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SYNOPSIS

Theoretical analyses are presented for the ultimate strength of long hinged reinforced concrete columns. Both concentric and eccentric loads are treated. The treatment of concentrically loaded columns is based on Engesser's tangent modulus formula and the treatment of eccentrically loaded columns is based on the principles advanced by Kármán. Both analyses utilize the stress-strain relationship for concrete determined by Hognestad from tests of short eccentrically loaded columns.

The analyses are compared with the test results of six experimental investigations carried out in the past. The test data, including 48 concentrically and 79 eccentrically loaded long hinged columns, and the analyses are in good agreement, thus indicating that the analyses with their basic assumptions yield reliable results.

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INTRODUCTION

Extensive investigations of short concentrically and eccentrically loaded columns lead in the past to the development of generally applicable analytical expressions for predicting the ultimate strength of short reinforced concrete columns. The experimental studies have shown that for short columns the effect of lateral deflections on the strength is small and that short columns are not in danger of buckling. Accordingly, the analytical expressions disregard both deflections and buckling. On the other hand, it is known that the strength of long columns may be reduced significantly by both of these factors. The analyses presented in this paper take them into account.

Reinforced concrete columns may fail either by material failure or by buckling. Material failure occurs by crushing of concrete in compression. Buckling occurs when the lateral deflection of the column increases without any increase of load.

Buckling of an eccentrically loaded hinged column is illustrated in Fig. 1 in which the moment resistance of a column and the external moment are shown as functions of the deflection at midheight. Figure 1 is drawn for a constant load P applied to a particular column at varying eccentricity. The moment resistance depends, therefore, only on the deflection of the column, while the external moment depends on the sum of the deflection and eccentricity.

The moment resistance is shown in Fig. 1 as the curve 1-2, the external moment as the straight line A-B for eccentricity e and as the straight line E-F for eccentricity e_{cr}; the two lines are parallel because their slope depends only on the magnitude of the load P which is constant. The straight line A-B intersects the curve 1-2 at points C and D for which the moment resistance is equal to the external moment. These two points represent two positions of equilibrium: stable equilibrium at C and unstable equilibrium at D. When the eccentricity of the load P is increased, points C and D move closer together. At the eccentricity e_{cr} line A-B becomes line E-F tangent to the curve 1-2. At an increase of the eccentricity beyond e_{cr} no position of equilibrium is possible. Thus, e_{cr} is the critical eccentricity corresponding to buckling of the column under the load P.

A column fails by material failure if the strength of the column material is exceeded before buckling is reached. For example, if the moment resistance curve in Fig. 1 ended at point C, the column would fail by material failure when the load P has reached eccentricity e.

NOTATION

The following notation is used throughout the paper:

A	=	a parameter given in Table 1
A_s	=	total area of longitudinal column reinforcement
B	=	a parameter given in Table 1
b	=	width of cross-section (Fig. 6)
d	=	total depth of cross-section (Fig. 6)
d'	=	effective depth of reinforcement (Fig. 6)
d''	=	distance between centroids of compression and tension reinforcement (Fig. 6)

e = end eccentricities of load for the case of $e_1 = e_2$ (Fig. 1)

e_1, e_2 = end eccentricities of load; e_2 is taken as the larger of the two values; e_2 is always taken as a positive quantity (Fig. 4)

e_{cr} = critical eccentricity

E_c = modulus of elasticity of concrete

E_s = modulus of elasticity of steel

E_t = tangent modulus of elasticity

f_c = concrete stress

f'_c = compressive strength of 6 x 12-in. concrete cylinders

f''_c = compressive strength of concrete in flexure (Fig. 2)

f'_{cu} = compressive strength of 8-in. concrete cubes

f'_{pr} = compressive strength of concrete prisms

f_s = steel stress

f_{s2} = steel stress corresponding to strain ϵ_2

f_{s3} = steel stress corresponding to strain ϵ_3

f_y = yield point of reinforcement

I = moment of inertia

ℓ = length of column (Fig. 4)

ℓ_1, ℓ_2 = lengths defined in Fig. 4

L = equivalent length defined in Fig. 4

M = bending moment

M_{col} = moment resistance of a column

p = $\frac{A_s}{bd}$, ratio of reinforcement

P = applied load

P_t = critical load computed by the tangent modulus theory

x = distance in the direction of column axis from the point of maximum column deflection (Fig. 4)

y = lateral deflections of column including the initial eccentricity (Fig. 4)

y_m = maximum lateral deflection including the initial eccentricity (Fig. 4)

ϵ_2, ϵ_3 = steel strains (Fig. 6)

ϵ = strain

ϵ_1, ϵ_4 = concrete strains (Fig. 6)

ϵ_o = compressive strain corresponding to maximum concrete stress (Fig. 2)

ϵ_u = ultimate concrete strain in flexure (Fig. 2)

ϵ_y = yield point strain of reinforcement

$\frac{1}{\rho}$ = curvature

$\frac{1}{\rho_m}$ = curvature at $x = 0$

ULTIMATE STRENGTH ANALYSIS

Basic Assumptions

The assumptions for the theories presented in this paper were generally chosen so as to give a lower limit for the ultimate strength of both concentrically and eccentrically loaded long reinforced concrete columns.

Stress-strain relationship for concrete. - The assumed stress-strain relationship for concrete in compression is shown in Fig. 2. It was derived by Hognestad (1) from the tests of short columns subjected to combined bending and axial load. It consists of a parabola and a sloping straight line.

The initial part of the curve in Fig. 2 is a second degree parabola expressed by the following equation:

$$f_c = f_c'' \left[\frac{2\epsilon}{\epsilon_o} - \left(\frac{\epsilon}{\epsilon_o}\right)^2 \right] \qquad (1)$$

The initial modulus of elasticity is taken as

$$E_c = 1,800,000 + 460\ f_c'' \ . \qquad (2)$$

The strain at the maximum stress is

$$\epsilon_o = 2\ f_c'' / E_c \ .$$

Between the strain ϵ_o and the ultimate strain ϵ_u, the stress-strain relationship is a descending straight line. The ultimate strain ϵ_u is taken as 0.0038.

The maximum stress f_c'' is taken as 0.85 times the cylinder strength f_c'. This relationship is believed to be correct only for vertically cast columns.

It is assumed further that concrete resists no tensile stresses. Since the concrete between the cracks carries some tensile stresses, the actual column deflections should be smaller than the computed ones. Thus, the assumption that no tensile stresses exist in the concrete should give somewhat lower ultimate loads than the actual.

The deflection of the column at buckling depends on the magnitude of strains in the concrete and steel. Since the buckling load depends on the deflection and since buckling can occur at almost any value of concrete strain, it is necessary to know the strains with a reasonable accuracy at all loads. Accordingly, a simplified stress block, such as a rectangle or a trapezoid, cannot yield generally accurate results and the use of a more complicated, but also more accurate, Hognestad's stress block is necessary.

Stress-strain relationship for reinforcing steel. - The stress-strain diagram for the reinforcing steel is assumed trapezoidal with the flat portion equal to the yield point stress f_y as shown in Fig. 3. The modulus of elasticity, E_s, is taken in all calculations as 30,000,000 psi.

Bernoulli's hypothesis. - Linear distribution of strains across the column cross-sections is assumed at all load levels.

Shape of deflected column. - It is assumed that the deflected shape of a column is a part of a cosine wave as shown in Fig. 4. This assumption is theoretically correct only for small lateral deflections and for constant modulus of elasticity along the column. Such conditions are satisfied only in concentrically loaded columns.

For eccentrically loaded columns, the use of the cosine wave assumption implies that the modulus of elasticity is constant along the column and equal to the average modulus at the section of failure. The stiffness at the section of failure is the smallest one; therefore, the actual deflections are smaller than the computed ones and the computed ultimate loads should be somewhat smaller than the actual.

Effect of shear on deflection. - For a solid cross-section, the effect of shear on deflections is small. It is disregarded in the theory presented herein.

Concentrically Loaded Columns

Buckling. - A concentrically loaded long compression member remains straight as long as the load remains below the buckling load. When a column subjected to an axial load lower than the critical is deflected slightly from its straight position of equilibrium, its moment resistance increases faster than the corresponding external moment due to the lateral deflection and the column springs back to its straight position. At the buckling or critical load, the increase in external moment due to increase in lateral deflection of the column is equal to the corresponding increase in the moment resistance. The column deflects without further increase of load.

The buckling load for a concentrically loaded column stressed below the proportional limit of the material was determined theoretically by Euler (2,3) in 1744. For columns failing by buckling after the proportional limit has been exceeded, two modifications of the Euler's formula were proposed in 1889: Engesser's tangent modulus formula (4,5) and Considère's double modulus formula (6). Shanley has shown by recent tests of aluminum columns (7,8,9) that the double modulus formula is based on an incorrect assumption and that the tangent modulus formula represents a lower limit for the buckling load of concentrically loaded columns.

It can be seen in Fig. 2 that the assumed stress-strain relationship for concrete has no proportional limit. Accordingly, the tangent modulus formula should be the most suitable for predicting the buckling strength of concentrically loaded reinforced concrete columns.

At any particular concentric load, the modulus of elasticity of the concrete and the moment of inertia are the same at all points.

Accordingly, the following differential equation may be written for small lateral deflections:

$$- \frac{d^2y}{dx^2} E_t I = M \qquad (3)$$

The solution of Eq. 3 gives the buckling load of a concentrically loaded column. The solution depends on the end conditions; for a column hinged at both ends it is

$$P_t = \frac{\pi^2 E_t I}{\ell^2} \qquad (4)$$

where P_t is the buckling load, E_t the tangent modulus of elasticity of concrete at buckling, I the moment of inertia of the transformed cross-section of the column and ℓ the length of the column.

Material failure. - An inspection of the assumed stress-strain relationships for concrete (Fig. 2) shows that a material failure occurs when the strain reaches the ultimate value for concrete $\epsilon_u = 0.0038$. In a concentrically loaded column, however, the material failure may be reached at a lower strain. For example, in a plain concrete column, the maximum material capacity corresponds to the strain ϵ_o; on the other hand, in a column reinforced with very large amount of steel having yield point larger than ϵ_o, the maximum material capacity is reached at yielding of the steel at a strain larger than ϵ_o. Therefore, the limiting value of the strain at failure of a concentrically loaded column depends on the strength of concrete, percentage of reinforcement and the yield point of reinforcement.

The material capacity of columns with reinforcement having a yield point lower than ϵ_o is always reached at the strain ϵ_o. For the practical range of concrete strengths, ϵ_o varies approximately from 1.2 to 2.3 percent. Thus for concentrically loaded columns reinforced with structural or intermediate grade steel, ϵ_o represents the limiting strain.

The material capacity of columns with reinforcement having a yield point higher than ϵ_o may be reached either at the strain ϵ_o or at a higher strain. Up to the strain of ϵ_o, the load carried by each material (i.e., steel and concrete) is increasing with increasing strain. After the strain ϵ_o is exceeded, however, the load carried by the concrete is decreasing while the load carried by the steel continues to increase. For low percentages of reinforcement, the loss in capacity of concrete cannot be compensated for by the increase of the load carried by the reinforcement so that material failure occurs at ϵ_o. On the other hand, for high percentages of steel, the loss in capacity of concrete is compensated for by the increase of the load carried by the reinforcement and the column is able to carry further increases of load beyond the strain ϵ_o.

It can be seen from the preceding discussion, that in most cases material capacity of concentrically loaded columns is reached at the strain ϵ_o. If present, the increase of capacity by straining the column beyond ϵ_o is usually small. It is assumed in the analysis, therefore, that for concentrically loaded columns ϵ_o represents the limiting value. In other words, if a concentrically loaded column does not buckle before reaching the unit shortening of ϵ_o, it fails by material failure.

Evaluation of ultimate load. - The tangent modulus of elasticity E_t in Eq. 4 may be evaluated from the stress-strain diagram for concrete (Fig. 2) as $df_c/d\epsilon$. Therefore, as long as the strain ϵ at ultimate load does not exceed ϵ_o, the tangent modulus is given by the following equation:

$$E_t = E_c \left(1 - \frac{\epsilon}{\epsilon_o}\right) \tag{5}$$

where E_c is the initial modulus of elasticity given by Eq. 2.

For rectangular reinforced concrete columns, the moment of inertia I of the transformed section may be written as:

$$I = \frac{bd^3}{12} + \frac{E_s p}{E_t} \frac{bd(d'')^2}{4} \tag{6}$$

where E_s/E_t is the modular ratio; the modular ratio varies with the load. For strains smaller than the yield point strain of the reinforcement ϵ_y, the modulus of elasticity E_s may be taken as 30×10^6 psi. For strains equal and larger than the yield point strain ϵ_y, E_s is equal to zero. Accordingly, the moment of inertia I decreases suddenly when the yield point strain is reached.

For strains smaller than ϵ_o, the relationship between the concentric load and the corresponding strains may be expressed as

$$\frac{P}{f''_c bd} = \frac{f_s p}{f''_c} + \frac{2\epsilon}{\epsilon_o} - \left(\frac{\epsilon}{\epsilon_o}\right)^2 \tag{7}$$

where f_s is equal to $E_s\epsilon$ for strains smaller than ϵ_y, and equal to f_y for strains equal or larger than ϵ_y.

Equations 4, 5, 6 and 7 contain four unknowns: P, E_t, I and ϵ. Thus, the buckling load may be evaluated by simultaneous solution of all four equations. If the resulting ϵ is larger than ϵ_o, the strength of the column is governed by material failure and its capacity may be evaluated from Eq. 7 by substituting $\epsilon = \epsilon_o$. Similarly, the critical length ℓ for a column of known cross-section and a known value of the load P may be found by a simultaneous solution of Eq. 4-7 for ℓ, E_t, I and ϵ.

Eccentrically Loaded Columns

Buckling. - Under an eccentric loading, a column deflects laterally; the lateral deflection increases with the load. If the stresses in the column remain below the proportional limit, at large deflections the load approaches the value of the buckling load for the same column loaded concentrically.

A general theory for determining the buckling load of an eccentrically loaded column in which the maximum stresses have exceeded the proportional limit of the material was proposed by Kármán (10) in 1910. The theory is based on the stress-strain relationship of the column material and on the assumption of linear strain distribution. The deflected shape of the column is determined by numerical integration along the column length. The validity of Kármán's theory was proved by tests of hinged rectangular steel columns reported by Chwalla (11) in 1934.

The determination of the exact shape of the deflected column by numerical integration is very laborious. Roš and Brunner (12) simplified Kármán's theory by assuming the deflected column shape as a half sine wave. Westergaard and Osgood (13) proposed the use of a part of a cosine wave as the deflected column shape. The sine wave assumption is inconsistent with the actual conditions in that it corresponds to zero curvature, i.e., an infinite stiffness, at the ends of the column. The cosine wave assumption is theoretically correct only at small lateral deflections as long as the stresses in the column remain below the proportional limit.

When the proportional limit is exceeded, the cosine wave assumption gives results on the safe side.

The basic work on buckling of eccentrically loaded reinforced concrete columns was carried out by Baumann (14) in 1930-33. Using Kármán's procedure in conjunction with experimentally determined properties of concrete, he found a good agreement between the tests and the theory. More recently, Ernst, Hromadik and Riveland (15) applied to reinforced concrete columns Kármán's method as modified by Westergaard and Osgood, limiting themselves to hinged columns with equal eccentricities at both column ends and using Hognestad's stress-strain relationship for concrete. The method presented herein is essentially the same as that presented by Ernst, Hromadik and Riveland but it is applicable to hinged columns both with equal and unequal eccentricities at the column ends.

An eccentrically loaded column becomes unstable when one of the following conditions is satisfied:

$$\frac{de}{dy_m} = 0, \ \frac{dP}{dy_m} = 0, \text{ or } \frac{dM}{dy_m} = 0.$$

If the relationship between the load P, moment M and strains is unique, as is assumed in this derivation, all three conditions give the same buckling load.

In the following, the eccentricities at the ends of the column are expressed in terms of the maximum total deflection y_m. The buckling load is then computed from the condition $de/dy_m = 0$.

A deflected hinged column is shown schematically in Fig. 4. Since it is assumed that the deflected shape of the eccentrically loaded column is a part of a cosine wave, it follows that:

$$y = y_m \cos \frac{\pi x}{L} \tag{8}$$

The eccentricities at the column ends are then expressed as:

$$e_1 = y_m \cos \frac{\pi \ell_1}{L} \tag{9}$$

and

$$e_2 = y_m \cos \frac{\pi \ell_2}{L} \tag{10}$$

Observing that $\ell_1 + \ell_2 = \ell$, rearranging and adding of Eq. 9 and 10 gives:

$$\frac{\pi}{L} = \frac{1}{\ell} \left(\arccos \frac{e_1}{y_m} + \arccos \frac{e_2}{y_m}\right) \tag{11}$$

The first derivative of Eq. 11 with respect to y_m includes expressions de_1/dy_m and de_2/dy_m which are equal to zero at buckling. After substituting from Eq. 9 and 10, the derivative of Eq. 11 may, therefore, be written as:

$$\frac{d(\frac{\pi}{L})}{dy_m} = \frac{1}{\ell y_m} \left(\cot \frac{\pi \ell_1}{L} + \cot \frac{\pi \ell_2}{L}\right)$$

or as

$$\frac{y_m}{\frac{\pi}{L}} \frac{d(\frac{\pi}{L})}{dy_m} = \frac{\cot \frac{\pi \ell_1}{L} + \cot \frac{\pi \ell_2}{L}}{\frac{\pi \ell}{L}} \tag{12}$$

Solution of Eq. 12 gives the critical slenderness ℓ, at which a column will buckle under an assumed load, and the corresponding values ℓ_1 and ℓ_2. The corresponding end eccentricities may then be computed from Eq. 9 and 10.

A graphical solution of Eq. 12 is shown in Fig. 5. In Fig. 5a, equation 12 is solved for several values of $\pi \ell/L$ and ℓ_1/ℓ. In Fig. 5b, equations 9 and 10 are solved for several values of the same two quantities. Fig. 5a and 5b are combined into Fig. 5c and 5d relating the left side of Eq. 12 to $\pi \ell/L$ and e_2/y_m, respectively, for several ratios of e_1/e_2.

The left side of Eq. 12 may be evaluated from the relationships between the curvature and the deflection of the column, and between the axial load, moment and deflection. The curvature and the maximum total deflection are related as follows:

$$\frac{1}{\rho} = - \frac{d^2y}{dx^2} = y_m\left(\frac{\pi}{L}\right)^2 \cos \frac{\pi x}{L} \qquad (13)$$

At the point of maximum moment

$$\frac{1}{\rho_m} = y_m\left(\frac{\pi}{L}\right)^2 \qquad (14)$$

Since the curvature may be expressed also as

$$\frac{1}{\rho_m} = \frac{\epsilon_4 - \epsilon_1}{d}$$

Eq. 14 may be written in the following form

$$\frac{\epsilon_4 - \epsilon_1}{d} = y_m\left(\frac{\pi}{L}\right)^2 \qquad (15)$$

where ϵ_4 and ϵ_1 are the strains at the section of maximum moment and d is the depth of the column (Fig. 6).

At the point of maximum moment, the relation between the load, moment and maximum total deflection may be expressed as

$$y_m = \frac{M_{col}}{P} \tag{16}$$

The relationship between P, M_{col} and the strains may be determined from statics and the stress-strain relationship for the particular material as is shown in a later part of this paper.

The critical eccentricity and the length of the column, for which an assumed load is the buckling load, may be determined from Eq. 15 and 16 with the aid of Fig. 5c,d. For a constant value of P, several corresponding values of $(\epsilon_4 - \epsilon_1)$ are chosen and the corresponding values of M_{col} are computed. For each value of $(\epsilon_4 - \epsilon_1)$ the deflection y_m is determined from Eq. 16 and the value of π/L is determined from Eq. 15. For each set of corresponding values of y_m and π/L, the quantity

$$\frac{y_m}{\frac{\pi}{L}} \frac{d(\frac{\pi}{L})}{dy_m}$$

is computed. Finally, Fig. 5c and 5d are entered with the vertical ordinate, and the critical length ℓ and eccentricity e_2 are evaluated from the horizontal ordinates.

If dimensionless ratios y_m/d, $\pi d/L$, $M_{col}/f''_c bd^2$ and $P/f''_c bd$ are used instead of y_m, π/L, M_{col} and P, respectively, the method of solution is unaffected since the right side of Eq. 12 remains unchanged. However, the results are obtained in terms of ratios ℓ/d and e_2/d instead of ℓ and e_2.

The equations derived above are a general solution for buckling of eccentrically loaded hinged columns. They are applicable to columns of any cross-section and of any material.

Relationships between loads, moments and strains. - The distribution of stresses in a reinforced concrete column is determined by two strains, the stress-strain relationships for concrete (Fig. 2) and steel (Fig. 3), and the cross-section of the column. The load P and moment M_{col} are expressed in terms of strains by summation of the stresses in the cross-section.

For rectangular cross-sections with symmetrical reinforcement illustrated in Fig. 6, the following equations may be derived:

$$\frac{P}{f_c''bd} = \frac{p(f_{s2} + f_{s3})}{2f_c''} + \frac{A}{\epsilon_4 - \epsilon_1} \qquad (17)$$

$$\frac{M_{col}}{f_c''bd^2} = \frac{p(f_{s3} - f_{s2})}{4f_c''}\frac{d''}{d} + \frac{B}{(\epsilon_4 - \epsilon_1)^2} - \frac{\epsilon_4 + \epsilon_1}{(\epsilon_4 - \epsilon_1)^2}\frac{A}{2} \qquad (18)$$

where parameters A, B and steel stresses f_{s2} and f_{s3} are known functions of concrete strains ϵ_1 and ϵ_4.

Formulas for parameters A and B are listed in Table 1. Four expressions are given for each parameter; each formula is applicable within certain limits for ϵ_4 and ϵ_1. Formulas for cases 1 and 3 refer to full cross-section in compression, while formulas for cases 2 and 4 refer to a cross-section partly in compression and partly in tension.

The steel stresses are related to steel strains ϵ_2 and ϵ_3 as $f_{s2} = E_s\epsilon_2$ and $f_{s3} = E_s\epsilon_3$, and strains ϵ_2 and ϵ_3 may be expressed as

$$\epsilon_2 = \epsilon_4 - (\epsilon_4 - \epsilon_1)\frac{d'}{d} \qquad (19)$$

$$\epsilon_3 = \epsilon_1 + (\epsilon_4 - \epsilon_1) \frac{d'}{d} \tag{20}$$

It should be noted from Fig. 3 that the steel stresses f_{s2} and f_{s3} have an upper limit equal to the yield point stress f_y.

In the derivation of Eq. 17 and 18, compressive strains were taken as positive quantities and tensile strains as negative quantities. The same sign convention should be followed in using these equations.

Material failure. - An eccentrically loaded column fails by crushing of concrete when the maximum strain ϵ_4 reaches the ultimate value for concrete ϵ_u. The load and moment at failure of such a column may be computed by solution of Eq. 9, 10, 15, 16, 17, 18, 19 and 20 after placing $\epsilon_4 = \epsilon_u$. This solution differs from the well-known equations for short columns (1) only in that it accounts automatically for the deflections of the column.

GRAPHS OF ULTIMATE LOADS

The procedure for eccentrically loaded columns described in the preceding section does not lend itself easily to a direct evaluation of the buckling load for a particular column. It is more suitable for correlating the ultimate loads to the critical lengths and eccentricities for a whole group of columns having the same properties of the cross-section and the same characteristics of loading other than the magnitude of the initial eccentricity. The ultimate load for a particular column of the group may then be obtained by interpolation.

It is not necessary to make a separate set of calculations for each particular column cross-section. If dimensionless quantities are used instead of the actual loads, moments and dimensions, one group pertains to all columns having the same properties of concrete, properties of steel, percentage of reinforcement, effective depth ratio d'/d and conditions of loading. For convenient use, the results may then be tabulated for each group or plotted as is illustrated in Fig. 7-10.

Twenty-eight dimensionless graphs were prepared for rectangular columns with symmetrical reinforcement; eight representative graphs are included in Fig. 7-10. The ultimate load reduced to the dimensionless quantity $P/f_c''bd$ is plotted as a function of the slenderness ratio ℓ/d for initial eccentricity ratios e_2/d ranging from 0 to 1.0. The slenderness ratio is covered in each graph in the range of 0 to 60. For f_y = 50,000 psi, e_1/e_2 = 1.0 and d'/d = 0.9, concrete strengths f_c' = 2000, 3000, 4000, 6000 psi and steel percentages p = 1.0, 2.0, 4.0 percent were included in all combinations, thus, giving 12 graphs. The selection of the characteristic quantities for the remaining 16 graphs was aimed in part at checking the theoretical analysis against the available test data and in part at evaluating the effects of several variables.

Effects of Variables

The eight representative graphs included in Fig. 7 through 10 illustrate the effects of slenderness ratio, eccentricity ratio, concrete strength, percentage of reinforcement, yield point of reinforcement, ratios of end eccentricities and long duration of loading. An inspection of the graphs shows that the slenderness ratio ℓ/d and the eccentricity ratio e_2/d are the most important variables. An increase of ℓ/d from 0 to 60 or an increase of e_2/d from 0 to 1.0 may reduce the ultimate load more than 80 percent. The effects of concrete strength f'_c and of the percentage of reinforcement p are illustrated in Fig. 7 and 8, respectively; it can be seen that the ultimate load increases with both of these variables less than in direct proportion. It may be also noted that the effects of e_2/d, f'_c and p on the strength of a long column are of about the same order of magnitude as for a short column.

The effect of the yield point of reinforcement f_y on the ultimate strength depends on the extent of yielding indicated in Fig. 9a by the dotted lines dividing the graph into five areas. In area I, the steel stresses at failure are below the yield point; thus, the yield point has no effect. In area II, all steel yields in compression; in area III, the steel yields in compression on one face of the column; in area IV, the steel yields in compression on one face and in tension on the other face; finally, in area V, the steel yields in tension on one face. It can be seen by comparing Fig. 9a and 9b, that in areas II - V, the ultimate load increases with increasing yield point.

The effect of the ratio of the end eccentricities can be seen by comparing Fig. 9b and 10a. The ultimate load is smallest for $e_1 = e_2$ and largest for $e_1 = -e_2$. In short columns with unequal eccentricities, there is no decrease in ultimate load with the increase of slenderness because such columns fail at the end sections for which the eccentricity at failure is equal to the initial eccentricity. The range of slenderness for which the ultimate load is constant increases with decreasing e_1/e_2 ratios and with increasing eccentricities.

The graphs in Fig. 7, 8, 9 and 10a represent the strength of columns subjected to short-time loading.* Earlier tests have shown that the maximum load which a short eccentrically loaded column can maintain indefinitely is about 10 percent lower than the corresponding short-time strength. The primary effects of creep appear to be a substantial increase of the maximum concrete strain at failure ϵ_u and a relatively small decrease of the maximum concrete stress f''_c (16). Neglecting the small effect of creep on concrete strength, the stress-strain relationship for long-time loading may be taken as shown in Fig. 2 but with the strain ordinates multiplied by a factor of 2. The ultimate loads may then be computed in the same manner as for short-time loading; Fig. 10b represents an ultimate load graph computed in this manner. A comparison of this graph with Fig. 9b illustrates the difference between the maximum load which a column can maintain for a short time and that which can be sustained indefinitely.

A quantitative investigation was made also of the effect of the effective depth ratio d'/d. A decrease of this ratio from 0.9 to 0.8 was found to decrease the ultimate load as much as 10 percent.

EXPERIMENTAL CHECK OF ANALYSES

The theoretical analyses of the strength of concentrically and eccentrically loaded long reinforced concrete columns presented in this paper are based on a set of assumptions derived from several investigations

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* A test to failure in a conventional testing machine carried out in a few hours.

of related problems. To check the accuracy of the analyses, the theoretical ultimate loads were compared with the results of tests of 48 concentrically and 79 eccentrically loaded long hinged columns. The test data were obtained in six independent investigations reported by Baumann (14) in 1934, by Thomas (17) in 1939, by Hanson and Rosenström (18) in 1947, by Rambøll (19) in 1951, by Ernst, Hromadik and Riveland (15) in 1953 and by Gehler and Hütter (20) in 1954. In addition, the analysis was checked against the tests of short columns reported by Hognestad (1) in 1951.

Experimental Data

All six investigations of long columns were carried out with square or rectangular tied columns reinforced with four symmetrically placed bars. The smallest cross-section was 3 x 3 in. and the largest one was 10 x 10 in. The length varied from 74 to 252 in. The concrete was made of Portland cement, sand and gravel. Some of the columns were cast in a horizontal, others in a vertical, position. The strength of concrete was determined by tests of cubes, prisms or cylinders cast, cured and tested together with the columns. In the investigation reported by Hanson and Rosenström, control prisms were cut out of the undisturbed ends of the columns after the test because control cubes indicated an unexpectedly low strength. The tests by Ernst, Hromadik and Riveland were the only investigation in which the strength of concrete was determined from 6 x 12-in. cylinders.

In Rambøll's and some of Baumann's tests, columns were tested in a horizontal position; in those cases, the effect of dead load was counteracted by a suspension system. The other columns were tested in a vertical position. In all tests, the load was applied through hinges usually made of knife edges.

In most eccentrically loaded columns, the load was applied at a certain eccentricity determined by direct measurements.* Of these, in 52 columns, the eccentricities were equal at both column ends ($e_1/e_2 = 1.0$) and in three columns tested by Baumann the eccentricity at one end was equal to zero ($e_1/e_2 = 0$). In 24 tests by Gehler and Hütter, the "eccentrically loaded" columns were subjected to a concentric axial load P and a transverse load H applied at the middepth of the specimen. The horizontal force was kept at one, two or three percent of the vertical load, so that the "initial eccentricity" at middepth was constant throughout the test and equal to

$$e = \frac{H\ell}{4P}$$

The test data provide a wide coverage of six major variables. The following ranges were covered by the 127 long columns:

1. Slenderness, $\ell/d = 11.7 - 40.7$
2. Concrete strength, $f'_c = 2130 - 6900$ psi
3. Ratio of reinforcement, $p = 0.005 - 0.050$
4. Yield point of reinforcement, $f_y = 28.9 - 53.8$ ksi
5. Effective depth, $d'/d = 0.77 - 0.91$
6. Eccentricity, $e_2 = 0 - 0.833$

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* In Thomas' tests, the initial eccentricities were determined also from strain and deflection measurements, but the directly measured initial eccentricities were used in this study.

Both the concentrically and eccentrically loaded columns covered separately almost complete ranges of variables 1 through 5. The minimum eccentricity for the eccentrically loaded columns was $e_2/d = 0.005$.

From the numerous existing tests of short columns, Series II and III of Hognestad's investigation (1) were selected for checking the equations presented in this paper. Columns in both series had a rectangular cross-section 10 x 10 in. reinforced symmetrically with 2.4 and 4.8 percent of longitudinal bars, respectively. All columns were 75 in. long and had equal end eccentricities, i.e., $e_2 = e_1$. The eccentricities varied from 0 to 12.5 inches. All columns were cast and tested in the same vertical position. The strength of concrete was determined by tests of 6 x 12-in. control cylinders.

The slenderness and eccentricity ratios and the ultimate loads reduced to the dimensionless quantity $P_{test}/f''_c bd$ are listed in Table 2 for short columns, in Table 3 for long concentrically loaded columns and in Table 4 for long eccentrically loaded columns.

Calculated Ultimate Loads

Tables 2, 3 and 4 contain also calculated ultimate loads reduced to the dimensionless quantity $P_{calc}/f''_c bd$. It was assumed in computing the theoretical loads for short concentrically loaded columns that the strain at failure ϵ is equal to ϵ_s and that the steel stress at failure f_s is equal to the yield point of reinforcement f_y. The ultimate loads were then computed directly from Eq. 7. In computing the ultimate loads for the short eccentrically loaded columns, the maximum strain in concrete at failure ϵ_4 was assumed equal to $\epsilon_u = 0.0038$. Setting $e_1 = e_2 = e$ and $\ell_1 = \ell_2 = \ell/2$, the eccentricity of load may be expressed from Eq. 9, 15 and 16 as

$$e = \frac{M_{col}}{P} \cos\left[\frac{\ell}{2d}\sqrt{(0.0038 - \epsilon_1)\frac{Pd}{M_{col}}}\right] \qquad (21)$$

For several chosen values of ϵ_1 and f'_c, the ratio M_{col}/P was computed from Eq. 17, 18, 19 and 20. The corresponding eccentricities e causing failure were then computed from Eq. 21. The resulting values were plotted in graphs of $P_{calc}/f'_c bd$ vs. e/d for each series as functions of the concrete strength f'_c. The ultimate loads for the test columns were obtained by entering the graphs with the known eccentricity and concrete strength. It should be noted that the eccentricity e is the eccentricity at the end of the column and, thus, not subject to change during the test.

The theoretical ultimate loads for <u>long concentrically</u> loaded columns were calculated directly from Eq. 4, 5, 6 and 7 for each test column.

The theoretical ultimate loads for <u>long eccentrically</u> loaded columns were obtained from the graphs of ultimate loads. This procedure involved several interpolations and for some columns also extrapolations; therefore, the resulting $P_{calc}/f''_c bd$ values in Table 4 are given only to two decimal places.

The calculated loads are based on the dimensions and material properties of the test columns as reported in the respective test reports. Data not given in the reports were estimated; of these, the most important was the assumption concerning the relationships between the strength of cylinders, prisms and cubes.

The stress-strain relationship assumed in the analysis for concrete is based on the strength f''_c. In computing the ultimate loads for vertically cast columns, this quantity was assumed related to the strength of 6 x 12-in. control cylinders as $f''_c = 0.85\ f'_c$. For horizontally cast columns, the relationship $f''_c = f'_c$ was assumed as indicated by earlier tests (16). Where cylinder strengths were not given, the following relationships were assumed between the strength of cylinders, prisms and cubes: $f'_c = f'_{pr}$ and $f'_c = 0.8\ f'_{cu}$.

Analyses vs. Tests

The ratios of the test to calculated ultimate loads for short columns are listed in the last column of Table 2. For all short concentrically loaded columns, the ratio is equal or smaller than 1.0 and, thus in agreement with Hognestad's observation of the presence of small accidental eccentricities (1). The average ratio for the short concentrically loaded columns is 0.901 and the standard deviation is 0.060. For all short eccentrically loaded columns, the ratios are close to 1.00; the over-all average is 1.000 and the standard deviation is 0.060. It should be noted, however, that a good over-all agreement between the theory and the Hognestad's test data should be expected in view of the fact that the properties of the concrete stress block shown in Fig. 2 were derived from the same investigation (1). On the other hand, the excellent agreement from group to group and for all individual columns is significant in judging the correctness of the theoretical equations and of the basic assumptions.

The analysis of the ultimate strength of long columns is compared with the test data in the last columns of Tables 3 and 4 and in Fig. 11 and 12 giving the ratio P_{test}/P_{calc} for every column. An examination of the tables and figures shows that the scatter is noticeably larger for the concentrically loaded columns; that of the 127 columns tested, six concentrically loaded columns were more than 20 percent weaker and none of the eccentrically loaded columns was more than 11 percent weaker than predicted by the theory; that the average ratio is higher for the eccentrically than for the concentrically loaded columns; and that no definite trends exist with any of the six major variables except as mentioned above.

The comparisons are summarized in Table 5. The average values for the individual investigations show that in all but one case the average is smaller for the concentric than for the eccentric loading. This result is identical with that for short columns tested by Hognestad, thus indicating that it is difficult to obtain concentric hinged loading. Furthermore, the averages for eccentrically loaded columns of every investigation exceed 1.00; this agrees with the fact that several assumptions of the theory represent a minimum condition as far as the ultimate strength is concerned. Finally, the higher standard over-all standard deviation for concentrically loaded columns reemphasizes the smaller spread in the P_{test}/P_{calc} ratios for eccentrically loaded columns. The poorer agreement for the concentrically loaded columns is understandable since differences and small unavoidable errors in testing techniques affect the test value of ultimate load more for a concentrically than for an eccentrically loaded column.

The over-all average value of the ratio of measured to computed ultimate loads of long columns is 1.066 and the standard deviation is 0.153. This, together with the reasonably good agreement for the individual investigations and absence of any definite trends with any major variable, leads to the conclusion that the ultimate load of hinged columns can be predicted accurately by the analyses presented in this paper.

CONCLUDING REMARKS

The ultimate load analyses presented herein were aimed primarily to improve the knowledge of the effect of deflections and buckling on the strength of reinforced concrete columns and to provide a tool for further studies.

The good agreement with the available test data indicates that the analyses are based on reasonable assumptions. Particularly, it serves as a further proof that Hognestad's stress block approximates well the actual stress-strain relationship for concrete subjected to combined axial compression and bending.

As a tool for further studies, the analyses permit the evaluation of the effects of various variables on the column strength. Such studies are indispensable in preparation of simplified design procedures.

And finally, the analyses permit construction of ultimate load tables and graphs, which can be used conveniently whenever it is necessary to evaluate accurately the ultimate loads of hinged reinforced concrete columns.

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$$\frac{P}{f_c'' bd} = \frac{p(f_{s2}+f_{s3})}{2f_c''} + \frac{A}{(\epsilon_4-\epsilon_1)} \qquad \frac{M_{col}}{f_c'' bd^2} = \frac{p(f_{s3}-f_{s2})}{4f_c''}\frac{d''}{d} + \frac{B}{(\epsilon_4-\epsilon_1)^2} - \frac{(\epsilon_4+\epsilon_1)}{(\epsilon_4-\epsilon_1)^2}\frac{A}{2}$$

Case 1 $\epsilon_4 < \epsilon_o$ $\epsilon_1 > 0$	$A = \frac{(\epsilon_4^2-\epsilon_1^2)}{\epsilon_o} - \frac{(\epsilon_4^3-\epsilon_1^3)}{3\epsilon_o^2}$ $\qquad B = \frac{2(\epsilon_4^3-\epsilon_1^3)}{3\epsilon_o} - \frac{(\epsilon_4^4-\epsilon_1^4)}{4\epsilon_o^2}$
Case 2 $\epsilon_4 < \epsilon_o$ $\epsilon_1 < 0$	$A = \frac{\epsilon_4^2}{\epsilon_o} - \frac{\epsilon_4^3}{3\epsilon_o^2}$ $\qquad B = \frac{2\epsilon_4^3}{3\epsilon_o} - \frac{\epsilon_4^4}{4\epsilon_o^2}$
Case 3 $\epsilon_o < \epsilon_4 < \epsilon_u$ $\epsilon_1 > 0$	$A = \frac{2\epsilon_o^3-3\epsilon_o\epsilon_1^2+\epsilon_1^3}{3\epsilon_o^2} + \frac{(\epsilon_u-0.85\epsilon_o)(\epsilon_4-\epsilon_o)-0.075(\epsilon_4^2-\epsilon_o^2)}{(\epsilon_u-\epsilon_o)}$ $B = \frac{2(\epsilon_o^3-\epsilon_1^3)}{3\epsilon_o} - \frac{(\epsilon_o^4-\epsilon_1^4)}{4\epsilon_o^2} + \frac{(\epsilon_u-0.85\epsilon_o)(\epsilon_4^2-\epsilon_o^2)-0.10(\epsilon_4^3-\epsilon_o^3)}{2(\epsilon_u-\epsilon_o)}$
Case 4 $\epsilon_o < \epsilon_4 < \epsilon_u$ $\epsilon_1 < 0$	$A = \frac{2}{3}\epsilon_o + \frac{(\epsilon_u-0.85\epsilon_o)(\epsilon_4-\epsilon_o)-0.075(\epsilon_4^2-\epsilon_o^2)}{(\epsilon_u-\epsilon_o)}$; $\quad B = \frac{5\epsilon_o^2}{12} + \frac{(\epsilon_u-0.85\epsilon_o)(\epsilon_4^2-\epsilon_o^2)-0.10(\epsilon_4^3-\epsilon_o^3)}{2(\epsilon_u-\epsilon_o)}$

Note: Compressive strains should be taken as positive quantities, tensile strains as negative quantities. The steel stresses $f_{s2} = E_s\epsilon_2$ and $f_{s3} = E_s\epsilon_3$ have an upper limit equal to the yield point stress f_y.

TABLE 2

HOGNESTAD'S TESTS OF SHORT COLUMNS (1)

b x d = 10 in. x 10 in., $\ell/d = 7.5$

Specimen	Concrete Strength f'_c, psi.	Eccentricity e_2/d	$\frac{P_{test}}{f''_c bd}$	$\frac{P_{calc}}{f''_c bd}$	$\frac{P_{test}}{P_{calc}}$
		Series II			
B-6a	4080	0	1.314	1.312	1.00
b	4040	0	1.224	1.315	0.93
C-6a	2020	0	1.308	1.630	0.80
b	1520	0	1.566	1.836	0.85
			Average	Col. 6	0.90
A-7a	5240	0.325	0.515	0.553	0.93
b	5810	0.25	0.575	0.622	0.92
B-7a	4080	0.25	0.738	0.689	1.07
b	4040	0.25	0.723	0.692	1.04
C-7a	1970	0.25	0.844	0.881	0.96
b	1520	0.25	0.983	0.922	1.07
			Average	Col. 7	1.00
A-8a	5520	0.50	0.345	0.359	0.96
b	5810	0.50	0.308	0.346	0.89
B-8a	4700	0.50	0.390	0.395	0.99
b	4260	0.50	0.403	0.414	0.97
C-8a	1820	0.50	0.639	0.612	1.04
b	1820	0.50	0.639	0.620	1.03
			Average	Col. 8	0.98
A-9a	5100	0.75	0.205	0.211	0.97
b	5170	0.75	0.208	0.208	1.00
B-9a	4700	0.75	0.235	0.226	1.04
b	4370	0.75	0.241	0.238	1.01
C-9a	1880	0.75	0.456	0.446	1.02
b	1730	0.75	0.446	0.460	0.97
			Average	Col. 9	1.00
A-10a	5100	1.25	0.106	0.102	1.04
b	5170	1.25	0.100	0.101	0.99
B-10a	4260	1.25	0.120	0.121	0.99
b	4370	1.25	0.119	0.119	1.00
C-10a	2300	1.25	0.227	0.222	1.02
b	1770	1.25	0.300	0.252	1.19
			Average	Col. 10	1.04
			Average	Series II	0.989
			Standard	Deviation	0.073

TABLE 2 (Concluded)

Specimen	Concrete Strength f'_c, psi.	Eccentricity, e_2/d	$\frac{P_{test}}{f''_c bd}$	$\frac{P_{calc}}{f''_c bd}$	$\frac{P_{test}}{P_{calc}}$
		Series III			
B-11a	3870	0	1.520	1.636	0.93
b	4010	0	1.422	1.614	0.88
C-11b	2070	0	2.006	2.189	0.92
			Average	Col. 11	0.91
A-12a	4150	0.25	0.892	0.848	1.05
b	5050	0.25	0.758	0.792	0.96
B-12a	4300	0.25	0.828	0.839	0.99
b	4010	0.25	0.833	0.856	0.97
C-12a	2300	0.25	1.286	1.147	1.12
b	2200	0.25	1.230	1.163	1.06
			Average	Col. 12	1.03
A-13a	5350	0.50	0.484	0.514	0.94
b	4850	0.50	0.510	0.541	0.94
B-13a	3580	0.50	0.592	0.639	0.93
b	4290	0.50	0.564	0.571	0.99
C-13a	2300	0.50	0.770	0.800	0.96
b	2070	0.50	0.778	0.829	0.94
			Average	Col. 13	0.95
A-14a	5350	0.75	0.312	0.339	0.92
b	5100	0.75	0.353	0.353	1.00
B-14a	3580	0.75	0.457	0.462	0.99
C-14a	1950	0.75	0.696	0.655	1.06
b	2070	0.75	0.591	0.641	0.92
			Average	Col. 14	0.98
A-15a	5100	1.25	0.203	0.184	1.10
b	4850	1.25	0.192	0.193	0.99
B-15a	3800	1.25	0.229	0.245	0.93
b	4630	1.25	0.214	0.201	1.06
C-15a	1950	1.25	0.437	0.426	1.03
b	2070	1.25	0.423	0.414	1.02
			Average	Col. 15	1.02
			Average	Series III	0.985
			Standard	Deviation	0.061

TABLE 3

TESTS OF CONCENTRICALLY LOADED LONG COLUMNS

Specimen	Concrete Strength f'_c, psi.	Cross-Section b x d in. x in.	Slenderness ℓ/d	$\frac{P_{test}}{f''_c bd}$	$\frac{P_{calc}}{f''_c bd}$	$\frac{P_{test}}{P_{calc}}$
		Baumann, 1930-1933 (14)				
I	2200	3.94 x 7.87	32.1	1.026	0.947	1.08
III	2330	5.51 x 5.51	22.9	1.285	1.221	1.05
V	3830	5.47 x 6.97	23.3	1.172	1.187	0.99
Va	3830	5.51 x 7.01	23.2	1.223	1.191	1.03
VI	3610	3.86 x 7.80	32.7	0.955	0.729	1.31
VIa	3610	3.94 x 7.87	32.1	0.950	0.748	1.27
VIII	4180	7.01 x 7.17	17.5	1.345	1.158	1.16
VIIIa	4180	7.09 x 7.09	15.5	1.531	1.182	1.30
3	4870	6.30 x 9.84	40.7	0.585	0.405	1.44
15	5630	6.34 x 9.72	40.4	0.418	0.386	1.08
				Average		1.171
				Standard Deviation		0.142
		Thomas, 1938 (17)				
LC1	3530	6.00 x 6.00	14.7	1.224	1.288	0.95
LC2	3840	6.00 x 6.00	20.7	1.043	1.125	0.93
				Average		0.940
		Hanson and Rosenstrom, 1945-1946 (18)				
1a	5270	5.16 x 7.13	25.8	0.973	0.713	1.36
2a	5980	5.20 x 7.17	25.6	0.833	0.693	1.20
3a	7110	5.31 x 7.20	25.1	0.750	0.673	1.11
				Average		1.223
		Rambøll, 1949-1950 (19)				
13	4860	5.59 x 7.13	13.0	0.790	1.064	0.74
14	4380	5.59 x 7.13	13.0	1.040	1.080	0.96
15	4220	5.79 x 7.13	12.6	0.983	1.090	0.90
16	4220	5.75 x 7.20	12.7	0.980	1.089	0.90
27	5010	5.55 x 7.17	20.6	0.768	0.897	0.86
28	4880	5.75 x 7.20	19.9	0.642	0.905	0.71
33	4700	5.63 x 7.20	30.1	0.686	0.619	1.11
				Average		0.883
				Standard Deviation		0.125

TABLE 3 (Concluded)

Specimen	Concrete Strength f'_c, psi.	Cross-Section b x d, in. x in.	Slenderness ℓ/d	$\frac{P_{test}}{f''_c bd}$	$\frac{P_{calc}}{f''_c bd}$	$\frac{P_{test}}{P_{calc}}$
		Gehler and Hütter, 1940-1942 and 1951-1952 (20)				
Ia	3270	5.50 x 6.30	40	0.562	0.517	1.09
				0.601		1.16
Ib	3920	5.50 x 6.30	30	0.749	0.701	1.07
				0.776		1.11
Ic	3670	5.50 x 6.30	25	1.033	0.871	1.19
				1.108		1.27
Id	3560	5.50 x 6.30	20	1.042	1.027	1.01
				1.182		1.15
Ie	3560	5.50 x 6.30	15	1.276	1.076	1.19
				1.214		1.13
IIa	3520	5.50 x 6.30	40	0.703	0.672	1.05
				0.754		1.12
1	4250	5.50 x 6.30	40	0.312	0.422	0.74
				0.352		0.83
2	4430	5.50 x 6.30	40	0.376	0.524	0.72
				0.491		0.94
3	4100	5.50 x 6.30	40	0.606	0.794	0.76
				0.531		0.67
4	2690	5.50 x 6.30	30	0.744	0.816	0.91
				0.717		0.88
5	3790	5.50 x 6.30	30	0.626	0.710	0.88
				0.658		0.93
6	4980	5.50 x 6.30	30	0.621	0.640	0.97
				0.621		0.97
				Average		0.989
				Standard Deviation		0.163
		Ernst, Hromadik and Riveland, 1952 (15)				
3	4040	6.00 x 6.00	15	1.046	1.149	0.91
4	4040	6.00 x 6.00	25	0.961	0.845	1.14
				Average		1.025

NOTE: Only specimens with ℓ/d larger than 10 are listed in this table.

TABLE 4

TESTS OF ECCENTRICALLY LOADED LONG COLUMNS

Specimen	Concrete Strength f'_c, psi.	Cross-Section d x b, in. x in.	Slenderness ℓ/d	Eccentricity Ratio e_2/d	$\frac{P_{test}}{f''_c bd}$	$\frac{P_{calc}}{f''_c bd}$	$\frac{P_{test}}{P_{calc}}$
			Baumann, 1930-1933 (14)				
Ia	2290	3.94 x 7.87	32.1	0.083	0.566	0.63	0.90
IIIa	2360	5.51 x 5.51	22.9	0.083	0.869	0.89	0.98
4	4670	9.84 x 9.84	12.0	0.166	0.562	0.63	0.89
5	4640	4.92 x 9.84	25.9	0.166	0.404	0.32	1.26
6	4670	6.30 x 9.84	40.7	0.166	0.206	0.15	1.37
7	3480	9.84 x 9.84	11.7	0.166	0.662	0.68	0.97
8	3480	4.96 x 9.84	25.6	0.166	0.366	0.36	1.02
9	3460	6.38 x 9.84	40.2	0.166	0.251	0.18	1.39
10	5100	9.88 x 9.96	11.7	0.333	0.364	0.41	0.89
11	5100	4.96 x 9.92	25.6	0.333	0.207	0.15	1.38
12	5070	6.38 x 9.84	40.2	0.333	0.094	0.09	1.04
13	5600	9.88 x 9.72	11.8	0.333	0.344	0.38	0.91
14	5600	4.96 x 9.76	25.6	0.333	0.159	0.14	1.14
19*	4070	5.11 x 9.84	24.7	0.166	0.501	0.51	0.98
26*	5070	9.84 x 9.92	12.6	0.166	0.707	0.70	1.01
32*	4680	9.84 x 9.84	12.6	0.166	0.786	0.79	0.99
				Average			1.070
				Standard Deviation			0.175
			Thomas, 1938 (17)				
LC3	3450	6.00 x 6.00	23.7	0.005	1.018	0.99	1.03
LC4	3440	6.00 x 6.00	26.7	0.005	0.994	0.89	1.12
LC8	3330	6.00 x 6.00	14.7	0.027	1.049	1.01	1.04
LC7	3810	6.00 x 6.00	20.7	0.037	0.894	0.88	1.02
LC6	3970	6.00 x 6.00	23.7	0.042	0.830	0.75	1.11
LC5	3640	6.00 x 6.00	26.7	0.047	0.921	0.74	1.24
LC12	3030	6.00 x 6.00	14.7	0.027	1.063	1.13	0.94
LC11	2700	6.00 x 6.00	20.7	0.037	1.139	1.01	1.13
LC10	2730	6.00 x 6.00	23.7	0.042	1.006	0.86	1.17
LC9R	2900	6.00 x 6.00	26.7	0.017	0.912	0.91	1.05
PLC1	2720	3.00 x 3.00	33.1	0.056	0.884	0.87	1.02
PLC2	2800	3.00 x 3.00	33.1	0.056	0.850	0.85	1.00
				Average			1.073
				Standard Deviation			0.063

TABLE 4 (Cont'd.)

Specimen	Concrete Strength f'_c, psi.	Cross-Section d x b, in. x in.	Slenderness ℓ/d	Eccentricity Ratio e_2/d	$\frac{P_{test}}{f''_c bd}$	$\frac{P_{calc}}{f''_c bd}$	$\frac{P_{test}}{P_{calc}}$
			Hanson and Rosenström, 1945-1946 (18)				
1b	5270	5.16 x 7.13	25.8	0.166	0.417	0.30	1.39
2b	5980	5.20 x 7.13	25.6	0.166	0.356	0.28	1.27
3b	7110	5.31 x 7.20	25.1	0.166	0.349	0.26	1.34
				Average			1.333
			Rambøll, 1949-1950 (19)				
17	4280	5.59 x 7.09	13.0	0.083	0.902	0.76	1.19
18	4030	5.67 x 7.13	12.8	0.083	0.866	0.77	1.12
19	4120	5.59 x 7.09	13.0	0.166	0.762	0.60	1.27
20	4150	5.63 x 7.17	12.9	0.166	0.805	0.60	1.34
21	3930	5.71 x 7.20	12.7	0.333	0.481	0.38	1.27
22	3970	5.67 x 7.17	12.8	0.333	0.502	0.38	1.32
23	4000	5.67 x 7.13	12.8	0.666	0.154	0.13	1.18
24	3750	5.67 x 7.13	12.8	0.666	0.164	0.14	1.17
25	4810	5.67 x 7.17	12.8	0.833	0.093	0.08	1.16
26	4560	5.55 x 7.13	13.1	0.833	0.098	0.08	1.16
29	5040	5.67 x 7.17	20.1	0.166	0.431	0.42	1.03
30	4630	5.63 x 7.17	20.3	0.333	0.278	0.24	1.16
31	4980	5.67 x 7.20	20.1	0.666	0.094	0.09	1.04
32	5040	5.59 x 7.20	20.4	0.833	0.074	0.07	1.06
34	5050	5.71 x 7.17	29.6	0.083	0.527	0.38	1.39
35	4540	5.67 x 7.20	29.9	0.166	0.336	0.33	1.02
36	4590	5.63 x 7.20	30.1	0.666	0.080	0.08	1.00
37	4650	5.71 x 7.17	29.6	0.333	0.164	0.15	1.02
38	5540	5.71 x 7.17	29.6	0.833	0.051	0.05	1.09
				Average			1.157
				Standard Deviation			0.118
			Gehler and Hütter, 1940-1942 and 1951-1952 (20)				
7	3020	5.50 x 6.30	15	0.0376	1.196	1.11	1.08
					1.134		1.02
8	4200	5.50 x 6.30	20	0.051	0.873	0.85	1.03
					0.810		0.95
9	4110	5.50 x 6.30	30	0.075	0.597	0.56	1.07
					0.555		0.99
10	3260	5.50 x 6.30	40	0.101	0.357	0.38	0.94
					0.388		1.02
11	3210	5.50 x 6.30	15	0.075	1.029	0.96	1.07
					1.117		1.16

TABLE 4 (Concluded)

Specimen	Concrete Strength f'_c, psi.	Cross-Section $d \times b$, in. x in.	Slenderness ℓ/d	Eccentricity Ratio e_2/d	$\frac{P_{test}}{f''_c bd}$	$\frac{P_{calc}}{f''_c bd}$	$\frac{P_{test}}{P_{calc}}$
12	4200	5.50 x 6.30	20	0.101	0.721	0.72	1.00
					0.735		1.02
13	4390	5.50 x 6.30	30	0.150	0.411	0.42	0.98
					0.418		1.00
14	2830	5.50 x 6.30	40	0.201	0.341	0.32	1.07
					0.306		0.96
15	3210	5.50 x 6.30	15	0.113	0.957	0.87	1.10
					1.043		1.20
16	3020	5.50 x 6.30	20	0.152	0.668	0.71	0.94
					0.777		1.09
17	3710	5.50 x 6.30	30	0.225	0.403	0.38	1.06
					0.358		0.94
18	2890	5.50 x 6.30	40	0.302	0.276	0.26	1.06
					0.266		1.02
					Average		1.032
					Standard Deviation		0.066
		Ernst, Hromadik and Riveland, 1952 (15)					
7	4040	6.00 x 6.00	15	0.125	0.761	0.74	1.03
8	4040	6.00 x 6.00	25	0.250	0.618	0.50	1.24
11	4040	6.00 x 6.00	15	0.250	0.554	0.52	1.07
12	4040	6.00 x 6.00	25	0.250	0.368	0.33	1.12
13	4040	6.00 x 6.00	25	0.375	0.236	0.24	0.98
					Average		1.088
					Standard Deviation		0.089

* End eccentricity $e_1 = 0$; in all other columns $e_1 = e_2$.

TABLE 5

SUMMARY OF P_{test}/P_{calc} FOR LONG COLUMNS

Investigation	Type of Loading	No. of Columns	Test Results, $\frac{P_{test}}{P_{calc}}$ Average	Standard Deviation
Baumann (14)	C[x]	10	1.171	0.142
	E[xx]	16	1.070	0.175
Thomas (17)	C	2	0.965	--
	E	12	1.073	0.063
Hanson and Rosenström (18)	C	3	1.123	--
	E	3	1.333	--
Ramb∅ll (19)	C	7	0.883	0.125
	E	19	1.157	0.118
Gehler and Hütter (20)	C	24	0.989	0.163
	E	24	1.032	0.066
Ernst, Hromadik and Riveland (15)	C	2	1.025	--
	E	5	1.088	0.089

Concentrically Loaded Columns, 48 specimens

Average P_{test}/P_{calc}	1.027
Standard Deviation	0.178

Eccentrically Loaded Columns, 79 specimens

Average P_{test}/P_{calc}	1.090
Standard Deviation	0.130

[x] C = concentrically loaded columns

[xx] E = eccentrically loaded columns

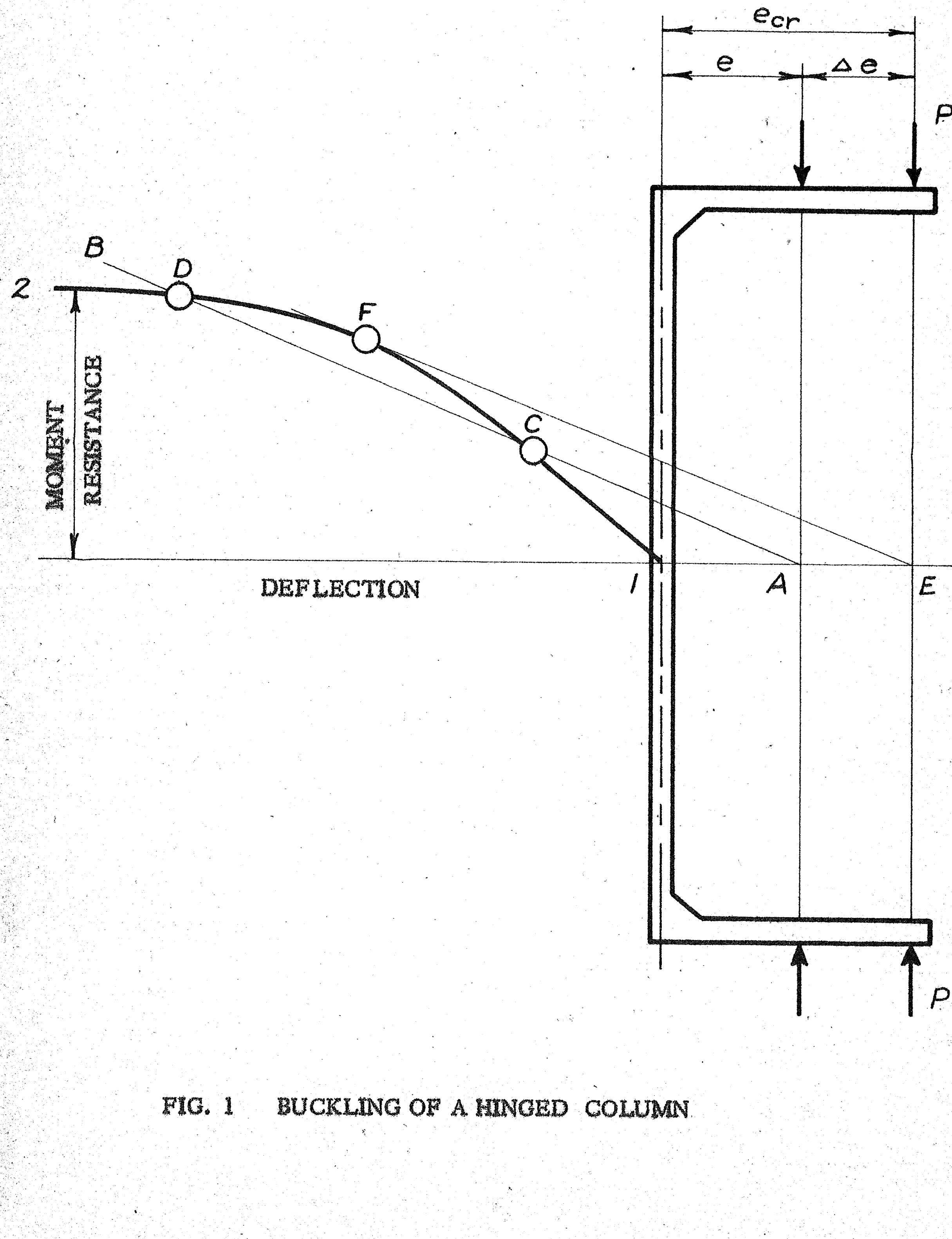

FIG. 1 BUCKLING OF A HINGED COLUMN

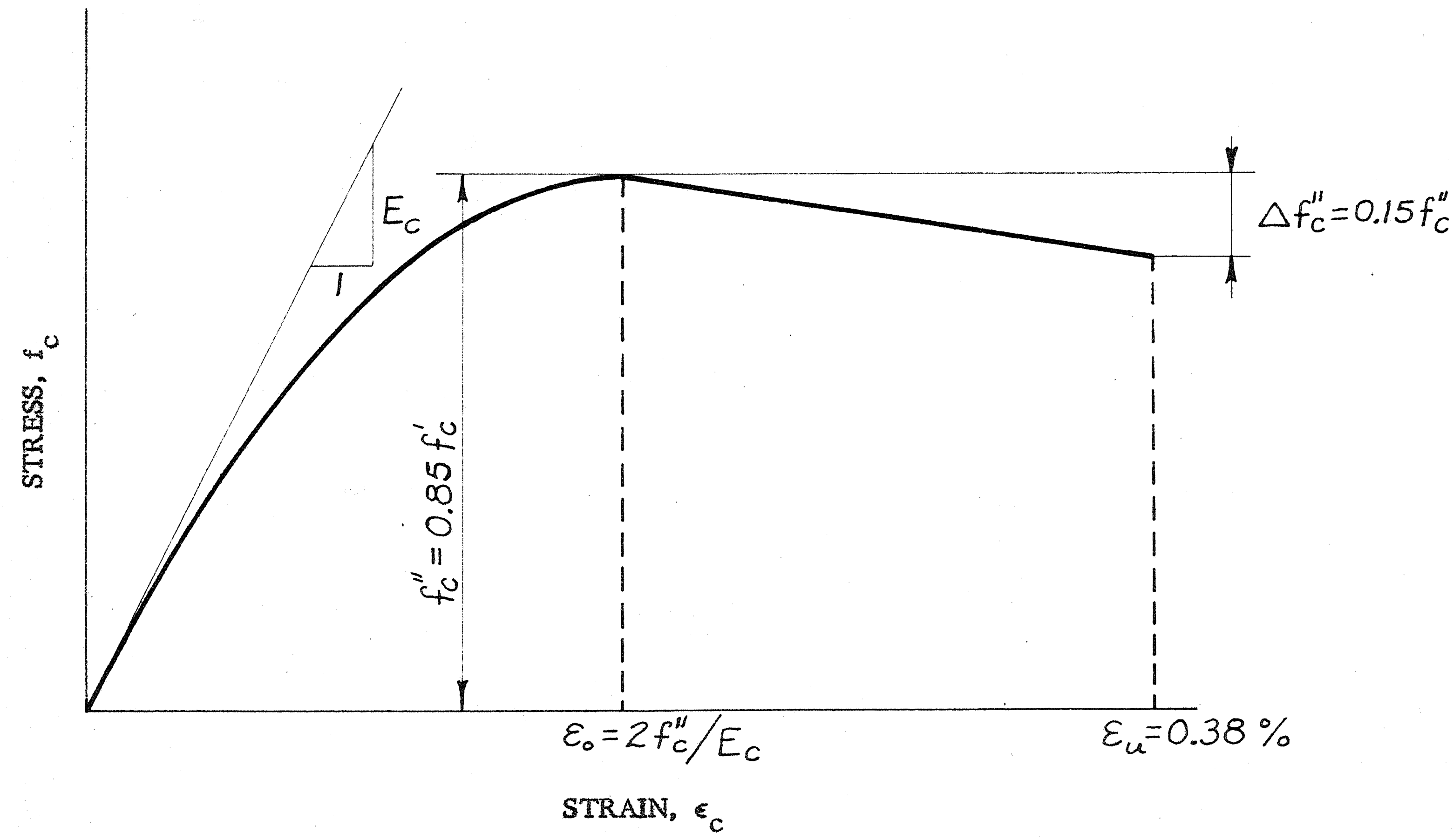

FIG. 2 ASSUMED STRESS-STRAIN RELATIONSHIP FOR CONCRETE

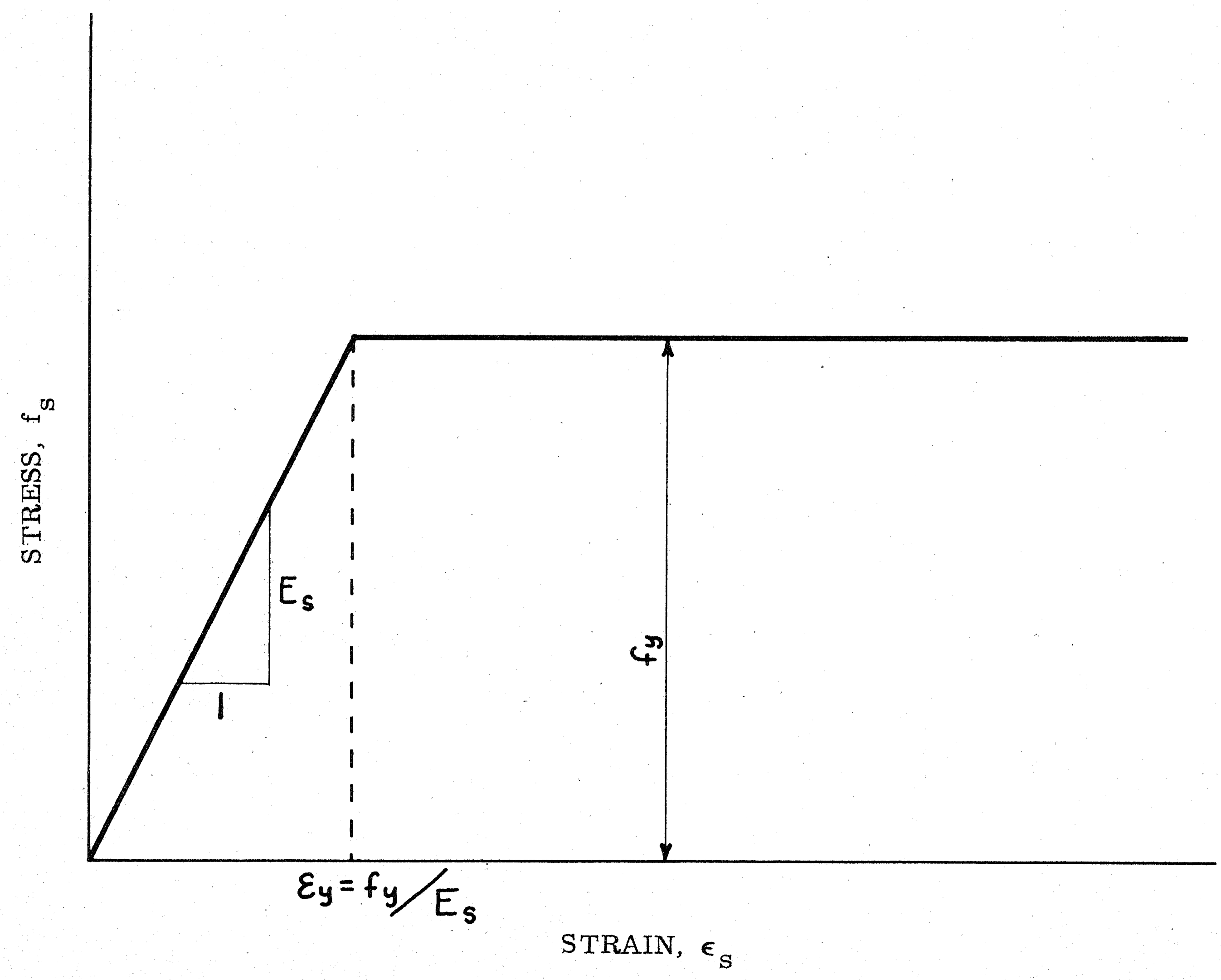

FIG. 3 ASSUMED STRESS-STRAIN RELATIONSHIP FOR REINFORCING STEEL

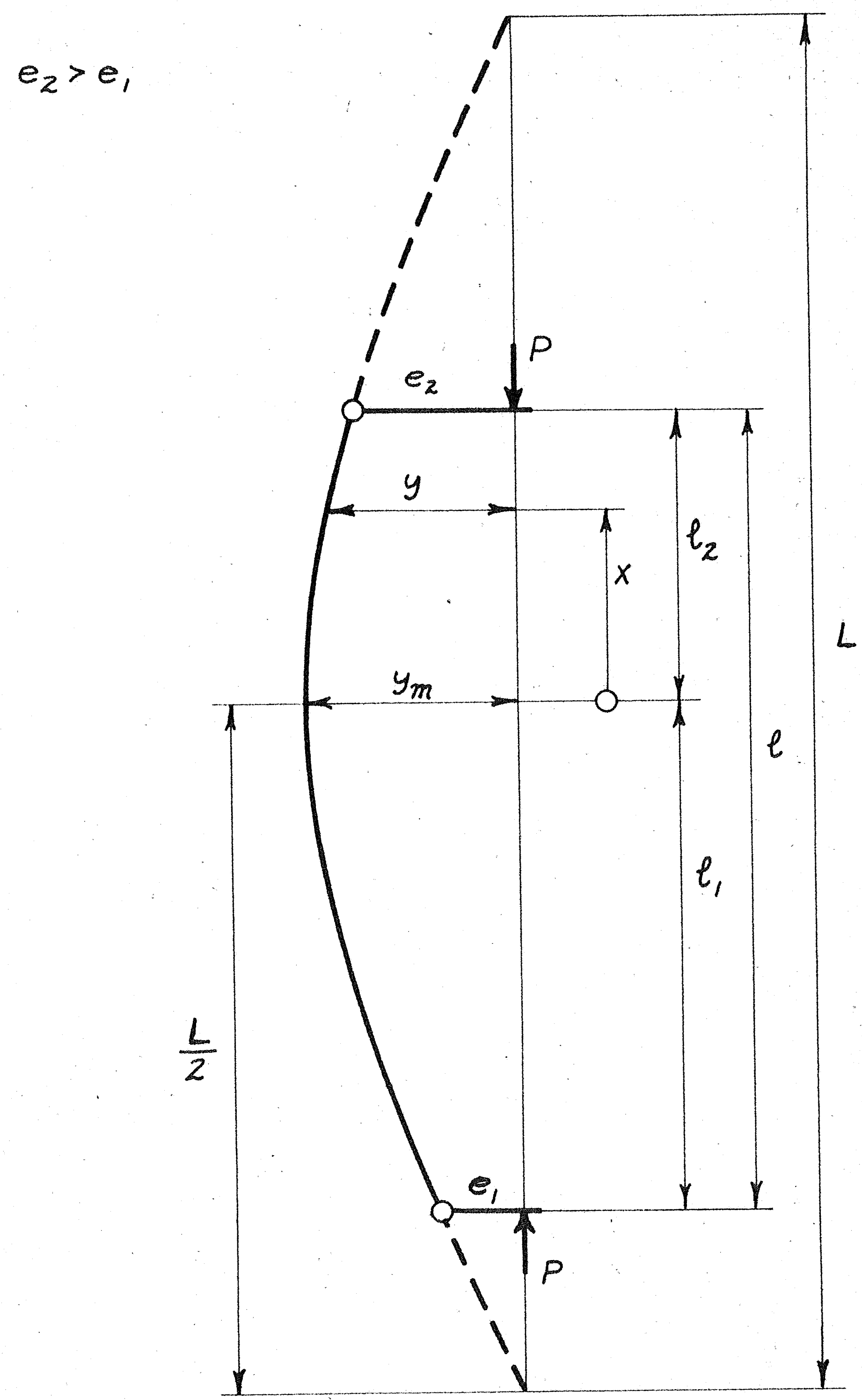

FIG. 4 DEFLECTED SHAPE OF AN ECCENTRICALLY LOADED HINGED COLUMN

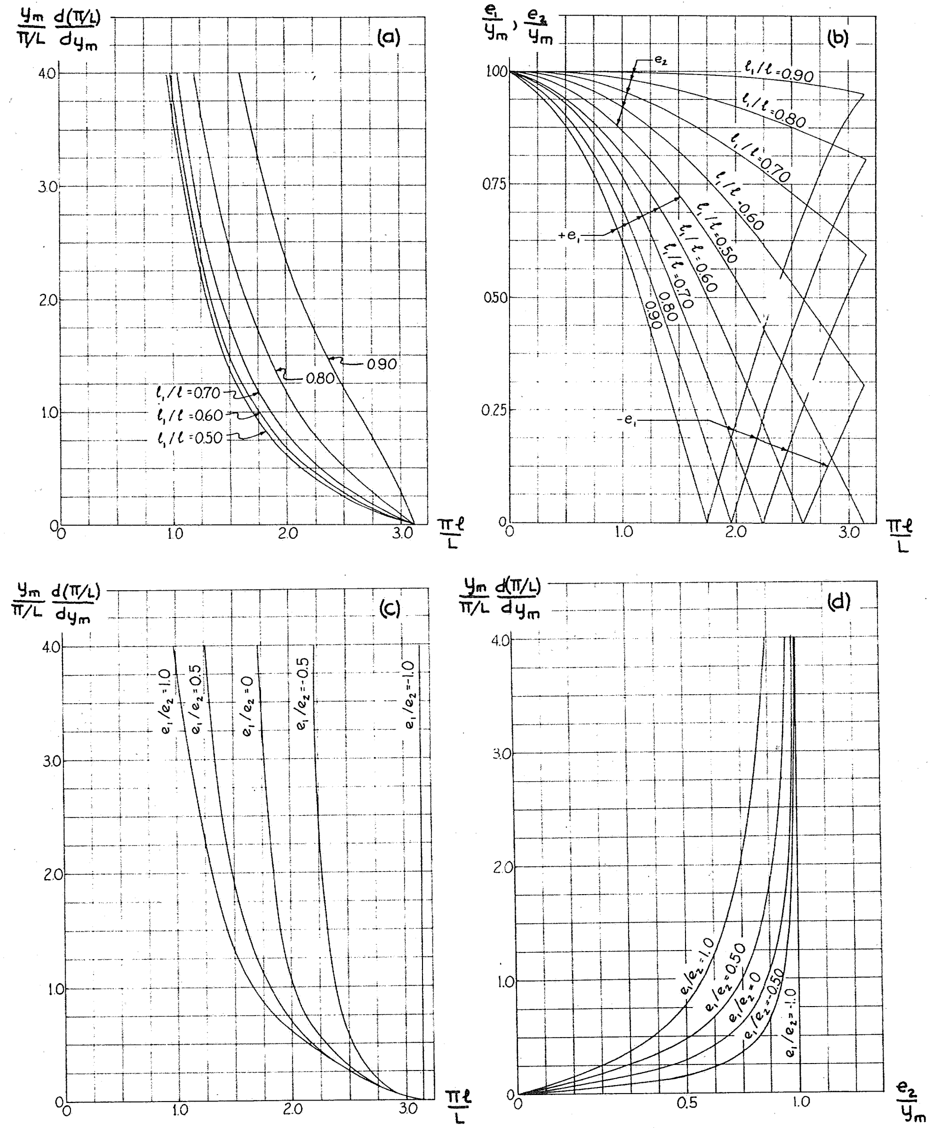

FIG. 5. SOLUTION OF EQ.

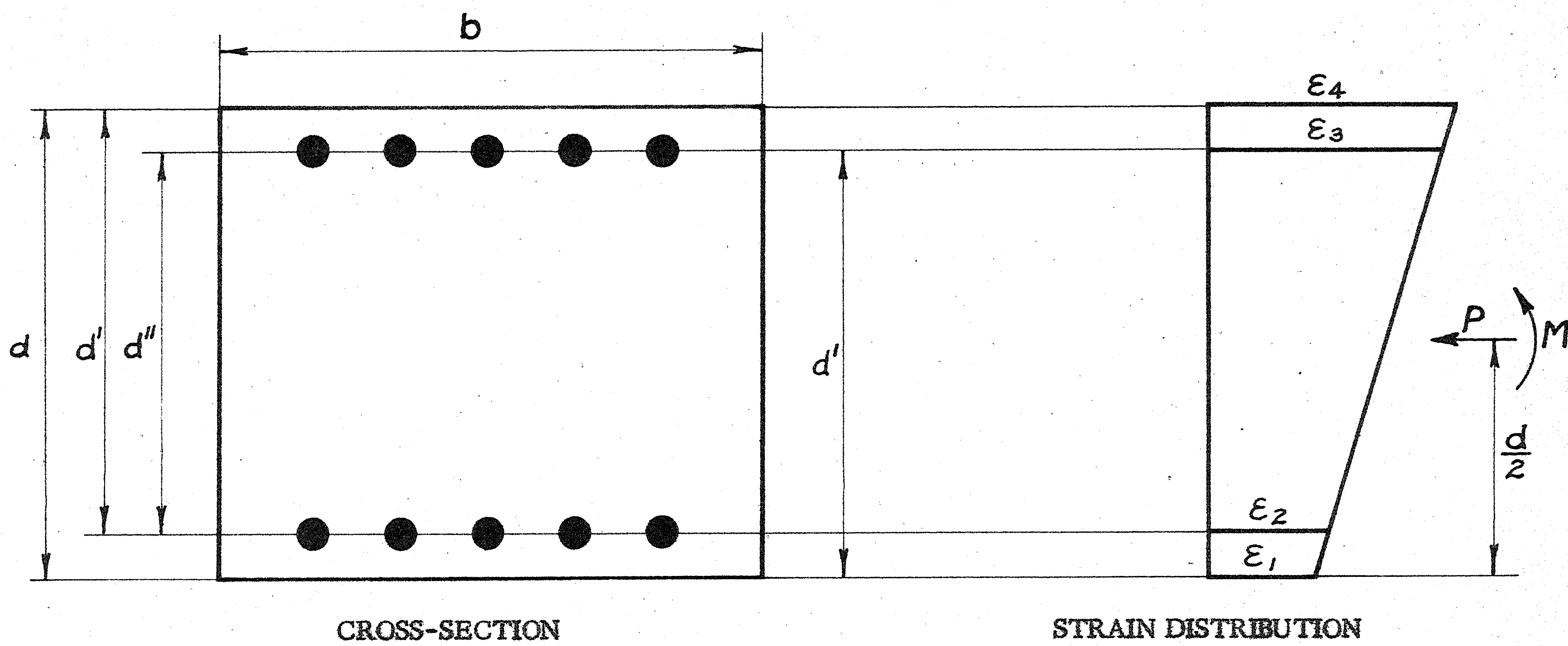

FIG. 6 DISTRIBUTION OF STRAIN IN A COLUMN

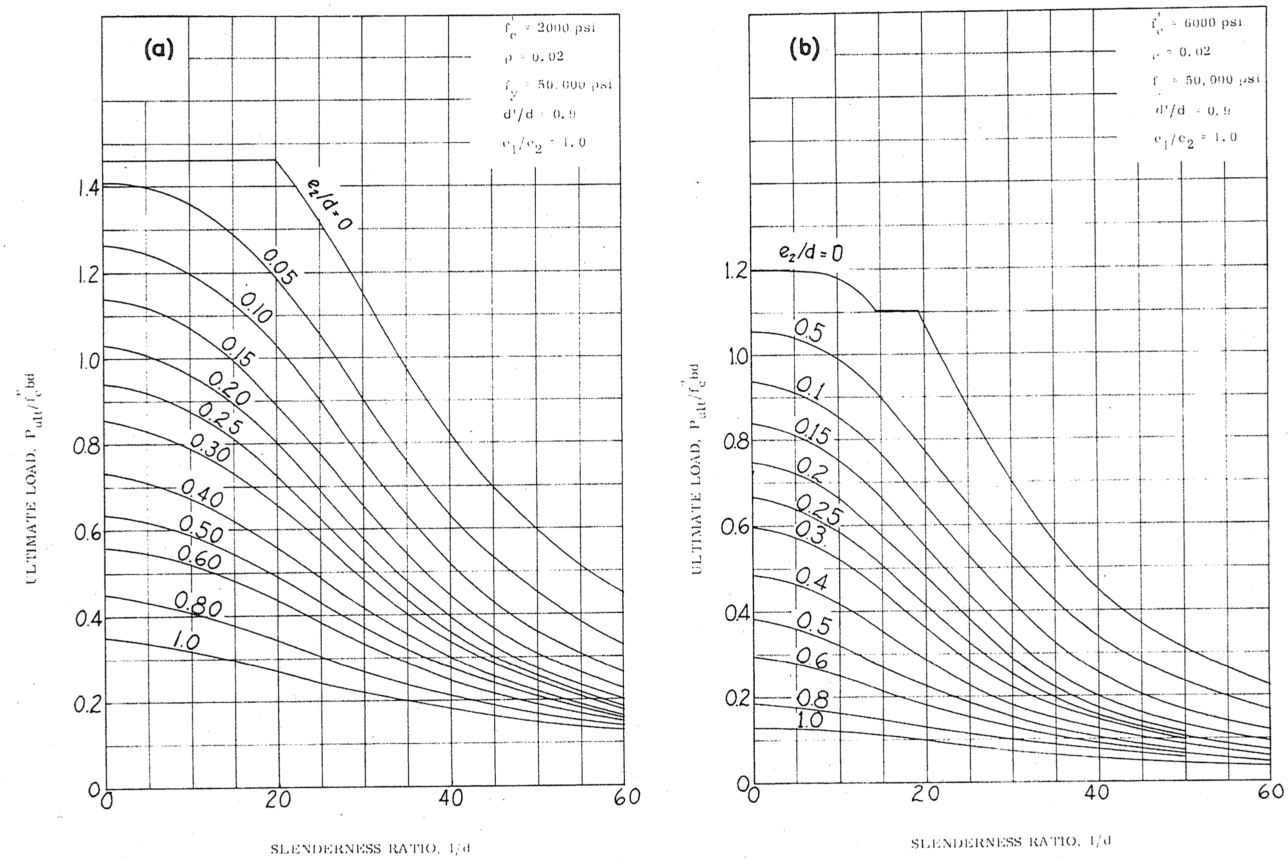

FIG. 7 ULTIMATE LOAD GRAPHS

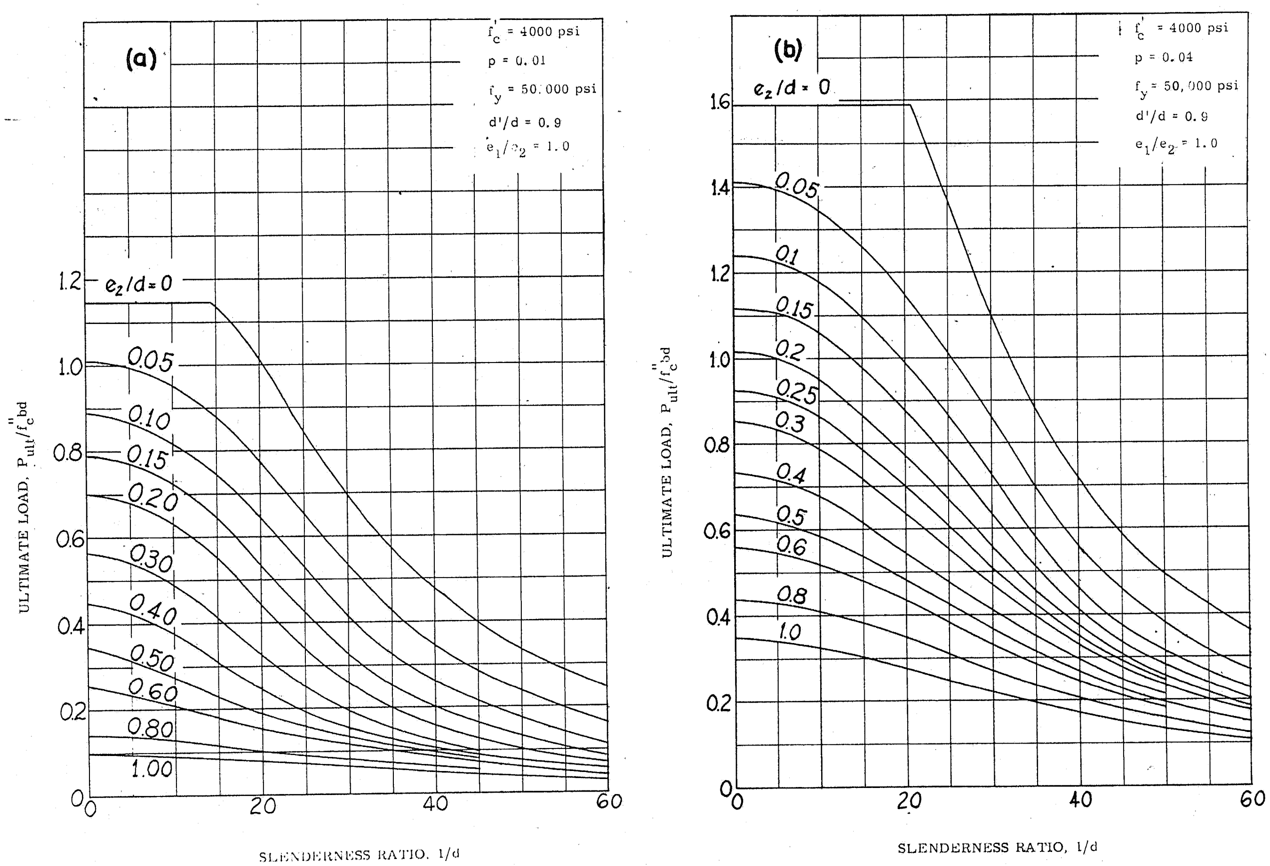

FIG. 8 ULTIMATE LOAD GRAPHS

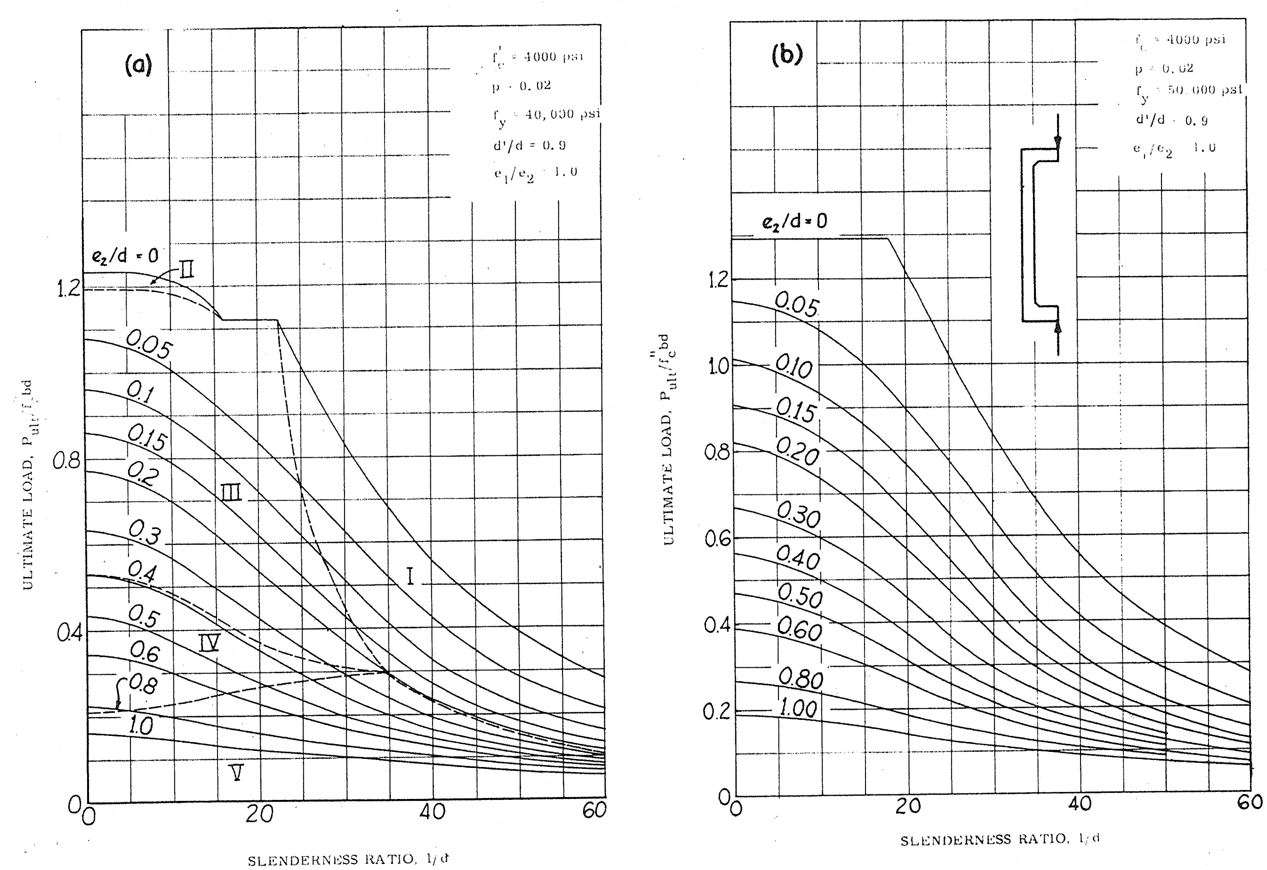

FIG. 9 ULTIMATE LOAD GRAPHS

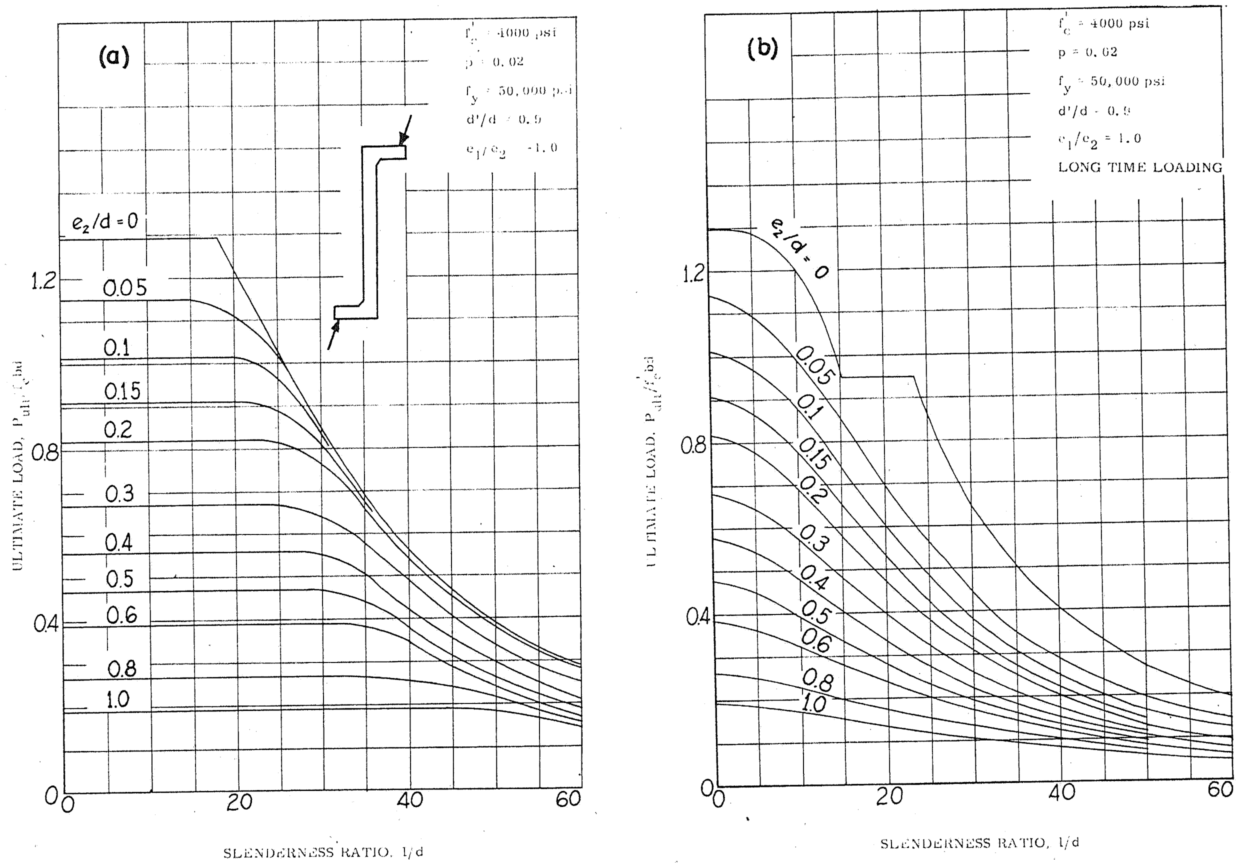

FIG. 1[illegible] ULTIMATE LOAD GRAPHS

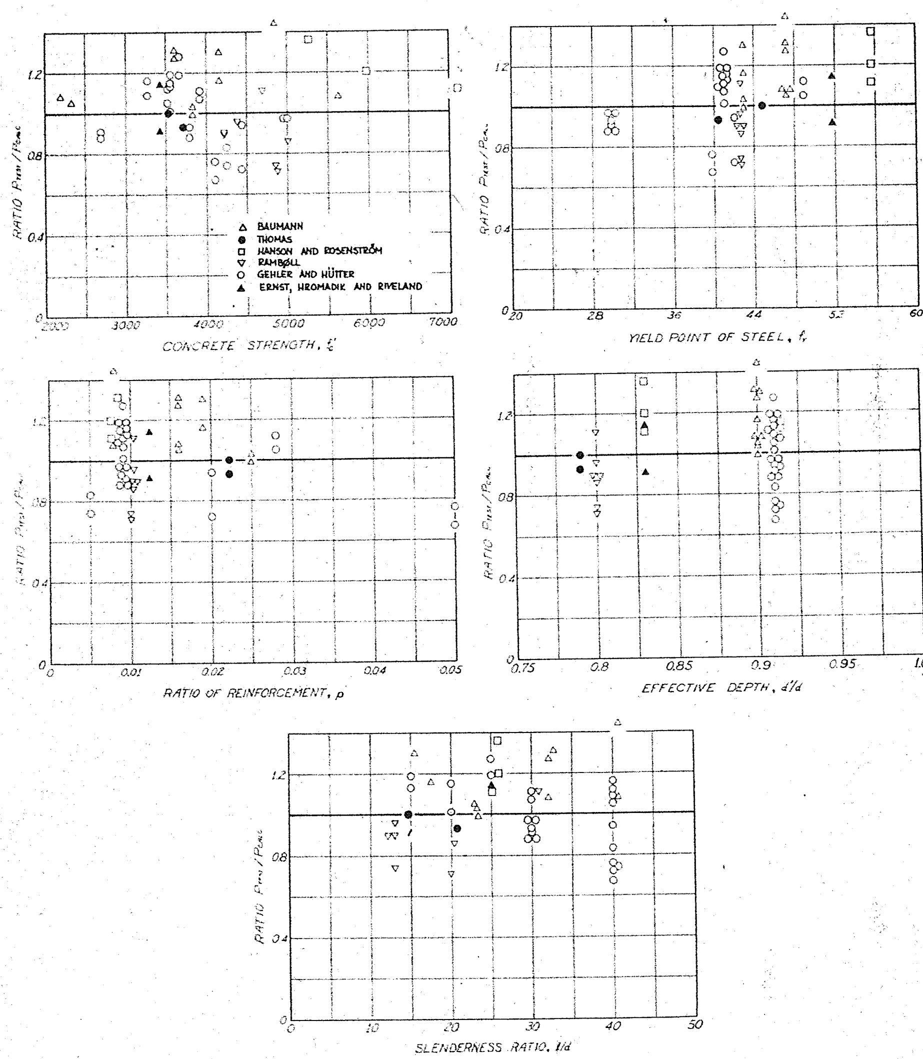

FIG 11. TEST RESULTS – CONCENTRICALLY LOADED COLUMNS

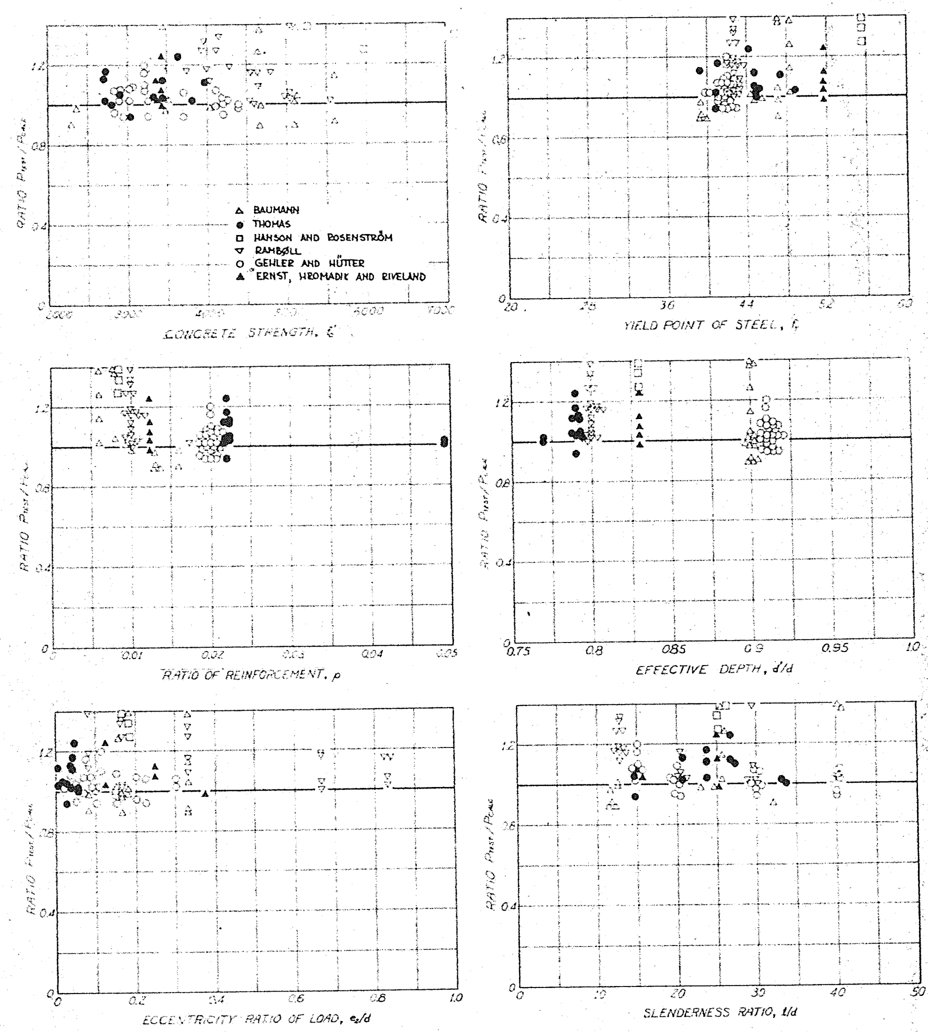

FIG 12. TEST RESULTS ECCENTRICALLY LOADED COLUMNS