f]\} \quad (4.12)$$

Then, the transformed LEQG admits an optimal solution if $\theta < \theta_s$. For $\theta < \theta_s$, let $Z_{s\theta}$ be the unique nonnegative definite solution of (4.10). Then, the optimal controller is given by

$$\tilde{u}_{s\theta}^* = \tilde{\mu}_{s\theta}^*(t, x_s(t)) = -\tilde{B}_0' Z_{s\theta} x_s(t) \quad (4.13)$$

Applying the inverse transformation of (4.4) to (4.13) we obtain:

$$u_{s\theta}^* = \mu_{s\theta}^*(t, x_s(t)) = (-B_1' Z_{s\theta} + B_2'(S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1}((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\theta} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})))x_s(t) \quad (4.14)$$

Fast subsystem

To obtain the fast subsystem: let $x_f := x_2 - \bar{x}_2$, $u_f := u - u_s$ and $\tau = \frac{t'-t}{\epsilon}$, where we take t to be frozen, and t' to vary on the same scale as t . We define the fast subsystem and the associated cost (as in the standard regulator problem; see [21]) by:

$$\frac{d}{d\tau} x_f^t = A_{22}(t)x_f^t + B_2(t)u_f^t; \quad x_f^t(0) = x_f(t) \quad (4.15)$$

$$J_{f\theta}^t(\mu_f^t) = \frac{2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^\infty (|x_f^t|^2 Q_{22}(t) + |u_f^t|^2) d\tau \right) \right] \right\} \right\} \quad (4.16)$$

This is a deterministic optimization problem, which admits a optimal controller that does not depend on the parameter θ :

$$\begin{aligned} u_f^{t*}(\tau) &= \mu_f^{t*}(x_f^t(\tau)) = -B_2'(t)Z_f(t)x_f^t(\tau) \\ \Rightarrow \mu_f^{t*}(t) &= \mu_f^{t*}(x_f^t(0)) = -B_2'(t)Z_f(t)x_f(t) = -B_2'(t)Z_f(t)(x_2(t) - \bar{x}_2(t)) \end{aligned} \quad (4.17)$$

where $Z_f(t)$ is the positive definite solution to the ARE ⁶

$$A_{22}'(t)Z_f + Z_f A_{22}(t) + Q_{22}(t) - Z_f S_{22}(t)Z_f = 0 \quad (4.18)$$

Substitute (4.1) and (4.14) into (4.17), to obtain

$$\begin{aligned} \mu_{f\theta}^*(t, x(t)) &= -B_2' Z_f x_2 - B_2' Z_f Q_{22}^{-1} (A_{12}' Z_{s\theta} + Q_{21} - A_{22}' (S_{22} + A_{22}Q_{22}^{-1}A_{22}')^{-1} \\ &\quad \cdot ((S_{21} + A_{22}Q_{22}^{-1}A_{12}')Z_{s\theta} - (A_{21} - A_{22}Q_{22}^{-1}Q_{21})))x_1(t) \end{aligned} \quad (4.19)$$

Also, to introduce the following Lyapunov equation, when the matrix $A_{22}(t)$ is Hurwitz:

$$A_{22}'(t)Z_{of} + Z_{of} A_{22}(t) + Q_{22}(t) = 0 \quad (4.20)$$

whose relevance to our problem will be seen shortly.

⁶This ARE admits a positive solution if the pair (A_{22}, B_2) is controllable.

4.2 Composite Controller

We now introduce the composite controller:

$$\mu_{c\theta}^*(t, x) = \mu_{s\theta}^*(t, x) + \mu_{f\theta}^*(t, x) \quad (4.21)$$

where $\mu_{s\theta}^*$ and $\mu_{f\theta}^*$ were defined by (4.14) and (4.19) respectively, for $\theta < \theta_s$. After some manipulations, this composite controller can be rewritten as

$$\mu_{c\theta}^*(t, x) = -B' \begin{bmatrix} Z_{s\theta} & 0 \\ Z_c & Z_f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (4.22)$$

where

$$Z_c := Z_f Q_{22}^{-1} (A'_{12} Z_{s\theta} + Q_{21}) - (I + Z_f Q_{22}^{-1} A'_{22}) (S_{22} + A_{22} Q_{22}^{-1} A'_{22})^{-1} \cdot ((S_{21} + A_{22} Q_{22}^{-1} A'_{12}) Z_{s\theta} - (A_{21} - A_{22} Q_{22}^{-1} Q_{21})) \quad (4.23)$$

4.3 Performances of the Suboptimal Controllers

Now, we are in a position to study the performances attained by the composite controller and the slow controller.

Theorem 3 *For the singularly perturbed system (2.1)–(2.3) with the cost function (2.2), let assumptions A1–A3 be satisfied, the pair $(A_{22}(t), B_2(t))$ be controllable for each $t \in [0, T]$, and the following condition hold:*

$Q_{f22} \leq Z_f(t_f)$, where $Z_f(t_f)$ is the solution to (4.18) at $t = t_f$, with θ fixed.

Then,

1. $\theta^*(\epsilon) \leq \theta_s$, asymptotically as $\epsilon \rightarrow 0^+$.
2. $\forall \theta < \theta_s$, $\exists \epsilon_\theta > 0$ such that $\forall \epsilon \in [0, \epsilon_\theta)$, the GRDE (2.7) admits a nonnegative definite solution, and consequently, the problem has an optimal solution, and the optimal cost for the problem can be approximated by

$$J_\theta^*(\epsilon) = x'_{10} Z_{s\theta}(t_0) x_{10} + \int_{t_0}^{t_f} \text{Tr}(G_1 G'_1 Z_{s\theta} + G_2 G'_2 Z_f) dt + O(\sqrt{\epsilon}) \quad (4.24)$$

3. $\forall \theta < \theta_s$, if we apply the composite controller $\mu_{c\theta}^*$ to the system, then $\exists \epsilon'_\theta > 0$ such that $\forall \epsilon \in [0, \epsilon'_\theta)$,

$$J_\theta^\epsilon := J_\theta(\mu_{c\theta}^*) = J_\theta^*(\epsilon) + O(\sqrt{\epsilon}) \quad (4.25)$$

4. $\forall \theta < \theta_s$, if, in addition, the matrix $A_{22}(t)$ is Hurwitz for every $t \in [t_0, t_f]$, and we apply the slow controller $\mu_{s\theta}^*$ to the system, then $\exists \tilde{\epsilon}_\theta > 0$ such that $\forall \epsilon \in [0, \tilde{\epsilon}_\theta)$,

$$J_\theta^s := J_\theta(\mu_{s\theta}^*) = J_\theta^*(\epsilon) + \int_{t_0}^{t_f} \text{Tr}(G_2 G'_2 (Z_{of} - Z_f)) dt + O(\sqrt{\epsilon}) \quad (4.26)$$

Proof Under the assumptions specified, the result 1) follows from Theorem 2 in [12]. The same Theorem says that there exists an $\epsilon_\theta > 0$, such that the full-order GRDE (2.7) admits a unique nonnegative definite solution, for $\epsilon \in [0, \epsilon_\theta)$, which can be approximated by

$$\tilde{Z} = \begin{bmatrix} Z_{s\theta}(t) + O(\sqrt{\epsilon}) & \epsilon(Z'_c(t) + Z'_{cb}(\tau)) + O(\epsilon^{3/2}) \\ \epsilon(Z_c(t) + Z_{cb}(\tau)) + O(\epsilon^{3/2}) & \epsilon(Z'_f(t) + Z'_{fb}(\tau)) + O(\epsilon^{3/2}) \end{bmatrix}$$

for all $t \in [t_0, t_f]$, where $Z_{cb}(\tau)$ and $Z_{fb}(\tau)$ are boundary layer correction terms, and as $\tau \rightarrow -\infty$, they converge to 0 exponentially in the τ time scale. Thus, the result 2) is also proved.

For 3) and 4), we substitute the controllers $\mu_{c\theta}^*$ and $\mu_{s\theta}^*$ into the full-order system to get the resulting control-free LEQG problems. The system, as well as the controllers, are in forms analogous to those of [12]. Also, the solution to the LEQG problem is the same as the solution to the H^∞ -optimal control problem, except that here we have to compute the cost incurred by the optimal controller. Thus, using the detailed derivations that led to Theorem 2 in [12], we can establish 3) and 4). \square

The infinite-horizon case

We take A, B, D, Q to be time-invariant, and $t_0 = 0, Q_f = 0$, and adopt the cost criterion (2.17). We first decompose the system into slow and fast subsystems as in the finite-horizon case. The slow subsystem is obtained by setting $\epsilon = 0$ in the state equation and the cost function, which leads to \bar{x}_2 as in (4.1), the slow dynamics (4.2) and the following cost function:

$$J_{s\theta\infty}(\mu_s) = \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \{ E \{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (|x_s|^2_{\bar{Q}_{11}} + x'_s Q_{12} \bar{x}_2 + \bar{x}'_2 Q_{21} x_s + |\bar{x}_2|^2_{\bar{Q}_{22}} + |u_s|^2) dt \right) \right] \} \} \quad (4.27)$$

We introduce the transformation (4.4), as in the finite-horizon case, to arrive at the infinite-horizon LEQG problem with state equation (4.5) and cost function

$$J_{s\theta\infty}(\theta) = \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (|x_s|^2_{\bar{Q}} + |\bar{u}_s|^2) dt \right) \right] \right\} \right\} \quad (4.28)$$

This problem admits a solution if the following GARE

$$\bar{A}'_0 \bar{Z}_s + \bar{Z}_s \bar{A}_0 - \bar{Z}_s S_0 \bar{Z}_s + \bar{Q} = 0 \quad (4.29)$$

admits minimal positive definite solution $Z_{s\theta}$, such that the matrix $\bar{A}_0 - S_0 Z_{s\theta}$ is Hurwitz. Let us define

$$\theta_{s\infty} := \sup \{ \theta \in \mathbf{R} : \text{the GARE (4.29) admits a nonnegative definite solution.} \} \quad (4.30)$$

Then, for $\theta < \theta_{s\infty}$, the optimal control for the slow subsystem is (4.13), which leads to (4.14) after the inverse transformation of (4.4).

The fast subsystem is the same as in the finite-horizon case, except that it is now independent of the frozen time instance. We let Z_f still be the solution to the ARE (4.18), and Z_{of} be the solution to the Lyapunov equation (4.20) when A_{22} is Hurwitz. The optimal fast controller is the same as (4.19). Thus, the composite controller can be formed in the same way as in the finite-horizon case, which leads to (4.22) with stationary matrices. This leads to the following theorem:

Theorem 4 For the singularly perturbed system (2.1)–(2.3) with the cost function (2.17), let assumptions A1–A3 be satisfied, the pairs (A_0, B_0) and (A_{22}, B_2) be controllable and the pair $(A_0, Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})$ be observable. Then,

$$1. \lim_{\epsilon \rightarrow 0^+} \theta_{\infty}^*(\epsilon) = \theta_{\infty}.$$

2. $\forall \theta < \theta_{\infty}$, $\exists \epsilon_{\theta} > 0$ such that $\forall \epsilon \in [0, \epsilon_{\theta})$, the GARE (2.18) admits a positive definite solution, and consequently, the problem has an optimal solution, and the optimal cost for the problem can be approximated by

$$J_{\theta\infty}^*(\epsilon) = \text{Tr}(G_1 G_1' Z_{s\theta} + G_2 G_2' Z_f) + O(\sqrt{\epsilon}) \quad (4.31)$$

3. $\forall \theta < \theta_{\infty}$, if we apply the composite controller $\mu_{c\theta}^*$ to the system, then $\exists \epsilon'_{\theta} > 0$ such that $\forall \epsilon \in [0, \epsilon'_{\theta})$,

$$J_{\theta\infty}^c := J_{\theta\infty}(\mu_{c\theta}^*) = J_{\theta\infty}^*(\epsilon) + O(\sqrt{\epsilon}) \quad (4.32)$$

4. $\forall \theta < \theta_{\infty}$, if, in addition, the matrix A_{22} is Hurwitz, and we apply the slow controller $\mu_{s\theta}^*$ to the system, then $\exists \bar{\epsilon}_{\theta} > 0$ such that $\forall \epsilon \in [0, \bar{\epsilon}_{\theta})$,

$$J_{\theta\infty}^s := J_{\theta\infty}(\mu_{s\theta}^*) = J_{\theta\infty}^*(\epsilon) + \text{Tr}(G_2 G_2' (Z_{of} - Z_f)) + O(\sqrt{\epsilon}) \quad (4.33)$$

Proof By Theorem 1 in [12], result 1) is follows, and there exists an $\epsilon_{\theta} > 0$, such that the full-order GARE (2.18) admits a positive definite solution, for $\epsilon \in [0, \epsilon_{\theta})$, which can be approximated by $\tilde{Z} = \begin{bmatrix} Z_{s\theta} + O(\sqrt{\epsilon}) & \epsilon Z_c' + O(\epsilon^{3/2}) \\ \epsilon Z_c + O(\epsilon^{3/2}) & \epsilon Z_f + O(\epsilon^{3/2}) \end{bmatrix}$. By Corollary 2 in [15], we have that the matrix $\tilde{A}_0 - S_0 Z_{s\theta}$ is Hurwitz for $\theta < \theta_{\infty}$, which leads to 2).

For 3) and 4), we substitute the controllers $\mu_{c\theta}^*$ and $\mu_{s\theta}^*$ into the full-order system to obtain the resulting control-free LEQG problems. By the same reasoning as in the finite-horizon case, we can use the detailed derivations that led to Theorem 1 of [12] to establish 3) and 4). \square

4.4 A Large Deviation Form

Now, we consider a large deviation form of the problem considered in section 2, which is the case when the system noise intensity asymptotically approaches zero. To formally illustrate this situation, we consider the following setup for the problem:

$$\begin{cases} dx_1 &= (A_{11}(t)x_1 + A_{12}(t)x_2) dt + B_1(t)u + \xi G_1(t) dw; & x_1(0) = x_{10} \\ \epsilon dx_2 &= (A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u) dt + \sqrt{\epsilon} \xi G_2(t) dw; & x_2(0) = x_{20} \end{cases} \quad (4.34)$$

with cost function:

$$J_{\theta}(\mu, \xi) = \frac{2\xi^2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2\xi^2} \left(x_{t_f}' Q_f x_{t_f} + \int_{t_0}^{t_f} (x' Q(t)x + u' u) dt \right) \right] \right\} \right\} \quad (4.35)$$

where ξ is small scalar parameter to be varied. We will study the solution as $\xi \rightarrow 0$. Note that the above problem is equivalent to the one considered in Section 2, if we introduce the substitutions:

$$\theta \leftarrow \frac{\theta}{\xi^2}; \quad G_{\epsilon} \leftarrow \xi G_{\epsilon} \quad (4.36)$$

Under assumptions A1–A3, we know that the optimal solution exists, for each fixed $\epsilon > 0$, if the GRDE (2.7) admits a nonnegative definite solution on $[t_0, t_f]$, and the optimal solution is given by (2.8), which does not depend on the value of the parameter ξ . Thus, $\forall \theta < \theta^*(\epsilon)$, where $\theta^*(\epsilon)$ is the quantity we defined in (2.9), the problem considered admits an optimal solution. The optimal cost is given by

$$J_\theta^*(\epsilon, \xi) = x_0' \tilde{Z}(0; \epsilon) x_0 + \xi^2 \int_{t_0}^{t_f} \text{Tr}(G_\epsilon G_\epsilon' \tilde{Z}(t; \epsilon)) dt \quad (4.37)$$

where $\tilde{Z}(t; \epsilon)$ is the solution to the GRDE (2.7). Hence, $J_\theta^*(\epsilon, \xi) \rightarrow x_0' \tilde{Z}(0; \epsilon) x_0$ as $\xi \rightarrow 0$.

Note that the solution depends on the value of ϵ explicitly. To obtain ϵ -free solutions, we decompose the system into slow and fast subsystems as in the previous subsection. The slow subsystem is obtained by setting $\epsilon = 0$ in the state dynamics, as well as in the cost function. After applying the transformation (4.4), we arrive at the standard LEQG problem (4.5)–(4.6), but under the substitutions (4.36). This problem admits an optimal solution if the GRDE (4.10) admits a nonnegative solution $Z_{s\theta}$ on $[t_0, t_f]$. We define the quantity θ_s in exactly the same way as in (4.12). Hence, $\forall \theta < \theta_s$, the slow LEQG problem admits an optimal control (4.13). Applying the inverse transformation of (4.4), we obtain the slow controller (4.14).

To obtain the fast subsystem, we use the same notation as in Subsection 4.1. The fast dynamics is the same as (4.15) and the cost function is the same as (4.16) under the substitutions (4.36). Then, the optimal control is given by (4.17), where Z_f is the positive definite solution to ARE (4.18). Substitution of (4.1) and (4.14) into (4.17) yields the fast controller, which is precisely (4.19). We also introduce the matrix Z_{of} to be the solution of Lyapunov equation (4.20).

Then, we form the composite controller as in (4.21), which leads to (4.22). We now summarize the result below, as a corollary to Theorem 3:

Corollary 1 *For the singularly perturbed system (4.34), (2.3) with cost function (4.35), let assumptions A1–A3 be satisfied, the pair $(A_{22}(t), B_2(t))$ be controllable for each $t \in [t_0, t_f]$, and the following condition hold:*

$Q_{f22} \leq Z_f(t_f)$, where $Z_f(t_f)$ is the solution to (4.18) at $t = t_f$, with θ fixed.

Then,

1. $\theta^*(\epsilon) \leq \theta_s$, asymptotically as $\epsilon \rightarrow 0^+$.
2. $\forall \theta < \theta_s$, $\exists \epsilon_\theta > 0$ such that $\forall \epsilon \in [0, \epsilon_\theta)$, the GRDE (2.7) admits a positive definite solution, and consequently, the problem has an optimal solution, and the optimal cost can be approximated by

$$J_\theta^*(\epsilon, \xi) = x_{10}' Z_{s\theta}(0) x_{10} + \xi^2 \int_{t_0}^{t_f} \text{Tr}(G_1 G_1' Z_{s\theta} + G_2 G_2' Z_f) dt + O(\sqrt{\epsilon}) \quad (4.38)$$

3. $\forall \theta < \theta_s$, if we apply the composite controller $\mu_{c\theta}^*$ to the system, then $\exists \epsilon'_\theta > 0$ such that $\forall \epsilon \in [0, \epsilon'_\theta)$,

$$J_\theta^c(\xi) := J_\theta(\mu_{c\theta}^*, \xi) = J_\theta^*(\epsilon, \xi) + O(\sqrt{\epsilon}) \quad (4.39)$$

4. $\forall \theta < \theta_s$, if, in addition, the matrix $A_{22}(t)$ is Hurwitz, $\forall t \in [0, T]$, and we apply the slow controller $\mu_{s\theta}^*$ to the system, then $\exists \tilde{\epsilon}_\theta > 0$ such that $\forall \epsilon \in [0, \tilde{\epsilon}_\theta)$,

$$J_{\theta}^*(\xi) := J_{\theta}(\mu_{s\theta}^*, \xi) = J_{\theta}^*(\epsilon, \xi) + O(\xi^2) + O(\sqrt{\epsilon}) \quad (4.40)$$

□

The infinite-horizon case

Consider the system described in (4.34), with A, B, D, Q time-invariant, and $t_0 = 0, Q_f = 0$. Furthermore, take the cost function to be as follows, to replace (4.35):

$$J_{\theta\infty}(\mu, \xi) = \lim_{t_f \rightarrow \infty} \frac{2\xi^2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2\xi^2} \left(\int_0^{t_f} (x'Q(t)x + u'R(t)u) dt \right) \right] \right\} \right\} \quad (4.41)$$

Assume that $(A_{\epsilon}, B_{\epsilon})$ is controllable, and (A_{ϵ}, Q) is observable for every $\epsilon > 0$. For each $\epsilon > 0$, the problem admits a solution if GARE (2.18) admits a minimal positive definite solution and the optimal controller is given by (2.19) independent of the parameter ξ as in the finite-horizon case. Note that the equivalence of this problem and the one considered in Section 2 is still true under the substitutions (4.36), where we define the quantity $\theta_{\infty}^*(\epsilon)$ the same way as in (2.20). The optimal cost is given by

$$J_{\theta\infty}^*(\epsilon, \xi) = \xi^2 \text{Tr}(G_{\epsilon} G'_{\epsilon} \tilde{Z}(t; \epsilon)) \quad (4.42)$$

where $\tilde{Z}(\epsilon)$ is the solution to the GARE (2.18). Hence, $J_{\theta\infty}^*(\epsilon, \xi) \rightarrow 0$ as $\xi \rightarrow 0$.

To obtain ϵ -free solutions to the problem, we decompose the system into slow and fast subsystems. The slow subsystem is obtained by setting $\epsilon = 0$ in the state dynamics, as well as in the cost function. After the same transformation as (4.4), we arrive at the standard LEQG problem with state equation (4.5) and cost function (4.28) under the substitutions (4.36). This problem admits an optimal solution if the GARE (4.29) admits a positive definite solution $Z_{s\theta}$, such that the matrix $\tilde{A}_0 - S_0 Z_{s\theta}$ is Hurwitz. We define $\theta_{s\infty}$ the same way as in (4.30). Then, for $\theta < \theta_{s\infty}$, the optimal control for the slow subsystem is (4.13), which leads to (4.14) after the inverse transformation of (4.4).

The fast subsystem is the same as in the finite-horizon case; thus, the optimal control is given by (4.17) where Z_f is the positive definite solution to ARE (4.18). Substitution of (4.1) and (4.14) into (4.17) yields the fast controller, which is exactly (4.19). We also introduce the matrix Z_{of} to be the solution of the Lyapunov equation (4.20).

Then, the composite controller is formed as in (4.21), leading to (4.22). To summarize:

Corollary 2 *For the singularly perturbed system (2.1)–(2.3) with the cost function (4.41), let assumptions A1–A3 be satisfied, the pairs (A_0, B_0) and (A_{22}, B_2) be controllable, and the pair $(A_0, Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})$ be observable. Then,*

1. $\lim_{\epsilon \rightarrow 0^+} \theta_{\infty}^*(\epsilon) = \theta_{s\infty}$.
2. $\forall \theta < \theta_{s\infty}, \exists \epsilon_{\theta} > 0$ such that $\forall \epsilon \in [0, \epsilon_{\theta})$, the GARE (2.18) admits a positive definite solution, and consequently, the problem has an optimal solution, and the optimal cost for the problem can be approximated by

$$J_{\theta\infty}^*(\epsilon, \xi) = \xi^2 (\text{Tr}(G_1 G'_1 Z_{s\theta} + G_2 G'_2 Z_f) + O(\sqrt{\epsilon})) \quad (4.43)$$

3. $\forall \theta < \theta_{s\infty}$, if we apply the composite controller $\mu_{c\theta}^*$ to the system, then $\exists \epsilon'_\theta > 0$ such that $\forall \epsilon \in [0, \epsilon'_\theta)$,

$$J_{\theta\infty}^\epsilon(\xi) := J_{\theta\infty}(\mu_{c\theta}^*, \xi) = J_{\theta\infty}^*(\epsilon, \xi) + O(\xi^2 \sqrt{\epsilon}) \quad (4.44)$$

4. $\forall \theta < \theta_{s\infty}$, if, in addition, the matrix A_{22} is Hurwitz, and we apply the slow controller $\mu_{s\theta}^*$ to the system, then $\exists \bar{\epsilon}_\theta > 0$ such that $\forall \epsilon \in [0, \bar{\epsilon}_\theta)$,

$$J_{\theta\infty}^s(\xi) := J_{\theta\infty}(\mu_{s\theta}^*, \xi) = J_{\theta\infty}^*(\epsilon, \xi) + O(\xi^2) \quad (4.45)$$

□

5 Model Simplification under Imperfect State Measurements

We now turn to the noisy state measurements case. As discussed before, the parameter values are taken as $\alpha = \beta = 1/2$ in this case. Again, we first decompose the system into slow and fast subsystems.

5.1 Time-Scale Decomposition

Slow subsystem

The slow subsystem is obtained by letting $dx_2 = 0$ and solving for $x_2 dt$ (to be denoted $\bar{x}_2 dt$) in terms of x_1, u, dw_t and under the working assumption A3:

$$\bar{x}_2 dt = -A_{22}^{-1}(\sqrt{\epsilon}A_{21}x_1 dt + B_2u dt + \sqrt{\epsilon}G_2 dw_t) \quad (5.1)$$

Using this in (2.1) and denoting $y_s := y$, we obtain:

$$\begin{aligned} dx_1 &= ((A_{11} + O(\sqrt{\epsilon}))x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u) dt + (G_1 + O(\sqrt{\epsilon}))dw_t; \\ x_1(t_0) &= x_{10} \end{aligned} \quad (5.2)$$

$$\begin{aligned} dy_{s1} &= ((C_{11} + O(\sqrt{\epsilon}))x_s - C_{12}A_{22}^{-1}B_2u) dt + (E_1 + \sqrt{\epsilon}C_{12}A_{22}^{-1}G_2)dw_t; \\ y_1(t_0) &= 0 \end{aligned} \quad (5.3)$$

$$\begin{aligned} dy_{s2} &= (\sqrt{\epsilon}(C_{21} - C_{22}A_{22}^{-1}A_{21})x_s - C_{22}A_{22}^{-1}B_2u) dt + \sqrt{\epsilon}(E_2 \\ &\quad - C_{22}A_{22}^{-1}G_2)dw_t; \quad y_2(t_0) = 0 \end{aligned} \quad (5.4)$$

$$\begin{aligned} J_{I\theta}(\mu_s) &= \frac{2}{\theta} \ln \{ E \{ \exp \left[\frac{\theta}{2} (|x_s(t_f)|_{Q_{11}}^2 + O(\epsilon) + \int_{t_0}^{t_f} (|x_s|_{Q_{11}}^2 - 2x'_s Q_{12} \bar{x}_2 + \bar{x}'_2 Q_{21} x_s \right. \\ &\quad \left. + |\bar{x}_2|_{Q_{22}}^2 + |u|^2) dt) \} \} \} \end{aligned} \quad (5.5)$$

For each $\epsilon > 0$, measurements (5.3) and (5.4) are equivalent to the vector measurement \tilde{y} , defined by

$$\tilde{y}_s := \begin{bmatrix} y_{s1} \\ \frac{1}{\sqrt{\epsilon}} y_{s2} \end{bmatrix} + C_{2\epsilon} A_{22}^{-1} B_2 u \quad (5.6)$$

Now set $\epsilon = 0$ in (5.2), (5.5) and (5.6) to arrive at the slow subproblem in terms of $x_s := x_1$, w_t , u and \tilde{y}_s , with $\bar{x}_2 = -A_{22}^{-1}B_2u$, reduced (slow) dynamics:

$$dx_s = (A_{11}x_s + B_0u)dt + G_1dw_t; \quad x_s(t_0) = x_{10}, \quad (5.7)$$

slow measurements:

$$d\tilde{y}_s = C_0x_sdt + E^\square dw_t; \quad \tilde{y}_s(t_0) = 0, \quad (5.8)$$

and reduced (slow) cost function:

$$J_{Is}(\mu_s) = \frac{2}{\theta} \ln \{ E \{ \exp \left[\frac{\theta}{2} (|x_s(t_f)|_{Q_{f11}}^2 + \int_{t_0}^{t_f} (|x_s|_{Q_{11}}^2 - 2x_s' Q_{12} A_{22}^{-1} B_2 u + |u|_{I+B_2' A_{22}^{-1} Q_{22} A_{22}^{-1} B_2}^2) dt) \right] \} \} \} \quad (5.9)$$

where

$$C_0 := C_1 - C_2 A_{22}^{-1} A_{21}; \quad E^\square := E - C_2 A_{22}^{-1} G_2$$

By Theorem 1, the solution to the slow LEQG problem exists if

(i) the following backward GRDE:

$$\dot{Z}_{s\theta} + \bar{A}_s' Z_{s\theta} + Z_{s\theta} \bar{A}_s - Z_{s\theta} \bar{S}_s Z_{s\theta} + \bar{Q}_s = 0; \quad Z_{s\theta}(t_f) = Q_{f11} \quad (5.10)$$

where

$$\begin{aligned} \bar{A}_s &:= A_{11} + B_0(I + B_2' A_{22}^{-1} Q_{22} A_{22}^{-1} B_2)^{-1} B_2' A_{22}^{-1} Q_{21} \\ \bar{S}_s(\theta) &:= B_0(I + B_2' A_{22}^{-1} Q_{22} A_{22}^{-1} B_2)^{-1} B_0' - \theta G_1 G_1' \\ \bar{Q}_s &:= Q_{11} - Q_{12} A_{22}^{-1} B_2(I + B_2' A_{22}^{-1} Q_{22} A_{22}^{-1} B_2)^{-1} B_2' A_{22}^{-1} Q_{21} \end{aligned}$$

admits a nonnegative definite solution $Z_{s\theta}(t)$ on $[t_0, t_f]$,

(ii) the following forward GRDE:

$$\dot{\Sigma}_{s\theta} = \bar{A}_s \Sigma_{s\theta} + \Sigma_{s\theta} \bar{A}_s' - \Sigma_{s\theta} \bar{R}_s \Sigma_{s\theta} + \bar{M}_s; \quad \Sigma_{s\theta}(t_0) = \Sigma_{011} \quad (5.11)$$

where

$$\begin{aligned} \bar{A}_s &:= A_{11} - G_1 E^\square N^{\square-1} C_0; \quad \bar{R}_s := C_0' N^{\square-1} C_0 - \theta Q_{11} \\ \bar{M}_s &:= G_1 G_1' - G_1 E^\square N^{\square-1} E^\square G_1'; \quad N^\square := E^\square E^\square \end{aligned}$$

admits a positive definite solution $\Sigma_{s\theta}$ on $[t_0, t_f]$, and

(iii) the solutions to (5.10) and (5.11) satisfy the spectral radius condition:

$$I - \theta \Sigma_{s\theta}(t) Z_{s\theta}(t) \quad \text{has only positive eigenvalues} \quad \forall t \in [t_0, t_f] \quad (5.12)$$

Hence, let us introduce the quantity:

$$\theta_{Is} := \sup \{ \theta \in \mathbb{R} : \text{the GRDEs (5.10) and (5.11) admit nonnegative definite solutions on } [t_0, t_f], \text{ and further satisfy (5.12).} \} \quad (5.13)$$

For $\theta < \theta_{Is}$, the slow LEQG problem admits an optimal controller, given by:

$$\bar{u}_{Is}^* = \bar{\mu}_{Is}^*(t, \tilde{y}_{s[t_0, t]}) = -(I + B_2' A_{22}^{-1} Q_{22} A_{22}^{-1} B_2)^{-1} (B_0' Z_{s\theta} - B_2' A_{22}^{-1} Q_{21}) \hat{x}_s \quad (5.14)$$

$$\begin{aligned} d\hat{x}_s &= (A_{11} + B_0 u + \theta G_1 G_1' Z_{s\theta}) \hat{x}_s dt + (I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1} (\Sigma_{s\theta} C_0' + G_1 E^\square) N^{\square-1} \\ &\quad (d\tilde{y}_s - C_0 \hat{x}_s dt - E^\square \theta G_1' Z_{s\theta} \hat{x}_s dt); \quad \hat{x}_s(t_0) = (I - \theta \Sigma_{s\theta}(t_0) Z_{s\theta}(t_0))^{-1} \bar{x}_{10} \end{aligned} \quad (5.15)$$

Fast subsystem

To obtain the fast subsystem: let $x_f := x_2 - \bar{x}_2 = x_2 + A_{22}^{-1} B_2 u_{I_s}^*$, $u_f := u - u_s$, $y_f := y - y_s$ and $\tau = \frac{t-t'}{\epsilon}$, where we take t to be frozen, and t' to vary on the same time scale as t . In terms of the equivalent measurements:

$$\tilde{y}_f := \begin{bmatrix} \frac{1}{\sqrt{\epsilon}} y'_{f1} & \frac{1}{\epsilon} y'_{f2} \end{bmatrix} \quad (5.16)$$

we define the fast subsystem and the associated cost, respectively, by:

$$dx_f^t = (A_{22}(t)x_f^t + B_2(t)u_f^t) d\tau + G_2 dw_\tau; \quad x_f^t(0) = x_f(t) \quad (5.17)$$

$$d\tilde{y}_f^t = C_2 x_f^t d\tau + E dw_\tau; \quad y_f^t(0) = 0 \quad (5.18)$$

$$J_{If\theta}^t(\mu_f^t) = E \left(\int_0^\infty (|x_f^t|_{Q_{22}(t)}^2 + |u_f^t|^2) d\tau \right) \quad (5.19)$$

This is a risk-neutral LQG problem, which is independent of the parameter θ . It admits an optimal controller:

$$u_{If}^{t*}(\tau) = \mu_{If}^{t*}(\tau, \tilde{y}_f^t(-\infty, \tau]) = -B_2'(t)Z_f(t)\hat{x}_f^t(\tau) \quad (5.20)$$

$$d\hat{x}_f^t = (A_{22} - S_{22}Z_f)\hat{x}_f^t d\tau + \Sigma_f C_2' N^{-1}(d\tilde{y}_f^t - C_2 \hat{x}_f^t d\tau) \quad (5.21)$$

where Z_f and Σ_f are the nonnegative definite solutions to the AREs:⁷

$$A_{22}'(t)Z_f + Z_f A_{22}(t) + Q_{22}(t) - Z_f S_{22}(t)Z_f = 0 \quad (5.22)$$

and

$$A_{22}(t)\Sigma_f + \Sigma_f A_{22}'(t) + G_2(t)G_2'(t) - \Sigma_f R_{22}(t)\Sigma_f = 0 \quad (5.23)$$

Transforming the control policy μ_{If}^{t*} back to the t time scale, we obtain the fast controller:

$$u_{If}^* = \mu_{If}^* = \mu_{If}^{t*}(x_f^t(0)) = -B_2'(t)Z_f(t)\hat{x}_f(t) \quad (5.24)$$

$$\epsilon d\hat{x}_f = (A_{22} - S_{22}Z_f)x_f dt + \Sigma_f C_2' N^{-1} \left(\begin{bmatrix} \sqrt{\epsilon} y_{f1} \\ y_{f2} \end{bmatrix} - C_2 \hat{x}_f dt \right); \quad \hat{x}_f(t_0) = x_f(t_0) \quad (5.25)$$

Also, we introduce two Lyapunov equations, when the matrix $A_{22}(t)$ is Hurwitz:

$$A_{22}'(t)Z_{of} + Z_{of} A_{22}(t) + Q_{22}(t) = 0 \quad (5.26)$$

$$A_{22}(t)\Sigma_{of} + \Sigma_{of} A_{22}'(t) + G_2(t)G_2'(t) = 0 \quad (5.27)$$

which, as we will shortly see, are relevant to the problem under consideration.

⁷These AREs admit positive definite solutions if the pair $(A_{22}(t), B_2(t))$ is controllable and the pair $(A_{22}(t), C_2(t))$ is observable.

5.2 Performance of Suboptimal Controllers

We now address the performance evaluation under the slow controller and the composite controller (to be defined), when applied to the full-order system, and the resulting degree of suboptimality with respect to the optimal performance of the full-order system. Toward this end, we first simplify the slow controller (5.14)–(5.15) using some straightforward algebraic manipulations that can be found in [13] (derived in the context of the corresponding H^∞ -optimal control problem):

$$\tilde{u}_{I_s}^* = \tilde{\mu}_{I_s}^*(t, \tilde{y}_s[t_0, t]) = -(B_1' Z_{s\theta} + B_2' V) \hat{x}_s, \quad (5.28)$$

$$d\hat{x}_s = (\bar{A}_s - \bar{S}_s Z_{s\theta}) \hat{x}_s dt + B_0(u - \tilde{u}_{I_s}^*) dt + (I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1} (\Sigma_{s\theta} C_1' + Y' C_2') N^{-1} (d\tilde{y}_s - (C_1 - C_2 A_{22}^{-1} (A_{21} + \theta G_2 G_1' Z_{s\theta})) \hat{x}_s dt); \quad \hat{x}_s(t_0) = (I - \theta \Sigma_{s\theta}(t_0) Z_{s\theta}(t_0))^{-1} \bar{x}_{10} \quad (5.29)$$

where

$$Y := Y_1 \Sigma_{s\theta} + Y_2; \quad V := V_1 Z_{s\theta} + V_2 \quad (5.30)$$

$$Y_1 := -(R_{22} + A_{22}' (G_2 G_2)^{-1} A_{22})^{-1} (R_{21} + A_{22}' (G_2 G_2)^{-1} A_{21}) \quad (5.31)$$

$$Y_2 := -(R_{22} + A_{22}' (G_2 G_2)^{-1} A_{22})^{-1} A_{22}' (G_2 G_2')^{-1} G_2 G_1 \quad (5.32)$$

$$V_1 := -(S_{22} + A_{22} Q_{22}^{-1} A_{22}')^{-1} (S_{21} + A_{22} Q_{22}^{-1} A_{12}') \quad (5.33)$$

$$V_2 := -(S_{22} + A_{22} Q_{22}^{-1} A_{22}')^{-1} A_{22} Q_{22}^{-1} Q_{21}$$

Substituting (5.6) into the above expression with $u = \tilde{u}_{I_s}^*$, we obtain the following alternative form for the slow controller:

$$u_{I_s}^* = \mu_{I_s}^*(t, y[t_0, t]) = -(B_1' Z_{s\theta} + B_2' V) \hat{x}_s^* \quad (5.34)$$

$$d\hat{x}_s^* = (\bar{A}_s - \bar{S}_s Z_{s\theta}) \hat{x}_s^* dt + (I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1} (\Sigma_{s\theta} C_1' + Y' C_2') N^{-1} \left(\left[dy_1' \quad \frac{1}{\sqrt{\epsilon}} dy_2' \right]' - (C_{2\epsilon} A_{22}^{-1} B_2 (B_1' Z_{s\theta} + B_2' V) + C_1 - C_2 A_{22}^{-1} (A_{21} + \theta G_2 G_1' Z_{s\theta})) \hat{x}_s^* dt \right); \quad \hat{x}_s^*(t_0) = (I - \theta \Sigma_{s\theta}(t_0) Z_{s\theta}(t_0))^{-1} \bar{x}_{10} \quad (5.35)$$

Before deriving an expression for the composite controller, let us introduce the notation:

$$X := X_1 \Sigma_{s\theta} + X_2; \quad U := U_1 Z_{s\theta} + U_2 \quad (5.36)$$

$$X_1 := (G_2 G_2)^{-1} A_{21} - (G_2 G_2)^{-1} A_{22} (R_{22} + A_{22}' (G_2 G_2)^{-1} A_{22})^{-1} (R_{21} + A_{22}' (G_2 G_2)^{-1} A_{21}) \quad (5.37)$$

$$X_2 := (G_2 G_2)^{-1} G_2 G_1' - (G_2 G_2)^{-1} A_{22} (R_{22} + A_{22}' (G_2 G_2)^{-1} A_{22})^{-1} A_{22}' (G_2 G_2')^{-1} G_2 G_1 \quad (5.38)$$

$$U_1 := Q_{22}^{-1} A_{12}' - Q_{22}^{-1} A_{22}' (S_{22} + A_{22} Q_{22}^{-1} A_{22}')^{-1} (S_{21} + A_{22} Q_{22}^{-1} A_{12}') \quad (5.39)$$

$$U_2 := Q_{22}^{-1} Q_{21} - Q_{22}^{-1} A_{22}' (S_{22} + A_{22} Q_{22}^{-1} A_{22}')^{-1} A_{22} Q_{22}^{-1} Q_{21}$$

Then, combining (5.24) and (5.34), we have the composite controller expressed as:

$$u_{I_c}^* = \mu_{I_c}^*(t, y[t_0, t]) := \mu_{I_s}^*(t, y[t_0, t]) + \mu_{I_f}^*(t, y[t_0, t]) = -(B_1' Z_{s\theta} + B_2' V) \hat{x}_s^c - B_2' Z_f \hat{x}_f^c \quad (5.40)$$

where a differential equation representation for \hat{x}_s^c can be obtained by substituting (5.6) with $u = u_{I_c}^*$ into (5.29):

$$d\hat{x}_s^c = (\bar{A}_s - \bar{S}_s Z_{s\theta}) \hat{x}_s^c dt - B_0 B_2' Z_f \hat{x}_f^c + (I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1} (\Sigma_{s\theta} C_1' + Y' C_2') N^{-1} \left(\left[dy_1' \quad \frac{1}{\sqrt{\epsilon}} dy_2' \right]' - (C_{2\epsilon} A_{22}^{-1} B_2 (B_1' Z_{s\theta} + B_2' V) + C_1 - C_2 A_{22}^{-1} (A_{21} + \theta G_2 G_1' Z_{s\theta})) \hat{x}_s^c dt - C_{2\epsilon} A_{22}^{-1} B_2 B_2' Z_f \hat{x}_f^c \right); \quad \hat{x}_s^c(t_0) = (I - \theta \Sigma_{s\theta}(t_0) Z_{s\theta}(t_0))^{-1} \bar{x}_{10} \quad (5.41)$$

To obtain the differential equation governing \hat{x}_f^c , we let

$$\begin{bmatrix} \sqrt{\epsilon} dy_{f1} \\ dy_{f2} \end{bmatrix} = \begin{bmatrix} \sqrt{\epsilon} dy_1 \\ dy_2 \end{bmatrix} - \sqrt{\epsilon}(C_{2\epsilon} A_{22}^{-1} B_2 (B_1' Z_{s\theta} + B_2' V) + C_1 \\ - C_2 A_{22}^{-1} (A_{21} + \theta G_2 G_1' Z_{s\theta})) \hat{x}_s^c dt)$$

and

$$\bar{x}_2 = -A_{22}^{-1} B_2 u_{I_s}^* = -U \hat{x}_s^c$$

and substitute the above into (5.25):

$$\begin{aligned} \epsilon d\hat{x}_f^c &= (A_{22} - S_{22} Z_f) \hat{x}_f^c dt + \Sigma_f C_2' N^{-1} \left(\begin{bmatrix} \sqrt{\epsilon} y_1' & y_2' \end{bmatrix} - \sqrt{\epsilon}(C_{2\epsilon} A_{22}^{-1} B_2 (B_1' Z_{s\theta} + B_2' V) \right. \\ &\quad \left. + C_1 - C_2 A_{22}^{-1} (A_{21} + \theta G_2 G_1' Z_{s\theta})) \hat{x}_s^c dt \right) - C_2 \hat{x}_f^c dt; \quad \hat{x}_f^c(t_0) = \bar{x}_{20} + U(t_0) \hat{x}_s^c(t_0) \end{aligned} \quad (5.42)$$

The infinite-horizon case

We take A, B, G, C, E, Q to be time-invariant, and $t_0 = 0, t_f = \infty, Q_f = 0$ and adopt the cost criterion (2.17). We first decompose the system into the slow and fast subsystems as in the finite-horizon case. For the slow subsystem, we can follow the same steps as in the finite-horizon case, to arrive at the slow dynamics and measurements, which are the same as (5.7) and (5.8), except that the system is now time-invariant. The cost function associated with this slow subsystem is

$$\begin{aligned} J_{I_s, \theta \infty}(\mu_{I_s}) &= \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \{ E \{ \exp \left[\frac{\theta}{2} \int_0^{t_f} (|x_s|^2_{Q_{11}} - 2x_s' Q_{12} A_{22}^{-1} B_2 u \right. \right. \\ &\quad \left. \left. + |u|^2_{I+B_2' A_{22}^{-1} Q_{22} A_{22}^{-1} B_2}) dt \right] \} \} \end{aligned} \quad (5.43)$$

This infinite-horizon LEQG problem admits an optimal solution, by Theorem 2, if
(i) the GARE:

$$\bar{A}_s' Z_{s\theta} + Z_{s\theta} \bar{A}_s - Z_{s\theta} \bar{S}_s Z_{s\theta} + \bar{Q}_s = 0 \quad (5.44)$$

admits a minimal positive definite solution $Z_{s\theta}$,

(ii) the GARE:

$$\bar{A}_s \Sigma_{s\theta} + \Sigma_{s\theta} \bar{A}_s' - \Sigma_{s\theta} \bar{R}_s \Sigma_{s\theta} + \bar{M}_s = 0 \quad (5.45)$$

admits a minimal positive definite solution $\Sigma_{s\theta}$, and

(iii) the solution of (5.44) and (5.45) satisfy the spectral radius condition:

$$I - \theta \Sigma_{s\theta} Z_{s\theta} \quad \text{has only positive eigenvalues.} \quad (5.46)$$

Introduce the following quantity, as the counterpart of (5.13):

$$\theta_{I_s, \infty} := \sup \{ \theta \in \mathbb{R} : \text{the GAREs (5.44) and (5.45) admit minimal positive definite solutions, and further satisfy (5.46).} \} \quad (5.47)$$

Then, for $\theta < \theta_{I_s, \infty}$, the optimal controller for the slow subsystem is precisely given by (5.14)–(5.15).

The fast subsystem is the same as (5.17)–(5.19), except that the system matrices are independent of t . It is again a risk-neutral LQG problem. The optimal controller is then given by (5.20)–(5.21), which depends on the solution to GAREs (5.22) and (5.23). Transforming it back to the t time

scale, yields the fast controller (5.24)–(5.25). We also introduce the two Lyapunov equations (5.26) and (5.27), which play a role in the performance evaluation in the sequel.

Further manipulations lead to the slow controller $\mu_{I_s}^*$, in exactly the same form as (5.34)–(5.35); and the composite controller $\mu_{I_c}^*$ given as in (5.40), (5.41) and (5.42).

The main results of this section are now given in Theorem 5 and 6 below, which provide expressions for the performances of the full-order system under full-order, slow and composite controllers, and establish their asymptotic optimality ($\text{ase} \rightarrow 0$), in the infinite and finite horizon cases, respectively.

Theorem 5 For the singularly perturbed system (2.1) with $\alpha = \beta = 1/2$, and under cost function (2.17):

1. For each $\epsilon > 0$, if the pairs (A_ϵ, B_ϵ) and (A_ϵ, G_ϵ) are controllable, the pairs (A_ϵ, C_ϵ) and (A_ϵ, Q) are observable, and the matrix N_ϵ is invertible, then $\forall \theta < \theta_{I_\infty}^*(\epsilon)$, the optimal cost for the full-order LEQG problem can be written as:

$$J_{I\theta\infty}^*(\epsilon) = \text{Tr}(\tilde{\Sigma}_\infty Q + (\tilde{\Pi}_\infty^{-1} + \theta(\tilde{Z}_\infty - \theta\tilde{Z}_\infty\tilde{\Sigma}_\infty\tilde{Z}_\infty))^{-1}((I - \theta\tilde{Z}_\infty\tilde{\Sigma}_\infty)Q \\ (I - \theta\tilde{\Sigma}_\infty\tilde{Z}_\infty) + \tilde{Z}_\infty B_\epsilon B_\epsilon' \tilde{Z}_\infty)) \quad (5.48)$$

where the matrix $\tilde{\Pi}_\infty$ is the unique positive definite solution to the following Lyapunov equation:

$$(A_\epsilon - S_\epsilon \tilde{Z}_\infty)\tilde{\Pi}_\infty + \tilde{\Pi}_\infty(A_\epsilon' - \tilde{Z}_\infty S_\epsilon') + (I - \theta\tilde{\Sigma}_\infty\tilde{Z}_\infty)^{-1}\tilde{\Sigma}_\infty C_\epsilon' N_\epsilon^{-1} C_\epsilon \\ (I - \theta\tilde{Z}_\infty\tilde{\Sigma}_\infty)^{-1} = 0 \quad (5.49)$$

2. Let assumption A3 be satisfied, the pairs (A_{11}, B_0) , $(A_{11}, G_1 G_1' - G_1 G_2' (G_2 G_2')^{-1} G_2 G_1')$ and (A_{22}, B_2) be controllable, and the pairs (A_{11}, C_0) , $(A_{11}, Q_{11} - Q_{12} Q_{22}^{-1} Q_{21})$ and (A_{22}, C_2) be observable. Then,

$$\text{i. } \lim_{\epsilon \rightarrow 0^+} \theta_{I_\infty}^*(\epsilon) = \theta_{I_{s\infty}}.$$

- ii. $\forall \theta < \theta_{I_{s\infty}}$, there exists $\epsilon_\theta > 0$, such that $\forall 0 < \epsilon \leq \epsilon_\theta$, the GAREs (2.18) and (2.22) admit minimal positive definite solutions, which can be approximated by

$$\tilde{Z}_\infty(\epsilon) = \begin{bmatrix} Z_{s\theta} + O(\sqrt{\epsilon}) & \epsilon(Z_f U + V)' + O(\epsilon^{3/2}) \\ \epsilon(Z_f U + V) + O(\epsilon^{3/2}) & \epsilon Z_f + O(\epsilon^{3/2}) \end{bmatrix} \quad (5.50)$$

and

$$\tilde{\Sigma}_\infty(\epsilon) = \begin{bmatrix} \Sigma_{s\theta} + O(\sqrt{\epsilon}) & \sqrt{\epsilon}(\Sigma_f X + Y)' + O(\epsilon) \\ \sqrt{\epsilon}(\Sigma_f X + Y) + O(\epsilon) & \Sigma_f + O(\sqrt{\epsilon}) \end{bmatrix} \quad (5.51)$$

Furthermore, $I - \theta\tilde{\Sigma}_\infty\tilde{Z}_\infty$ has only positive eigenvalues.

- iii. $\forall \theta < \theta_{I_{s\infty}}$, there exists $\tilde{\epsilon}_\theta \in (0, \epsilon_\theta]$, such that $\forall 0 < \epsilon \leq \tilde{\epsilon}_\theta$, the Lyapunov equation (5.49) admits a positive definite solution, which can be approximated by

$$\tilde{\Pi}_\infty(\epsilon) = \begin{bmatrix} I & 0 \\ -U & I \end{bmatrix} \begin{bmatrix} \Pi_{s\theta} + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Pi_f + O(\sqrt{\epsilon}) \end{bmatrix} \begin{bmatrix} I & -U' \\ 0 & I \end{bmatrix} \quad (5.52)$$

where $\Pi_{s\theta}$ is the unique positive definite solution to the following Lyapunov equation:

$$(\bar{A}_s - \bar{S}_s Z_{s\theta})\Pi_{s\theta} + \Pi_{s\theta}(\bar{A}_s' - Z_{s\theta} \bar{S}_s') + (I - \theta\Sigma_{s\theta} Z_{s\theta})^{-1}((\Sigma_{s\theta} C_1' + Y' C_2') \\ \cdot N^{-1}(C_1 \Sigma_{s\theta} + C_2 Y) + X' G_2 G_2' X)(I - \theta Z_{s\theta} \Sigma_{s\theta})^{-1} = 0 \quad (5.53)$$

and Π_f is the unique positive definite solution to the following Lyapunov equation:

$$(A_{22} - B_2 B_2' Z_f) \Pi_f + \Pi_f (A_{22}' - Z_f B_2 B_2') + \Sigma_f C_2' N^{-1} C_2 \Sigma_f = 0. \quad (5.54)$$

iv. $\forall \theta < \theta_{I\infty}, \forall 0 < \epsilon \leq \bar{\epsilon}_\theta$, the optimal cost for the full-order LEQG problem can be approximated by:

$$\begin{aligned} J_{I\theta\infty}^*(\epsilon) = & \text{Tr}(\Sigma_{s\theta} Q_{11} + (\Pi_{s\theta}^{-1} + \theta(Z_{s\theta} - \theta Z_{s\theta} \Sigma_{s\theta} Z_{s\theta}))^{-1}((Z_{s\theta} B_1 + V' B_2) \\ & (B_1' Z_{s\theta} + B_2' V) + (I - \theta Z_{s\theta} \Sigma_{s\theta}) Q_{11} (I - \theta Z_{s\theta} \Sigma_{s\theta}) - (I - \theta Z_{s\theta} \Sigma_{s\theta}) Q_{12} U \\ & - U' Q_{21} (I - \theta \Sigma_{s\theta} Z_{s\theta}) + U' Q_{22} U) + \Sigma_f Q_{22} \\ & + \Pi_f (Q_{22} + Z_f B_2 B_2' Z_f)) + O(\sqrt{\epsilon}) \end{aligned} \quad (5.55)$$

v. $\forall \theta < \theta_{I\infty}$, if the composite controller $\mu_{I\theta\infty}^*$ is applied to the system, then $\exists \epsilon'_\theta > 0$ such that $\forall \epsilon \in [0, \epsilon'_\theta)$,

$$J_{I\theta\infty}^\epsilon := J_{I\theta\infty}(\mu_{I\theta\infty}^*) = J_{I\theta\infty}^*(\epsilon) + O(\sqrt{\epsilon}) \quad (5.56)$$

vi. $\forall \theta < \theta_{I\infty}$, if, in addition, the matrix A_{22} is Hurwitz, and the slow controller $\mu_{I\theta\infty}^*$ is applied to the system, then $\exists \hat{\epsilon}_\theta > 0$ such that $\forall \epsilon \in [0, \hat{\epsilon}_\theta)$,

$$\begin{aligned} J_{I\theta\infty}^\theta := J_{I\theta\infty}(\mu_{I\theta\infty}^*) = & J_{I\theta\infty}^*(\epsilon) + \text{Tr}(\Sigma_{of} Q_{22} - \Sigma_f Q_{22} \\ & - \Pi_f (Q_{22} + Z_f B_2 B_2' Z_f)) + O(\sqrt{\epsilon}) \end{aligned} \quad (5.57)$$

Proof We first substitute the optimal controller (2.25) and (2.26) into the full-order system, for any $\theta < \theta_{I\infty}^*(\epsilon)$, to obtain the following control-free infinite-horizon LEQG problem in terms of $x^e := [x' \hat{x}']'$ and w :

$$\begin{aligned} dx^e = & \begin{bmatrix} A_\epsilon & -B_\epsilon B_\epsilon' \tilde{Z}_\infty \\ (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} \tilde{\Sigma}_\infty C_\epsilon' N_\epsilon^{-1} C_\epsilon & A_\epsilon - S_\epsilon \tilde{Z}_\infty - (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} \tilde{\Sigma}_\infty C_\epsilon' N_\epsilon^{-1} C_\epsilon \end{bmatrix} \\ & \cdot x^e dt + \begin{bmatrix} G_\epsilon \\ (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} \tilde{\Sigma}_\infty C_\epsilon' N_\epsilon^{-1} E_\epsilon \end{bmatrix} dw_t \\ := & F_\epsilon^e x^e dt + G_\epsilon^e dw_t \end{aligned} \quad (5.58)$$

$$J_{I\theta\infty}^* = \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \int_0^{t_f} x^{e'} H^e x^e dt \right] \right\} \right\} \quad (5.59)$$

where

$$H^e := \begin{bmatrix} Q & 0 \\ 0 & \tilde{Z}_\infty B_\epsilon B_\epsilon' \tilde{Z}_\infty \end{bmatrix} \quad (5.60)$$

and $x^e(t_0)$ is a Gaussian random variable with mean and variance:

$$E x^e(t_0) = \begin{bmatrix} \bar{x}_0 \\ (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} \bar{x}_0 \end{bmatrix} =: \bar{x}_0^e; \quad \text{Var}(x^e(t_0)) = \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix} =: \Theta_0^e$$

To compute $J_{I\theta\infty}^*$ explicitly, we associate with above system a fictitious measurement:

$$dy^e = dv_t; \quad y^e(t_0) = 0 \quad (5.61)$$

where v_t , or y^e , is a standard Wiener process independent of the initial condition and w_t .

Then, the two GAREs associated with this problem are:

$$F_\epsilon^e \tilde{\Xi}_\infty + \tilde{\Xi}_\infty F_\epsilon^e + \tilde{\Xi}_\infty \theta G_\epsilon^e G_\epsilon^{e'} \tilde{\Xi}_\infty + H^e = 0 \quad (5.62)$$

and

$$F_\epsilon^e \tilde{\Theta}_\infty + \tilde{\Theta}_\infty F_\epsilon^{e'} + \tilde{\Theta}_\infty \theta H^e \tilde{\Theta}_\infty + G_\epsilon^e G_\epsilon^{e'} = 0 \quad (5.63)$$

It is shown in Appendix B that the minimal solutions to these GAREs are

$$\tilde{\Xi}_\infty = \left[\begin{array}{cc} \tilde{Z}_\infty^{-1} & \tilde{Z}_\infty^{-1} \\ \tilde{Z}_\infty^{-1} & (\tilde{Z}_\infty - \theta \tilde{Z}_\infty \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} + \tilde{\Delta}_\infty^{-1} \end{array} \right]^{-1} \quad (5.64)$$

$$\tilde{\Theta}_\infty = \left[\begin{array}{cc} \tilde{\Sigma}_\infty^{-1} & -\tilde{\Sigma}_\infty^{-1} + \theta \tilde{Z}_\infty \\ -\tilde{\Sigma}_\infty^{-1} + \theta \tilde{Z}_\infty & \tilde{\Sigma}_\infty^{-1} - \theta \tilde{Z}_\infty + \tilde{\Pi}_\infty^{-1} \end{array} \right]^{-1} \quad (5.65)$$

where $\tilde{\Delta}_\infty$ is the unique positive definite solution to the Lyapunov equation:

$$(I - \theta \tilde{Z}_\infty \tilde{\Sigma}_\infty)(A_\epsilon' - R_\epsilon \tilde{\Sigma}_\infty)(I - \theta \tilde{Z}_\infty \tilde{\Sigma}_\infty)^{-1} \tilde{\Delta}_\infty + \tilde{\Delta}_\infty (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty)^{-1} (A_\epsilon - \tilde{\Sigma}_\infty R_\epsilon) \\ (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty) + \tilde{Z}_\infty B_\epsilon B_\epsilon' \tilde{Z}_\infty = 0 \quad (5.66)$$

Furthermore, the matrices $F_\epsilon^e + \theta G_\epsilon^e G_\epsilon^{e'} \tilde{\Xi}_\infty$ and $F_\epsilon^{e'} + \theta H^e \tilde{\Theta}_\infty$ are Hurwitz.

We note that $\frac{1}{\theta} \tilde{\Theta}_\infty^{-1}$ is the maximal solution to the GARE (5.62), since $F_\epsilon^e + \theta G_\epsilon^e G_\epsilon^{e'} \tilde{\Theta}_\infty$ is an anti-stable matrix. Thus, by Theorem 5 of [22], we have $\frac{1}{\theta} \tilde{\Theta}_\infty^{-1} > \tilde{\Xi}_\infty$. It is easily seen that $\tilde{\Theta}_\infty > \Theta_0^e$. Hence, we obtain

$$J_{I\theta\infty}^* = \text{Tr}(\tilde{\Theta}_\infty H^e)$$

in view of Theorem 2. Using a matrix inversion identity, we obtain (5.48). Hence, part 1) is proven.

The GARE (2.18) also arises in the singularly perturbed H^∞ -optimal control problem, whose approximate solution has been studied extensively in [12]. Here, due to the factor $\sqrt{\epsilon}$ multiplying A_{21} and G_2 , these two matrices do not play any role in the GARE (5.44), nor in the zeroth order approximation of the solution. Thus, GARE (2.18) admits a minimal positive definite solution, which is approximated by (5.50), for sufficiently small ϵ if $\theta < \theta_{I\infty}$.

To study the behavior of the solution of GARE (2.22), we first partition $\tilde{\Sigma}_\infty$ as follows (in a way consistent with the given partitioning on Σ_0):

$$\tilde{\Sigma}_\infty := \left[\begin{array}{cc} \tilde{\Sigma}_{\infty 11} & \sqrt{\epsilon} \tilde{\Sigma}_{\infty 12} \\ \sqrt{\epsilon} \tilde{\Sigma}_{\infty 21} & \tilde{\Sigma}_{\infty 22} \end{array} \right] \quad (5.67)$$

where $\tilde{\Sigma}_{\infty 12} = \tilde{\Sigma}_{\infty 21}'$. Then, substituting of this structure into GARE (2.22), we obtain the following coupled matrix Riccati equations for the matrices $\tilde{\Sigma}_{\infty 11}$, $\tilde{\Sigma}_{\infty 12}$ and $\tilde{\Sigma}_{\infty 22}$:

$$A_{11} \tilde{\Sigma}_{\infty 11} + \sqrt{\epsilon} A_{12} \tilde{\Sigma}_{\infty 12}' + \tilde{\Sigma}_{\infty 11} A_{11}' + \sqrt{\epsilon} \tilde{\Sigma}_{\infty 12} A_{12}' + G_1 G_1' - \tilde{\Sigma}_{\infty 11} R_{\epsilon 11} \tilde{\Sigma}_{\infty 11} \\ - \tilde{\Sigma}_{\infty 12} R_{\epsilon 21} \tilde{\Sigma}_{\infty 11} - \tilde{\Sigma}_{\infty 11} R_{\epsilon 12} \tilde{\Sigma}_{\infty 12}' - \tilde{\Sigma}_{\infty 12} R_{\epsilon 22} \tilde{\Sigma}_{\infty 12}' = 0 \quad (5.68)$$

$$\epsilon A_{11} \tilde{\Sigma}_{\infty 12} + \sqrt{\epsilon} A_{12} \tilde{\Sigma}_{\infty 22} + \tilde{\Sigma}_{\infty 11} A_{21}' + \tilde{\Sigma}_{\infty 12} A_{22}' + G_1 G_2' - \epsilon \tilde{\Sigma}_{\infty 11} R_{\epsilon 11} \tilde{\Sigma}_{\infty 12} \\ - \epsilon \tilde{\Sigma}_{\infty 12} R_{\epsilon 21} \tilde{\Sigma}_{\infty 12} - \tilde{\Sigma}_{\infty 11} R_{\epsilon 12} \tilde{\Sigma}_{\infty 22} - \tilde{\Sigma}_{\infty 12} R_{\epsilon 22} \tilde{\Sigma}_{\infty 22} = 0 \quad (5.69)$$

$$\epsilon A_{21} \tilde{\Sigma}_{\infty 12} + A_{22} \tilde{\Sigma}_{\infty 22} + \epsilon \tilde{\Sigma}_{\infty 12}' A_{21}' + \tilde{\Sigma}_{\infty 22} A_{22}' + G_2 G_2' - \epsilon^2 \tilde{\Sigma}_{\infty 12}' R_{\epsilon 11} \tilde{\Sigma}_{\infty 12} \\ - \epsilon \tilde{\Sigma}_{\infty 22} R_{\epsilon 21} \tilde{\Sigma}_{\infty 12} - \epsilon \tilde{\Sigma}_{\infty 12}' R_{\epsilon 12} \tilde{\Sigma}_{\infty 22} - \tilde{\Sigma}_{\infty 22} R_{\epsilon 22} \tilde{\Sigma}_{\infty 22} = 0 \quad (5.70)$$

The above set of equations are the same as (2.26)–(2.28) of [13] for $\epsilon \rightarrow 0$ (except for certain obvious modifications), which permits us to apply the results of [13] to the present case. Hence, for $\theta < \theta_{I\infty}$, (5.68)–(5.70) admit solutions for sufficiently small ϵ , which can be approximated by

$\tilde{\Sigma}_{\infty 11} = \Sigma_{s\theta} + O(\sqrt{\epsilon})$, $\tilde{\Sigma}_{\infty 12} = X'\Sigma_f + Y' + O(\sqrt{\epsilon})$ and $\tilde{\Sigma}_{\infty 22} = \Sigma_f + O(\sqrt{\epsilon})$. Thus, the solution to (2.22) can be approximated by (5.51), for sufficiently small ϵ and for $\theta < \theta_{I\infty}$.

Furthermore, for $\theta < \theta_{I\infty}$ and sufficiently small ϵ , the matrix $I - \theta\tilde{\Sigma}_{\infty}\tilde{Z}_{\infty}$ can be approximated by

$$\begin{bmatrix} I - \theta\Sigma_{s\theta}Z_{s\theta} + O(\sqrt{\epsilon}) & O(\epsilon) \\ O(\sqrt{\epsilon}) & I + O(\sqrt{\epsilon}) \end{bmatrix}$$

Hence, it can have only positive eigenvalues. Thus, part 2.ii) is proved.

Fix any $\theta > \theta_{I\infty}$; then, either one of the GAREs (5.44) and (5.45) does not admit any positive definite solution, or the matrix $I - \theta\Sigma_{s\theta}Z_{s\theta}$ has at least one negative eigenvalue. The former implies that one of the GAREs (2.18) and (2.22) does not admit any positive definite solution for sufficiently small ϵ , by the result of [12], which further implies that $\theta > \theta_{I\infty}^*(\epsilon)$. The latter implies that the matrix $I - \theta\tilde{\Sigma}_{\infty}\tilde{Z}_{\infty}$ has at least one negative eigenvalue for sufficiently small ϵ , which again implies that $\theta > \theta_{I\infty}^*(\epsilon)$. Hence, $\theta > \theta_{I\infty}^*(\epsilon)$ for sufficiently small ϵ , $\forall \theta > \theta_{I\infty}$. Thus, part 2.i) is also proved.

Let $T = \begin{bmatrix} I & 0 \\ U & I \end{bmatrix}$, and $\Pi_{\infty} = T\tilde{\Pi}_{\infty}T'$. Then, premultiplying (5.53) by T and postmultiplying it by T' yields the following Lyapunov equation for Π_{∞} :

$$T(A_{\epsilon} - S_{\epsilon}\tilde{Z}_{\infty})T^{-1}\Pi_{\infty} + \Pi_{\infty}T^{-1}(A'_{\epsilon} - \tilde{Z}_{\infty}S'_{\epsilon})T' + T(I - \theta\tilde{\Sigma}_{\infty}\tilde{Z}_{\infty})^{-1}\tilde{\Sigma}_{\infty}C'_{\epsilon}N_{\epsilon}^{-1}C_{\epsilon}(I - \theta\tilde{Z}_{\infty}\tilde{\Sigma}_{\infty})^{-1}T' = 0 \quad (5.71)$$

Note the following approximations for $\epsilon \in (0, \epsilon_0]$, which are easily obtained in view of the approximations for \tilde{Z}_{∞} and $\tilde{\Sigma}_{\infty}$:

$$T(A_{\epsilon} - S_{\epsilon}\tilde{Z}_{\infty})T^{-1} = \begin{bmatrix} \bar{A}_s - \bar{S}_sZ_{s\theta} & O(1) \\ O(\frac{1}{\sqrt{\epsilon}}) & \frac{1}{\epsilon}(A_{22} - B_2B'_2Z_f) + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \quad (5.72)$$

$$\begin{aligned} T(I - \theta\tilde{\Sigma}_{\infty}\tilde{Z}_{\infty})^{-1}\tilde{\Sigma}_{\infty}C'_{\epsilon}N_{\epsilon}^{-1}C_{\epsilon}(I - \theta\tilde{Z}_{\infty}\tilde{\Sigma}_{\infty})^{-1}T' \\ = \begin{bmatrix} L_s + O(\sqrt{\epsilon}) & O(\frac{1}{\sqrt{\epsilon}}) \\ O(\frac{1}{\sqrt{\epsilon}}) & \frac{1}{\epsilon}\Sigma_fC'_2N^{-1}C_2\Sigma_f + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \end{aligned} \quad (5.73)$$

$$L_s = (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}((\Sigma_{s\theta}C'_1 + Y'C'_2)N^{-1}(C_1\Sigma_{s\theta} + C_2Y) + X'G_2G'_2X)(I - \theta Z_{s\theta}\Sigma_{s\theta})^{-1} \quad (5.74)$$

Suppose that Π_{∞} takes the form $\begin{bmatrix} \Pi_{11\infty} & \sqrt{\epsilon}\Pi_{12\infty} \\ \sqrt{\epsilon}\Pi_{21\infty} & \Pi_{22\infty} \end{bmatrix}$, where $\Pi_{21\infty} = \Pi'_{12\infty}$, and substitute it into the Lyapunov equation (5.71) to arrive at the following equations for $\Pi_{11\infty}$, $\Pi_{12\infty}$ and $\Pi_{22\infty}$:

$$(\bar{A}_s - \bar{S}_sZ_{s\theta})\Pi_{11\infty} + \Pi_{11\infty}(\bar{A}_s - \bar{S}_sZ_{s\theta})' + L_s + O(\sqrt{\epsilon}) = 0 \quad (5.75)$$

$$\Pi_{11\infty}O(1) + \Pi_{12\infty}(A_{22} - B_2B'_2Z_f) + O(1) + O(\sqrt{\epsilon}) = 0 \quad (5.76)$$

$$(A_{22} - B_2B'_2Z_f)\Pi_{22\infty} + \Pi_{22\infty}(A_{22} - B_2B'_2Z_f)' + \Sigma_fC'_2N^{-1}C_2\Sigma_f + O(\sqrt{\epsilon}) = 0 \quad (5.77)$$

Then, it follows that $\Pi_{11\infty} = \Pi_{s\theta}$, $\Pi_{22\infty} = \Pi_f$ and some $\Pi_{12\infty}$ (which exists) solve equations (5.75)–(5.77) at $\epsilon = 0$. By a further application of the implicit function theorem [23] as in the proof of Theorem 1 of [12], the solutions to (5.75)–(5.77) are approximated by $\Pi_{11\infty} = \Pi_{s\theta} + O(\sqrt{\epsilon})$, $\Pi_{22\infty} = \Pi_f + O(\sqrt{\epsilon})$ and $\Pi_{12\infty} = O(1)$, for sufficiently small ϵ . This then completes the proof of part 2.iii).

A mere substitution of (5.50), (5.51) and (5.52) into (5.48) yields the desired result (5.55) (detailed algebraic manipulations can be found in Appendix B), which proves part 2.iv).

Now substitute the composite controller μ_{Ic}^* into the full-order system to arrive at an infinite-horizon control-free LEQG problem. Let

$$x_f^c := x_s + U\hat{x}_s^c \quad (5.78)$$

$$\tilde{x}^c := [x_1', \hat{x}_s^c + \sqrt{\epsilon}x_f^c X(I - \theta Z_{s\theta}\Sigma_{s\theta})^{-1}, x_f^c, \hat{x}_f^c] \quad (5.79)$$

In terms of the state variable \tilde{x}^c , this LEQG problem can be written as:

$$\begin{aligned} d\tilde{x}^c &= \begin{bmatrix} F_{11}^c + O(\sqrt{\epsilon}) & O(1) \\ O(\frac{1}{\sqrt{\epsilon}}) + \frac{1}{\epsilon}F_{22}^c + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \tilde{x}^c dt + \begin{bmatrix} G_1^c + O(\sqrt{\epsilon}) \\ \frac{1}{\sqrt{\epsilon}}G_2^c + O(1) \end{bmatrix} dw_t \\ &:= F_{\epsilon}^c \tilde{x}^c dt + G_{\epsilon}^c dw_t \end{aligned} \quad (5.80)$$

$$J_{I\theta\infty}^c := \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \int_0^{t_f} \tilde{x}^c (H^c + O(\sqrt{\epsilon})) \tilde{x}^c dt \right] \right\} \right\} \quad (5.81)$$

where

$$F_{11}^c = \begin{bmatrix} A_{11} \\ (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}((\Sigma_{s\theta}C_1' + Y'C_2')N^{-1}C_1 + X'A_{21}) \\ -B_1B_1'Z_{s\theta} - B_1B_2'V - A_{12}U \\ F_{cs} \end{bmatrix} \quad (5.82)$$

$$F_{cs} = \bar{A}_s - \bar{S}_s Z_{s\theta} - (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}((\Sigma_{s\theta}C_1' + Y'C_2')N^{-1}C_1 + X'A_{21} + \theta X'G_2G_1'Z_{s\theta}) \quad (5.83)$$

$$F_{22}^c = \begin{bmatrix} A_{22} & -B_2B_2'Z_f \\ \Sigma_f C_2' N^{-1} C_2 & A_{22} - B_2B_2'Z_f - \Sigma_f C_2' N^{-1} C_2 \end{bmatrix} \quad (5.84)$$

$$G_1^c = \begin{bmatrix} G_1 \\ (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}(X'G_2 + (\Sigma_{s\theta}C_1' + Y'C_2')N^{-1}E) \end{bmatrix} \quad (5.85)$$

$$G_2^c = \begin{bmatrix} G_2 \\ \Sigma_f C_2' N^{-1} E \end{bmatrix} \quad (5.86)$$

$$H^c = \begin{bmatrix} H_{11}^c & O(1) \\ O(1) & H_{22}^c \end{bmatrix} \quad (5.87)$$

$$H_{11}^c = \begin{bmatrix} Q_{11} & -Q_{12}U \\ -U'Q_{21} & (Z_{s\theta}B_1 + V'B_2)(B_1Z_{s\theta} + B_2'V) + U'Q_{22}U \end{bmatrix} \quad (5.88)$$

$$H_{22}^c = \begin{bmatrix} Q_{22} & 0 \\ 0 & Z_f B_2 B_2' Z_f \end{bmatrix} \quad (5.89)$$

The derivation of the above is fairly straightforward in view of Lemma 1.

The initial state $\tilde{x}^c(0)$ is a Gaussian random vector with mean \bar{x}_0^c and covariance Σ_0^c , given by:

$$\begin{aligned} \bar{x}_0^c &:= \begin{bmatrix} \bar{x}_{10} \\ (I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}\bar{x}_{10} + O(\sqrt{\epsilon}) \\ \bar{x}_{20} + U(I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}\bar{x}_{10} \\ \bar{x}_{20} + U(I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}\bar{x}_{10} \end{bmatrix} \\ \Sigma_0^c &:= \begin{bmatrix} \begin{bmatrix} \Sigma_{011} & 0 \\ 0 & 0 \end{bmatrix} + O(\epsilon) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \begin{bmatrix} \Sigma_{022} & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \end{aligned}$$

To evaluate the cost $J_{I\theta\infty}^c$, we associate a fictitious measurement (5.61) with this LEQG problem, where v_t , or y^e , is a standard Wiener process independent of the initial state and w_t .

Then, the two GAREs associated with this problem are:

$$F_\epsilon^{c'} \tilde{\Xi}_\infty^c + \tilde{\Xi}_\infty^c F_\epsilon^c + \tilde{\Xi}_\infty^c \theta G_\epsilon^c G_\epsilon^{c'} \tilde{\Xi}_\infty^c + H^c = 0 \quad (5.90)$$

and

$$F_\epsilon^c \tilde{\Theta}_\infty^c + \tilde{\Theta}_\infty^c F_\epsilon^{c'} + \tilde{\Theta}_\infty^c \theta H^c \tilde{\Theta}_\infty^c + G_\epsilon^c G_\epsilon^{c'} = 0 \quad (5.91)$$

It is shown in Appendix B that the minimal solutions to these GAREs exist for $\theta < \theta_{I\infty}$ and sufficiently small ϵ , and can be approximated by

$$\tilde{\Xi}_\infty^c = \begin{bmatrix} \tilde{\Xi}_{11}^c + O(\sqrt{\epsilon}) & O(\epsilon) \\ O(\epsilon) & \epsilon \tilde{\Xi}_{22}^c + O(\epsilon^{3/2}) \end{bmatrix} \quad (5.92)$$

$$\tilde{\Xi}_{11}^c = \begin{bmatrix} Z_{s\theta}^{-1} & Z_{s\theta}^{-1} \\ Z_{s\theta}^{-1} & (Z_{s\theta} - \theta Z_{s\theta} \Sigma_{s\theta} Z_{s\theta})^{-1} + \Delta_{s\theta}^{-1} \end{bmatrix}^{-1} \quad (5.93)$$

$$\tilde{\Xi}_{22}^c = \begin{bmatrix} Z_f + \Delta_f & -\Delta_f \\ -\Delta_f & \Delta_f \end{bmatrix} \quad (5.94)$$

$$\Delta_{s\theta}(I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1}(\tilde{A}_s - \Sigma_{s\theta} \tilde{R}_s)(I - \theta \Sigma_{s\theta} Z_{s\theta}) + (I - \theta Z_{s\theta} \Sigma_{s\theta})(\tilde{A}'_s - \tilde{R}_s \Sigma_{s\theta})(I - \theta Z_{s\theta} \Sigma_{s\theta})^{-1} \Delta_{s\theta} + (Z_{s\theta} B_1 + V' B_2)(B_1 Z_{s\theta} + B'_2 V) + U' Q_{22} U = 0 \quad (5.95)$$

$$\Delta_f(A_{22} - \Sigma_f R_{22}) + \Delta_f(A'_{22} - R_{22} \Sigma_f) + Z_f B_2 B'_2 Z_f = 0 \quad (5.96)$$

$$\tilde{\Theta}_\infty^c = \begin{bmatrix} \tilde{\Theta}_{11}^c + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \tilde{\Theta}_{22}^c + O(\sqrt{\epsilon}) \end{bmatrix} \quad (5.97)$$

$$\tilde{\Theta}_{11}^c = \begin{bmatrix} \Sigma_{s\theta}^{-1} & \theta Z_{s\theta} - \Sigma_{s\theta}^{-1} \\ \theta Z_{s\theta} - \Sigma_{s\theta}^{-1} & \Sigma_{s\theta}^{-1} - \theta Z_{s\theta} + \Pi_{s\theta}^{-1} \end{bmatrix}^{-1} \quad (5.98)$$

$$\tilde{\Theta}_{22}^c = \begin{bmatrix} \Sigma_f + \Pi & \Pi_f \\ \Pi_f & \Pi_f \end{bmatrix} \quad (5.99)$$

Furthermore, the matrices $F_\epsilon^c + \theta G_\epsilon^c G_\epsilon^{c'} \tilde{\Xi}_\infty^c$ and $F_\epsilon^{c'} + \theta H^c \tilde{\Theta}_\infty^c$ are Hurwitz. By Theorem 5 of [22], the matrix $I - \theta \tilde{\Theta}_\infty^c \tilde{\Xi}_\infty^c$ has only positive eigenvalues.

Obviously $\tilde{\Theta}_\infty^c > \Theta_0^c$. Hence, by Theorem 2, $J_{I\theta\infty}^c = \text{Tr}(\tilde{\Theta}_\infty^c H^c)$. Some straightforward algebraic manipulations lead to part 2.v).

Now substitute the slow controller $\mu_{I_s}^*$ into the full-order system to arrive at an infinite-horizon control-free LEQG problem. Let

$$x_f^s := x_2 + U \hat{x}_s^s \quad (5.100)$$

$$\tilde{x}^s := [x_1', \hat{x}_s^{s'} + \sqrt{\epsilon} x_f^{s'} X (I - \theta Z_{s\theta} \Sigma_{s\theta})^{-1}, x_f^{s'}] \quad (5.101)$$

In terms of the state variable \tilde{x}^s , this LEQG problem can be written as:

$$\begin{aligned} d\tilde{x}^s &= \begin{bmatrix} F_{11}^c + O(\sqrt{\epsilon}) & O(1) \\ O(\frac{1}{\sqrt{\epsilon}}) + & \frac{1}{\epsilon} A_{22} + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \tilde{x}^s dt + \begin{bmatrix} G_1^c + O(\sqrt{\epsilon}) \\ \frac{1}{\sqrt{\epsilon}} G_2 + O(1) \end{bmatrix} dw_t \\ &:= F_\epsilon^s \tilde{x}^s dt + G_\epsilon^s dw_t \end{aligned} \quad (5.102)$$

$$J_{I\theta\infty}^s := \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \int_0^{t_f} \tilde{x}^{c'} (H^s + O(\sqrt{\epsilon})) \tilde{x}^c dt \right] \right\} \right\} \quad (5.103)$$

where

$$H^s = \begin{bmatrix} H_{11}^c & O(1) \\ O(1) & Q_{22} \end{bmatrix} \quad (5.104)$$

and F_{11}^c , G_1^c and H_{11}^c are as defined in (5.82), (5.85) and (5.88). The derivation for the above are straightforward in view of Lemma 1.

The initial state $\bar{x}^s(0)$ is a Gaussian random vector with mean \bar{x}_0^s and covariance Σ_0^s , given by:

$$\bar{x}_0^s := \begin{bmatrix} \bar{x}_{10} \\ (I - \theta \Sigma_\theta Z_{s\theta})^{-1} \bar{x}_{10} + O(\sqrt{\epsilon}) \\ \bar{x}_{20} + U(I - \theta \Sigma_\theta Z_{s\theta})^{-1} \bar{x}_{10} \end{bmatrix}; \quad \Sigma_0^s := \begin{bmatrix} \begin{bmatrix} \Sigma_{011} & 0 \\ 0 & 0 \end{bmatrix} + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Sigma_{022} + O(\sqrt{\epsilon}) \end{bmatrix}$$

To evaluate the cost $J_{I\theta\infty}^s$, we associate (as in the composite controller case) a fictitious measurement (5.61) with this LEQG problem, where v_t , or y^e , is a standard Wiener process independent of the initial state and w_t .

Then, the two GAREs associated with this problem are:

$$F_\epsilon^s \tilde{\Xi}_\infty^s + \tilde{\Xi}_\infty^s F_\epsilon^s + \tilde{\Xi}_\infty^s \theta G_\epsilon^s G_\epsilon^{s'} \tilde{\Xi}_\infty^s + H^s = 0 \quad (5.105)$$

and

$$F_\epsilon^s \tilde{\Theta}_\infty^s + \tilde{\Theta}_\infty^s F_\epsilon^{s'} + \tilde{\Theta}_\infty^s \theta H^s \tilde{\Theta}_\infty^s + G_\epsilon^s G_\epsilon^{s'} = 0 \quad (5.106)$$

It is shown in Appendix B that, if A_{22} is Hurwitz, the minimal solutions to these GAREs exist for $\theta < \theta_{I\infty}$ and sufficiently small ϵ , and can be approximated by

$$\tilde{\Xi}_\infty^s = \begin{bmatrix} \tilde{\Xi}_{11}^c + O(\sqrt{\epsilon}) & O(\epsilon) \\ O(\epsilon) & \epsilon Z_{of} + O(\epsilon^{3/2}) \end{bmatrix} \quad (5.107)$$

$$\tilde{\Theta}_\infty^s = \begin{bmatrix} \tilde{\Theta}_{11}^c + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Sigma_{of} + O(\sqrt{\epsilon}) \end{bmatrix} \quad (5.108)$$

where $\tilde{\Xi}_{11}^c$ and $\tilde{\Theta}_{11}^c$ are as defined in (5.93) and (5.98). Furthermore, the matrices $F_\epsilon^s + \theta G_\epsilon^s G_\epsilon^{s'} \tilde{\Xi}_\infty^s$ and $F_\epsilon^{s'} + \theta H^s \tilde{\Theta}_\infty^s$ are Hurwitz. By Theorem 5 of [22], the matrix $I - \theta \tilde{\Theta}_\infty^s \tilde{\Xi}_\infty^s$ has only positive eigenvalues.

Hence, by Theorem 2, $J_{I\theta\infty}^s = \text{Tr}(\tilde{\Theta}_\infty^s H^s)$. Some straightforward algebraic manipulations lead to (5.57). This completes the proof of the Theorem. \square

The finite-horizon case

In this subsection, we establish the finite-horizon counterpart of Theorem 5.

Theorem 6 For the singularly perturbed system (2.1) with $\alpha = \beta = 1/2$, and under the cost function (2.2):

1. For each $\epsilon > 0$, if $N_\epsilon(t)$ is invertible $\forall t \in [t_f, t_f]$, then $\forall \theta < \theta_I^*(\epsilon)$, the optimal cost for the full-order LEQG problem can be written as:

$$\begin{aligned} J_{I\theta}^s(\epsilon) = & \bar{x}_0' \tilde{Z}(t_0) (I - \theta \Sigma_0 \tilde{Z}(t_0))^{-1} \bar{x}_0 + \int_{t_0}^{t_f} \text{Tr}(\tilde{\Sigma} \dot{Q} + (\tilde{\Pi}^{-1} + \theta(\tilde{Z} - \theta \tilde{Z} \tilde{\Sigma} \tilde{Z}))^{-1} \\ & ((I - \theta \tilde{Z} \tilde{\Sigma}) Q (I - \theta \tilde{\Sigma} \tilde{Z}) + \tilde{Z} B_\epsilon B_\epsilon' \tilde{Z}) dt - \frac{1}{\theta} (\ln(\det(I - \theta \tilde{\Sigma}(t_f) Q_f)) \\ & - \ln(\det(I + \theta \tilde{\Pi}(\tilde{Z} - \theta Q_f \tilde{\Sigma}(t_f) Q_f)))) \end{aligned} \quad (5.109)$$

where the matrix $\tilde{\Pi}$ is the unique nonnegative definite solution to the following Lyapunov differential equation:

$$\dot{\tilde{\Pi}} = (A_\epsilon - S_\epsilon \tilde{Z})\tilde{\Pi} + \tilde{\Pi}(A'_\epsilon - \tilde{Z}'S'_\epsilon) + (I - \theta \tilde{\Sigma} \tilde{Z})^{-1} \tilde{\Sigma} C'_\epsilon N_\epsilon^{-1} C_\epsilon \tilde{\Sigma} (I - \theta \tilde{Z} \tilde{\Sigma})^{-1} \quad (5.110)$$

with $\tilde{\Pi}(t_0) = 0$.

2. Let assumptions A1-A3 be satisfied, the pair (A_{22}, B_2) be controllable, and the pair (A_{22}, C_2) be observable $\forall t \in [t_0, t_f]$. Then,

i. $\limsup_{\epsilon \rightarrow 0^+} \theta_I^*(\epsilon) \leq \theta_{I_s}$.

ii. $\forall \theta < \theta_{I_s}$, there exists $\epsilon_\theta > 0$, such that $\forall 0 < \epsilon \leq \epsilon_\theta$, the GRDEs (2.7) and (2.11) admit nonnegative definite solutions on $[t_0, t_f]$, which can be approximated by

$$\tilde{Z}(t; \epsilon) = \begin{bmatrix} Z_{s\theta} + O(\sqrt{\epsilon}) & \epsilon(Z_f U + V + Z_{cb})' + O(\epsilon^{3/2}) \\ \epsilon(Z_f U + V + Z_{cb}) + O(\epsilon^{3/2}) & \epsilon(Z_f + Z_{fb}) + O(\epsilon^{3/2}) \end{bmatrix} \quad (5.111)$$

and

$$\tilde{\Sigma}(t; \epsilon) = \begin{bmatrix} \Sigma_{s\theta} + O(\sqrt{\epsilon}) & \sqrt{\epsilon}(\Sigma_f X + Y + \Sigma_{cb})' + O(\epsilon) \\ \sqrt{\epsilon}(\Sigma_f X + Y + \Sigma_{cb}) + O(\epsilon) & \Sigma_f + \Sigma_{fb} + O(\sqrt{\epsilon}) \end{bmatrix} \quad (5.112)$$

where $Z_{cb}(\tau)$, $Z_{fb}(\tau)$ are boundary layer terms at t_f , and $\Sigma_{cb}(\tau)$ and $\Sigma_{fb}(\tau)$ are boundary layer terms at t_0 , and they converge to zero exponentially in the τ time scale.

Furthermore, $I - \theta \tilde{\Sigma} \tilde{Z}$ has only positive eigenvalues $\forall t \in [t_0, t_f]$.

iii. $\forall \theta < \theta_{I_s}$, there exists $\bar{\epsilon}_\theta \in (0, \epsilon_\theta]$, such that $\forall \epsilon$, $0 < \epsilon \leq \bar{\epsilon}_\theta$, the Lyapunov equation (5.110) admits a unique nonnegative definite solution on $[t_0, t_f]$, which can be approximated by

$$\tilde{\Pi}(\epsilon) = \begin{bmatrix} I & 0 \\ -U & I \end{bmatrix} \begin{bmatrix} \Pi_{s\theta} + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Pi_f + \Pi_{fb} + O(\sqrt{\epsilon}) \end{bmatrix} \begin{bmatrix} I & -U' \\ 0 & I \end{bmatrix} \quad (5.113)$$

on the time interval $[t_0, t_f + \epsilon \ln \epsilon]$, and by

$$\tilde{\Pi}(\epsilon) = \begin{bmatrix} \Pi_{s\theta} + O(\sqrt{\epsilon}) & O(1) \\ O(1) & O(1) \end{bmatrix} \quad (5.114)$$

on the time interval $[t_f + \epsilon \ln \epsilon, t_f]$, where $\Pi_{s\theta}$ is the unique nonnegative definite solution to the following Lyapunov differential equation:

$$\dot{\Pi}_{s\theta} = (\bar{A}_s - \bar{S}_s Z_{s\theta})\Pi_{s\theta} + \Pi_{s\theta}(\bar{A}'_s - Z_{s\theta} \bar{S}'_s) + (I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1} ((\Sigma_{s\theta} C'_1 + Y' C'_2) N^{-1} (C_1 \Sigma_{s\theta} + C_2 Y) + X' G_2 G'_2 X) (I - \theta Z_{s\theta} \Sigma_{s\theta})^{-1}; \quad \Pi_{s\theta}(t_0) = 0 \quad (5.115)$$

and Π_f is the unique positive definite solution to the Lyapunov equation (5.54), and Π_{fb} is a boundary layer term at t_0 , which converges to zero exponentially in the τ time scale.

iv. $\forall \theta < \theta_{I_s}$, $\forall 0 < \epsilon \leq \bar{\epsilon}_\theta$, the optimal cost for the full-order LEQG problem can be approximated by:

$$\begin{aligned} J_{I\theta}^*(\epsilon) = & \bar{x}'_{10} Z_{s\theta}(t_0) (I - \theta \Sigma_{011} Z_{s\theta}(t_0))^{-1} \bar{x}_{10} + \int_{t_0}^{t_f} \text{Tr}(\Sigma_{s\theta} Q_{11} + (\Pi_{s\theta}^{-1} + \theta(Z_{s\theta} \\ & - \theta Z_{s\theta} \Sigma_{s\theta} Z_{s\theta}))^{-1} ((Z_{s\theta} B_1 + V' B_2)(B'_1 Z_{s\theta} + B'_2 V) + (I - \theta Z_{s\theta} \Sigma_{s\theta}) Q_{11} \\ & (I - \theta \Sigma_{s\theta} Z_{s\theta}) - (I - \theta Z_{s\theta} \Sigma_{s\theta}) Q_{12} U - U' Q_{21} (I - \theta \Sigma_{s\theta} Z_{s\theta}) + U' Q_{22} U) \\ & + \Sigma_f Q_{22} + \Pi_f (Q_{22} + Z_f B_2 B'_2 Z_f)) dt - \frac{1}{\theta} (\ln(\det(I - \Sigma_{s\theta}(t_f) Q_{f11})) \\ & - \ln(\det(I + \theta \Pi_{s\theta}(t_f) (Q_{f11} - \theta Q_{f11} \Sigma_{s\theta}(t_f) Q_{f11})))) + O(\sqrt{\epsilon}) \end{aligned} \quad (5.116)$$

v. $\forall \theta < \theta_{I\theta}$, if the composite controller $\mu_{I\theta}^*$ is applied to the system, then $\exists \epsilon'_\theta > 0$ such that $\forall \epsilon \in [0, \epsilon'_\theta)$,

$$J_{I\theta}^c := J_{I\theta}(\mu_{I\theta}^*) = J_{I\theta}^*(\epsilon) + O(\sqrt{\epsilon}) \quad (5.117)$$

vi. $\forall \theta < \theta_{I\theta}$, if, in addition, the matrix A_{22} is Hurwitz for $t \in [t_0, t_f]$, and the slow controller $\mu_{I\theta}^*$ is applied to the system, then $\exists \hat{\epsilon}_\theta > 0$ such that $\forall \epsilon \in [0, \hat{\epsilon}_\theta)$,

$$J_{I\theta}^s := J_{I\theta}(\mu_{I\theta}^*) = J_{I\theta}^*(\epsilon) + \int_{t_0}^{t_f} \text{Tr}(\Sigma_{of} Q_{22} - \Sigma_f Q_{22} - \Pi_f(Q_{22} + Z_f B_2 B_2' Z_f)) dt + O(\sqrt{\epsilon}) \quad (5.118)$$

Proof We first substitute the optimal controller (2.14) and (2.15) into the full-order system, for any $\theta < \theta_{I\theta}^*(\epsilon)$, to obtain a control-free finite-horizon LEQG problem. In terms of x^e and w , the LEQG problem has the state dynamics (5.58) (with time-varying coefficient matrices) and the following cost function:

$$J_{I\theta}^* = \frac{2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(x^{e'}(t_f) Q_f^e x^e(t_f) + \int_{t_0}^{t_f} x^{e'} H^e x^e dt \right) \right] \right\} \right\} \quad (5.119)$$

where

$$Q_f^e := \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix} \quad (5.120)$$

and $x^e(t_0)$ is a Gaussian random vector with mean and covariance given respectively by:

$$E x^e(t_0) = \begin{bmatrix} \bar{x}_0 \\ (I - \theta \bar{Z}(t_0) \Sigma_0)^{-1} \bar{x}_0 \end{bmatrix} =: \bar{x}_0^e; \quad \text{Var}(x^e(t_0)) = \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix} =: \Theta_0^e$$

To evaluate $J_{I\theta}^*$, we associate with the above system a fictitious measurement (5.61), where v_t , or y^e , is a standard Wiener process independent of the initial state and w_t .

Then, the two GRDEs associated with this problem are:

$$\dot{\tilde{\Xi}} + F_\epsilon^{e'} \tilde{\Xi} + \tilde{\Xi} F_\epsilon^e + \tilde{\Xi} \theta G_\epsilon^e G_\epsilon^{e'} \tilde{\Xi} + H^e = 0; \quad \tilde{\Xi}(t_f) = Q_f^e \quad (5.121)$$

and

$$\dot{\tilde{\Theta}} = F_\epsilon^e \tilde{\Theta}_\infty + \tilde{\Theta}_\infty F_\epsilon^{e'} + \tilde{\Theta}_\infty \theta H^e \tilde{\Theta}_\infty + G_\epsilon^e G_\epsilon^{e'}; \quad \tilde{\Theta}(t_0) = \Theta_0^e \quad (5.122)$$

It is shown in Appendix B that the nonnegative definite solutions to these GRDEs are, respectively,

$$\tilde{\Xi} = \begin{bmatrix} \tilde{Z}^{-1} & \tilde{Z}^{-1} \\ \tilde{Z}^{-1} & (\tilde{Z} - \theta \tilde{Z} \tilde{\Sigma} \tilde{Z})^{-1} + \tilde{\Delta}^{-1} \end{bmatrix}^{-1} \quad (5.123)$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Sigma}^{-1} & -\tilde{\Sigma}^{-1} + \theta \tilde{Z} \\ -\tilde{\Sigma}^{-1} + \theta \tilde{Z} & \tilde{\Sigma}^{-1} - \theta \tilde{Z} + \tilde{\Pi}^{-1} \end{bmatrix}^{-1} \quad (5.124)$$

where $\tilde{\Delta}$ is the unique nonnegative definite solution to the Lyapunov differential equation:

$$\begin{aligned} \dot{\tilde{\Delta}} + ((I - \theta \tilde{Z} \tilde{\Sigma})(A_\epsilon' - R_\epsilon \tilde{\Sigma})(I - \theta \tilde{Z} \tilde{\Sigma})^{-1} + \frac{d}{dt}((I - \theta \tilde{\Sigma} \tilde{Z})^{-1})(I - \theta \tilde{\Sigma} \tilde{Z})) \tilde{\Delta} + \tilde{\Delta}((I - \theta \tilde{\Sigma} \tilde{Z})^{-1}(A_\epsilon - \tilde{\Sigma} R_\epsilon)(I - \theta \tilde{\Sigma} \tilde{Z}) + \frac{d}{dt}((I - \theta \tilde{\Sigma} \tilde{Z})^{-1})(I - \theta \tilde{\Sigma} \tilde{Z})) \\ + \tilde{Z} B_\epsilon B_\epsilon' \tilde{Z} = 0; \quad \tilde{\Delta}(t_f) = 0 \end{aligned} \quad (5.125)$$

and these solutions further satisfy the spectral radius condition. Hence, we obtain

$$J_{I\theta}^* = \bar{x}_0^e \bar{\Xi}(t_0)(I - \theta \Theta_0^e \bar{\Xi}(t_0))^{-1} \bar{x}_0^e + \int_{t_0}^{t_f} \text{Tr}(\bar{\Theta}_\infty H^e) dt - \frac{1}{\theta} \ln(\det(I - \theta \bar{\Theta}(t_f) Q_f^e))$$

by Theorem 1. After some straightforward algebraic manipulations, we obtain (5.109), which completes the proof of part 1).

The approximate solution to GRDE (2.7) has already been studied in [12]. Here, due to the factor $\sqrt{\epsilon}$ multiplying A_{21} and G_2 , these two matrices do not play any role in the GRDE (5.10), as well as in the zeroth order approximation of the solution. Thus, GRDE (2.7) admits a unique nonnegative definite solution, which is approximated by (5.111), for sufficiently small ϵ if $\theta < \theta_{Is}$.

To study the behavior of the solution of GRDE (2.11), we first partition $\bar{\Sigma}$ as in (5.67) and substitute it into GRDE (2.11). This leads to the following matrix differential equations for the matrices $\bar{\Sigma}_{11}$, $\bar{\Sigma}_{12}$ and $\bar{\Sigma}_{22}$:

$$\begin{aligned} \dot{\bar{\Sigma}}_{11} = & A_{11} \bar{\Sigma}_{11} + \sqrt{\epsilon} A_{12} \bar{\Sigma}'_{12} + \bar{\Sigma}_{11} A'_{11} + \sqrt{\epsilon} \bar{\Sigma}_{12} A'_{12} + G_1 G'_1 - \bar{\Sigma}_{11} R_{e11} \bar{\Sigma}_{11} \\ & - \bar{\Sigma}_{12} R_{e21} \bar{\Sigma}_{11} - \bar{\Sigma}_{11} R_{e12} \bar{\Sigma}'_{12} - \bar{\Sigma}_{12} R_{e22} \bar{\Sigma}'_{12}; \quad \bar{\Sigma}_{11}(t_0) = \Sigma_{011} \end{aligned} \quad (5.126)$$

$$\begin{aligned} \epsilon \dot{\bar{\Sigma}}_{12} = & \epsilon A_{11} \bar{\Sigma}_{12} + \sqrt{\epsilon} A_{12} \bar{\Sigma}_{22} + \bar{\Sigma}_{11} A'_{21} + \bar{\Sigma}_{12} A'_{22} + G_1 G'_2 - \epsilon \bar{\Sigma}_{11} R_{e11} \bar{\Sigma}_{12} \\ & - \epsilon \bar{\Sigma}_{12} R_{e21} \bar{\Sigma}_{12} - \bar{\Sigma}_{11} R_{e12} \bar{\Sigma}_{22} - \bar{\Sigma}_{12} R_{e22} \bar{\Sigma}_{22}; \quad \bar{\Sigma}_{12}(t_0) = \Sigma_{012} \end{aligned} \quad (5.127)$$

$$\begin{aligned} \epsilon \dot{\bar{\Sigma}}_{22} = & \epsilon A_{21} \bar{\Sigma}_{12} + A_{22} \bar{\Sigma}_{22} + \epsilon \bar{\Sigma}'_{12} A'_{21} + \bar{\Sigma}_{22} A'_{22} + G_2 G'_2 - \epsilon^2 \bar{\Sigma}'_{12} R_{e11} \bar{\Sigma}_{12} \\ & - \epsilon \bar{\Sigma}_{22} R_{e21} \bar{\Sigma}_{12} - \epsilon \bar{\Sigma}'_{12} R_{e12} \bar{\Sigma}_{22} - \bar{\Sigma}_{22} R_{e22} \bar{\Sigma}_{22}; \quad \bar{\Sigma}_{22}(t_0) = \Sigma_{022} \end{aligned} \quad (5.128)$$

The above set of equations are the same as (2.16)–(2.18) of [13] for $\epsilon \rightarrow 0$ (except for certain obvious modifications), which permits us to apply the results of [13]. Hence, for $\theta < \theta_{Is}$, (5.126)–(5.128) admit unique solutions for sufficiently small ϵ , which can be approximated by $\bar{\Sigma}_{11} = \Sigma_{s\theta} + O(\sqrt{\epsilon})$, $\bar{\Sigma}_{12} = X' \Sigma_f + Y' + \Sigma_{cb} + O(\sqrt{\epsilon})$ and $\bar{\Sigma}_{22} = \Sigma_f + \Sigma_{fb} + O(\sqrt{\epsilon})$. Thus, the solution to (2.11) can be approximated by (5.112), for sufficiently small ϵ and $\theta < \theta_{Is}$.

Furthermore, for $\theta < \theta_{Is}$ and sufficiently small ϵ , the matrix $I - \theta \bar{\Sigma} \bar{Z}$ can be approximated by

$$\begin{bmatrix} I - \theta \Sigma_{s\theta} Z_{s\theta} + O(\sqrt{\epsilon}) & \dot{O}(\epsilon) \\ O(\sqrt{\epsilon}) & I + O(\sqrt{\epsilon}) \end{bmatrix}$$

Hence, it can have only positive eigenvalues $\forall t \in [t_0, t_f]$. Thus, parts 2.i), ii) are proved.

To derive an approximate expression for $J_{I\theta}^*$, we first obtain an approximate form for $\bar{\Pi}$ on the time interval $[t_0, t_f]$, as in the infinite-horizon case. Let $T = \begin{bmatrix} I & 0 \\ U & I \end{bmatrix}$, and $\Pi = T \bar{\Pi} T'$. Then, Π satisfies the following Lyapunov differential equation:

$$\begin{aligned} \dot{\Pi} = & (T(A_\epsilon - S_\epsilon \bar{Z})T^{-1} + \dot{T}T^{-1})\Pi + \Pi(T^{-1}(A'_\epsilon - \bar{Z}'S_\epsilon)T' + T^{-1}\dot{T}') + T(I - \theta \bar{\Sigma} \bar{Z})^{-1} \\ & \bar{\Sigma} C'_\epsilon N_\epsilon^{-1} C_\epsilon (I - \theta \bar{Z} \bar{\Sigma})^{-1} T'; \quad \Pi(t_0) = 0 \end{aligned} \quad (5.129)$$

Note the following approximations for $\epsilon \in (0, \epsilon_\theta]$ and $t \in [t_0, t_f + \epsilon \ln \epsilon]$, which are again easily obtained in view of the approximations for \bar{Z} and $\bar{\Sigma}$:

$$T(A_\epsilon - S_\epsilon \bar{Z})T^{-1} + \dot{T}T^{-1} = \begin{bmatrix} \bar{A}_s - \bar{S}_s Z_{s\theta} & O(1) \\ O(\frac{1}{\sqrt{\epsilon}}) & \frac{1}{\epsilon} (A_{22} - B_2 B'_2 Z_f) + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \quad (5.130)$$

$$\begin{aligned} T(I - \theta \bar{\Sigma} \bar{Z})^{-1} \bar{\Sigma} C'_\epsilon N_\epsilon^{-1} C_\epsilon (I - \theta \bar{Z} \bar{\Sigma})^{-1} T' \\ = \begin{bmatrix} L_s + L_{sb} + O(\sqrt{\epsilon}) & \frac{1}{\sqrt{\epsilon}} (L'_{cb} + O(1)) \\ \frac{1}{\sqrt{\epsilon}} (L_{cb} + O(1)) & \frac{1}{\epsilon} \Sigma_f C'_2 N^{-1} C_2 \Sigma_f + L_{fb} + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \end{aligned} \quad (5.131)$$

$$\begin{aligned} L_s = & (I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1} ((\Sigma_{s\theta} C'_1 + Y' C'_2) N^{-1} (C_1 \Sigma_{s\theta} + C_2 Y) + X' G_2 G'_2 X) \\ & (I - \theta Z_{s\theta} \Sigma_{s\theta})^{-1} \end{aligned} \quad (5.132)$$

where L_{sb} , L_{cb} and L_{fb} are boundary layer terms at t_0 , and they converge to zero exponentially in the τ time scale.

Suppose that Π takes the form $\begin{bmatrix} \Pi_{11} & \sqrt{\epsilon}\Pi_{12} \\ \sqrt{\epsilon}\Pi_{21} & \Pi_{22} \end{bmatrix}$, where $\Pi_{21} = \Pi'_{12}$, and substitute it into the Lyapunov equation (5.129) to arrive at the following differential equations for Π_{11} , Π_{12} and Π_{22} on $[t_0, t_f + \epsilon \ln \epsilon]$:

$$\dot{\Pi}_{11} = (\bar{A}_s - \bar{S}_s Z_{s\theta})\Pi_{11} + \Pi_{11}(\bar{A}_s - \bar{S}_s Z_{s\theta})' + L_s + L_{sb} + O(\sqrt{\epsilon})$$

$$\epsilon \dot{\Pi}_{12} = \Pi_{11}O(1) + \Pi_{12}(A_{22} - B_2 B'_2 Z_f) + O(1) + O(\sqrt{\epsilon})$$

$$\epsilon \dot{\Pi}_{22} = (A_{22} - B_2 B'_2 Z_f)\Pi_{22} + \Pi_{22}(A_{22} - B_2 B'_2 Z_f)' + \Sigma_f C'_2 N^{-1} C_2 \Sigma_f + L_{fb} + O(\sqrt{\epsilon})$$

If there are no boundary layer terms L_{sb} , L_{cb} and L_{fb} , then, clearly $\Pi_{11} = \Pi_{s\theta}$, $\Pi_{22} = \Pi_f$ and some Π_{12} (which exists) solve (5.75)–(5.77) at $\epsilon = 0$; by a further application of the implicit function theorem as in the proof of Theorem 2 of [12], the solution to the above set of equations can be approximated by $\Pi_{11} = \Pi_{s\theta} + O(\sqrt{\epsilon})$, $\Pi_{22} = \Pi_f + \Pi_{fb} + O(\sqrt{\epsilon})$ and $\Pi_{12} = O(1)$, for sufficiently small ϵ .

We now note that with the boundary layer terms L_{sb} , L_{cb} and L_{fb} , the above approximation forms for Π_{11} , Π_{12} and Π_{22} are equally valid. The reason is the following. The matrix $A_{22} - B_2 B'_2 Z_f$ is Hurwitz for each $t \in [t_0, t_f]$ (see [24]). Since they converge to zero exponentially in the τ time scale, the boundary layer terms L_{cb} and L_{fb} will induce only additional boundary layer correction terms to the solutions of Π_{12} and Π_{22} , which also converge to zero exponentially in the τ time scale. The term L_{sb} will induce only $O(\epsilon)$ correction terms to Π_{11} . Then, the matrix $\tilde{\Pi}$ is approximated by (5.113) on the time interval $[t_0, t_f + \epsilon \ln \epsilon]$.

On the time interval $[t_f + \epsilon \ln \epsilon, t_f]$, we note the following approximation forms:

$$A_\epsilon - S_\epsilon \tilde{Z} = \begin{bmatrix} O(1) & O(1) \\ O(\frac{1}{\sqrt{\epsilon}}) & \frac{1}{\epsilon}(A_{22} - B_2 B'_2 Z_f - B_2 B'_2 Z_{fb}) + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \quad (5.133)$$

$$\begin{aligned} (I - \theta \tilde{Z} \tilde{Z})^{-1} \tilde{\Sigma} C'_\epsilon N_\epsilon^{-1} C_\epsilon (I - \theta \tilde{Z} \tilde{Z})^{-1} \\ = \begin{bmatrix} L_s + O(\sqrt{\epsilon}) & O(\frac{1}{\sqrt{\epsilon}}) \\ O(\frac{1}{\sqrt{\epsilon}}) & \frac{1}{\epsilon} \Sigma_f C'_2 N^{-1} C_2 \Sigma_f + O(\frac{1}{\sqrt{\epsilon}}) \end{bmatrix} \end{aligned} \quad (5.134)$$

Take $\tilde{\Pi}$ to be of the form $\begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} \end{bmatrix}$, where $\tilde{\Pi}_{12} = \tilde{\Pi}'_{22}$. Substitution of this structure into (5.110), yields the following differential equations for $\tilde{\Pi}_{11}$, $\tilde{\Pi}_{12}$ and $\tilde{\Pi}_{22}$:

$$\dot{\tilde{\Pi}}_{11} = O(1)\tilde{\Pi}_{11} + O(1)\tilde{\Pi}_{21} + \tilde{\Pi}_{11}O(1) + \tilde{\Pi}_{12}O(1) + O(1) \quad (5.135)$$

$$\epsilon \dot{\tilde{\Pi}}_{12} = \tilde{\Pi}_{11}O(1) + \tilde{\Pi}_{12}(A_{22} - B_2 B'_2 Z_f - B_2 B'_2 Z_{fb})' + O(\sqrt{\epsilon}) \quad (5.136)$$

$$\begin{aligned} \epsilon \dot{\tilde{\Pi}}_{22} = O(1)\tilde{\Pi}_{12} + (A_{22} - B_2 B'_2 Z_f - B_2 B'_2 Z_{fb})\tilde{\Pi}_{22} + \tilde{\Pi}_{21}O(1) + \tilde{\Pi}_{22}(A_{22} - B_2 B'_2 Z_f \\ - B_2 B'_2 Z_{fb})' + O(1) \end{aligned} \quad (5.137)$$

Since Z_{fb} converges to zero exponentially in the τ time scale, there exists a t_a , where $t_f - t_a = O(\epsilon)$, such that, $\forall t \in [t_f + \epsilon \ln \epsilon, t_a]$, all eigenvalues of the matrix $A_{22} - B_2 B'_2 Z_f - B_2 B'_2 Z_{fb}$ have negative real parts that are strictly less than an ϵ -independent constant $\lambda_m < 0$. Thus, by the linearity of the differential equations (5.135)–(5.137), we conclude that $\tilde{\Pi}_{11}$, $\tilde{\Pi}_{12}$ and $\tilde{\Pi}_{22}$ are of $O(1)$ on $[t_f + \epsilon \ln \epsilon, t_a]$.

On the time interval $[t_a, t_f]$, the state transition matrix for the system of differential equations (5.135)–(5.137) is of $O(1)$, and the input to this system is of order $O(\frac{1}{\epsilon})$. Thus, the growth for $\tilde{\Pi}_{11}$, $\tilde{\Pi}_{12}$ and $\tilde{\Pi}_{22}$ on this time interval is of $O(1)$.

Hence, the matrices $\tilde{\Pi}_{11}$, $\tilde{\Pi}_{12}$ and $\tilde{\Pi}_{22}$ are $O(1)$ on $[t_f + \epsilon \ln \epsilon, t_f]$. Thus, $\dot{\tilde{\Pi}}_{11}$ is of $O(1)$ on this time interval, which implies that

$$\tilde{\Pi}_{11}(t) = \tilde{\Pi}_{11}(t_f + \epsilon \ln \epsilon) + O(\sqrt{\epsilon}) = \Pi_{s\theta}(t_f + \epsilon \ln \epsilon) + O(\sqrt{\epsilon}) = \Pi_{s\theta}(t) + O(\sqrt{\epsilon})$$

This then proves part 2.iii).

A mere substitution of (5.111), (5.112), (5.113) and (5.114) into (5.109) yields the desired result (5.116) (detailed algebraic manipulations can be found in Appendix B), which proves part 2.iv).

To prove part 2.v), we substitute the composite controller μ_{jc}^* into the full-order system to arrive at a control-free LEQG problem. Let x_f^c and \tilde{x}^c be defined as in (5.78) and (5.79). In terms of the state variable \tilde{x}^c , this LEQG problem can be expressed as:

$$d\tilde{x}^c = F_\epsilon^c \tilde{x}^c dt + G_\epsilon^c dw_t \quad (5.138)$$

$$J_{f\theta}^c := \frac{2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\tilde{x}^{c'}(t_f) Q_f^c \tilde{x}^c(t_f) + \int_{t_0}^{t_f} \tilde{x}^{c'}(H^c + O(\sqrt{\epsilon})) \tilde{x}^c dt \right) \right] \right\} \right\} \quad (5.139)$$

where

$$Q_f^c = \begin{bmatrix} \begin{bmatrix} Q_{f11} & 0 \\ 0 & 0 \end{bmatrix} + O(\epsilon) & O(\epsilon) \\ O(\epsilon) & \begin{bmatrix} \epsilon Q_{f22} & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \quad (5.140)$$

with initial state $\tilde{x}^c(t_0)$ being a Gaussian random vector with mean \bar{x}_0^c and covariance Σ_0^c given by:

$$\bar{x}_0^c := \begin{bmatrix} \bar{x}_{10} \\ (I - \theta \Sigma_{011} Z_{s\theta}(t_0))^{-1} \bar{x}_{10} + O(\sqrt{\epsilon}) \\ \bar{x}_{20} + U(t_0)(I - \theta \Sigma_{011} Z_{s\theta}(t_0))^{-1} \bar{x}_{10} \\ \bar{x}_{20} + U(t_0)(I - \theta \Sigma_{011} Z_{s\theta}(t_0))^{-1} \bar{x}_{10} \end{bmatrix}$$

$$\Sigma_0^c := \begin{bmatrix} \begin{bmatrix} \Sigma_{011} & 0 \\ 0 & 0 \end{bmatrix} + O(\epsilon) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \begin{bmatrix} \Sigma_{022} & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

Again, the derivation of the above involves a simple application of Lemma 1.

To evaluate $J_{f\theta}^c$, we associate a fictitious measurement (5.61) with this LEQG problem, where v_t , or y^c , is a standard Wiener process independent of the initial condition and w_t .

Then, the two GRDEs associated with this problem are:

$$\dot{\tilde{\Xi}}^c + F_\epsilon^c \tilde{\Xi}^c + \tilde{\Xi}^c F_\epsilon^{c'} + \tilde{\Xi}^c \theta G_\epsilon^c G_\epsilon^{c'} \tilde{\Xi}^c + H^c = 0; \quad \tilde{\Xi}^c(t_f) = Q_f^c \quad (5.141)$$

and

$$\dot{\tilde{\Theta}}^c = F_\epsilon^c \tilde{\Theta}^c + \tilde{\Theta}^c F_\epsilon^{c'} + \tilde{\Theta}^c \theta H^c \tilde{\Theta}^c + G_\epsilon^c G_\epsilon^{c'}; \quad \tilde{\Theta}^c(t_0) = \Sigma_0^c \quad (5.142)$$

It is shown in Appendix B that the nonnegative definite solutions to these GRDEs exist on $[t_0, t_f]$ for $\theta < \theta_{Is}$ and for sufficiently small ϵ , and can be approximated by

$$\tilde{\Xi}^c = \begin{bmatrix} \tilde{\Xi}_{11}^c + O(\sqrt{\epsilon}) & O(\epsilon) \\ O(\epsilon) & \epsilon(\tilde{\Xi}_{22}^c + \tilde{\Xi}_{fb}^c) + O(\epsilon^{3/2}) \end{bmatrix} \quad (5.143)$$

$$\tilde{\Theta}^c = \begin{bmatrix} \tilde{\Theta}_{11}^c + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \tilde{\Theta}_{22}^c + \tilde{\Theta}_{fb}^c + O(\sqrt{\epsilon}) \end{bmatrix} \quad (5.144)$$

$$(5.145)$$

where $\tilde{\Xi}_{11}^c$, $\tilde{\Xi}_{22}^c$, $\tilde{\Theta}_{11}^c$ and $\tilde{\Theta}_{22}^c$ are as defined by (5.93), (5.94), (5.98) and (5.99), except that $\Delta_{s\theta}$ now satisfies the Lyapunov differential equation:

$$\begin{aligned} \dot{\Delta}_{s\theta} + \Delta_{s\theta}((I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1}(\bar{A}_s - \Sigma_{s\theta}\bar{R}_s)(I - \theta\Sigma_{s\theta}Z_{s\theta}) + \frac{d}{dt}((I - \theta\Sigma_{s\theta}Z_{s\theta})^{-1})(I \\ - \theta\Sigma_{s\theta}Z_{s\theta})) + ((I - \theta Z_{s\theta}\Sigma_{s\theta})(\bar{A}'_s - \bar{R}_s\Sigma_{s\theta})(I - \theta Z_{s\theta}\Sigma_{s\theta})^{-1} + (I - \theta Z_{s\theta}\Sigma_{s\theta})\frac{d}{dt}((I \\ - \theta Z_{s\theta}\Sigma_{s\theta})^{-1}))\Delta_{s\theta} + (Z_{s\theta}B_1 + V'B_2)(B_1Z_{s\theta} + B'_2V) + U'Q_{22}U = 0 \end{aligned} \quad (5.146)$$

with terminal condition $\Delta_{s\theta}(t_f) = 0$, and $\tilde{\Xi}_{f_b}^c$ and $\tilde{\Theta}_{f_b}^c$ are some boundary layer terms at t_f and t_0 , respectively, and they converge to zero exponentially in the τ time scale.

Furthermore, as it is shown in Appendix B, the matrix $I - \theta\tilde{\Theta}\tilde{\Xi}$ can have only positive eigenvalues for all $t \in [t_0, t_f]$ and for sufficiently small ϵ .

Hence, by Theorem 1, the cost $J_{I\theta}^c$ is can be expressed by

$$\bar{x}_0^c \tilde{\Xi}^c(t_0)(I - \theta\Sigma_0^c \tilde{\Xi}^c(t_0))^{-1} \bar{x}_0^c + \int_{t_0}^{t_f} \text{Tr}(\tilde{\Theta}^c H^c) dt - \frac{1}{\theta} \ln(\det(I - \theta\tilde{\Theta}^c(t_f)Q_f^c))$$

Some straightforward algebraic manipulations lead to part 2.v).

Finally, we substitute the slow controller $\mu_{I_s}^*$ into the full-order system dynamics to arrive at a finite-horizon control-free LEQG problem. Let x_f^s and \bar{x}^s be defined as in (5.100) and (5.101). In terms of the state variable \bar{x}^s , this LEQG problem can be written as:

$$d\bar{x}^s = F_\epsilon^s \bar{x}^s dt + G_\epsilon^s dw_t \quad (5.147)$$

$$J_{I\theta}^s := \frac{2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\bar{x}^{s'}(t_f) Q_f^s \bar{x}^s(t_f) + \int_{t_0}^{t_f} \bar{x}^{c'}(H^s + O(\sqrt{\epsilon})) \bar{x}^c dt \right) \right] \right\} \right\} \quad (5.148)$$

where

$$Q_f^s = \begin{bmatrix} \begin{bmatrix} Q_{f11} & 0 \\ 0 & 0 \end{bmatrix} + O(\epsilon) & O(\epsilon) \\ O(\epsilon) & \epsilon Q_{f22} \end{bmatrix} \quad (5.149)$$

with initial state $\bar{x}^s(t_0)$ being a Gaussian random vector, with mean \bar{x}_0^s and covariance Σ_0^s given by:

$$\bar{x}_0^s := \begin{bmatrix} \bar{x}_{10} \\ (I - \theta\Sigma_{011}Z_{s\theta}(t_0))^{-1}\bar{x}_{10} + O(\sqrt{\epsilon}) \\ \bar{x}_{20} + U(t_0)(I - \theta\Sigma_{011}Z_{s\theta}(t_0))^{-1}\bar{x}_{10} \end{bmatrix}; \quad \Sigma_0^s := \begin{bmatrix} \begin{bmatrix} \Sigma_{011} & 0 \\ 0 & 0 \end{bmatrix} + v(\epsilon) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Sigma_{022} \end{bmatrix}$$

The derivation for the above is straightforward in view of Lemma 1.

To compute the cost $J_{I\theta}^s$ explicitly, we again associate a fictitious measurement (5.61) with this LEQG problem, where v_t , or y^c , is a standard Wiener process independent of the initial state and w_t .

Then, the two GRDEs associated with this problem are:

$$\dot{\tilde{\Xi}}^s + F_\epsilon^{s'} \tilde{\Xi}^s + \tilde{\Xi}^s F_\epsilon^s + \tilde{\Xi}^s \theta G_\epsilon^s G_\epsilon^{s'} \tilde{\Xi}^s + H^s = 0; \quad \tilde{\Xi}^s(t_f) = Q_f^s \quad (5.150)$$

and

$$\dot{\tilde{\Theta}}^s = F_\epsilon^s \tilde{\Theta}^s + \tilde{\Theta}^s F_\epsilon^{s'} + \tilde{\Theta}^s \theta H^s \tilde{\Theta}^s + G_\epsilon^s G_\epsilon^{s'}; \quad \tilde{\Theta}^s(t_0) = \Sigma_0^s \quad (5.151)$$

It is shown in Appendix B that, if $A_{22}(t)$ is Hurwitz $\forall t \in [t_0, t_f]$, then these GRDEs admit nonnegative definite solutions on $[t_0, t_f]$ for $\theta < \theta_{I_s}$ and for sufficiently small ϵ , which can be approximated by

$$\tilde{\Xi}^s = \begin{bmatrix} \tilde{\Xi}_{11}^s + O(\sqrt{\epsilon}) & O(\epsilon) \\ O(\epsilon) & \epsilon(Z_{of} + \tilde{\Xi}_{fb}^s) + O(\epsilon^{3/2}) \end{bmatrix} \quad (5.152)$$

$$\tilde{\Theta}^s = \begin{bmatrix} \tilde{\Theta}_{11}^s + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Sigma_{of} + \tilde{\Theta}_{fb}^s + O(\sqrt{\epsilon}) \end{bmatrix} \quad (5.153)$$

where $\tilde{\Xi}_{fb}^s$ and $\tilde{\Theta}_{fb}^s$ are some boundary layer terms at t_f and t_0 , respectively, and they converge to zero exponentially in the τ time scale.

Furthermore, the matrix $I - \theta \tilde{\Theta} \tilde{\Xi}$ has only positive eigenvalues for all $t \in [t_0, t_f]$ and for sufficiently small ϵ .

Hence, by Theorem 1, the cost $J_{I\theta}^s$ is can be expressed by

$$\bar{x}_0' \tilde{\Xi}^s(t_0) (I - \theta \Sigma_0^s \tilde{\Xi}^s(t_0))^{-1} \bar{x}_0 + \int_{t_0}^{t_f} \text{Tr}(\tilde{\Theta}^s H^s) dt - \frac{1}{\theta} \ln(\det(I - \theta \tilde{\Theta}^s(t_f) Q_f^s))$$

Some straightforward algebraic manipulations leads to part 2.vi).

This completes the proof of the Theorem. \square

5.3 A Large Deviation Form

We again consider a large deviation form of the problem under noisy state measurements. The system under consideration is

$$\begin{cases} dx_1 = (A_{11}(t)x_1 + A_{12}(t)x_2 + B_1(t)u_t) dt + \xi G_1(t) dw_t; & x_1(t_0) = x_{10} \\ \epsilon dx_2 = (\epsilon^{1/2} A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u_t) dt + \epsilon^{1/2} \xi G_2(t) dw_t; & x_2(t_0) = x_{20} \\ dy_1 = (C_{11}(t)x_1 + C_{12}(t)x_2) dt + \xi E_1(t) dv_t; & y_1(t_0) = 0 \\ dy_2 = (\epsilon^{1/2} C_{21}(t)x_1 + C_{22}(t)x_2) dt + \epsilon^{1/2} \xi E_2(t) dv_t; & y_2(t_0) = 0 \end{cases} \quad (5.154)$$

with cost function

$$J_{I\theta}(\mu_I, \xi) = \frac{2\xi^2}{\theta} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2\xi^2} \left(x_{t_f}' Q_f x_{t_f} + \int_{t_0}^{t_f} (x' Q(t) x + u' u) dt \right) \right] \right\} \right\} \quad (5.155)$$

where the initial state x_0 is a Gaussian random variable with mean \bar{x}_0 and variance $\xi^2 \Sigma_0$, and ξ is a small scalar parameter to be varied. We will again study the solution as the parameter $\xi \rightarrow 0$. This problem is equivalent to the one considered in Section 3, if we introduce the following substitutions:

$$\theta \leftarrow \frac{\theta}{\xi^2}; \quad G_\epsilon \leftarrow \xi G; \quad E_\epsilon \leftarrow \xi E; \quad \Sigma_0 \leftarrow \xi^2 \Sigma_0 \quad (5.156)$$

Define the quantity $\theta_I^*(\epsilon)$ exactly as in (2.13). Then, for any $\theta < \theta_I^*(\epsilon)$, the problem admits an optimal controller, given by (2.14) and (2.15).

The optimal solution to the full-order problem again depends on the value of ϵ explicitly. To obtain ϵ -free solutions, we decompose the system into slow and fast subsystems. The slow subsystem can be obtained in the same way as before. Under the substitution law (5.156), the slow LEQG problem is again described by (5.7), (5.8) and (5.9). This leads to a definition of θ_{I_s} exactly as in (5.13). For any $\theta < \theta_{I_s}$, the optimal controller for the slow subproblem is exactly (5.14) and (5.15). The fast subsystem is again (5.17), (5.18) and (5.19), under the substitution law (5.156). Thus, the fast controller is exactly the same as (5.24)–(5.25). Hence, we can form the slow and

composite controllers $\mu_{I_s}^*$ and $\mu_{I_c}^*$ as before. The slow controller $\mu_{I_s}^*$ is as in (5.34) and (5.35), and the composite controller $\mu_{I_c}^*$ is exactly given by (5.40), (5.41) and (5.42).

This all then leads to the following corollary to Theorem 6:

Corollary 3 For the singularly perturbed system (5.154) under the cost function (5.155):

1. For each $\epsilon > 0$, if $N_\epsilon(t)$ is invertible $\forall t \in [t_f, t_f]$, then, $\forall \theta < \theta_I^*(\epsilon)$, the optimal cost for the full-order LEQG problem can be rewritten as:

$$J_{I\theta}^*(\epsilon; \xi) = \bar{x}_0' \bar{Z}(t_0) (I - \theta \Sigma_0 \bar{Z}(t_0))^{-1} \bar{x}_0 + O(\xi^2) \quad (5.157)$$

2. Let assumptions A1-A3 be satisfied, the pair (A_{22}, B_2) be controllable, and the pair (A_{22}, C_2) be observable $\forall t \in [t_0, t_f]$. Then,

- i. $\limsup_{\epsilon \rightarrow 0+} \theta_I^*(\epsilon) \leq \theta_{I_s}$.

- ii. $\forall \theta < \theta_{I_s}$, $\forall 0 < \epsilon \leq \bar{\epsilon}_\theta$, the optimal cost for the full-order LEQG problem can be approximated by:

$$J_{I\theta}^*(\epsilon; \xi) = \bar{x}_{10}' Z_{s\theta}(t_0) (I - \theta Q_{011} Z_{s\theta}(t_0))^{-1} \bar{x}_{10} + O(\xi^2) + O(\sqrt{\epsilon}) \quad (5.158)$$

- iii. $\forall \theta < \theta_{I_s}$, if the composite controller $\mu_{I_{c\theta}}^*$ is applied to the system, then $\exists \epsilon'_\theta > 0$ such that $\forall \epsilon \in [0, \epsilon'_\theta]$,

$$J_{I\theta}^c := J_{I\theta}(\mu_{I_{c\theta}}^*) = J_{I\theta}^*(\epsilon; \xi) + O(\sqrt{\epsilon}) + O(\xi^2 \sqrt{\epsilon}) \quad (5.159)$$

- iv. $\forall \theta < \theta_{I_s}$, if, in addition, the matrix A_{22} is Hurwitz for $t \in [t_0, t_f]$, and the slow controller $\mu_{I_{s\theta}}^*$ is applied to the system, then $\exists \hat{\epsilon}_\theta > 0$ such that $\forall \epsilon \in [0, \hat{\epsilon}_\theta]$,

$$J_{I\theta}^s := J_{I\theta}(\mu_{I_{s\theta}}^*) = J_{I\theta}^*(\epsilon; \xi) + O(\xi^2) + O(\sqrt{\epsilon}) \quad (5.160)$$

□

Infinite-Horizon Case

We take A, B, G, C, E, Q to be time-invariant, and $t_0 = 0, t_f = \infty, Q_f = 0$. The system dynamics is still described by (5.154). We associate with this system the performance index:

$$J_{I\theta\infty}(\mu_{I\infty}, \xi) = \lim_{t_f \rightarrow \infty} \frac{2\xi^2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2\xi^2} \int_0^{t_f} (x' Q(t) x + u' u) dt \right] \right\} \right\} \quad (5.161)$$

Then, the counterpart of Corollary 3 (which is corollary to Theorem 5) is:

Corollary 4 For the singularly perturbed system (5.154) under the cost function (5.161):

1. For each $\epsilon > 0$, if the pairs (A_ϵ, B_ϵ) and (A_ϵ, G_ϵ) are controllable, the pairs (A_ϵ, C_ϵ) and (A_ϵ, Q) are observable, and the matrix N_ϵ is invertible, then, $\forall \theta < \theta_{I\infty}^*(\epsilon)$, the optimal cost for the full-order LEQG problem can be written as:

$$J_{I\theta\infty}^*(\epsilon; \xi) = O(\xi^2) \quad (5.162)$$

2. Let assumption A3 be satisfied, the pairs (A_{11}, B_0) , $(A_{11}, G_1 G_1' - G_1 G_2' (G_2 G_2')^{-1} G_2 G_1')$ and (A_{22}, B_2) be controllable, and the pairs (A_{11}, C_0) , $(A_{11}, Q_{11} - Q_{12} Q_{22}^{-1} Q_{21})$ and (A_{22}, C_2) be observable. Then,

i. $\lim_{\epsilon \rightarrow 0^+} \theta_{I\infty}^*(\epsilon) = \theta_{I\infty}$.

ii. $\forall \theta < \theta_{I\infty}$, if the composite controller $\mu_{Ic\theta}^*$ is applied to the system, then $\exists \epsilon'_\theta > 0$ such that $\forall \epsilon \in [0, \epsilon'_\theta)$,

$$J_{I\theta\infty}^c := J_{I\theta\infty}(\mu_{Ic\theta}^*) = J_{I\theta\infty}^*(\epsilon; \xi) + O(\xi^2 \sqrt{\epsilon}) \quad (5.163)$$

iii. $\forall \theta < \theta_{I\infty}$, if, in addition, the matrix A_{22} is Hurwitz, and the slow controller $\mu_{Is\theta}^*$ is applied to the system, then $\exists \hat{\epsilon}_\theta > 0$ such that $\forall \epsilon \in [0, \hat{\epsilon}_\theta)$,

$$J_{I\theta\infty}^s := J_{I\theta\infty}(\mu_{Is\theta}^*) = O(\xi^2) \quad (5.164)$$

□

6 Examples

We present here three numerical results for the infinite-horizon case, one for perfect state measurements and two for noisy state measurements. As stressed earlier, the quantities $\theta_{s\infty}$ and $\theta_{I\infty}$ play important roles in the computation of approximate values for $\theta_\infty^*(\epsilon)$ and $\theta_{I\infty}^*(\epsilon)$.

Example 1

Consider the system

$$\begin{bmatrix} dx_1 \\ \epsilon dx_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u dt + \begin{bmatrix} 1 \\ \sqrt{\epsilon} \end{bmatrix} dw_t \quad (6.1)$$

$$J_{\theta\infty} = \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (2x_1^2 + 2x_1x_2 + x_2^2 + u'u) dt \right) \right] \right\} \right\} \quad (6.2)$$

By using a particular search algorithm, we can compute the quantity:

$$\theta_{s\infty} = 1.8892$$

We next compute the maximum allowable θ level for the full-order system (6.1)–(6.2) for different fixed values of ϵ .

$\theta_\infty^*(\epsilon)$	1.5616	1.9842	1.9274	1.9019	1.8930	1.8903
ϵ	1	0.1	0.01	0.001	0.0001	0.00001

Note that as $\epsilon \rightarrow 0$, $\theta_\infty^*(\epsilon) \rightarrow \theta_{s\infty}$.

Now, we choose $\theta = 1.6 < \theta_{s\infty}$, and design the suboptimal controllers for the system based on this value of θ :

$$\mu_s^*(x_1) = -1.7633x_1; \quad \mu_c^*(x_1, x_2) = -3.3249x_1 - 0.61803x_2$$

Then, we apply these controllers, μ_s^* and μ_c^* to system (6.1)–(6.2) and obtain the corresponding performance levels J_s and J_c , respectively. These values are tabulated below along with the optimal cost level for different values of ϵ :

ϵ	1	0.1	0.01	0.001	0.0001	0.00001
$J_\infty^*(\epsilon)$	∞	3.0434	2.1560	1.9818	1.9350	1.9209
$J_c(\epsilon)$	∞	∞	2.1638	1.9828	1.9351	1.9209
$J_s(\epsilon)$	∞	∞	∞	2.2150	2.1366	2.1151

We also compute the cost level at $\epsilon = 0$:

$$J_{\infty}^*(0) = J_c(0) = 1.9146; \quad J_s(0) = 2.1056$$

We see that the composite controller asymptotically achieves the optimal performance level as $\epsilon \rightarrow 0$, but the slow controller achieves a suboptimal but finite performance level asymptotically, which is consistent with the result of Theorem 4. Also, the composite controller appears to be more robust than the slow one with respect to changes in the value of ϵ . \diamond

Example 2

Consider the system

$$\begin{bmatrix} dx_1 \\ \epsilon dx_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \sqrt{\epsilon} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u dt + \begin{bmatrix} 1 \\ \sqrt{\epsilon} \end{bmatrix} dw_t \quad (6.3)$$

$$\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \sqrt{\epsilon} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\epsilon} \end{bmatrix} dv_t \quad (6.4)$$

$$J_{I\theta\infty} = \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \{ E \{ \exp \left[\frac{\theta}{2} \int_0^{t_f} (2x_1^2 + 2x_1x_2 + x_2^2 + u'u) dt \right] \} \} \quad (6.5)$$

The maximum allowable θ level for the slow subsystem is

$$\theta_{I\infty} = 1.4212$$

We can also compute the maximum allowable θ level $\theta_{I\infty}^*(\epsilon)$ of the system (6.3)–(6.5) for different values of ϵ .

$\theta_{I\infty}^*(\epsilon)$	1.4133	1.4189	1.4204	1.4209	1.4211
ϵ	0.001	10^{-4}	10^{-5}	10^{-6}	10^{-7}

Note again that as $\epsilon \rightarrow 0$, $\theta_{I\infty}^*(\epsilon) \rightarrow \theta_{I\infty}$.

Now, we choose $\theta = 1 < \theta_{I\infty}$, and design the slow and composite controllers for the corresponding cost function:

$$\mu_{I_s}^* = -\hat{x}_s^s; \quad \mu_{I_c}^* = -\hat{x}_s^c - 0.61803\hat{x}_f^c$$

where

$$\begin{aligned} d\hat{x}_s^s &= -5.6833\hat{x}_s^s dt + \begin{bmatrix} 0.44721 & 0.84721/\sqrt{\epsilon} \end{bmatrix} \begin{bmatrix} dy_1 + 2\hat{x}_s^s dt \\ dy_2 + 4\hat{x}_s^s dt \end{bmatrix} \\ \begin{bmatrix} d\hat{x}_s^c \\ \epsilon d\hat{x}_f^c \end{bmatrix} &= \begin{bmatrix} -5.6833 & 2.0944/\sqrt{\epsilon} - 1.3013 \\ -3.0902\sqrt{\epsilon} & -3.4721 \end{bmatrix} \begin{bmatrix} \hat{x}_s^c \\ \hat{x}_f^c \end{bmatrix} dt \\ &\quad + \begin{bmatrix} 0.44721 & 0.84721/\sqrt{\epsilon} \\ 0 & 0.61803 \end{bmatrix} \begin{bmatrix} dy_1 + 2\hat{x}_s^c dt \\ dy_2 + 4\hat{x}_s^c dt \end{bmatrix} \end{aligned}$$

Then, we apply $\mu_{I_s}^*$ and $\mu_{I_c}^*$ to system (6.3)–(6.5) and obtain the corresponding performance levels J_{I_s} and J_{I_c} . They are tabulated below along with the optimal cost level, for different values of ϵ :

ϵ	0.001	10^{-4}	10^{-5}	10^{-6}	10^{-7}
$J_{I\infty}^*(\epsilon)$	3.9236	3.7409	3.6873	3.6707	3.6655
$J_{Ic}(\epsilon)$	∞	∞	3.6921	3.6710	3.6655
$J_{Is}(\epsilon)$	∞	3.9147	3.7691	3.7361	3.7390

We can also compute the optimal cost level at $\epsilon = 0$:

$$J_{I\infty}^*(0) = J_{Ic}(0) = 3.6631; \quad J_{Is}(0) = 3.7361$$

We see that the composite controller asymptotically achieves the optimal performance level as $\epsilon \rightarrow 0$, but the slow controller achieves a suboptimal but finite performance level asymptotically, again consistent with the statement of Theorem 5. We also note that in this case the composite controller is more sensitive than the slow one to changes in the value of ϵ . A possible explanation for this behavior is the following: since the quantity $J_{Is}(0)$ is very close to $J_{Ic}(0)$, this means there is little for the fast controller to do to improve the performance of the overall system. Furthermore, since the fast controller is an LQG design for the fast subsystem, the closed-loop fast subsystem under such a controller may not exhibit better performance, in the H^∞ sense, than the open-loop fast subsystem. \diamond

Example 3

Consider the system

$$\begin{bmatrix} dx_1 \\ \epsilon dx_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \sqrt{\epsilon} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u dt + \begin{bmatrix} 1 \\ 2\sqrt{\epsilon} \end{bmatrix} dw_t \quad (6.6)$$

$$\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2\sqrt{\epsilon} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\epsilon} \end{bmatrix} dv_t \quad (6.7)$$

$$J_{I\theta\infty} = \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \{ E \{ \exp \left[\frac{\theta}{2} \int_0^{t_f} (3x_1^2 + 2x_1x_2 + 2x_2^2 + u'u) dt \right] \} \} \quad (6.8)$$

The maximum allowable θ level for the slow subsystem is

$$\theta_{Is\infty} = 6.3756$$

We can also compute the maximum allowable θ level $\theta_{I\infty}^*(\epsilon)$ of the system (6.6)–(6.8) for different values of ϵ .

$\theta_{I\infty}^*(\epsilon)$	6.8401	6.8836	6.5390	6.4289	6.3923
ϵ	0.01	0.0015	10^{-4}	10^{-5}	10^{-6}

Note again that as $\epsilon \rightarrow 0$, $\theta_{I\infty}^*(\epsilon) \rightarrow \theta_{Is\infty}$.

Now, we choose $\theta = 5 < \theta_{Is\infty}$, and design the slow and composite controllers for the corresponding cost function:

$$\mu_{Is}^* = -0.87591\hat{x}_s^s; \quad \mu_{Ic}^* = -0.87591\hat{x}_s^c - 0.73205\hat{x}_f^c$$

where

$$\begin{aligned} d\hat{x}_s^* &= -5.5564\hat{x}_s^* dt + \begin{bmatrix} 0.010525 & 0.24449/\sqrt{\epsilon} \end{bmatrix} \begin{bmatrix} dy_1 + 0.87591\hat{x}_s^* dt \\ dy_2 + 1.7518\hat{x}_s^* dt \end{bmatrix} \\ \begin{bmatrix} d\hat{x}_s^c \\ \epsilon d\hat{x}_f^c \end{bmatrix} &= \begin{bmatrix} -5.5564 & 0.35796/\sqrt{\epsilon} - 2.1884 \\ -23.191\sqrt{\epsilon} & -4.8552 \end{bmatrix} \begin{bmatrix} \hat{x}_s^c \\ \hat{x}_f^c \end{bmatrix} dt \\ &\quad + \begin{bmatrix} 0.010525 & 0.24449/\sqrt{\epsilon} \\ 0 & 1.5616 \end{bmatrix} \begin{bmatrix} dy_1 + 0.87591\hat{x}_s^c dt \\ dy_2 + 1.7518\hat{x}_s^c dt \end{bmatrix} \end{aligned}$$

Then, we apply $\mu_{I_s}^*$ and $\mu_{I_c}^*$ to system (6.6)–(6.8) and obtain the corresponding performance levels J_{I_s} and J_{I_c} . They are tabulated below along with the optimal cost level, for different values of ϵ :

ϵ	0.01	0.0015	10^{-4}	10^{-5}	10^{-6}
$J_{I_\infty}^*(\epsilon)$	4.6326	4.0662	3.9404	3.9210	3.9157
$J_{I_c}(\epsilon)$	∞	4.1528	3.9408	3.9210	3.9157
$J_{I_s}(\epsilon)$	∞	∞	4.6194	4.5808	4.5710

We can also compute the cost level at $\epsilon = 0$:

$$J_{I_\infty}^*(0) = J_{I_c}(0) = 3.9134; \quad J_{I_s}(0) = 4.5668$$

Again the composite controller asymptotically achieves the optimal performance level as $\epsilon \rightarrow 0$, but the slow controller achieves a suboptimal but finite performance level asymptotically. Contrary to what was observed in Example 2, here the composite controller is less sensitive to changes in the value of ϵ than the slow one. \diamond

7 Conclusion

In this paper, we have presented a model reduction technique for the LEQG problem for linear singularly perturbed systems under perfect and imperfect state measurements, in both the finite and infinite horizon cases. We have obtained a time-scale decomposition procedure that decomposes the full-order problem into appropriate slow and fast lower-order subproblems, the optimal solutions to which yield slow and fast controllers, respectively. An appropriate combination of these then yields a composite controller, which is shown to achieve asymptotically the optimal performance level for the full-order system as the singular perturbation parameter $\epsilon \rightarrow 0$. It has also been shown that when the fast subsystem is open loop stable, the slow controller can asymptotically achieve some finite (but not optimal) performance level whenever the full-order problem admits a solution. There is a clear positive gap between the asymptotic performance level a slow controller can achieve and the asymptotic performance level achieved by a full-order optimal controller, and the paper provides a characterization of this performance loss. This indicates that there is a tradeoff between controller simplicity (due to model reduction) and loss of performance. In a large deviation context, i.e., when the intensity of the noise in the system dynamics goes to zero, however, the slow controller can asymptotically achieve the optimal performance level for the full-order system, provided that the fast subsystem is open-loop stable.

To obtain the optimal solution to the slow subsystem arrived at as a result of model reduction, it has turned out that one needs to develop a theory for the general LEQG problem with general cost structure (with cross terms) and correlation between system and measurement noises. Since

the LEQG problem has not been solved in the literature in such a full generality, we have also provided in the paper (Section 3) a clean and complete solution to this problem in both finite and infinite horizons via a different line from that of [5]. This solution is exactly the central solution of the corresponding H^∞ -optimal control problem, and our line of proof would be useful even for the standard LEQG problem since it relaxes some of the assumptions made in [5].

One possible nontrivial extension of the results of this paper would be the derivation of higher-order correction terms. The composite controller constructed in the paper achieves a performance level that is $O(\sqrt{\epsilon})$ close to the optimal one. This, however, may not be sufficient in some applications. Hence, high-order correction terms for the composite controller is of some interest. Another extension would be to the multiple time scale problems, so as to obtain the counterparts of [25] which deals with the H^∞ -optimal control problem. One other challenging extension would be to the nonlinear case, under both regular [26] and singular [17] perturbations.

Appendix

A

Consider the infinite-horizon LEQG problem under perfect state measurements, with the system dynamics

$$dx = (Ax + Bu_t)dt + Gdw_t; \quad x(0) = x_0 \quad (\text{A.1})$$

where x is the n -dimensional state vector; u_t is the p -dimensional control input, w_t is an r -dimensional vector-valued standard Wiener process, and A, B, G are constant matrices. The control input u_t is generated by a closed-loop control policy μ , according to

$$u(t) = \mu(t, x_{[0,t]}) \quad (\text{A.2})$$

where $\mu \in \mathcal{M}$ is an admissible control. Associated with this system, we introduce the cost function

$$J_{\theta\infty}(\mu) = \lim_{t_f \rightarrow \infty} \frac{2}{\theta t_f} \ln \left\{ E \left\{ \exp \left[\frac{\theta}{2} \left(\int_0^{t_f} (x'Qx + u'u) dt \right) \right] \right\} \right\} \quad (\text{A.3})$$

Then, we have the following result:

Theorem 7 *Consider the infinite-horizon LEQG problem above, with the pair (A, B) controllable and the pair (A, Q) observable. Furthermore, let the following GARE*

$$A'Z + ZA - Z(BB' - \theta GG')Z + Q = 0 \quad (\text{A.4})$$

admit a minimal positive definite solution Z , such that the matrix $A - (BB' - \theta GG')Z$ is Hurwitz. Then, the optimal controller is given by

$$u^* = \mu^*(x) = -B'Zx \quad (\text{A.5})$$

with the associated cost being

$$J_{\theta\infty}^* = \inf_{\mu \in \mathcal{M}} J_{\theta\infty}(\mu) = J_{\theta\infty}(\mu^*) = \text{Tr}(GG'Z) \quad (\text{A.6})$$

Proof The proof follows readily from the results for the finite-horizon case, given in [1]. \square

B

Verification of (3.7)

By the definition of \hat{x} and the filter equation (3.6), we have the following:

$$\begin{aligned} d\hat{x} &= (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}[(A + \theta\tilde{\Sigma}Q + (\tilde{\Sigma}C' + L)N^{-1}C)\hat{x} dt + (B + \theta\tilde{\Sigma}P)u dt + (\tilde{\Sigma}C' + L) \\ &\quad N^{-1}dy] - (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\frac{d}{dt}(I - \theta\tilde{\Sigma}\tilde{Z})(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\hat{x} dt = (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}[(\tilde{A} - \tilde{\Sigma}\tilde{R} \\ &\quad - \theta(\tilde{A}\tilde{\Sigma} - \tilde{\Sigma}\tilde{R}\tilde{\Sigma})\tilde{Z} + \theta\tilde{\Sigma}\tilde{Z} + \theta\dot{\tilde{\Sigma}}\tilde{Z})\hat{x} dt + (B + \theta\tilde{\Sigma}P)u dt + (\tilde{\Sigma}C' + L)N^{-1}dy] \\ &= (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(\tilde{A} - \tilde{\Sigma}\tilde{R} + \theta(\tilde{\Sigma}\tilde{A}' + \tilde{M})\tilde{Z} - \theta\tilde{\Sigma}(\tilde{A}'\tilde{Z} + \tilde{Z}\tilde{A} - \tilde{Z}\tilde{S}\tilde{Z} + \tilde{Q}))\hat{x} dt \\ &\quad + (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}[(B + \theta\tilde{\Sigma}P)u dt + (\tilde{\Sigma}C' + L)N^{-1}dy] \end{aligned}$$

Regrouping terms in the above equation yields (3.7).

Verification of (3.11)

To show the identity (3.11), we note that

$$\tilde{\Psi} = (I - \theta\tilde{Z}\tilde{\Sigma})^{-1}\tilde{Z}$$

Thus, we have the following:

$$\begin{aligned} \dot{\tilde{\Psi}} + \tilde{\Psi}(A + \theta\tilde{\Sigma}Q) + (A + \theta\tilde{\Sigma}Q)'\tilde{\Psi} &= \theta\tilde{Z}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(\tilde{\Sigma}\dot{\tilde{Z}} + \dot{\tilde{\Sigma}}\tilde{Z})(I - \theta\tilde{\Sigma}\tilde{Z})^{-1} + \dot{\tilde{Z}}(I \\ &\quad - \theta\tilde{\Sigma}\tilde{Z})^{-1} + \tilde{\Psi}(A + \theta\tilde{\Sigma}Q) + (A + \theta\tilde{\Sigma}Q)'\tilde{\Psi} = \theta\tilde{\Psi}\dot{\tilde{\Sigma}}\tilde{\Psi} + (I - \theta\tilde{Z}\tilde{\Sigma})^{-1}\dot{\tilde{Z}}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1} + \tilde{\Psi} \\ &\quad \cdot (A + \theta\tilde{\Sigma}Q) + (A' + \theta Q\tilde{\Sigma})\tilde{\Psi} = \theta\tilde{\Psi}(\tilde{\Sigma}\tilde{A}' + \tilde{A}\tilde{\Sigma} - \tilde{\Sigma}\tilde{R}\tilde{\Sigma} + \tilde{M})\tilde{\Psi} - (I - \theta\tilde{Z}\tilde{\Sigma})^{-1}(\tilde{A}'\tilde{Z} \\ &\quad + \tilde{Z}\tilde{A} - \tilde{Z}\tilde{S}\tilde{Z} + \tilde{Q})(I - \theta\tilde{\Sigma}\tilde{Z})^{-1} + \tilde{\Psi}(A + \theta\tilde{\Sigma}Q) + (A' + \theta Q\tilde{\Sigma})\tilde{\Psi} \end{aligned}$$

It is easy to show the following identities:

$$\begin{aligned} \theta\tilde{\Psi}\tilde{A}\tilde{\Sigma}\tilde{\Psi} - \tilde{\Psi}\tilde{A}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1} + \tilde{\Psi}A &= -\theta\tilde{\Psi}LN^{-1}C\tilde{\Sigma}\tilde{\Psi} + \tilde{\Psi}BR^{-1}P'(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}; \\ \theta\tilde{\Psi}\tilde{M}\tilde{\Psi} + \theta\tilde{\Psi}\tilde{S}\tilde{\Psi} &= -\theta\tilde{\Psi}LN^{-1}L'\tilde{\Psi} + \theta\tilde{\Psi}BR^{-1}B'\tilde{\Psi}; \\ -\theta\tilde{\Psi}\tilde{\Sigma}\tilde{R}\tilde{\Sigma}\tilde{\Psi} - (I - \theta\tilde{Z}\tilde{\Sigma})^{-1}\tilde{Q}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1} &+ \theta\tilde{\Psi}\tilde{\Sigma}Q + \theta Q\tilde{\Sigma}\tilde{\Psi} = -\theta\tilde{\Psi}\tilde{\Sigma}C'N^{-1}C\tilde{\Sigma}\tilde{\Psi} \\ &+ (I - \theta\tilde{Z}\tilde{\Sigma})^{-1}PR^{-1}P'(I - \theta\tilde{\Sigma}\tilde{Z})^{-1} - Q \end{aligned}$$

Using these identities, we arrive at the following:

$$\begin{aligned} \dot{\tilde{\Psi}} + \tilde{\Psi}(A + \theta\tilde{\Sigma}Q) + (A + \theta\tilde{\Sigma}Q)'\tilde{\Psi} &= -Q - \theta\tilde{\Psi}(\tilde{\Sigma}C' + L)N^{-1}(C\tilde{\Sigma} + L')\tilde{\Psi} \\ &+ ((I - \theta\tilde{Z}\tilde{\Sigma})^{-1}P + \tilde{\Psi}B)R^{-1}(P'(I - \theta\tilde{\Sigma}\tilde{Z})^{-1} + B'\tilde{\Psi}) \end{aligned}$$

This proves identity (3.11).

Verification for (3.12)

We note the following equality:

$$\tilde{\Sigma}^{-1}\varepsilon + \tilde{\Psi}\hat{x} = \tilde{\Sigma}^{-1}x - (\tilde{\Sigma}^{-1} - \tilde{\Psi})\hat{x} = \tilde{\Sigma}^{-1}x - \tilde{\Sigma}^{-1}(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\hat{x} = \tilde{\Sigma}^{-1}e$$

It is straightforward to obtain the following expression for the differential for Υ by using the above equality:

$$\begin{aligned} d\Upsilon = & \text{Tr}(E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Psi}(\tilde{\Sigma}C' + L)N^{-1}E + (G' - E'N^{-1}(C\tilde{\Sigma} + L'))\frac{1}{\theta}\tilde{\Sigma}^{-1}(G \\ & - (\tilde{\Sigma}C' + L)N^{-1}E))dt - (x'Qx + 2x'Pu + u'Ru)dt + |\tilde{u}|_R^2dt + \frac{2}{\theta}(\epsilon'\tilde{\Sigma}^{-1}G - \epsilon'\tilde{\Sigma}^{-1} \\ & (\tilde{\Sigma}C' + L)N^{-1}Edw_t - \frac{1}{\theta}|C\epsilon - \theta(C\tilde{\Sigma} + L')\tilde{\Psi}\tilde{x}|_{N^{-1}}^2dt - \frac{1}{\theta}\epsilon'\tilde{\Sigma}^{-1}\tilde{M}\tilde{\Sigma}^{-1}\epsilon dt \end{aligned}$$

Note the following equality:

$$\begin{aligned} |G'\tilde{\Sigma}^{-1}\epsilon - E'N^{-1}(C\tilde{\Sigma} + L')\tilde{\Sigma}^{-1}\epsilon|^2 &= |(G' - E'N^{-1}L')\tilde{\Sigma}^{-1}\epsilon - E'N^{-1}(C\epsilon - \theta(C\tilde{\Sigma} + L') \\ & \tilde{\Psi}\tilde{x})|^2 = |C\epsilon - \theta(C\tilde{\Sigma} + L')\tilde{\Psi}\tilde{x}|_{N^{-1}}^2 + |(G' - E'N^{-1}L')\tilde{\Sigma}^{-1}\epsilon|^2 \\ &= |C\epsilon - \theta(C\tilde{\Sigma} + L')\tilde{\Psi}\tilde{x}|^2 + \epsilon'\tilde{\Sigma}^{-1}\tilde{M}\tilde{\Sigma}^{-1}\epsilon \end{aligned}$$

This leads to the Equation (3.12).

Verification of (5.64), (5.65), (5.123) and (5.124)

Let

$$\bar{X}i := \begin{bmatrix} \frac{1}{\theta}\tilde{\Sigma}^{-1} & -\frac{1}{\theta}\tilde{\Sigma}^{-1} + \tilde{Z} \\ -\frac{1}{\theta}\tilde{\Sigma}^{-1} + \tilde{Z} & \frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z} \end{bmatrix}$$

We first show that $\bar{X}i$ is a solution to GRDE (5.121):

$$\begin{aligned} \text{11-block of LHS} &= \frac{1}{\theta}\frac{d}{dt}(\tilde{\Sigma}^{-1}) + \frac{1}{\theta}\tilde{\Sigma}^{-1}A_\epsilon - \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon + \frac{1}{\theta}A_\epsilon\tilde{\Sigma}^{-1} - \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon \\ &+ \frac{1}{\theta}\tilde{\Sigma}^{-1}G_\epsilon G'_\epsilon\tilde{\Sigma}^{-1} + \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon + Q = 0 \\ \text{12-block of LHS} &= -\frac{1}{\theta}\frac{d}{dt}(\tilde{\Sigma}^{-1}) + \frac{d}{dt}(\tilde{Z}) - \frac{1}{\theta}\tilde{\Sigma}^{-1}B_\epsilon B'_\epsilon\tilde{Z} - (\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z})(A_\epsilon - S_\epsilon\tilde{Z}) \\ &+ \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon - A'_\epsilon(\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z}) + \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon - \tilde{\Sigma}^{-1}G_\epsilon G'_\epsilon(\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z}) - \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon \\ &= -\frac{1}{\theta}\frac{d}{dt}(\tilde{\Sigma}^{-1}) - \frac{1}{\theta}\tilde{\Sigma}^{-1}B_\epsilon B'_\epsilon\tilde{Z} - \frac{1}{\theta}\tilde{\Sigma}^{-1}(A_\epsilon - S_\epsilon\tilde{Z}) + \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon - \frac{1}{\theta}A'_\epsilon\tilde{\Sigma}^{-1} \\ &- \tilde{\Sigma}^{-1}G_\epsilon G'_\epsilon(\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z}) - Q = 0 \\ \text{22-block of LHS} &= \frac{1}{\theta}\frac{d}{dt}(\tilde{\Sigma}^{-1}) - \dot{\tilde{Z}} + (\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z})B_\epsilon B'_\epsilon\tilde{Z} + (\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z})(A_\epsilon - S_\epsilon\tilde{Z}) \\ &- \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon + \tilde{Z}B_\epsilon B'_\epsilon(\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z}) + (A'_\epsilon - \tilde{Z}S_\epsilon)(\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z}) - \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon \\ &+ (\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z})\theta G_\epsilon G'_\epsilon(\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z}) + \frac{1}{\theta}C'_\epsilon N_\epsilon^{-1}C_\epsilon + \tilde{Z}B_\epsilon B'_\epsilon\tilde{Z} \\ &= \frac{1}{\theta}\frac{d}{dt}(\tilde{\Sigma}^{-1}) - \dot{\tilde{Z}} + (\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z})B_\epsilon B'_\epsilon\tilde{Z} + \frac{1}{\theta}\tilde{\Sigma}^{-1}(A_\epsilon - S_\epsilon\tilde{Z}) - \tilde{Z}A_\epsilon + \tilde{Z}S_\epsilon\tilde{Z} + \tilde{Z}B_\epsilon \\ &B'_\epsilon(\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z}) + (A'_\epsilon - \tilde{Z}S_\epsilon)\frac{1}{\theta}\tilde{\Sigma}^{-1} - A'_\epsilon\tilde{Z} + \tilde{Z}S_\epsilon\tilde{Z} - \frac{1}{\theta}R_\epsilon - Q \\ &+ (\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z})\theta G_\epsilon G'_\epsilon(\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z}) + \tilde{Z}B_\epsilon B'_\epsilon\tilde{Z} \\ &= \frac{1}{\theta}\frac{d}{dt}(\tilde{\Sigma}^{-1}) + \frac{1}{\theta}\tilde{\Sigma}^{-1}B_\epsilon B'_\epsilon\tilde{Z} + \frac{1}{\theta}\tilde{\Sigma}^{-1}A_\epsilon - \frac{1}{\theta}\tilde{\Sigma}^{-1}S_\epsilon\tilde{Z} + \frac{1}{\theta}\tilde{Z}B_\epsilon B'_\epsilon\tilde{\Sigma}^{-1} + \frac{1}{\theta}A'_\epsilon\tilde{\Sigma}^{-1} \\ &- \frac{1}{\theta}\tilde{Z}S_\epsilon\tilde{\Sigma}^{-1} - \frac{1}{\theta}R_\epsilon + (\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z})\theta G_\epsilon G'_\epsilon(\frac{1}{\theta}\tilde{\Sigma}^{-1} - \tilde{Z}) - \theta\tilde{Z})G_\epsilon G'_\epsilon\tilde{Z} \\ &= \frac{1}{\theta}\frac{d}{dt}(\tilde{\Sigma}^{-1}) + \frac{1}{\theta}\tilde{\Sigma}^{-1}A_\epsilon + \frac{1}{\theta}A'_\epsilon\tilde{\Sigma}^{-1} - \frac{1}{\theta}R_\epsilon + \frac{1}{\theta}\tilde{\Sigma}^{-1}G_\epsilon G'_\epsilon\tilde{\Sigma}^{-1} = 0 \end{aligned}$$

To prove that $\tilde{\Xi}$ defined by (5.123) is the solution to GRDE (5.121), we will first show that $\tilde{\Xi}^{-1}$ is a solution to the GRDE:

$$-\frac{d}{dt}(\tilde{\Xi}^{-1}) + \tilde{\Xi}^{-1}F_\epsilon' + F_\epsilon'\tilde{\Xi}^{-1} + \theta G_\epsilon'G_\epsilon' + \tilde{\Xi}^{-1}H'\tilde{\Xi}^{-1} = 0 \quad (\text{B.1})$$

assuming for the moment that $Q_f > 0$ and $\tilde{\Delta}(t_f) > 0$ (which guarantees the invertibility of \tilde{Z} and $\tilde{\Delta}$, as well as $\tilde{\Xi}$). Then, $\tilde{\Xi}$ is the solution to GRDE (5.121) by a limiting argument, in view of the uniqueness of the solution. Assuming the invertibility of Q_f and $\tilde{\Delta}(t_f) > 0$,

$$\tilde{\Xi}^{-1} = \begin{bmatrix} \tilde{Z}^{-1} & \tilde{Z}^{-1} \\ \tilde{Z}^{-1} & (\tilde{Z} - \theta\tilde{Z}\tilde{\Sigma}\tilde{Z})^{-1} + \tilde{\Delta}^{-1} \end{bmatrix}$$

Let $\Xi_{22} := (\tilde{Z} - \theta\tilde{Z}\tilde{\Sigma}\tilde{Z})^{-1} + \tilde{\Delta}^{-1}$. We show that the above satisfies GRDE (B.1) block by block.

$$\begin{aligned} \text{11-block of LHS} &= -\frac{d}{dt}(\tilde{Z}^{-1}) + A_\epsilon\tilde{Z}^{-1} - B_\epsilon B_\epsilon' + \tilde{Z}^{-1}A_\epsilon' - B_\epsilon B_\epsilon' + \tilde{Z}^{-1}Q\tilde{Z}^{-1} + B_\epsilon B_\epsilon' \\ &\quad + \theta G_\epsilon'G_\epsilon' = 0 \end{aligned}$$

$$\begin{aligned} \text{12-block of LHS} &= -\frac{d}{dt}(\tilde{Z}^{-1}) + A_\epsilon\tilde{Z}^{-1} - B_\epsilon B_\epsilon'\tilde{Z}\Xi_{22} + \tilde{Z}^{-1}C_\epsilon'N_\epsilon^{-1}C_\epsilon\tilde{\Sigma}(I - \theta\tilde{Z}\tilde{\Sigma})^{-1} \\ &\quad + \tilde{Z}^{-1}(A_\epsilon - S_\epsilon\tilde{Z} - (I - \theta\tilde{Z}\tilde{\Sigma})^{-1}\tilde{\Sigma}C_\epsilon'N_\epsilon^{-1}C_\epsilon)' + \tilde{Z}^{-1}Q\tilde{Z}^{-1} + B_\epsilon B_\epsilon'\tilde{Z}\Xi_{22} \\ &= -\frac{d}{dt}(\tilde{Z}^{-1}) + A_\epsilon\tilde{Z}^{-1} + \tilde{Z}^{-1}A_\epsilon' - S_\epsilon + \tilde{Z}^{-1}Q\tilde{Z}^{-1} = 0 \end{aligned}$$

For the 22-block of (B.1), the proof is quite involved. We obtain the GRDE that is satisfied by Ξ_{22} :

$$\begin{aligned} -\dot{\Xi}_{22} &+ (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma}C_\epsilon'N_\epsilon^{-1}C_\epsilon\tilde{Z}^{-1} + (A_\epsilon - S_\epsilon\tilde{Z} - (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma}C_\epsilon'N_\epsilon^{-1}C_\epsilon)\Xi_{22} \\ &+ \tilde{Z}^{-1}C_\epsilon'N_\epsilon^{-1}C_\epsilon\tilde{\Sigma}(I - \theta\tilde{Z}\tilde{\Sigma})^{-1} + \Xi_{22}(A_\epsilon - S_\epsilon\tilde{Z} - (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma}C_\epsilon'N_\epsilon^{-1}C_\epsilon)' \\ &+ \tilde{Z}^{-1}Q\tilde{Z}^{-1} + \Xi_{22}\tilde{Z}B_\epsilon B_\epsilon'\tilde{Z}\Xi_{22} + \theta(I - \theta\tilde{\Sigma}\tilde{Z})^{-1}\tilde{\Sigma}C_\epsilon'N_\epsilon^{-1}C_\epsilon\tilde{\Sigma}(I - \theta\tilde{Z}\tilde{\Sigma})^{-1} = 0 \end{aligned}$$

The above GRDE will be written in the following compact form:

$$-\dot{\Xi}_{22} + \check{F}\Xi_{22} + \Xi_{22}\check{F}' + \Xi_{22}\check{G}\Xi_{22} + \check{H} = 0 \quad (\text{B.2})$$

where the definitions of \check{F} , \check{G} and \check{H} should be clear from the context.

Note that $\tilde{\Xi}^{-1}$ satisfies GRDE (B.1). By a simple application of matrix inversion identities, we obtain

$$\tilde{\Xi}^{-1} = \begin{bmatrix} \tilde{Z}^{-1} & \tilde{Z}^{-1} \\ \tilde{Z}^{-1} & (\tilde{Z} - \theta\tilde{Z}\tilde{\Sigma}\tilde{Z})^{-1} \end{bmatrix}$$

Then, clearly $\tilde{\Xi}_{22} := (\tilde{Z} - \theta\tilde{Z}\tilde{\Sigma}\tilde{Z})^{-1}$ satisfies the GRDE (B.2).

Let us evaluate the matrix $\check{F} + \tilde{\Xi}_{22}\check{G}$. We start with:

$$\begin{aligned} (I - \theta\tilde{\Sigma}\tilde{Z})(\check{F} + \tilde{\Xi}_{22}\check{G}) &= (I - \theta\tilde{\Sigma}\tilde{Z})(A_\epsilon - S_\epsilon\tilde{Z}) - \tilde{\Sigma}C_\epsilon'N_\epsilon^{-1}C_\epsilon + B_\epsilon B_\epsilon'\tilde{Z} = A_\epsilon + \theta G_\epsilon'G_\epsilon'\tilde{Z} \\ &\quad - \theta\tilde{\Sigma}(\tilde{Z}A_\epsilon - \tilde{Z}S_\epsilon\tilde{Z}) - \tilde{\Sigma}C_\epsilon'N_\epsilon^{-1}C_\epsilon = A_\epsilon + \theta G_\epsilon'G_\epsilon'\tilde{Z} + \theta\tilde{\Sigma}(A_\epsilon\tilde{Z} + \frac{d}{dt}\tilde{Z}) - \tilde{\Sigma}R_\epsilon \\ &= A_\epsilon - \tilde{\Sigma}R_\epsilon + \theta(G_\epsilon'G_\epsilon' + \tilde{\Sigma}A_\epsilon')\tilde{Z} + \theta\tilde{\Sigma}\frac{d}{dt}\tilde{Z} = A_\epsilon - \tilde{\Sigma}R_\epsilon - \theta(A_\epsilon - \tilde{\Sigma}R_\epsilon)\tilde{\Sigma}\tilde{Z} \\ &\quad + \theta\tilde{\Sigma}\frac{d}{dt}\tilde{Z} + \theta\frac{d}{dt}\tilde{\Sigma}\tilde{Z} = (A_\epsilon - \tilde{\Sigma}R_\epsilon)(I - \theta\tilde{\Sigma}\tilde{Z}) - \frac{d}{dt}(I - \theta\tilde{\Sigma}\tilde{Z}) \end{aligned}$$

Hence, we have

$$\check{F} + \tilde{\Xi}_{22}\check{G} = (I - \theta\tilde{\Sigma}\tilde{Z})^{-1}(A_\epsilon - \tilde{\Sigma}R_\epsilon)(I - \theta\tilde{\Sigma}\tilde{Z}) + \frac{d}{dt}((I - \theta\tilde{\Sigma}\tilde{Z})^{-1})(I - \theta\tilde{\Sigma}\tilde{Z})$$

Note that the Lyapunov equation (5.125) can be rewritten as:

$$\dot{\tilde{\Delta}} + (\tilde{F} + \tilde{\Xi}_{22}\tilde{G})'\tilde{\Delta} + \tilde{\Delta}(\tilde{F} + \tilde{\Xi}_{22}\tilde{G}) + \tilde{G} = 0$$

Then, we have

$$-\frac{d}{dt}(\tilde{\Delta}^{-1}) + \tilde{\Delta}^{-1}(\tilde{F} + \tilde{\Xi}_{22}\tilde{G})' + (\tilde{F} + \tilde{\Xi}_{22}\tilde{G})\tilde{\Delta}^{-1} + \tilde{\Delta}^{-1}\tilde{G}\tilde{\Delta}^{-1} = 0$$

Furthermore, we have the equality:

$$-\dot{\tilde{\Xi}}_{22} + \tilde{F}\tilde{\Xi}_{22} + \tilde{\Xi}_{22}\tilde{F}' + \tilde{\Xi}_{22}\tilde{G}\tilde{\Xi}_{22} + \tilde{H} - \frac{d}{dt}(\tilde{\Delta}^{-1}) + \tilde{\Delta}^{-1}(\tilde{F} + \tilde{\Xi}_{22}\tilde{G})' + (\tilde{F} + \tilde{\Xi}_{22}\tilde{G})\tilde{\Delta}^{-1} + \tilde{\Delta}^{-1}\tilde{G}\tilde{\Delta}^{-1} = 0$$

which can be rewritten as:

$$-\frac{d}{dt}(\tilde{\Delta}^{-1} + \tilde{\Xi}_{22}) + \tilde{F}(\tilde{\Xi}_{22} + \tilde{\Delta}^{-1}) + (\tilde{\Xi}_{22} + \tilde{\Delta}^{-1})\tilde{F}' + (\tilde{\Delta}^{-1} + \tilde{\Xi}_{22})\tilde{G}(\tilde{\Delta}^{-1} + \tilde{\Xi}_{22}) + \tilde{H} = 0$$

This shows that $\tilde{\Xi}_{22}$ is a solution of GRDE (B.2). Hence, we have proved that (5.123) is the solution to GRDE (5.121).

In the infinite horizon case, the GARE (5.62) admits a positive definite solution (5.64), since $\tilde{\Delta} \rightarrow \tilde{\Delta}_\infty$ as $t_f \rightarrow \infty$. To show that it is the minimal one of such solutions, we simply compute the matrix $F_\epsilon^e + \tilde{\Xi}_\infty^{-1}H^e$:

$$F_\epsilon^e + \tilde{\Xi}_\infty^{-1}H^e = \begin{bmatrix} A_\epsilon + \tilde{Z}_\infty^{-1}Q & 0 \\ * & \tilde{F} + \tilde{\Xi}_{\infty 22}\tilde{G} \end{bmatrix}$$

where $*$ denotes arbitrary term of no interest. It is clear that the matrix $A_\epsilon + \tilde{Z}_\infty^{-1}Q$ is anti-stable. By the derivation above, the matrix $\tilde{F} + \tilde{\Xi}_{\infty 22}\tilde{G} = (I - \theta\tilde{\Sigma}_\infty\tilde{Z}_\infty)^{-1}(A_\epsilon - \tilde{\Sigma}_\infty R_\epsilon)(I - \theta\tilde{\Sigma}_\infty\tilde{Z}_\infty)$ is Hurwitz. Then, by Theorem 5 of [22], the matrix $\tilde{F} + \tilde{\Xi}_{\infty 22}\tilde{G}$ is anti-stable. This implies that the matrix $F_\epsilon^e + \tilde{\Xi}_\infty^{-1}H^e$ is anti-stable, and hence, the matrix $F_\epsilon^e + \theta G_\epsilon^e G_\epsilon^{e'} \tilde{\Xi}_\infty$ is Hurwitz. Then, $\tilde{\Xi}_\infty$ is the minimal solution to the GARE.

To prove that $\tilde{\Theta}$ defined by (5.124) is the solution to GRDE (5.122), we will show that $\tilde{\Theta}^{-1}$ is a solution to the GRDE:

$$\frac{d}{dt}(\tilde{\Theta}^{-1}) + F_\epsilon^{e'}\tilde{\Theta}^{-1} + \tilde{\Theta}^{-1}F_\epsilon^e + \tilde{\Theta}^{-1}G_\epsilon^e G_\epsilon^{e'}\tilde{\Theta}^{-1} + \theta H^e = 0 \quad (\text{B.3})$$

assuming for the moment that $Q_f > 0$ and $\tilde{\Pi}(t_0) > 0$ (which guarantees the invertibility of \tilde{Z} and $\tilde{\Pi}$, as well as $\tilde{\Theta}$). Then, $\tilde{\Theta}$ is the solution to GRDE (5.122) by a limiting argument, in view of the uniqueness of the solution. Note that $\theta\tilde{\Xi}$ satisfies the GRDE (B.3). By a proof similar to the one for $\tilde{\Xi}^{-1}$, we can prove that (5.124) is the solution to GRDE (5.122), and (5.65) is the minimal solution to GARE (5.63).

It is straightforward to see that

$$\tilde{\Xi}(t_0)^{-1} - \theta\tilde{\Theta}_0^e = \begin{bmatrix} \tilde{Z}(t_0)^{-1} - \theta\tilde{\Sigma}_0 & \tilde{Z}(t_0)^{-1} \\ \tilde{Z}(t_0)^{-1} & \tilde{Z}(t_0)^{-1}(\tilde{Z}(t_0)^{-1} - \theta\tilde{\Sigma}_0)^{-1}\tilde{Z}(t_0)^{-1} + \tilde{\Delta}(t_0)^{-1} \end{bmatrix} > 0$$

since $\tilde{Z}(t_0)^{-1} - \theta\tilde{\Sigma}_0 > 0$ and $\tilde{\Delta}(t_0)^{-1} > 0$, if $Q_f > 0$ and $\tilde{\Delta}(t_f) > 0$. Note that $\tilde{\Xi}^{-1}$ and $\theta\tilde{\Theta}$ satisfy the same GRDE, but with different initial conditions. Hence, $\tilde{\Xi}^{-1} - \theta\tilde{\Theta} > 0$ for all $t \in [t_0, t_f]$. Then, we have that the matrix $I - \theta\tilde{\Theta}(t)\tilde{\Xi}(t)$ has only positive eigenvalues (increasing as Q_f and $\tilde{\Delta}(t_f)$ decrease to singular nonnegative definite matrices), for each t by a limiting argument.

Verification of (5.55) and (5.116)

In view of (5.48), we have

$$J_{I\theta\infty}^*(\epsilon) = \text{Tr}(\tilde{\Sigma}_\infty Q + (\Pi_\infty^{-1} + \theta T^{-1'}(\tilde{Z}_\infty - \theta \tilde{Z}_\infty \tilde{\Sigma}_\infty \tilde{Z}_\infty)T^{-1})^{-1}T^{-1'}((I - \theta \tilde{Z}_\infty \tilde{\Sigma}_\infty)Q \\ (I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty) + \tilde{Z}_\infty B_\epsilon B_\epsilon' \tilde{Z}_\infty)T^{-1})$$

Note that

$$\tilde{Z}_\infty - \theta \tilde{Z}_\infty \tilde{\Sigma}_\infty \tilde{Z}_\infty = \begin{bmatrix} Z_{s\theta} - \theta Z_{s\theta} \Sigma_{s\theta} Z_{s\theta} + O(\sqrt{\epsilon}) & O(\epsilon) \\ O(\epsilon) & O(\epsilon) \end{bmatrix}$$

Hence, we have the approximation:

$$(\Pi_\infty^{-1} + \theta T^{-1'}(\tilde{Z}_\infty - \theta \tilde{Z}_\infty \tilde{\Sigma}_\infty \tilde{Z}_\infty)T^{-1})^{-1} \\ = \begin{bmatrix} (\Pi_{s\theta}^{-1} + \theta(Z_{s\theta} - \theta Z_{s\theta} \Sigma_{s\theta} Z_{s\theta}))^{-1} + O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ O(\sqrt{\epsilon}) & \Pi_f + O(\sqrt{\epsilon}) \end{bmatrix}$$

In view of the approximations for \tilde{Z}_∞ and $\tilde{\Sigma}_\infty$, we have that

$$\begin{aligned} \text{Tr}(\tilde{\Sigma}_\infty Q) &= \text{Tr}(\Sigma_{s\theta} Q_{11} + \Sigma_f Q_{22}) + O(\sqrt{\epsilon}) \\ (I - \theta \tilde{Z}_\infty \tilde{\Sigma}_\infty)Q(I - \theta \tilde{\Sigma}_\infty \tilde{Z}_\infty) \\ &= \begin{bmatrix} (I - \theta Z_{s\theta} \Sigma_{s\theta})Q_{11}(I - \theta \Sigma_{s\theta} Z_{s\theta}) + O(\sqrt{\epsilon}) & (I - \theta Z_{s\theta} \Sigma_{s\theta})Q_{12} + O(\sqrt{\epsilon}) \\ Q_{21}(I - \theta \Sigma_{s\theta} Z_{s\theta}) + O(\sqrt{\epsilon}) & Q_{22} + O(\sqrt{\epsilon}) \end{bmatrix} \\ \tilde{Z}_\infty B_\epsilon B_\epsilon' \tilde{Z}_\infty &= \begin{bmatrix} (Z_{s\theta} B_1 + (V' + U' Z_f) B_2)(B_1' Z_{s\theta} + B_2'(V + Z_f U)) + O(\sqrt{\epsilon}) \\ Z_f B_2(B_1' Z_{s\theta} + B_2'(V + Z_f U)) + O(\sqrt{\epsilon}) \\ (Z_{s\theta} B_1 + (V' + U' Z_f) B_2)B_2' Z_f + O(\sqrt{\epsilon}) \\ Z_f B_2 B_2' Z_f + O(\sqrt{\epsilon}) \end{bmatrix} \end{aligned}$$

Using all of these approximations in (5.48) yields easily (5.55).

In the finite-horizon case, a derivation similar to above leads to (5.116).

A Useful Lemma

In order to prove the optimality for the suboptimal controllers, we need the following lemma in most of our derivations to follow.

Lemma 1

- | | |
|--|---|
| 1. $A_{22}U_1 + B_2 B_2' V_1 = -B_2 B_1'$ | 2. $A_{22}U_2 + B_2 B_2' V_2 = 0$ |
| 3. $Q_{22}U_2 - A_{22}'V_2 = Q_{21}$ | 4. $Q_{22}U_1 - A_{22}'V_1 = A_{12}'$ |
| 5. $A_{22}'X_1 + C_2'N^{-1}C_2Y_1 = -C_2'N^{-1}C_1$ | 6. $A_{22}'X_2 + C_2'N^{-1}C_2Y_2 = 0$ |
| 7. $G_2G_2'X_2 - A_{22}Y_2 = G_2G_1'$ | 8. $G_2G_2'X_1 - A_{22}Y_1 = A_{21}$ |
| 9. $A_{22}U + B_2 B_2' V = -B_2 B_1' Z_{s\gamma}$ | 10. $Q_{22}U - A_{22}'V = Q_{21} + A_{12}'Z_{s\gamma}$ |
| 11. $A_{22}'X + C_2'N^{-1}C_2Y = -C_2'N^{-1}C_1\Sigma_{s\gamma}$ | 12. $G_2G_2'X - A_{22}Y = G_2G_1' + A_{21}\Sigma_{s\gamma}$ |

□

Verification of (5.92), (5.97), (5.143) and (5.144)

Assume that $\tilde{\Xi}^c$ and $\tilde{\Theta}^c$ are in the following forms:

$$\tilde{\Xi}^c = \begin{bmatrix} \Xi_{11}^c & \epsilon \Xi_{12}^c \\ \epsilon \Xi_{21}^c & \Xi_{22}^c \end{bmatrix}; \quad \tilde{\Theta}^c = \begin{bmatrix} \Theta_{11}^c & \sqrt{\epsilon} \Theta_{12}^c \\ \sqrt{\epsilon} \Theta_{21}^c & \Theta_{22}^c \end{bmatrix}$$

Substitute these forms into GRDEs (5.141) and (5.142) to obtain differential equations for Ξ_{11}^c , Ξ_{12}^c , Ξ_{22}^c , Θ_{11}^c , Θ_{12}^c and Θ_{22}^c :

$$\begin{aligned} \dot{\Xi}_{11}^c + \Xi_{11}^c F_{11}^c + F_{11}^c \Xi_{11}^c + \Xi_{11}^c \theta G_1^c G_1^{c'} \Xi_{11}^c + H_{11}^c + O(\sqrt{\epsilon}) &= 0 \\ \epsilon \dot{\Xi}_{12}^c + \Xi_{12}^c F_{22}^c + \Xi_{11}^c O(1) + O(1) + O(\sqrt{\epsilon}) &= 0 \\ \epsilon \dot{\Xi}_{22}^c + \Xi_{22}^c F_{22}^c + F_{22}^c \Xi_{22}^c + H_{22}^c + O(\sqrt{\epsilon}) &= 0 \end{aligned}$$

and

$$\begin{aligned} \dot{\Theta}_{11}^c &= F_{11}^c \Theta_{11}^c + \Theta_{11}^c F_{11}^{c'} + \Theta_{11}^c \theta H_{11}^c \Theta_{11}^c + G_1^c G_1^{c'} + O(\sqrt{\epsilon}) \\ \epsilon \dot{\Theta}_{12}^c &= F_{22}^c \Theta_{12}^c + O(1) \Theta_{11}^c + O(1) + O(\sqrt{\epsilon}) \\ \epsilon \dot{\Theta}_{22}^c &= F_{22}^c \Theta_{22}^c + \Theta_{22}^c F_{22}^{c'} + G_2^c G_2^{c'} \end{aligned}$$

It is clear that the matrix F_{22}^c is Hurwitz. We need only to show that

$$\dot{\tilde{\Xi}}_{11}^c + \tilde{\Xi}_{11}^c F_{11}^c + F_{11}^c \tilde{\Xi}_{11}^c + \tilde{\Xi}_{11}^c \theta G_1^c G_1^{c'} \tilde{\Xi}_{11}^c + H_{11}^c = 0 \quad (\text{B.4})$$

$$\tilde{\Xi}_{22}^c F_{22}^c + F_{22}^c \tilde{\Xi}_{22}^c + H_{22}^c = 0 \quad (\text{B.5})$$

and

$$\dot{\tilde{\Theta}}_{11}^c = F_{11}^c \tilde{\Theta}_{11}^c + \tilde{\Theta}_{11}^c F_{11}^{c'} + \tilde{\Theta}_{11}^c \theta H_{11}^c \tilde{\Theta}_{11}^c + G_1^c G_1^{c'} \quad (\text{B.6})$$

$$F_{22}^c \tilde{\Theta}_{22}^c + \tilde{\Theta}_{22}^c F_{22}^{c'} + G_2^c G_2^{c'} = 0 \quad (\text{B.7})$$

Then, by an application of implicit function Theorem as in the proof for Theorem 2 in [12], it follows that (5.143) and (5.144) are the solutions to GRDEs (5.141) and (5.142), respectively.

We first show that $\tilde{\Xi}_{22}^c$ is the unique positive definite solution to GARE (B.5) for each $t \in [t_0, t_f]$. This can be done block by block:

$$\begin{aligned} \text{22-block of LHS} &= +\Delta_f B_2 B_2' Z_f + \Delta_f (A_{22} - B_2 B_2' Z_f - \Sigma_f C_2' N^{-1} C_2) + Z_f B_2 B_2' \Delta_f \\ &\quad + (A_{22} - B_2 B_2' Z_f - \Sigma_f C_2' N^{-1} C_2)' \Delta_f + Z_f B_2 B_2' Z_f \\ &= \Delta_f (A_{22} - \Sigma_f C_2' N^{-1} C_2) + (A_{22} - \Sigma_f C_2' N^{-1} C_2)' \Delta_f + Z_f B_2 B_2' Z_f = 0 \end{aligned}$$

$$\begin{aligned} \text{12-block of LHS} &= -(Z_f + \Delta_f) B_2 B_2' Z_f - \Delta_f (A_{22} - B_2 B_2' Z_f - \Sigma_f C_2' N^{-1} C_2) - A' \Delta_f \\ &\quad + C_2' N^{-1} C_2 \Sigma_f \Delta_f = -Z_f B_2 B_2' Z_f - \Delta_f (A_{22} - \Sigma_f C_2' N^{-1} C_2) \\ &\quad - (A' - C_2' N^{-1} C_2 \Sigma_f) \Delta_f = 0 \end{aligned}$$

$$\begin{aligned} \text{11-block of LHS} &= (Z_f + \Delta_f) A - \Delta_f \Sigma_f C_2' N^{-1} C_2 + A' (Z_f + \Delta_f) - C_2' N^{-1} C_2 \Sigma_f \Delta_f \\ &\quad + Q_{22} = (\Delta_f (A - \Sigma_f C_2' N^{-1} C_2) + (A' - C_2' N^{-1} C_2 \Sigma_f) \Delta_f + Z_f B_2 B_2' Z_f) = 0 \end{aligned}$$

By duality, we can prove that $\tilde{\Theta}_{22}^c$ is the unique positive definite solution to GARE (B.7) for each $t \in [t_0, t_f]$.

Toward proving (B.4) and (B.6), we first note that the matrices \bar{A}_s , \bar{S}_s , \bar{Q}_s , \bar{A}_s , \bar{R}_s and \bar{M}_s can be rewritten as:

$$\begin{aligned} \bar{A}_s &= A_{11} - A_{12} Q_{22}^{-1} Q_{21} + (S_{12} + A_{12} Q_{22}^{-1} A_{22}') (S_{22} + A_{22} Q_{22}^{-1} A_{22}')^{-1} A_{22} Q_{22}^{-1} Q_{21} \\ \bar{S}_s &= S_{11} + A_{12} Q_{22}^{-1} A_{12}' - (S_{12} + A_{12} Q_{22}^{-1} A_{22}') (S_{22} + A_{22} Q_{22}^{-1} A_{22}')^{-1} \end{aligned}$$

$$\begin{aligned}
& \cdot (S_{21} + A_{22}Q_{22}^{-1}A'_{12}) \\
\bar{Q}_s &= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} + Q_{12}Q_{22}^{-1}A'_{22}(S_{22} + A_{22}Q_{22}^{-1}A'_{22})^{-1}A_{22}Q_{22}^{-1}Q_{21} \\
\bar{A}_s &= A_{11} - G_1G'_2(G_2G'_2)^{-1}A_{21} + G_1G'_2(G_2G'_2)^{-1}A_{22} \\
& \quad \cdot (R_{22} + A'_{22}(G_2G_2)^{-1}A_{22})^{-1}(R_{21} + A'_{22}(G_2G_2)^{-1}A_{21}) \\
\bar{R}_s &= R_{11} + A'_{21}(G_2G'_2)^{-1}A_{21} - (R_{12} + A'_{21}(G_2G_2)^{-1}A_{22}) \\
& \quad \cdot (R_{22} + A'_{22}(G_2G_2)^{-1}A_{22})^{-1}(R_{21} + A'_{22}(G_2G_2)^{-1}A_{21}) \\
\bar{M}_s &= G_1G'_1 - G_1G'_2(G_2G'_2)^{-1}G_2G'_1 + G_1G'_2(G_2G'_2)^{-1}A_{22} \\
& \quad \cdot (R_{22} + A'_{22}(G_2G_2)^{-1}A_{22})^{-1}A'_{22}(G_2G'_2)^{-1}G_2G_1
\end{aligned}$$

Now, we show that $\bar{\Xi}_s$ is a positive definite solution to GRDE (B.4), where

$$\bar{\Xi}_s := \begin{bmatrix} \frac{1}{\theta}\Sigma_{s\theta}^{-1} & -\frac{1}{\theta}\Sigma_{s\theta}^{-1} + Z_{s\theta} \\ -\frac{1}{\theta}\Sigma_{s\theta}^{-1} + Z_{s\theta} & \frac{1}{\theta}\Sigma_{s\theta}^{-1} - Z_{s\theta} \end{bmatrix}$$

Again, we will show this block by block. For the 11-block of GRDE (B.4), we have

$$\begin{aligned}
\text{LHS} &= \frac{1}{\theta} \frac{d}{dt}(\Sigma_{s\theta}^{-1}) + \frac{1}{\theta}\Sigma_{s\theta}^{-1}A_{11} - \frac{1}{\theta}\Sigma_{s\theta}^{-1}((\Sigma_{s\theta}C'_1 + Y'C'_2)N^{-1}C_1 + X'A_{21}) + \frac{1}{\theta}A'_{11}\Sigma_{s\theta}^{-1} \\
& \quad - \frac{1}{\theta}((\Sigma_{s\theta}C'_1 + Y'C'_2)N^{-1}C_1 + X'A_{21})'\Sigma_{s\theta}^{-1} + \frac{1}{\theta}\Sigma_{s\theta}^{-1}G_1G'_1\Sigma_{s\theta}^{-1} - \frac{1}{\theta}\Sigma_{s\theta}^{-1}G_1G'_2X\Sigma_{s\theta}^{-1} - \frac{1}{\theta} \\
& \quad \cdot \Sigma_{s\theta}^{-1}X'G_2G'_1\Sigma_{s\theta}^{-1} + \frac{1}{\theta}\Sigma_{s\theta}^{-1}(X'G_2G'_2X + (\Sigma_{s\theta}C'_1 + Y'C'_2)N^{-1}(C_1\Sigma_{s\theta} + C_2Y))\Sigma_{s\theta}^{-1} + Q_{11} \\
&= \frac{1}{\theta} \frac{d}{dt}(\Sigma_{s\theta}^{-1}) + \frac{1}{\theta}\Sigma_{s\theta}^{-1}A_{11} + \frac{1}{\theta}A'_{11}\Sigma_{s\theta}^{-1} - \frac{1}{\theta}R_{11} + \frac{1}{\theta}\Sigma_{s\theta}^{-1}G_1G'_1\Sigma_{s\theta}^{-1} - \frac{1}{\theta}\Sigma_{s\theta}^{-1}(X'A_{21}\Sigma_{s\theta} \\
& \quad + \Sigma_{s\theta}A_{21}X + G_1G'_2X + X'G_2G'_1 - X'G_2G'_2X - Y'C'_2N^{-1}C_2Y)\Sigma_{s\theta}^{-1} \\
&= \frac{1}{\theta} \frac{d}{dt}(\Sigma_{s\theta}^{-1}) + \frac{1}{\theta}\Sigma_{s\theta}^{-1}A_{11} + \frac{1}{\theta}A'_{11}\Sigma_{s\theta}^{-1} - \frac{1}{\theta}R_{11} + \frac{1}{\theta}\Sigma_{s\theta}^{-1}G_1G'_1\Sigma_{s\theta}^{-1} - \frac{1}{\theta}\Sigma_{s\theta}^{-1}(X'(A_{21}\Sigma_{s\theta} \\
& \quad + G_2G'_1) + (X'G_2G'_2 - Y'A'_{22})X - X'G_2G'_2X - Y'C'_2N^{-1}C_2Y)\Sigma_{s\theta}^{-1} \\
&= \frac{1}{\theta} \frac{d}{dt}(\Sigma_{s\theta}^{-1}) + \frac{1}{\theta}\Sigma_{s\theta}^{-1}A_{11} + \frac{1}{\theta}A'_{11}\Sigma_{s\theta}^{-1} - \frac{1}{\theta}R_{11} + \frac{1}{\theta}\Sigma_{s\theta}^{-1}G_1G'_1\Sigma_{s\theta}^{-1} - \frac{1}{\theta}\Sigma_{s\theta}^{-1}(X'(A_{21}\Sigma_{s\theta} \\
& \quad + G_2G'_1) + Y'C'_2N^{-1}C_1)\Sigma_{s\theta}^{-1} = \frac{1}{\theta} \frac{d}{dt}(\Sigma_{s\theta}^{-1}) + \Sigma_{s\theta}^{-1}\bar{A}_s + \bar{A}'_s\Sigma_{s\theta}^{-1} - \bar{R}_s + \Sigma_{s\theta}^{-1}\bar{M}_s\Sigma_{s\theta}^{-1} = 0
\end{aligned}$$

For the 12-block, the following algebraic manipulations verify the result:

$$\begin{aligned}
\text{LHS} &= -\frac{1}{\theta} \frac{d}{dt}(\Sigma_{s\theta}^{-1}) + \dot{Z}_{s\theta} - A'_{11}(\frac{1}{\theta}\Sigma_{s\theta}^{-1} - Z_{s\theta}) + (C'_1N^{-1}(C_1\Sigma_{s\theta} + C_2Y) + A'_{21}X)\Sigma_{s\theta}^{-1} \\
& \quad - \frac{1}{\theta}\Sigma_{s\theta}^{-1}(B_1B'_1Z_{s\theta} + B_1B'_2V + A_{12}U) - (\frac{1}{\theta}\Sigma_{s\theta}^{-1} - Z_{s\theta})(\bar{A}_s - \bar{S}_sZ_{s\theta}) + \frac{1}{\theta}\Sigma_{s\theta}^{-1}((\Sigma_{s\theta}C'_1 \\
& \quad + Y'C'_2)N^{-1}C_1 + X'A_{21} + \theta X'G_2G'_1Z_{s\theta}) - \Sigma_{s\theta}^{-1}G_1G'_1(\frac{1}{\theta}\Sigma_{s\theta}^{-1} - Z_{s\theta}) + \frac{1}{\theta}\Sigma_{s\theta}^{-1}G_1G'_2X \\
& \quad \Sigma_{s\theta}^{-1} + \Sigma_{s\theta}^{-1}X'G_2G'_1(\frac{1}{\theta}\Sigma_{s\theta}^{-1} - Z_{s\theta}) - \frac{1}{\theta}\Sigma_{s\theta}^{-1}(X'G_2G'_2X + (\Sigma_{s\theta}C'_1 + Y'C'_2)N^{-1} \\
& \quad (C_1\Sigma_{s\theta} + C_2Y))\Sigma_{s\theta}^{-1} - Q_{12}U \\
&= -\frac{1}{\theta} \frac{d}{dt}(\Sigma_{s\theta}^{-1}) + (\dot{Z}_{s\theta} + Z_{s\theta}\bar{A}_s - Z_{s\theta}\bar{S}_sZ_{s\theta} + A'_{11}Z_{s\theta} - Q_{12}U) - A'_{11}\frac{1}{\theta}\Sigma_{s\theta}^{-1} - \frac{1}{\theta}\Sigma_{s\theta}^{-1}G_1 \\
& \quad \cdot G'_1\Sigma_{s\theta}^{-1} - \frac{1}{\theta}\Sigma_{s\theta}^{-1}(B_1B'_1Z_{s\theta} + B_1B'_2V + A_{12}U + \bar{A}_s - \bar{S}_sZ_{s\theta} - \theta G_1G'_1Z_{s\theta}) \\
& \quad + C'_1N^{-1}C_1 + \frac{1}{\theta}\Sigma_{s\theta}^{-1}Y'C'_2N^{-1}C_1 + \frac{1}{\theta}\Sigma_{s\theta}^{-1}X'A_{21} + \frac{1}{\theta}\Sigma_{s\theta}^{-1}X'G_2G'_1\Sigma_{s\theta}^{-1} \\
&= -\frac{1}{\theta} \frac{d}{dt}(\Sigma_{s\theta}^{-1}) - \theta Q_{11} + C'_1N^{-1}C_1 - A'_{11}\frac{1}{\theta}\Sigma_{s\theta}^{-1} - \frac{1}{\theta}\Sigma_{s\theta}^{-1}G_1G'_1\Sigma_{s\theta}^{-1} - \frac{1}{\theta}\Sigma_{s\theta}^{-1}A_{11}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\theta} \Sigma_{s\theta}^{-1} Y' C_2' N^{-1} C_1 + \frac{1}{\theta} \Sigma_{s\theta}^{-1} X' A_{21} + \frac{1}{\theta} \Sigma_{s\theta}^{-1} X' G_2 G_1' \Sigma_{s\theta}^{-1} \\
& = \frac{1}{\theta} \frac{d}{dt} (\Sigma_{s\theta}^{-1}) + \Sigma_{s\theta}^{-1} \bar{A}_s + \bar{A}_s' \Sigma_{s\theta}^{-1} - \bar{R}_s + \Sigma_{s\theta}^{-1} \bar{M}_s \Sigma_{s\theta}^{-1} = 0
\end{aligned}$$

For the 22-block, the derivation is as follows:

$$\begin{aligned}
\text{LHS} &= \frac{1}{\theta} \frac{d}{dt} (\Sigma_{s\theta}^{-1}) - \dot{Z}_{s\theta} + \left(\frac{1}{\theta} \Sigma_{s\theta}^{-1} - Z_{s\theta} \right) (\bar{A}_s - \bar{S}_s Z_{s\theta}) - \frac{1}{\theta} \Sigma_{s\theta}^{-1} ((\Sigma_{s\theta} C_1' + Y' C_2') N^{-1} C_1 \\
& + X' A_{21} + \theta X' G_2 G_1' Z_{s\theta}) + \left(\frac{1}{\theta} \Sigma_{s\theta}^{-1} - Z_{s\theta} \right) (B_1 B_1' Z_{s\theta} + B_1 B_2' V + A_{12} U) + (\bar{A}_s' - Z_{s\theta} \\
& \cdot \bar{S}_s') \left(\frac{1}{\theta} \Sigma_{s\theta}^{-1} - Z_{s\theta} \right) - ((\Sigma_{s\theta} C_1' + Y' C_2') N^{-1} C_1 + X' A_{21} + \theta X' G_2 G_1' Z_{s\theta})' \frac{1}{\theta} \Sigma_{s\theta}^{-1} + (Z_{s\theta} \\
& \cdot B_1 B_1' + V' B_2 B_1' + U' A_{12}') \left(\frac{1}{\theta} \Sigma_{s\theta}^{-1} - Z_{s\theta} \right) + (Z_{s\theta} B_1 + V' B_2) (B_1' Z_{s\theta} + B_2' V) + U' \\
& \cdot Q_{22} U + \left(\frac{1}{\theta} \Sigma_{s\theta}^{-1} - Z_{s\theta} \right) \theta G_1 G_1' \left(\frac{1}{\theta} \Sigma_{s\theta}^{-1} - Z_{s\theta} \right) - \left(\frac{1}{\theta} \Sigma_{s\theta}^{-1} - Z_{s\theta} \right) G_1 G_2' X \Sigma_{s\theta}^{-1} - \Sigma_{s\theta}^{-1} X' \\
& G_2 G_1' \left(\frac{1}{\theta} \Sigma_{s\theta}^{-1} - Z_{s\theta} \right) + \frac{1}{\theta} \Sigma_{s\theta}^{-1} (X' G_2 G_2' X + (\Sigma_{s\theta} C_1' + Y' C_2') N^{-1} (C_1 \Sigma_{s\theta} + C_2 Y)) \Sigma_{s\theta}^{-1} \\
& = \frac{1}{\theta} \frac{d}{dt} (\Sigma_{s\theta}^{-1}) - \dot{Z}_{s\theta} - Z_{s\theta} \bar{A}_s + Z_{s\theta} \bar{S}_s Z_{s\theta} - \bar{A}_s' Z_{s\theta} + Z_{s\theta} \bar{S}_s' Z_{s\theta} + \theta Z_{s\theta} G_1 G_1' Z_{s\theta} - Z_{s\theta} B_1 \\
& \cdot B_1' Z_{s\theta} - Z_{s\theta} A_{12} U - U' A_{12}' Z_{s\theta} + V' B_2 B_2' V + U' Q_{22} U + \frac{1}{\theta} \Sigma_{s\theta}^{-1} (\bar{A}_s - \bar{S}_s Z_{s\theta} + B_1 B_1' \\
& \cdot Z_{s\theta} + B_1 B_2' V + A_{12} U - \theta G_1 G_1' Z_{s\theta}) + (\bar{A}_s' - Z_{s\theta} \bar{S}_s' + Z_{s\theta} B_1 B_1' + V' B_2 B_1' + U' A_{12}' \\
& - \theta Z_{s\theta} G_1 G_1') \frac{1}{\theta} \Sigma_{s\theta}^{-1} - \frac{1}{\theta} C_1' N^{-1} C_1 + \frac{1}{\theta} \Sigma_{s\theta}^{-1} G_1 G_1' \Sigma_{s\theta}^{-1} - \frac{1}{\theta} \Sigma_{s\theta}^{-1} (\Sigma_{s\theta} A_{21}' X \\
& + X' A_{21} \Sigma_{s\theta} + G_1 G_2' X + X' G_2 G_1' - X' G_2 G_2' X - Y' C_2' N^{-1} C_2 Y) \Sigma_{s\theta}^{-1} \\
& = \frac{1}{\theta} \frac{d}{dt} (\Sigma_{s\theta}^{-1}) + \bar{Q}_s + Z_{s\theta} \bar{S}_s Z_{s\theta} - Z_{s\theta} S_{11} Z_{s\theta} - Z_{s\theta} A_{12} U - Z_{s\theta} B_1 B_2' V + U' Q_{22} \\
& + \frac{1}{\theta} \Sigma_{s\theta}^{-1} A_{11} + \frac{1}{\theta} A_{11}' \Sigma_{s\theta}^{-1} - \frac{1}{\theta} C_1' N^{-1} C_1 + \frac{1}{\theta} \Sigma_{s\theta}^{-1} G_1 G_1' \Sigma_{s\theta}^{-1} - \frac{1}{\theta} \Sigma_{s\theta}^{-1} (\Sigma_{s\theta} A_{21}' X \\
& + G_1 G_2' X + \Sigma_{s\theta} C_1' N^{-1} C_2 Y) \Sigma_{s\theta}^{-1} \\
& = \frac{1}{\theta} \frac{d}{dt} (\Sigma_{s\theta}^{-1}) + Q_{11} + \frac{1}{\theta} \Sigma_{s\theta}^{-1} A_{11} + \frac{1}{\theta} A_{11}' \Sigma_{s\theta}^{-1} - \frac{1}{\theta} C_1' N^{-1} C_1 + \frac{1}{\theta} \Sigma_{s\theta}^{-1} G_1 G_1' \Sigma_{s\theta}^{-1} - \frac{1}{\theta} \Sigma_{s\theta}^{-1} \\
& (\Sigma_{s\theta} A_{21}' X + G_1 G_2' X + \Sigma_{s\theta} C_1' N^{-1} C_2 Y) \Sigma_{s\theta}^{-1} \\
& = \frac{1}{\theta} \frac{d}{dt} (\Sigma_{s\theta}^{-1}) + \Sigma_{s\theta}^{-1} \bar{A}_s + \bar{A}_s' \Sigma_{s\theta}^{-1} - \bar{R}_s + \Sigma_{s\theta}^{-1} \bar{M}_s \Sigma_{s\theta}^{-1} = 0
\end{aligned}$$

To show that $\tilde{\Xi}_{11}^c$ is the solution to GRDE (B.4), we will show equivalently that $\tilde{\Xi}_{11}^{c-1}$ satisfies the GRDE:

$$-\frac{d}{dt} (\tilde{\Xi}_{11}^{c-1}) + F_{11}^c \tilde{\Xi}_{11}^{c-1} + \tilde{\Xi}_{11}^{c-1} F_{11}^{c'} + \theta G_1^c G_1^{c'} + \tilde{\Xi}_{11}^{c-1} H_{11}^c \tilde{\Xi}_{11}^{c-1} = 0 \quad (\text{B.8})$$

assuming for the moment that $Q_{f11} > 0$ and $\Delta_{s\theta}(t_f) > 0$ (which guarantees the invertibility of $Z_{s\theta}$ and $\Delta_{s\theta}$, as well as $\tilde{\Xi}_{11}^c$). Then, by a limiting argument, $\tilde{\Xi}_{11}^c$ is the solution to GRDE (B.4), in view of the uniqueness of the solution.

Assuming that $Q_{f11} > 0$ and $\Delta_{s\theta}(t_f) > 0$, it is clear that $\tilde{\Xi}_s^{-1}$ is a solution to GRDE (B.8). Let $\Xi_{s22} := (Z_{s\theta} - \theta Z_{s\theta} \Sigma_{s\theta} Z_{s\theta})^{-1} + \Delta_{s\theta}^{-1}$. It is straightforward to show that $\tilde{\Xi}_{11}^{c-1}$ satisfies GRDE (B.8) for 11 12 and 21 sub-blocks. For the 22-block of GRDE (B.8), we obtain the following GRDE:

$$\begin{aligned}
& -\dot{\Xi}_{s22} + (I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1} ((\Sigma_{s\theta} C_1' + Y' C_2') N^{-1} C_1 + X' A_{21}) Z_{s\theta}^{-1} + F_{cs} \Xi_{s22} + Z_{s\theta}^{-1} ((\Sigma_{s\theta} C_1' \\
& + Y' C_2') N^{-1} C_1 + X' A_{21})' (I - \theta Z_{s\theta} \Sigma_{s\theta})^{-1} + \Xi_{s22} F_{cs}' + Z_{s\theta}^{-1} Q_{11} Z_{s\theta}^{-1} - Z_{s\theta}^{-1} Q_{12} U \Xi_{s22} \\
& - \Xi_{s22} U' Q_{21} Z_{s\theta}^{-1} + \Xi_{s22} ((Z_{s\theta} B_1 + V' B_2) (B_1' Z_{s\theta} + B_2' V) + U' Q_{22} U) \Xi_{s22} + (I - \theta \Sigma_{s\theta} \\
& \cdot Z_{s\theta})^{-1} (X' G_2 G_2' X + (\Sigma_{s\theta} C_1' + Y' C_2') N^{-1} (C_1 \Sigma_{s\theta} + C_2 Y)) (I - \theta Z_{s\theta} \Sigma_{s\theta})^{-1} = 0
\end{aligned}$$

which will be denoted by the following compact form:

$$-\dot{\Xi}_{s,22} + \check{F}_s \Xi_{s,22} + \Xi_{s,22} \check{F}_s' + \Xi_{s,22} \check{G}_s \Xi_{s,22} + \check{H}_s = 0 \quad (\text{B.9})$$

where the matrices \check{F}_s , \check{G}_s and \check{H}_s are clearly defined from the context.

Since

$$\bar{\Xi}_s^{-1} = \begin{bmatrix} Z_{s\theta}^{-1} & Z_{s\theta}^{-1} \\ Z_{s\theta}^{-1} & (Z_{s\theta} - \theta Z_{s\theta} \Sigma_{s\theta} Z_{s\theta})^{-1} \end{bmatrix}$$

and $\bar{\Xi}_s^{-1}$ is a solution to the GRDE (B.8), then $\bar{\Xi}_{s,22} := (Z_{s\theta} - \theta Z_{s\theta} \Sigma_{s\theta} Z_{s\theta})^{-1}$ is a solution to the GRDE (B.9). The matrix $\check{F}_s + \bar{\Xi}_{s,22} \check{G}_s$ can be evaluated as follows:

$$\begin{aligned} (I - \theta \Sigma_{s\theta} Z_{s\theta})(\check{F}_s + \bar{\Xi}_{s,22} \check{G}_s) &= (I - \theta \Sigma_{s\theta} Z_{s\theta})(\bar{A}_s - \bar{S}_s Z_{s\theta} - Z_{s\theta}^{-1} Q_{12} U) + Z_{s\theta}^{-1} ((Z_{s\theta} B_1 \\ &+ V' B_2)(B_1' Z_{s\theta} + B_2' V) + U' Q_{22} U) - (\Sigma_{s\theta} C_1' + Y' C_2') N^{-1} C_1 - X' A_{21} - \theta X' G_2 G_1' \\ &\cdot Z_{s\theta} = \bar{A}_s - \bar{S}_s Z_{s\theta} + B_1 B_1' Z_{s\theta} + B_1 B_2' V - Z_{s\theta}^{-1} Q_{12} U + Z_{s\theta}^{-1} V' B_2 B_1' Z_{s\theta} + Z_{s\theta}^{-1} V' B_2 \\ &\cdot B_2' V + Z_{s\theta}^{-1} U' Q_{22} U - \theta \Sigma_{s\theta} (Z_{s\theta} \bar{A}_s - Z_{s\theta} \bar{S}_s Z_{s\theta} - Q_{12} U) - \Sigma_{s\theta} C_1' N^{-1} C_1 - Y' C_2' N^{-1} C_1 \\ &- X' A_{21} - \theta X' G_2 G_1' Z_{s\theta} = A_{11} + \theta G_1 G_1' Z_{s\theta} + Z_{s\theta}^{-1} V' (A_{22} U + B_2 B_1' Z_{s\theta} + B_2 B_2' V) \\ &+ \theta \Sigma_{s\theta} (\dot{Z}_{s\theta} + \bar{A}_s Z_{s\theta} + \bar{Q}_s + Q_{12} U) - \Sigma_{s\theta} C_1' N^{-1} C_1 - Y' C_2' N^{-1} C_1 - X' A_{21} - \theta X' G_2 G_1' \\ &\cdot Z_{s\theta} = \theta \Sigma_{s\theta} \dot{Z}_{s\theta} + A_{11} + \theta \Sigma_{s\theta} Q_{11} - \Sigma_{s\theta} C_1' N^{-1} C_1 - Y' C_2' N^{-1} C_1 - X' A_{21} + (\theta G_1 G_1' \\ &+ \theta \Sigma_{s\theta} A_{11} - \theta X' G_2 G_1') Z_{s\theta} = \theta \Sigma_{s\theta} \dot{Z}_{s\theta} + \bar{A}_s - \Sigma_{s\theta} \bar{R}_s + \theta (\Sigma_{s\theta} \bar{A}_s' + \bar{M}_s) Z_{s\theta} \\ &= -\frac{d}{dt} (I - \theta \Sigma_{s\theta} Z_{s\theta}) + (\bar{A}_s - \Sigma_{s\theta} \bar{R}_s) (I - \theta \Sigma_{s\theta} Z_{s\theta}) \end{aligned}$$

Hence, we have

$$\check{F}_s + \bar{\Xi}_{s,22} \check{G}_s = (I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1} (\bar{A}_s - \Sigma_{s\theta} \bar{R}_s) (I - \theta \Sigma_{s\theta} Z_{s\theta}) + \frac{d}{dt} ((I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1}) \cdot (I - \theta \Sigma_{s\theta} Z_{s\theta})$$

From this it is easy to see that $\bar{\Xi}_{s,22}$ is a solution to GRDE (B.9).

In the infinite horizon case, the GARE (5.90) admits a positive definite solution (5.92), since $\Delta_{s\theta}$ converges as $t_f \rightarrow \infty$. To show that it is the minimal one of such solutions, we need only to show that $\bar{\Xi}_{11}^c$ is the minimal positive definite solution to the GARE:

$$\bar{\Xi}_{11}^c F_{11}^c + F_{11}^c \bar{\Xi}_{11}^c + \bar{\Xi}_{11}^c \theta G_1^c G_1^{c'} \bar{\Xi}_{11}^c + H_{11}^c = 0$$

Compute the matrix $F_{11}^c + \bar{\Xi}_{11}^c{}^{-1} H_{11}^c$ as follows:

$$F_{11}^c + \bar{\Xi}_{11}^c{}^{-1} H_{11}^c = \begin{bmatrix} A_{11} + Z_{s\theta}^{-1} Q_{11} - Z_{s\theta}^{-1} U' Q_{21} & 0 \\ * & \check{F}_s + \bar{\Xi}_{s,22} \check{G}_s \end{bmatrix}$$

where $*$ denotes an arbitrary term of no interest to us here. It is clear that the matrix $A_{11} + Z_{s\theta}^{-1} Q_{11} - Z_{s\theta}^{-1} U' Q_{21} = \bar{A}_s + Z_{s\theta}^{-1} \bar{Q}_s$ is anti-stable. By the derivation above, the matrix $\check{F}_s + \bar{\Xi}_{s,22} \check{G}_s = (I - \theta \Sigma_{s\theta} Z_{s\theta})^{-1} (\bar{A}_s - \Sigma_{s\theta} \bar{R}_s) (I - \theta \Sigma_{s\theta} Z_{s\theta})$ is Hurwitz. Then, by Theorem 5 of [22], the matrix $\check{F}_s + \bar{\Xi}_{s,22} \check{G}_s$ is anti-stable. This implies that the matrix $F_{11}^c + \bar{\Xi}_{11}^c{}^{-1} H_{11}^c$ is anti-stable, and hence, the matrix $F_{11}^c + \theta G_1^c G_1^{c'} \bar{\Xi}_{11}^c$ is Hurwitz. Then, $\bar{\Xi}_{11}^c$ is the minimal solution to the GARE.

By a similar argument, we can prove that (5.144) is the solution to GRDE (5.142), and (5.97) is the minimal solution to GARE (5.91).

To show the spectral radius condition, i.e., that $I - \theta \tilde{\Theta}^c \tilde{\Xi}^c$ has only positive eigenvalues for each $t \in [t_0, t_f]$, it will be sufficient to prove that $I - \theta \tilde{\Theta}_{11}^c \tilde{\Xi}_{11}^c$ has only positive eigenvalues for each $t \in [t_0, t_f]$, in view of the approximations for $\tilde{\Theta}^c$ and $\tilde{\Xi}^c$.

It is straightforward to see that

$$\begin{aligned} & \tilde{\Xi}_{11}^c(t_0)^{-1} - \theta \tilde{\Theta}_{11}^c(t_0) \\ &= \begin{bmatrix} Z_{s\theta}(t_0)^{-1} - \theta \Sigma_{11} 0 & Z_{s\theta}(t_0)^{-1} \\ Z_{s\theta}(t_0)^{-1} & Z_{s\theta}(t_0)^{-1} (Z_{s\theta}(t_0)^{-1} - \theta \Sigma_{011})^{-1} Z_{s\theta}(t_0)^{-1} + \Delta_{s\theta}(t_0)^{-1} \end{bmatrix} > 0 \end{aligned}$$

since $Z_{s\theta}(t_0)^{-1} - \theta \Sigma_{011} > 0$ and $\Delta_{s\theta}(t_0)^{-1} > 0$, if $Q_{f11} > 0$ and $\Delta_{s\theta}(t_f) > 0$. Note that $\tilde{\Xi}_{11}^c^{-1}$ and $\theta \tilde{\Theta}_{11}^c$ satisfies the same GRDE, except with different initial conditions. Hence, $\tilde{\Xi}_{11}^c^{-1} - \theta \tilde{\Theta}_{11}^c > 0$ for all $t \in [t_0, t_f]$. Then, we have that the matrix $I - \theta \tilde{\Theta}_{11}^c(t) \tilde{\Xi}_{11}^c(t)$ has only positive eigenvalues (increasing as Q_f and $\tilde{\Delta}(t_f)$ decreases to singular nonnegative definite matrices), for each t by a limiting argument.

Verification for (5.107), (5.108), (5.152) and (5.153)

The proof for these results exactly parallels the proof for (5.92), (5.97), (5.143) and (5.144) described in detail in the previous subsection. It is easy to see that the detailed intermediate results derived in the previous subsection can be directly applied to this case. Hence, we omit the details of the proof here.

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