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Dynamics of Missile Launchers
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ANALYSIS OF MISSILE LAUNCHERS

Part J Phase 1

(Four-Degree-of-Freedom Multiple Launcher)

by

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This is the 1th part of the first phase report on University of Illinois Project No. 46-22-60-304, "Launcher Dynamics Study". The analysis was performed under contract No. DA-11-070-508-ORD-593. This particular report is an investigation of a four-degree of freedom multiple launcher including blast force effects.

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Respectfully submitted
University of Illinois

A handwritten signature in cursive script, reading "M. Stippes".

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LAUNCHER DYNAMICS STUDY

Part J Phase 1

Four-Degree-of-Freedom Multiple Launcher

ABSTRACT:

This report contains an analysis of a four degree of freedom model multiple launching system. The launcher pivots about a point and the motion is resisted by torsional springs. All of the missiles are parallel to one another in the launcher and are to be fired one by one in a pre-arranged order. The equations of motion of the system including the effects of blast are then established. In the second part, the effect of the blast force of the missile on the launcher face is considered.

TABLE OF CONTENTS

List of Symbols

Chapter		Page
I	Introduction	1
II	Mathematical Description of the Problem	3
	1. Coordinate System	3
	2. Generalized Coordinates	3
	3. Transformation of Coordinates	4
	4. Position and Velocity Relations	5
	5. Kinetic and Potential Energy of the System	8
	6. Equations of Motion	10
	7. Free Motion of the Launcher	13
III	Computation of the Blast Force	16

List of Symbols

(A, B, C)	reference system fixed to the earth
(X, Y, Z)	coordinates fixed in space whose Euler angles relative to (A, B, C) are $(\gamma_e, \gamma_a, \theta)$
(x, y, z)	coordinates fixed to the box at the pivot point O .
(x_m, y_m, z_m)	coordinates fixed to the mass center of the moving missile.
η	distance from mass center of the moving missile to xz -plane
γ_e	angle of elevation
γ_a	traverse angle of the launcher
$(\theta_1, \theta_2, \theta_3)$	rotation of (x, y, z) relative to (X, Y, Z)
(o, y_b, z_b)	coordinates of O_B the mass center of the launcher, in (x, y, z)
(x_m, η, z_m)	coordinates of O_M mass center of the moving missile, in (x, y, z)
(x_i, η_o, z_i)	coordinates of O_I , mass center of the unfired missile in (x, y, z)
M_b, M_m	mass of the box and the missile respectively
M'_m	mass of the unfired missile $M_m = M'_m$ in numerical value
$I_{11}, I_{12}, \text{ etc.}$	mass moments of inertia of the launcher with respect to (x, y, z)
$J_{11}, J_{12}, \text{ etc.}$	mass moments of inertia of the firing missile with respect to (x_m, y_m, z_m)
$J'_{11}, J'_{12}, \text{ etc.}$	mass moments of inertia of the unfired missile in the launcher with respect to (x'_m, y'_m, z'_m)
W_b, W_m	weight of the box and the missile respectively
W'_m	weight of the unfired missile $W_m = W'_m$ in numerical value
i	the reverse of the order of firing i.e. the number of missiles left in the launcher
$F(t)$	thrust force of the missile
\mathcal{F}	dissipation function

T	blast force on the launcher
$\beta_{11}, \beta_{12}, \text{ etc.}$	torsional spring constants
$\underline{c}_{11}, \underline{c}_{12}, \text{ etc.}$	damping coefficients
(X_T, η_b, Z_T)	point of application of the blast force in (x, y, z) . η_b is the constant thickness of the launcher along y -axis. X_T, Z_T change with time when the missile leaves the launcher

INTRODUCTION

This launching system differs from the others because it contains a number of missiles which can be fired successively in a launcher of box type. Referring to Fig. 1, we notice that the launcher is fixed at O where motion is resisted by torsional springs and dash pots. The launching tubes are parallel and are open at both ends. We also assume that there is no interference between exhaust cones of successive firings. In other words, at each time we consider the effect of one missile only.

The following information can be obtained from this report:

- (1) Equations of motion of the system before and after the missile leaves the launcher.
- (2) The linear displacement, velocity and acceleration of the mass center of the missile at end of guidance.
- (3) The displacements (angular) of the launcher.
- (4) An expression for computing the magnitude and point of application of the blast force acting on a part of a ring section.

Chapter II

MATHEMATICAL DESCRIPTION OF THE PROBLEM

1. Coordinate System:

(A, B, C)	A fixed reference system on the earth
(X, Y, Z)	Another fixed reference system on the earth. Y is parallel to the initial direction of the launcher tubes. This set of axes is obtained by rotating (A, B, C) through a traverse angle γ_a about the C -axis, followed by a rotation of γ_e about the X -axis (the new position of the A -axis after the rotation γ_a)
(x, y, z)	This set of axes is fixed in the launcher at O and moves with it. The set is coincident with (X, Y, Z) initially.
(x_m, y_m, z_m)	A set of axes fixed at the mass center of the missile which is being fired. These axes move with the missile.
(x'_m, y'_m, z'_m)	Set of axes fixed at the mass center of an arbitrary unfired missile. These axes are always parallel to (x, y, z)
η	Since the missile always moves parallel to the tube, one coordinate is enough to describe completely its motion relative to the launcher. η is measured from the mass center of the missile to the xz -plane.

2. Generalized Coordinates

This system has four degrees of freedom.

The orientation of the launcher may be described by three successive rotations $\theta_1, \theta_2, \theta_3$ as shown in Fig. 1. The position of a moving missile with respect to the

launcher is expressed by η . Thus the generalized coordinates are $\theta_1, \theta_2, \theta_3, \eta$.

3. Transformation of Coordinates

Following the same line of reasoning in TAM Report 139, p. 16, we obtain the transformation matrices

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = [S] \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (1a)$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = [T] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (2a)$$

where

$$[S] = \begin{bmatrix} \ell_1 & -m_1 \ell_2 & m_1 m_2 \\ m_1 & \ell_1 \ell_2 & -\ell_1 m_2 \\ 0 & m_2 & \ell_2 \end{bmatrix} \quad (1b)$$

$$m_1 = \sin \gamma_a, \quad \ell_1 = \cos \gamma_a$$

$$m_2 = \sin \gamma_e, \quad \ell_2 = \cos \gamma_e$$

$$[T] = \begin{bmatrix} 1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2} & -\theta_3 & \theta_2 \\ \theta_3 + \theta_1 \theta_2 & 1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} & -\theta_1 \\ -\theta_2 + \theta_1 \theta_3 & \theta_1 + \theta_2 \theta_3 & 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \end{bmatrix} \quad (2b)$$

and

$$[S][T] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (3)$$

where

$$a_{11} = \ell_1 \left(1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2} \right) - m_1 \ell_2 (\theta_3 + \theta_1 \theta_2) + m_1 m_2 (-\theta_2 + \theta_1 \theta_3)$$

$$a_{12} = -\ell_1 \theta_3 - m_1 \ell_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + m_1 m_2 (\theta_1 + \theta_2 \theta_3)$$

$$a_{13} = \ell_1 \theta_2 + m_1 \ell_2 \theta_1 + m_1 m_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right)$$

$$a_{21} = m_1 \left(1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_1 \ell_2 (\theta_3 + \theta_1 \theta_2) - \ell_1 m_2 (-\theta_2 + \theta_1 \theta_3)$$

$$a_{22} = -m_1 \theta_3 + \ell_1 \ell_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) - \ell_1 m_2 (\theta_1 + \theta_2 \theta_3)$$

$$a_{23} = m_1 \theta_2 - \ell_1 \ell_2 \theta_1 - \ell_1 m_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right)$$

$$a_{31} = m_2 (\theta_3 + \theta_1 \theta_2) + \ell_2 (-\theta_2 + \theta_1 \theta_3)$$

$$a_{32} = m_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3)$$

$$a_{33} = -m_2 \theta_1 + \ell_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right)$$

4. Position and Velocity Relations

O_B : mass center of the launcher

From Eqs. (1a), (2a), we obtain $\overline{O O_B}$ in (A, B, C) as

$$\begin{bmatrix} A_b \\ B_b \\ C_b \end{bmatrix} = [S] [T] \begin{bmatrix} 0 \\ y_b \\ z_b \end{bmatrix}$$

Then

$$C_b = Y_b \left[m_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3) \right] \\ + Z_b \left[-m_2 \theta_1 + \ell_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \right] \quad (4)$$

O_M : mass center of the firing missile

Following the same procedure and taking the velocity components in (X, Y, Z) we obtain

$$\begin{bmatrix} V_{mx} \\ V_{my} \\ V_{mz} \end{bmatrix} = [\dot{T}] \begin{bmatrix} x_m \\ \eta \\ z_m \end{bmatrix} + [T] \begin{bmatrix} 0 \\ \dot{\eta} \\ 0 \end{bmatrix} \quad (5a)$$

$$V_{mx} = -x_m (\theta_2 \dot{\theta}_2 + \theta_3 \dot{\theta}_3) - \eta \dot{\theta}_3 + z_m \dot{\theta}_2 - \dot{\eta} \theta_3 \\ V_{my} = x_m (\dot{\theta}_3 + \dot{\theta}_1 \theta_2 + \theta_1 \dot{\theta}_2) - \eta (\theta_1 \dot{\theta}_1 + \theta_3 \dot{\theta}_3) - z_m \dot{\theta}_1 + \dot{\eta} \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) \\ V_{mz} = x_m (-\dot{\theta}_2 + \dot{\theta}_1 \theta_3 + \theta_1 \dot{\theta}_3) + \eta (\dot{\theta}_1 + \dot{\theta}_2 \theta_3 + \theta_2 \dot{\theta}_3) - z_m (\theta_1 \dot{\theta}_1 + \theta_2 \dot{\theta}_2) \\ + \dot{\eta} (\theta_1 + \theta_2 \theta_3) \quad (5b)$$

$$C_m = x_m \left[m_2 (\theta_3 + \theta_1 \theta_2) + \ell_2 (-\theta_2 + \theta_1 \theta_3) \right] \\ + \eta \left[m_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3) \right] \\ + z_m \left[-m_2 \theta_1 + \ell_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \right] \quad (6)$$

O = mass center of an arbitrary unfired missile.

We assume here that i is the reverse of the order of firing. i lies between 1 and k . The $(k+1)^{st}$ missile is the one being fired. For every given i , we can locate the missile by the use of (x_i, η_o, z_i) . Note here η_o is the same for all unfired missiles.

$$\begin{bmatrix} V_{ix} \\ V_{iy} \\ V_{iz} \end{bmatrix} = [T] \begin{bmatrix} x_i \\ \eta_o \\ z_i \end{bmatrix} \quad (7a)$$

$$V_{ix} = -x_i (\dot{\theta}_2 \dot{\theta}_2 + \dot{\theta}_3 \dot{\theta}_3) - \eta_o \dot{\theta}_3 + z_i \dot{\theta}_2$$

$$V_{iy} = x_i (\dot{\theta}_3 + \dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_1 \dot{\theta}_2) - \eta_o (\dot{\theta}_1 \dot{\theta}_1 + \dot{\theta}_3 \dot{\theta}_3) - z_i \dot{\theta}_1 \quad (7b)$$

$$V_{iz} = x_i (-\dot{\theta}_2 + \dot{\theta}_1 \dot{\theta}_3 + \dot{\theta}_1 \dot{\theta}_3) + \eta_o (\dot{\theta}_1 + \dot{\theta}_2 \dot{\theta}_3 + \dot{\theta}_2 \dot{\theta}_3) - z_i (\dot{\theta}_1 \dot{\theta}_1 + \dot{\theta}_2 \dot{\theta}_2)$$

$$\begin{aligned} C_i = x_i [m_2 (\theta_3 + \theta_1 \theta_2) + \ell_2 (-\theta_2 + \theta_1 \theta_3)] \\ + \eta_o [m_2 (1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2}) + \ell_2 (\theta_1 + \theta_2 \theta_3)] \\ + z_i [-m_2 \theta_1 + \ell_2 (1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2})] \end{aligned} \quad (8)$$

i ranges from 1 to k

$F(t)$: thrust force

We assume here that the thrust force acts on O_M and along the longitudinal axis of the missile

From

$$\begin{bmatrix} X_m \\ Y_m \\ Z_m \end{bmatrix} = [T] \begin{bmatrix} x_m \\ \eta \\ z_m \end{bmatrix} \quad (9a)$$

we obtain

$$\begin{aligned} X_m &= x_m \left(1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2}\right) - \eta \theta_3 + z_m \theta_2 \\ Y_m &= x_m (\theta_3 + \theta_1 \theta_2) + \eta \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2}\right) - z_m \theta_1 \\ Z_m &= x_m (-\theta_2 + \theta_1 \theta_3) + \eta (\theta_1 + \theta_2 \theta_3) + z_m \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2}\right) \end{aligned} \quad (9b)$$

The components of F are

$$\begin{aligned} F_x &= -F \theta_3 \\ F_y &= F \\ F_z &= F \theta_1 \end{aligned} \quad (10)$$

5. Kinetic and Potential Energy of the System

Kinetic Energy:

$$T = T_b + \sum T'_m + T_m \quad (11)$$

where T_b , T_m , $\sum T'_m$ denote the kinetic energy of the launcher, of the firing missile and of the unfired missiles respectively.

$$T_b = \frac{1}{2} (I_{11} \dot{\theta}_1^2 + I_{22} \dot{\theta}_2^2 + I_{33} \dot{\theta}_3^2 - 2 I_{23} \dot{\theta}_2 \dot{\theta}_3) \quad (11a)$$

I_{12} , I_{13} vanish because of the symmetry of the launcher

$$T_m = \frac{1}{2} [M_m (V_{mx}^2 + V_{my}^2 + V_{mz}^2) + J_{11} \dot{\theta}_1^2 + J_{22} \dot{\theta}_2^2 + J_{33} \dot{\theta}_3^2] \quad (11b)$$

$J_{12} = J_{13} = J_{23} = 0$ due to symmetry of the missile

$$\sum T'_m = \sum_{i=1}^K \frac{1}{2} \left[M'_m (V_{ix}^2 + V_{iy}^2 + V_{iz}^2) + J'_{11} \dot{\theta}_1^2 + J'_{22} \dot{\theta}_2^2 + J'_{33} \dot{\theta}_3^2 \right] \quad (11c)$$

Potential Energy

$$V = V_b + \sum V'_m + V_m + V_s \quad (12)$$

where V_b , V_m , $\sum V'_m$, V_s denote the potential energy of the launcher, of the firing missile, of the unfired missiles, and of the spring, respectively.

$$\begin{aligned} V_b &= W_b C_b \\ &= W_b \left\{ y_b \left[m_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3) \right] \right. \\ &\quad \left. + z_b \left[-m_2 \theta_1 + \ell_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \right] \right\} \end{aligned} \quad (12a)$$

$$\begin{aligned} V_m &= W_m C_m \\ &= W_m \left\{ x_m \left[m_2 (\theta_3 + \theta_1 \theta_2) + \ell_2 (-\theta_2 + \theta_1 \theta_3) \right] \right. \\ &\quad + \eta \left[m_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3) \right] \\ &\quad \left. + z_m \left[-m_2 \theta_1 + \ell_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \right] \right\} \end{aligned} \quad (12b)$$

$$\begin{aligned}
\Sigma V'_m &= \sum_{i=1}^k W'_m C_i \\
&= W'_m \sum_{i=1}^k \left\{ x_i \left[m_2 (\theta_3 + \theta_1 \theta_2) + \ell_2 (-\theta_2 + \theta_1 \theta_3) \right] \right. \\
&\quad \left. + z_i \left[-m_2 \theta_1 + \ell_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \right] \right\} \\
&\quad + W'_m k \eta_o \left[m_2 \left(1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3) \right] \quad (12c)
\end{aligned}$$

$$\begin{aligned}
V_s &= \frac{1}{2} (\beta_{11} \theta_1^2 + \beta_{22} \theta_2^2 + \beta_{33} \theta_3^2 + 2\beta_{12} \theta_1 \theta_2 \\
&\quad + 2\beta_{13} \theta_1 \theta_3 + 2\beta_{23} \theta_2 \theta_3) \quad (12d)
\end{aligned}$$

Dissipation due to dash pots

$$\begin{aligned}
\mathcal{F} &= \frac{1}{2} (c_{11} \dot{\theta}_1^2 + c_{22} \dot{\theta}_2^2 + c_{33} \dot{\theta}_3^2 + 2c_{12} \dot{\theta}_1 \dot{\theta}_2 \\
&\quad + 2c_{13} \dot{\theta}_1 \dot{\theta}_3 + 2c_{23} \dot{\theta}_2 \dot{\theta}_3) \quad (13)
\end{aligned}$$

6. Equations of Motion

If θ_i ($i = 1, 2, 3, 4$) denotes the generalized coordinates $(\theta_1, \theta_2, \theta_3, \eta)$, Lagrange's equations of motion are :

$$\begin{aligned}
\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{\theta}_i} \right] - \frac{\partial T}{\partial \theta_i} &= - \frac{\partial V}{\partial \theta_i} - \frac{\partial \mathcal{F}}{\partial \dot{\theta}_i} + F_x \frac{\partial X_m}{\partial \theta_i} + F_y \frac{\partial Y_m}{\partial \theta_i} \\
&\quad + F_z \frac{\partial Z_m}{\partial \theta_i} \quad (14)
\end{aligned}$$

The above equation yields four differential equations. Three of these may be written in indicial form as follows:

$$\begin{aligned}
a_{ij} \ddot{\theta}_j + b_{ij} \dot{\theta}_j + c_{ij} \theta_j + d_i &= 0 \quad (15) \\
(i, j &= 1, 2, 3)
\end{aligned}$$

where

$$a_{11} = I_{11} + M_m (Z_m^2 + \eta^2) + J_{11} + M'_m \sum_{i=1}^k (z_i^2 + \eta_o^2) + k J'_{11}$$

$$a_{12} = -M_m x_m \eta - M'_m \eta_o \sum_{i=1}^k x_i$$

$$a_{13} = -M_m x_m z_m - M'_m \sum_{i=1}^k x_i z_i$$

$$a_{21} = -M_m x_m \eta - M'_m \eta_o \sum_{i=1}^k x_i$$

$$a_{22} = I_{22} + M_m (z_m^2 + x_m^2) + J_{22} + M'_m \sum_{i=1}^k (x_i^2 + z_i^2) + k J'_{22}$$

$$a_{23} = -I_{23} - M_m z_m \eta - M'_m \sum_{i=1}^k z_i \eta_o$$

$$a_{31} = -M_m x_m z_m - M'_m \sum_{i=1}^k x_i z_i$$

$$a_{32} = -I_{23} - M_m z_m \eta - M'_m \eta_o \sum_{i=1}^k z_i$$

$$a_{33} = I_{33} + M_m (\eta^2 + x_m^2) + J_{33} + M'_m \sum_{i=1}^k (\eta_o^2 + x_i^2) + k J'_{33}$$

$$b_{11} = 2 M_m \eta \dot{\eta} + c_{11}$$

$$b_{12} = c_{12}$$

$$b_{13} = c_{13}$$

$$b_{21} = -2 M_m x_m \dot{\eta} + c_{12}$$

$$b_{22} = c_{22}$$

$$b_{23} = -2 M_m z_m \dot{\eta} + c_{13}$$

$$b_{31} = c_{13}$$

$$b_{32} = c_{23}$$

$$b_{33} = 2 M_m \eta \dot{\eta} + c_{33}$$

$$c_{11} = -W_b (y_b m_2 + z_b \ell_2) - W_m (\eta m_2 + z_m \ell_2) - W'_m \sum_{i=1}^k z_i \ell_2 \\ - W'_m k \eta_o m_2 + \beta_{11}$$

$$c_{12} = M_m x_m \ddot{\eta} + W_m x_m m_2 + W'_m \sum_{i=1}^k x_i m_2 + \beta_{12} = F x_m$$

$$c_{13} = W_m x_m \ell_2 + W'_m \sum_{i=1}^k x_i \ell_2 + \beta_{13}$$

$$c_{21} = W_m x_m m_2 + W'_m m_2 \sum_{i=1}^k x_i + \beta_{12}$$

$$c_{22} = -W_b z_b \ell_2 - W_m z_m \ell_2 - W'_m \sum_{i=1}^k z_i \ell_2 + \beta_{22}$$

$$c_{23} = W_b y_b \ell_2 + W_m \eta \ell_2 + W'_m k \eta_o \ell_2 + \beta_{23} - M_m z_m \ddot{\eta} + F z_m$$

$$c_{31} = W_m x_m \ell_2 + W'_m \sum_{i=1}^k x_i \ell_2 + \beta_{13}$$

$$c_{32} = W_b \ell_2 + W_m \eta \ell_2 + W'_m k \eta_o \ell_2 + \beta_{23}$$

$$c_{33} = -W_b y_b m_2 - W_m m_2 \eta - W'_m k \eta_o m_2 + \beta_{33}$$

$$d_1 = -M_m z_m \ddot{\eta} + W_b (y_b \ell_2 - z_b m_2) + W_m (\eta \ell_2 - z_m m_2) - W'_m \sum_{i=1}^k z_i m_2 \\ + W'_m k \eta_o \ell_2 + F z_m$$

$$d_2 = -W_m x_m \ell_2 - W'_m \sum_{i=1}^k x_i \ell_2$$

$$d_3 = M_m x_m \ddot{\eta} + W_m x_m m_2 + W'_m \sum_{i=1}^k x_i m_2 - F x_m$$

The fourth equation is

$$-M_m z_m \ddot{\theta}_1 + M_m x_m \ddot{\theta}_3 + W_m \ell_2 \theta_1 + M_m \ddot{\eta} + W_m m_2 - F = 0 \quad (16)$$

7. Free Motion of the Launcher

After the missile leaves the launcher, the latter will assume free motion.

Eq. (16) becomes trivial and vanishes identically. The required equations of motion can be obtained by putting

$$\eta = \eta_0, \quad M_m = 0, \quad W_m = 0, \quad J_{11} = J_{12} \dots = 0$$

$$x_m = X_T, \quad z_m = Z_T, \quad F = -T$$

The last term, $F = -T$, is due to the exhaust of the missile leaving the launcher.

Also note X_T, Z_T are quite different from x_m, z_m .

The equations are

$$a'_{ij} \ddot{\theta}_j + b'_{ij} \dot{\theta}_j + c'_{ij} \theta_j + d'_i = 0 \quad (17)$$

$$i, j = 1, 2, 3$$

where

$$a'_{11} = I_{11} + M'_m \sum_{i=1}^k (z_i^2 + \eta_0^2) + k J'_{11}$$

$$a'_{12} = -M'_m \eta_0 \sum_{i=1}^k x_i$$

$$a'_{13} = -M'_m \sum_{i=1}^k x_i z_i$$

$$a'_{21} = -M'_m \eta_0 \sum_{i=1}^k x_i$$

$$a'_{22} = I_{22} + M'_m \sum_{i=1}^k (x_i^2 + z_i^2) + k J'_{22}$$

$$a'_{23} = -I_{23} - M'_m \sum_{i=1}^k z_i \eta_0$$

$$a'_{31} = -M'_m \sum_{i=1}^k x_i z_i$$

$$a'_{32} = -I_{23} - M'_m \eta_o \sum_{i=1}^k z_i$$

$$a'_{33} = I_{33} + M'_m \sum_{i=1}^k (\eta_o^2 + x_i^2) + k J'_{33}$$

$$b'_{11} = c_{11}$$

$$b'_{12} = c_{12}$$

$$b'_{13} = c_{13}$$

$$b'_{21} = c_{12}$$

$$b'_{22} = c_{22}$$

$$b'_{23} = c_{23}$$

$$b'_{31} = c_{13}$$

$$b'_{32} = c_{23}$$

$$b'_{33} = c_{33}$$

$$c'_{11} = -W_b (y_b m_2 + z_b \ell_2) - W'_m \sum_{i=1}^k z_i \ell_2 - W'_m k \eta_o m_2 + \beta_{11}$$

$$c'_{12} = W'_m \sum_{i=1}^k x_i m_2 + \beta_{12} + T X_T$$

$$c'_{13} = W'_m \sum_{i=1}^k x_i \ell_2 + \beta_{13}$$

$$c'_{21} = W'_m m_2 \sum_{i=1}^k x_i + \beta_{12}$$

$$c'_{22} = -W_b z_b \ell_2 - W'_m \sum_{i=1}^k z_i \ell_2 + \beta_{22}$$

$$c'_{23} = W_b y_b \ell_2 + W'_m k \eta_o \ell_2 + \beta_{23} - T Z_T$$

$$c'_{31} = W'_m \sum_{i=1}^k x_i \ell_2 + \beta_{13}$$

$$c'_{32} = W_b \ell_2 + W'_m k \eta_o \ell_2 + \beta_{23}$$

$$c'_{33} = -W_b y_b m_2 - W'_m k \eta_o m_2 + \beta_{33}$$

$$d'_1 = W_b (y_b \ell_2 - z_b m_2) - W'_m \sum_{i=1}^k z_i m_2 + W'_m k \eta_o \ell - T Z_T$$

$$d'_2 = -W'_m \sum_{i=1}^k x_i \ell_2$$

$$d'_3 = W'_m \sum_{i=1}^k x_i m_2 + T X_T$$

Chapter III

COMPUTATION OF THE BLAST FORCE

To calculate the blast force acting on the launcher due to the exhaust field of the rocket, we use an approximate value for the stagnation pressure the coordinates of which are fixed in the rocket. This force will be incorrect due to the velocity of the rocket; however, since we are also neglecting interaction between the exhaust jet and the launcher, we will consider it sufficiently accurate to adopt this point of view. The calculation is based on the method outlined in the following reference:

"Method of Determining Stagnation Temperature-Pressure Distribution in Exhaust Field of Rocket Using Underexpanded Convergent-Divergent Nozzle", Report Number 58-875 of Rock Island Arsenal Research & Development Division Design Engineering Branch.

The following notation will be used in this section:

S	the distance measured from the nozzle exit of the missile along its axis of symmetry
r	the distance from a point to the axis of symmetry of the missile
$p(S, r)$	total gage pressure at point (S, r)
P_m	total gage pressure at the axis of symmetry
P_e	pressure at the nozzle exit

$$P_e = P_c \left[\frac{\frac{\gamma+1}{2} M^2}{1 + \frac{\gamma-1}{2} M^2} \right]^{\frac{\gamma}{\gamma-1}} \left[\frac{2\gamma}{\gamma+1} M^2 - \frac{\gamma-1}{\gamma+1} \right]^{\frac{1}{1-\gamma}} - P_a$$

P_c	rocket motor chamber pressure
γ	ratio of specific heats

M Mach number of gas flow at exit of nozzle. May be computed from:

$$\frac{A_t}{A_e} = M \left[\frac{\gamma + 1}{2 + (\gamma - 1) M^2} \right]^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

A_t area of nozzle at throat

A_e area of nozzle at exit

P_a atmospheric pressure

S_o for $S \leq S_o$, $P_m = P_e$

d_e diameter of nozzle exit

r_e radius of nozzle exit

$$k_1 = P_a \left[\left(\frac{\gamma + 1}{2} \right)^{\frac{\gamma}{\gamma - 1}} - 1 \right] \left[16 (M - 1) d_e \right]^{-k_2}$$

$$k_2 = - (1.4 + 0.437 M)$$

$$k_3 = d_e (S/d_e)_1$$

$(S/d_e)_1$ is given by Fig. 2

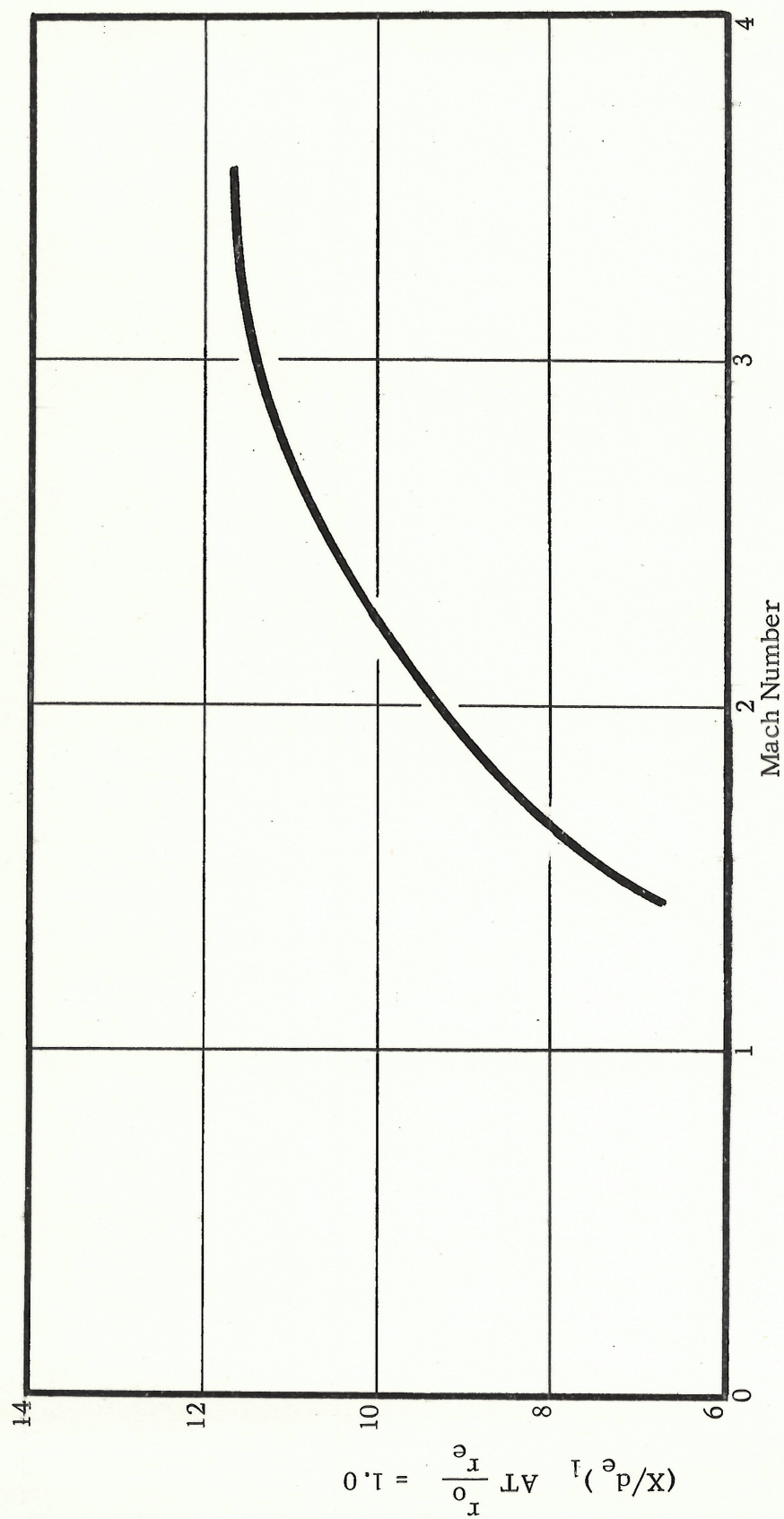
r_o the reference point where the pressure falls to 1/4 of its value on the axis. r_o is given by

$$r_o = r_e \quad \text{if} \quad \frac{S/d_e}{(S/d_e)_1} < 1$$

$$r_o = r_e \left[\frac{S/d_e}{(S/d_e)_1} \right]^{1.16} \quad \text{if} \quad \frac{S/d_e}{(S/d_e)_1} \geq 1$$

x_s, z_s rectangular coordinates in the plane (surface of the launcher) normal to S at the center of the hole parallel to the z-axis

R_o, R_i outer and inner radius of a given ring section about the center of the hole



CROSSPLOT OF SPREADING CHARACTERISTICS

versus

MACH NUMBER

FIG 2

ϕ_1, ϕ_2	angles measured with respect to x_s ; they are the circumferential locations of the ring section
T_i	the blast force on the above ring section
R_e	effective radius of a ring section
M_x, M_z	moments about x_s, z_s , respectively, of the ring section
$\text{Erf}(x)$	error function $= \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2} d\lambda$

The pressure at any point (S, r) is given by

$$P(S, r) = a e^{-b^2 r^2} \quad (18)$$

There are four cases for the above expression with different a, b

$$\text{i} \quad 0 < S < S_o, \quad \frac{S}{k_3} < 1$$

$$a = P_e$$

$$b^2 = \left(\frac{1.15}{r_e} \right)^2$$

$$\text{ii} \quad 0 < S < S_o, \quad \frac{S}{k_3} \geq 1$$

$$a = P_e$$

$$b^2 = \left(\frac{1.15}{r_e} \right)^2 (k_3)^{2.32} (S)^{-2.32}$$

$$\text{iii} \quad S > S_o, \quad \frac{S}{k_3} < 1$$

$$a = k_1 S^{k_2}$$

$$b^2 = \left(\frac{1.15}{r_e} \right)^2$$

$$\text{iv} \quad S > S_o, \quad \frac{S}{k_3} \geq 1$$

$$a = k_1 S^{k_2}$$

$$b^2 = \left(\frac{1.15}{r_e} \right)^2 (k_3)^{2.32} (S)^{-2.32}$$

Using Eq. (18) we obtain

$$T_i = \int_{\varphi_1}^{\varphi_2} \int_{R_i}^{R_o} p \, dA = \int_{\varphi_1}^{\varphi_2} \int_{R_i}^{R_o} p \, r \, dr \, d\phi$$

or

$$T_i = \frac{a}{2b^2} (\phi_2 - \phi_1) (e^{-b^2 R_i^2} - e^{-b^2 R_o^2}) \quad (19)$$

$$R_e = \frac{\int_{\phi_1}^{\phi_2} \int_{R_i}^{R_o} r p dA}{\int_{\phi_1}^{\phi_2} \int_{R_i}^{R_o} p dA}$$

or

$$R_e = \frac{\left\{ \frac{\sqrt{\pi}}{2b} [\text{Erf}(bR_o) - \text{Erf}(bR_i)] - (R_o e^{-b^2 R_o^2} - R_i e^{-b^2 R_i^2}) \right\}}{e^{-b^2 R_i^2} - e^{-b^2 R_o^2}} \quad (20)$$

$$dM_z = -R_e \cos \phi \int_{R_i}^{R_o} p r dr d\phi \quad \frac{2\pi}{2\pi}$$

$$= -R_e \cos \phi \frac{d\phi}{2\pi} \int_{R_i}^{R_o} p 2\pi r dr$$

$$M_z = - \frac{T_i R_e}{\phi_2 - \phi_1} (\sin \phi_2 - \sin \phi_1) \quad (21)$$

Similarly

$$M_x = \frac{T_i R_e}{\phi_2 - \phi_1} (\cos \phi_2 - \cos \phi_1) \quad (22)$$

From Eqs. (19), (21), (22), with all the required data, we can find the magnitude and point of application of the force acting on any given part of a ring section at a given time. Eqs. (15), (16), should yield the accelerations and velocities of the launcher and missile as the missile is on the point of leaving the launcher. S may be expressed as a function of time, i. e. $S = S(t)$. This must be obtained from the rocket motion after firing.

At a given time t , the effective area on the launcher face covered by the exhaust cone (use $r = r_0$ for its computation) is known. We divide this area into a number of suitably chosen ring sections with respect to the center of the hole for the particular rocket. The magnitude and point of application of the blast force on each ring section are computed; then we can find the magnitude and location of the resultant force. This will give T and X_T, Z_T in Eq. (17).

Note that in the calculation of the effective areas, the effect due to holes on the launcher face is neglected.

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