DECENTRALIZED STACKELBERG STRATEGIES FOR INTERCONNECTED STOCHASTIC DYNAMIC SYSTEMS

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ABSTRACT

A two-level sequential decision formulation for the control of interconnected stochastic linear discrete-time systems is investigated. An interconnection of several systems is considered, whereby each subsystem has a decision maker and an associated quadratic cost function. One of the decision makers is designated as leader or coordinator and his control strategies are to be chosen prior to those of the others. The information available to each decision maker may be different from those of the others. The second level decision makers are regarded as followers in the context of Stackelberg strategies. Their strategies are in accordance with the Nash equilibrium concept except that the coordinator's strategy is known to all of them. The coordinator chooses his strategy under the assumption that the followers will fully exploit the prior announcement of his strategy. Recursive equations for determining the control laws for each subsystem are derived for various types of information structures. Centralized information is considered first. Finally feedback Stackelberg strategies are derived for the more realistic but more complicated (from a design computation viewpoint) situation where the subsystem control laws are based only on local subsystem measurements.

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1. Introduction

In this paper we investigate a sequential decision approach to the control of an interconnection of several subsystems. Associated with each subsystem is a decision-maker or controller and a performance criterion function or cost function. A framework for studying strategies for the control of such systems is non-zero N-person differential games [1-7]. Various solution concepts for defining optimality have been proposed and examined. One of the most widely studied solution concepts is the Cournot or Nash strategy [1-3] whereby the decision-makers simultaneously minimize their respective cost functions with respect to their individual controls. At equilibrium when all the decision-makers apply their Nash strategies, the cost function of any subsystem is at a minimum with respect to the control for that subsystem.

A sequential decision solution concept was first studied by Stackelberg [21] in the context of a static economic problem with two decision-makers. In [22,23,10] the Stackelberg concept was developed for two-person dynamic games with perfect information. Three types of Stackelberg strategies were investigated in [23,23,10]: open-loop, closed-loop, and feedback. In general, the open-loop and closed-loop Stackelberg strategies do not satisfy the principle of optimality but the feedback strategy and the more general equilibrium strategy [11] are defined to satisfy the principle of optimality. Open-loop Stackelberg strategies were considered in [24] for two groups of players where the players in each group use Nash strategies with respect to each other but each group plays according to the open-loop Stackelberg concept with respect to the other group.

All these strategies are for deterministic dynamic games. In [12] the feedback Stackelberg solution concept is extended to stochastic two-person dynamic games.

The approach to be explicitly developed in this paper is based on the coordination solution concept suggested in [20] for deterministic systems. We allow stochastic disturbances in the dynamic process model and in the measurement model, as in [12] but several second-level decision makers or followers are present as in [20]. Several types of information structure are considered. Explicit recursion formulas for the design of the feedback Stackelberg controllers for the coordinator and the followers are presented. The strategies are adaptive to changes in information available at each stage and they satisfy the principle of optimality. The strategies of the second level decision-makers are equilibrium Nash strategies with respect to each other and in addition, they take into account the known strategy of the coordinator. The coordinator chooses his strategy with the full anticipation that the other decision makers will take the coordinator strategy into account in minimizing their individual cost functions.

2. Problem Formulation

Consider M discrete-time linear subsystems, each modeled by

$$x^i(k+1) = A^{io}(k)x^o(k) + A^{ii}(k)x^i(k) + \sum_{\substack{j=1 \\ i\neq j}}^{M} A^{ij}(k)x^j(k) + B^i(k)u^i(k) + \theta^i(k). \quad (1)$$

The measurement of each subsystem is given by

$$z^i(k) = H^{io}(k)x^o(k) + H^{ii}(k)x^i(k) + \sum_{\substack{j=1 \\ i\neq j}}^{M} H^{ij}(k)x^j(k) + \xi^i(k) \qquad i = 1,\dots,M; \quad (2)$$

where x^i is the n^i-dimensional state vector of the i-th subsystem, u^i is the m^i-dimensional local control vector DM^i for the i-th subsystem, z^i is the ℓ^i-dimensional measured output vector for the i-th subsystem. $\{x^i(0);\ \theta^i(k) \in R^{ni};\ \xi^i(k) \in R^{\ell i};\ i=1,\ldots,M \quad k=0,\ldots,N-1\}$ are mutually independent Gaussian random vectors with known means and covariances.

$$E\{x^i(0)\} = 0 \quad ; \qquad Cov\{x^i(0)\} = \Sigma^i(0)$$

$$E\{\theta^i(k)\} = 0 \quad ; \qquad Cov\{\theta^i(k)\} = \Theta^i(k)$$

$$E\{\xi^i(k)\} = 0 \quad ; \qquad Cov\{\xi^i(k)\} = \Xi^i(k)$$

Each subsystem seeks to minimize the expected value of its cost function

$$J^i(u^i) = \frac{1}{2} x^{iT}(N)K^{ii}(N)x^i(N) + \frac{1}{2}\sum_{k=0}^{N-1}[x^{iT}(k)Q^{ii}(k)x^i(k) + u^{iT}(k)R^{ii}(k)u^i(k)] \qquad i=1,\ldots,M \tag{3}$$

where K^{ii}, G^{ii}, and R^{ii} are all positive-definite.

In addition to the M-subsystems, we assume that we have a coordinator subsystem modeled by

$$x^o(k+1) = A^o(k)x^o(k) + \sum_{i=1}^{M} A^{oi}(k)x^i(k) + \theta^o(k) \tag{4}$$

and the measurement of the coordinator subsystem is given by

$$z^o(k) = H^o(k)x^o(k) + \sum_{i=1}^{M} H^{oi}(k)x^i(k) + \xi^o(k) \tag{5}$$

where x^o is the n^o-dimensional state vector of the coordinator subsystem, u^o is an m^o-dimensional control vector chosen by the coordinator DM^o, z^o is the ℓ^o-dimensional measured output vector of the coordinator subsystem.

$\{x^o(0);\ \theta^o(k) \in R^{no};\ \xi^o(k) \in R^{\ell o};\ k = 0,\ldots,N-1\}$ are mutually independent with the random vectors of each subsystem.

$$E\{x^o(0)\} = 0 \quad ; \qquad Cov\{x^o(0)\} = \Sigma^o(0)$$

$$E\{\theta^o(k)\} = 0 \quad ; \qquad Cov\{\theta^o(k)\} = \Theta^o(k)$$

$$E\{\xi^o(k)\} = 0 \quad ; \qquad Cov\{\xi^o(k)\} = \Xi^o(k)$$

The coordinator chooses u^o to minimize the expected value of the cost function

$$J^o(u^o) = \frac{1}{2} x^{oT}(N)K^o(N)x^o(N) + \frac{1}{2} \sum_{i=1}^{M} x^{iT}(N)K^{oi}(N)x^i(N)$$
$$+ \frac{1}{2} \sum_{k=0}^{N-1} [x^{oT}(k)Q^o(k)x^o(k) + u^{oT}(k)R^o(k)u^o(k) + \sum_{i=1}^{M} x^{iT}(k)Q^{oi}(k)x^i(k)] \qquad (6)$$

where K^o, K^{oi}, Q^o, R^o, Q^{oi} are all positive definite.

The Stackelberg approach [20] to the coordination of the subsystems is to consider DM^o as a leader and DM^i as followers. We imagine that DM^o provides DM^i the exact knowledge of all decisions made by the coordinator and each DM^i minimizes J^i with respect to u^i for each given decision of DM^o assuming that the other subsystems will do the same. With this assumption the subsystems play Nash among themselves. The coordinator then minimizes J^o with respect to u^o, considering that the decisions from the subsystems result from choices of u^i which minimize J^i for $i = 1,\ldots,M$. Additionally, the information sets include exact knowledge of the system dynamic DM^o, DM^i, the measurements and the cost-functionals. The statistics of the random elements for all k are also included.

The optimal feedback Stackelberg approach to the 2-level coordination of the subsystems [20] is described by the following procedure: At each stage, the coordinator computes the subsystems' expected reactions to his decision, based on minimizing the subsystems' expected cost-to-go assuming that all second level decision makers will use their optimal feedback Stackelberg strategies in the future. The coordinator then seeks to minimize his expected cost-to-go assuming that the subsystems will respond as expected. Each subsystem then uses the coordinator's decision to compute his optimal decision, assuming that the other subsystems will do the same. These expectations are conditioned on the information sets available to each subsystem.

The information set consists of exact knowledge of the system dynamics, the measurement rules and the cost functionals of all decision makers. Additionally, it includes exact knowledge of all decisions made by each player up to stage k-1 and the statistics of random elements $\theta^i(k)$, $\xi^i(k)$, $i=0,1,\ldots,M$ for all k. Also, the Stackelberg nature of the game implies that the followers' information contains the exact value of the leader's decision at time k, $u^o(k)$.

Let arg min f(k) denote the value of u at which f(k) achieves its absolute minimum. Then the equations that define these optimal solutions are as follows:

$$u_o^i(u^o,k) = \arg\min_{u^i} E\{J^i(u^i,x^i,k)\,|\,Z^i(k)\} \tag{7}$$

$$u^{o*}(k) = \arg\min_{u^o} E\{J^o(u^o,x^o,x^i,k)\,|\,Z^o(k)\} \tag{8}$$

$$u^{i*}(k) = u_o^i(u^{o*},k). \tag{9}$$

The optimal cost-to-go at each stage are

$$J^{i*}(k) = E\{J^i(u^i,x^i,k)|Z^i(k),u^i = u^{i*},u^o = u^o\} \quad i=1,\ldots,M \tag{10}$$

$$J^{o*}(k) = E\{J^o(u^o,x^o,x^i,k)|Z^o(k),u^o = u^{o*},u^i = u^{i*}\} \quad i=1,\ldots,M. \tag{11}$$

Stochastic dynamic programming can be used to obtain the solutions.

Two possible cases will be considered in this paper. First, when the information is centralized, several classes of information structures are discussed. One is when all decision makers have perfect system state measurement. Another is when the information of all the followers are identical and the coordinator's information contains the followers' information. Second, we will constrain each controller to be in decentralized structure and the i-th subsystem including the coordinator knows only its own measurement.

3. Coordination with Centralized Information

In general the coordinator has some information from each subsystem and, in turn makes some decisions that will influence the dynamic response of the lower-level subsystems. By definition of Stackelberg strategies [10] all decisions made by the coordinator are known to the second level decision makers. However, some information may or may not be available to the coordinator and lower-level subsystems. When the information sets are centralized, either the coordinator and the lower-level subsystems have perfect information of state, or the lower-level subsystems have the same measurement but the information set of the coordinator consists of his own measurement and the lower-level subsystems' measurement. Several particular cases of this problem are examined. Let us examine a system with one coordinator and two second level decision makers. Consider the augmented system

$$x(k+1) = A(k)x(k) + B^o(k)u^o(k) + B^1(k)u^1(k) + B^2(k)u^2(k) + v(k) \qquad (12)$$

where $x^T(k) = [x^{oT}(k)\ x^{1T}(k)\ x^{2T}(k)]$

$v^T(k) = [\theta^{oT}(k)\ \theta^{1T}(k)\ \theta^{2T}(k)]$

$x(0)$ and $v(k)$ are Gaussian random vectors with zero mean and covariance $\Sigma(0)$ and $\Lambda(k)$, and the measurement of each subsystem is

$$z^i(k) = H^i(k)x(k) + \xi^i(k) \qquad i = 0,1,2 \qquad (13)$$

the quadratic cost is

$$J^i(u^i) = \frac{1}{2}\, x^T(N)K^i(N)x(N) + \frac{1}{2} \sum_{k=0}^{N-1} [x^T(k)Q^i(k)x(k) + u^{iT}(k)R^i(k)u^i(k)]$$

$$i = 0,1,2 \qquad (14)$$

3.1 Perfect Information

Suppose all subsystems have perfect information of the states, i.e., $z^i(k) = x(k), i = 0,1,2$. Assume that the expected cost-to-go at stage k is

$$V^i(k) = \frac{1}{2}\, x^T(k)S^i(k)x(k) + \frac{1}{2}\gamma^i(k), \qquad i = 0,1,2 \qquad (15)$$

for some deterministic matrix $S^i(k)$ and scalar function $\gamma^i(k)$. Using dynamic programming as shown in Appendix 1 the optimal strategies are

$$u^{o*}(k) = -L^o(k)x(k) \qquad (16)$$

$$u^i_o(k) = -\Delta^i(k)[A(k)x(k) + B^o(k)u^o(k)], \qquad i = 1,2 \qquad (17)$$

where

$$L^o(k) = [R^o(k) + \hat{B}^T(k)S^o(k+1)\hat{B}(k)]^{-1}\hat{B}^T(k)S^o(k+1)\hat{A}(k)$$

$$\Delta^i(k) = [I - L^i(k)B^j(k)L^j(k)B^i(k)]^{-1}(L^i(k) - L^i(k)B^j(k)L^j(k))$$

$$i = 1,2, \; j = 1,2, \; i \neq j$$

$$\hat{A}(k) = A(k) - B^1(k)\Delta^1(k)A(k) - B^2(k)\Delta^2(k)A(k)$$

$$\hat{B}(k) = B^0(k) - B^1(k)\Delta^1(k)B^0(k) - B^2(k)\Delta^2(k)B^0(k)$$

$$L^i(k) = [R^i(k) + B^{iT}(k)S^i(k+1)B^i(k)]^{-1}B^{iT}(k)S^i(k+1)$$

Assuming that the indicated inverses exist the other quantities are obtained from

$$S^o(k) = Q^o(k) + \hat{A}^T(k)S^o(k+1)\hat{A}(k) - L^{oT}(k)[R^o(k) + \hat{B}^T(k)S^o(k+1)\hat{B}(k)]L^o(k) \quad (18)$$

$$S^o(N) = K^o(N) \quad (19)$$

$$\gamma^o(k) = \gamma^o(k+1) + \mathrm{tr}S^o(k+1)\Lambda(k) \quad (20)$$

$$\gamma^o(N) = 0 \quad (21)$$

$$S^i(k) = Q^i(k) + [A(k) - B^o(k)L^o(k)]^T\Delta^{iT}(k)R^i(k)\Delta^i(k)[A(k) - B^o(k)L^o(k)]$$
$$+ [\hat{A}(k) - \hat{B}(k)L^o(k)]^TS^i(k+1)[\hat{A}(k) - \hat{B}(k)L^o(k)], \quad i = 1,2 \quad (22)$$

$$S^i(N) = K^i(N), \quad i = 1,2 \quad (23)$$

$$\gamma^i(k) = \gamma^i(k+1) + \mathrm{tr}S^i(k+1)\Lambda(k), \quad i = 1,2 \quad (24)$$

$$\gamma^i(N) = 0, \quad i = 1,2 \quad (25)$$

These equations can be solved backwards in time. In summary we have the following calculations: Starting at $k = N-1$, $S^o(N)$, $S^i(N)$, $i = 1,2$ are given.

1. Compute $L^i(k)$, $i = 1,2$
2. Compute $\Delta^i(k)$, $i = 1,2$
3. Compute $\hat{A}(k)$, $\hat{B}(k)$
4. Compute $L^o(k)$
5. Compute $S^o(k)$, $S^i(k)$, $i = 1,2$
6. $k \to k-1$ and go to 1. Stop when $k = 0$.

Note that the control laws for the coordinator and the i-th subsystem involve perfect measurement of the state.

3.2 Coordination with Nested Information Structure

3.2.1 Incomplete Information for Coordinator and Subsystem

Consider the case where the information of the state is incomplete. At each stage, in addition to their own estimates, the optimal strategies would include terms involving an estimate of the other subsystems' estimates of the state in the future. This leads to estimators of much larger dimension than the system itself. For a special case of the stochastic problem, consider the case where each subsystem has the same measurement

$$(z^1(k) = z^2(k) = z(k) = H(k)x(k) + \xi(k))$$

and the coordinator knows both his measurement and all subsystems measurements. So for any k, $Z^o(k) \supset Z(k)$, implying that the information sets are nested. We also have to assume that there is no information transfer among subsystems through their controls [12]. The optimal strategies for this case are derived in Appendix 2 as

$$u_o^i(k) = -\Delta^i(k)(A(k)\hat{x}(k) + B^o(k)u^o(k)), \quad i = 1,2 \tag{26}$$

$$u^{o*}(k) = -\Delta^o(k)Y(k)\hat{x}^o(k) - \Delta^o(k)M(k)[\hat{x}(k) - \hat{x}^o(k)] \tag{27}$$

$$J^{o*}(k) = \frac{1}{2}\begin{bmatrix} \hat{x}^o(k) \\ \hat{x}(k)-\hat{x}^o(k) \end{bmatrix}^T \begin{bmatrix} S^A(k) & S^B(k) \\ S^{BT}(k) & S^C(k) \end{bmatrix} \begin{bmatrix} \hat{x}^o(k) \\ \hat{x}(k)-\hat{x}^o(k) \end{bmatrix} + \frac{1}{2}\gamma^o(k) \tag{28}$$

$$J^{i*}(k) = \frac{1}{2}\hat{x}^T(k)S^i(k)\hat{x}(k) + \frac{1}{2}\gamma^i(k) \qquad i = 1,2 \tag{29}$$

where

$$\hat{x}(k) = E\{x(k)|z^i(k)\}, \ \hat{x}^o(k) = E\{x(k)|z^o(k)\}$$

$\Delta^i(k)$, $\hat{A}(k)$, $\hat{B}(k)$, and $L^i(k)$ are defined as in the perfect information case with $S^A(k)$ replacing $S^o(k)$. In addition we have

$$S^A(k) = Q^o(k) + A^T(k)(I-G(k))^T S^A(k+1)(I-G(k))A(k) - Y^T(k)\Delta^o(k)Y(k) \tag{30}$$

$$\begin{aligned} S^B(k) = {} & A^T(k)(I-G(k))^T S^B(k+1)(I-G(k))A(k) \\ & + A^T(k)(I-G(k))^T(S^B(k+1) - S^A(k+1))G(k)A(k) \\ & - A^T(k)(I-G(k))^T S^B(k+1)K(k+1)H(k+1)A(k) \\ & - Y^T(k)\Delta^o(k)M(k) \end{aligned} \tag{31}$$

$$\begin{aligned} S^C(k) = {} & -M^T(k)\Delta^o(k)M(k) + A^T(k)G^T(k)S^A(k+1)G(k)A(k) \\ & + A^T(k)[I - K(k+1)H(k+1)]^T S^C(k+1)[I - K(k+1)H(k+1)]A(k) \\ & + A^T(k)(S^B(k+1)K(k+1)H(k+1) - S^B(k+1))G(k)A(k) \\ & - A^T(k)G^T(k)(S^B(k+1) - S^B(k+1)K(k+1)H(k+1))A(k) \end{aligned} \tag{32}$$

$$Y(k) = \hat{B}(k)S^A(k+1)[I-G(k)]A(k)$$

$$\begin{aligned} M(k) = {} & \hat{B}(k)S^A(k+1)G(k)A(k) + \hat{B}^T(k)(S^B(k+1) - S^A(k+1))A(k) \\ & - B^T(k)S^B(k+1)K(k+1)H(k+1)A(k) \end{aligned}$$

$$G(k) = B^1(k)\Delta^1(k) + B^2(k)\Delta^2(k)$$

$$\Delta^{o}(k) = [R^{o}(k) + \hat{B}(k)S^{A}(k+1)\hat{B}(k)]^{-1}$$

$$K^{i}(k+1) = P^{i}(k+1/k)H^{iT}(k+1)[H^{i}(k+1)P^{i}(k+1/k)H^{iT}(k+1) + \Xi(k+1)]^{-1}$$

$$P^{i}(k+1/k) = A(k+1)P^{i}(k/k)A^{T}(k+1) + \Lambda(k)$$

$$P^{i}(k+1/k+1) = [I - K^{i}(k+1)H^{i}(k+1)]P^{i}(k+1/k)$$

$$P^{i}(0/0) = \Sigma(0)$$

for $i = 0,1,2$ and where $H^{i} = H$ for $i = 1,2$.

$$\gamma^{o}(k) = \gamma^{o}(k+1) + trQ^{o}(k)P^{o}(k/k) + tr[K^{o}(k+1)[H^{o}(k+1)P^{o}(k+1/k)H^{oT}(k+1)$$
$$+ \Xi^{o}(k)]K^{oT}(k+1)(S^{A}(k+1) + S^{C}(k+1) - 2S^{B}(k+1))]$$
$$+ 2trP^{o}(k+1/k)K(k+1)H(k+1)(S^{B}(k+1) - S^{C}(k+1))$$
$$+ trK(k+1)[H(k+1)P^{o}(k+1/k)H^{T}(k+1) + \Xi(k+1)]K^{T}(k+1)S^{C}(k+1) \qquad (33)$$

$$S^{i}(k) = Q^{i}(k) + (A(k) + \hat{B}(k)\Delta^{o}(k)Y(k))^{T}S^{i}(k+1)(A(k) + \hat{B}(k)\Delta^{o}(k)Y(k))$$
$$+ (\Delta^{i}(k)A(k) + B^{o}(k)\Delta^{o}(k)Y(k)^{T}R^{i}(k)\Delta^{i}(k)A(k) + B^{o}(k)\Delta^{o}(k)Y(k))$$
$$i = 1,2 \qquad (34)$$

$$\gamma^{i}(k) = \gamma^{i}(k+1) + trQ^{i}(k)P(k/k) + trS^{i}(k+1)K(k+1)$$
$$+ trS^{i}(k+1)K(k+1)[H(k+1)P(k+1/k)H^{T}(k+1) + \Xi(k+1)]K^{T}(k+1)$$
$$+ tr[P(k/k) - P^{o}(k/k)](M(k) - Y(k))^{T}\Delta^{oT}(k)(B^{oT}(k)R^{i}(k)B^{o}(k)$$
$$+ B^{T}S^{i}(k+1)B)\Delta^{o}(k)(M(k) - Y(k)). \qquad (35)$$

The recursive equations (30) and (34) are identical to equations (18) and (22) in the perfect information case, with the same initial conditions, so that the solution $S^A(k)$ and $S^i(k)$ in (30) and (34) are equal to $S^o(k)$ and $S^i(k)$ in (18) and (22). Thus, as far as the followers are concerned, they play a "separation principle" strategy which consists of the optimal deterministic feedback law of their best estimate of the state. The leader strategy includes his own estimate and a term involving a difference in estimates. When both estimates are the same, the leader also plays as in the "separation principle."

3.2.2 Perfect Information for Coordinator

Consider the problem in which coordinator has perfect state measurement while the lower level subsystems have available only noisy output measurements. In addition, we assume that conditions are such that the coordinator can deduce exactly the lower level subsystems' state estimators, and the lower level subsystems have the same noisy measurement, i.e., $Z(k) = Z^2(k) = Z(k)$.

When the coordinator has perfect state measurement and can deduce exactly the state of the lower level subsystems' state estimator, i.e., $H^o(k) \equiv I$ and $\xi^o(k) \equiv 0$, also $Z^o(k) \supset Z(k)$. The problem is of "nested information" type except the coordinator does not have to estimate its own state ($E[x(k)/Z^o(k)] = x(k)$).

The control law of the coordinator is

$$u^{o*}(k) = -\Delta^o(k)Y(k)x(k) - \Delta^o(k)M(k)(\hat{x}(k) - x(k)) \tag{36}$$

and the control laws of the lower level subsystems are

$$u_o^i(k) = -\Delta^i(k)[A(k)\hat{x}(k) + B^o u^o(k)], \quad i = 1,2 \tag{37}$$

where $E[x(k)/Z(k)] = \hat{x}(k)$. The optimal cost-to-go is

$$J^{o*}(k) = \frac{1}{2} \begin{bmatrix} x(k) \\ \hat{x}(k)-x(k) \end{bmatrix}^T \begin{bmatrix} S^A(k) & S^B(k) \\ S^{BT}(k) & S^C(k) \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k)-x(k) \end{bmatrix} + \frac{1}{2}\gamma^o(k) \tag{38}$$

$$J^{i*}(k) = \frac{1}{2}\hat{x}^T(k)S^i(k)\hat{x}(k) + \frac{1}{2}\gamma^i(k) \qquad i = 1,2 \tag{39}$$

where all matrices are the same as in 3.2.1 case.

3.2.3 No Measurements for Subsystems

Consider the problem in which the coordinator has a noisy measurement, while the lower level subsystems have no measurement available to them and are restricted to using only a priori information.

When the lower level subsystems have no measurements, i.e., $H^i(k) \equiv 0$ (null matrix) and $Z^i(k) \equiv Z^i(0)$ for all k, the problem is also of nested information type. The control law of the coordinator is

$$u^o(k) = -\Delta^i(k)Y(k)\hat{x}^o(k) - \Delta^o(k)M(k)(\hat{x}(k) - \hat{x}^o(k) \tag{40}$$

and the control laws of the lower level subsystems is

$$u_o^i(k) = -\Delta^i(k)[A\hat{x}(k) + B^o(k)u^o(k)], \quad i = 1,2 \tag{41}$$

where $E[x(k)/Z^o(k)] = \hat{x}^o(k)$, $E[x(k)/Z(k)] = \hat{x}(k)$.

The optimal cost-to-go is

$$J^{o*}(k) = \frac{1}{2}\begin{bmatrix} \hat{x}^o \\ \hat{x}(k)-\hat{x}^o(k) \end{bmatrix}^T \begin{bmatrix} S^A(k) & S^B(k) \\ S^{BT}(k) & S^C(k) \end{bmatrix} \begin{bmatrix} \hat{x}^o(k) \\ \hat{x}(k)-\hat{x}^o(k) \end{bmatrix} + \frac{1}{2}\gamma^o(k) \tag{42}$$

$$J^{i*}(k) = \frac{1}{2}\hat{x}^T(k)S^i(k)\hat{x}(k) + \frac{1}{2}\gamma^i(k), \quad i = 1,2 \tag{43}$$

where all matrices are the same as in 3.2.1.

Substitution of (40) and (41) into the system equation gives

$$\begin{aligned} x(k+1) = {} & A(k)x(k) - (B^1(k)\Delta^1(k)A(k) + B^2(k)\Delta^2(k)A(k))\hat{x}(k) \\ & - (B^o(k) - B^1(k)\Delta^1(k)B^o(k) - B^2(k)\Delta^2(k)B^o(k))\Delta^o(k)Y(k)\hat{x}(k) \\ & - (B^o(k) - B^1(k)\Delta^1(k)B^o(k) - B^2(k)\Delta^2(k)B^o(k))\Delta^o(k)Y(k) \\ & (\hat{x}(k) - \hat{x}^o(k). \end{aligned} \tag{44}$$

It follows that the optimal estimate of the states by the lower level subsystems, given only a priori information, i.e., no output measurement, is given by

$$\begin{aligned} \hat{x}(k+1) = {} & [A(k) - B^1(k)\Delta^1(k)A(k) - B^2(k)\Delta^2(k)A(k) \\ & - (B^o(k) - B^1(k)\Delta^1(k)B^o(k) - B^2(k)\Delta^2(k)B^o(k)\Delta^o(k)Y(k)]\hat{x}(k) \end{aligned} \tag{45}$$

with initial condition $\hat{x}(0|0) = \bar{x}(0)$.

In addition, when $\bar{x}(0) = 0$, then $\hat{x}(k/k) = 0$ so that

$$u^{o*}(k) = -\Delta^o(k)[Y(k) - M(k)]\hat{x}^o(k) \tag{46}$$

and

$$u_o^i(k) = -\Delta^i(k)B^o(k)u^o(k), \quad i = 1,2 \tag{47}$$

4. Constrained Decentralized Structure

It may be desirable to have a control policy that is simpler to implement than the optimal policy. Satisfactory control of a high-order linear system may often be achieved using relatively fewer system measurements and a controller of low order. This has been the motivation for a number of optimal designs, using output feedback or dynamic controllers of a specified order. For recent work in this field we refer the reader to [13]-[19].

4.1 Decentralized Control with Instantaneous Output Feedback

Consider the stochastic problem where a restriction is placed on the control of the i-th subsystem and the coordinator at any instant to be a linear transformation of the measurement at that instant. Also, there is no information transfer among subsystems through their controls. This simplifies the problem since a filter is no longer used to estimate the state. Then

$$u^i(k) = F^i(k)z^i(k), \quad i = 0,1,2, \quad k = 0,1,\ldots,N-1 \tag{48}$$

where $F^i(k)$ is to be determined to minimize the expected value of $J^i(u^i)$.

Consider the augmented system (12) and the measurement

$$z^i(k) = H^i(k)x(k) + \xi^i(k), \quad i = 0,1,2. \tag{49}$$

Then
$$u^i(k) = F^i(k)H^i(k)x(k) + F^i(k)\xi^i(k), \quad i=0,1,2 \tag{50}$$

and
$$x(k+1) = (A(k) + \sum_{i=0}^{2} B^i(k)F^i(k)H^i(k))x(k) + \sum_{i=0}^{2} B^i(k)F^i(k)\xi^i(k) + v_k. \tag{51}$$

Define
$$\Sigma(k) = E\{x(k)x^T(k)\}$$

and note that $x(k)$ depends on $\xi^i(k)$ for $i=0,1,\ldots,k-1$ only, implying that $E\{x(k)v^T(k)\} = 0$. Then the recursive equation for $\Sigma(k)$ is

$$\Sigma(k+1) = (A(k) + \sum_{i=0}^{2} B^i(k)F^i(k)H^i(k))\Sigma(k)(A(k) + \sum_{i=0}^{2} B^i(k)F^i(k)H^i(k))^T + \sum_{i=0}^{2} B^i(k)F^i(k)\Xi^i(k)F^{iT}(k)B^{iT}(k) + \Lambda(k). \tag{52}$$

Lemma 4.1.1: If a linear system described by (12) is controlled using a linear control policy (48) then the expected cost (14) can be expressed as

$$\begin{aligned} E[J^i(k)] &= \frac{1}{2} E[x^T(k)S^i(k)x(k)] + \frac{1}{2} \sum_{\ell=k+1}^{N} \mathrm{tr}\, S^i(\ell)\Lambda(\ell-1) \\ &+ \frac{1}{2} \sum_{\ell=k+1}^{N} \{\mathrm{tr}\, F^{iT}(\ell-1)(R^i(\ell-1) + B^{iT}S^i(\ell)B^{iT}(\ell-1))F^i(\ell-1)\Xi^i(\ell-1) \\ &+ \sum_{\substack{j=0 \\ i\neq j}}^{2} F^{jT}(\ell-1)B^{jT}(\ell-1)S^i(\ell)B^j(\ell-1)F^j(\ell-1)\Xi^j(\ell-1)\}, \quad i = 1,2 \end{aligned} \tag{53}$$

where

$$\begin{aligned} S^i(k) &= Q^i(k) + H^{iT}(k)F^{iT}(k)R^i(k)F^i(k)H^i(k) + \\ &+ (A(k) + \sum_{j=0}^{2} B^j(k)F^j(k)H^j(k))^T S^i(k+1)(A(k) + \sum_{j=0}^{2} B^j(k)F^j(k)H^j(k)) \quad i = 1,2 \end{aligned} \tag{54}$$

$$S^i(N) = K^i(N) \qquad i = 1,2. \tag{55}$$

Proof: The proof is by induction.

Consider the augmented system (12) and the cost criterion (14). The assumption obviously holds for $k = N$. For any k

$$\begin{aligned} E[J^i(k)] &= E\{\frac{1}{2} \sum_{\ell=k}^{N-1} \{x^T(\ell)Q^i(\ell)x(\ell) + u^{iT}(\ell)R^i(\ell)u^i(\ell)\} + \frac{1}{2} E\{x^T(N)K^i(N)x(N)\} \\ &= E[J^i(k+1)] + E\{\frac{1}{2} x^T(k)Q^i(k)x(k) + \frac{1}{2} u^{iT}(k)R^i(k)u^i(k)\}, \quad i = 1,2 \end{aligned} \tag{56}$$

with $k = k+1$ using (53) in (56) and after some algebra the assumption holds for $k = k+1$. Thus (53) holds for $k - 0,1,\ldots,N$.

For $i = 1,2$, apply dynamic programming and at each step set the derivative of the remaining cost with respect to each element of $F^i(k)$ equal to zero. Thus in terms of $M^i(k)$, $Y^i(k)$, $\Gamma^i(k)$, and $T^i(k)$ which are defined in (70), (71), (72), and (73), we have

$$F^{1*}(k) = M^1(k)[A(k) + B^o(k)F^o(k)H^o(k)]Y^1(k)$$
$$+ M^1(k)B^2(k)F^2(k)H^2(k)Y^1(k) \tag{57}$$

$$F^{2*}(k) = M^2(k)[A(k) + B^o(k)F^o(k)H^o(k)]Y^2(k)$$
$$+ M^2(k)B^1(k)F^1(k)H^1(k)Y^2(k) \tag{58}$$

or

$$F^{1*}(k) = \Gamma^1(k)[A(k) + B^o(k)F^o(k)H^o(k)]T^1(k) \tag{59}$$

$$F^{2*}(k) = \Gamma^2(k)[A(k) + B^o(k)F^o(k)H^o(k)]T^2(k). \tag{60}$$

<u>Lemma 4.1.2</u>: If a linear system described by (12) is controlled using a linear control policy (48) then the expected cost (14) for $i = 0$ can be expressed as

$$E[J^o(k)] = \frac{1}{2} E[x^T(k)S^o(k)x(k)] + \frac{1}{2}\Big[\sum_{i=k+1}^{N} \mathrm{tr}S^o(i)\Lambda(i-1)$$
$$+ \mathrm{tr}F^{oT}(i-1)[R^o(i) + B^{oT}(i)S^o(i)B^o(i)]F^o(i-1)\Xi^o(i-1)$$
$$+ \sum_{j=1}^{2} \mathrm{tr}F^{j*T}(i-1)B^{jT}(i)S^o(k)B^j(k)F^{j*}(i-1)\Xi^j(i-1)\Big] \tag{64}$$

where

$$S^o(k) = Q^o(k) + H^{oT}(k)F^{oT}(k)R^o(k)F^o(k)H^o(k) + [A(k) + B^o(k)F^o(k)H^o(k)$$
$$+ B^1(k)F^{1*}(k)H^1(k) + B^2(k)F^{2*}(k)H^2(k)]^T S^o(k+1)[A(k) + B^o(k)F^o(k)H^o(k)$$
$$+ B^1(k)F^{1*}(k)H^1(k) + B^2(k)F^{2*}(k)H^2(k)] \tag{62}$$

$$S^o(N) = K^o(N). \tag{63}$$

The proof is the same as given in Lemma 4.1.1.

At each step the necessary condition for a minimum is that the derivative of the remaining cost with respect to each element of $F^o(k)$ must equal zero.

$$F^{o}(k) = -[R^{o}(k) + (B^{o}(k) + B^{1}(k)\Gamma^{1}(k)B^{o}(k) + B^{2}(k)\Gamma^{2}(k)B^{o}(k)^{T}S^{o}(k+1)(B^{o}(k)$$
$$+ B^{1}(k)\Gamma^{1}(k)B^{o}(k) + B^{2}(k)\Gamma^{2}(k)B^{o}(k))]^{-1}\{(B^{o}(k) + B^{1}(k)\Gamma^{1}(k)B^{o}(k)$$
$$+ B^{2}(k)\Gamma^{2}(k)B^{o}(k))^{T}S^{o}(k+1)[A(k) + B^{1}(k)\Gamma^{1}(k)A(k)T^{1}(k)H^{1}(k)$$
$$+ B^{2}(k)\Gamma^{2}(k)A(k)T^{2}(k)H^{2}(k)]\Sigma(k)[H^{o}(k) + H^{o}(k)T^{1}(k)H^{1}(k)$$
$$+ H^{o}(k)T^{2}(k)H^{2}(k)]^{T} + (B^{1}(k)\Gamma^{1}(k)B^{o}(k)$$
$$+ B^{2}(k)\Gamma^{2}(k)B^{o}(k))^{T}S^{o}(k+1)[B^{1}(k)\Gamma^{1}(k)A(k)T^{1}\Xi^{1}(k)(H^{o}(k)T^{1}(k))^{T}$$
$$+ B^{2}(k)\Gamma^{2}(k)A(k)T^{2}\Xi^{2}(k)(H^{o}(k)T^{2}(k))^{T}\}[H^{o}(k)\Sigma(k)H^{oT}(k) + (H^{o}(k)$$
$$+ H^{o}(k)T^{1}(k)H^{1}(k) + H^{o}(k)T^{2}(k)H^{2}(k))\Sigma(k)(H^{o}(k) + H^{o}(k)T^{1}(k)H^{1}(k)$$
$$+ H^{o}(k)T^{2}(k)H^{2}(k))^{T} + (\Xi^{o}(k) + H^{o}(k)T^{1}\Xi^{1}(k)(H^{o}(k)T^{1}(k))^{T}$$
$$+ H^{o}(k)T^{2}\Xi^{2}(k)(H^{o}(k)T^{2}(k))^{T}]^{-1}. \quad (64)$$

Theorem 1: The sequences $\{F^{i}(k)\}$ $i = 0,1,2$; $k = 0,1,\ldots,N-1$ of the coordinator and i-th subsystem that minimize $E\{J^{i}(u^{i})\}$ $i = 0,1,2$ subject to the constraint (48) are given by equations (59), (60) and (64) where it is assumed that the required inverse matrices exist and

1. $$\Sigma(k+1) = [A(k) + \sum_{i=0}^{2} B^{i}(k)F^{i}(k)H^{i}(k)]\Sigma(k)[A(k) + \sum_{i=0}^{2} B^{i}(k)F^{i}(k)H^{i}(k)]^{T}$$
$$+ \sum_{i=0}^{2} B^{i}(k)F^{i}(k)\Xi^{i}(k)F^{iT}(k)B^{iT}(k) + \Lambda(k). \quad (65)$$

   $\Sigma(0)$ is given.

2. $$S^{i}(k) = Q^{i}(k) + H^{iT}(k)F^{iT}(k)R^{i}(k)F^{i}(k)H^{i}(k)$$
$$+ [A(k) + \sum_{j=0}^{2} B^{j}(k)F^{j}(k)H^{j}(k)]^{T}S^{i}(k+1)[A(k) + \sum_{j=0}^{2} B^{j}(k)F^{j}(k)H^{j}(k)]$$
$$i = 1,2 \quad (66)$$

$$S^{i}(N) = K^{i}(N), \quad i = 1,2 \quad (67)$$

3. $S^o(k) = Q^o(k) + H^{oT}(k)F^{oT}(k)R^o(k)F^o(k)H^o(k) + [A(k) + B^o(k)F^o(k)H^o(k) + B^1(k)F^{1*}(k)H^1(k) + B^2(k)F^{2*}(k)H^2(k)]^T S^o(k+1)[A(k) + B^o(k)F^o(k)H^o(k) + B^1(k)F^{1*}(k)H(k) + B^2(k)F^{2*}(k)H^2(k)]$. (68)

$$S^o(N) = K^o(N). \tag{69}$$

Also,

$$\Gamma^i(k) = [I - M^i(k)B^j(k)M^j(k)B^i(k)]^{-1}[M^i(k) + M^i(k)B^j(k)M^j(k)] \qquad i = 1,2,\ j = 1,2,\ i \neq j \tag{70}$$

$$T^i(k) = [Y^i(k) + Y^j(k)H^j(k)Y^i(k)][I - H^i(k)Y^j(k)H^j(k)Y^i(k)]^{-1} \qquad i = 1,2,\ j = 1,2,\ i \neq j \tag{71}$$

$$M^i(k) = -[R^i(k) + B^{iT}S^i(k+1)B^i]^{-1}B^{iT}S^i(k+1), \quad i = 1,2 \tag{72}$$

$$Y^i(k) = \Sigma(k)H^{iT}(k)[H^i(k)\Sigma(k)H^{iT}(k) + \Xi^i(k)]^{-1}, \quad i = 1,2. \tag{73}$$

The sequences $\{F^i(k)\}, i = 0,1,2;\ k = 0,1,\ldots,N-1$ of the coordinator and the i-th subsystem are the solution to the discrete two-point boundary value problem. Note that (65), (66) and (67) are recursive relationships for generating $\Sigma(k)$ and $S^i(k)$, $i = 0,1,2$ except (65) which is a forward equation and (66) and (67) which are backward equations, and all depend on the sequences $\{F^i(k)\}$, $i = 0,1,2$. But the sequences $\{F^i(k)\}$, $i = 0,1,2$ as given by (57), (58) and (64) depend on both sequences $\{S^i(k)\}$ $i = 0,1,2$ and $\Sigma(k)$. Thus unless either $\{F^i(k)\}$ or $\{S^i(k)\}$ and $\{\Sigma(k)\}$ are known no simple calculation will solve the problem. We suggest the following simple procedure to solve the equations:

1. Make an initial guess for the gain $\{F^o_j(k)\}$ and $\{F^i_j(k)\}$, $i = 1,2$; $k = 0,1,\ldots,N-1$. Let $j = 0$.
2. Use $\{F^o_j(k)\}$ and $\{F^i_j(k)\}$ to solve (65) forward in time to determine $\{\Sigma_j(k)\}$ with $\Sigma_j(0) = \Sigma(0)$.

3. Use $\{F_j^o(k)\}$ and $\{F_j^i(k)\}$ to solve (66) and (68) backward in time to determine $\{S_j^i(k)\}$, i = 1,2 and $\{S_j^o(k)\}$ with $S_j^i(N) = K^i(N)$, i = 1,2 and $S_j^o(N) = K^o(N)$.
4. Use $\{\Sigma_j(k)\}$ and $\{S_j^o(k)\}$ in (64) to determine $\{F_{j+1}^o(k)\}$.
5. Use $\{\Sigma_j(k)\}$, $\{S_j^i(k)\}$, i = 1,2 and $\{F_{j+1}^o(k)\}$ in (59) and (60) to determine $\{F_{j+1}^i(k)\}$ i = 1,2. Let j = i + 1.
6. Repeat (2)-(5) until the desired degree of convergence is reached.

So far no convergence conditions for this algorithm have been found, but as with most algorithms of this type it is expected that convergence depends on the initial guess.

4.2 Decentralized Control with Dynamic Output Feedback

Consider the stochastic problem where a dynamic controller of a specified order for the i-th subsystem and the coordinator described by

$$w^i(k+1) = D^i(k)w^i(k) + \bar{M}^i(k)z^i(k), \quad i = 0,1,2 \tag{74}$$

where $w^i \in R^{si}$ is the state vector of the controllers used. Then

$$u^i(k) = N^i(k)w^i(k) + F^i(k)z^i(k), \quad i = 0,1,2 \tag{75}$$

also

$$z^i(k) = H^i(k)x(k) + \xi^i(k), \quad i = 0,1,2. \tag{76}$$

For a given integer $s^i (0 \leq s^i \leq n)$, find matrices $N^i(k)$, $F^i(k)$, $D^i(k)$ and $\bar{M}^i(k)$ such that the corresponding expected cost $E\{J^i(u^i)\}$ will be minimum. Note that if $s^i = 0$ the controller is reduced to

$$u^i(k) = F^i(k)z^i(k), \quad i = 0,1,2$$

and if $s^i = n$, an optimal solution is obtained.

The solution of the problem is obtained through the following steps. First, we consider the augmented system formed by the i-th subsystem combined with the state of the controller. Second, we transform the stochastic optimization problem to a deterministic one. Third, use dynamic programming to obtain the recursive equations.

Consider the augmented state vector

$$\tilde{x}^T(k) = [x^T(k) \quad w^{oT}(k) \quad w^{1T}(k) \quad w^{2T}(k)]$$

then

$$\tilde{x}(k+1) = (\tilde{A}(k) + \tilde{B}^o(k)\tilde{F}^o(k)\tilde{H}^o(k) + \tilde{B}^1(k)\tilde{F}^1(k)\tilde{H}^1(k) + \tilde{B}^2(k)\tilde{F}^2(k)\tilde{H}^2(k))\tilde{x}(k) + \sum_{i=0}^{2} \tilde{B}^i(k)\tilde{F}^i(k)\tilde{I}^i(k)\xi^i(k) + \tilde{I}(k)v(k) \tag{77}$$

$$u^i(k) = N^i(k)w^i(k) + F^i(k)H^i(k)x(k) + F^i(k)\xi^i(k) \qquad i = 0,1,2$$

where

$$\tilde{F}^i(k) = \begin{bmatrix} F^i(k) & N^i(k) \\ \bar{M}^i(k) & D^i(k) \end{bmatrix}$$

$$\tilde{I} = \begin{bmatrix} I \\ \hline 0 \end{bmatrix} \begin{matrix} \updownarrow n \\ \updownarrow s^o+s^1+s^2 \end{matrix} \quad ; \qquad T = [\, I \mid 0 \,] \quad \begin{matrix} \leftarrow n \rightarrow \mid \leftarrow s^o+s^1+s^2 \rightarrow \end{matrix}$$

then

$$u^i(k) = T\tilde{F}^i(k)\tilde{H}^i\tilde{x}(k) + T\tilde{F}^i\tilde{I}^i\xi^i(k), \quad i = 0,1,2. \tag{78}$$

Define

$$\tilde{\Sigma}(k) = E[\tilde{x}(k)\tilde{x}^T(k)]. \tag{79}$$

Note that the augmented system (77) and controller (78) are of the same form as (51) and (49). The following theorem can be derived using the same argument as Theorem 1.

Theorem 2: The sequences $\{\tilde{F}^i(k)\}, i = 0,1,2;\ k = 0,1,\ldots,N-1$ of the coordinator and the i-th subsystem that minimize $E\{J^i(u^i)\}, i = 0,1,2$ subject to the constraint (44) are given by

$$\tilde{F}^{o}(k) = -[\tilde{R}^{o} + (\tilde{B}^{o} + \tilde{B}^{1}\Gamma^{1}\tilde{B}^{o} + \tilde{B}^{2}\Gamma^{2}\tilde{B}^{o})^{T}\tilde{S}^{o}(k+1)(\tilde{B}^{o} + \tilde{B}^{1}\Gamma^{1}\tilde{B}^{o}$$
$$+ \tilde{B}^{2}\Gamma^{2}\tilde{B}^{o})]^{-1}\{(\tilde{B}^{o} + \tilde{B}^{1}\Gamma^{1}\tilde{B}^{o} + \tilde{B}^{2}\Gamma^{2}\tilde{B}^{o})^{T}\tilde{S}^{o}(k+1)[\tilde{A} + \tilde{B}^{1}\Gamma^{1}\tilde{A}\tilde{T}^{1}\tilde{H}^{1}$$
$$+ \tilde{B}^{2}\Gamma^{2}\tilde{A}\tilde{T}^{2}\tilde{H}^{2}]\tilde{\Sigma}(k)[\tilde{H}^{o} + \tilde{H}^{o}\tilde{T}^{1}\tilde{H}^{1} + \tilde{H}^{o}\tilde{T}^{2}\tilde{H}^{2}]^{T} + (\tilde{B}^{1}\Gamma^{1}\tilde{B}^{o}$$
$$+ \tilde{B}^{2}\Gamma^{2}\tilde{B}^{o})^{T}S^{o}(k+1)(\tilde{B}^{1}\Gamma^{1}\tilde{A}\Xi^{1}\tilde{H}^{oY}\tilde{T}^{1T} + \tilde{B}^{2}\Gamma^{2}\tilde{A}\Xi^{2}\tilde{H}^{oT}\tilde{T}^{2T})\}[\tilde{H}^{o}\tilde{\Sigma}(k)\tilde{H}^{oT} + (\tilde{H}^{o} + \tilde{H}^{o}\tilde{T}^{1}\tilde{H}^{1}$$
$$+ \tilde{H}^{o}\tilde{T}^{2}\tilde{H}^{2})\tilde{\Sigma}(k)(\tilde{H}^{o} + \tilde{H}^{o}\tilde{T}^{1}\tilde{H}^{1} + \tilde{H}^{o}\tilde{T}^{2}\tilde{H}^{2})^{T} + (\Xi^{o} + \tilde{H}^{o}\Xi^{1}\tilde{H}^{oT}\tilde{T}^{1T}$$
$$+ \tilde{H}^{o}\Xi^{2}\tilde{H}^{oT}\tilde{T}^{2T})]^{-1}. \qquad (80)$$

$$\tilde{F}^{i*}(k) = -[\tilde{R}^{i} + \tilde{B}^{iT}\tilde{S}^{i}(k+1)\tilde{B}^{i}]^{-1}\tilde{B}^{iT}\tilde{S}^{i}(k+1)[\tilde{A}$$
$$+ \sum_{\substack{j=0 \\ i\neq j}}^{2} \tilde{B}^{j}\tilde{F}^{j}(k)\tilde{H}^{j}]\tilde{\Sigma}(k)\tilde{H}^{iT}[\tilde{H}^{i}\tilde{\Sigma}(k)\tilde{H}^{iT} + \Xi^{i}]^{-1}, \quad i = 1,2 \qquad (81)$$

where it is assumed that the required inverse matrices exist and where

1. $$\tilde{\Sigma}(k+1) = (\tilde{A} + \tilde{B}^{o}\tilde{F}^{o}(k)\tilde{H}^{o} + \tilde{B}^{1}\tilde{F}^{1}(k)\tilde{H}^{1} + \tilde{B}^{2}\tilde{F}^{2}(k)\tilde{H}^{2})\tilde{\Sigma}(k)(\tilde{A}$$
$$+ \tilde{B}^{o}\tilde{F}^{o}(k)\tilde{H}^{o} + \tilde{B}^{1}\tilde{F}^{1}(k)\tilde{H}^{1} + \tilde{B}^{2}\tilde{F}^{2}(k)\tilde{H}^{2})^{T}$$
$$+ \sum_{i=0}^{2} \tilde{B}^{i}\tilde{F}^{i}(k)\Xi^{i}(k)\tilde{F}^{iT}(k)\tilde{B}^{iT} + \tilde{\Lambda}(k) \qquad (82)$$

$\tilde{\Sigma}(0)$ is given.

2. $$\tilde{S}^{i}(k) = \tilde{Q}^{i} + \tilde{H}^{iT}\tilde{F}^{iT}(k)\tilde{R}^{i}\tilde{F}^{i}(k)\tilde{H}^{i} + (\tilde{A} + \tilde{B}^{o}\tilde{F}^{o}(k)\tilde{H}^{o} + \tilde{B}^{i}\tilde{F}^{i}(k)\tilde{H}^{i}$$
$$+ \tilde{B}^{j}\tilde{F}^{j}(k)\tilde{H}^{j})^{T}\tilde{S}^{i}(k+1)(\tilde{A} + \tilde{B}^{o}\tilde{F}^{o}(k)\tilde{H}^{o} + \tilde{B}^{i}\tilde{F}^{i}(k)\tilde{H}^{i} + \tilde{B}^{j}\tilde{F}^{j}(k)\tilde{H}^{j})$$
$$i = 1,2,\ j = 1,2 \quad i \neq j \qquad (83)$$

$\tilde{S}^{i}(N) = \tilde{K}^{i}(N)$.

3. $$\tilde{S}^{o}(k) = \tilde{Q}^{o} + \tilde{H}^{oT}\tilde{F}^{oT}(k)\tilde{R}^{o}\tilde{F}^{o}(k)\tilde{H}^{o} + (\tilde{A} + \tilde{B}^{o}\tilde{F}^{o}(k)\tilde{H}^{o} + \tilde{B}^{1}\tilde{F}^{1*}(k)\tilde{H}^{1}$$
$$+ \tilde{B}^{2}\tilde{F}^{2*}(k)\tilde{H}^{2})^{T}\tilde{S}^{o}(k+1)(\tilde{A} + \tilde{B}^{o}\tilde{F}^{o}(k)\tilde{H}^{o} + \tilde{B}^{1}\tilde{F}^{1*}(k)\tilde{H}^{1} + \tilde{B}^{2}\tilde{F}^{2*}(k)\tilde{H}^{2}) \qquad (84)$$

$\tilde{S}^{o}(N) = \tilde{K}^{o}(N)$.

Also,

$$\Gamma^{i}(k) = [I - M^{i}\tilde{B}^{j}M^{j}\tilde{B}^{i}]^{-1}[M^{i} + M^{i}\tilde{B}^{j}M^{j}] \qquad i = 1,2,\ j = 1,2,\ i \neq j$$

$$T^{i}(k) = [Y^{i} + Y^{j}H^{j}Y^{i}][I - H^{i}Y^{j}H^{j}Y^{i}]^{-1} \qquad i = 1,2,\ j = 1,2,\ i \neq j$$

$$M^i(k) = -[\tilde{R}^i + \tilde{B}^{iT}\tilde{S}^i(k+1)\tilde{B}^i]^{-1}\tilde{B}^{iT}\tilde{S}^i(k+1), \qquad i = 1,2$$

$$Y^i(k) = \tilde{\Sigma}(k)\tilde{H}^{iT}[\tilde{H}^i\tilde{\Sigma}(k)\tilde{H}^{iT} + \Xi^i]^{-1}, \qquad i = 1,2$$

Again, the sequences $\{\tilde{F}^i(k)\}$, $i = 0,1,2$; $k = 0,1,\ldots,N-1$ of the coordinator and the i-th subsystem are the solutions to the discrete two-point boundary value problem as the previous one but the equations are more complicated to solve.

In the case where either the coordinator has noise in its measurement or the lower level subsystems have no noise in their measurement, and want to use output feedback, they can do so by reducing the dimension of their controller to zero.

5. Conclusions

The control of an interconnected set of linear discrete-time stochastic systems has been considered. The organizational form of the system permits one decision maker to be the coordinator or leader and the decision makers for the other subsystems are all followers with respect to the coordinator, but they use the Nash strategy with respect to other second level decision makers. Both centralized and decentralized control structures were considered. As in single decision maker control problems with output feedback constraints, decentralization constraints generally lead to two-point boundary value problems. Explicit recursive formulas for these two-point boundary value problems have been derived. The sequential decision approach seems to be a natural one when the cost function associated with one decision maker has a more global significance compared to the others. This decision maker takes the role of a coordinator and leader.

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Appendix 1.

Consider augmented system (12)

$$x(k+1) = A(k)x(k) + B^0(k)u^0(k) + B^1(k)u^1(k) + B^2(k)u^2(k) + v(k) \qquad \text{(A.1.1)}$$

then

$$E[x(k+1)/z(k) = x] = Ax(k) + B^0u^0(k) + B^1u^1(k) + B^2u^2(k) \qquad \text{(A.1.2)}$$

and quadratic cost (14)

$$J^i(u^i) = \frac{1}{2}x^T(N)K^i(N)x(N) + \frac{1}{2}\sum_{k=0}^{N-1}[x^T(k)Q^i(k)x(k) + u^{iT}(k)R^i(k)u^i(k)] \qquad \text{(A.1.3)}$$

Assume that the expected cost to go at stage k is

$$V^i(k) = \frac{1}{2}x^T(k)S^i(k)x(k) + \frac{1}{2}\gamma^i(k), \qquad i = 1,2 \qquad \text{(A.1.4)}$$

then

$$V^i(k) = \min_{u_k^i} E[\frac{1}{2}x^T(k)Q^i(k)x(k) + \frac{1}{2}u^{iT}(k)R^i(k)u^i(k) + V^i(k+1)/x(k)] \qquad i = 1,2$$

$$= \min_{u_k^i} [\frac{1}{2}x^T(k)Q^i(k)x(k) + \frac{1}{2}u^{iT}(k)R^i(k)u^i(k) + E[V^i(k+1)/x(k)]. \qquad \text{(A.1.5)}$$

When k = N

$$V^i(N) = \frac{1}{2}x^T(N)K^i(N)x(N), \qquad i = 1,2. \qquad \text{(A.1.6)}$$

Lemma 1: Let x be normal with mean m and covariance R, then

$$E[x^TSx] = m^TSm + \operatorname{tr} SR .$$

From Lemma 1 and (A.1.4) with k = k+1

$$E[V^i(k+1)/x(k)] = \frac{1}{2}(Ax(k) + B^iu^i(k) + B^ju^j(k) + B^ou^o(k))^TS^i(k+1)(Ax(k) + B^iu^i(k) + B^ju^j(k) + B^ou^o(k)) + \frac{1}{2}\operatorname{tr}S^i(k+1)\Lambda(k) + \frac{1}{2}\gamma^i(k+1)$$
$$i = 1,2,\ i \neq j \qquad \text{(A.1.7)}$$

Using (A.1.7) in (A.1.5) to obtain $u^i(k)$ that minimize the expected value of the cost function

$$u^i(k) = -[R^i + B^{iT}S^i(k+1)B^i]^{-1}B^{iT}S^i(k+1)[Ax(k) + B^j u^j(k) + B^o u^o(k)] \qquad (A.1.8)$$

$$i = 1,2, \; i \neq j$$

Let

$$L^i(k) = [R^i + B^{iT}S^i(k+1)B^i]^{-1}B^{iT}S^i(k+1). \qquad (A.1.9)$$

Then (A.1.8) becomes

$$u^i(k) = -L^i(k)[Ax(k) + B^j u^j(k) + B^o u^o(k)], i = 1,2, \; i \neq j \qquad (A.1.10)$$

For 2-subsystems solve for $u^1(k)$ and $u^2(k)$

$$u_o^1(k) = -\Delta^1(k)(Ax(k) + B^o u^o(k)) \qquad (A.1.11)$$

and

$$u_o^2(k) = -\Delta^2(k)(Ax(k) + B^o u^o(k)) \qquad (A.1.12)$$

where

$$\Delta^i(k) = [I - L^i B^j L^j B^i]^{-1}[L^i - L^i B^j L^j], \quad i = 1,2, i \neq j. \qquad (A.1.13)$$

Using (A.1.11) and (A.1.12) in (A.1.1) and defining

$$\hat{A}(k) = A + B^1\Delta^1 A + B^2\Delta^2 A \qquad (A.1.14)$$

$$\hat{B}(k) = B^0 - B^1\Delta^1 B^0 - B^2\Delta^2 B^0. \qquad (A.1.15)$$

We have

$$x(k+1) = \hat{A}(k)x(k) + \hat{B}(k)u^0(k) + v_k. \qquad (A.1.16)$$

Now

$$V^0(k) = \frac{1}{2} x^T(k)S^0(k)x(k) + \frac{1}{2}\gamma^0(k). \qquad (A.1.17)$$

Then

$$V^0(k) = \min_{u^o(k)} E[\frac{1}{2} x^T(k)Q^0(k)x(k) + \frac{1}{2} u^{0T}(k)R^0(k)u^0(k) + V^0(k+1)/x(k)]$$

$$= \min_{u^o(k)} [\frac{1}{2} x^T(k)Q^0(k)x(k) + \frac{1}{2} u^{0T}(k)R^0(k)u^0(k) + $$

$$+ E[V^0(k+1)/x(k)]] \qquad (A.1.18)$$

At k = N,

$$V^0(N) = \frac{1}{2} x^T(N)K^0(N)x(N). \qquad (A.1.19)$$

Using Lemma 1, (A.1.16), and (A.1.17) we have

$$E[V^0(k+1)/x(k)] = \frac{1}{2}\,(\hat{A}(k)x(k)+\hat{B}(k)u^0(k))^T S^0(k+1)(\hat{A}(k)x(k) + \hat{B}(k)u^0(k)) + \frac{1}{2}\,\mathrm{tr}S^0(k+1)\Lambda(k) + \frac{1}{2}\,\gamma^0(k+1). \qquad (A.1.20)$$

Using (A.1.20) in (A.1.18) we obtain

$$u^{o*}(k) = -[R^0+\hat{B}^T S^0(k+1)\hat{B}]^{-1}\hat{B}^T S^0(k+1)\hat{A}x(k). \qquad (A.1.21)$$

Let

$$L^0(k) = [R^0+\hat{B}^T S^0(k+1)\hat{B}]^{-1}\hat{B}^T S^0(k+1)\hat{A}. \qquad (A.1.22)$$

Then

$$u^{o*}(k) = -L^0(k)x(k). \qquad (A.1.23)$$

To obtain recursive equation for $S^0(k)$, use (A.1.23) in (A.1.18) and after some algebra

$$S^0(k) = Q^0(k)+\hat{A}^T S^0(k+1)\hat{A} - L^{0T}[R^0+\hat{B}^T S^0(k+1)\hat{B}]L^0 \qquad (A.1.24)$$

$$S^0(N) = K^0(N) \qquad (A.1.25)$$

$$\gamma^0(k) = \gamma^0(k+1)+\mathrm{tr}\; S^0(k+1)\Lambda(k) \qquad (A.1.26)$$

$$\gamma^0(N) = 0. \qquad (A.1.27)$$

To obtain recursive equations for $S^i(k)$ $i=1,2$, use (A.1.23), (A.1.11), (A.1.12), and (A.1.5). After some algebra

$$S^i(k) = Q^i(k)+(A-B^0L^0)^T\Delta^{iT}(k)R^i\Delta^i(k)(A-B^0L^0)+(\hat{A}-\hat{B}L^0)S^i(k+1)(\hat{A}-\hat{B}L^0) \qquad i=1,2 \qquad (A.1.28)$$

$$S^i(N) = K^i(N), \qquad i=1,2 \qquad (A.1.29)$$

$$\gamma^i(k) = \gamma^i(k+1)+\mathrm{tr}\; S^i(k+1)\Lambda(k), \qquad i=1,2 \qquad (A.1.30)$$

$$\gamma^i(N) = 0, \qquad i=1,2 \qquad (A.1.31)$$

Appendix 2.

Given a stochastic Markov sequence of state vector $\{x(k)\}$

$$x(k+1) = A(k)x(k) + B^0(k)u^0(k) + B^1(k)u^1(k) + B^2(k)u^2(k) + v(k) \quad (A.2.1)$$

where $u^i(k)$, $i = 0,1,2$ are deterministic inputs, $v(k)$ random, and measurements given by

$$z^1(k) = z^2(k) = H\ (k)x(k) + \xi\ (k) \quad (A.2.2)$$

$$Z^0(k) \supset Z^1(k); z^0(k) = H^0(k)x(k) + \xi^0(k). \quad (A.2.3)$$

The assumptions are the same as given in Section 2. Define

$$z^*(k) = [z^{1T}(0),\ldots,z^{1T}(k)]^T \quad (A.2.4)$$

$$z^{0*}(k) = [z^{0T}(0),\ldots,z^{0T}(k)]^T \quad (A.2.5)$$

$$\hat{x}(k) = E[x(k)/z^*(k)] \quad (A.2.6)$$

$$\hat{x}^0(k) = E[x(k)/z^{0*}(k)] \quad (A.2.7)$$

$$P(k/k) = E\{x(k)-\hat{x}(k))(x(k)-\hat{x}(k))'/z^*(k)\} \quad (A.2.8)$$

$$\hat{x}(k+1/k) = E[x(k+1)/z^*(k)]. \quad (A.2.9)$$

The recursive relations define the conditional expectations for lower level assumptions given by

$$\hat{x}(k+1/k) = A(k)\hat{x}(k) + B^0(k)u^0(k) + B^1(k)u^1(k) + B^2(k)u^2(k) \quad (A.2.10)$$

$$P(k+1/k) = A(k+1)P(k/k)A^T(k+1) + \Lambda(k) \quad (A.2.11)$$

$$\hat{x}(k+1) = \hat{x}(k+1/k) + K(k+1)[z(k+1) - H(k+1)\hat{x}(k+1/k)] \quad (A.2.12)$$

$$K(k+1) = P(k+1/k)H^T(k+1)[H(k+1)P(k+1/k)H^T(k+1) + \Xi(k+1)]^{-1} \quad (A.2.13)$$

$$P(k+1/k+1) = [I - K(k+1)H(k+1)]P(k+1/k) \quad (A.2.14)$$

$$P(0/0) = \Sigma(0). \quad (A.2.15)$$

Also

$$E[\hat{x}(k+1)/z^*(k)] = \bar{x}(k+1) = A\hat{x}(k) + B^0u^0(k) + B^1u^1(k) + B^2u^2(k) \tag{A.2.16}$$

$$Cov[\hat{x}(k+1)/z^*(k)] = K(k+1)[H(k+1)P(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1). \tag{A.2.17}$$

The recursive relation defining the conditional expectations for the coordinator subsystem is given by

$$\hat{x}^o(k+1) = \hat{x}^o(k+1/k) + K^o(k+1)[z^o(k+1) - H^o(k+1)\hat{x}^o(k+1/k)]$$

$$K^o(k+1) = P^o(k+1/k)H^{oT}(k+1)[H^o(k+1)P^o(k+1/k)H^{oT}(k+1) + \Xi(k+1)]$$

$$P^o(k+1/k) = A(k+1)P^o(k/k)A^T(k+1) + \Lambda(k)$$

$$P^o(k+1/k+1) = [I - K^o(k+1)H^o(k+1)]P^o(k+1/k)$$

$$P^o(0/0) = \Sigma(0).$$

Also

$$E[\hat{x}^o(k+1)/z^{o*}(k)] = A(k)\hat{x}^o(k) + B^o(k)u^o(k) + B^1(k)u^1(k) + B^2(k)u^2(k)$$

$$Cov[x^o(k+1)/z^{o*}] = K^o(k+1)[H^o(k+1)P^o(k+1/k)H^{oT}(k+1) + \Xi^o(k+1)]K^{oT}(k+1).$$

Assume at stage k the cost-to-go for the i-th subsystem is

$$J^{i*}(k) = \frac{1}{2}\hat{x}^T(k)S^i(k)\hat{x}(k) + \frac{1}{2}\gamma^i(k). \tag{A.2.18}$$

The optimal strategies for subsystem i are given by

$$u^i(k) = \arg\min_{u^i(k)} E[\frac{1}{2}x^T(k)Q^i(k)x(k) + \frac{1}{2}u^{iT}(k)R^i(k)u^i(k) + J^{i*}(k+1)/z^*(k)] \tag{A.2.19}$$

At $k = N$

$$\begin{aligned} J^{i*}(N) &= E[\frac{1}{2}x^T(N)K^i(N)x(N)/z^*(N)] \\ &= \frac{1}{2}\hat{x}^T(N)K^i(N)\hat{x}(N) + \frac{1}{2}trK^i(N)P^i(N). \end{aligned} \tag{A.2.20}$$

Using Lemma 1 in Appendix 1

$$\begin{aligned} u^i(k) = \arg\min_{u^i(k)} [&\frac{1}{2}\hat{x}^T(k)Q^i(k)\hat{x}(k) + \frac{1}{2}trQ^i(k)P^i(k) + \frac{1}{2}u^{iT}(k)R^i(k)u^i(k) \\ &+ \frac{1}{2}[A\hat{x}(k) + B^0u^0(k) + B^iu^i(k) + B^ju^j(k)]^T S^i(k+1)[A\hat{x}(k) \end{aligned}$$

$$+ B^0u^0(k) + B^iu^i(k) + B^ju^j(k)]$$

$$+\frac{1}{2}\,\mathrm{tr}S^i(k+1)K^i(k+1)[H^i(k+1)P^i(k+1/k)H^{iT}(k+1)$$

$$+ \Xi^i(k+1)]K^{iT}(k+1) + \frac{1}{2}\,\gamma^i(k+1)]. \quad (A.2.21)$$

The minimizing control $u^i(k)$ is

$$u^i(k) = -[R^i(k) + B^{iT}S^i(k+1)B^i]^{-1}B^{iT}S^i(k+1)[A\hat{x}(k) + B^0u^0(k) + B^ju^j(k)]. \quad (A.2.22)$$

Recall the definition of $L^i(k)$ in (A.1.9)

$$L^i(k) = [R^i(k) + B^{iT}S^i(k+1)B^i]^{-1}B^{iT}S^i(k+1). \quad (A.2.23)$$

Then

$$u^i(k) = -L^i(k)[A\hat{x}(k) + B^0u^0(k) + B^ju^j(k)]. \quad (A.2.24)$$

For 2-subsystem solve for $u^1(k)$ and $u^2(k)$

$$u^1(k) = -\Delta^1(k)[A\hat{x}(k) + B^0u^0(k)] \quad (A.2.25)$$

$$u^2(k) = -\Delta^2(k)[A\hat{x}(k) + B^0u^0(k)] \quad (A.2.26)$$

where

$$\Delta^i(k) = [I - L^iB^jL^jB^i]^{-1}[L^i - L^iB^jL^j],\ i = 1,2,\ i \neq j \quad (A.2.27)$$

Assume that at stage k the cost-to-go for the coordinator subsystem is

$$J^{*0}(k) = \frac{1}{2}\begin{bmatrix}\hat{x}^0(k)\\ \hat{x}(k)-\hat{x}^0(k)\end{bmatrix}^T\begin{bmatrix}S^A(k) & S^B(k)\\ S^{BT}(k) & S^C(k)\end{bmatrix}\begin{bmatrix}\hat{x}^0(k)\\ \hat{x}(k)-\hat{x}^0(k)\end{bmatrix} + \frac{1}{2}\,\gamma^o(k). \quad (A.2.28)$$

At $k = N$,

$$J^{*0}(N) = \frac{1}{2}\,\hat{x}^{0T}(N)K^0(N)\hat{x}^0(N) + \frac{1}{2}\,\mathrm{tr}K^0(N)P^0(N).$$

Then

$$u^{o*}(k) = \underset{u^o(k)}{\arg\min}\ E[\frac{1}{2}\,x^T(k)Q^0(k)x(k) + \frac{1}{2}\,u^{0T}(k)R^0(k)u^0(k) + J^{o*}(k+1)/z^{o*}(k)]. \quad (A.2.29)$$

For any matrix Γ [12]

$$E\{\hat{x}^{0T}(k+1)\Gamma\hat{x}(k+1)/z^{o*}(k)\} = E[\{\hat{x}^0(k+1/k) + \tilde{K}(k+1)[z(k+1) - H(k+1)\hat{x}^0(k+1/k)]\}^T$$

$$\Gamma\{\hat{x}(k+1/k) + K(k+1)[z(k+1) - H(k+1)\hat{x}(k+1)/k)]\}/z^{*0}(k)\} \quad (A.2.30)$$

where

$$\tilde{K}(k+1) = P^0(k+1/k)H^T(k+1)[H(k+1)P^0(k+1/k)H^T(k+1) + \Xi(k+1)]^{-1} \quad (A.2.31)$$

$$E\{\hat{x}^{0T}(k+1)\Gamma\hat{x}(k+1)/z^{o*}(k)\} = \bar{x}^0(k+1)^T\Gamma\bar{x}(k+1) + \bar{x}^{0T}(k+1)\Gamma K(k+1)H(k+1)(\bar{x}^0(k+1) - \bar{x}(k+1)) + \text{tr } P^0(k+1/k)\Gamma K(k+1)H(k+1) \quad (A.2.32)$$

$$E[\hat{x}(k+1)\Gamma\hat{x}(k+1)/z^{0*}(k)] = E[\{\hat{x}(k+1/k) + K(k+1)[z(k+1) - H(k+1)\hat{x}(k+1/k)]\}^T\Gamma \{\hat{x}(k+1/k) + K(k+1)[z(k+1) - H(k+1)\hat{x}(k+1/k)]\}/z^{*0}(k)\}$$

$$= \bar{x}^T(k+1)\Gamma\bar{x}(k+1) + 2\bar{x}^T(k+1)\Gamma K(k+1)H(k+1)(\bar{x}^0(k+1) - \bar{x}(k+1))$$
$$+ \text{tr}\{\Gamma K(k+1)[H(k+1)P^0(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1)$$
$$+ (\bar{x}(k+1) - \bar{x}^0(k+1))^T H^T(k+1)K^T(k+1)\Gamma K(k+1)H(k+1)(\bar{x}(k+1) - \bar{x}^0(k+1)). \quad (A.2.33)$$

Expand (A.2.29) using (A.2.32) and (A.2.33)

$$u^{0*}(k) = \arg\min_{u^o(k)}[\tfrac{1}{2}\hat{x}^{0T}(k)Q^0(k)\hat{x}^0(k) + \tfrac{1}{2}u^{0T}(k)R^0(k)u^0(k) + \tfrac{1}{2}\text{tr}Q^0(k)P^0(k)$$
$$+ \tfrac{1}{2}\bar{x}^{0T}(k+1)(S^A + S^C - 2S^B)\bar{x}^0(k+1) + \bar{x}^{0T}(k+1)(S^B - S^C)\bar{x}(k+1)$$
$$+ \bar{x}^{0T}(k+1)(S^B - S^C)K(k+1)H(k+1)(\bar{x}^0(k+1) - \bar{x}(k+1))$$
$$+ \tfrac{1}{2}\bar{x}^T(k+1)S^C\bar{x}(k+1) + \bar{x}^T(k+1)S^C K(k+1)H(k+1)(\bar{x}^0(k+1) - \bar{x}(k+1))$$
$$+ \tfrac{1}{2}(\bar{x}(k+1) - \bar{x}^0(k+1))^T H^T(k+1)K^T(k+1)S^C K(k+1)H(k+1)(\bar{x}(k+1) - \bar{x}^0(k+1)) + \tfrac{1}{2}\gamma^0(k)$$
$$+ \tfrac{1}{2}\text{tr}\{K^0(k+1)[H^0(k+1)P^0(k+1/k)H^{0T}(k+1) + \Xi^0(k+1)]K^{0T}(k+1)(S^A + S^C - 2S^B)\}$$
$$+ \tfrac{1}{2}\text{tr}2P^0(k+1/k)K(k+1)H(k+1)(S^B - S^C) + \tfrac{1}{2}\text{tr}K(k+1)[H(k+1)P^0(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1)S^C. \quad (A.2.34)$$

Recall that

$$\bar{x}^0(k+1) = A(k)\hat{x}^0(k) - (B^1(k)\Delta^1(k)A(k) + B^2\Delta^2(k)A(k))\hat{x}(k) + \hat{B}(k)u^0(k) \quad (A.2.35)$$

where

$$\hat{B}(k) = B^0(k) - B^1(k)\Delta^1(k)B^0(k) - B^2(k)\Delta^2(k)B^0(k). \quad (A.2.36)$$

Let

$$G(k) = B^1(k)\Delta^1(k) + B^2(k)\Delta^2(k) \tag{A.2.37}$$

then (A.2.35) becomes

$$\bar{x}^0(k+1) = (I-G(k))A(k)\hat{x}^0(k) - G(k)A(k)(\hat{x}(k)-\hat{x}^0(k)) + \hat{B}(k)u^0(k) \tag{A.2.38}$$

and

$$\begin{aligned}\bar{x}(k+1) &= (I-G(k))A(k)\hat{x}^0(k) \\ &\quad + (I-G(k))A(k)(\hat{x}(k)-\hat{x}^0(k)) + \hat{B}(k)u^0(k)\end{aligned} \tag{A.2.39}$$

$$\bar{x}(k+1) - \bar{x}^0(k+1) = A(k)(\hat{x}(k)-\hat{x}^0(k)). \tag{A.2.40}$$

Substitute (A.2.40) in (A.2.34) and differentiating $u^{0*}(k)$ is given by

$$u^{0*}(k) = -\Delta^0(k)Y(k)\hat{x}^0(k) - \Delta^0(k)M(k)[\hat{x}(k)-\hat{x}^0(k)] \tag{A.2.41}$$

where

$$\Delta^0(k) = [R^0(k) + \hat{B}^T(k)S^A(k+1)\hat{B}(k)]^{-1}$$

$$Y(k) = \hat{B}(k)S^A(k+1)[I-G(k)]A(k)$$

$$\begin{aligned}M(k) &= \hat{B}^T(k)S^A(k+1)G(k)A(k) + \hat{B}^T(k)(S^B(k+1)-S^A(k+1))A(k) \\ &\quad - \hat{B}^T(k)S^B(k+1)K(k+1)H(k+1)A(k).\end{aligned}$$

The recursive equations for S^A, S^B, S^C, $\gamma^0(k)$ are obtained by substituting $u^{0*}(k)$ back in (A.2.40)

$$S^A(k) = Q^0(k) + A^T(k)(I-G(k))^T S^A(k+1)(I-G(k))A(k) - Y^T(k)\Delta^0(k)Y(k) \tag{A.2.42}$$

$$\begin{aligned}S^B(k) &= A^T(k)(I-G(k))^T S^B(k+1)(I-G(k))A(k) \\ &\quad + A^T(k)(I-G(k))^T(S^B(k+1)-S^A(k+1))G(k)A(k) \\ &\quad - A^T(k)(I-G(k))^T S^B(k+1)K(k+1)H(k+1)A(k) - Y^T(k)\Delta^0(k)M(k)\end{aligned} \tag{A.2.43}$$

$$\begin{aligned}S^C(k) &= -M^T(k)\Delta^0(k)M(k) + A^T(k)G^T(k)S^A(k+1)G(k)A(k) \\ &\quad + A^T(k)[I-K(k+1)H(k+1)]^T S^C(k+1)[I-K(k+1)H(k+1)]A(k)\end{aligned}$$

$$+ A^T(k)(S^B(k+1)K(k+1)H(k+1)-S^B(k+1))G(k)A(k)$$

$$-A^T(k)G^T(k)(S^B(k+1)-S^B(k+1)K(k+1)H(k+1))A(k) \qquad (A.2.44)$$

$$\gamma^0(k) = \gamma^0(k+1) + tr\ Q^0(k)P^0(k) + tr[K^0(k+1)[H^0(k+1)P^0(k+1/k)H^{0T}(k+1) +$$

$$+ \Xi^0(k)]K^{0T}(k+1)(S^A(k+1) + S^C(k+1) - 2S^B(k+1))]$$

$$+ 2tr\ P^0(k+1/k)K(k+1)H(k+1)(S^B(k+1)-S^C(k+1))$$

$$+ tr\ K(k+1)[H(k+1)P^0(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1)S^C(k+1). \qquad (A.2.45)$$

To obtain the recursive equation for $S^i(k)$ of the i-th subsystem, substitute $u^{0*}(k)$, $u_o^i(k)$ back in (A.2.21)

$$S^i(k) = Q^i(k) + (A(k) + \hat{B}(k)\Delta^0(k)Y(k))^T S^i(k+1)(A(k) + \hat{B}(k)\Delta^0(k)Y(k))$$

$$+ (\Delta^i(k)A(k) + B^0(k)\Delta^0(k)Y(k)^T R^i(k)(\Delta^i(k)A(k) + B^0(k)\Delta^0(k)Y(k))$$

$$i = 1,2 \qquad (A.2.46)$$

$$\gamma^i(k) = \gamma^i(k+1) + tr\ Q^i(k)P(k) + tr\ S^i(k+1)K(k+1)$$

$$+ tr\ S^i(k+1)K(k+1)[H(k+1)P(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1)$$

$$+ tr[P(k/k)-P^0(k/k)](M(k)-Y(k))^T\Delta^{0T}(k)(B^{0T}(k)R^i(k)B^0(k)$$

$$+ \hat{B}^T S^i(k+1)\hat{B})\Delta^0(k)(M(k)-Y(k)) \qquad (A.2.47)$$

All these strategies are for deterministic dynamic games. In [12] the feedback Stackelberg solution concept is extended to stochastic two-person dynamic games.

The approach to be explicitly developed in this paper is based on the coordination solution concept suggested in [20] for deterministic systems. We allow stochastic disturbances in the dynamic process model and in the measurement model, as in [12] but several second-level decision makers or followers are present as in [20]. Several types of information structure are considered. Explicit recursion formulas for the design of the feedback Stackelberg controllers for the coordinator and the followers are presented. The strategies are adaptive to changes in information available at each stage and they satisfy the principle of optimality. The strategies of the second level decision-makers are equilibrium Nash strategies with respect to each other and in addition, they take into account the known strategy of the coordinator. The coordinator chooses his strategy with the full anticipation that the other decision makers will take the coordinator strategy into account in minimizing their individual cost functions.

2. Problem Formulation

Consider M discrete-time linear subsystems, each modeled by

$$x^i(k+1) = A^{io}(k)x^o(k) + A^{ii}(k)x^i(k) + \sum_{\substack{j=1 \\ i\neq j}}^{M} A^{ij}(k)x^j(k) + B^i(k)u^i(k) + \theta^i(k). \quad (1)$$

The measurement of each subsystem is given by

$$z^i(k) = H^{io}(k)x^o(k) + H^{ii}(k)x^i(k) + \sum_{\substack{j=1 \\ i\neq j}}^{M} H^{ij}(k)x^j(k) + \xi^i(k) \qquad i = 1,\ldots,M; \quad (2)$$

where x^i is the n^i-dimensional state vector of the i-th subsystem, u^i is the m^i-dimensional local control vector DM^i for the i-th subsystem, z^i is the ℓ^i-dimensional measured output vector for the i-th subsystem. $\{x^i(0);\ \theta^i(k) \in R^{ni};\ \xi^i(k) \in R^{\ell i};\ i=1,\ldots,M\ \ k=0,\ldots,N-1\}$ are mutually independent Gaussian random vectors with known means and covariances.

$$E\{x^i(0)\} = 0 \quad ; \qquad Cov\{x^i(0)\} = \Sigma^i(0)$$
$$E\{\theta^i(k)\} = 0 \quad ; \qquad Cov\{\theta^i(k)\} = \Theta^i(k)$$
$$E\{\xi^i(k)\} = 0 \quad ; \qquad Cov\{\xi^i(k)\} = \Xi^i(k)$$

Each subsystem seeks to minimize the expected value of its cost function

$$J^i(u^i) = \frac{1}{2}\, x^{iT}(N)K^{ii}(N)x^i(N) + \frac{1}{2}\sum_{k=0}^{N-1}[x^{iT}(k)Q^{ii}(k)x^i(k) + u^{iT}(k)R^{ii}(k)u^i(k)]$$
$$i = 1,\ldots,M \qquad (3)$$

where K^{ii}, G^{ii}, and R^{ii} are all positive-definite.

In addition to the M-subsystems, we assume that we have a coordinator subsystem modeled by

$$x^o(k+1) = A^o(k)x^o(k) + \sum_{i=1}^{M} A^{oi}(k)x^i(k) + \theta^o(k) \qquad (4)$$

and the measurement of the coordinator subsystem is given by

$$z^o(k) = H^o(k)x^o(k) + \sum_{i=1}^{M} H^{oi}(k)x^i(k) + \xi^o(k) \qquad (5)$$

where x^o is the n^o-dimensional state vector of the coordinator subsystem, u^o is an m^o-dimensional control vector chosen by the coordinator DM^o, z^o is the ℓ^o-dimensional measured output vector of the coordinator subsystem.

$\{x^o(0);\ \theta^o(k)\in R^{n_o};\ \xi^o(k)\in R^{\ell_o};\ k=0,\dots,N-1\}$ are mutually independent with the random vectors of each subsystem.

$$E\{x^o(0)\} = 0 \quad ; \qquad Cov\{x^o(0)\} = \Sigma^o(0)$$
$$E\{\theta^o(k)\} = 0 \quad ; \qquad Cov\{\theta^o(k)\} = \Theta^o(k)$$
$$E\{\xi^o(k)\} = 0 \quad ; \qquad Cov\{\xi^o(k)\} = \Xi^o(k)$$

The coordinator chooses u^o to minimize the expected value of the cost function

$$J^o(u^o) = \frac{1}{2}\, x^{oT}(N)K^o(N)x^o(N) + \frac{1}{2}\sum_{i=1}^{M} x^{iT}(N)K^{oi}(N)x^i(N)$$
$$+ \frac{1}{2}\sum_{k=0}^{N-1}\Big[x^{oT}(k)Q^o(k)x^o(k) + u^{oT}(k)R^o(k)u^o(k) + \sum_{i=1}^{M} x^{iT}(k)Q^{oi}(k)x^i(k)\Big] \tag{6}$$

where K^o, K^{oi}, Q^o, R^o, Q^{oi} are all positive definite.

The Stackelberg approach [20] to the coordination of the subsystems is to consider DM^o as a leader and DM^i as followers. We imagine that DM^o provides DM^i the exact knowledge of all decisions made by the coordinator and each DM^i minimizes J^i with respect to u^i for each given decision of DM^o assuming that the other subsystems will do the same. With this assumption the subsystems play Nash among themselves. The coordinator then minimizes J^o with respect to u^o, considering that the decisions from the subsystems result from choices of u^i which minimize J^i for $i=1,\dots,M$. Additionally, the information sets include exact knowledge of the system dynamic DM^o, DM^i, the measurements and the cost-functionals. The statistics of the random elements for all k are also included.

where

$$L^o(k) = [R^o(k) + \hat{B}^T(k)S^o(k+1)\hat{B}(k)]^{-1}\hat{B}^T(k)S^o(k+1)\hat{A}(k)$$

$$\Delta^i(k) = [I - L^i(k)B^j(k)L^j(k)B^i(k)]^{-1}(L^i(k) - L^i(k)B^j(k)L^j(k))$$

$$i = 1,2, \ j = 1,2, \ i \neq j$$

$$\hat{A}(k) = A(k) - B^1(k)\Delta^1(k)A(k) - B^2(k)\Delta^2(k)A(k)$$

$$\hat{B}(k) = B^0(k) - B^1(k)\Delta^1(k)B^0(k) - B^2(k)\Delta^2(k)B^0(k)$$

$$L^i(k) = [R^i(k) + B^{iT}(k)S^i(k+1)B^i(k)]^{-1}B^{iT}(k)S^i(k+1)$$

Assuming that the indicated inverses exist the other quantities are obtained from

$$S^o(k) = Q^o(k) + \hat{A}^T(k)S^o(k+1)\hat{A}(k) - L^{oT}(k)[R^o(k) + \hat{B}^T(k)S^o(k+1)\hat{B}(k)]L^o(k) \quad (18)$$

$$S^o(N) = K^o(N) \quad (19)$$

$$\gamma^o(k) = \gamma^o(k+1) + \mathrm{tr}S^o(k+1)\Lambda(k) \quad (20)$$

$$\gamma^o(N) = 0 \quad (21)$$

$$S^i(k) = Q^i(k) + [A(k) - B^o(k)L^o(k)]^T\Delta^{iT}(k)R^i(k)\Delta^i(k)[A(k) - B^o(k)L^o(k)]$$
$$+ [\hat{A}(k) - \hat{B}(k)L^o(k)]^T S^i(k+1)[\hat{A}(k) - \hat{B}(k)L^o(k)], \quad i = 1,2 \quad (22)$$

$$S^i(N) = K^i(N), \quad i = 1,2 \quad (23)$$

$$\gamma^i(k) = \gamma^i(k+1) + \mathrm{tr}S^i(k+1)\Lambda(k), \quad i = 1,2 \quad (24)$$

$$\gamma^i(N) = 0, \quad i = 1,2 \quad (25)$$

$$u_o^i(k) = -\Delta^i(k)(A(k)\hat{x}(k) + B^o(k)u^o(k)), \quad i = 1,2 \tag{26}$$

$$u^{o*}(k) = -\Delta^o(k)Y(k)\hat{x}^o(k) - \Delta^o(k)M(k)[\hat{x}(k) - \hat{x}^o(k)] \tag{27}$$

$$J^{o*}(k) = \frac{1}{2}\begin{bmatrix} \hat{x}^o(k) \\ \hat{x}(k)-\hat{x}^o(k) \end{bmatrix}^T \begin{bmatrix} S^A(k) & S^B(k) \\ S^{BT}(k) & S^C(k) \end{bmatrix} \begin{bmatrix} \hat{x}^o(k) \\ \hat{x}(k)-\hat{x}^o(k) \end{bmatrix} + \frac{1}{2}\gamma^o(k) \tag{28}$$

$$J^{i*}(k) = \frac{1}{2}\hat{x}^T(k)S^i(k)\hat{x}(k) + \frac{1}{2}\gamma^i(k) \qquad i = 1,2 \tag{29}$$

where

$$\hat{x}(k) = E\{x(k)|z^i(k)\}, \ \hat{x}^o(k) = E\{x(k)|z^o(k)\}$$

$\Delta^i(k)$, $\hat{A}(k)$, $\hat{B}(k)$, and $L^i(k)$ are defined as in the perfect information case with $S^A(k)$ replacing $S^o(k)$. In addition we have

$$S^A(k) = Q^o(k) + A^T(k)(I-G(k))^T S^A(k+1)(I-G(k))A(k) - Y^T(k)\Delta^o(k)Y(k) \tag{30}$$

$$\begin{aligned} S^B(k) = {} & A^T(k)(I-G(k))^T S^B(k+1)(I-G(k))A(k) \\ & + A^T(k)(I-G(k))^T(S^B(k+1) - S^A(k+1))G(k)A(k) \\ & - A^T(k)(I-G(k))^T S^B(k+1)K(k+1)H(k+1)A(k) \\ & - Y^T(k)\Delta^o(k)M(k) \end{aligned} \tag{31}$$

$$\begin{aligned} S^C(k) = {} & -M^T(k)\Delta^o(k)M(k) + A^T(k)G^T(k)S^A(k+1)G(k)A(k) \\ & + A^T(k)[I - K(k+1)H(k+1)]^T S^C(k+1)[I - K(k+1)H(k+1)]A(k) \\ & + A^T(k)(S^B(k+1)K(k+1)H(k+1) - S^B(k+1))G(k)A(k) \\ & - A^T(k)G^T(k)(S^B(k+1) - S^B(k+1)K(k+1)H(k+1))A(k) \end{aligned} \tag{32}$$

$$Y(k) = \hat{B}(k)S^A(k+1)[I-G(k)]A(k)$$

$$\begin{aligned} M(k) = {} & \hat{B}(k)S^A(k+1)G(k)A(k) + \hat{B}^T(k)(S^B(k+1) - S^A(k+1))A(k) \\ & - B^T(k)S^B(k+1)K(k+1)H(k+1)A(k) \end{aligned}$$

$$G(k) = B^1(k)\Delta^1(k) + B^2(k)\Delta^2(k)$$

4. Constrained Decentralized Structure

It may be desirable to have a control policy that is simpler to implement than the optimal policy. Satisfactory control of a high-order linear system may often be achieved using relatively fewer system measurements and a controller of low order. This has been the motivation for a number of optimal designs, using output feedback or dynamic controllers of a specified order. For recent work in this field we refer the reader to [13]-[19].

4.1 Decentralized Control with Instantaneous Output Feedback

Consider the stochastic problem where a restriction is placed on the control of the i-th subsystem and the coordinator at any instant to be a linear transformation of the measurement at that instant. Also, there is no information transfer among subsystems through their controls. This simplifies the problem since a filter is no longer used to estimate the state. Then

$$u^i(k) = F^i(k)z^i(k), \quad i = 0,1,2, \quad k = 0,1,\dots,N-1 \tag{48}$$

where $F^i(k)$ is to be determined to minimize the expected value of $J^i(u^i)$.

Consider the augmented system (12) and the measurement

$$z^i(k) = H^i(k)x(k) + \xi^i(k), \quad i = 0,1,2. \tag{49}$$

Then

$$u^i(k) = F^i(k)H^i(k)x(k) + F^i(k)\xi^i(k), \quad i=0,1,2 \tag{50}$$

and

$$x(k+1) = (A(k) + \sum_{i=0}^{2} B^i(k)F^i(k)H^i(k))x(k) + \sum_{i=0}^{2} B^i(k)F^i(k)\xi^i(k) + v_k. \tag{51}$$

Define

$$\Sigma(k) = E\{x(k)x^T(k)\}$$

and note that $x(k)$ depends on $\xi^i(k)$ for $i=0,1,\dots,k-1$ only, implying that $E\{x(k)v^T(k)\} = 0$. Then the recursive equation for $\Sigma(k)$ is follows:

$$\Sigma(k+1) = (A(k) + \sum_{i=0}^{2} B^i(k)F^i(k)H^i(k))\Sigma(k)(A(k) + \sum_{i=0}^{2} B^i(k)F^i(k)H^i(k))^T$$
$$+ \sum_{i=0}^{2} B^i(k)F^i(k)\Xi^i(k)F^{iT}(k)B^{iT}(k) + \Lambda(k). \quad (52)$$

Lemma 4.1.1: If a linear system described by (12) is controlled using a linear control policy (48) then the expected cost (14) can be expressed as

$$E[J^i(k)] = \frac{1}{2} E[x^T(k)S^i(k)x(k)] + \frac{1}{2} \sum_{\ell=k+1}^{N} \mathrm{tr}\, S^i(\ell)\Lambda(\ell-1)$$
$$+ \frac{1}{2} \sum_{\ell=k+1}^{N} \{\mathrm{tr}\, F^{iT}(\ell-1)(R^i(\ell-1) + B^{iT}S^i(\ell)B^{iT}(\ell-1))F^i(\ell-1)\Xi^i(\ell-1)$$
$$+ \sum_{\substack{j=0 \\ i\neq j}}^{2} F^{jT}(\ell-1)B^{jT}(\ell-1)S^i(\ell)B^j(\ell-1)F^j(\ell-1)\Xi^j(\ell-1)\}, \quad i = 1,2 \quad (53)$$

where

$$S^i(k) = Q^i(k) + H^{iT}(k)F^{iT}(k)R^i(k)F^i(k)H^i(k) +$$
$$+ (A(k) + \sum_{j=0}^{2} B^j(k)F^j(k)H^j(k))^T S^i(k+1)(A(k) + \sum_{j=0}^{2} B^j(k)F^j(k)H^j(k)) \quad (54)$$
$$i = 1,2$$

$$S^i(N) = K^i(N) \qquad i = 1,2. \quad (55)$$

Proof: The proof is by induction.

Consider the augmented system (12) and the cost criterion (14). The assumption obviously holds for $k = N$. For any k

$$E[J^i(k)] = E\{\frac{1}{2} \sum_{\ell=k}^{N-1} \{x^T(\ell)Q^i(\ell)x(\ell) + u^{iT}(\ell)R^i(\ell)u^i(\ell)\} + \frac{1}{2} E\{x^T(N)K^i(N)x(N)\}$$
$$= E[J^i(k+1)] + E\{\frac{1}{2} x^T(k)Q^i(k)x(k) + \frac{1}{2} u^{iT}(k)R^i(k)u^i(k)\}, i = 1,2 \quad (56)$$

with $k = k+1$ using (53) in (56) and after some algebra the assumption holds for $k = k+1$. Thus (53) holds for $k = 0,1,\ldots,N$.

For $i = 1,2$, apply dynamic programming and at each step set the derivative of the remaining cost with respect to each element of $F^i(k)$ equal to zero. Thus in terms of $M^i(k)$, $Y^i(k)$, $\Gamma^i(k)$, and $T^i(k)$ which are defined in (70), (71), (72), and (73), we have

Appendix 2.

Given a stochastic Markov sequence of state vector $\{x(k)\}$

$$x(k+1) = A(k)x(k) + B^0(k)u^0(k) + B^1(k)u^1(k) + B^2(k)u^2(k) + v(k) \tag{A.2.1}$$

where $u^i(k)$, $i = 0,1,2$ are deterministic inputs, $v(k)$ random, and measurements given by

$$z^1(k) = z^2(k) = H^1(k)x(k) + \xi^1(k) \tag{A.2.2}$$

$$Z^0(k) \supset Z^1(k); z^0(k) = H^0(k)x(k) + \xi^0(k). \tag{A.2.3}$$

The assumptions are the same as given in Section 2. Define

$$z^*(k) = [z^{1T}(0),\ldots,z^{1T}(k)]^T \tag{A.2.4}$$

$$z^{0*}(k) = [z^{0T}(0),\ldots,z^{0T}(k)]^T \tag{A.2.5}$$

$$\hat{x}(k) = E[x(k)/z^*(k)] \tag{A.2.6}$$

$$\hat{x}^0(k) = E[x(k)/z^{0*}(k)] \tag{A.2.7}$$

$$P(k/k) = E\{(x(k)-\hat{x}(k))(x(k)-\hat{x}(k))'/z^*(k)\} \tag{A.2.8}$$

$$\hat{x}(k+1/k) = E[x(k+1)/z^*(k)]. \tag{A.2.9}$$

The recursive relations define the conditional expectations for lower level assumptions given by

$$\hat{x}(k+1/k) = A(k)\hat{x}(k) + B^0(k)u^0(k) + B^1(k)u^1(k) + B^2(k)u^2(k) \tag{A.2.10}$$

$$P(k+1/k) = A(k+1)P(k/k)A^T(k+1) + \Lambda(k) \tag{A.2.11}$$

$$\hat{x}(k+1) = \hat{x}(k+1/k) + K(k+1)[z(k+1) - H(k+1)\hat{x}(k+1/k)] \tag{A.2.12}$$

$$K(k+1) = P(k+1/k)H^T(k+1)[H(k+1)P(k+1/k)H^T(k+1) + \Xi(k+1)]^{-1} \tag{A.2.13}$$

$$P(k+1/k+1) = [I - K(k+1)H(k+1)]P(k+1/k) \tag{A.2.14}$$

$$P(0/0) = \Sigma(0). \tag{A.2.15}$$

Also

$$E[\hat{x}(k+1)/z^*(k)] = \hat{\bar{x}}(k+1) = A\hat{x}(k) + B^0u^0(k) + B^1u^1(k) + B^2u^2(k) \qquad (A.2.16)$$

$$Cov[\hat{x}(k+1)/z^*(k)] = K(k+1)[H(k+1)P(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1). \qquad (A.2.17)$$

The recursive relation defining the conditional expectations for the coordinator subsystem is given by

$$\hat{x}^o(k+1) = \hat{x}^o(k+1/k) + K^o(k+1)[z^o(k+1) - H^o(k+1)\hat{x}^o(k+1/k)]$$

$$K^o(k+1) = P^o(k+1/k)H^{oT}(k+1)[H^o(k+1)P^o(k+1/k)H^{oT}(k+1) + \Xi(k+1)]$$

$$P^o(k+1/k) = A(k+1)P^o(k/k)A^T(k+1) + \Lambda(k)$$

$$P^o(k+1/k+1) = [I - K^o(k+1)H^o(k+1)]P^o(k+1/k)$$

$$P^o(0/0) = \Sigma(0).$$

Also

$$E[\hat{x}^o(k+1)/z^{o*}(k)] = A(k)\hat{x}^o(k) + B^o(k)u^o(k) + B^1(k)u^1(k) + B^2(k)u^2(k)$$

$$Cov[x^o(k+1)/z^{o*}] = K^o(k+1)[H^o(k+1)P^o(k+1/k)H^{oT}(k+1) + \Xi^o(k+1)]K^{oT}(k+1).$$

Assume at stage k the cost-to-go for the i-th subsystem is

$$J^{i*}(k) = \frac{1}{2}\hat{x}^T(k)S^i(k)\hat{x}(k) + \frac{1}{2}\gamma^i(k). \qquad (A.2.18)$$

The optimal strategies for subsystem i are given by

$$u^i(k) = \arg\min_{u^i(k)} E[\frac{1}{2}x^T(k)Q^i(k)x(k) + \frac{1}{2}u^{iT}(k)R^i(k)u^i(k) + J^{i*}(k+1)/z^*(k)] \qquad (A.2.19)$$

At $k = N$

$$J^{i*}(N) = E[\frac{1}{2}x^T(N)K^i(N)x(N)/z^*(N)]$$
$$= \frac{1}{2}\hat{x}^T(N)K^i(N)\hat{x}(N) + \frac{1}{2}\mathrm{tr}K^i(N)P^i(N). \qquad (A.2.20)$$

Using Lemma 1 in Appendix 1

$$u^i(k) = \arg\min_{u^i(k)} [\frac{1}{2}\hat{x}^T(k)Q^i(k)\hat{x}(k) + \frac{1}{2}\mathrm{tr}Q^i(k)P^i(k) + \frac{1}{2}u^{iT}(k)R^i(k)u^i(k)$$
$$+ \frac{1}{2}[A\hat{x}(k) + B^0u^0(k) + B^iu^i(k) + B^ju^j(k)]^TS^i(k+1)[A\hat{x}(k)$$

$$+ B^0u^0(k) + B^iu^i(k) + B^ju^j(k)]$$

$$+\frac{1}{2}\,\mathrm{tr}S^i(k+1)K^i(k+1)[H^i(k+1)P^i(k+1/k)H^{iT}(k+1)$$

$$+ \Xi^i(k+1)]K^{iT}(k+1) + \frac{1}{2}\,\gamma^i(k+1)]. \tag{A.2.21}$$

The minimizing control $u^i(k)$ is

$$u^i(k) = -[R^i(k) + B^{iT}S^i(k+1)B^i]^{-1}B^{iT}S^i(k+1)[A\hat{x}(k) + B^0u^0(k) + B^ju^j(k)]. \tag{A.2.22}$$

Recall the definition of $L^i(k)$ in (A.1.9)

$$L^i(k) = [R^i(k) + B^{iT}S^i(k+1)B^i]^{-1}B^{iT}S^i(k+1). \tag{A.2.23}$$

Then

$$u^i(k) = -L^i(k)[A\hat{x}(k) + B^0u^0(k) + B^ju^j(k)]. \tag{A.2.24}$$

For 2-subsystem solve for $u^1(k)$ and $u^2(k)$

$$u^1(k) = -\Delta^1(k)[A\hat{x}(k) + B^0u^0(k)] \tag{A.2.25}$$

$$u^2(k) = -\Delta^2(k)[A\hat{x}(k) + B^0u^0(k)] \tag{A.2.26}$$

where

$$\Delta^i(k) = [I - L^iB^jL^jB^i]^{-1}[L^i - L^iB^jL^j],\ i = 1,2,\ i \neq j \tag{A.2.27}$$

Assume that at stage k the cost-to-go for the coordinator subsystem is

$$J^{*0}(k) = \frac{1}{2}\begin{bmatrix}\hat{x}^0(k)\\ \hat{x}(k)-\hat{x}^0(k)\end{bmatrix}^T\begin{bmatrix}S^A(k) & S^B(k)\\ S^{BT}(k) & S^C(k)\end{bmatrix}\begin{bmatrix}\hat{x}^0(k)\\ \hat{x}(k)-\hat{x}^0(k)\end{bmatrix} + \frac{1}{2}\,\gamma^0(k). \tag{A.2.28}$$

At k = N,

$$J^{*0}(N) = \frac{1}{2}\,\hat{x}^{0T}(N)K^0(N)\hat{x}^0(N) + \frac{1}{2}\,\mathrm{tr}K^0(N)P^0(N).$$

Then

$$u^{o*}(k) = \arg\min_{u^o(k)} E[\frac{1}{2}\,x^T(k)Q^0(k)x(k) + \frac{1}{2}\,u^{0T}(k)R^0(k)u^0(k) + J^{o*}(k+1)/z^{o*}(k)]. \tag{A.2.29}$$

For any matrix Γ [12]

$$E\{\hat{x}^{0T}(k+1)\Gamma\hat{x}(k+1)/z^{o*}(k)\} = E[\{\hat{x}^0(k+1/k) + \tilde{K}(k+1)[z(k+1) - H(k+1)\hat{x}^0(k+1/k)]\}^T$$

$$\Gamma\{\hat{x}(k+1/k) + K(k+1)[z(k+1) - H(k+1)\hat{x}(k+1)/k)]\}/z^{*0}(k)\} \tag{A.2.30}$$

where

$$\tilde{K}(k+1) = P^0(k+1/k)H^T(k+1)[H(k+1)P^0(k+1/k)H^T(k+1) + \Xi(k+1)]^{-1} \quad (A.2.31)$$

$$E\{\hat{x}^{0T}(k+1)\Gamma\hat{x}(k+1)/z^{0*}(k)\} = \bar{x}^0(k+1)^T\Gamma\bar{x}(k+1) + \bar{x}^{0T}(k+1)\Gamma K(k+1)H(k+1)(\bar{x}^0(k+1) - \bar{x}(k+1)) + \mathrm{tr}\, P^0(k+1/k)\Gamma K(k+1)H(k+1) \quad (A.2.32)$$

$$E[\hat{x}(k+1)\Gamma\hat{x}(k+1)/z^{0*}(k)] = E[\{\hat{x}(k+1/k) + K(k+1)[z(k+1)-H(k+1)\hat{x}(k+1/k)]\}^T\Gamma \{\hat{x}(k+1/k) + K(k+1)[z(k+1)-H(k+1)\hat{x}(k+1/k)]\}/z^{*0}(k)\}$$

$$= \bar{x}^T(k+1)\Gamma\bar{x}(k+1) + 2\bar{x}^T(k+1)\Gamma K(k+1)H(k+1)(\bar{x}^0(k+1)-\bar{x}(k+1))$$

$$+ \mathrm{tr}\{\Gamma K(k+1)[H(k+1)P^0(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1)$$

$$+ (\bar{x}(k+1)-\bar{x}^0(k+1))^T H^T(k+1)K^T(k+1)\Gamma K(k+1)H(k+1)(\bar{x}(k+1)-\bar{x}^0(k+1)). \quad (A.2.33)$$

Expand (A.2.29) using (A.2.32) and (A.2.33)

$$u^{0*}(k) = \arg\min_{u^0(k)}[\tfrac{1}{2}\hat{x}^{0T}(k)Q^0(k)\hat{x}^0(k) + \tfrac{1}{2}u^{0T}(k)R^0(k)u^0(k) + \tfrac{1}{2}\mathrm{tr}Q^0(k)P^0(k)$$

$$+ \tfrac{1}{2}\bar{x}^{0T}(k+1)(S^A + S^C - 2S^B)\bar{x}^0(k+1) + \bar{x}^{0T}(k+1)(S^B - S^C)\bar{x}(k+1)$$

$$+ \bar{x}^{0T}(k+1)(S^B - S^C)K(k+1)H(k+1)(\bar{x}^0(k+1) - \bar{x}(k+1))$$

$$+ \tfrac{1}{2}\bar{x}^T(k+1)S^C\bar{x}(k+1) + \bar{x}^T(k+1)S^C K(k+1)H(k+1)(\bar{x}^0(k+1) - \bar{x}(k+1))$$

$$+ \tfrac{1}{2}(\bar{x}(k+1) - \bar{x}^0(k+1))^T H^T(k+1)K^T(k+1)S^C K(k+1)H(k+1)(\bar{x}(k+1) - \bar{x}^0(k+1)) + \tfrac{1}{2}\gamma^0(k)$$

$$+ \tfrac{1}{2}\mathrm{tr}\{K^0(k+1)[H^0(k+1)P^0(k+1/k)H^{0T}(k+1) + \Xi^0(k+1)]K^{0T}(k+1)(S^A + S^C - 2S^B)\}$$

$$+ \tfrac{1}{2}\mathrm{tr}2P^0(k+1/k)K(k+1)H(k+1)(S^B - S^C) + \tfrac{1}{2}\mathrm{tr}K(k+1)[H(k+1)P^0(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1)S^C. \quad (A.2.34)$$

Recall that

$$\bar{x}^0(k+1) = A(k)\hat{x}^0(k) - (B^1(k)\Delta^1(k)A(k) + B^2\Delta^2(k)A(k))\hat{x}(k) + \hat{B}(k)u^0(k) \quad (A.2.35)$$

where

$$\hat{B}(k) = B^0(k) - B^1(k)\Delta^1(k)B^0(k) - B^2(k)\Delta^2(k)B^0(k). \quad (A.2.36)$$

Let

$$G(k) = B^1(k)\Delta^1(k) + B^2(k)\Delta^2(k) \qquad (A.2.37)$$

then (A.2.35) becomes

$$\bar{x}^0(k+1) = (I-G(k))A(k)\hat{x}^0(k) - G(k)A(k)(\hat{x}(k)-\hat{x}^0(k)) + \hat{B}(k)u^0(k) \qquad (A.2.38)$$

and

$$\bar{x}(k+1) = (I-G(k))A(k)\hat{x}^0(k)$$
$$+ (I-G(k))A(k)(\hat{x}(k)-\hat{x}^0(k)) + \hat{B}(k)u^0(k) \qquad (A.2.39)$$

$$\bar{x}(k+1) - \bar{x}^0(k+1) = A(k)(\hat{x}(k)-\hat{x}^0(k)). \qquad (A.2.40)$$

Substitute (A.2.40) in (A.2.34) and differentiating $u^{0*}(k)$ is given by

$$u^{0*}(k) = -\Delta^0(k)Y(k)\hat{x}^0(k) - \Delta^0(k)M(k)[\hat{x}(k)-\hat{x}^0(k)] \qquad (A.2.41)$$

where

$$\Delta^0(k) = [R^0(k) + \hat{B}^T(k)S^A(k+1)\hat{B}(k)]^{-1}$$

$$Y(k) = \hat{B}(k)S^A(k+1)[I-G(k)]A(k)$$

$$M(k) = \hat{B}^T(k)S^A(k+1)G(k)A(k) + \hat{B}^T(k)(S^B(k+1)-S^A(k+1))A(k)$$
$$-\hat{B}^T(k)S^B(k+1)K(k+1)H(k+1)A(k).$$

The recursive equations for S^A, S^B, S^C, $\gamma^0(k)$ are obtained by substituting $u^{0*}(k)$ back in (A.2.40)

$$S^A(k) = Q^0(k) + A^T(k)(I-G(k))^T S^A(k+1)(I-G(k))A(k) - Y^T(k)\Delta^0(k)Y(k) \qquad (A.2.42)$$

$$S^B(k) = A^T(k)(I-G(k))^T S^B(k+1)(I-G(k))A(k)$$
$$+ A^T(k)(I-G(k))^T(S^B(k+1)-S^A(k+1))G(k)A(k)$$
$$-A^T(k)(I-G(k))^T S^B(k+1)K(k+1)H(k+1)A(k) - Y^T(k)\Delta^0(k)M(k) \qquad (A.2.43)$$

$$S^C(k) = -M^T(k)\Delta^0(k)M(k) + A^T(k)G^T(k)S^A(k+1)G(k)A(k)$$
$$+ A^T(k)[I-K(k+1)H(k+1)]^T S^C(k+1)[I-K(k+1)H(k+1)]A(k)$$

$$+ A^T(k)(S^B(k+1)K(k+1)H(k+1)-S^B(k+1))G(k)A(k)$$

$$-A^T(k)G^T(k)(S^B(k+1)-S^B(k+1)K(k+1)H(k+1))A(k) \qquad (A.2.44)$$

$$\gamma^0(k) = \gamma^0(k+1) + \mathrm{tr}\ Q^0(k)P^0(k) + \mathrm{tr}[K^0(k+1)[H^0(k+1)P^0(k+1/k)H^{0T}(k+1) +$$

$$+ \Xi^0(k)]K^{0T}(k+1)(S^A(k+1) + S^C(k+1) - 2S^B(k+1))]$$

$$+ 2\mathrm{tr}\ P^0(k+1/k)K(k+1)H(k+1)(S^B(k+1)-S^C(k+1))$$

$$+ \mathrm{tr}\ K(k+1)[H(k+1)P^0(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1)S^C(k+1). \qquad (A.2.45)$$

To obtain the recursive equation for $S^i(k)$ of the i-th subsystem, substitute $u^{0*}(k)$, $u_o^i(k)$ back in (A.2.21)

$$S^i(k) = Q^i(k) + (A(k) + \hat{B}(k)\Delta^0(k)Y(k))^T S^i(k+1)(A(k) + \hat{B}(k)\Delta^0(k)Y(k))$$

$$+ (\Delta^i(k)A(k) + B^0(k)\Delta^0(k)Y(k)^T R^i(k)(\Delta^i(k)A(k) + B^0(k)\Delta^0(k)Y(k))$$

$$i = 1,2 \qquad (A.2.46)$$

$$\gamma^i(k) = \gamma^i(k+1) + \mathrm{tr}\ Q^i(k)P(k) + \mathrm{tr}\ S^i(k+1)K(k+1)$$

$$+ \mathrm{tr}\ S^i(k+1)K(k+1)[H(k+1)P(k+1/k)H^T(k+1) + \Xi(k+1)]K^T(k+1)$$

$$+ \mathrm{tr}[P(k/k)-P^0(k/k)](M(k)-Y(k))^T\Delta^{0T}(k)(B^{0T}(k)R^i(k)B^0(k)$$

$$+ \hat{B}^T S^i(k+1)\hat{B})\Delta^0(k)(M(k)-Y(k)) \qquad (A.2.47)$$