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STRAIN ENERGY EXPRESSION FOR LARGE DEFORMATIONS OF ISOTROPIC ELASTIC SHELLS SUBJECTED TO ARBITRARY TEMPERATURE DISTRIBUTION

by

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1. INTRODUCTION

Important applications of shells occur in present day high-speed airframes, in jet engines, in rockets, and in atomic energy technology. Particularly, the problem of thermal stresses in shells that undergo large deflections is of interest.

This report presents a strain energy expression for large deflections of isotropic, homogeneous, elastic shells, including the effects of heating. With this strain energy expression, thermal buckling and post-buckling behavior of heated shells may be treated.

Since geometrical concepts are used extensively in the development of the theory, a brief review of the geometry of surfaces and of shells is presented.

Three basic assumptions are employed in the development of the theory. It is assumed that the shell is made of elastic, isotropic, homogeneous material. Stress components normal to the shell middle surface are assumed to be negligible compared to other stress components. Finally, to establish the dependence of the displacement vector on the thickness coordinate, the Kirchhoff-Love assumption is envoked; that is, normals to the undeformed middle surface are assumed to remain straight, normal, and inextensional under deformation. The usual small deflection restriction on displacement components (neglection of quantities which are of second or higher order in the displacement components of a point off the middle surface is discarded. Power series expansions through the thickness of the shell are avoided.

2. NOTATIONS

(x, y, z)	Arbitrary shell coordinates; coordinates (x, y) are measured in the middle surface of a shell, coordinate z is perpendicular to the middle surface.
u, v, w	Denote components of displacement along the tangents to (x, y, z) coordinate lines, respectively, of a point on the middle surface of the shell.
(E, F, G)	Coefficients in the first fundamental form of a surface (See Eq. 4). E and G are also used to denote Young's modulus and the shear modulus, respectively. No confusion should result from this dual use.
D	Surface parameter defined by Eq. (6).
A, B	Surface parameters for orthogonal surface coordinates (See Eq. 7).
$\overline{r}$	Position vector of a point in a surface measured with respect to (X, Y, Z) reference axis.
$\hat{n}$	Unit normal vector to a surface. The positive sense of $\hat{n}$ is defined by Eq. (9).
$\overline{R}$	Denotes a surface $z = \text{const.}$ (See Eq. 10).
(H, I, J)	Denote coefficients of the first fundamental form of the surface defined by $\overline{R}$ (See Eqs. 11 and 12). $K = \sqrt{HJ - I^2}$
*	Denotes quantities referred to the deformed state.
$\epsilon_x, \epsilon_y, \gamma_{xy}$	Denote strain components referred to (x, y) coordinate system.
k_1, k_2	Denote curvatures of normal sections in direction of lines of principal curvature. r_1, r_2 are the corresponding radii of curvature. (See Eq. 20).
k_g, k_m	Denote Gaussian curvature and mean curvature, respectively. (See Eqs. 21 and 22).
e, f, g	Denote coefficients of second fundamental form. (See Eqs. 23 and 24).
$\mathcal{E}, \mathcal{F}, \mathcal{D}$	Denote coefficients of third fundamental form. (See Eqs. 25 and 26).
L, M, N; L_i, M_i, N_i	Denote functions of (x, y). For example, See Eqs. (33) and (39).
U, V, W	For special cases u, v, w are denoted by U, V, W, respectively.

P_i, Q_i, R_i	Denote first degree terms in (U, V, W) in the expressions for strain components. (For example, see Eqs. 57 and 63).
p_i, q_i, r_i	Denote second degree terms in (U, V, W) in the expressions for strain components (For example, seeEqs. 57 and 63).
$\mathcal{V}_0$	Strain energy density.
$\mathcal{V}$	Total strain energy.
h	Shell thickness.
ν	Poisson's ratio.
k	Thermal coefficient of linear expansion.
T	Temperature distribution, $T = T(x, y)$

3. RESUME OF GEOMETRY OF SHELLS

Surface Theory. A surface may be represented parametrically by the equations

$$X = X(x, y), \quad Y = Y(x, y), \quad Z = Z(x, y) \tag{1}$$

where (x, y) are surface coordinates (1)[1]. A point in the surface is given by the relation

$$\bar{r} = \hat{i}\, X + \hat{j}\, Y + \hat{k}\, Z \tag{2}$$

where $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors directed along positive X, Y, Z axes, respectively. Hence, the square of the distance ds between two neighboring points on the middle surface is

$$ds^2 = d\bar{r} \cdot d\bar{r} = E\, dx^2 + 2F\, dx\, dy + G\, dy^2 \tag{3}$$

where

$$E = \bar{r}_x \cdot \bar{r}_x, \quad F = \bar{r}_x \cdot \bar{r}_y, \quad G = \bar{r}_y \cdot \bar{r}_y \tag{4}$$

Subscripts x and y denote partial derivatives with respect to x and y.

The quantity $E\, dx^2 + 2F\, dx\, dy + G\, dy^2$ is called the "first fundamental form" of a surface.

Equations (1), (2), and (4) yield

$$\begin{aligned} E &= X_x^{\,2} + Y_x^{\,2} + Z_x^{\,2} \\ F &= X_x X_y + Y_x Y_y + Z_x Z_y \\ G &= X_y^{\,2} + Y_y^{\,2} + Z_y^{\,2} \end{aligned} \tag{5}$$

These equations show that E and G are always positive. Also, since ds^2 is positive, by Eqs. (3) and (5),

$$D = \sqrt{E\, G - F^2} > 0 \tag{6}$$

---

[1] Numbers in parentheses refer to references listed in the Bibliography.

For orthogonal surface coordinates, $F = 0$. Hence, since E and G are always positive, the following notation is convenient for orthogonal surface coordinates:

$$E = A^2, \qquad G = B^2 \tag{7}$$

where A and B are positive functions of x and y.

Since $d\bar{r} = \bar{r}_x \, dx + \bar{r}_y \, dy$, the vectors $\bar{r}_x$, $\bar{r}_y$ are tangent to (x, y) coordinate lines, respectively. Hence, unit vectors $\hat{x}$, $\hat{y}$, tangent respectively to (x, y) coordinates lines are given by the relations

$$\hat{x} = \frac{\bar{r}_x}{|\bar{r}_x|} = \frac{\bar{r}_x}{A}$$

$$\hat{y} = \frac{\bar{r}_y}{|\bar{r}_y|} = \frac{\bar{r}_y}{B} \tag{8}$$

The vector $\bar{r}_x \times \bar{r}_y$ is normal to the surface. By Eqs. (2), (5), and (6), the magnitude of $\bar{r}_x \times \bar{r}_y$ is D. Hence, the unit normal vector of the surface is

$$\hat{n} = \frac{\bar{r}_x \times \bar{r}_y}{D} \tag{9}$$

Equation (9) defines the positive sense of $\hat{n}$. Hence, if the symbols (x, y) are interchanged, the positive sense of $\hat{n}$ is reversed.

Geometric Representation of Shells. Let a surface be represented parametrically by Eqs. (1), where (x, y) are orthogonal surface coordinates. Let $\pm z$ be measured from this surface (called the middle surface of the shell); positive z is measured in the positive sense of the surface normal n (see Eq. 9). Let the free surfaces of an undeformed shell be defined by the surfaces $z = \pm h/2$, where in general h may be a function of (x, y). If the shell has constant thickness, h = const.

The surface z = const in the undeformed shell is represented by the equation

$$\bar{R} = \bar{r} + z\hat{n} \tag{10}$$

where $\bar{r}$ and $\hat{n}$ are defined by Eqs. (2) and (9). Equation (10) expresses the surface z = const. in terms of the surface coordinates (x, y) of the middle surface and the coordinate z. For z = 0, Eq. (10) represents the middle surface of the shell. The vectors $\bar{R}_x$, $\bar{R}_y$, $\bar{R}_z$ are tangent respectively to (x, y, z) coordinate lines at z = const. Since, for orthogonal surface coordinates, the lines of principal curvature on the middle surface are coordinates lines,

$$\overline{R}_x \cdot \overline{R}_y = \overline{R}_x \cdot \overline{R}_z = \overline{R}_y \cdot \overline{R}_z = 0$$

Accordingly, the coordinates (x, y, z) are orthogonal, and the coordinate surfaces (x = const, y = const, z = const) form a triply orthogonal system of surfaces. Since coordinate lines and coordinate surfaces are generally curved, the coordinates (x, y, z) are called orthogonal curvilinear coordinates, or briefly, shell coordinates.

The square of the distance ds between two neighboring points in the surface z = const in the undeformed shell is

$$ds^2 = d\overline{R} \cdot d\overline{R} = H\,dx^2 + 2I\,dx\,dy + J\,dy^2 \tag{11}$$

where the scalar functions H, I, and J are defined by the relations

$$H = \overline{R}_x \cdot \overline{R}_x, \qquad I = \overline{R}_x \cdot \overline{R}_y, \qquad J = \overline{R}_y \cdot \overline{R}_y \tag{12}$$

Subscripts x and y denote differentiation with respect to x and y. For orthogonal shell coordinates, I = 0. Hence, $K = \sqrt{HJ - I^2} = \sqrt{HJ}$.

4. STRAIN COMPONENTS

Consider a shell deformed under the action of loads. Let (u, v, w) denote components of displacement along the tangents to (x, y, z) coordinate lines, respectively. Then the equation of the deformed middle surface of the shell is

$$\bar{r}^* = \bar{r} + u\hat{x} + v\hat{y} + w\hat{n} \tag{13}$$

where $(\hat{x}, \hat{y}, \hat{n})$ are unit vectors tangent respectively to (x, y, z) coordinate lines and where the asterisk denotes the deformed state. Hence, the coefficients of the first fundamental form for the deformed middle surface are

$$E^* = \bar{r}^*_x \cdot \bar{r}^*_x, \quad F^* = \bar{r}^*_x \cdot \bar{r}^*_y, \quad G^* = \bar{r}^*_y \cdot \bar{r}^*_y \tag{14}$$

Also,

$$D^* = \sqrt{E^*G^* - F^{*2}} \tag{15}$$

and

$$\hat{n}^* = \frac{\bar{r}^*_x \times \bar{r}^*_y}{D^*} \tag{16}$$

To determine the equation of the deformed surface z = const, we employ the Kirchhoff-Love assumption; that is, we assume normals to the undeformed surface remain straight, normal, and inextensional. (This assumption implies that $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$. See Ref. 2.) Then the equation for the deformed surface z = const is

$$\bar{R}^* = \bar{r}^* + z\hat{n}^* \tag{17}$$

The scalar functions of surface theory that correspond to Eq. (17) are

$$H^* = \bar{R}^*_x \cdot \bar{R}^*_x, \quad I^* = \bar{R}^*_x \cdot \bar{R}^*_y, \quad J^* = \bar{R}^*_y \cdot \bar{R}^*_y$$

$$K^* = \sqrt{H^* J^* - I^{*2}} \tag{18}$$

Novozhilov (3) has given general equations for strain components. Rewriting his equations in terms of Eqs. (12) and (18), we obtain for orthogonal shell coordinates

$$2\epsilon_x = \frac{H^*}{H} - 1, \quad 2\epsilon_y = \frac{J^*}{J} - 1, \quad \gamma_{xy} = \frac{I^*}{HJ} \tag{19}$$

With Eqs. (10), (12), (13), (16), (17), and (18), Eqs. (19) express the strain components at a general point in an arbitrary shell in terms of the displacement components (u, v, w) of the corresponding point in the middle surface of the shell. Taylor (4) has employed the above general theory in the problem of buckling of a right-circular cone.

5. STRAIN COMPONENTS IN TERMS OF DISPLACEMENT COMPONENTS.

The reduction of Eqs. (19) to expressions in (u, v, w) is lengthy. However, considerable simplification is achieved by using the following relationships (1).

Theorem of Rodriques:

For orthogonal shell coordinates,

$$\hat{n}_x = -k_1 \bar{r}_x = -\frac{1}{r_1} \bar{r}_x$$

$$\hat{n}_y = -k_2 \bar{r}_y = -\frac{1}{r_2} \bar{r}_y \qquad (20)$$

where k_1, k_2 are curvatures of normal sections in the direction of lines of principal curvature and r_1, r_2 are the corresponding radii of curvature.

Gaussian Curvature:

$$k_g = k_1 k_2 = \frac{1}{r_1} \frac{1}{r_2} \qquad (21)$$

Mean Curvature

$$k_m = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad (22)$$

Second Fundamental Form of Surfaces:

$$e\,dx^2 + 2f\,dx\,dy + g\,dy^2 = -d\bar{r} \cdot d\hat{n} \qquad (23)$$

where

$$e = \bar{r}_{xx} \cdot \hat{n} = \bar{r}_{xx} \cdot \frac{(\bar{r}_x \times \bar{r}_y)}{D}$$

$$f = \bar{r}_{xy} \cdot \hat{n} = \bar{r}_{xy} \cdot \frac{(\bar{r}_x \times \bar{r}_y)}{D} \qquad (24)$$

$$g = \bar{r}_{yy} \cdot \hat{n} = \bar{r}_{yy} \cdot \frac{(\bar{r}_x \times \bar{r}_y)}{D}$$

For orthogonal surface coordinates,

$$e = \frac{E}{r_1} = \frac{A^2}{r_1}, \quad g = \frac{G}{r_2} = \frac{B^2}{r_2}, \quad f = 0 \tag{24a}$$

Third Fundamental Form of Surfaces:

$$\mathcal{E}\, dx^2 + 2\,\mathcal{F}\, dx\, dy + \mathcal{G}\, dy^2 = d\hat{n} \cdot d\hat{n} \tag{25}$$

where

$$\begin{aligned} \mathcal{E} &= -k_g E + 2k_m e \\ \mathcal{F} &= -k_g F + 2k_m f \\ \mathcal{G} &= -k_g G + 2k_m g \end{aligned} \tag{26}$$

For orthogonal surface coordinates,

$$\begin{aligned} \mathcal{E} &= \frac{E}{r_1^2} = \frac{A^2}{r_1^2} = \frac{e}{r_1} \\ \mathcal{F} &= 0 \\ \mathcal{G} &= \frac{G}{r_2^2} = \frac{B^2}{r_2^2} = \frac{g}{r_2} \end{aligned} \tag{26a}$$

Scalar Products:

Some useful scalar products are listed in Table 1 on page 11.

In Table 1, subscripts x and y denote derivatives with respect to x and y. For orthogonal surface coordinates, $E = A^2$, $G = B^2$, $F = f = \mathcal{F} = 0$.

The scalar product of a vector in the left column of Table 1 with a vector in the top row of Table 1 is obtained simply as illustrated by the following example:

$$\hat{n}_x \cdot \bar{r}_y = -f$$

With the preceding relationships, the strain-displacement equations may be derived as follows:

By Eqs. (12), (10), (24a), and (26a), and Table 1, we find

TABLE 1

SCALAR PRODUCTS

$\leftrightarrow$	$\overline{r}_x$	$\overline{r}_y$	$\hat{n}$	$\hat{n}_x$	$\hat{n}_y$	$\overline{r}_{xx}$	$\overline{r}_{xy}$	$\overline{r}_{yy}$
$\overline{r}_x$	E	F	0	- e	- f	$\frac{1}{2}E_x$	$\frac{1}{2}E_y$	$F_y - \frac{1}{2}G_x$
$\overline{r}_y$	F	G	0	- f	- g	$F_x - \frac{1}{2}E_y$	$\frac{1}{2}G_x$	$\frac{1}{2}G_y$
$\hat{n}$	0	0	1	0	0	e	f	g
$\hat{n}_x$	- e	- f	0	$\mathcal{E}$	$\mathcal{F}$			
$\hat{n}_y$	- f	- g	0	$\mathcal{F}$	$\mathcal{G}$			
$\overline{r}_{xx}$	$\frac{1}{2}E_x$	$F_x - \frac{1}{2}E_y$	e					
$\overline{r}_{xy}$	$\frac{1}{2}E_y$	$\frac{1}{2}G_x$	f					
$\overline{r}_{yy}$	$F_y - \frac{1}{2}G_x$	$\frac{1}{2}G_y$	g					

$$H = A^2 \left(1 - \frac{z}{r_1}\right)^2 , \quad I = 0, \quad J = B^2\left(1 - \frac{z}{r_2}\right)^2 \tag{27}$$

Differentiations of Eq. (17) with respect to x and with respect to y yield.

$$\begin{aligned} \overline{R^*_x} &= \overline{r^*_x} + z\,\widehat{n^*_x} \\ \overline{R^*_y} &= \overline{r^*_y} + z\,\widehat{n^*_y} \end{aligned} \tag{28}$$

Equations (14), (18) and (28) yield

$$
\begin{aligned}
H^* &= E^* + 2z\, \overline{r}^*_x \cdot \hat{n}^*_x + z^2\, \hat{n}^*_x \cdot \hat{n}^*_x \\
I^* &= F^* + z\,(\overline{r}^*_x \cdot \hat{n}^*_y + \overline{r}^*_y \cdot \hat{n}^*_x) + z^2\, \hat{n}^*_x \cdot \hat{n}^*_y \\
J^* &= G^* + 2z\, \overline{r}^*_y \cdot \hat{n}^*_y + z^2\, \hat{n}^*_y \cdot \hat{n}^*_y
\end{aligned}
\qquad (29)
$$

By Eqs. (8) and (13), we obtain

$$
\overline{r}^* = \overline{r} + u\,\frac{\overline{r}_x}{A} + v\,\frac{\overline{r}_y}{B} + w\,\hat{n} \qquad (30)
$$

With Eq. (20), differentiation of Eq. (30) yields

$$
\overline{r}^*_x = \left(A + u_x - \frac{uA_x}{A} - \frac{wA}{r_1}\right)\frac{\overline{r}_x}{A} + \left(v_x - \frac{vB_x}{B}\right)\frac{\overline{r}_y}{B} + u\,\frac{\overline{r}_{xx}}{A} + v\,\frac{\overline{r}_{xy}}{B} + w_x\,\hat{n}
$$

$$
\overline{r}^*_y = \left(u_y - \frac{uA_y}{A}\right)\frac{\overline{r}_x}{A} + \left(B + v_y - \frac{vB_y}{B} - \frac{Bw}{r_2}\right)\frac{\overline{r}_y}{B} + u\,\frac{\overline{r}_{xy}}{A} + v\,\frac{\overline{r}_{yy}}{B} + w_y\,\hat{n}
$$

By the theory of differential geometry, $\overline{r}_{xx}$, $\overline{r}_{xy}$, $\overline{r}_{yy}$ may be expressed in terms of $\overline{r}_x$, $\overline{r}_y$, and $\hat{n}$. For orthogonal surface coordinates, the following relations exist (1):

$$
\begin{aligned}
\overline{r}_{xx} &= A_x\,\frac{\overline{r}_x}{A} - \frac{A\,A_y}{B}\,\frac{\overline{r}_y}{B} + \frac{A^2}{r_1}\,\hat{n} \\
\overline{r}_{xy} &= A_y\,\frac{\overline{r}_x}{A} + B_x\,\frac{\overline{r}_y}{B} \\
\overline{r}_{yy} &= -\frac{B\,B_x}{A}\,\frac{\overline{r}_x}{A} + B_y\,\frac{\overline{r}_y}{B} + \frac{B^2}{r_2}\,\hat{n}
\end{aligned}
\qquad (31)
$$

Substituting Eq. (31) into the above expressions for $\overline{r}^*_x$ and $\overline{r}^*_y$, we obtain

$$\overline{r^*_x} = A\left[(1+L_1)\frac{\overline{r_x}}{A} + M_1\frac{\overline{r_y}}{B} + N_1\hat{n}\right]$$

$$\overline{r^*_y} = B\left[L_2\frac{\overline{r_x}}{A} + (1+M_2)\frac{\overline{r_y}}{B} + N_2\hat{n}\right] \qquad (32)$$

where

$$L_1 = \frac{u_x}{A} + \frac{v A_y}{AB} - \frac{w}{r_1}$$

$$M_1 = \frac{v_x}{A} - \frac{u A_y}{AB}$$

$$N_1 = \frac{w_x}{A} + \frac{u}{r_1}$$

$$L_2 = \frac{u_y}{B} - \frac{v B_x}{AB} \qquad (33)$$

$$M_2 = \frac{v_y}{B} + \frac{u B_x}{AB} - \frac{w}{r_2}$$

$$N_2 = \frac{w_y}{B} + \frac{v}{r_2}$$

The terms L_1, M_1, N_1, L_2, M_2, N_2 are related to the strain components $(\epsilon'_x, \epsilon'_y, \gamma'_{xy})$ of the middle surface as follows (2):

$$\epsilon'_x = L_1 + \frac{1}{2}(L_1^2 + M_1^2 + N_1^2)$$

$$\epsilon'_y = M_2 + \frac{1}{2}(L_2^2 + M_2^2 + N_2^2) \qquad (34)$$

$$\gamma'_{xy} = M_1 + L_2 + L_1 L_2 + M_1 M_2 + N_1 N_2$$

For small deflections, quadratic terms in Eq. (34) may be neglected.

By Eqs. (14) and (32), we obtain

$$E^* = A^2 \left[(1 + L_1)^2 + M_1^{\ 2} + N_1^{\ 2} \right]$$

$$F^* = AB \left[(1 + L_1)\, L_2 + M_1\, (1 + M_2) + N_1\, N_2 \right] \tag{35}$$

$$G^* = B^2 \left[L_2^{\ 2} + (1 + M_2)^2 + N_2^{\ 2} \right]$$

Hence, Eqs. (15) and (35) yield

$$\begin{aligned} D^{*2} = A^2B^2 \Big\{ & \left[(1 + L_1)(1 + M_2) - L_2\, M_1 \right]^2 \\ & + \left[(1 + L_1)\, N_2 - L_2\, N_1 \right]^2 \\ & + \left[(1 + M_2)\, N_1 - M_1\, N_2 \right]^2 \Big\} \end{aligned} \tag{36}$$

For large deformation of circular cylinders, Eqs. (35) and (36) reduce to the results obtained in (5). For small deformations of an arbitrary shell, Eqs. (35) and (36) agree with the results of (2).

By Eqs. (16) and (32), we obtain

$$\begin{aligned} D^* \hat{n}^* = AB \Big\{ & \left[M_1\, N_2 - N_1\, (1 + M_2) \right] \frac{\overline{r}_x}{A} \\ & + \left[L_2\, N_1 - N_2\, (1 + L_1) \right] \frac{\overline{r}_y}{B} \\ & + \left[(1 + L_1)(1 + M_2) - L_2\, M_1 \right] \hat{n} \Big\} \end{aligned} \tag{37}$$

Let us rewrite Eq. (37) in the form

$$D^* \hat{n}^* = L \frac{\overline{r}_x}{A} + M \frac{\overline{r}_y}{B} + N \hat{n} \tag{38}$$

where

$$L = AB\left[M_1 N_2 - N_1(1+M_2)\right]$$

$$M = AB\left[L_2 N_1 - N_2(1+L_1)\right] \qquad (39)$$

$$N = AB\left[(1+L_1)(1+M_2) - L_2 M_1\right]$$

Differentiations of Eq. (38) with respect to x and with respect to y yield

$$D^*_x \hat{n}^* + D^* \hat{n}^*_x = L_x \frac{\bar{r}_x}{A} + L \frac{\bar{r}_{xx}}{A} - L \frac{A_x}{A} \cdot \frac{\bar{r}_x}{A}$$

$$+ M_x \frac{\bar{r}_y}{B} + M \frac{\bar{r}_{xy}}{B} - M \frac{B_x}{B} \cdot \frac{\bar{r}_y}{B}$$

$$+ N_x \hat{n} + N \hat{n}_x$$

$$D^*_y \hat{n}^* + D^* \hat{n}^*_y = L_y \frac{\bar{r}_x}{A} + L \frac{\bar{r}_{xy}}{A} - \frac{L A_y}{A} \cdot \frac{\bar{r}_x}{A}$$

$$+ M_y \frac{\bar{r}_y}{B} + M \frac{\bar{r}_{yy}}{B} - M \frac{B_y}{B} \cdot \frac{\bar{r}_y}{B}$$

$$+ N_y \hat{n} + N \hat{n}_y$$

Substituting Eqs. (20) and (31) into these equations, we obtain

$$D^*_x \hat{n}^* + D^* \hat{n}^*_x = \left(L_x + M \frac{A_y}{B} - N \frac{A}{r_1}\right) \frac{\bar{r}_x}{A}$$

$$+ \left(M_x - L \frac{A_y}{B}\right) \frac{\bar{r}_y}{B} + \left(N_x + L \frac{A}{r_1}\right) \hat{n}$$

$$D^*_y \hat{n}^* + D^* \hat{n}^*_y = \left(L_y - M \frac{B_x}{A}\right) \frac{\bar{r}_x}{A} + \left(M_y + L \frac{B_x}{A} - N \frac{B}{r_2}\right) \frac{\bar{r}_y}{B} \qquad (40)$$

$$+ \left(N_y + M \frac{B}{r_2}\right) \hat{n}$$

Hence, since $\bar{r}^*_x \cdot \hat{n}^* = \bar{r}^*_y \cdot \hat{n}^* = 0$, the scalar vector products of Eqs. (32) and (40) yield

$$D^* \overline{r^*_x} \cdot \widehat{n^*_x} = A \left[(1 + L_1)(L_x + M \frac{A_y}{B} - N \frac{A}{r_1}) + M_1 (M_x - L \frac{A_y}{B}) + N_1 (N_x + L \frac{A}{r_1}) \right]$$

$$D^* \overline{r^*_x} \cdot \widehat{n^*_y} = A \left[(1 + L_1)(L_y - M \frac{B_x}{A}) + M_1 (M_y + L \frac{B_x}{A} - N \frac{B}{r_2}) + N_1 (N_y + M \frac{B}{r_2}) \right]$$

$$D^* \overline{r^*_y} \cdot \widehat{n^*_x} = B \left[L_2 (L_x + M \frac{A_y}{B} - N \frac{A}{r_1}) + (1 + M_2)(M_x - L \frac{A_y}{B}) + N_2 (N_x + L \frac{A}{r_1}) \right] \quad (41)$$

$$D^* \overline{r^*_y} \cdot \widehat{n^*_y} = B \left[L_2 (L_y - M \frac{B_x}{A}) + (1 + M_2)(M_y + L \frac{B_x}{A} - N \frac{B}{r_2}) + N_2 (N_y + M \frac{B}{r_2}) \right]$$

Substituting Eq. (37) into Eq. (40) and rearranging, we obtain, with the notations of Eq. (39),

$$D^* \widehat{n^*_x} = (L_x + M \frac{A_y}{B} - N \frac{A}{r_1} - L \frac{D^*_x}{D^*}) \frac{\overline{r}_x}{A} + (M_x - L \frac{A_y}{B} - M \frac{D^*_x}{D^*}) \frac{\overline{r}_y}{B} + (N_x + L \frac{A}{r_1} - N \frac{D^*_x}{D^*}) \widehat{n} \quad (42)$$

$$D^* \widehat{n^*_y} = (L_y - M \frac{B_x}{A} - L \frac{D^*_y}{D^*}) \frac{\overline{r}_x}{A} + (M_y + L \frac{B_x}{A} - N \frac{B}{r_2} - M \frac{D^*_y}{D^*}) \frac{\overline{r}_y}{B} + (N_y + M \frac{B}{r_2} - N \frac{D^*_y}{D^*}) \widehat{n}$$

Hence, by Eq. (42), we obtain

$$D^{*2}\,\widehat{n^*_x} \bullet \widehat{n^*_x} = A^2 \Big[(L_3 - \frac{LD^*_x}{AD^*})^2 + (M_3 - \frac{M\,D^*_x}{AD^*})^2 + (N_3 - \frac{ND^*_x}{AD^*})^2 \Big]$$

$$D^{*2}\,\widehat{n^*_x} \cdot \widehat{n^*_y} = AB \Big[(L_3 - \frac{LD^*_x}{AD^*})(L_4 - \frac{LD^*_y}{BD^*}) + (M_3 - \frac{M\,D^*_x}{AD^*})(M_4 - \frac{M\,D^*_y}{BD^*}) + (N_3 - \frac{ND^*_x}{AD^*})(N_4 - \frac{N\,D^*_y}{BD^*}) \Big] \qquad (43)$$

$$D^{*2}\,\widehat{n^*_y} \cdot \widehat{n^*_y} = B^2 \Big[(L_4 - \frac{L\,D^*_y}{BD^*})^2 + (M_4 - \frac{M\,D^*_y}{BD^*})^2 + (N_4 - \frac{N\,D^*_y}{BD^*})^2 \Big]$$

where

$$L_3 = \frac{L_x}{A} + \frac{M\,A_y}{AB} - \frac{N}{r_1}$$

$$M_3 = \frac{M_x}{A} - \frac{L\,A_y}{AB}$$

$$N_3 = \frac{N_x}{A} + \frac{L}{r_1}$$

$$L_4 = \frac{L_y}{B} - \frac{M\,B_x}{AB} \qquad (44)$$

$$M_4 = \frac{M_y}{B} + \frac{L\,B_x}{AB} - \frac{N}{r_2}$$

$$N_4 = \frac{N_y}{B} + \frac{M}{r_2}$$

Hence, by Eqs. (29), (35), (41), and (43),

$$H^* = A^2 \left[(1 + L_1 + \frac{z}{D^*} L_3)^2 + (M_1 + \frac{z}{D^*} M_3)^2 + (N_1 + \frac{z}{D^*} N_3)^2\right] + \frac{z^2}{D^{*2}} \frac{D^*_x}{D^*} \left[\frac{D^*_x}{D^*} (L^2 + M^2 + N^2) - 2(LL_x + MM_x + NN_x)\right]$$

$$I^* = AB \left[(1 + L_1)(L_2 + \frac{z}{D^*} L_4) + (1 + M_2)(M_1 + \frac{z}{D^*} M_3) + N_1 (N_2 + \frac{z}{D^*} N_4)\right] + AB \frac{z}{D^*} \left[L_3 (L_2 + \frac{z}{D^*} L_4) + M_4 (M_1 + \frac{z}{D^*} M_3) + N_3 (N_2 + \frac{z}{D^*} N_4)\right] + \frac{z^2}{D^{*2}} \left[\frac{D^*_x D^*_y}{D^{*2}} (L^2 + M^2 + N^2) - \frac{D^*_x}{D^*} (LL_y + MM_y + NN_y) - \frac{D^*_y}{D^*} (LL_x + MM_x + NN_x)\right]$$

$$J^* = B^2 \left[(L_2 + \frac{z}{D^*} L_4)^2 + (1 + M_2 + \frac{z}{D^*} M_4)^2 + (N_2 + \frac{z}{D^*} N_4)^2\right] + \frac{z^2}{D^{*2}} \frac{D^*_y}{D^*} \left[\frac{D^*_y}{D^*} (L^2 + M^2 + N^2) - 2(LL_y + MM_y + NN_y)\right]$$

However, since $\hat{n}^* \cdot \hat{n}^* = 1$, Eq. (38) yields $D^{*2} = L^2 + M^2 + N^2$.

Hence, differentiation yields

$$D^* D^*_x = LL_x + MM_x + NN_x$$

$$D^* D^*_y = LL_y + MM_y + NN_y$$

Substitution of these relations into the above expressions for H^*, I^*, J^* yields

$$H^* = A^2 \left[(1 + L_1 + \frac{z}{D^*} L_3)^2 + (M_1 + \frac{z}{D^*} M_3)^2 + (N_1 + \frac{z}{D^*} N_3)^2 \right]$$

$$- \frac{z^2}{D^{*2}} D^{*2}_x$$

$$I^* = AB \left[(1 + L_1 + \frac{z}{D^*} L_3)(L_2 + \frac{z}{D^*} L_4) + (1 + M_2 \right.$$

$$\left. + \frac{z}{D^*} M_x)(M_1 + \frac{z}{D^*} M_3) + (N_1 + \frac{z}{D^*} N_3)(N_2 + \frac{z}{D^*} N_4) \right] \qquad (45)$$

$$- \frac{z^2}{D^{*2}} D^*_x D^*_y$$

$$J^* = B^2 \left[(L_2 + \frac{z}{D^*} L_4)^2 + (1 + M_2 + \frac{z}{D^*} M_4)^2 + (N_2 + \frac{z}{D^*} M_4)^2 \right]$$

$$- \frac{z^2}{D^{*2}} D^{*\,2}_y$$

Hence, by Eqs. (19), (21), (22), (27), (34), and (45), the strain components are

$$\epsilon_x = \epsilon'_x + \frac{z}{r_1} \left[1 + 2\epsilon'_x + \frac{r_1}{D^*} (L_3 + L_1 L_3 + M_1 M_3 + N_1 N_3) \right]$$

$$+ \frac{3}{2} (\frac{z}{r_1})^2 \left[1 + 2\epsilon'_x + \frac{4}{3} \frac{r_1}{D^*} (L_3 + L_1 L_3 + M_1 M_3 + N_1 N_3) \right.$$

$$\left. + \frac{1}{3} (\frac{r_1}{D^*})^2 (L_3^{\,2} + M_3^{\,2} + N_3^{\,2} - \frac{D^{*2}_x}{A^2}) \right]$$

$$\epsilon_y = \epsilon'_y + \frac{z}{r_2} \left[1 + 2\epsilon'_y + \frac{r_2}{D^*} (M_4 + L_2 L_4 + M_2 M_4 + N_2 N_4) \right] \qquad (46)$$

$$+ \frac{3}{2} (\frac{z}{r_2})^2 \left[1 + 2\epsilon'_y + \frac{4r_2}{3D^*} (M_4 + L_2 L_4 + M_2 M_4 + N_2 N_4) \right.$$

$$\left. + \frac{1}{3} (\frac{r_2}{D^*})^2 (L_4^{\,2} + M_4^{\,2} + N_4^{\,2} - \frac{D^{*\,2}_y}{B^2}) \right]$$

$$\gamma_{xy} = \gamma'_{xy} + z\,k_m \Big[2\gamma'_{xy} + \frac{1}{D^* k_m} (L_4 + L_1 L_4 + M_1 M_4$$

$$+ N_1 N_4 + M_3 + L_2 L_3 + M_2 M_3 + N_2 N_3)\Big]$$

$$+ z^2 (4k_m^{\,2} - k_g) \Big[\gamma'_{xy} + \frac{2\,k_m}{(4\,k^2_m - k_g)\,D^*} (L_4 + L_1 L_4$$

$$+ M_1 M_4 + N_1 N_4 + M_3 + L_2 L_3 + M_2 M_3 + N_2 N_3)$$

$$+ \frac{1}{(4k_m^{\,2} - k_g)\,D^{*2}} (L_3 L_4 + M_3 M_4 + N_3 N_4$$

$$- \frac{D^*_x D^*_y}{AB})\Big]$$

6. STRAIN ENERGY FOR LARGE DEFLECTIONS OF A CIRCULAR CYLINDRICAL SHELL

If (X, Y, Z) are rectangular coordinates, the middle surface of an arbitrary cylindrical shell is defined by the equations

$$\begin{aligned} X &= x \\ Y &= a \cos y \\ Z &= a \sin y \end{aligned} \qquad (47)$$

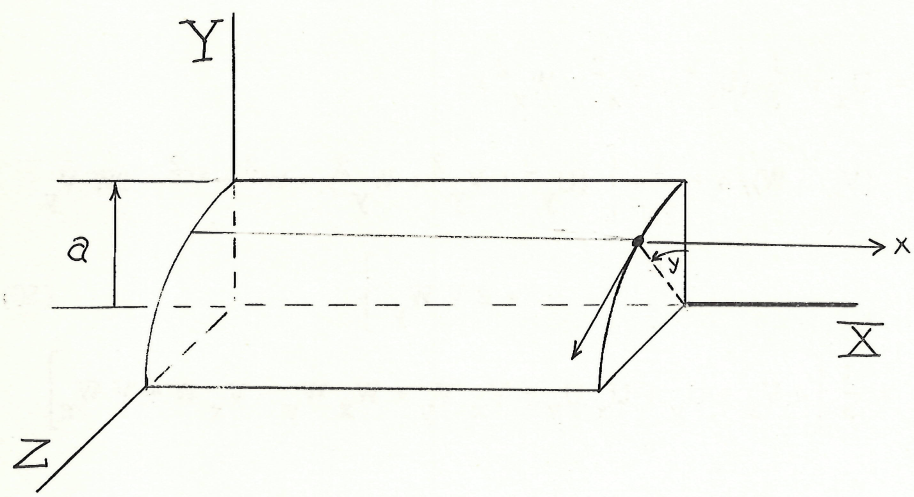

FIG. 1

where (x, y) are axial and circumferential coordinates of the middle surface, respectively. (See Fig. 1).

By Eqs. (5), (7), (20), and (21), (22)

$$E = 1 \qquad A = 1 \qquad \frac{1}{r_1} = 0 \qquad k_g = 0$$

$$F = 0 \qquad B = a \qquad \frac{1}{r_2} = a \qquad k_m = \frac{1}{2a}$$

$$G = a^2 \qquad D = a$$

By Eqs. (33), where for special cases we denote (u, v, w) by (U, V, W), respectively,

$$\begin{aligned} L_1 &= U_x \\ M_1 &= V_x \\ N_1 &= W_x \end{aligned} \tag{48}$$

$$\begin{aligned} L_2 &= U_y/a \\ M_2 &= (V_y - W)/a \\ N_2 &= (W_y + V)/a \end{aligned} \tag{49}$$

By Eqs. (34),

$$\epsilon'_x = U_x + \frac{1}{2}(U_x^{\,2} + V_x^{\,2} + W_x^{\,2})$$

$$\epsilon'_y = (V_y - W)/a + \frac{1}{2a^2}\Big[(U_y^{\,2} + V_y^{\,2} + W_y^{\,2}) + W^2 + V^2 - 2W\,V_y + 2\,V\,W_y\Big] \tag{50}$$

$$\gamma'_{xy} = \frac{1}{a}\Big[a\,V_x + U_y + U_x\,U_y + V_x\,V_y + W_x\,W_y - V_x\,W + V\,W_x\Big]$$

By Eqs. (39),

$$\begin{aligned} L &= -\,aW_x + V_x\,W_y + V_x\,V - W_x\,V_y + W_x\,W \\ M &= -\,W_y - V + U_y\,W_x - W_y\,U_x - V\,U_x \\ N &= a + V_y - W + a\,U_x + U_x\,V_y - U_x\,W - U_y\,V_x \end{aligned} \tag{51}$$

Differentrating Eq. (51) with respect to x,

$$L_x = -\,a\,W_{xx} + V_{xx}\,W_y + V_x\,W_{xy} + V_{xx}\,V + V_x^{\,2} - W_{xx}\,V_y - W_x\,V_{xy} + W_{xx}\,W + W_x^{\,2}$$

(52)

$$M_x = -\,W_{xy} - V_x + U_{xy}\,W_x + U_y\,W_{xx} - W_{yx}\,U_x - W_y\,U_{xx} - V_x\,U_x - V\,U_{xx}$$

$$N_x = V_{xy} - W_x + a\,U_{xx} + U_{xx} V_y + U_x V_{xy} - U_{xx} W - U_x W_x - U_{xy} V_x$$
$$- U_y V_{xx}$$

Differentiating Eq. (51) with respect to y,

$$L_y = - a\,W_{xy} + V_{xy} W_y + V_x W_{yy} + V_{xy} V + V_x V_y - W_{xy} V_y - W_x V_{yy}$$
$$+ W_{xy} W + W_x W_y$$

$$M_y = - W_{yy} - V_y + U_{yy} W_x + U_y W_{xy} - W_{yy} U_x - W_y U_{xy} - V_y U_x - V\,U_{xy} \quad (53)$$

$$N_y = V_{yy} - W_y + a\,U_{xy} + U_{xy} V_y + U_x V_{yy} - U_{xy} W - U_x W_y - U_{yy} V_x$$
$$- U_y V_{xy}$$

By Eq. (44),

$$L_3 = L_x$$
$$M_3 = M_x \quad (54)$$
$$N_3 = N_x$$

$$L_4 = L_y/a$$
$$M_4 = (M_y - N)/a \quad (55)$$
$$N_4 = (N_y + M)/a$$

By the equations (See Art. 5),

$$D^{*2} = L^2 + M^2 + N^2$$
$$D^* D^*_x = L\,L_x + M\,M_x + N\,N_x$$
$$D^* D^*_y = L\,L_y + M\,M_y + N\,N_y$$

we obtain

$$D^*_x/D^* = \frac{L\,L_x + M\,M_x + N\,N_x}{L^2 + M^2 + N^2}$$

$$D^*_y/D^* = \frac{L\,L_y + M\,M_y + N\,N_y}{L^2 + M^2 + N^2}$$

Hence, by Eq. (46), the strain components are

$$\epsilon_x = \epsilon'_x + z\left(\frac{L_3 + L_1\,L_3 + M_1\,M_3 + N_1\,N_3}{\sqrt{L^2 + M^2 + N^2}}\right)$$
$$+ \frac{z^2}{2}\left[\frac{L_3{}^2 + M_3{}^2 + N_3{}^2}{L^2 + M^2 + N^2} - \left(\frac{L\,L_x + M\,M_x + N\,N_x}{L^2 + M^2 + N^2}\right)^2\right]$$

$$\epsilon_y = \epsilon'_y + z\left(\frac{1 + 2\epsilon'_y}{a} + \frac{M_4 + L_2\,L_4 + M_2\,M_4 + N_2\,N_4}{\sqrt{L^2 + M^2 + N^2}}\right)$$
$$+ \frac{3z^2}{2a^2}\left[1 + 2\epsilon'_y + \frac{4a}{3}\;\frac{M_4 + L_2\,L_4 + M_2\,M_4 + N_2\,N_4}{\sqrt{L^2 + M^2 + N^2}}\right. \qquad (56)$$
$$\left. + \frac{a^2}{3}\left(\frac{L_4{}^2 + M_4{}^2 + N_4{}^2}{L^2 + M^2 + N^2}\right) - \frac{1}{3}\left(\frac{L\,L_y + M\,M_y + N\,N_y}{L^2 + M_2 + N^2}\right)^2\right]$$

$$\gamma_{xy} = \gamma'_{xy} + \frac{z}{a}\left[\gamma'_{xy} + a\left(\frac{L_4 + L_1\,L_4 + M_1\,M_4 + N_1N_4 + M_3 + L_2L_3 + M_2M_3 + N_2N_3}{\sqrt{L^2 + M^2 + N^2}}\right)\right]$$
$$+ \frac{z^2}{a^2}\left[\gamma'_{xy} + a\left(\frac{L_4 + L_1L_4 + M_1M_4 + N_1N_4 + M_3 + L_2L_3 + M_2M_3 + N_2N_3}{\sqrt{L^2 + M^2 + N^2}}\right)\right.$$
$$\left. + a^2\left(\frac{L_3L_4 + M_3M_4 + N_3N_4}{L^2 + M^2 + N^2}\right) - a\,\frac{(LL_x + MM_x + NN_x)(LL_y + MM_y + NN_y)}{(L^2 + M^2 + N^2)^2}\right]$$

Now, let

$$\epsilon_x = (P_1 + p_1) + z\,(Q_1 + q_1) + z^2\,(R_1 + r_1)$$

$$\epsilon_y = (P_2 + p_2) + z\,(Q_2 + q_2) + z^2\,(R_2 + r_2)$$

$$\gamma_{xy} = (P_3 + p_3) + z\,(Q_3 + q_3) + z^2\,(R_3 + r_3)$$

where P_i, Q_i, R_i denote first degree terms in u, v, w, and p_i, q_i, r_i denote second degree terms.

The strain energy density $\mathcal{V}_0$ is

$$\mathcal{V}_0 = \frac{G}{1 - \nu}\Big[\,\epsilon_x^{\,2} + \epsilon_y^{\,2} + 2\,\epsilon_x\,\epsilon_y + \tfrac{1}{2}\,(1 - \nu)\,\gamma_{xy}^{2} - 2\,(1 + \nu)(\epsilon_x + \epsilon_y)\,K\,T\Big] \tag{58}$$

and the total strain energy $\mathcal{V}$ is

$$\mathcal{V} = \iiint \mathcal{V}_0 \; r dr d\theta dz \tag{59}$$

Substituting Eq. (56) into Eq. (58), and then substituting the resulting expression for $\mathcal{V}_0$ into Eq. (59) and integrating with respect to z, collecting in powers of h, and keeping the temperature terms separate, we obtain the following results:

$$\epsilon_x^{\,2} = (P_1 + p_1)^2 + 2(P_1 + p_1)(Q_1 + q_1)\,z + \left[2(R_1 + r_1)(P_1 + p_1) + (Q_1 + q_1)^2\right] z^2 + 2(Q_1 + q_1)(R_1 + r_1)\,z^3 + (R_1 + r_1)^2\,z^4$$

$$\epsilon_y^{\,2} = (P_2 + p_2)^2 + 2(P_2 + p_2)(Q_2 + q_2)\,z + \left[2(R_2 + r_2)(P_2 + p_2) + (Q_2 + q_2)^2\right] z^2 + 2(Q_2 + q_2)(R_2 + r_2)\,z^3 + (R_2 + r_2)^2\,z^4$$

$$\gamma_{xy}^{2} = (P_3 + p_3)^2 + 2(P_3 + p_3)(Q_3 + q_3)\, z$$

$$+ \left[2(R_3 + r_3)(P_3 + p_3) + (Q_3 + q_3)^2 \right] z^2$$

$$+ 2(Q_3 + q_3)(R_3 + r_3)\, z^3 + (R_3 + r_3)^2\, z^4$$

$$\epsilon_x \epsilon_y = (P_1 + p_1)(P_2 + p_2) + \left[(P_1 + p_1)(Q_2 + q_2) + (P_2 + p_2)(Q_1 + q_1) \right] z$$

$$+ \left[(P_1 + p_1)(R_2 + r_2) + (P_2 + p_2)(R_1 + r_1) + (Q_1 + q_1)(Q_2 + q_2) \right] z^2$$

$$+ \left[(Q_1 + q_1)(R_2 + r_2) + (Q_2 + q_2)(R_1 + r_1) \right] z^3$$

$$+ (R_1 + r_1)(R_2 + r_2)\, z^4$$

$$\epsilon_x + \epsilon_y = (P_1 + P_2 + p_1 + p_2) + (Q_1 + Q_2 + q_1 + q_2)\, z + (R_1 + R_2 + r_1 + r_2)\, z^2 \tag{60}$$

and

$$\int_{-h/2}^{h/2} \epsilon_x^{2}\, dz = (P_1 + p_1)^2 h + \frac{h^3}{12} \left[2(R_1 + r_1)(P_1 + p_1) + (Q_1 + q_1)^2 \right]$$

$$+ \frac{h^5}{80} (R_1 + r_1)^2$$

$$\int_{-h/2}^{h/2} \epsilon_y^{2}\, dz = h(P_2 + p_2)^2 + \frac{h^3}{12} \left[2(R_2 + r_2)(P_2 + p_2) + (Q_2 + q_2)^2 \right]$$

$$+ \frac{h^5}{80} (R_2 + r_2)^2$$

$$\int_{-h/2}^{h/2} \gamma_{xy}^{\,2}\, dz = h\,(P_3 + p_3)^2 + \frac{h^3}{12}\left[2\,(R_3 + r_3)\,(P_3 + p_3) + (Q_3 + q_3)^2\right] + \frac{h^5}{80}\,(R_3 + r_3)^2$$

$$\int_{-h/2}^{h/2} \epsilon_x\,\epsilon_y = h\,(P_1 + p_1)\,(P_2 + p_2) + \frac{h^3}{12}\left[\,(P_1 + p_1)\,(R_2 + r_2) + (P_2 + p_2)\,(R_1 + r_1) + (Q_1 + q_1)\,(Q_2 + q_2)\right] + \frac{h^5}{80}\,(R_1 + r_1)\,(R_2 + r_2)$$

$$\int_{-h/2}^{h/2} (\epsilon_x + \epsilon_y)\, dz = h\,(P_1 + P_2 + p_1 + p_2) + \frac{h^3}{12}\,(R_1 + R_2 + r_1 + r_2)$$

Consequently,

$$\int \mathcal{V}_o\, dz = \frac{G}{1-\nu}\Bigg\{$$

$$h\,\Big[(P_1^{\,2} + P_2^{\,2} + \tfrac{1}{2}(1-\nu)\,P_3^{\,2} + 2\nu\,P_1P_2) + (2\,P_1\,p_1 + 2\,P_2\,p_2 + (1-\nu)\,P_3\,p_3$$

$$+ 2\nu\,P_1\,p_2 + 2\nu\,P_2\,p_1) + (p_1^{\,2} + p_2^{\,2} + \tfrac{1}{2}(1-\nu)\,p_3^{\,2} + 2\nu\,p_1\,p_2)\Big]$$

$$+ \frac{h^3}{12}\Big[\,(Q_1^{\,2} + 2\,R_1\,P_1 + Q_2^{\,2} + 2\,R_2\,P_2 + \tfrac{1}{2}(1-\nu)\,Q_3^{\,2} + (1-\nu)\,R_3\,P_3$$

$$+ 2\nu\,P_1\,R_2 + 2\nu\,P_2\,R_1 + 2\nu\,Q_1\,Q_2) + (2Q_1\,q_1 + 2\,R_1\,p_1 + 2\,P_1\,r_1 + 2Q_2\,q_2$$

$$+ 2\,R_2\,p_2 + 2\,P_2\,r_2 + (1-\nu)\,Q_3\,q_3 + (1-\nu)\,R_3\,p_3 + (1-\nu)\,P_3\,r_3 + 2\nu\,P_1\,r_2$$

$$+ 2\nu\,R_2\,p_1 + 2\nu\,p_2\,r_1 + 2\nu\,R_1\,p_2 + 2\nu\,Q_1\,q_2 + 2\nu\,Q_2\,q_1) + (q_1^{\,2} + 2\,r_1\,p_1$$

$$+ q_2^2 + 2 r_2 p_2 + \frac{1}{2}(1-\nu) q_3^2 + (1-\nu) r_3 p_3 + 2\nu p_1 r_2 + 2\nu p_2 r_1 \qquad (62)$$

$$+ 2\nu q_1 q_2)\Big] + \frac{h^5}{80} \Big[(R_1^2 + R_2^2 + \frac{1}{2}(1-\nu) R_3^2 + 2\nu R_1 R_2) + (2R_1 r_1 + 2 R_2 r_2$$

$$+ (1-\nu) R_3 r_3 + 2\nu R_1 r_2 + 2\nu R_2 r_1) + (r_1^2 + r_2^2 + \frac{1}{2}(1-\nu) r_3^2 + 2\nu r_1 r_2)\Big]$$

$$- 2(1+\nu)\Big[h(P_1 + P_2 + p_1 + p_2) + \frac{h^3}{12}(R_1 + R_2 + r_1 + r_2)\Big] KT \Big\}$$

where, from Eq. (56), and (57),

$$P_1 = U_x$$

$$p_1 = \frac{1}{2}(U_x^2 + V_x^2 + W_x^2)$$

$$Q_1 = - W_{xx}$$

$$q_1 = \frac{1}{a}\left[a U_{xx} W_x + V_{xx}(W_y + V)\right]$$

$$R_1 = 0$$

$$r_1 = \frac{1}{2a^2}\left[a^2 W_{xx}^2 + (W_{xy} + V_x)^2 \right]$$

$$(63)$$

$$P_2 = \frac{1}{a}(V_y - W)$$

$$p_2 = \frac{1}{2a^2}\left[U_y^2 + (V_y - W)^2 + (W_y + V)^2 \right]$$

$$Q_2 = \frac{1}{a^2}(- W - W_{yy})$$

$$q_2 = \frac{1}{a^3}\Big[U_y^2 + \frac{1}{2} a^2 W_x^2 + a U_{yy} W_x + \frac{1}{2}(W_y + V)^2 + (W_y + V)(V_{yy} - W_y)$$

$$+ (V_y - W)^2 \Big]$$

$$R_2 = \frac{1}{a^3} (- W - W_{yy})$$

$$V_2 = \frac{1}{2a^4} \Big[3 U_y{}^2 + 3(V_y - W)^2 + 3 (W_y + V)^2 + 10a\, U_{yy} W_x + 5a^2 W_x{}^2$$

$$+ 10 (W_y + V) (V_{yy} - W_y) + a^2 W_{xy}{}^2 - 2a\, U_y W_{xy} + (W_{yy} + V_y)^2$$

$$- 2 (V_y - W) (W_{yy} + V_y) \Big]$$

$$P_3 = \frac{1}{a} (a V_x + U_y)$$

$$p_3 = \frac{1}{a} \Big[U_x U_y + V_x (V_y - W) + W_x (W_y + V) \Big]$$

$$Q_3 = \frac{1}{a^2} (U_y - a V_x - 2a W_{xy})$$

$$q_3 = \frac{1}{a^2} \Big[U_y U_x + 2a\, U_{xy} W_x + V_x (V_y - W) + (W_y + V) (2V_{xy} - W_x) \Big]$$

$$R_3 = \frac{1}{a^3} (U_y - a W_{xy})$$

$$r_3 = \frac{1}{a^3} \Big[a^2 W_{xx} W_{xy} - a U_y W_{xx} + a W_x U_{xy} + U_y U_x + V_x (V_y - W)$$

$$+ V_{xy} (W_y + V) + (W_{xy} + V_x) (W_{yy} + W) \Big]$$

7. STRAIN ENERGY FOR LARGE DEFLECTIONS OF SPHERICAL SHELLS

If (X, Y, Z) are rectangular coordinates, the middle surface of a spherical shell with center at origin is given by the equations

$$
\begin{aligned}
X &= a \sin x \cos y \\
Y &= a \sin x \sin y \qquad\qquad (64) \\
Z &= a \cos x
\end{aligned}
$$

where a is the radius of the middle surface x is the colatitudes and y is the longitude (See Fig. 2)

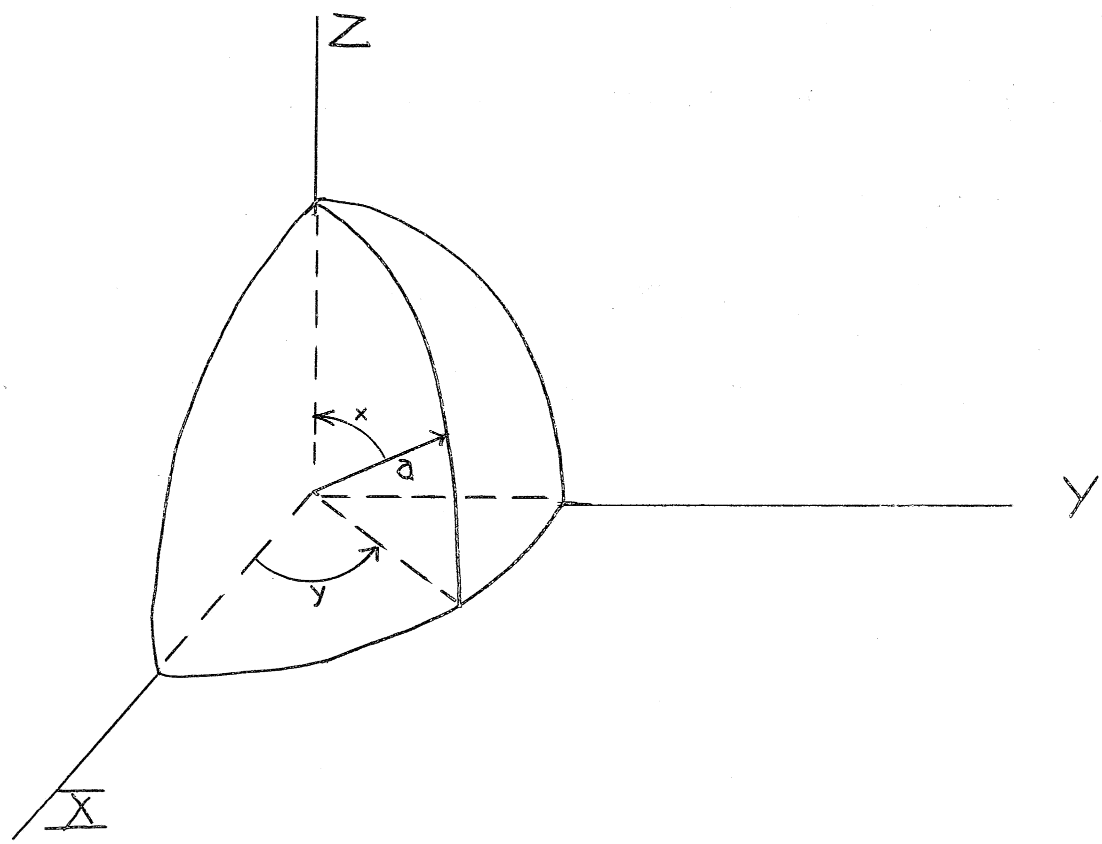

FIG. 2

By Eq. (5), (7), (20), and (21), (22)

$$E = a^2 \qquad F = 0 \qquad G = a^2 \sin^2 x$$

$$A = a \qquad B = a \sin x \qquad D = a^2 \sin x$$

$$r_1 = r_2 = -a \qquad k_g = \frac{1}{a^2} \qquad k_m = -\frac{1}{a}$$

By Eq. (33), where for special cases we denote (u, v, w) by (U, V, W), respectively,

$$L_1 = \frac{1}{a}(U_x + W)$$

$$M_1 = \frac{1}{a} V_x \tag{64}$$

$$N_1 = \frac{1}{a}(W_x - U)$$

$$L_2 = \frac{1}{a \sin x}(U_y - V \cos x)$$

$$M_2 = \frac{1}{a \sin x}(V_y + U \cos x + W \sin x) \tag{65}$$

$$N_2 = \frac{1}{a \sin x}(W_y - V \sin x)$$

By Eq. (34),

$$\epsilon'_x = \frac{1}{a}(U_x + W) + \frac{1}{2a^2}\left[(U_x + W)^2 + V_x^{\ 2} + (W_x - U)^2\right]$$

$$\epsilon'_y = \frac{1}{a \sin x}(V_y + U \cos x + W \sin x) + \frac{1}{2a^2 \sin^2 x}\left[(U_y - V \cos x)^2 + (V_y + U \cos x + W \sin x)^2 + (W_y - V \sin x)^2\right]$$

$$\gamma'_{xy} = \frac{V_x}{a} + \frac{1}{a \sin x}(U_y - V \cos x) + \frac{1}{a^2 \sin x}\left[(U_x + W)(U_y - V \cos x) + V_x (V_y + U \cos x + W \sin x) + (W_x - U)(W_y - V \sin x)\right]$$

By Eq. (39),

$$L = a \sin x (U - W_x) + V_x (W_y - V \sin x) - (W_x - U)(V_y + U \cos x + W \sin x)$$

$$M = a (V \sin x - W_y) + (W_x - U)(U_y - V \cos x) - (U_x + W)(W_y - V \sin x) \qquad (66)$$

$$N = a^2 \sin x + a \sin x (U_x + W) + a (V_y + U \cos x + W \sin x) - V_x (U_y - V \cos x)$$
$$+ (U_x + W)(V_y + U \cos x + W \sin x)$$

Differentiating L, M, N, with respect to x and y, we find

$$L_x = a \cos x (U - W_x) + a \sin x (U_x - W_{xx}) + V_{xx} (W_y - V \sin x)$$
$$+ V_x (W_{xy} - V_x \sin x - V \cos x) - (W_{xx} - U_x)(V_y + U \cos x + W \sin x)$$
$$- (W_x - U)(V_{xy} + U_x \sin x - U \sin x + W_x \sin x + W \cos x)$$

$$M_x = a (V_x \sin x + V \cos x - W_{xy}) + (W_{xx} - U_x)(U_y - V \cos x)$$
$$+ (W_x - U)(U_{xy} - V_x \cos x + V \sin x) - (U_{xx} + W_x)(W_y - V \sin x)$$
$$- (U_x + W)(W_{xy} - V_x \sin x - V \cos x)$$

$$N_x = a^2 \cos x + 2a \cos x (U_x + W) + a \sin x (U_{xx} + W_x) + a (V_{xy} - U \sin x$$
$$+ W_x \sin x) - V_{xx} (U_y - V \cos x) - V_x (U_{xy} + V \sin x - V_x \cos x)$$
$$+ (U_{xx} + W_x)(V_y + U \cos x + W \sin x) + (U_x + W)(V_{xy} + U_x \cos x$$
$$- U \sin x + W_x \sin x + W \cos x) \qquad (67)$$

and

$$L_y = a \sin x (U_y - W_{xy}) + V_{xy} (W_y - V \sin x) + V_x (W_{yy} - V_y \sin x)$$

$$- (W_{xy} - U_y) (V_y + U \cos x + W \sin x) - (W_x - U) (V_{yy} + U_y \cos x$$

$$+ W_y \sin x)$$

$$M_y = a (V_y \sin x - W_{yy}) + (W_{xy} - U_y) (U_y - V \cos x) + (W_x - U) (U_{yy} - V_y \cos x)$$

$$- (U_{xy} + W_y) (W_y - V \sin x) - (U_x + W) (W_{yy} - V_y \sin x)$$

$$N_y = a \sin x (U_{xy} + W_y) + a (V_{yy} + U_y \cos x + W_y \sin x) - V_{xy} (U_y - V \cos x)$$

$$- V_x (U_{yy} - V_y \cos x) + (U_{xy} + W_y) (V_y + U \cos x + W \sin x)$$

$$+ (U_x + W) (V_{yy} + U_y \cos x + W_y \sin x) \qquad (68)$$

By Eq. (44),

$$L_3 = (L_x + N)/a$$

$$M_3 = M_x/a$$

$$N_3 = (N_x - L)/a \qquad (69)$$

$$L_4 = \frac{1}{a \sin x} (L_y - M \cos x)$$

$$M_4 = \frac{1}{a \sin x} (M_y + N \sin x + L \cos x) \qquad (70)$$

$$N_4 = \frac{1}{a \sin x} (N_y - M \sin x)$$

From the equations (See Art. 5)

$$D^{*2} = L^2 + M^2 + N^2$$

$$D^* D^*_x = LL_x + MM_x + NN_x$$

$$D^* D^*_y = LL_y + MM_y + NN_y$$

we get

$$\frac{D^*_x}{D^*} = \frac{LL_x + MM_x + NN_x}{L^2 + M^2 + N^2}$$

$$\frac{D^*_y}{D^*} = \frac{LL_y + MM_y + NN_y}{L^2 + M^2 + N^2}$$

Hence, by Eq. (46), the strain components are

$$\epsilon_x = \epsilon'_x + z\left(\frac{L_3 + L_1 L_3 + M_1 M_3 + N_1 N_3}{\sqrt{L^2 + M^2 + N^2}} - \frac{1 + \epsilon'_x}{a} \right)$$

$$+ z^2 \left[\frac{3}{2a^2} (1 + 2\epsilon'_x) - \frac{2}{a} \left(\frac{L_3 + L_1 L_3 + M_1 M_3 + N_1 N_3}{\sqrt{L^2 + M^2 + N^2}} \right) \right.$$

$$\left. + \frac{1}{2} \left(\frac{L_3^2 + M_3^2 + N_3^2}{L^2 + M^2 + N^2} \right) - \frac{1}{2a^2} \left(\frac{LL_x + MM_x + NN_x}{L^2 + M^2 + N^2} \right)^2 \right]$$

$$\epsilon_y = \epsilon'_y + z \left[\frac{M_4 + L_2 L_4 + M_2 M_4 + N_2 N_4}{\sqrt{L^2 + M^2 + N^2}} - \frac{1 + 2\epsilon'_y}{a} \right]$$

$$+ z^2 \left[\frac{3}{2a^2} (1 + 2\epsilon'_y) - \frac{2}{a} \frac{M_4 + L_2 L_4 + M_2 M_4 + N_2 N_4}{\sqrt{L^2 + M^2 + N^2}} \right.$$

$$\left. + \frac{1}{2} \left(\frac{L_4^2 + M_4^2 + N_4^2}{L^2 + M^2 + N^2} \right) - \frac{1}{2a^2 \sin^2 x} \left(\frac{LL_y + MM_y + NN_y}{L^2 + M^2 + N^2} \right)^2 \right]$$

$$\gamma_{xy} = \gamma'_{xy} + z\left[\frac{L_4 + L_1L_4 + M_1M_4 + N_1N_4 + M_3 + L_2L_3 + M_2M_3 + N_2N_3}{\sqrt{L^2 + M^2 + N^2}} - \frac{2}{a}\gamma'_{xy}\right]$$

$$+ z^2\left[\frac{3}{a^2}(\gamma'_{xy}) - \frac{2}{a}\;\frac{L_4 + L_1L_4 + M_1M_4 + N_1N_4 + M_3 + L_2L_3 + M_2M_3 + N_2N_3}{\sqrt{L^2 + M^2 + N^2}}\right.$$

$$\left. + \frac{L_3L_4 + M_3M_4 + N_3N_4}{L^2 + M^2 + N^2} - \frac{1}{a^2 \sin x}\left(\frac{LL_x + MM_x + NN_x}{L^2 + M^2 + N^2}\right)\left(\frac{LL_y + MM_y + NN_y}{L^2 + M^2 + N^2}\right)\right]$$

Now, let

$$\epsilon_x = (P_1 + p_1) + z(Q_1 + q_1) + z^2(R_1 + r_1)$$

$$\epsilon_y = (P_2 + p_2) + z(Q_2 + q_2) + z^2(R_2 + r_2) \qquad (72)$$

$$\gamma_{xy} = (P_3 + p_3) + z(Q_3 + q_3) + z^2(R_3 + r_3)$$

where, by Eqs. (71) and (72),

$$P_1 = \frac{1}{a}(U_x + W)$$

$$p_1 = \frac{1}{2a^2}\left[(U_x + W)^2 + V_x^{\,2} + (W_x - u)^2\right] \qquad (73)$$

$$Q_1 = \frac{-1}{a^2}(W + W_{xx})$$

$$q_1 = \frac{1}{a^3 \sin^2 x}\Big[\sin x\, V_{xx}(W_y - V \sin x) + \sin^2 x\,(W_x - U)(U_{xx} + W_x)$$

$$- \frac{1}{2}\sin^2 x\,(U - W_x)^2 - \frac{1}{2}(V \sin x - W_y)^2 - \sin^2 x\,(U_x + W)^2$$

$$- \sin^2 x\, V_x^{\,2}\Big]$$

$$R_1 = \frac{1}{a^3 \sin x} \left[(W + W_{xx}) \sin x + (1 - \cos x)(U_{xx} + W_x) \right]$$

$$r_1 = \frac{1}{2a^4 \sin^4 x} \Big\{ 7 \sin^4 x (U_x + W)^2 + 3 \sin^4 x V_x^2 - 2 \sin^3 x V_{xx} (W_y - V \sin x)$$

$$- 2 \sin^3 x \cos x (U - W_x)(U_x - W_{xx}) - 2 \sin x \cos x (U \sin x - W_y)(V_x \sin x + V \cos x - W_{xy})$$

$$+ 2 \sin^3 x \cos x V_x V - 4 \cos^2 x \sin x (U_x + W)(V_y + U \cos x + W \sin x)$$

$$- 2 \cos^2 x \sin x V_x (U_y - V \cos x) - \sin^2 x \cos^2 x (U_x + W)^2$$

$$+ 2 \sin^2 x \cos x (U_x + W)(V_{xy} - U \sin x + W_x \sin x) + \cos^2 x (V_y + U \cos x + W \sin x)^2$$

$$- 2 \sin^3 x V_x U_y + 2 \sin^3 x (U_x - W_{xx})(V_y + U \cos x + W \sin x)$$

$$+ 2 \sin^3 x V_x (W_{xy} - V_x \sin x) + 2 \sin^3 x (U_x + W)(V_y + U \cos x + W \sin x)$$

$$+ \sin^4 x W_{xx}^2 + 4 \sin^3 x \cos x U (U_x + W) - 2 \sin^3 x \cos x W_{xx} U$$

$$- 2 \sin^3 x \cos x W_{xx} (U - W_x) + 4 \sin^3 x V_y (U_x + W) - 2 \sin^3 x V_y W_{xx} + \cos^2 x \sin^2 x U^2$$

$$+ 2 \cos^2 x \sin^2 x U (U - W_x) + 2 \cos x \sin^2 x V_y U + 2 \cos x \sin^2 x (U - W_y) V_y$$

$$+ \sin^2 x V_y^2 + \sin^2 x (V_x \sin x + V \cos x - W_{xy})^2 - 2 \cos x \sin^2 x V_x W_y$$

$$+ 2 \cos x \sin^2 x (W_x - U)(V_y + U \cos x + W \sin x) + 2 \sin^3 x \cos x (U - W_x)(U_x + W)$$

$$+ 2 \sin^4 x (U_{xx} + W_x)(U - W_x) + \sin^4 x (U - W_x)^2 - 8 \sin^2 x (U_x + W)^2$$

$$+ 4 \sin^3 x W_{xx} (V_y + U \cos x + W \sin x) - 4 \sin^2 x V_y (V_y + U \cos x + W \sin x)$$

$$- 4 \sin^3 x V_y (U_x + W) - 4 \sin^2 x (U_{xx} + W_x)(V_y + U \cos x + W \sin x)$$

$$- 4 \sin^3 x \,(U_{xx} + W_x)\,(U_x + W) - 4 \cos x \sin x \, V_{xy}\,(V_y + U \cos x + W \sin x)$$

$$- 4 \cos x \sin^2 x \, V_{xy}\,(U_x + W) - 4 \cos x \sin^2 x \, W_x\,(V_y + U \cos x + W \sin x)$$

$$- 4 \cos^3 x \, W_x\,(U_x + W) - 10 \sin x\,(U_x + W)\,(V_y + U \cos x + W \sin x)$$

$$+ 2 \sin x \, V_x\,(U_y - V \cos x) + (V \sin x - W_y)^2 + 3\,(V_y + U_x \sin x + U \cos x + 2\,W \sin x)^2 \Big\}$$

$$P_2 = \frac{1}{a \sin x} \quad (V_y + U \cos x + W \sin x)$$

$$p_2 = \frac{1}{2a^2 \sin^2 x} \Big[(U_y - V \cos x)^2 + (V_y + U \cos x + W \sin x)^2$$

$$+ (W_y - V \sin x)^2 \Big]$$

$$Q_2 = \frac{-1}{a^2 \sin^2 x} \Big[\sin x \cos x \, W_x + W_{yy} + \sin^2 x \, W \Big]$$

$$q_2 = \frac{1}{2a^3 \sin^2 x} \Big[2 \cos x \, V_x\,(W_y - V \sin x)$$

$$+ 2\,(W_x - U)\,(U_{yy} - V_y \cos x) + \frac{2}{\sin x}\,(W_y - V \sin x)\,(V_{yy} + U_y \cos x + W_y \sin x)$$

$$+ 2 \cot x\,(U - W_x)\,(V_y + U \cos x + W \sin x) - 2 \cot x\,(V \sin x - W_y)\,(U_y - V \cos x)$$

$$- 2\,(V_y + U \cos x + W \sin x)\,(U \cos x + W \sin x) - 2\,V_y\,(V_y + U \cos x + W \sin x)$$

$$- 2 \sin x \cos x\,(U_x + W)\,(U - W_x) - \sin^2 x\,(U - W_x)^2 - (V \sin x - W_y)^2$$

$$- 2\,(U_y - V \cos x)^2 \Big]$$

$$R_2 = \frac{1}{a^3 \sin^2 x} \Big[\sin^2 x \, W + W_{yy} + \sin x \cos x \, W_x \Big]$$

$$r_2 = \frac{1}{2a^4 \sin^4 x} \Big\{ 3 \sin^2 x\,(U_y - V \cos x)^2 + 5 \sin^2 x\,(W_y - V \sin x)^2$$

$$- 4 \sin^2 x\,(U_x \sin x - W_x \cos x)\,(V_y + U \cos x + W \sin x) + 4 \sin x \, W_{yy}\,(V_y$$

$$+ U\cos x + W\sin x) - 4\cos x\sin x\,(U - W_x)(V_y + U\cos x + W\sin x)$$

$$+ 4\cos x\sin x\,(V\sin x - W_y)(U_y - V\cos x) + \sin^2 x\,(U_y - W_{xy})^2$$

$$- 2\sin x\cos x\,(U_y - W_{xy})(V\sin x - W_y) + \cos^2 x\,(V\sin x - W_y)^2 + 2\sin^2 x\,(U_{xy} + W_y)$$

$$+ 2\sin x\,(U_{xy} + V\sin x)(V_{yy} + U_y\cos x + W_y\sin x) + 2\sin^2 x\,(U_y - V\cos x)(W_{xy}$$

$$- U_y - \sin x\,V_x) - 2\sin^2 x\,(\cos x\,V_x - U_{xy} - W_y)(W_y - V\sin x)$$

$$- 2\sin^2 x\,(W_x - U)(U_{yy} - V_y\cos x) + 2\sin^2 x\,(U_x + W)(W_{yy} - V_y\sin x) + 4\sin^3 x\cos x$$

$$(U - W_x)(V_y + U\cos x + W\sin x) + 2\sin^2 x\,(U_x + W)(V_y\sin x - W_{yy})$$

$$+ 6\sin x\,(V_y + U\cos x + W\sin x)(V_y\sin x - W_{yy}) + (V_y\sin x - W_{yy})^2$$

$$+ 2\sin x\cos x\,(U - W_x)(V_y\sin x - W_{yy}) + 3\sin^2 x\,(U - W_x)^2 - 8\sin^3 x\,(U_x + W)$$

$$(V_y + U\cos x + W\sin x) - 6\sin^4 x\,(U_x + W)^2 - x\sin^2 x\,(V_y + U\cos x + W\sin x)^2$$

$$- 4\sin x\,(U_y\sin x - W_{yy})(V_y + U\cos x + W\sin x) + 2\sin^3 x\cos x\,(U - W_x)(U_x + W)$$

$$- 2\sin^3 x\,V_x\,(U_y - V\cos x) - 2\,(U_{xy} + W_y)(U_{yy} + U_y\cos x)$$

$$P_3 = \frac{V_x}{a} + \frac{1}{a\sin x}\,(U_y - V\cos x)$$

$$p_3 = \frac{1}{a^2\sin x}\Big[(U_x + W)(U_y - V\cos x) + V_x\,(V_y + U\cos x + W\sin x)$$

$$+ (W_x - U)(W_y - V\sin x)\Big]$$

$$Q_3 = \frac{2}{a^2\sin^2 x}\Big[\cos x\,(W_y - V\sin x) + \sin x\,(V\cos x - W_{xy})\Big]$$

$$
\begin{aligned}
q_3 = \frac{1}{a^3 \sin^2 x} \Big[& 2 V_{xy} (W_y - V \sin x) - 2 \cos x (W_x - U) (U_y - V \cos x) \\
& - 2 \cos x (U_x + W) (W_y - V \sin x) + \sin x (W_x - U) (2 U_{xy} - 2V_x \cos x \\
& + 2 W_y + V \sin x - V \cos x) - 2 \cot x (W_y - V \sin x) (V_y + U \cos x \\
& + W \sin x) - 2 \sin x (U_x + W) (U_y - V \cos x) - 2 \sin x V_x (V_y \\
& + U \cos x + W \sin x) \Big]
\end{aligned}
$$

$$
R_3 = \frac{1}{a^3 \sin^2 x} \Big[3 \sin x W_{xy} - \sin^2 x V_x - \sin x \cos x V - 2 \cos x W_y \Big]
$$

$$
\begin{aligned}
r_3 = \frac{1}{a^4 \sin^4 x} \Big\{ & 3 \sin^3 x (U_x + W) (U_y - V \cos x) + 3 \sin^3 x V_x (V_y + U \cos x \\
& W \sin x) - \sin^3 x (W_x - U) (W_y - V \sin x) - 4 \sin^2 x V_{xy} (W_y - V \sin x) + 4 \sin^2 x \cos x \\
& \cdot (U_x + W) (W_y - V \sin x) - 2 \sin^3 x (W_x - U) (U_{xy} - 2 V_x \cos x + V \sin x - V \cos x) \\
& + \sin^2 x V_{xy} (W_y - V \sin x) + 2 \sin^2 x \cos x (W_x - U) (U_y - V \cos x) + \sin^3 x (U_y \\
& - W_{xy}) (U_x - W_{xx}) - \cos x \sin x (V \sin x - W_y) (V_y + U \cos x + W \sin x) \\
& - \cos^2 x \sin x (V \sin x - W_y) (U - W_x) - \cos x \sin^2 x (V \sin x - W_y) (U_x - W_{xx}) \\
& + \sin x (V_y \sin x - W_{yy}) (V \cos x - W_{xy}) + \cos x \sin x (U_{xy} + W_y) (V_y + U \cos x \\
& + W \sin x) - \cos x \sin^3 x V (U_x + W) - V \sin^4 x (U_{xx} + W_x) + \sin^2 V_{xy} (U_{xy} \\
& + 2W_y - V \sin x) + 2 \sin^4 x V (U - W_x) - \sin^3 x (U_y - W_{xy}) (U_x + W) - 2 U_{xy} \cos x \sin x \\
& (V_y + U \cos x + W \sin x) - \cos x (V \sin x - W_y) (V_y \sin x - W_{yy}) - \sin^2 x (U_{xy} \\
& + W_y) (V_{xy} - U \sin x + W_x \sin x) + \cos x \sin x (U_y - W_{xy}) (V_y + U \cos x + W \sin x)
\end{aligned}
$$

$- \sin^2 x \cos x \, W_y \, (U_x + W)$

Substitution of Eqs. (73) into Eq. (62) yields $\int \mathcal{V}_o \, dz$ for a spherical shell.

8. STRAIN ENERGY FOR LARGE DEFLECTIONS OF RIGHT-CIRCULAR CONICAL SHELLS

If (X, Y, Z) are rectangular coordinates, the middle surface of a conic shell with the vertex at the origin is given by the equation

$$X = x \cos a$$
$$Y = x \sin a \cos y \qquad (74)$$
$$Z = x \sin a \sin y$$

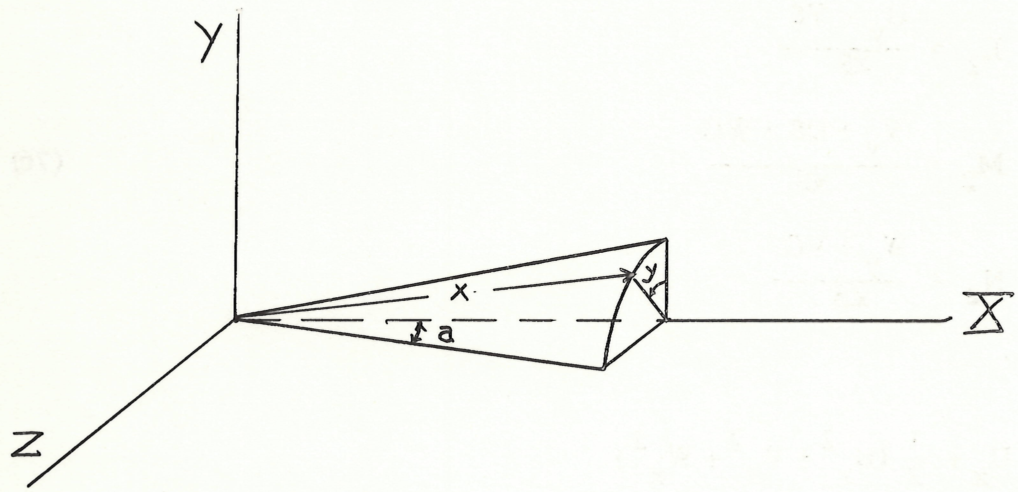

FIG. 3

where x and y are shell coordinates indicated in Fig. (3).

$$E = 1 \qquad F = 0 \qquad G = x^2 \sin^2 a$$
$$A = 1 \qquad B = x \sin a \qquad D = x \sin a$$

and

$$\frac{1}{r_1} = 0 \qquad r_2 = x \tan a$$

$$k_g = 0 \qquad k_m = \frac{1}{2x \tan a}$$

Let

$\sin a = S \qquad \cos a = C \qquad \tan a = T$

then

$G = x^2S^2, \qquad B = xS, \qquad D = xS, \qquad r_2 = xT, \qquad k_m = \frac{1}{2xT}$

By Eq. (33), where for special cases (u, v, w) are denoted by (U, V, W) respectively,

$$\begin{aligned} L_1 &= U_x \\ M_1 &= V_x \\ N_1 &= W_x \end{aligned} \tag{75}$$

$$\begin{aligned} L_2 &= \frac{U_y - VS}{xS} \\ M_2 &= \frac{V_y + US - WC}{xS} \\ N_2 &= \frac{W_y + VC}{xS} \end{aligned} \tag{76}$$

By Eq. (34),

$$\begin{aligned} \epsilon'_x &= U_x + \frac{1}{2}(U_x^{\ 2} + V_x^{\ 2} + W_x^{\ 2}) \\ \epsilon'_y &= \frac{1}{Sx}(V_y + SU - CW) + \frac{1}{2x^2S^2}\Big[(U_y - SV)^2 + (V_y + SU - CW)^2 \\ &\quad + (W_y + CV)^2\Big] \\ \gamma'_{xy} &= V_x + \frac{1}{xS}(U_y - SV) + \frac{1}{xS}\Big[U_x(U_y - SV) + V_x(V_y + SU - CW) \\ &\quad + W_x(W_y + CV)\Big] \end{aligned} \tag{77}$$

By Eq. (39),

$$L = - Sx\, W_x + (W_y + CV)\, V_x - (V_y + SU - CW)\, W_x$$

$$M = - (W_y + CV) + (U_y - SV)\, W_x - (W_y + CV)\, U_x \qquad (78)$$

$$N = Sx + Sx\, U_x + (V_y + SU - CW) + (V_y + SU - CW)\, U_x - (U_y - SV)\, V_x$$

Differentiating L, M, N, with respect to x and y, we obtain

$$L_x = - SW_x - Sx\, W_{xx} + (W_{xy} + CV_x)\, V_x + (W_y + CV)\, V_{xx} - W_{xx} (V_y + SU - CW) - W_x (V_{xy} + SU_x - CW_x)$$

$$M_x = - W_{xy} - CV_x + W_{xx} (U_y - SV) + W_x (U_{xy} - SV_x) - U_{xx} (W_y + CV) - U_x (W_{xy} + CV_x) \qquad (79)$$

$$N_x = S + SU_x + Sx\, U_{xx} + (V_{xy} + SU_x - CW_x) + U_{xx} (V_y + SU - CW) + U_x (V_{xy} + SU_x - CW_x) - V_{xx} (U_y - SV) - V_x (U_{xy} - SV_x)$$

and

$$L_y = Sx\, W_{xy} + (W_{yy} + CV_y)\, V_x + (W_y + CV)\, V_{xy} - (V_{yy} + SU_y - CW_y)\, W_x - (V_y + SU - CW)\, W_{xy}$$

$$M_y = - (W_{yy} + CV_y) + (U_{yy} - SV_y)\, W_x + (U_y - SV)\, W_{xy} - (W_{yy} + CV_y)\, U_x - (W_y + CV)\, U_{xy} \qquad (80)$$

$$N_y = Sx\, U_{xy} + (V_{yy} + SU_y - CW_y) + (V_{yy} + SU_y - CW_y)\, U_x + (V_y + SU - CW)\, U_{xy} - (U_{yy} - SV_y)\, V_x - (U_y - SV)\, V_{xy}$$

By Eq. (44),

$$L_3 = L_x$$
$$M_3 = M_x \qquad (81)$$
$$N_3 = N_x$$

and

$$L_4 = \frac{1}{xS}\Big[S(W_y + CV) - Sx\,W_{xy} + (W_{yy} + CV_y)\,V_x + (W_y + CV)\,V_{xy}$$
$$- (V_{yy} + 2S\,U_y - CW_y - SV)\,W_x - (V_y + SU - CW)\,W_{xy} + S(W_y + CV)\,U_x\Big]$$

$$M_4 = \frac{1}{xS}\Big[- CSx - CSx\,U_x - C(V_y + SU - CW) - (W_{yy} + CV_y) - S^2 x\,W_x$$
$$+ (U_{yy} - 2S\,V_y - S^2U + CSW)\,W_x + (U_y - SV)\,W_{xy} - (W_{yy} + 2\,CV_y$$
$$+ CSU - C^2W)\,U_x - (W_y + CV)\,U_{xy} + (SW_y + CU_y)\,V_x\Big] \qquad (82)$$

$$N_4 = \frac{1}{xS}\Big[Sx\,U_{xy} + (V_{yy} + SU_y - CW_y) - C(W_y + CV) + V_{yy} + SU_y - 2CW_y$$
$$- C^2V)\,U_x + (V_y + SU - CW)\,U_{xy} - (U_{yy} - SV_y)\,V_x - (U_y - SV)\,V_{xy}$$
$$+ C(U_y - SV)\,W_x\Big]$$

From the equations (See Art. 5)

$$D^{*2} = L^2 + M^2 + N^2$$

$$D^*\,D^*_x = LL_x + MM_x + NN_x$$

$$D^*\,D^*_y = LL_y + MM_y + NN_y$$

we obtain

$$\frac{D^*_x}{D^*} = \frac{LL_x + MM_x + NN_x}{L^2 + M^2 + N^2}$$

$$\frac{D^*_y}{D^*} = \frac{LL_y + MM_y + NN_y}{L^2 + M^2 + N^2}$$

Hence, by Eq. (46), the strain components are

$$\begin{aligned}
\epsilon_x &= \epsilon'_x + \frac{z}{r_1}\left[1 + 2\epsilon'_x + \frac{r_1}{D^*}(L_3 + L_1L_3 + M_1M_3 + N_1N_3)\right] \\
&+ \frac{3}{2}\left(\frac{z}{r_1}\right)^2\left[1 + 2\epsilon'_x + \frac{4}{3}\frac{r_1}{D^*}(L_3 + L_1L_3 + M_1M_3 + N_1N_3)\right. \\
&\left. + \frac{1}{3}\left(\frac{r_1}{D^*}\right)^2\left(L_3^2 + M_3^2 + N_3^2 - \frac{D^{*2}_x}{A^2}\right)\right] \\
&= \epsilon'_x + z\left(\frac{L_3 + L_1L_3 + M_1M_3 + N_1N_3}{\sqrt{L^2 + M^2 + N^2}}\right) \\
&+ \frac{z^2}{2}\left[\frac{L_3^2 + M_3^2 + N_3^2}{L^2 + M^2 + N^2} - \left(\frac{LL_x + MM_x + NN_x}{L^2 + M^2 + N^2}\right)^2\right]
\end{aligned} \tag{83}$$

$$\begin{aligned}
\epsilon_y &= \epsilon'_y + \frac{z}{r_2}\left[1 + 2\epsilon'_y + \frac{r_2}{D^*}(M_4 + L_2L_4 + M_2M_4 + N_2N_4)\right] \\
&+ \frac{3}{2}\left(\frac{z}{r_2}\right)^2\left[1 + 2\epsilon'_y + \frac{4}{3}\frac{r_2}{D^*}(M_4 + L_2L_4 + M_2M_4 + N_2N_4)\right. \\
&\left. + \frac{1}{3}\left(\frac{r_2}{D^*}\right)^2\left(L_4^2 + M_4^2 + N_4^2 - \frac{D^{*2}_y}{B^2}\right)\right] \\
&= \epsilon'_y + \frac{z}{xT}\left[1 + 2\epsilon'_y + xT\frac{M_4 + L_2L_4 + M_2M_4 + N_2N_4}{\sqrt{L^2 + M^2 + N^2}}\right] \\
&+ \frac{3z^2}{3(xT)^2}\left[1 + 2\epsilon'_y + \frac{4xT}{3}\frac{M_4 + L_2L_4 + M_2M_4 + N_2N_4}{\sqrt{L^2 + M^2 + N^2}}\right.
\end{aligned}$$

$$\gamma_{xy} = \gamma'_{xy} + \frac{z}{xT}\left[\gamma'_{xy} + xT\,\frac{L_4 + L_1L_4 + M_1M_4 + N_1N_4 + M_3 + L_2L_3 + M_2M_3 + N_2N_3}{\sqrt{L^2 + M^2 + N^2}}\right]$$

$$+ \frac{z^2}{x^2T^2}\left[\gamma'_{xy} + xT\,\frac{L_4 + L_1L_4 + M_1M_4 + N_1N_4 + M_3 + L_2L_3 + M_2M_3 + N_2N_3}{\sqrt{L^2 + M^2 + N^2}}\right.$$

$$+ x^2T^2\left(\frac{L_3L_4 + M_3M_4 + N_3N_4}{L^2 + M^2 + N^2}\right)$$

$$\left. - \frac{xT}{C}\left(\frac{LL_x + MM_x + NN_x}{L^2 + M^2 + N^2}\right)\left(\frac{LL_y + MM_y + NN_y}{L^2 + M^2 + N^2}\right)\right]$$

Now, let

$$\epsilon_x = (P_1 + p_1) + z\,(Q_1 + q_1) + z^2\,(R_1 + r_1)$$

$$\epsilon_y = (P_2 + p_2) + z\,(Q_2 + q_2) + z^2\,(R_2 + r_2) \qquad (84)$$

$$\gamma_{xy} = (P_3 + p_3) + z\,(Q_3 + q_3) + z^2\,(R_3 + r_3)$$

where, by Eqs. (83) and (84),

$$P_1 = U_x$$

$$p_1 = \frac{1}{2}\,(U_x^{\;2} + V_x^{\;2} + W_x^{\;2})$$

$$Q_1 = -\,W_{xx}$$

$$q_1 = U_{xx}\,W_x + \frac{1}{Sx}\,(W_y + CV)\,V_{xx}$$

$$R_1 = -\frac{2}{Sx^3}\left[\,Sx\,U_x + (V_y + SU - CW)\right] \qquad (85)$$

$$r_1 = \frac{1}{2S^2x^4}\left[\,S^2\,x^2\,(W_x + x\,W_{xx})^2 + x^2\,(W_{xy} + CV_x)^2\right.$$

$$- 2Sx^2 (U_{xy} - SV_x) V_x - 2Sx^2 (U_y - SV) V_{xx} - 6 S^2 x^3 U_x U_{xx} - 4x^2 SU_x (V_{xy}$$

$$+ SU_x - CW_x) - 4 Sx U_x (V_y + SU - CW) - 4S x^2 (V_y + SU - CW) U_{xx}$$

$$- 6 x (V_y + SU - CW) (V_{xy} + SU_x - CW_x) - S^2 x^2 W_x{}^2 - (W_y + CV)^2$$

$$+ 2S x V_x (U_y - SV) + 2 (V_y + SU - CW)^2 - 4 S^2 x^2 U_x{}^2 \Big]$$

$$P_2 = \frac{1}{Sx} (V_y + SU - CW)$$

$$p_2 = \frac{1}{2x^2 S^2} \Big[(U_y - SV)^2 + (V_y + SU - CW)^2 + (W_y + CV)^2 \Big]$$

$$Q_2 = \frac{1}{S^2 x^2} \Big[C (V_y + SU - CW) - S^2 x W_x - (W_{yy} + C V_y) \Big]$$

$$q_2 = \frac{1}{x^3 S^3} \Big[C (U_y - SV)^2 + C (V_y + SU - CW)^2 + \frac{C}{2} (W_y + CV)^2$$

$$+ x S W_x (U_{yy} - SV_y) + x^2 S^2 V_x W_y + S (U_y - SV) (W_y + CV) + (W_y + CV) (V_{yy}$$

$$+ SU_y - CW_y) - S^2 x W_x (V_y + SU - CW) + S^3 x^2 W_x U_x + \frac{CS^2 x^2}{2} W_x{}^2 \Big]$$

$$R_2 = \frac{1}{S^3 x^3} \Big[C^2 S x U_x - C S^2 x W_x - C (W_{yy} + C V_y) + 2 C^2 (V_y + SU - CW) \Big]$$

$$r_2 = \frac{1}{2x^4 S^4} \Big[3C^2 (U_y - SV)^2 + (W_y + CV)^2 + 2CS^2 x (W_y - CV) V_x$$

$$+ 4 CS (U_y - SV) (W_y + CV) + 2C (W_y + CV) (V_{yy} + SU_y - CW_y) - 4CS^2 x W_x$$

$$(V_y + SU - CW) + S^2 x^2 W_x{}^2 - 2S^2 x W_{xy} (W_y + CV) + S^2 x^2 W_{xy}{}^2$$

$$- 2SC x (U_{yy} - SV_y) W_x - 2CS x (U_y - SV) W_{xy} + 2 CS^3 x^2 U_x W_x$$

$$+ (W_{yy} + CV_y)^2 + 2 S^2 x W_x (W_{yy} + CV_y) - 2C (W_{yy} + CV_y) (V_y + SU - CW)$$

$$+ 3C^2 (V_y + SU - CW)^2 \Big]$$

$$P_3 = V_x + \frac{1}{xS} (U_y - SV)$$

$$p_3 = \frac{1}{xS} \left[U_x (U_y - SV) + V_x (V_y + SU - CW) + W_x (W_y + CV) \right]$$

$$Q_3 = \frac{1}{x^2 S^2} \left[C (U_y - SV) + 2S (W_y + CV) - 2 S x W_{xy} - CS x V_x \right]$$

$$q_3 = \frac{1}{x^2 S^2} \left[C U_x (U_y - SV) + C V_x (V_y + SU - CW) + 2 V_{xy} (W_y + CV) \right.$$
$$- CW_x (W_y + CV) + 2S x W_x (U_{xy} - SV_x) - 2 S W_x (U_y - SV) - \frac{2}{x}$$
$$\left. \cdot (W_y + CV) (V_y + SU - CW) \right]$$

$$R_3 = \frac{1}{x^3 S^3} \left[C^2 (U_y - SV) + CS (W_y + CV) - CS x W_{xy} \right]$$

$$r_3 = \frac{1}{x^3 S^3} \left[- (W_y + CV) W_x - S^2 x W_{xx} (W_y + CV) + S^2 x^2 W_{xy} W_{xx} - CSx (U_y - SV) W_{xx} \right.$$
$$+ (W_{xy} + CV_x) (W_{yy} + CV_y) + S^2 (x W_x (W_{xy} + CV_x) - C (W_y + CV) (V_{xy} + SU_x$$
$$- CW_x) - C (W_{xy} + CV_x) (V_y + SU - CW) - 2S^2 x U_x U_{xy} - 2S U_{xy} (V_y + SU - CW)$$
$$- 2SU_x (V_{yy} + SU_y - CW_y) - \frac{2}{x} (V_{yy} + SU_y - CW_y) (V_y + SU - CW)$$
$$+ 2C V_{xy} (W_y + CV) + CS x W_x (U_{xy} - S V_x) - CS W_x (U_y - SV)$$
$$- \frac{1}{x} (W_y + CV) (W_{yy} + CV_y)$$

Substitution of Eq. (85) into Eq. (62) yields $\int \mathcal{V}_0 \, dz$ for a conical shell.

9. SUMMARY

Expressions for the strain components of an arbitrary point in an elastic isotropic thin shell of any shape are given by Eqs. (46) in terms of the strain components of the middle surface and functions of (x, y, z). These results are specialized for the circular cylindrical shell in Eqs. (57) and (63). Similar results for the sphere and for the right-circular cone are given by Eqs. (72) and (73) and by Eqs. (84) and (85), respectively.

A strain energy expression for a circular cylindrical shell subjected to a temperature distribution that is an arbitrary function of middle surface coordinates (x, y) is given by Eq. (62). Similar expressions for a spherical shell and for a right-circular conical shell may be obtained by Eqs. (62) and (73) and by Eqs. (62) and (85), respectively.

In general, the practical use of these expressions is formidable even with the use of digital computers. For equilibrium studies of very thin shells, it is questionable that terms beyond h^3 are significant. However, in questions of thermal buckling and post-buckling behavior, higher degree terms in h may be significant.

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