MACHINE LEARNING FOR PRICING EUROPEAN BASKET OPTIONS

BY

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Actuarial Science
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2019

Urbana, Illinois

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ABSTRACT

In this thesis, we show how to deploy machine learning techniques such as Gaussian process regression to approximate the European basket option prices. For the underlying asset of European basket option, we assume it follows multivariate Black & Scholes model, and we can derive the PDE for the option price. In order to deal with the curse of dimensionality, we assume that the basket consists of several comonotonic groups, in each comonotonic group, the stock prices are driven by a single random source. Then we can derive an approximation for the price of European basket option. Next, we introduce the finite difference scheme to price the European basket option for given parameters such as risk-free interest rates, maturities and so on. However, for approximating the European basket option for different risk-free interest rates, maturities and strikes, using finite difference scheme to get corresponding approximations costs much time. Hence, in order to save time for approximating the European basket option for different risk-free interest rates, maturities and strikes, we deploy Gaussian process regression to fit the training set produced by finite difference scheme and after comparing the results, we can conclude that the errors are often well within reasonable limits and hence very acceptable from a practical point of view and the Gaussian process regression truly save much time.

ACKNOWLEDGMENTS

I would first like to thank my thesis advisor Prof. Linders and Prof. Chong of the Math Department at University of Illinois at Urbana-Champaign. I really appreciate their guidance and patience. I have learned a lot from them.

I would also like to thank Kara and Patrick, they give me some useful insights for this thesis.

At last, thank Marci Blocher, staff of the math department, she helps me a lot with this thesis.

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CHAPTER 1

INTRODUCTION

A basket option is a common financial product that diversifies the market risk. It is a contingent claim written on a basket of stocks, and in the simplest form, the *stock basket* is the weighted sum of the stock prices. A European basket derivative is a basket option with a single payoff at maturity. The stock prices in the basket are assumed to be dependent, hence one can employ a multivariate stock price model to express the dynamics of the stock prices in the basket. By assuming a multivariate stock price model, we can derive the partial differential equation of a European basket option. Hence, the European basket option price can be derived theoretically by solving the corresponding partial differential equation. However, the PDE is hard to solve even if we can derive a solution, it consists of a multi-dimensional integral and evaluates such an integral is impossible for high dimensions. Indeed, one can use the numerical methods to approximate prices for the basket derivative; see e.g. Reisinger and Wittum (2007) [7], Leentvaar and Oosterlee (2008) [4].

In order to approximate the price of an European basket derivative fast and efficiently, Linders and Hanbali (2019) [2] proposed a PDE-based approach to approximate the price of European basket derivatives. By deploying the concept of comonotonicity, which is an extreme positive dependence structure of stock prices in the basket. The stock prices are assumed to be driven by a single random source, one can manage the dimensionality issue effectively. Then a one-dimensional PDE of the European basket derivative price can be derived given the assumption of the comonotonic market, hence the PDE approach can still be used to approximate the basket price. However it can not provide a realistic basket derivative prices. Therefore, they consider an appropriate transformation of the marginal distribution of stock prices to derive a close approximation for the real basket.

In this thesis, we still employ the concept of comonotonicity, but we assume the *stock basket* is composed of several comonotonic groups. Note that a major assumption in Linders and Hanbali (2019) [2] is that all correlations are the same and equals to 1, by using the approach with the comonotonic groups, we can allow for different correlations. In each comonotonic group, the stock prices are driven by a single random source, and the different random sources for different groups are correlated. By considering such an assumption, one can still derive an approximation for a European basket derivative based on PDE approach. However, we still can not prove that this approximation is an upper bound for the European basket option price. Hence, we regard the derivative price under this assumption as an approximation of the accurate European basket option price.

We consider the *stock basket* consisting of 2 comonotonic groups, for given risk-free rate r, maturity T and strike k, we can use finite difference scheme to derive the approximation for a European basket derivative price. However, if we want to approximate the European basket prices with different risk-free rates, maturities and strikes, the calculations need to be repeated many times, which takes much time to get the corresponding approximations. In addition, since the approximations derived by finite difference scheme can be summarized in a few parameters. Moreover, the dynamics of stock prices are modelled with also a limited set of parameters(risk-free interest rate, volatilies, etc.), hence it provides an ideal situation for deploying machine learning techniques. Therefore, We illustrate that we can arrive at speed-ups of 2.7×10^6 of magnitude by deploying Gaussian process regression. However, the increasing of the speed leads to the decreasing of the accuracy. But we show that the loss of accuracy is still within reasonable interval $[4.3593 \times 10^{-7}, 2.7 \times 10^{-3}]$.

CHAPTER 2

EUROPEAN BASKET OPTIONS IN THE MULTIVARIATE BLACK & SCHOLES MODEL

2.1 The multivariate Black & Scholes model

Consider an efficient and frictionless market with n non-dividend paying stocks, labeled from 1 to n. At time $t \geq 0$, we use $S_i(t)$ to denote the time-t price of stock i. Under the real-world probability measure $\mathbb{P}$, where the 'real world' means that we measure the probability based on historical data of the multivariate stock prices, for $i = 1, 2, ..., n$, the stock price dynamics in the multivariate Black & Scholes setting are described as following:

$$S_i(t) = S_i(0) + \int_0^t \mu_i S_i(s)\mathrm{d}s + \int_0^t \sigma_i S_i(s)\mathrm{d}B_i(s), \tag{2.1}$$

where μ_i is the drift of the stock i, σ_i is the volatility of the stock i, $\underline{B}(t) = (B_1(t), B_2(t), ..., B_n(t))$ and $\{\underline{B}(t)|t \geq 0\}$, is correlated n-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the filtration $(\mathcal{F}_t)_{t\geq 0}$ which records the 'past behavior' of the multivariate stock price process. The correlation between the stocks i and j is denoted by $\rho_{i,j}$, and satisfies:

$$\mathbb{E}[\mathrm{d}B_i(t), \mathrm{d}B_j(t)] = \rho_{i,j}\mathrm{d}t,$$

where $i, j = 1, 2, ..., m$.

We use $S(t)$ to denote the time-t price of the *stock basket*, which is the weighted sum of the n different stocks:

$$S(t) = \omega_1 S_1(t) + \omega_2 S_2(t) + \cdots + \omega_n S_n(t), \tag{2.2}$$

where the weights ω_i, for $i = 1, 2, ..., n$, are assumed to be positive constants.

2.2 Pricing European basket derivatives in the multivariate Black & Scholes model

Consider a European basket derivative with maturity T and pay-off function H, which is a contingent claim with pay-off determined by $H(S(T))$. We denote the arbitrage-free price of the European basket derivative at time $t \geq 0$ by $V(t, S_1(t), S_2(t), ..., S_n(t))$. If there is no confusion, we use V or $V(t, S_1, S_2, ..., S_n)$ to simplify the notation.

For an n-dimension European basket option, where the basket has n individual stocks $S_1, S_2, ..., S_n$ as the components, given the multivariate Black & Scholes setting, the time-t price of the European type derivative denoted by $V(t, S_1, S_2, ..., S_n)$ satisfies the following partial differential equation (**PDE**):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{i,j}\omega_i\omega_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + r\sum_{i=1}^{n} S_i \frac{\partial V}{\partial S_i} = 0, \tag{2.3}$$

where the $(\sigma_i\sigma_j\rho_{i,j})_{1<i,j<n}$ is the variance-covariance matrix of the stock prices and r is the risk-free interest rate, which is constant and deterministic. Given appropriate boundary and final conditions, the time-t price of the European basket option is the solution of the **PDE** (2.3). In order to get the analytic solution, we need to solve an n-dimension integration problem. However, when n is large enough, it becomes an impossible mission to solve such a high dimensional integral problem. Indeed, one can choose finite difference scheme to numerically solve the **PDE** (2.3), nevertheless, such a finite difference scheme requires to construct a time grid and an n-dimensional stock price grids, which takes much time for pricing and calibrating process. Therefore, pricing basket derivatives using **PDE** methods is still an open problem.

CHAPTER 3

CONVEX UPPER BOUND FOR EUROPEAN BASKET OPTION WITH AN EXTREME POSITIVE DEPENDENCE STRUCTURE

In order to price the European basket option with high dimensions more efficiently, we first derive an upper bound for the price of the European basket option. We obtain an upper bound by assuming all stocks in the basket move in the same direction. In other words, we assume the *stock basket* to be comonotonic, since the pay-off function H of the derivative is a *convex* function, hence it can derive the upper bound of the basket derivative by deploying an extreme positive dependence structure.

Construct an artificial frictionless comonotonic financial market with n individual stocks traded. In this comonotonic market, the time-t price of stock i is denoted by $S_i^c(t)$. Hence, for $t > 0$ and $i = 1, 2, ..., n$, the dynamics of the n individual stocks are described as the following stochastic integral equation (**SIE**):

$$S_i^c(t) = S_i(0) + \int_0^t \mu_i S_i^c(s)\mathrm{d}s + \int_0^t \sigma_i S_i^c(s)\mathrm{d}B(s), \tag{3.1}$$

where $B = \{B(t)|t \geq 0\}$ is a standard Brownian motion and $S_i(0)$ are the time-0 prices of stock i in the basket, for $i = 1, 2, ..., n$. Note that the stock prices are driven by a single random source B, which means that if one stock price will turn out to be large at time T, all other stocks will be large too, this corresponds with a comonotonic dependence structure. We have the time-t stock prices vector $\underline{X(t)} = (S_1^c(t), S_2^c(t), ..., S_n^c(t))$, with marginal distributions denoted by $F_{S_i^c}$, $i = 1, 2, ..., n$. It is comonotonic, which means that:

$$(S_1^c, S_2^c, ..., S_n^c) \stackrel{d}{=} (F_{S_1^c}^{-1}(U), F_{S_2^c}^{-1}(U), ..., F_{S_n^c}^{-1}(U),$$

where U is a uniform $(0, 1)$ r.v.

In this comonotonic market, for $t \geq 0$, the corresponding time-t basket price denoted by $S^c(t)$ can be expressed as the following:

$$S^c(t) = \omega_1 S_1^c(t) + \omega_2 S_2^c(t) + \cdots + \omega_n S_n^c(t). \tag{3.2}$$

Consider the European basket option with *convex* pay-off function H and maturity T, written on the comonotonic basket $S^c(t)$. Referred to Linders and Hanbali (2019) [2], since the market is comonotonic, the time-t price of the European basket option is precisely determined by the time-t basket price, hence we can denote the time-t price of the European basket derivative by $V^c(t, S^c(t))$. According to Kaas et al (2000) [3], one can prove that the comonotonic basket option price $V^c(t, S^c(t))$ is an upper bound of the original price V.

Referred to Linders and Hanbali (2019), the following Lemma shows that the comonotonic basket $S^c = \{S^c(t)|t \geq 0\}$ follows an Ito process with time-dependent drift and volatility.

Lemma 3.1. *Consider the comonotonic market described in (3.1). The stochastic integral equation of the comonotonic basket S^c defined in (3.2) is given by:*

$$\mathrm{d}S^c(t) = \mu^c(s)\mathrm{d}s + \sigma^c(s)\mathrm{d}B(s), \tag{3.3}$$

where

$$\mu^c(s) = \sum_{i=1}^{n} \mu_i \omega_i S_I^c(s) \quad \textit{and} \quad \sigma^c(s) = \sum_{i=1}^{n} \sigma_i \omega_i S_i^c(s). \tag{3.4}$$

We assume that the market is arbitrage free. Therefore for $i = 1, 2..., n$, there exists a parameter $\lambda \in \mathbb{R}$ satisfying:

$$\lambda = \frac{\mu_i - r}{\sigma_i}. \tag{3.5}$$

Notice that the λ is named the market price of risk, which represents the reward obtained for holding a unit of volatility. As a result of the comonotonic assumption, the volatility of each stock is driven by the same Brownian motion. Hence the market price of risk is the same for every stock in the basket.

We consider the process $B_\lambda = \{B_\lambda(t)|t \geq 0\}$ and:

$$\mathrm{d}B_\lambda(t) = \mathrm{d}B(t) + \lambda \mathrm{d}t. \tag{3.6}$$

The B_λ is a shifted Brownian motion. By plugging the relation $r = \mu_i - \lambda\sigma_i$ in expression (6) and replacing B by B_λ, we can derive that:

$$\mathrm{d}S^c(t) = rS^c(t)\mathrm{d}t + \sigma^c(t)\mathrm{d}B_\lambda(t), \tag{3.7}$$

hence the basket S^c can be described as follows:

$$S^c(t) = \sum_{i=1}^{n} \omega_i S_i(0) e^{(r-\frac{1}{2}\sigma_i^2)t+\sigma B(t)}. \tag{3.8}$$

Consider the European basket derivative we described above, the comonotonic option price is driven by a single Brownian motion B_λ, the time-t stock price $S_i(t)$ and the time-t comonotonic price $S^c(t)$ is determined by the realization of B_λ, hence we can denote the time-t comonotonic European basket derivative price by V_λ^c. According to Linders and Hanbali (2019) [2], the following theorem describes the **PDE** for the derivative price V_λ^c.

Theorem 3.1. *The derivative price V_λ^c with payoff at time T given in (3.10) satisfies the following back-ward partial differential equation:*

$$\frac{\partial V_\lambda^c}{\partial t} + \frac{1}{2}\frac{\partial^2 V_\lambda^c}{\partial B_\lambda} - rV_\lambda^c = 0, \tag{3.9}$$

where the final condition is given by:

$$V_\lambda^c(T, B_\lambda) = H\left(\sum_{i=1}^{n} \omega_i S_i(0) e^{(r-\frac{1}{2}\sigma_i^2)T+\sigma_i B_\lambda}\right). \tag{3.10}$$

Moreover, in Linders and Hanbali (2019) [2], the following theorem shows that there exists such an explicit solution for (3.9) which satisfies the final condition (3.10).

Theorem 3.2. *The time-t price of the comonotonic European basket derivative with pay-off function H denoted by V_λ^c can be given by:*

$$V_\lambda^c(t, B_\lambda) = e^{-r(T-t)} \int_{-\infty}^{\infty} H\left(\sum_{i=1}^{n} \omega_i S_i^c(t) e^{(r-\frac{1}{2}\sigma_i^2)(T-t)+\sigma_i y}\right) \Phi_{T-t}(y)\mathrm{d}y,$$

where $S_i^c(t)$ is the time-t price of the comonotonic stock i, which is described as follows:

$$S_i^c(t) = S_i(0)e^{(r-\frac{1}{2})t+\sigma_i B_\lambda(t)},$$

and Φ_{T-t} *is the density function of a normal distribution with mean 0 and variance* $T-t$*:*

$$\Phi_{T-t}(y) = \frac{e^{-\frac{y^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}.$$

By deploying the assumption of a comonotonic financial market, we can derive a one dimensional **PDE** (3.9) for the comonotonic European basket derivative price V_λ^c. Hence, introducing the concept of the comonotonicity leads to useful dimension reduction of the pricing problem. However, working in the comonotonic market means that we do not derive the realistic basket derivative price. Referred to Linders and Hanbali (2019) [2], we consider an appropriate transformation of marginal volatilities for stock price process, which provides a lower bound for the European basket derivative price, but it is closer to the realistic basket derivative price compared with the upper bound derived by assuming the comonotonicity of the *stock basket*.

CHAPTER 4

CONVEX LOWER BOUND FOR EUROPEAN BASKET OPTION WITH AN EXTREME POSITIVE DEPENDENCE STRUCTURE

In this chapter, we derive a lower bound for European basket option with an extreme positive dependence structure by deploying an appropriate transformation of the marginal distributions.

Consider an artificial market with n different stocks, labeled from 1 to n, the dynamics of the time-t price of stock i denoted by $S_i^l(t)$ are assumed to be:

$$S_i^l(t) = S_i(0) + \int_0^t \mu_i S_i^l(s)\mathrm{d}s + \int_0^t \nu_i \sigma_i S_i^l(s)\mathrm{d}B(s), \tag{4.1}$$

where $i = 1, 2..., n$ and $0 \leq \nu_i \leq 1$. Notice that the time-t stock prices have different distributions compared with the real stock prices, the stock price process $S_i^l = \{S_i^l(t)|t \geq 0\}$ are less volatile than the realistic stock price process. Since $\nu_i > 0$, the individual stock prices S_1^l, S_2^l,..., S_n^l are still comonotonic.

The corresponding time-t basket price is denoted by $S^l(t)$ and:

$$S^l(t) = \sum_{i=1}^{n} \omega_i(t) S_i^l(t). \tag{4.2}$$

The distribution of time-t stock price S_i depends on different choices of ν_i, $i = 1, 2, ..., m$. The time-t price of a basket derivative with convex pay-off function H, written on the basket S^l, is denoted by $V^l(t, S^l)$. For factors ν_i, where $i = 1, 2, ..., n$, referred to Linders and Hanbali (2019) [2], the following lemma provides reliable conditions for parameters ν_i to derive a lower bound V^l for the European basket derivative.

Lemma 4.1. *The factors ν_i, for $i = 1, 2, 3, ..., n$, which are in the dynamics of the stock prices given in (4.1), are assumed to be:*

$$\nu_i = \frac{\sum_{j=1}^{n} \beta_j \rho_{i,j} \sigma_j}{\sqrt{\sum_{k=1}^{n} \sum_{j=1}^{n} \beta_k \beta_j \rho_{k,j} \sigma_k \sigma_j}}, \tag{4.3}$$

with $\beta_j > 0$, for $j = 1, 2, ..., n$. Then there always exists a choice for $\beta_j > 0$, $j = 1, 2, ..., n$ to satisfy $0 \leq \nu_i \leq 1$, such that:

$$V^l(t, S^l(t)) \leq V(t, S(t))$$

For details about the selection of β_j and a proof of Lemma (4.1), we can refer to Kaas et al (2000) [3], Deelstra et al (2004) [1] and Linders and Stassen (2016) [5].

We can write the dynamics of the stock i as the following stochastic differential equation (**SDE**):

$$\mathrm{d}S_i^l(t) = \mu_i S_i^l(t)\mathrm{d}t + \nu_i \sigma_i S_i^l(t)\mathrm{d}B_i(t). \tag{4.4}$$

From **SDE** (4.4), we can notice that the volatility for the stock i is $\nu_i\sigma_i$. Hence, we can derive the partial differential equation for the lower bound V^l by using the same method of deriving the **PDE** of the upper bound given in (3.9). Hence we still consider shifted Brownian motions $B_{\lambda^l} = \{B_{\lambda^l}(t)|t \geq 0\}$, with drifts $\lambda^l = \frac{\mu_i - r}{\nu_i \sigma_i}$, $i = 1, 2, ..., n$. Therefore, the partial differential equation for the lower bound of the European basket derivative is given as:

$$\frac{\partial V^l}{\partial t} + \frac{1}{2}\frac{\partial^2 V^l}{\partial B_{\lambda^l}} - rV^l = 0, \tag{4.5}$$

Note that the time-t price of the European basket derivative written on the basket S^l with pay-off function H and maturity T is the solution for **PDE** (4.5), with the final condition given as following:

$$V^l(T, B_{\lambda^l}) = H\left(\sum_{i=1}^{n} \omega_i S_i(0) e^{(r - \frac{1}{2}\nu_i \sigma_i^2)T + \nu_i \sigma_i B_{\lambda^l}(T)}\right). \tag{4.6}$$

Therefore, the explicit solution of PDE (4.5) with the final condition (4.6) is the time-t price of the European basket derivative written on the *stock basket* S^l, and it is a lower bound for the European basket derivative price.

Note that we derive this lower bound by assuming the *stock basket* is driven by a single Brownian motion. In order to diversify the risk, in next chapter, we consider a partial dependence structure to derive an approximation for the price of a European basket option.

CHAPTER 5

APPROXIMATION FOR EUROPEAN BASKET OPTION PRICE WITH PARTIAL DEPENDENCE STRUCTURE

We still consider an efficient and frictionless market with n non-dividend paying stocks. We divide these n stocks into m groups, $m \in \mathbb{N}$ and $2 \leq m \leq n$. In group i, for $i = 1, 2, ..., m$, there are n_i stocks, hence, $\sum_{i=1}^{m} n_i = n$. For $i = 1, 2, ..., m$, $j = 1, 2, ..., n_i$, the dynamics given the multivariate Black & Scholes setting are assumed to be:

$$S_{ij}(t) = S_{ij}(0) + \int_0^t \mu_{ij} S_{ij}(s) \mathrm{d}s + \int_0^t \sigma_{ij} S_{ij}(s) \mathrm{d}B_{ij}(s),$$

where the μ_{ij} and σ_{ij} are the drift and volatility of stock j in group i, and $\overline{B} = \{B_{ij}(t) | i = 1, 2, ..., m, j = 1, 2, ..., n_i, t \geq 0\}$ is correlated n-dimensional Brownian motion with $\mathbb{E}[\mathrm{d}B_{ij}(t)\mathrm{d}B_{ka}(t)] = \rho_{ij,ka}$, for $i = 1, 2, ..., m$, $k = 1, 2, ..., m$, $j = 1, 2, ..., n_i$, $a = 1, 2, ..., n_k$.

In partial dependence structure, the individual stocks in each group are assumed to move in the same direction which means that different stocks in the same group are comonotonic. In the comonotonic group i, n_i stocks are traded and we denote the price of stock j in group i at time $t \geq 0$ by$S^c_{ij}(t)$, for $j = 1, 2, 3, ..., n_i$, $i = 1, 2, ..., m$, and under real world probability measure $\mathbb{P}$, the stochastic differential equation of the stock S^c_{ij} (**SDE**) is assumed to be:

$$\mathrm{d}S^c_{ij}(t) = \mu_{ij} S^c_{ij}(t)\mathrm{d}t + \sigma_{ij} S^c_{ij}(t)\mathrm{d}B_i(t), \tag{5.1}$$

for $j = 1, 2, ..., n_i$, $i = 1, 2, ..., m$, where $B_i = \{B_i(t) | \ t \geq 0\}$ is a standard Brownian motion. And ρ_{ik} is the correlation between group i and group k, which satisfies:

$$\mathbb{E}[\mathrm{d}B_i(t)\mathrm{d}B_k(t)] = \rho_{ik}\mathrm{d}t.$$

With the dynamics of stock j in group i given by (5.1). And for different group i and group k, we assume $\rho_{ij,ka} = \rho_{i,k}$, for $j = 1, 2, ..., n_i$, $a = 1, 2, ..., n_k$. The time-t price of the stock basket based on this assumption denoted by

$S^u(t)$ is given by:

$$S^u(t) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \omega_{ij}(t) S^c_{ij}(t), \tag{5.2}$$

where the weights ω_{ij}, for $i = 1, 2, ...m$, $j = 1, 2, ..., n_i$ are assumed to be positive constants.

Consider the weighted sum of stock prices in group i at time t, denoted by $X^c_i(t)$ and:

$$X^c_i(t) = \sum_{j=1}^{n_i} \omega_{ij}(t) S^c_{ij}(t). \tag{5.3}$$

Therefore, the time-t price of the *stock basket* denoted by $S^u(t)$ can be described as:

$$S^u(t) = \sum_{i=1}^{m} X^c_i(t). \tag{5.4}$$

In the following lemma, we show that the comonotonic group $X^c_i = \{X^c_i(t)|t \geq 0\}$ defined in (5.3) follows an Ito process with time-dependent drift and volatility given by (5.6).

Lemma 5.1. *The dynamics of the comonotonic group* $X^c_i = \{X^c_i(t)|t \geq 0\}$ *can be given by:*

$$X^c_i(t) = X^c_i(0) + \int_0^t \mu^c_i(t)\mathrm{d}t + \int_0^t \sigma^c_i(t)\mathrm{d}B_i(t), \tag{5.5}$$

where

$$\begin{aligned} \mu^c_i(t) &= \sum_{j=1}^{n_i} \omega_{ij}\mu_{ij} S_{ij}(t), \\ \sigma^c_i(t) &= \sum_{j=1}^{n_i} \omega_{ij}\sigma_{ij} S_{ij}(t). \end{aligned} \tag{5.6}$$

Proof. For $i = 1, 2, ..., m$, $j = 1, 2, ..., n_i$, the stochastic integral equation **SIE** of the stock j in group i is given by:

$$S^c_{ij}(t) = S_{ij}(0) + \int_0^t \mu_{ij} S_{ij}(s)\mathrm{d}s + \int_0^t \sigma_{ij} S_{ij}(s)\mathrm{d}s \tag{5.7}$$

Since the comonotonic groups X^c_i, $i = 1, 2, ..., m$, are given by (5.3), hence we can derive the stochastic integral equation of the comonotonic group X^c_i

directly by using the sum of the **SIE** (5.7) of each stock in the comonotonic group. □

The time-t price of the *stock basket* $S^u(t)$ is the sum of the comonotonic groups $X_i^c(t)$, for $i = 1, 2, ..., m$, hence we can write the stochastic integral equation (**SIE**) of the stock basket S^u directly from (5.4) and (5.5)

$$S^u(t) = S(0) + \sum_{i=1}^{m} \int_0^t \mu_i^c(t)\mathrm{d}t + \sum_{i=1}^{m} \int_0^t \sigma_i^c(t)\mathrm{d}B_i(t). \tag{5.8}$$

We can use the following stochastic differential equation to describe the dynamics of the *stock basket*:

$$\mathrm{d}S^u(t) = \sum_{i=1}^{m} \mu_i^c(t)\mathrm{d}t + \sum_{i=1}^{m} \sigma_i^c(t)\mathrm{d}B_i(t). \tag{5.9}$$

Note that the stock basket is driven by random sources B_i, for $i = 1, 2, ..., m$.

The market we defined above is assumed to be arbitrage-free, this absence of arbitrage opportunities is equivalent with the existence of a parameter $\lambda_i \in \mathbb{R}$ for comonotonic group i, which satisfies:

$$\lambda_i = \frac{\mu_{ij} - r}{\sigma_{ij}}, \quad \text{for} \quad i = 1, 2, ..., m,$$

where $j = 1, 2, ..., n_i$. For comonotonic group i, the market price of risk λ_i is the same for each stock j in group i, the reason is that the volatility of each stock j in group i is driven by the same Brownian motion B_i and hence a unit of volatility of stock j in group i is fundamentally the same as a unit of volatility of any other stock in group i.

Define the process $B_{\lambda_i} = \{B_{\lambda_i}(t) | t \geq 0\}$ for comonotonic group i, satisfies

$$\mathrm{d}B_{\lambda_i}(t) = \mathrm{d}B_i(t) + \lambda_i \mathrm{d}t, \tag{5.10}$$

which is a Brownian motion for group i with drift λ_i. Hence, we can derive the stochastic differential equation of comonotonic group ${}^iS^c$ by replacing the B_i by B_{λ_i} defined in (5.10) and using the relation $r = \mu_{ij} - \lambda_i \sigma_{ij}$:

$$\mathrm{d}X_i^c(t) = rX_i^c(t)\mathrm{d}t + \sigma_i^c(t)\mathrm{d}B_{\lambda_i}(t),$$

and:

$$X_i^c(t) = \sum_{j=1}^{n_i} \omega_{ij} S_{ij}(0) e^{(r-\frac{1}{2}\sigma_{ij}^2)t+\sigma_{ij}B_{\lambda_i}(t)}.$$

Therefore, we can derive that:

$$\mathrm{d}S^u(t) = rS^u(t)\mathrm{d}t + \sum_{i=1}^{m} \sigma_i^c(t)\mathrm{d}B_{\lambda_i}(t), \tag{5.11}$$

where

$$S^u(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \omega_{ij} S_{ij}(0) e^{(r-\frac{1}{2}\sigma_{ij}^2)t+\sigma_{ij}B_{\lambda_i}(t)}. \tag{5.12}$$

By Girsanov's theorem, which is stated below, for $i = 1, 2, ..., m$, since the drift of B_{λ_i} equals to the market price of risk, there exist a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the stochastic process B_{λ_i} is a standard Brownian motion under measure $\mathbb{Q}$. In the following parts of the thesis, we consider **SDE** (5.11) as the dynamic of the *stock basket*, but we still work under real world probability measure $\mathbb{P}$, which means that B_{λ_i}, for $i = 1, 2, ..., m$, are still Brownian motions with drift λ_i.

Theorem 5.1. *Girsanov's Theorem: Let $\gamma = \{\gamma_t : t \in [0, T]\}$ be a $\{\mathcal{F}_t\}$-predictable process such that:*

$$\mathbb{E}[e^{\frac{1}{2}\int_0^T \gamma_t^2 \mathrm{d}t}] < -\infty.$$

There exists a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that:
1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$;
2. $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = e^{-\int_0^T \gamma_t \mathrm{d}(t) - \frac{1}{2}\int_0^T \gamma_t^2 \mathrm{d}t}$;
3. The process $\tilde{B}(t) = B(t) + \int_0^t \gamma_s \mathrm{d}s$ is a $(\{\mathcal{F}_t\}, \mathbb{Q})$-Brownian motion.
Note that $\gamma = \{\gamma_t : t \in [0, T]\}$ here is a process called the market price of risk .

Since the dynamic of the *stock basket* is given by (5.11), hence the *stock basket* is driven by random sources B_{λ_i}, for $i = 1, 2, ..., m$. The realizations of B_{λ_i} at time t determine the time-t *stock basket* price $S^u(t)$ in (5.12) and $S_{ij}^c(t)$, for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n_i$. Moreover, the price of a European type basket derivative written on the *stock basket* $S^u = \{S^u(t)|t \geq 0\}$ with pay-off function H and maturity T can be expressed by a function of time t and $B_{\lambda_i}(t)$, for $i = 1, 2, ..., m$. Hence we use $V^u(t, B_{\lambda_1}, B_{\lambda_2}, ..., B_{\lambda_m})$ to denote the time-t price of the European basket derivative and we use V^u to simplify the notation, if there is no confusion. Referred to Linders and Yang (2017) [6], if different comonotonic groups are independent, i.e. $\rho_{ik} = 0$, for different group i and j, one can prove that V^u is an upper bound of the realistic basket derivative price and:

$$V \leq V^u \leq V_\lambda^c$$

In order to derive the partial differential equation of European basket option V^u, we consider a hedging portfolio $\pi = \{(\pi_1(t), ..., \pi_{m+1}(t)|t \geq 0\}$ to replicate the European basket derivative V^u, where $\pi_1(t)$ to $\pi_m(t)$ denote the number of units invested respectively in the m different comonotonic groups at time t and $\pi_{m+1}(t)$ denote the number of units invested at time t in a risk-free asset $D(t)$, we denote the value of the portfolio by $\pi(t)$ and:

$$\pi(t) = \sum_{i=1}^{m} \pi_i(t)^i S^c(t) + \pi_{m+1}(t)D(t),$$

where the dynamic of $D(t)$ can be described as:

$$\mathrm{d}D(t) = rD(t)\mathrm{d}t.$$

The following theorem shows that the time-t derivative price $V^u(t)$ with final condition given by (5.14) satisfies a backward partial differential equation (5.13) and there also exists a unique self-financing hedging portfolio replicating the basket derivative V^u.

Theorem 5.2. *The partial differential equation of the European basket derivative V^u with pay-off given in (5.14) can be described as:*

$$\frac{\partial V^u}{\partial t} + \sum_{i=1}^{m} \frac{1}{2} \frac{\partial^2 V^u}{\partial B_{\lambda_i}{}^2} + \sum_{i=1}^{m} \sum_{k=i+1}^{m} \rho_{ik} \frac{\partial^2 V^u}{\partial B_{\lambda_i} \partial B_{\lambda_k}} - rV^u = 0. \tag{5.13}$$

where the final condition is given by:

$$V^u(T, B_{\lambda_1}, ..., B_{\lambda_m}) = H\left(\sum_{i=1}^{m} \sum_{j=1}^{n_i} \omega_{ij} S_{ij}(0) e^{(r - \frac{1}{2}\sigma_{ij}^2)T + \sigma_{ij} B_{\lambda_i}(T)} \right). \tag{5.14}$$

The unique, self-financing, hedging strategy $\pi = \{(\pi_1(t), ..., \pi_{m+1}(t)|t \geq 0\}$ of *this European basket derivative is given by:*

$$\pi_i(t) = \frac{1}{{}^i\sigma^c(t)} \frac{\partial V^u}{\partial B_{\lambda_i}}, for, i = 1, 2, ..., m.$$
$$\pi_{m+1}(t) = \frac{1}{D(t)}(V^u(t, B_{\lambda_1}, B_{\lambda_2}, ..., B_{\lambda_m}) - \sum_{i=1}^{m} \pi_i(t))^i S^c(t)).$$

Proof. Since π is a replicating strategy, we can derive that:

$$\pi(t) = V^u(t, B_{\lambda_1}, B_{\lambda_2}, ..., B_{\lambda_m}). \tag{5.15}$$

We can write the $\pi_{m+1}(t)$ directly from (5.15) and (5) as the following:

$$()\pi_{m+1}(t) = \frac{1}{D(t)}(V^u(t, B_{\lambda_1}, B_{\lambda_2}, ..., B_{\lambda_m}) - \sum_{i=1}^{m} \pi_i(t))^i S^c(t)). \tag{5.16}$$

By the self-financing condition, we can derive that:

$$\mathrm{d}\pi(t) = \sum_{i=1}^{m} \pi_i(t) \mathrm{d}^i S^c(t) + \pi_{m+1}(t) \mathrm{d}D(t).$$

By plugging the equation (5.16), we can get the following:

$$\mathrm{d}\pi(t) = rV^u(t, B_{\lambda_1}, B_{\lambda_2}, ..., B_{\lambda_m})\mathrm{d}t + \sum_{i=1}^{m} \pi_i(t)^i \sigma^c(t) \mathrm{d}B_{\lambda_i}(t). \tag{5.17}$$

On the other hand, for $V^u(t, B_{\lambda_1}, B_{\lambda_2}, ..., B_{\lambda_m})$, we apply Ito's Lemma to derive:

$$\begin{aligned} \mathrm{d}V^u(t, B_{\lambda_1}, B_{\lambda_2}, ..., B_{\lambda_m}) = \Bigg(\frac{\partial V^u}{\partial t} + \sum_{i=1}^{m} \frac{1}{2} \frac{\partial^2 V^u}{\partial {B_{\lambda_i}}^2} \\ + \sum_{i=1}^{m} \sum_{k=i+1}^{m} \rho_{ik} \frac{\partial^2 V^u}{\partial B_{\lambda_i} \partial B_{\lambda_k}} \Bigg) \mathrm{d}t + \sum_{i=1}^{m} \frac{\partial V^u}{\partial B_{\lambda_i}} \mathrm{d}B_{\lambda_i}. \end{aligned} \tag{5.18}$$

Define the stochastic process $Z(t) = \pi(t) - V^u(t, B_{\lambda_1}, B_{\lambda_2}, ..., B_{\lambda_m})$, then it follows from (5.17) and (5.18) that:

$$\mathrm{d}Z = \left(\frac{\partial V^u}{\partial t} + \sum_{i=1}^{m} \frac{1}{2}\frac{\partial^2 V^u}{\partial {B_{\lambda_i}}^2} + \sum_{i=1}^{m}\sum_{k=i+1}^{m} \rho_{ik}\frac{\partial^2 V^u}{\partial B_{\lambda_i}\partial B_{\lambda_k}} \right.$$
$$\left. - rV^u\right)\mathrm{d}t + \sum_{i=1}^{m}\left(\frac{\partial V^u}{\partial B_{\lambda_i}} - \pi_i(t)^i\sigma^c(t)\right)\mathrm{d}B_{\lambda_i}.$$

Since (5.15) holds, we can derive that $\mathrm{d}Z = 0$. Then the uniqueness of the canonical decomposition for continuous semi-martingales leads to the partial differential equation given by (5.13) must be hold. □

Comparing the **PDE** (5.13) with the **PDE** (2.3), we can note that taking advantage of the partial dependence structure leads to dimension reduction about the **PDE** of the European basket derivative. However, working in the partial dependence structure does not provide the realistic derivative price, it can just provide an approximation to the European derivative price. This approximation should be an upper bound, but it still needs to be proved. In next chapter, we consider $m = 2$, and we use finite difference scheme to numerically derive the approximation of the European basket derivative price.

CHAPTER 6

THE PRICE OF THE EUROPEAN BASKET DERIVATIVES WRITTEN ON STOCK BASKET WITH 2 COMONOTONIC GROUPS

6.1 The PDE for the European basket derivative

Consider the *stock basket* defined in (5.4) given $m = 2$, the stocks in the same group are comonotonic. Assume that:

$$\mathbb{E}[\mathrm{d}B_1(t)\mathrm{d}B_2(t)] = \rho \mathrm{d}t.$$

Since the basket consists of the 2 comonotonic groups, the time-t basket derivative price denoted by $V^u(t, B_{\lambda_1}, B_{\lambda_2})$ written on the basket we defined above satisfies the following **PDE**, which is derived directly from **PDE** (5.13) by using the relation $m = 2$:

$$\frac{\partial V^u}{\partial t} + \sum_{i=1}^{2} \frac{1}{2} \frac{\partial^2 V^u}{\partial B_{\lambda_i}^2} + \rho \frac{\partial^2 V^u}{\partial B_{\lambda_1} \partial B_{\lambda_2}} - rV^u = 0, \tag{6.1}$$

where the V^u is the simplified notation of $V^u(t, B_{\lambda_1}, B_{\lambda_2})$ and the final condition is given by:

$$V^u(T, B_{\lambda_1}, B_{\lambda_2}) = H\left(\sum_{i=1}^{2} \sum_{j=1}^{n_i} \omega_{ij} S_{ij}(0) e^{(r - \frac{1}{2}\sigma_{ij}^2)t + \sigma_{ij} B_{\lambda_i}(t)} \right). \tag{6.2}$$

Our aim is to derive the price of the European type basket derivative written on the stock basket with 2 comonotonic groups, hence we need to derive the explicit solution for the backward **PDE** (6.1) with final condition (6.2). However, in order to get the solution of the 2-dimensional partial differential equation, we need to solve 2-dimensional integral problem, it is not easy to derive the solution of such an integral. Instead, we use the finite difference scheme to get the price of the European basket derivative written on the *stock basket* with 2 comonotonic groups.

6.2 Finite difference scheme for the European basket derivative

In the remainder of this section, we take advantage of the following explicit finite difference method to numerically obtain a solution:
Take the time step δt and:

$$t_k = T - k\delta t.$$

Take also a grid with step size δB_1 for B_{λ_1}:

$$b_{1i} = (i - I_1)\delta B_1.$$

And take a grid with step size δB_2 for B_{λ_2}:

$$b_{2j} = (j - J_1)\delta B_2,$$

where I_1, J_1, I, J are some integers. The value for the comonotonic basket derivative price $V^u(t_k, b_{1i}, b_{2j})$ of the contract at the grid point (t_k, b_{1i}, b_{2j}) is denoted by $V_{i,j}^k$. In the following theorem, we show the explicit expression for $V_{i,j}^{k+1}$ given the values $V_{i+1,j+1}^k$, $V_{i,j+1}^k$, $V_{i+1,j}^k$, $V_{i-1,j}^k$, $V_{i,j-1}^k$.

Theorem 6.1. *Given the values $V_{i+1,j+1}^k$, $V_{i,j+1}^k$, $V_{i+1,j}^k$, $V_{i-1,j}^k$, $V_{i,j-1}^k$, we can derive the expression for $V_{i,j}^{k+1}$, which is described as:*

$$\begin{aligned} V_{i,j}^{k+1} = \rho\frac{\delta t}{\delta B_1 \delta B_2} V_{i+1,j+1}^k + \left(\frac{1}{2}\frac{\delta t}{\delta B_2^2} - \rho\frac{\delta t}{\delta B_1 \delta B_2}\right) V_{i,j+1}^k \\ + \left(\frac{1}{2}\frac{\delta t}{\delta B_1^2} - \rho\frac{\delta t}{\delta B_1 \delta B_2}\right) V_{i+1,j}^k + \frac{1}{2}\frac{\delta t}{\delta B_1^2} V_{i-1,j}^k \\ + \frac{1}{2}\frac{\delta t}{\delta B_2^2} V_{i,j-1}^k + \left(1 - r\delta t - \frac{\delta t}{\delta B_1^2} - \frac{\delta t}{\delta B_2^2} + \rho\frac{\delta t}{\delta B_1 \delta B_2}\right) V_{i,j}^k, \end{aligned} \tag{6.3}$$

Proof. The partial derivatives in $(t, B_{\lambda_1}, B_{\lambda_2})$ can be approximated as followings:

$$\frac{\partial V^u}{\partial t} = \frac{V_{i,j}^k - V_{i,j}^{k+1}}{\delta t} + O(\delta t); \tag{6.4}$$

$$\frac{\partial^2 V^u}{\partial B_{\lambda_1}^2} = \frac{V_{i+1,j}^k - 2V_{i,j}^k + V_{i-1,j}^k}{\delta B_1^2} + O(\delta B_1^2); \tag{6.5}$$

$$\frac{\partial^2 V^u}{\partial B_{\lambda_2}^2} = \frac{V_{i,j+1}^k - 2V_{i,j}^k + V_{i,j-1}^k}{\delta B_2^2} + O(\delta B_2^2); \tag{6.6}$$

$$\frac{\partial^2 V^u}{\partial B_{\lambda_1} \partial B_{\lambda_2}} = \frac{V_{i+1,j+1}^k - V_{i,j+1}^k - V_{i+1,j}^k + V_{i,j}^k}{\delta B_1 \delta B_2}. \tag{6.7}$$

By plugging the (6.4), (6.5), (6.6), (6.7) into the PDE (6.1) we can derive that:

$$\begin{aligned} -rV_{i,j}^k + \frac{V_{i,j}^k - V_{i,j}^{k+1}}{\delta t} + \frac{1}{2}\frac{V_{i+1,j}^k - 2V_{i,j}^k + V_{i-1,j}^k}{\delta B_1^2} \\ +\frac{1}{2}\frac{V_{i,j+1}^k - 2V_{i,j}^k + V_{i,j-1}^k}{\delta B_2^2} + \rho\frac{V_{i+1,j+1}^k - V_{i,j+1}^k - V_{i,j+1}^k + V_{i,j}^k}{\delta B_1 \delta B_2} = 0, \end{aligned} \tag{6.8}$$

where the final condition $V_{i,j}^0$ characterizing the contract follows from the pay-off function:

$$V_{i,j}^0 = H\left(\sum_{a=1}^{n_1} \omega_{1a} S_{1a}(0) e^{(r-\frac{1}{2}\sigma_{1a}^2)T + \sigma_{1a} b_{1i}} + \sum_{k=1}^{n_2} \omega_{2k} S_{2k}(0) e^{(r-\frac{1}{2}\sigma_{2k}^2)T + \sigma_{2k} b_{2j}}\right). \tag{6.9}$$

By considering the appropriate boundary conditions, we can derive the explicit expression for $V_{i,j}^{k+1}$ shown in (6.3), for $i = 1, 3, ..., I-1$, $j = 1, 3, ..., J-1$ and $k = 0, 1, ..., N-1$. □

For given risk-free interest rate r, maturity T and strike K, we can use the finite difference scheme we described above to numerically solve the approximation V^u, but if we consider different risk-free rates r, different maturities T and different strikes K, we need to recompute the derivative prices by finite difference scheme. Hence, it takes much time to recalculate the derivative prices since the calculations need to be repeated many times. On the other hand, the derivative to be priced by finite difference scheme based on few parameters, such as risk-free rate r, strike K and maturity T. Referred to J. De Spiegeleer et al (2018) [8], for the purpose of saving time, one can employ Gaussian process regression to price basket options given the variance gamma setting. Hence, in next chapter, we consider deploying Gaussian process regression to finite difference scheme to speed up the pricing process. However, the increasing of the speed leads to some decreasing of the accuracy. We show the error is still in a reasonable limits.

CHAPTER 7

GAUSSIAN PROCESS REGRESSION FOR APPROXIMATING THE PRICE OF A EUROPEAN BASKET DERIVATIVE

7.1 Gaussian process

We first review the definition of the Gaussian process. A stochastic process $X = \{X_t | t \in A\}$ is said to be Gaussian process if and only if for every finite set of indices $t_1, ..., t_k$ in the index set A, $X_{t_1,...,t_k} = (X_{t_1}, X_{t_2}, ..., X_{t_k})$ is a multivariate Gaussian random variable. A random vector $X = (X_1, ..., X_k)^T$ is said to be a multivariate Gaussian random variable if there exists $\mu \in \mathbb{R}^k$, $B \in \mathbb{R}^{k\times l}$ such that $X = BZ + \mu$ for $Z = (Z_1, Z_2, ..., Z_l)$, $Z_n \sim \mathcal{N}(0, 1)$, $n = 1, 2, ..., l$, i.i.d.

7.2 Gaussian process regression

Consider a brief review of the Gaussian process regression (GPR). One can refer to Raussen and Williams (2006) [7] to get a detailed explanation. Unlike many supervised machine learning algorithms that learn exact value for the parameters in a function, GPR is a Bayesian approach infers a probability distribution over all possible values. We first introduce how the Bayesian approach works. We consider a training set of observations $(X, y) = \{(x_i, y_i)|i = 1, ..., n\}$, where each x_i is an input vector of dimension d and y_i is the corresponding output. We first assume a linear function $y = x\beta + \epsilon$, where x is the input and y is the corresponding output. AS a Bayesian approach, it needs to specify a prior distribution, $p(\beta)$ for the parameter β and calculate $p(\beta|y, X)$ based on the observed data considering the Bayes' Rule:

$$p(\beta|y, X) = \frac{p(y|X, \beta)p(\beta)}{p(y|X)}. \tag{7.1}$$

The updated distribution $p(\beta|y, X)$ is called the posterior distribution, which incorporates the information from both the prior distribution and the observed data. In order to get predictions at a new input $x^\star$, the distribution for the prediction $f^\star$ can be calculated by weighting all possible predictions by the calculated posterior distribution given in (7.1):

$$p(f^\star|x^\star, y, X) = \int_\beta p(f^\star|x^\star, \beta)p(\beta|y, X)\mathrm{d}\beta$$

Note that the prior and likelihood are often assumed to be Gaussian, we can derive a Gaussian distribution for the predictive distribution given this assumption, hence we can use its corresponding mean to obtain a point prediction.

Gaussian process regression is a non-parametric, which means that it is not limited by functional form, and it is a Bayesian approach. Hence rather than calculating the probability distribution of parameters of a specific function, GPR calculates the probability distribution over all admissible functions that fit the training data. However, as a Bayesian approach, we need to specify a prior on the function space, calculate the posterior using the training data, and compute the predictive posterior distribution on the new inputs.

We still consider a training set of observations $(X, y) = \{(x_i, y_i)|i = 1, ..., n\}$, where each x_i is an input vector of dimension d and y_i is the corresponding output. In GPR, the relation between inputs and outputs is assumed to be given by:

$$y_i = f(x_i) + \varepsilon_i,$$

where $f(x) = \{f(x_1), f(x_2), ..., f(x_n)\}$ is assumed to be Gaussian process prior and $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$, for $i = 1, 2, ..., n$, are i.i.d. random variables representing the noise in the data. To be more concrete, as the definition of the Gaussian process, the index set A here is the labels of the training set inputs, any subset of $(f(X_1), f(X_2), ..., f(X_n))$ are joint Gaussian distributed. And the process $f(x)$ is unambiguously determined by its mean function $m(x)$ and a covariance or kernel function $k(x, x')$. The mean function is usually assumed to be zero and we often assume the square exponential kernel as the kernel function. For a sample $(X, f) = \{(x_i, f_i)|i = 1, ..., n\}$ generated from $f(x)$, we have:

$$f \sim \mathcal{N}(0, K(X, X)),$$

where $K(X,X)$ is the covariance matrix, and defined as:

$$K(X,X) = \begin{bmatrix} k(x_1,x_1) & \cdots & k(x_1,x_n) \\ k(x_2,x_1) & \cdots & k(x_2,x_n) \\ \vdots & \ddots & \vdots \\ k(x_n,x_1) & \cdots & k(x_n,x_n) \end{bmatrix}.$$

Next, we consider a $n^\star \times d$ testing input matrix $X^\star$, which consists of $n^\star$ observations, and we denote the corresponding vectors of unknown function values by $f^\star$, then the joint distribution of training outputs y and function values $f^\star$ is multivariate normal distribution:

$$\begin{bmatrix} y \\ f^* \end{bmatrix} = \mathcal{N}\left(0, \begin{bmatrix} K(X,X)+\sigma_n^2 I & K(X,X^*) \\ K(X^*,X) & K(X^*,X^*) \end{bmatrix}\right).$$

Since we choose a Gaussian process prior, calculating the predictive distribution $f^*|X^*,X,y$ is manageable, and leads to a normal distribution that can be completely determined by the mean $m(x)$ and covariance $k(x,x')$:

$$\begin{aligned} f^*|X^*,X,y \sim \mathcal{N}(K(X^*,X)[K(X,X)+\sigma_n^2 I]^{-1}y, \\ K(X^*,X^* - K(X^*,X)[K(X,X)+\sigma_n^2 I]^{-1}K(X,X^*)). \end{aligned} \tag{7.2}$$

Note that the predictions of testing set are the mean of the distribution given in (7.2).

The kernel function we used is the squared exponential (SE) kernel which is given by:

$$\begin{aligned} k(x,x') &= \sigma_f^2 e^{\frac{-|x-x'|^2}{2l^2}} \\ &= \sigma_f^2 e^{\frac{-\sum_{k=1} d(x_k - x'_k)^2}{2l^2}}, \end{aligned}$$

where σ_f^2 is the highest possible covariance and l is a length-scale parameter that determines the smoothness of the fit. We estimate them by considering the training data and maximizing the marginal (log-)likelihood. The marginal (log-)likelihood is given by:

$$-0.5\log(\det(K(X,X))) - 0.5y^T K(X,X)^{-1}y$$

There are several main advantages of Gaussian process regression. First, GPR can take fuller account of the uncertainties related to models and parameter values. Second, GPR is able to involve the prior information. Furthermore, after training the GPR model, it is fast to get the predictions of new inputs.

For approximating a European basket derivative price, given different risk rates r, maturities T and strikes K, using finite difference scheme takes much time to derive the corresponding approximations. On the other hand, the finite difference scheme can be summarized to several parameters (e.g. risk-free rate r and etc.), which is appropriate for deploying Gaussian process regression.

7.3 Gaussian process regression for getting the approximation of the European basket option

We use an example to illustrate how to deploy GPR to get the approximation of the European basket derivative when we consider different risk-free rates r, maturities T and strikes K. Consider a European-type basket call option, written on a four-stock basket with parameters shown in Table 7.1.

Table 7.1: The four-basket call option

stocks	1	2	3	4
Volatilities	0.02	0.03	0.03	0.05
Spot prices	100	100	100	100
Weights	0.25	0.25	0.25	0.25
Maturity	1			
Risk-free rate	0.05			

In addition, the correlations are defined as: $\rho_{i,j} = 0.3$, for $i = 1, 2$, $j = 3, 4$. $\rho_{1,2} = 0.8$, $\rho_{3,4} = 0.9$. Moreover, we assume there are two groups, the first group consists of stock 1 and stock 2, and the second group contains stock 3 and stock 4.

The model procedure starts by constructing a training set (X, y). The input matrix X consists of 1000 combinations of different risk-free rates r, strikes K and maturities T. Random parameter combinations are sampled uniformly over the ranges given in Table 7.2. Then we use finite difference scheme described in chapter 6 to get the approximations V^u as the approximations of European basket derivative prices, which will be the training outputs y.

Table 7.2: Parameter ranges for the training set

parameter	min	max	number
r	0.01	0.06	1000
T	$\frac{5}{6}$	$\frac{7}{6}$	1000
K	80	120	1000

We also construct a testing set $(X_\star, y_\star)$ with size equals to 1000 by the similar method of constructing the training set. And the parameter combinations $X_\star$ are sampled uniformly over the ranges given in Table 7.3.

Table 7.3: Parameter ranges for the testing set

parameter	min	max	number
r	0.02	0.05	1000
T	$\frac{11}{12}$	$\frac{13}{12}$	1000
K	90	110	1000

For training set, we use GPR to do regression and then we deploy this fitted model to test matrix $X_\star$ and compare the results and time of GPR with finite difference scheme respectively.

The pricing performance is measured in terms of maximum, minimum and average absolute errors (MAE, MME and AAE), for both the training set (in-sample prediction) and a testing set (out-of-sample prediction), which are defined as:

$$\text{MAE} = \max |\text{EBO}_{\text{FD}}(i) - \text{EBO}_{\text{GPR}}(i)|,$$

$$\text{MME} = \min |\text{EBO}_{\text{FD}}(i) - \text{EBO}_{\text{GPR}}(i)|,$$

$$\text{AAE} = \frac{\sum_{i=1}^{n} |\text{EBO}_{\text{FD}}(i) - \text{EBO}_{\text{GPR}}(i)|}{n},$$

where the $\mathrm{EBO}_{\mathrm{FD}}$ represents the approximation of the European basket option price by finite difference scheme, and the $\mathrm{EBO}_{\mathrm{GPR}}$ represents the approximation of the European basket option price by GPR.

We compare finite difference scheme performance with the Gaussian process regression performance in Table 7.4.

Table 7.4

	training set	testing set
size	1000	1000
MAE	0.0011	0.0027
MME	6.7348×10^{-8}	4.3592×10^{-7}
AAE	3.3854×10^{-5}	2.9746×10^{-5}

In addition, using the finite difference scheme to get testing outputs costs around 9 hours 22 mins (MacBook Pro 13-inch, 2018, Four Thunderbolt 3 Ports) and if we use GPR to get the predictions of the testing set, it just costs 0.032216 seconds. (MacBook Pro 13-inch, 2018, Four Thunderbolt 3 Ports).

Next, we scatter plot the prices obtained under the finite difference scheme versus the GPR obtained estimates. In a perfect estimation, we would observe a straight line $y = x$, In the following picture, we can see that our 1000 out-of sample predictions by GPR hardly deviates from this straight line.

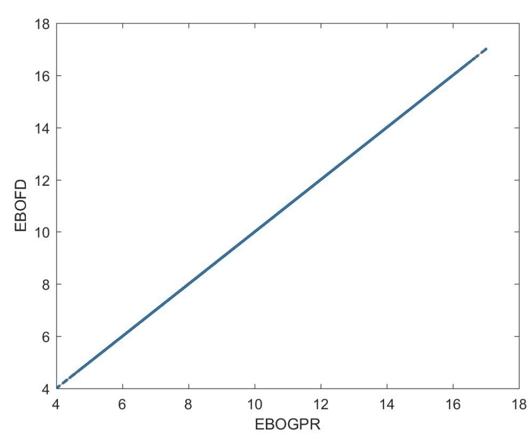

Figure 7.1: Out of sample predictions with GPR

From Tabel 7.4 and Figure 7.1, it shows that the reduced accuracy of predictions by deploying the GPR is well within reliable limits $[4.3593 \times 10^{-7}, 2.7 \times 10^{-3}]$. And it truly saves much time to get the results.

CHAPTER 8

CONCLUSION

We propose an approximation of pricing European basket option with partial dependence structure by deploying Gaussian process regression, we reduce the initial multi-dimensional problem to a 2-dimensional problem, then for given risk-free interests, strikes, maturities, we numerically get the approximations of the European basket option prices by finite difference scheme. For saving more time to get the approximations given different parameters, we employ Gaussian process regression to speed up our methods. We use an example to illustrate the effectiveness of Gaussian process regression and show the loss of accuracy can be accepted.

However, we do not prove that the approximation of a European basket derivative price derived by assuming a partial dependence structure is an upper bound for the realistic European basket derivative price, thus for further application, we still need to find the relation between this approximation and the accurate European basket derivative price. Moreover, we need to consider more different size of training set and testing set to measure the performance of Gaussian process regression. In addition, Gaussian process regression does not perform well in the margin, hence we construct the testing set with a smaller range of parameters compared with the training set. As a result, we still need to explore further on the marginal situations.

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