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ANALYTICAL SOLUTION OF THERMAL STRESSES FOR A VISCOELASTIC HOLLOW CYLINDER

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ABSTRACT

A solution is presented for the thermal stresses in an isotropic, homogenious, viscoelastic, hollow cylinder enclosed by a rigid casing. The governing equation, which is of the integral-differential form, is integrated numerically. The resulting partial differential equation is solved exactly subjected to the imposed boundary conditions. For comparison, a simple four-parameter viscoelastic material is chosen since an independent analytical solution is available. A second example is presented in order to demonstrate the capability of this solution in handling large number of viscoelastic parameters which are usually required to characterize real solid propellant and other polymeric structural materials.

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LIST OF SYMBOLS

A, a	Coefficients of material properties parameters given by Eq. (2.22b)
b, c	Outer and inner radii of thick-walled cylinder
d, D_o	Coefficients of material properties parameters given by Eq. (2.22)
E_{jj}	An arbitrary response function
e_{ij}	Deviatoric strain tensor
G_1, G_2	Shear moduli
h	Thickness of the cylinder
K	Bulk modulus
P_i, Q_i, R_i, S_i	Material properties coefficients given by Eqs. (2.10)
r, θ, z	Cylindrical coordinates
s_{ij}	Deviatoric stress tensor
T	Temperature
t	Time
t_j	Time at the j-th interval
u, v, w	Radial, circumferential, and axial displacement
$x, y, \hat{z}$	Material properties coefficients given by Eqs. (2.20)
α	Coefficient of linear thermal expansion
$\epsilon_r, \epsilon_\theta, \epsilon_z$	Radial, circumferential and axial strain
ζ, η	Uniaxial and bulk compliances
η_2	Coefficient of shear viscosity
κ	Diffusivity
ξ_{ij}	Series defined by Eq. (2.15c)

$\sigma_r, \sigma_\theta, \sigma_z$	Radial, circumferential and axial stress
$(\Delta\sigma_r)_j$	Change in radial stress from time t_{j-1} to t_j
τ_1, τ_2	Relaxation times
ϕ, ψ	Functions defining viscoelastic properties

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1. INTRODUCTION

In the recent years, the problem of thermal stress analysis for a linear, viscoelastic, hollow cylinder has been studied quite extensively [1].* The motivation for these studies have been the growing need for the structural analysis of solid propellant materials. The common technique of solving a viscoelastic structural problem, when a direct analytical method of solution is not possible, is by means of a Laplace or Fourier integral transformation of the governing equations in the time variable. Essentially, the transformation reduces the viscoelastic problem to a mathematically analogous elastic problem. After obtaining the solution to the associated elastic problem, the exact viscoelastic solution is found by the inverse transform provided that the inverse transform is possible. However, upon including the temperature dependent mechanical properties of a realistic solid propellant, where the temperature is both a function of time and space, and the complex geometries and load conditions, the solution in the transformed plane is usually complicated and it is not possible to perform the inverse transformation. When this approach fails, one usually attacks the viscoelastic structural problem from the numerical viewpoint.

The alternative approach of solution is to employ numerical methods [1,3]. The field equations which are formulated from a set of integral-differential stress-strain relations, are numerically

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*Numbers in the square brackets indicate references at the end of the thesis.

integrated over the time intervals which are divided into a number of relatively small intervals. Even though this approach can handle a large class of viscoelastic properties and the thermal effect of these properties can be included quite easily, the difficulty of the dependence of the past history in the viscoelastic material response still has to be resolved. The difficulty lies in the fact that the response of the viscoelastic materials depends on the past history effect and hence it is necessary to take in the account of all the previous responses at each time interval. Even when calculations are performed on an automatic digital computer, a large amount of the past history effects of the solution has to be stored and the amount increases as the solution progresses. Often this amount exceeds the storage space of a computer, especially, when the temperature is an arbitrary function of both the spatial and the time variables and the material properties are functions of the temperature.

Recently, a numerical procedure has been developed to overcome the aforementioned difficulty [3]. The method consists of integrating the governing equations numerically and then replacing the resulting partial differential equation in the radial variable by a difference equation. In overcoming the computer storage problem, a combination of integral and exponential series is used to characterize the temperature-dependent material properties.

The purpose of this investigation is to extend the solution obtained in Reference 3 such that an analytical solution is obtained for the spatial-variable dependence. The method still involves a numerical integration on the governing equation but the differential equation at each time interval are solved exactly in an analytical form. Thus the problem reduces to calculation of an elastic solution for each time interval. For the purpose of illustration, two examples are presented. A simple four-parameter viscoelastic material is chosen for the first example. For this case, an analytical solution is available and therefore comparison with the present method is possible. The second example consists of the same geometrical configuration but more general viscoelastic material properties are used. The properties are such that they characterize real solid propellant materials and, thus, this second example illustrates the broad capability of the present approach.

2. ANALYSIS

Consider an isotropic, homogeneous, viscoelastic, hollow cylinder with rigid case, as shown in Figure 1. The inner and outer radii are denoted by c and b, respectively. The analysis will consider a relatively long cylinder with simple geometrical configuration; thus, the plane-strain conditions will be imposed in the axial direction.

The working coordinate system will be cylindrical, as shown in Figure 1. Then, the viscoelastic strain-stress relations assume the following form

$$\epsilon_r = \int_0^t \left[\psi(t-t') \frac{\partial \sigma_r}{\partial t'} + \phi(t-t') \left\{ \frac{\partial \sigma_\theta}{\partial t'} + \frac{\partial \sigma_z}{\partial t'} \right\} \right] dt' + \alpha T(r,t) \quad (2.1a)$$

$$\epsilon_\theta = \int_0^t \left[\psi(t-t') \frac{\partial \sigma_\theta}{\partial t'} + \phi(t-t') \left\{ \frac{\partial \sigma_r}{\partial t'} + \frac{\partial \sigma_z}{\partial t'} \right\} \right] dt' + \alpha T(r,t) \quad (2.1b)$$

$$\epsilon_z = \int_0^t \left[\psi(t-t') \frac{\partial \sigma_z}{\partial t'} + \phi(t-t') \left\{ \frac{\partial \sigma_z}{\partial t'} + \frac{\partial \sigma_\theta}{\partial t'} \right\} \right] dt' + \alpha T(r,t) \quad (2.1c)$$

where α represents the thermal coefficient of expansion and it is assumed that the temperature, T, is an arbitrary function of the radial coordinate r and time t. With this radial-dependent temperature, the loading condition is axially symmetric. Equation (2.1) is the integral representations of the linear viscoelastic constitutive equations with thermal expansion being included. The functions $\phi(t)$ and $\psi(t)$ represent the viscoelastic material properties. It is

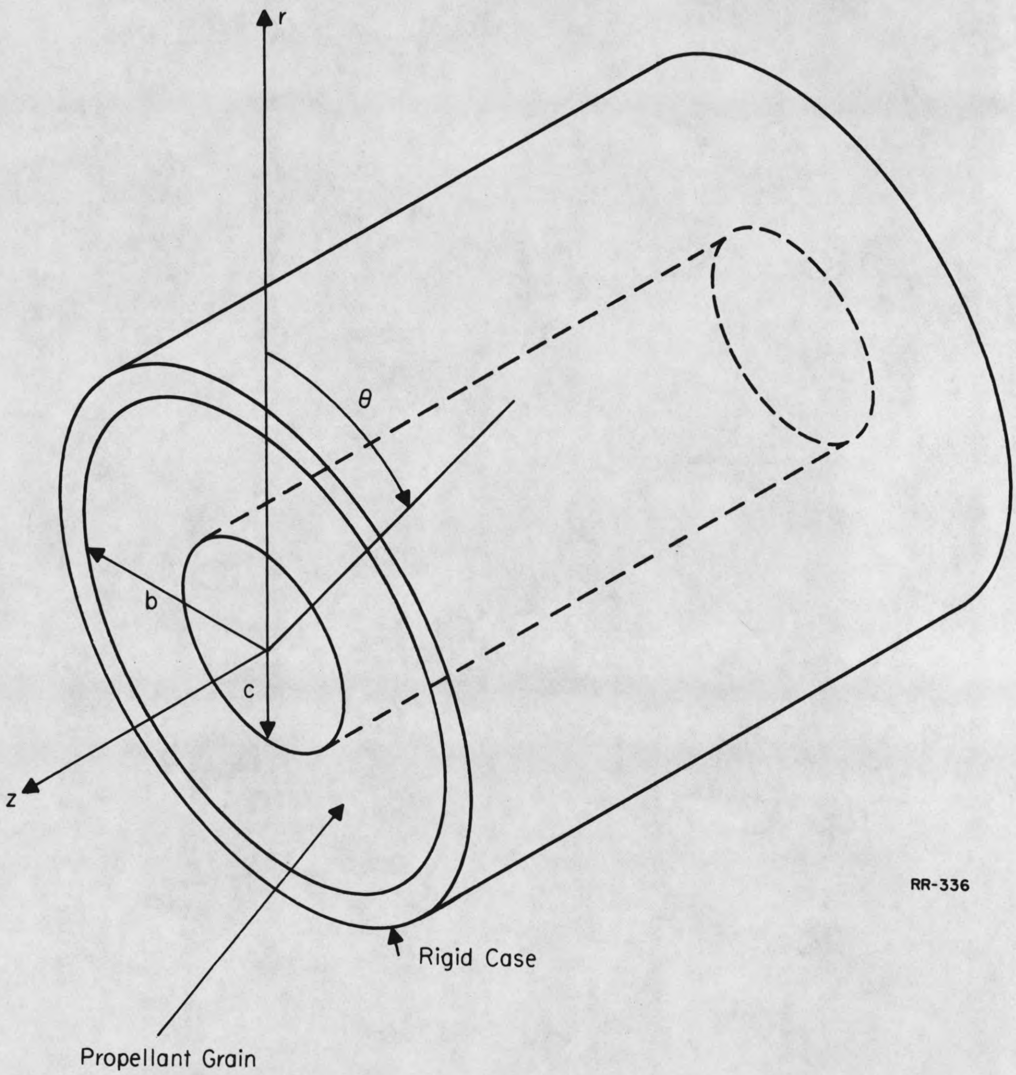

Figure 1. Solid propellant grain configuration and the coordinate system.

advantageous to express these two viscoelastic response functions in terms of the uniaxial compliance $\zeta(t)$ and the bulk compliance $\eta(t)$; the following relations can be easily shown to exist

$$\zeta(t) = \psi(t),$$

$$\frac{1}{3}\eta(t) = \psi(t) + 2\phi(t). \qquad (2.2)$$

Because of the plane strain and axial symmetry, the equilibrium condition for the stresses reduces to only one equation

$$\sigma_r = \sigma_\theta + r\frac{\partial\sigma_r}{\partial r} = 0. \qquad (2.3)$$

The above equation represents the equilibrium in the radial direction in cylindrical coordinates. The strain-displacement relations are

$$\epsilon_r = \frac{\partial u}{\partial r}, \qquad (2.4a)$$

$$\epsilon_\theta = \frac{u}{r}, \qquad (2.4b)$$

$$\epsilon_z = \frac{\partial w}{\partial z}. \qquad (2.4c)$$

Thus, together with the strain-stress relations, Eq. (2.1), they complete the set of governing field equations. As in the problem of shrinkage, the case bounding the solid propellant grain is assumed to be infinitively rigid, since the stiffness of the case is much greater than the stiffness of the propellant. Then, together with the plane-strain assumption, the strain in the z-direction is equal to zero for all time. Therefore, it follows from Eq. (2.1c) that

$$\int_0^t \left[\phi(t-t') \left(\frac{\partial\sigma_r}{\partial t'} + \frac{\partial\sigma_\theta}{\partial t'} \right) + \psi(t-t') \frac{\partial\sigma_z}{\partial t'} + \frac{\partial\alpha T(r,t')}{\partial t'} \right] dt' = 0. \qquad (2.5)$$

For a highly imcompressible viscoelastic material, the function $\phi(t)/\psi(t)$ is nearly a constant for all time and therefore it is possible to assume that this ratio $\phi(t)/\psi(t)$ can be multiplied through a time integral and taken inside of the integral sign. Using this assumption, it is possible to solve for $\frac{\partial\sigma_z}{\partial t'}$ from the above equation. Substituting the resulting expression for $\frac{\partial\sigma_z}{\partial t'}$ into equations (2.1a) and (2.1b), we have

$$\epsilon_r = \int_0^t \left[\left\{ \psi(t-t') - \frac{\phi^2(t-t')}{\psi(t-t')} \right\} \frac{\partial\sigma_r}{\partial t'} + \left\{ \phi(t-t') - \frac{\phi^2(t-t')}{\psi(t-t')} \right\} \frac{\partial\sigma_\theta}{\partial t'} + \left\{ 1 - \frac{\phi(t-t')}{\psi(t-t')} \right\} \frac{\partial\alpha T(r,t')}{\partial t'} \right] dt' \qquad (2.6a)$$

$$\epsilon_\theta = \int_0^t \left[\left\{ \phi(t-t') - \frac{\phi^2(t-t')}{\psi(t-t')} \right\} \frac{\partial\sigma_r}{\partial t'} + \left\{ \psi(t-t') - \frac{\phi^2(t-t')}{\psi(t-t')} \right\} \frac{\partial\sigma_\theta}{\partial t'} + \left\{ 1 - \frac{\phi(t-t')}{\psi(t-t')} \right\} \frac{\partial\alpha T(r,t')}{\partial t'} \right] dt'. \qquad (2.6b)$$

Turning to the equilibrium equation given by Eq. (2.3), it is possible to eliminate the circumferential stress, σ_θ, from the above equations. After performing this operation, we obtain

$$\epsilon_r = \int_0^t \left[\left\{ \psi(t-t') + \phi(t-t') - \frac{2\phi^2(t-t')}{\psi(t-t')} \right\} \frac{\partial\sigma_r}{\partial t'} + \left\{ \phi(t-t') - \frac{\phi^2(t-t')}{\psi(t-t')} \right\} r \frac{\partial}{\partial t'} \left(\frac{\partial\sigma_r}{\partial t'} \right) + \left\{ 1 - \frac{\phi(t-t')}{\psi(t-t')} \right\} \frac{\partial\alpha T(r,t')}{\partial t'} \right] dt' \qquad (2.7a)$$

$$\epsilon_\theta = \int_o^t \Big[\Big\{ \psi(t-t') + \phi(t-t') - \frac{2\phi^2(t-t')}{\psi(t-t')} \Big\} \frac{\partial \sigma_r}{\partial t'}$$

$$\Big\{ \psi(t-t') - \frac{\phi^2(t-t')}{\psi(t-t')} \Big\} r \frac{\partial}{\partial t'} \left(\frac{\partial \sigma_r}{\partial t'} \right) + \Big\{ 1 - \frac{\phi(t-t')}{\psi(t-t')} \Big\} \frac{\partial}{\partial t'} \alpha T(r,t') \Big] dt' \tag{2.7b}$$

which are expressed in terms of the radial stress σ_r only. The governing equation of this problem is the following compatibility equation

$$\epsilon_r = \epsilon_\theta + r \frac{\partial \epsilon_\theta}{\partial r} . \tag{2.8}$$

Thus, by substituting the corresponding radial and circumferential strains,given by Eq. (2.7a) and Eq. (2.7b) respectively, we have

$$\int_o^t \Big[\Big\{ \psi(t-t') - \frac{\phi^2(t-t')}{\psi(t-t')} \Big\} \Big\{ \frac{\partial}{\partial t'} \left(r^2 \frac{\partial^2 \sigma_r}{\partial r^2} \right) + 3 \frac{\partial}{\partial t'} \left(r \frac{\partial \sigma_r}{\partial r} \right) \Big\}$$

$$+ \Big\{ 1 - \frac{\phi(t-t')}{\psi(t-t')} \Big\} \frac{\partial}{\partial t'} \Big\{ r \frac{\partial \alpha T(r,t')}{\partial r} \Big\} \Big] dt' = 0 \tag{2.9}$$

where the material response functions $\phi(t)$ and $\psi(t)$ are now expanded in exponential series (see reference 1) as follows

$$\psi(t) = P_o + \sum_{i=1}^{q} P_i e^{-t/\tau_i} , \tag{2.10a}$$

$$\phi(t) = Q_o + \sum_{i=1}^{q} Q_i e^{-t/\tau_i} , \tag{2.10b}$$

$$\frac{\phi^2(t)}{\psi(t)} = R_o + \sum_{i=1}^{q} R_i e^{-t/\tau_i} , \tag{2.10c}$$

$$\frac{\phi(t)}{\psi(t)} = S_o + \sum_{i=1}^{q} S_i e^{-t/\tau_i}, \qquad (2.10d)$$

where P_i, Q_i, R_i, S_i $(i=0,1,2,...,q)$ and τ_i $(i=1,2,...,q)$ are parameters defining the mechanical properties of a given viscoelastic material.

The problem now reduces to solving for the radial stress, σ_r, from the integral-differential equation, given by Eq. (2.9), subjected to boundary conditions i) at the inner radius, the radial stress vanishes and ii) at the outer radius, the circumferential strain is zero. However, it is generally not possible to find an analytical solution to Eq. (2.9) for some realistic linear viscoelastic materials even in the case where there is no temperature dependence for the material properties. In view of this difficulty, the method of solution of Eq. (2.9) will consist of (a) performing a numerical integration with respect to the time variable, and (b) analytically solving the resulting partial differential equation with the imposed boundary conditions.

In Eq. (2.9), the integrand is a product of two time-dependent functions and it has a general form

$$\hat{I}(t) = \int_o^t E(t-t') \frac{\partial\hat{\sigma}}{\partial t'} dt'. \qquad (2.11)$$

The time scale is divided into a grid in which each interval has a length Δt. The continuous function $\hat{\sigma}$ is now replaced by a step function which is constant over each time interval. Using this representation, Eq. (2.7) can be written, for some discrete value $t=t_j$ $(j=1,2,\text{etc.})$, as follows

$$\int_o^t E(t-t') \frac{\partial\hat{\sigma}}{\partial t'} dt' = \Delta t \left\{ E_{j1}\left(\frac{\partial\hat{\sigma}}{\partial t'}\right)_1 + E_{j2}\left(\frac{\partial\hat{\sigma}}{\partial t'}\right)_2 + ... + E_{jj}\left(\frac{\partial\hat{\sigma}}{\partial t'}\right)_j \right\} \tag{2.12}$$

where, from Eq. (2.10), $E_{jj} = E(t_j - t_{j-1})$ and $t_o = 0$. The subscripts denote the quantities evaluated at a particular time interval. With the aid of finite difference technique, it is possible to assume $\left(\frac{\partial\hat{\sigma}}{\partial t'}\right)_j \doteq \left(\frac{\Delta\hat{\sigma}_j}{\Delta t}\right)$; then

$$\int_o^t E(t-t') \frac{\partial\hat{\sigma}}{\partial t'} dt' = E_{j1}\Delta\hat{\sigma}_1 + E_{j2}\Delta\hat{\sigma}_2 + ... + E_{jj}\Delta\hat{\sigma}_j$$

$$= E_{jj}\Delta\hat{\sigma}_j + \sum_{k=1}^{j-1} E_{jk}\Delta\hat{\sigma}_k \tag{2.13}$$

where $\Delta\hat{\sigma}_k$ denotes the change in $\hat{\sigma}$ from the time $t=t_{k-1}$ to $t=t_k$.

Equation (2.13) is equivalent to replacing $E(t-t')$ by a constant value $E(t_j - t_{j-1})$ over each interval $t_{k-1} \le t \le t_k$. Corresponding to this, we assume that $\hat{\sigma}$ has a constant slope over each interval, i.e., $\hat{\sigma}$ is replaced by a piece-wise continuous function; then, $\frac{\partial\hat{\sigma}}{\partial t'}$ assumes a constant value over each interval. Thus, this integration procedure is equivalent to multiplying two step functions. It is possible to express $\hat{I}(t)$ as

$$\hat{I}(t) = E_o\hat{\sigma}_{j-1} + \sum_{i=1}^{q} E_i\xi_{ij} + E_{jj}\Delta\hat{\sigma}_j \tag{2.14}$$

where we have introduced the definitions

$$\hat{\sigma}_{j-1} = \sum_{k=1}^{j-1} \Delta\hat{\sigma}_k \tag{2.15a}$$

$$E_{jk} = E_o + \sum_{i=1}^{q} E_i e^{-(t_j - t_k)/\tau_i} \quad (2.15b)$$

$$\xi_{ij} = \sum_{k=1}^{j-1} e^{-(t_j - t_k)/\tau_i} \Delta\hat{\sigma}_k. \quad (2.15c)$$

Now, consider $\xi_{i,j-1}$, that is, the immediate previous time interval at $t=t_{j-1}$

$$\xi_{i,j-1} = \sum_{k=1}^{j-2} e^{-(t_{j-1} - t_{k-1})/\tau_i} \Delta\hat{\sigma}_k. \quad (2.16)$$

Comparing the two ξ's, ξ_{ij} and $\xi_{i,j-1}$, it can be easily shown that

$$\xi_{ij} = (\xi_{i,j-1} + \Delta\hat{\sigma}_{j-1}) e^{-\Delta t/\tau_i}. \quad (2.17)$$

This above recursive formula in ξ_{ij} permits one to calculate these ξ's at a discrete time interval, $t=t_j$, in terms of the corresponding values at the previous time interval, $t=t_{j-1}$. The importance of this result in relation to the outcome of the solution for this analysis is that it eliminates storing information from all previous intervals except one. In actual calculations this reduces considerably the amount of information which has to be stored.

With this step-wise integration scheme in mind, Eq. (2.7b) assumes the following form

$$(t_o)_j = (S_x)_j + (S_y)_j + (S_{\hat{z}})_j + x(\Delta\sigma_r)_j + yr \frac{\partial(\Delta\sigma_r)_j}{\partial r} + \hat{z}\Delta\alpha T_j(r) \quad (2.18)$$

where

$$(S_x)_j = (P_o+Q_o)(\sigma_r)_{j-1} + \sum_{i=1}^{q}(P_i+Q_i)\sum_{k=1}^{j-1} e^{-(t_j-t_k)/\tau_i}(\Delta\sigma_r)_k, \tag{2.19a}$$

$$(S_y)_j = (-P_o+R_o)\left(r\frac{\partial\sigma_r}{\partial r}\right)_{j-1} + \sum_{i=1}^{q}(-P_i+R_i)\sum_{k=1}^{j-1} e^{-(t_j-t_k)/\tau_i}\left(r\frac{\partial\Delta\sigma_r}{\partial r}\right)_k, \tag{2.19b}$$

$$(S_{\hat{z}})_j = (1-S_o)\alpha T_{j-1}(r) + \sum_{i=1}^{q}(-S_i)\sum_{k=1}^{j-1} e^{-(t_j-t_k)/\tau_i}\Delta\alpha T_k(r). \tag{2.19c}$$

It should be noted that the above series all have their own recursive relations of the form given by Eq. (2.17). Also, Eq. (2.18) can be used in order to satisfy the second boundary condition. The coefficients in Eq. (2.18) are defined as follows

$$x = (P_o+Q_o-2R_o) + \sum_{i=1}^{q}(P_i+Q_i-2R_i), \tag{2.20a}$$

$$y = (-P_o+R_o) + \sum_{i=1}^{q}(-P_i+R_i), \tag{2.20b}$$

$$\hat{z} = (1-S_o) + \sum_{i=1}^{q}(-S_i). \tag{2.20c}$$

Similarly, Eq. (2.9) can be written as

$$a\frac{1}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial(\Delta\sigma_r)_i}{\partial r}\right) + N_j = 0 \tag{2.21}$$

where

$$N_j = A_o\frac{1}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial\sigma_r}{\partial r}\right)_{j-1} + \sum_{i=1}^{q} A_i \sum_{k=1}^{j-1} e^{-(t_j-t_k)/\tau_i}\frac{1}{r}\frac{\partial}{\partial r}\left(r^3\frac{\partial\Delta\sigma_r}{\partial r}\right)_k$$

$$+ D_o\, r \frac{\partial}{\partial r} \alpha T_{j-1}(r) + \sum_{i=1}^{q} D_i \sum_{k=1}^{j-1} e^{-(t_j - t_k)/\tau_i} \left(r \frac{\partial \Delta\alpha T(r)}{\partial r} \right)_k$$

$$+ d\, r \frac{\partial}{\partial r} \Delta\alpha T_j(r) \qquad (2.22a)$$

and, by definition

$$a = A_o + \sum_{i=1}^{q} A_i, (A_o = P_o + R_o,\ A_i = P_i + R_i)$$

$$d = -1 + S_o + \sum_{i=1}^{q} S_i, (D_o = -1 + S_o). \qquad (2.22b)$$

The problem now reduces to finding a solution to Eq. (2.21) subjected to the following boundary conditions

i) at $r = c$, the inner radius, $\sigma_r = 0$,

ii) at $r = b$, the outer radius, $\epsilon_\theta = 0$.

Condition i) imposes that the grain should have zero surface traction on the inner surface; condition ii) implies that there is no radial displacement on the outer surface, as for an infinitely rigid case.

Upon integrating Eq. (2.21) directly, we have

$$(\Delta\sigma_r)_j = -\frac{1}{a} \int_c^r \frac{dr}{r^3} \int_c^r N_j\, r\, dr + Y_j \int_c^r \frac{dr}{r^3} + Z_j \qquad (2.23)$$

where Y_j and Z_j are constants of integration; c denotes the inner radius and r is an arbitrary radial point. From boundary condition i), it can be easily deduced that the integration constant Z_j is zero. By applying the method of integration-by-parts twice, Eq. (2.23) becomes

$$(\Delta\sigma_r)_j = -\frac{A_o}{a}(\sigma_r)_{j-1} - \frac{1}{a}\sum_{i=1}^{q} A_i \sum_{k=1}^{j-1} e^{t_k/\tau_i}(\Delta\sigma_r)_k e^{-t_j/\tau_i} + \frac{Y_j}{2c^2}\left(1-\frac{c^2}{r^2}\right)$$

$$-\frac{1}{a}\sum_{i=1}^{q} D_i \sum_{k=1}^{j-1} e^{t_k/\tau_i}\left[(\Delta I)_k + \frac{1}{2}\Delta\alpha T_k(c)\left(1-\frac{c^2}{r^2}\right)\right]e^{-t_j/\tau_i}$$

$$-\frac{D_o}{a} I_j - \frac{(d-D_o)}{a}(\Delta I)_j + \frac{1}{2a}\left\{D_o\alpha T_j(c)\right.$$

$$\left. + (d-D_o)\Delta\alpha T_j(c)\right\}\left(1-\frac{c^2}{r^2}\right) \tag{2.24}$$

where

$$I_j = \frac{1}{r^2}\int_c^r \alpha T_j(r)\,r\,dr \tag{2.25a}$$

$$(\Delta I)_j = \frac{1}{r^2}\int_c^r \Delta\alpha T_j(r)\,r\,dr. \tag{2.25b}$$

Equation (2.24) is the required solution for this problem. It remains to obtain the boundary value Y_j. However, it is advantageous to write Eq. (2.24) in a more compact form. Then, by writing out explicitly a few expressions of $(\Delta\sigma_r)_j$ for different time intervals, that is to say, j=1,2,3,etc., we can conclude that $(\sigma_r)_{j-1}$ and $(\sigma_r)_k$ assume the following forms

$$(\sigma_r)_{j-1} = -\frac{D_o}{a} I_{j-1} + \frac{1}{2a}\left\{D_o\alpha T_{j-1}(c) + (d-D_o)\Delta\alpha T_{j-1}(c)\right\}\left(1-\frac{c^2}{r^2}\right)$$

$$+ \frac{Y_{j-1}}{2c^2}\left(1-\frac{c^2}{r^2}\right) - \frac{1}{A}\sum_{i=1}^{q} D_i \sum_{k=1}^{j-1} e^{t_k/\tau_i}(\Delta I)_k e^{-t_j/\tau_i} \tag{2.26}$$

$$(\Delta\sigma_r)_k = -\frac{D_o}{a} I_k + \frac{1}{2a}\left\{D_o\,\alpha T_k(c) + (d-D_o)\Delta\alpha T_k(c)\right\}\left(1 - \frac{c^2}{r^2}\right)$$

$$+ \frac{Y_k}{2c^2}\left(1 - \frac{c^2}{r^2}\right). \tag{2.27}$$

Substituting the above equations into Eq. (2.24) and collecting terms, we obtain

$$(\Delta\sigma_r)_j = M'_j\left(1 - \frac{c^2}{r^2}\right) + L_j \tag{2.28}$$

where

$$M'_j = -(U_1)_{j-1} + \frac{Y_j}{2c^2} - (S_1)_j + (U_2)_{j-1} + (U_3)_{j-1} + (U_4)_j$$

$$- (S_2)_j - (S_3)_j - (S_4)_j \tag{2.29a}$$

$$L_j = (U_5)_{j-1} - (U_6)_j + (S_5)_j \tag{2.29b}$$

where the S's are given by the following series

$$(S_1)_j = \frac{1}{2ac^2} \sum_{i=1}^{q} A_i \sum_{k=1}^{j-1} e^{t_k/\tau_i}\, Y_k e^{-t_j/\tau_i} \tag{2.30a}$$

$$(S_2)_j = \frac{D_o}{2a^2} \sum_{i=1}^{q} A_i \sum_{k=1}^{j-1} e^{t_k/\tau_i}\, \alpha T_k(c) e^{-t_j/\tau_i} \tag{2.30b}$$

$$(S_3)_j = \frac{(d-D_o)}{2a^2} \sum_{i=1}^{q} A_i \sum_{k=1}^{j-1} e^{t_k/\tau_i}\, \Delta\alpha T_k(c) e^{-t_j/\tau_i} \tag{2.30c}$$

$$(S_4)_j = \frac{1}{2a} \sum_{i=1}^{q} D_i \sum_{k=1}^{j-1} e^{t_k/\tau_i}\, \Delta\alpha T_k(c) e^{-t_j/\tau_i} \tag{2.30d}$$

$$(S_5)_j = \frac{D_o}{a^2} \sum_{i=1}^{q} A_i \sum_{k=1}^{j-1} e^{t_k/\tau_i} (I)_k \, e^{-t_j/\tau_i} \tag{2.30e}$$

and the U's are given by the following terms

$$(U_1)_{j-1} = \frac{A_o}{2ac^2} Y_{j-1} \tag{2.31a}$$

$$(U_2)_{j-1} = \frac{D_o}{2a}\left(1 - \frac{A_o}{a}\right) \alpha T_{j-1}(c) \tag{2.31b}$$

$$(U_3)_{j-1} = \frac{A_o}{2a^2} (D_o - d) \Delta\alpha T_{j-1}(c) \tag{2.31c}$$

$$(U_4)_j = \frac{d}{2a} \Delta\alpha T_j(c) \tag{2.31d}$$

$$(U_5)_{j-1} = \frac{D_o}{a}\left(\frac{A_o}{a} - 1\right) I_{j-1} \tag{2.31e}$$

$$(U_6)_j = \frac{d}{a} (\Delta I)_j . \tag{2.31f}$$

Equation (2.29a) contains the unknown integration constant Y_j which can be evaluated from the boundary condition ii). It should be pointed out that M_j' is independent of the radial coordinate.

Each series in Eq. (2.30) has a corresponding recursive formula. Let

$$\sum_{k=1}^{j-1} e^{t_k/\tau_i} \, Y_k e^{-t_j/\tau_i} = \tilde{Y}_{ij} \tag{2.32a}$$

$$\sum_{k=1}^{j-1} e^{t_k/\tau_i} \, \alpha T_k(c) e^{-t_j/\tau_i} = \tilde{T}_{ij} \tag{2.32b}$$

$$\sum_{k=1}^{j-1} e^{t_k/\tau_i} I_k e^{-t_j/\tau_i} = \tilde{I}_{ij}. \qquad (2.32c)$$

Then, using Eq. (2.17), the recursive formulae are given by

$$\tilde{Y}_{ij} = (\tilde{Y}_{i,j-1} + Y_{j-1}) e^{-\Delta t/\tau_i} \qquad (2.33a)$$

$$\tilde{T}_{ij} = (\tilde{T}_{i,j-1} + \alpha T_{j-1}(c)) e^{-\Delta t/\tau_i} \qquad (2.33b)$$

$$\tilde{I}_{ij} = (\tilde{I}_{i,j-1} + I_{j-1}) e^{-\Delta t/\tau_i}. \qquad (2.33c)$$

These series given by Eq. (2.30) contain the viscoelastic history effects. When actual computations are made on a automatic digital computer large amount of the past history information will be needed and often the computer will have exhausted the necessary storage space before an answer can be obtained. Therefore, without the recursive relations given by Eq. (2.33), it will be impossible to obtain a solution.

Now, we proceed to evaluate Y_j, the other boundary value, from the boundary condition ii). Thus, at r = b, setting Eq. (2.18) equal to zero and solving for $(\Delta\sigma_r)_j$, we have

$$[(\Delta\sigma_r)_j]_{r=b} = -\frac{1}{x}\left[(S_x)_j + (S_y)_j + (S_{\hat{z}})_j + yr\frac{\partial}{\partial r}(\Delta\sigma_r)_j + \hat{z}\Delta\alpha T_j(r)\right]_{r=b} \qquad (2.34)$$

where subscript r=b denotes the quantity is evaluated at r=b. Differentiating Eq. (2.28), we get

$$\left[r \frac{\partial}{\partial r}(\Delta\sigma_r)_j \right]_{r=b} = 2M'_j \frac{c^2}{b^2} + \left[r \frac{\partial L_j}{\partial r} \right]_{r=b} . \tag{2.35}$$

Thus, with the aid of Eq. (2.28) and Eq. (2.34), together with Eq. (2.35), it is possible to eliminate Y_j and express M'_j as

$$M_j = \frac{- x[L_j]_{r=b} - y\left[b \frac{\partial}{\partial r} L_j \right]_{r=b} - [(S_x)_j + (S_y)_j + (S_{\hat{z}})_j + \hat{z}\alpha T_j(r)]_{r=b}}{\left[x + (2y-x) \frac{c^2}{b^2} \right]} \tag{2.36}$$

Finally, the radial stress can be expressed as

$$(\Delta\sigma_r)_j = M_j \left(1 - \frac{c^2}{r^2} \right) + L_j \tag{2.37}$$

where M_j and L_j are given by Eq. (2.36) and (2.29b) respectively.

When actual calculations are performed, the solution for the radial stress could be obtained by updating the values from the preceeding interval, i.e., $(\Delta\sigma_r)_j = (\sigma_r)_j - (\sigma_r)_{j-1}$. For checking purposes, at time $\Delta t = 0$, the radial stress solution reduces to the well-known Lamé solution of the elastic case; thus, hand calculations can be performed easily. Because of the existence of the recursion relations for all the series, the viscoelastic history effects do not have to be summed over all the previous time intervals. It is only necessary to update the corresponding quantities from the preceeding interval. This eliminates the storage problem when actual calculations are made on a digital computer in contrast to the fact that a large amount of information is needed to account for the time history if such a recursive formula is not available. In order to illustrate this analysis, two examples are given in the following sections. The computations for these examples are performed on a CONTROL DATA 1604 automatic digital computer machine.

3. EXAMPLE A

Because of the availibility of an independent analytical result, this example is chosen in order to obtain comparison with the present analysis. The independent analytical result is taken from reference 2, which is designated as I'(2) in that reference. The chosen model is a simple four-parameter one with a viscoelastic response given by the parameters G_1, G_2, and τ_2, and the volumetric bulk modulus, K. (Figure 2). These parameters are given by the following definitions:

$$\frac{G_1}{G_2} = 0.1 \tag{3.1}$$

$$\frac{G_1 + G_2}{k} = 0.0201342 \tag{3.2}$$

$$\tau_2 = \frac{\eta_2}{G_2}. \tag{3.3}$$

We need to get relationships between the viscoelastic functions $\phi(t)$ and $\psi(t)$ and the parameters defined in this example. In this example, the deviatoric strain e_{ij} is related to the deviatoric stress s_{ij} by

$$e_{ij} = s_{ij}\left\{\frac{1}{2G_1}\left[1 - \frac{1}{1 + G_1/G_2}\, e^{-\frac{G}{G_1 + G_2}\cdot\frac{t}{\tau_2}}\right]\right\}. \tag{3.4}$$

By the definition of the deviatoric quantities, together with the definitions for this analysis, it can be shown that

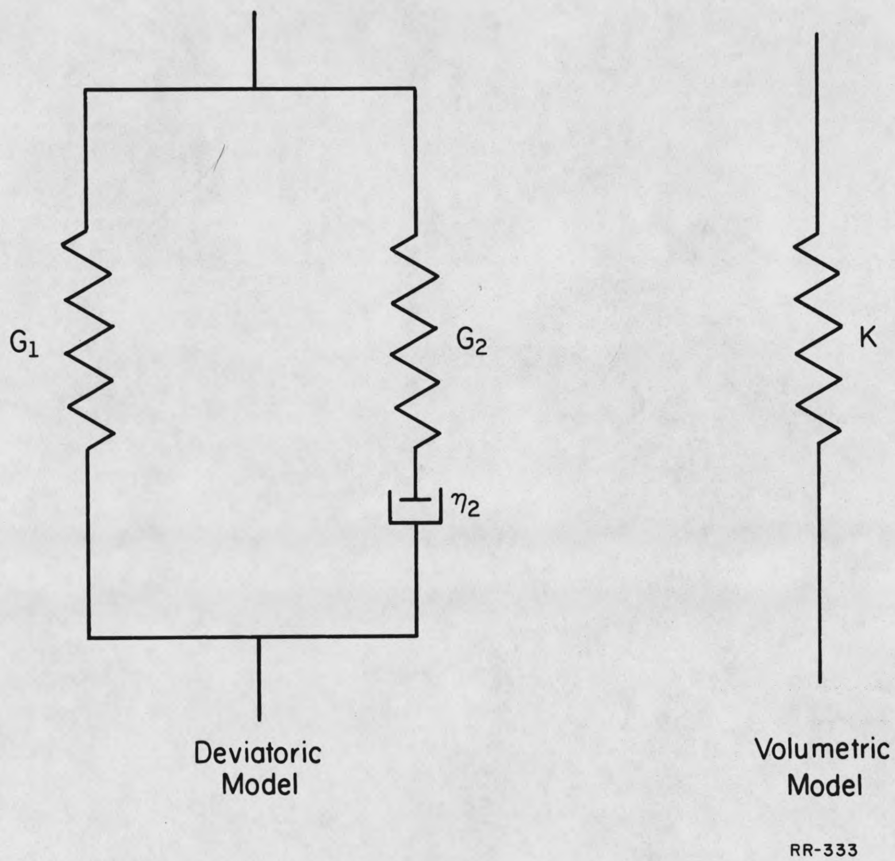

Figure 2. Viscoelastic models of the mechanical properties for Example A.

$$\psi(t) - \phi(t) = \frac{1}{2G_1}\left[1 - \frac{1}{1 + \frac{G_1}{G_2}} e^{-\frac{G_1}{G_1 + G_2}\frac{t}{\tau_2}}\right]. \tag{3.5}$$

Similarly, with the definition of the elastic bulk modulus, we have

$$\psi(t) + 2\phi(t) = \frac{1}{3K} . \tag{3.6}$$

Using the formulae (3.1) and (3.2), it can be easily shown that

$$K\psi(t) = 182.2224 - 165.555e^{-\lambda t'} \tag{3.7}$$

$$K\phi(t) = -90.9444 + 82.7775e^{-\lambda t'} \tag{3.8}$$

where

$$\lambda = \frac{G}{G_1 + G_2}$$

$$t' = \frac{t}{\tau_2} .$$

Consider the ratio

$$\frac{\phi(t)}{\psi(t)} = \frac{182.2224 - 165.555\, e^{-\lambda t'}}{-90.9444 + 82.7775\, e^{-\lambda t'}} . \tag{3.9}$$

Since the value of $\frac{\phi(t)}{\psi(t)}$ given by the above quation is very close to 0.5 for all time, we can approximate it by

$$\frac{\phi(t)}{\psi(t)} = S_o + S_1 e^{-\lambda t'} . \tag{3.10}$$

By matching Equations (3.9) and (3.10) at $t = 0$ and $t = \infty$, we find that $S_o = -.499085$ and $S_1 = 0.00909$. Now

$$\frac{K\phi^2(t)}{\psi(t)} = \left[S_o + S_1 e^{-\lambda t'}\right]\left[-90.9444 + 82.7775\ e^{-\lambda t'}\right]$$

$$= R_o + R_1 e^{-\lambda t'} + R_2 e^{-2\lambda t'} \qquad (3.11)$$

Equating coefficients on both sides of Eq. (3.11) we obtain $R_o = 45.3889$, $R_1 = -42.1396$ and $R_2 = 0.752447$.

All the necessary property functions for this analysis are now available and the viscoelastic parameters defining the model for this example are listed in Table I. The temperature distribution which is only a function of time is given by

$$\alpha T(t') = 1 - e^{-t'}. \qquad (3.12)$$

The results are given by Figures 3, 4 and 5. Figure 3 gives the radial stress at the outer surface of the cylinder as a function of time. Similarly, Figure 4 gives the circumferential stress at the outer radius. The radii are taken to be 1 and 3 for the inner and the outer radii respectively. Figure 5 presents the radial and circumferential strains at the inner radius. The corresponding analytical results are also given in the respective figures. The size of the time interval, Δt, is taken to be 0.05.

Table I. Viscoelastic Parameters for Example A

i	P_i	Q_i	R_i	S_i	τ_i
0	182.2224	-90.9444	45.3889	-.499085	------
1	-165.5550	82.7775	-42.1396	.009090	11.0
2	0	0	.7524	0	5.5

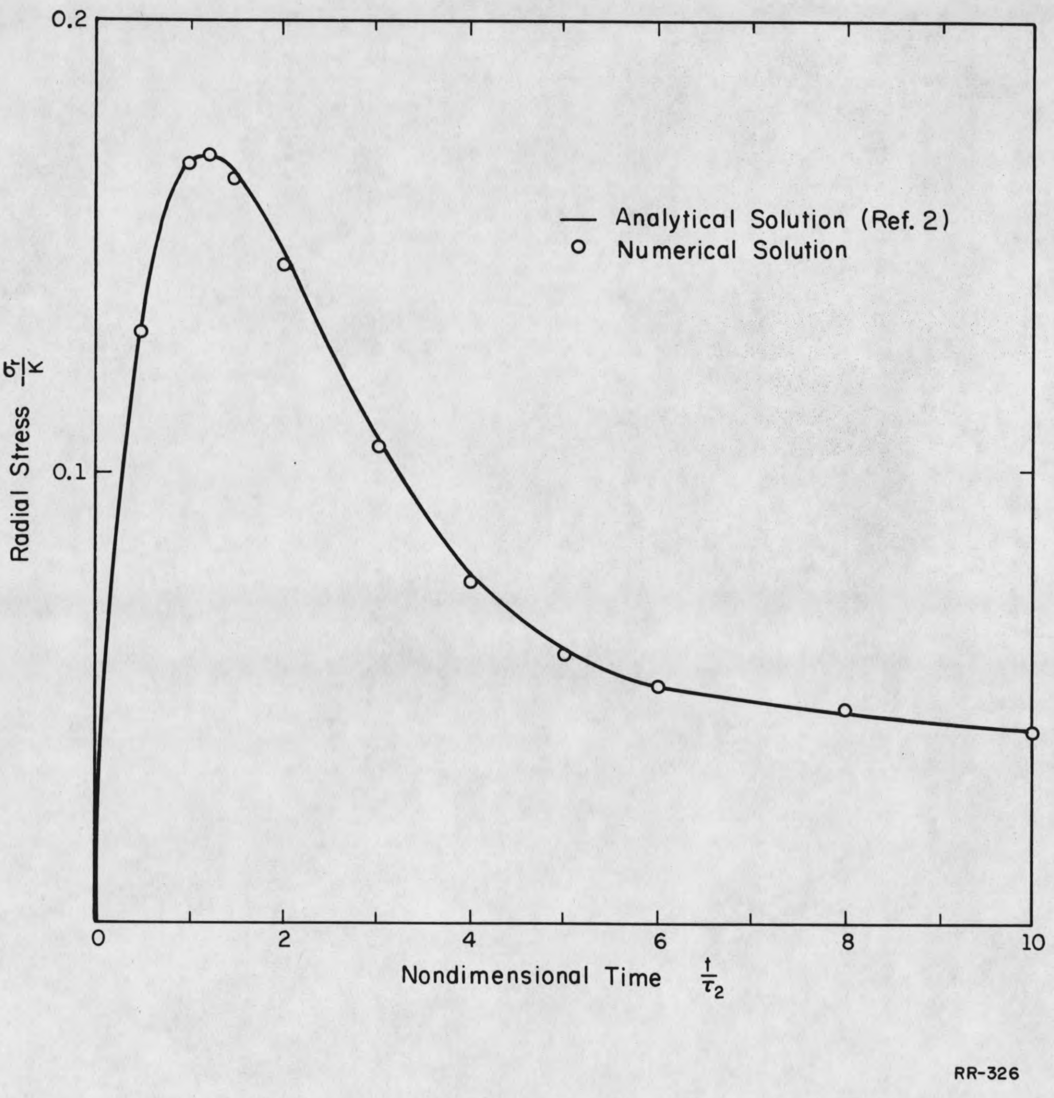

Figure 3. Radial stress at r = 3.

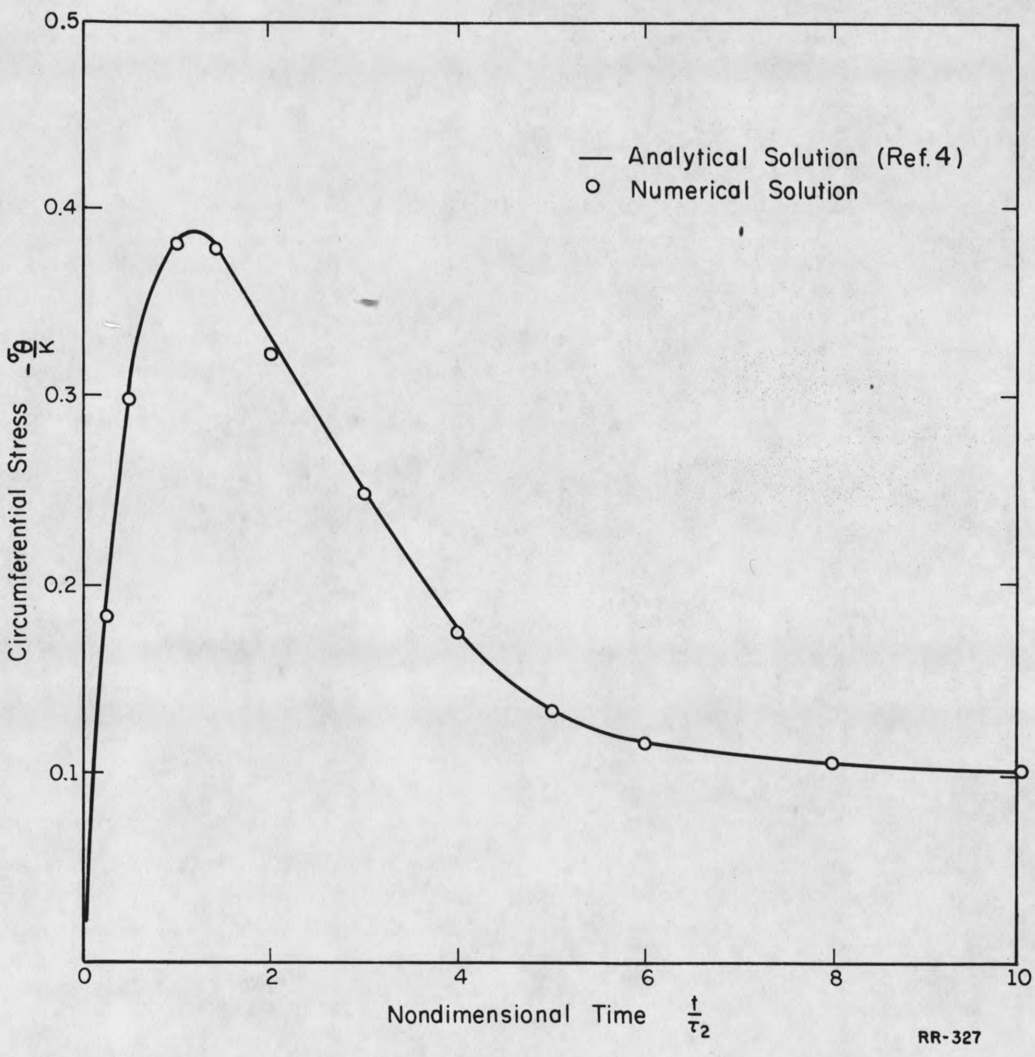

Figure 4. Circumferential stress at r = 1.

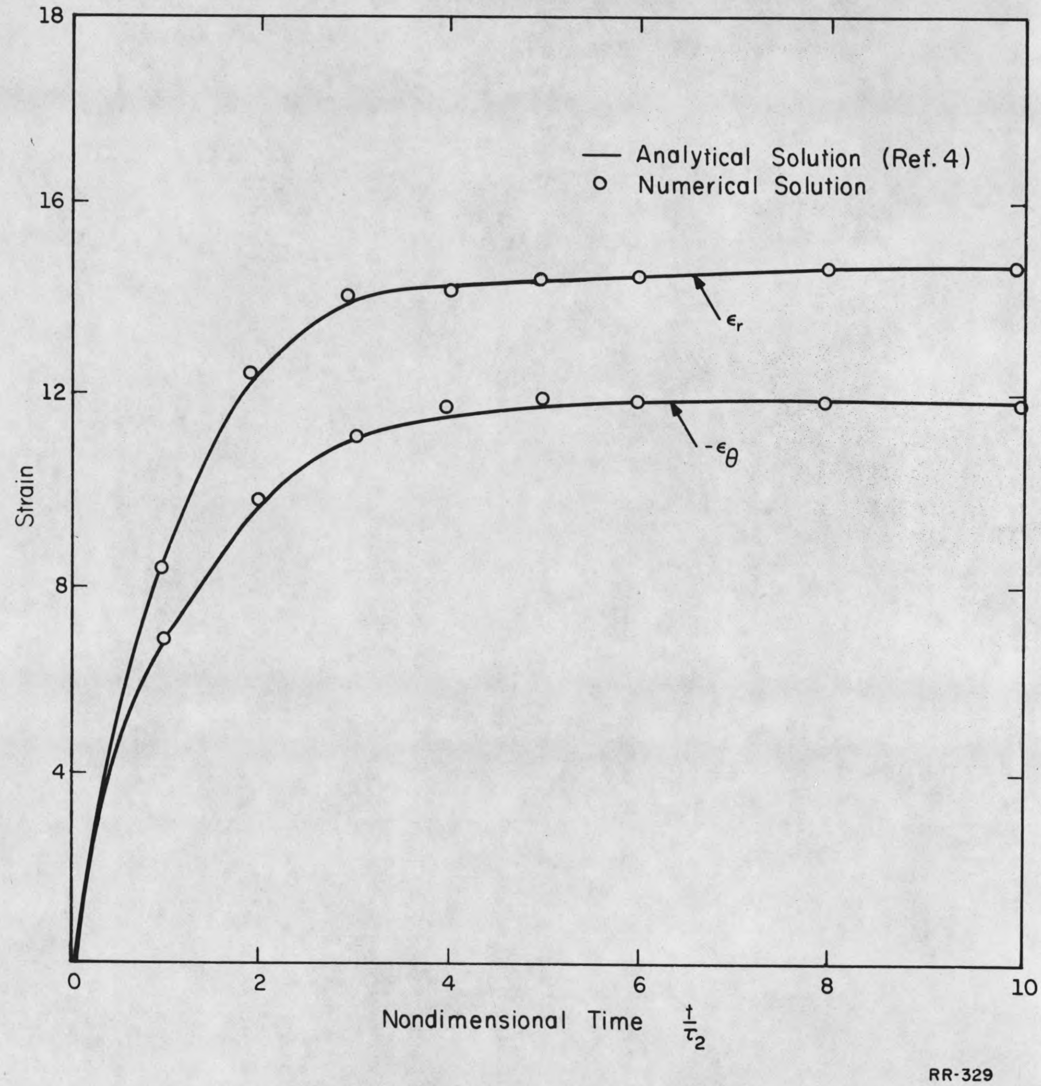

Figure 5. Radial and circumferential strains at r = 1.

4. EXAMPLE B

The purpose of this example is to demonstrate the capability of this analysis in handling some realistic solid propellant materials with temperature dependence. The temperature distribution, which is a function of both the radial coordinate and the time variable, is chosen to be

$$\frac{T}{T_o} = 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)} e^{-(2n+1)^2\pi^2\Gamma/4} \cos[(2n+1)\Omega/2] \quad (4.1)$$

where $\Omega = \frac{r}{h}$,

$\Gamma = \frac{\kappa t}{h^2}$, non-dimensional time,

κ = diffusitivity,

T_o = reference temperature.

The above equation corresponds to the temperature distribution in a slab which is a close approximation in a thick-walled cylinder. The glassy and rubber values of the Poisson's ratios are assumed to be 0.49 and 0.499 respectively. Using these Poisson's ratios, the coefficients P_i, Q_i, R_i and S_i (i=0,1,2,...,q=12) defined by Eqs. (2.10), were calculated and listed in Table 2. The following relations were used in the calculation

Table II. Viscoelastic Parameters for Example B

i	P_i	Q_i	R_i	S_i	τ_i
0	$.117650^{-1}$†	$-.587060^{-2}$	$.292940^{-2}$	$-.499$	------
1	$-.910466^{-3}$	$.454952^{-3}$	$-.231424^{-3}$	$.750^{-3}$	$.2^{8}$
2	$-.457169^{-3}$	$.232836^{-3}$	$-.120588^{-3}$	$.750^{-3}$	$.2^{7}$
3	$-.116824^{-2}$	$.581261^{-3}$	$-.294452^{-3}$	$.750^{-3}$	$.2^{6}$
4	$-.177649^{-2}$	$.879304^{-3}$	$-.443175^{-3}$	$.750^{-3}$	$.2^{5}$
5	$-.172307^{-2}$	$.853128^{-3}$	$-.430114^{-3}$	$.750^{-3}$	$.2^{4}$
6	$-.201635^{-2}$	$.996835^{-3}$	$-.501824^{-3}$	$.750^{-3}$	$.2^{3}$
7	$-.168213^{-2}$	$.833067^{-3}$	$-.420103^{-3}$	$.750^{-3}$	$.2^{2}$
8	$-.105839^{-2}$	$.527435^{-3}$	$-.267539^{-3}$	$.750^{-3}$	$.2^{1}$
9	$-.565823^{-3}$	$.286077^{-3}$	$-.147155^{-3}$	$.750^{-3}$	$.2$
10	$-.217360^{-3}$	$.115330^{-3}$	$-.619526^{-4}$	$.750^{-3}$	$.2^{-1}$
11	$-.965246^{-4}$	$.561206^{-4}$	$-.324071^{-4}$	$.750^{-3}$	$.2^{-2}$
12	$-.439487^{-4}$	$.303584^{-4}$	$-.195518^{-4}$	$.750^{-3}$	$.2^{-3}$

† $.117650^{-1} = .117650 \times 10^{-1}$.

$$Q_o = -0.499P_o$$

$$Q_i = -0.49P_i + 0.75\times10^{-3}$$

$$S_o = -0.499$$

$$S_i = 0.75\times10^{-3}$$

$$R_o = S_oQ_o$$

$$R_i = S_oQ_i + Q_oS_i. \tag{4.2}$$

The reference temperature is taken to be 77^oF, and the coefficient of thermal expansion is $5.25\times10^{-5}in/in/^oF$. The chosen temperature distribution starts from zero and then increases as time goes on until it approaches $T/T_o = 1$, as shown in Figures 6 and 7 for $\Omega = 0.1$ and $\Omega = 0.3$ respectively. The radial temperature distribution at various times is presented in Figure 8. Figures 9, 10 and 11 show the circumferential stress at $\Omega = 0.1$ and the radial stress at $\Omega = 0.3$ versus Γ, the non-dimensional time, for the grid sizes, Δt's, 0.01, 0.05 and 0.1 respectively.

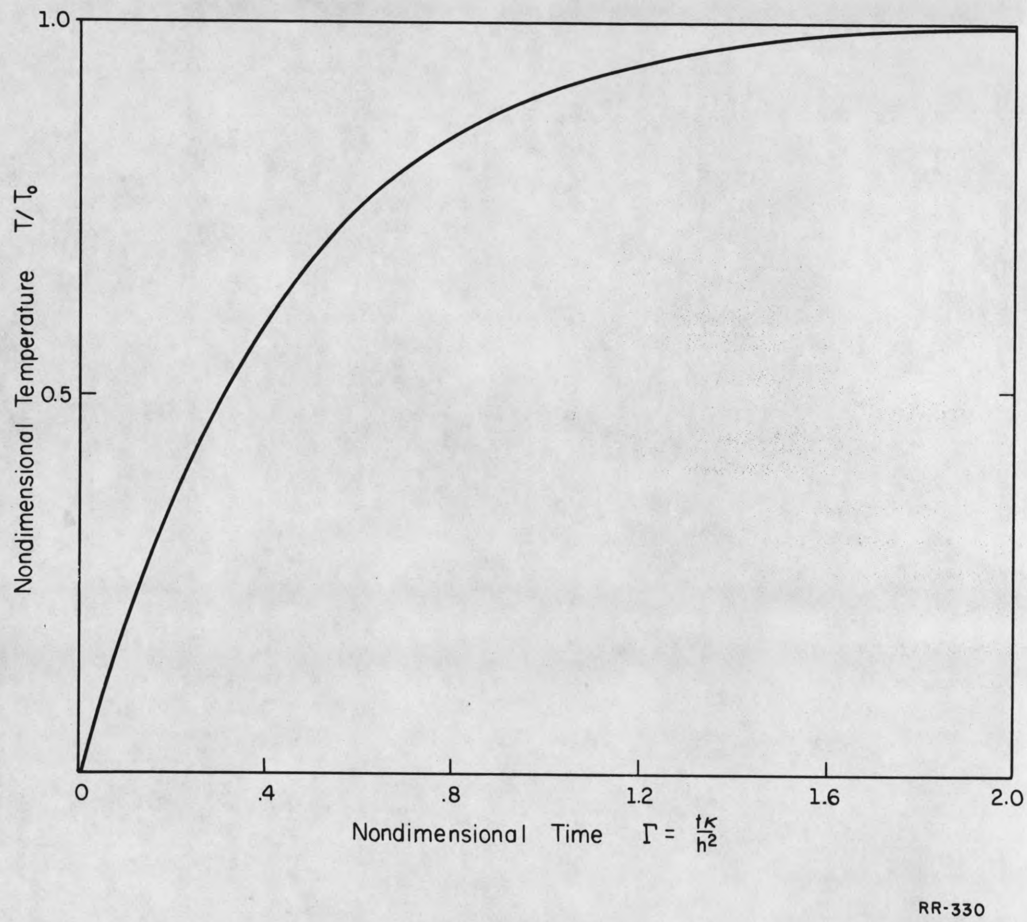

Figure 6. Temperature distribution at $\Omega = \frac{r}{h} = 0.1$.

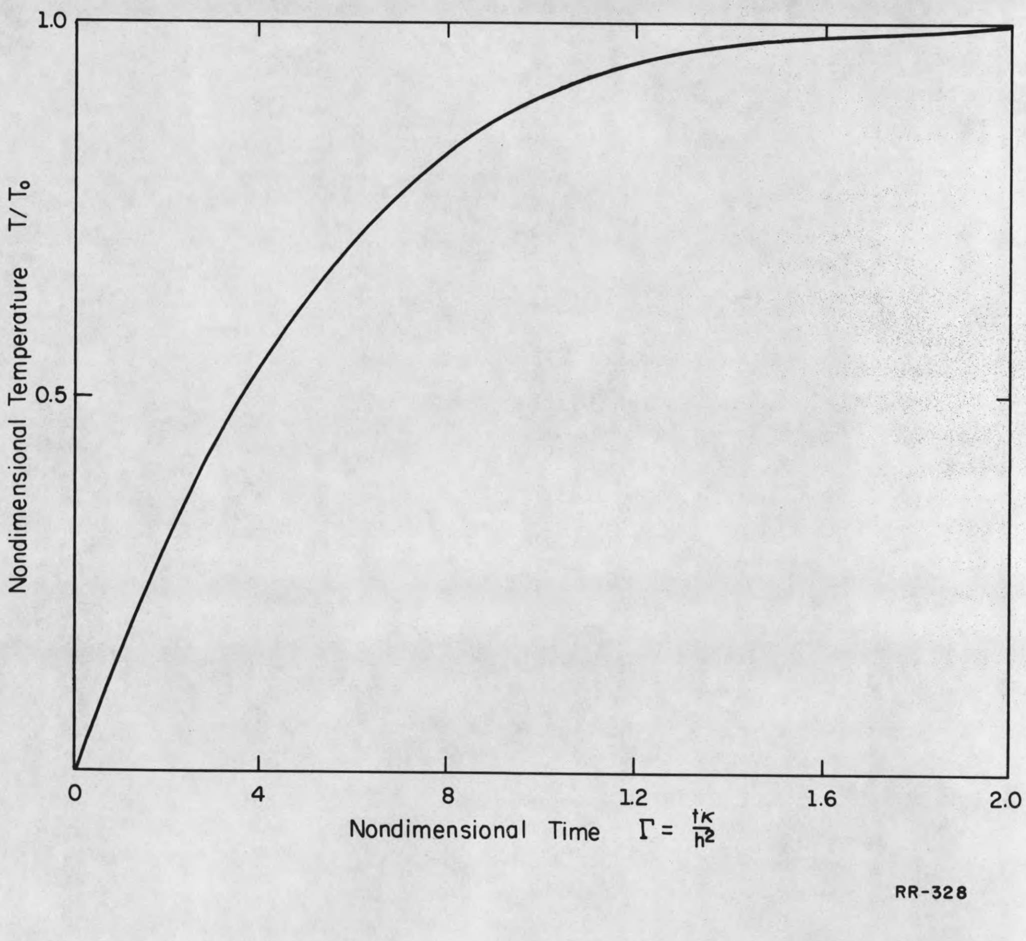

Figure 7. Temperature distribution at $\Omega = \frac{r}{h} = 0.3$.

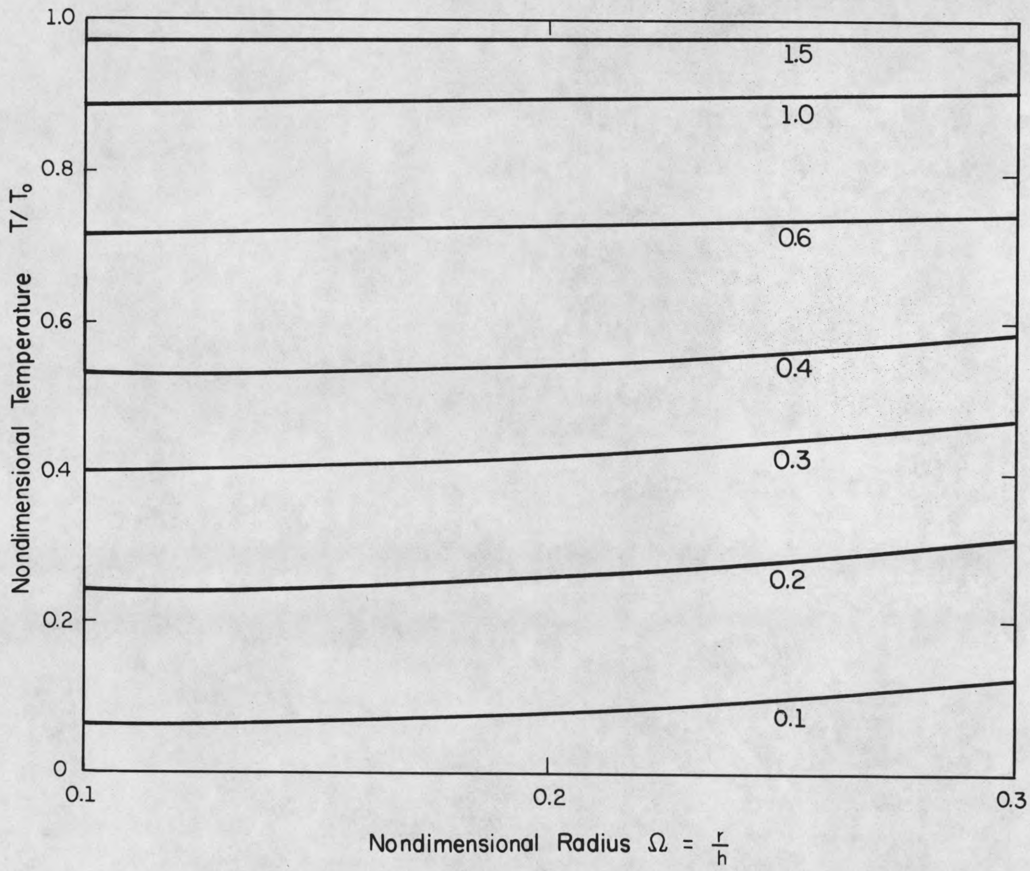

Figure 8. Temperature distribution at various times in the thick-walled cylinder. The numbers on the curves are the values of $\Gamma = t\kappa/h^2$.

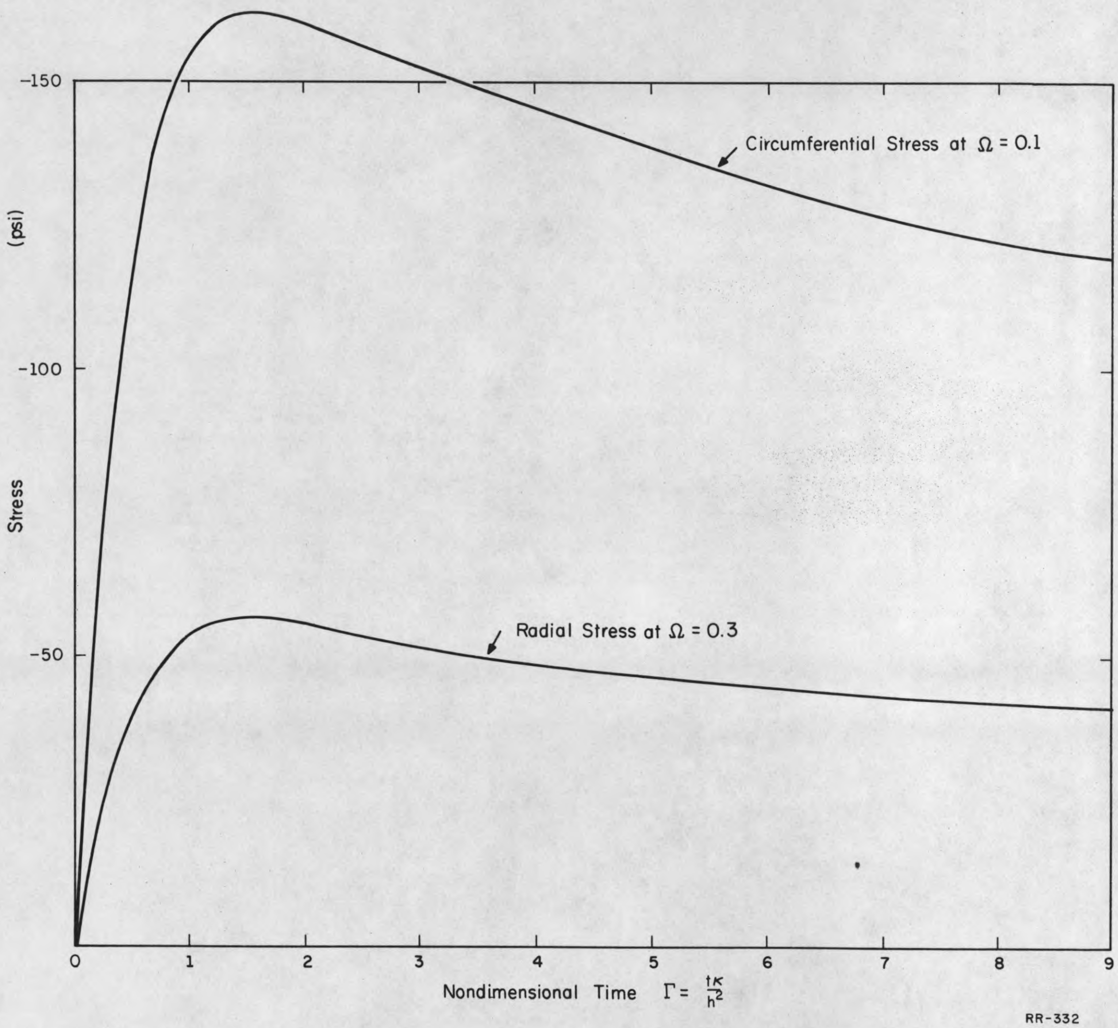

Figure 9. Circumferential and radial stresses at $\Omega = 0.1$ and $\Omega = 0.3$ with $\Delta t = 0.01$.

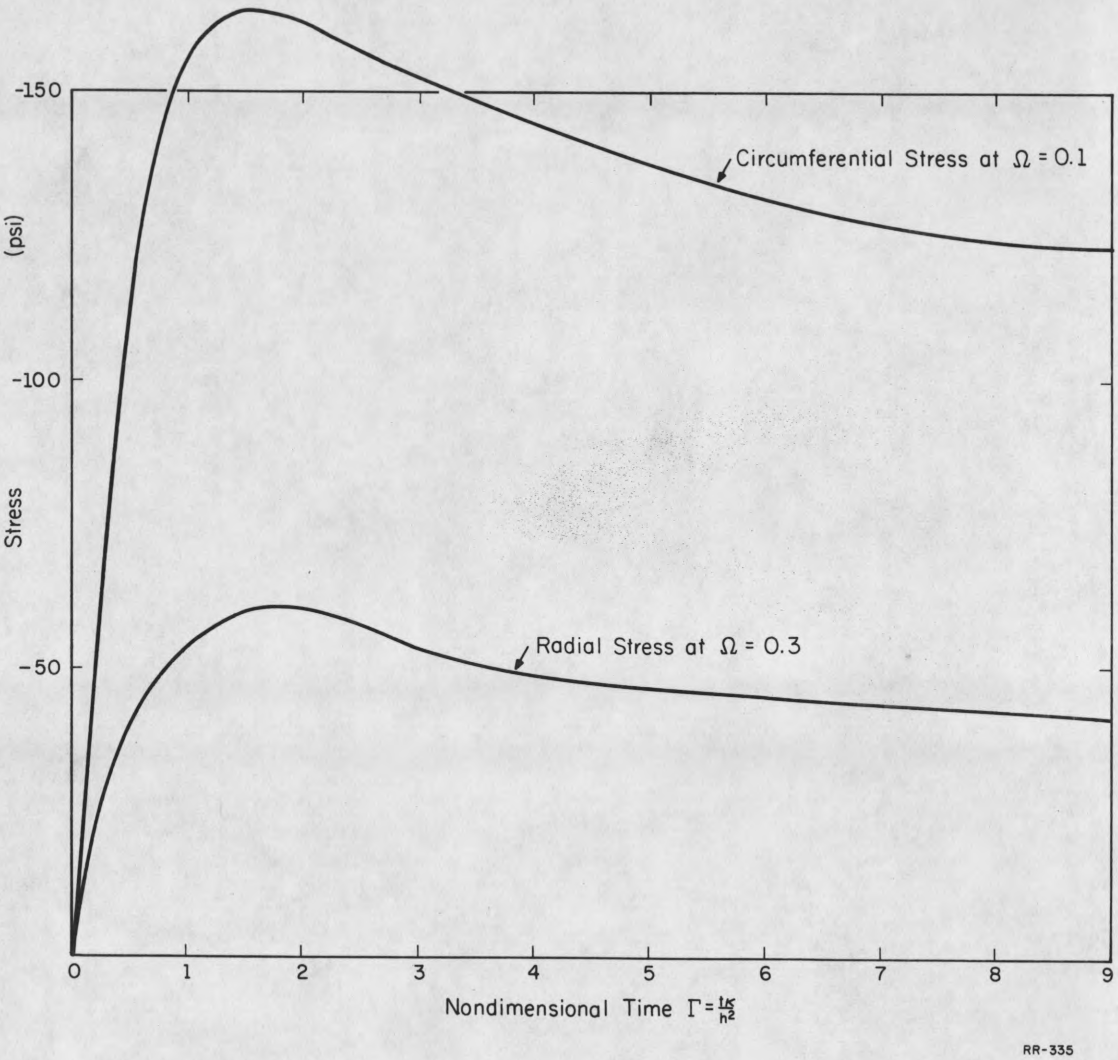

Figure 10. Circumferential and radial stresses at Ω = 0.1 and Ω = 0.3 with Δt = 0.05.

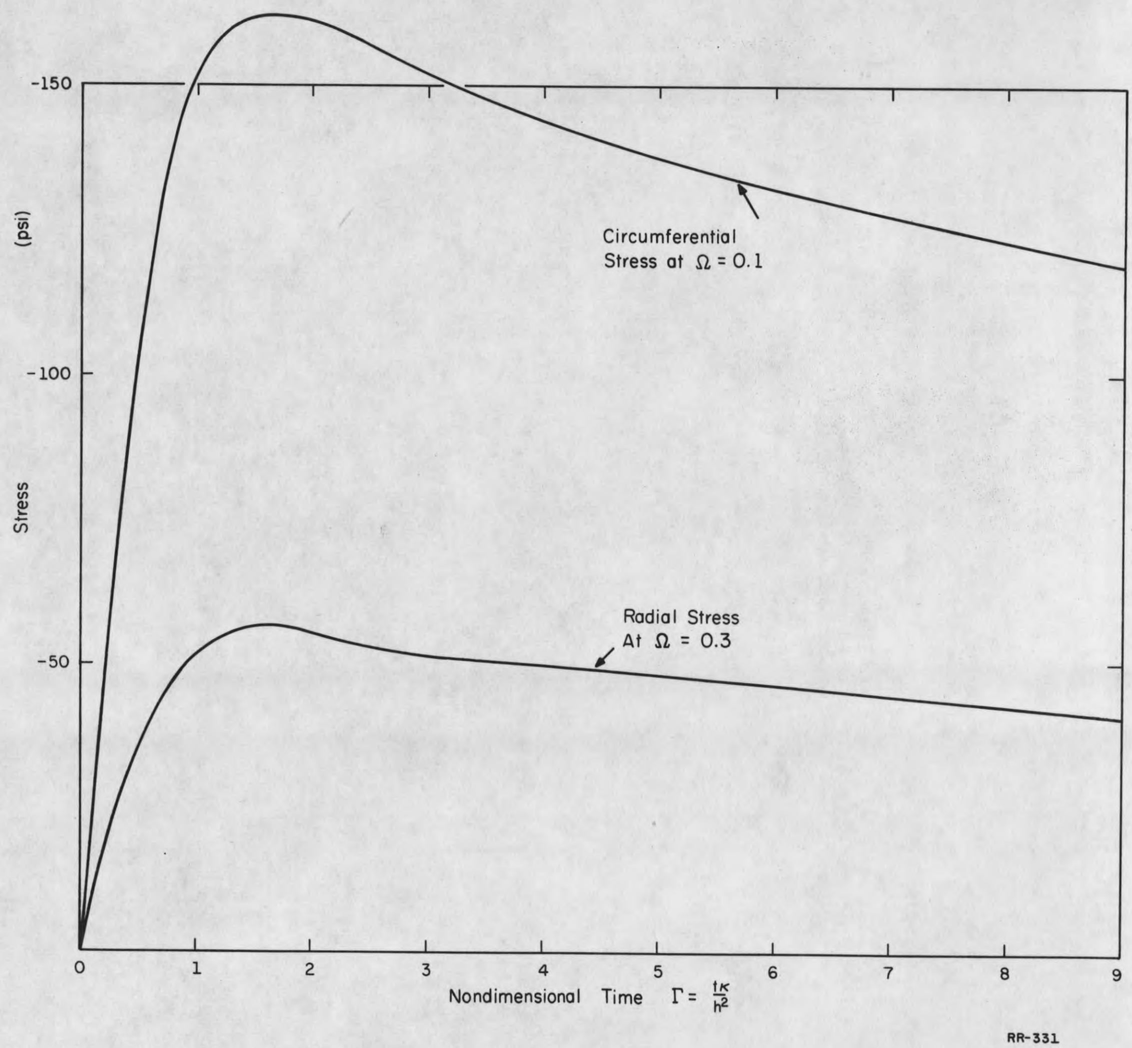

Figure 11. Circumferential and radial stresses at $\Omega = 0.1$ and $\Omega = 0.3$ with $\Delta t = 0.1$.

7. CONCLUSIONS

From Eq. (2.37), the stresses could be calculated quite easily because for a given temperature distribution the terms L_j are constant over each interval and therefore they have to be evaluated only once at the beginning of the calculations. The objective of Example A was to check the solution of this analysis against the result obtained in reference 2. The results from Example A indicated that this approach yielded reasonable results via computer calculations as compared with the analytical solution obtained in reference 2. Figures 3, 4 and 5 showed that the comparison between the stresses and the strains and the corresponding analytical results were extremely close.

It should be mentioned that the analysis was performed in terms of the change-in-stress $(\Delta\sigma_r)_j$ rather than the total-stress $(\sigma_r)_j$ because the latter approach would lead to an instability when the solution approached the steady-state condition. The reason for this was a repeated arithmatical step in which a difference of nearly equal numbers had to be taken. This resulted in accumulation of large errors which in turn led to the instability.

The choice of the grid size, Δt, in Example B had very small influence on the accuracy of the solution as shown by Figures 9, 10 and 11. For $\Delta t = 0.01$, given by Figure 9, the circumferential stress σ_θ reached a maximum of 162 psi at $\Gamma = 1.6$ and decreased to 119 psi at $\Gamma = 9.0$. Comparing with $\Delta t = 0.1$, the same stress had a maximum of 162.4 psi at $\Gamma = 1.7$ and decreased to 118.5 psi at $\Gamma = 9.0$ as presented in Figure 11. Similar comparisons showed that the results were extremely close to each other for different Δt's. This implied that the largest value of Δt considered, i.e. $\Delta t = 0.1$, would be most desirable because it required the least running time of the computer.

In conclusion, it may be emphasized that this analysis is capable of handling realistic viscoelastic materials. It presents a closed form solution in the spatial variable and computes the stresses as an elastic solution for each time interval.

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A solution is presented for the thermal stresses in an isotropic, homogenious, viscoelastic, hollow cylinder enclosed by a rigid casing. The governing equation, which is of the integral-differential form, is integrated numerically. The resulting partial differntial equation is solved exactly subjected to the imposed boundary conditions. For comparision, a simple four-parameter viscoelastic material is chosen since an independent analytical solution is available. A second example is presented in order to demonstrate the capability of this solution in handling large number of viscoelastic parameters which are usually required to characterize real solid propellant and other polymeric structural materials.

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