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A FINITE ELEMENT STRESS ANALYSIS
OF A CRACK IN A BI-MATERIAL PLATE

by

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ABSTRACT

A stress analysis is obtained for the problem of a crack in one material of a bi-material plate located perpendicular to the material interface. A numerical solution is obtained using finite element techniques to obtain the force displacement relationships. Knowing this, a work integral method is used to determine the stress intensity factors for the crack. Since the work integral is independent of path, the path of integration can be chosen far enough away from the crack tip to avoid the complications of the crack tip singularity.

The problem is studied for a number of cases where the crack length to plate width ratio, distance from crack tip to material interface, and the ratio of material constants were varied as parameters.

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LIST OF SYMBOLS

| | | |
|---------------------|---|---|
| a | = | half of the crack length |
| b | = | distance from crack tip to material interface |
| E_1 | = | elastic modulus of material containing crack |
| E_2 | = | elastic modulus of second material |
| ν_1 | = | Poisson's ratio of material containing crack |
| ν_2 | = | Poisson's ratio of second material |
| K | = | stress intensity factor |
| K_I | = | stress intensity factor for opening mode of crack extension |
| K_{II} | = | stress intensity factor for in-plane shear mode of crack extension |
| C_1 | = | constant |
| σ | = | normal stress |
| C_2 | = | constant |
| τ | = | shear stress |
| K_{Ic} | = | critical value of K_I |
| K_{IIc} | = | critical value of K_{II} |
| \mathcal{G} | = | strain energy release rate |
| \mathcal{G}_I | = | strain energy release rate for opening mode of crack extension |
| \mathcal{G}_{II} | = | strain energy release rate for in-plane shear mode of crack extension |
| \mathcal{G}_{Ic} | = | critical value of \mathcal{G}_I |
| \mathcal{G}_{IIc} | = | critical value of \mathcal{G}_{II} |
| β | = | bielastic constant |
| r | = | polar coordinate |
| λ | = | ratio of crack length to plate width ($\frac{2a}{w}$) |

- \underline{f} = generalized force matrix for an element
 \underline{x} = generalized displacement matrix for an element
 \underline{k} = stiffness matrix of an element
 \underline{F} = generalized force matrix for a plate
 \underline{X} = generalized displacement stiffness matrix of a plate
 \underline{K} = stiffness matrix of a plate
 u = horizontal displacement
 v = vertical displacement
 $\underline{\epsilon}$ = two-dimensional strain matrix
 $\underline{\sigma}$ = two-dimensional stress matrix
 W = strain energy density
 J_w = portion of path independent work integral used to determine k
 J_T = portion of path independent work integral used to determine k
 P = load
 W = $W_1 + W_2$, total bi-material plate width
 W_1 = width of material containing crack
 W_2 = width of second material
 B = plate thickness = 1
 C = stress intensity factor correction
 C_λ = portion of correction C due to finite plate width
 C_m = portion of correction C due to the influence of the second material on the crack tip stress field.

I. INTRODUCTION

A. Statement of Problem

In recent years, the use of composite materials and adhesive or bonded joints has been given considerable attention for structural applications. More insight is needed, however, to understand how these materials behave under applied loads. In the area of composites, there has been much published [1-7]¹ using a "strength of materials" approach to characterize the material's strength and develop the equations which govern its behavior, and these have been of much use for design applications. The appearance of a flaw or crack in a material could lead to a mode of failure where fracture occurs with nominal cross-section stresses well below the material yield stress. The behavior of cracks in plates composed of two different materials bonded together has been analytically treated for the case of a crack lying on or touching the interface. These will be discussed in a later section.

Most of the work in fracture of composites [8-10] treats the composite as a macroscopically homogeneous anisotropic material. However, to obtain additional information concerning fracture of composite members and bonded joints, the problem of a crack in a bi-material plate should be studied. The purpose of this paper is to analyze the stresses in this type of cracked structure. The crack is of length $2a$ and runs perpendicular to the material interface with the closest crack tip at a distance b from the interface (see Fig. 1). The boundary conditions are a uniform vertical displacement of the top and bottom of the plate and plate sides are traction free. This could represent either a portion of a bonded joint or a section of composite as shown in Fig. 2.

The behavior of the crack tip stress field, which controls fracture, is studied by solving the linear elastic crack tip stress analysis problem numerically using a finite

¹Numbers in brackets refer to list of references at the end of the paper

element procedure. The change in the crack tip stress field with the ratios $2a/b$, E_1/E_2 , and ν_1/ν_2 , where E_1 and E_2 are the elastic moduli of the two materials and ν_1 and ν_2 the corresponding Poisson's ratio, is presented. The effect of the crack on the stresses along the material interface is noted.

B. Fracture Mechanics

Fracture mechanics is the study of fracture due to the propagation of small cracks or flaws developed in the material due to manufacturing, processing, or service. Fracture frequently occurs when nominal stress levels are below the yield strength of the material. When the combination of nominal stress and crack length reach a critical value, the crack will propagate rapidly through the material and fracture occurs. This crack behavior is discussed by the use of a parameter called the stress intensity factor, K , which is based on linear elastic crack tip stress analysis.

For problems of in-plane remote loading, two stress intensity factors are defined, one for each of two different modes of crack extension. The first mode of crack extension, known as mode I or the opening mode, is depicted in Fig. 3a. For this mode, the stress intensity factor, denoted K_I , has the value, for remote loading:

$$K_I = C_1 \sigma \sqrt{\pi a} \quad (1)$$

where C_1 = a constant dependent on the geometry and boundary conditions of the system,

σ = the average gross section normal stress applied perpendicular to the crack,

a = half crack length.

The second mode of crack extension, known as mode II or in-plane shear is depicted in Fig. 3b and has a stress intensity factor, K_{II} , computed for remote

loading from

$$K_{II} = C_2 \tau \sqrt{\pi a} \quad (2)$$

where C_2 = a constant dependent on the geometry and boundary conditions of the system,

τ = the average gross section shear stress applied to the system,

a = half crack length.

The critical values of the stress intensity factors are denoted by a subscript c, i.e. K_{Ic} and K_{IIc} , which are material properties in plane strain. It can be shown [11] that the stress intensity factors are related to the rate of change in strain energy at the onset of rapid crack extension. These relationships are, for plane stress,

$$\begin{aligned} K_I^2 &= E \mathcal{G}_I \\ K_{II}^2 &= E \mathcal{G}_{II} \end{aligned} \quad (3)$$

and for plane strain

$$\begin{aligned} K_I^2 &= E \mathcal{G}_I (1 - \nu^2) \\ K_{II}^2 &= E \mathcal{G}_{II} (1 - \nu^2) \end{aligned} \quad (4)$$

where \mathcal{G}_I = the rate of change in strain energy with respect to an increment of crack growth for mode I,

\mathcal{G}_{II} = the rate of change in strain energy with respect to an increment of crack growth for mode II,

E = the elastic modulus of the material,

ν = Poisson's ratio.

The critical values associated with K_{Ic} and K_{IIc} are designated \mathcal{G}_{Ic} and \mathcal{G}_{IIc} . As long as the stress intensity factor is greater than K_{Ic} or K_{IIc} , the fracture toughness, the crack will be self-propagating. Knowledge of K_{Ic} and K_{IIc} is experimentally determined, however one must know K_I and K_{II} to determine when failure

will occur. Exact forms of Eq. (1) and (2) can be obtained from the elasticity solution, of a particular boundary value problem, when one is available. These have been obtained for many crack shapes and loading geometries and can be found in the literature, for example Paris and Sih [12].

C. Review of Related Publications

The analytical work of interest to this paper starts with Muskhelishvili [13] who solved the problem of a vertically loaded stamp of one material pressing on a half plane of another. Although this is not a crack problem, it is a two-material problem and the method and form of the solution are similar to those problems of cracks in plates of dissimilar materials. Muskhelishvili points out [13] that the stresses near the ends of the stamp contain an oscillatory characteristic. This is due to the presence of terms of the form:

$$\sigma \sim \frac{1}{r^{1/2}} \left(\frac{\sin}{\cos} \right) [\beta \log r^*] \quad (5)$$

where r^* is a function of the distance from the ends of the stamp.

The above expression is a function only of position, geometry, and material constants. Thus the oscillatory character is fixed for a given crack geometry and set of material constants, and cannot be changed by varying the intensity of the load.

This oscillatory behavior arises through the use of complex potentials. In choosing Goursat functions which satisfy the boundary conditions, the following solution appears:

$$\sigma_y + i\tau_{xy} \sim r^{-1/2 + i\beta} \quad (6)$$

Upon taking the real and imaginary parts of Eq. (6) the oscillatory characteristic is introduced.

Williams [14] later solved the problem of a crack on the interface of two bonded dissimilar materials using an eigenfunction technique and obtained results similar to

those of Muskhelishvili's previous results for the stamp problem in that the stresses were again of the form of Eq. (5). From the eigenfunction expansion, Williams got solutions of the form

$$\sigma \sim r^{\lambda} \quad (\lambda = -1/2 \pm i\beta) \quad (7)$$

where λ is a complex number for the solution of bi-material problems. Since λ satisfies real boundary conditions, the real and the imaginary parts of λ are both solutions. Thus Eq. (7) is expressed as

$$\sigma \sim 1/2 \left[r^{\lambda} + r^{\bar{\lambda}} \right]$$

and

$$\sigma \sim \frac{1}{2i} \left[r^{\lambda} - r^{\bar{\lambda}} \right] \quad (8)$$

which reduces to the oscillatory form of Eq. (5).

After Williams, Erdogan [15] studied the problem of cracks on an interface of bonded dissimilar materials, using a method similar to Muskhelishvili's solution of a stamp on a half plane. The Goursat functions are again chosen in such a manner that Eq. (6) holds and thus the oscillatory stress of Eq. (5) is obtained. Erdogan carries his solution further and determines the stress intensity factors. It is interesting to note that for a single bond between two semi-infinite cracks in an infinite plate, loaded with concentrated loads at infinity, the stress intensity factors are independent of material constants, while for more than one bond separated by finite length cracks, the stress intensity factors become dependent on a bi-elastic constant, β , which is a function of the elastic moduli and Poisson's ratios of the materials. This bi-elastic constant is also the value of β in Eqs. (5), (6) and (7). Except for the special case of one bond, the dependence of K on \underline{c} is of a form similar to Eq. (5):

$$K \sim \left(\frac{\sin}{\cos} \right) \left[\beta \log \underline{c} \right] \quad (9)$$

where \underline{c} is a function of the crack length and the distance between the cracks.

Rice and Sih [16,17], expanded Williams' work by using the eigenvalue expansion technique to determine the Goursat functions from which complex variable methods are used to solve the problem. As could probably be expected, the stress solutions again fall into the form of Eq. (5). The solution is carried out in [17] to the determination of the stress intensity factors. Rice and Sih's results for a single bond appear to contradict that of Erdogan [16] since Rice and Sih's stress intensity formulas depend on the bi-elastic constant in the form of Eq. (9). Rice and Sih's solution, however, is for one semi-infinite crack loaded with concentrated forces acting on the crack face, or wedge forces. This is different than Erdogan's problem of one bond of finite length. Unpublished work by the author leads him to believe that if Erdogan's results were for a concentrated load at a finite distance from the crack, the dependence of the solution on the bi-elastic constant would appear. However, removing the loads to infinity does remove the dependence on the bi-elastic constant. Rice and Sih's result cannot be compared to Erdogan's since removing the load to infinity in Rice and Sih's work results in a problem that does not have a solution. Also, in Rice and Sih's result, the value of the expression for r^* in Eq. (9) does not appear to be a dimensionless product. It can be shown using the theorem of dimensional analysis that r^* must be dimensionless to be dimensionally homogeneous as defined in [18]. Thus Rice and Sih's results present a fundamental question that appears to remain unresolved.

Other publications [19, 20, 21] dealing with the problem of cracks on the interface between two dissimilar materials, for various load and plate geometries, follow the procedures of Muskhelishvili or Williams. They have the same oscillatory stress characteristic given by Eq. (5), and the form of the stress intensity given in Eq. (9).

No additional work is known to the author in the analytical area of dissimilar bonded materials with cracks. In another publication by Zak and Williams [22] for the solution of the problem of a crack with one tip on an interface and the crack running

perpendicular to the interface. The results show that the stresses have a form given by Eq. (7), however λ is no longer a complex number, but is still dependent on the bi-elastic constant. This means that the stress at the crack tip does have a real singularity, but its strength is dependent on the bi-elastic constant. The author is unaware of any attempts to interpret this result in the concepts of fracture mechanics.

All analytical attempts that the author is aware of thus far have been for special cases of cracks on or touching the bi-material interface. While the solutions have very important academic value, the fact that they do represent special cases and have complex solutions that are hard to work with leave them difficult to use for engineering problems. To obtain close form solutions for the case where cracks are close to but not on or touching the interface would indeed be a difficult task. It appears important to develop a numerical technique to handle all these problems with the use of the computer. This is the goal in this paper.

To avoid the problem of the oscillatory stress character, the crack is removed from contact with the interface. Once the tip is in a homogeneous one material environment, it is assumed that it once again exhibits a square root singularity.

II. ANALYSIS

A. Finite Element Procedure

The problem to be analyzed is that of a crack in a two-dimensional plate, with the crack running perpendicular to a bi-material interface where the leading tip is a distance b away from the interface. The plate is loaded in such a way that the top and bottom of the plate are subjected to a uniform displacement (see Fig. 1). The crack is a mathematical crack, that is, a slit having no thickness in the plate and the surface of the crack is traction free. The interface is also a mathematical one in that the two material edges share a common line across which the tractions and displacements are continuous. The stress intensity factors are found from the stress field in the plate obtained by making a numerical analysis using the finite element method.

The use of finite elements allows one to study the approximate behavior of a continuum by treating it as a structure composed of a number of elements interconnected by a finite, rather than an infinite, number of points. Forces are considered to act through these nodal points to other elements in such a way that they may be considered representative of the actual stress distribution acting on the boundaries of the element. Thus, knowing the force displacement relationship for the individual elements, the behavior of the plate composed of assembled elements can be determined by the techniques of structural analysis.

The force displacement relationships are found by determining a stiffness matrix such that

$$\underline{f} = \underline{k}\underline{x} \quad (10)$$

where \underline{f} is a $(n \times 1)$ matrix representing the generalized forces acting on the element nodes. The matrix \underline{k} is the stiffness matrix of the element. It is a measure of the elements' resistance to the applied forces, analogous to a spring stiffness in

elementary mechanics. The matrix \underline{x} is a $(n \times 1)$ matrix of the generalized nodal displacements caused by the forces \underline{f} .

Once the values of \underline{k} are known for all the elements, they can be assembled using structural analysis to obtain a set of linear algebraic equations which define the force displacement characteristics of the plate. This set of equations is

$$\underline{F} = \underline{K} \underline{X} \quad (11)$$

where \underline{F} is the $(m \times 1)$ matrix representing the forces acting on all the nodes in the plate, \underline{K} is the $(m \times m)$ matrix representing the stiffness of the plate, and \underline{X} is the $(m \times 1)$ matrix representing the displacements of all the nodes in the plate.

From equilibrium, the forces acting on any node in the interior of the plate must be zero. On the boundary either the forces acting on, or the displacement of, every node must be assigned. Where the displacements are specified, the corresponding equations are eliminated from Eq. (11), leaving a system of equations with unknown displacements, the matrices \underline{F} and \underline{K} being known. This system is then solved for the unknown displacements of the nodes. Then the stresses and strains in the plate may be computed.

The element stiffness matrix \underline{k} is determined from an energy analysis. First, however, an assumption has to be made about the displacements within the element. For the analysis used in this paper, it is assumed that if u is the horizontal displacement at some point (x, y) in the plate and v is the corresponding vertical displacement, they have the form

$$\begin{aligned} u &= A_1 + A_2x + A_3y + A_4x^2 + A_5xy + A_6y^2 = f_u(x, y) \\ v &= A_7 + A_8x + A_9y + A_{10}x^2 + A_{11}xy + A_{12}y^2 = f_v(x, y) \end{aligned} \quad (12)$$

The twelve constants for each element may be determined by specifying twelve nodal displacements in the element. In the analysis used here the elements are chosen as triangles with nodes at the vertices of the triangle and at mid-points of the triangles' legs. Each node is allowed to have a vertical and horizontal component of displacement (see Fig. 4). Thus, knowing the coordinates (x_i, y_i) of each node, i , the twelve A 's in Eq. (12) can be found by solving

$$\begin{aligned} u_i(x_i, y_i) &= f_u(x, y) \\ v_i(x_i, y_i) &= f_v(x, y) \end{aligned} \quad (i = 1, \dots, 6) \quad (13)$$

The strain displacement relations are given as [23]

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned} \quad (14)$$

From Eqs. (12), (13) and (14), one may obtain a system of equations which defines the strains in the element as a function of x , y , x_i , y_i , and the displacements. This relation is given by

$$\underline{\epsilon} = \underline{B} \underline{x} \quad (15)$$

$$\text{where } \underline{\epsilon} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

\underline{x} is the (12×1) matrix composed of the nodal displacements, and \underline{B} is a (3×12) matrix which is a function of x , y , x_i and y_i , $i = 1, \dots, 6$ (see Appendix A for

the derivation of \underline{B}). For two dimensional problems, Hooke's law can be expressed as

$$\underline{\sigma} = \underline{D} \underline{\epsilon} \quad (16)$$

where $\underline{\epsilon}$ is as defined above,

$$\underline{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}, \quad \underline{D} \text{ (plane stress)} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

E = elastic modulus

ν = Poisson's ratio

Once \underline{B} and \underline{D} have been determined for an element, the stiffness \underline{k} is determined from [22]

$$\underline{k} = \int \underline{B}^T \underline{D} \underline{B} dA \quad (17)$$

where integration is over the area of the element.

Upon determining \underline{k} , \underline{K} can be assembled from structural analysis techniques. Knowing \underline{K} , one then goes back to Eq. (11) and solves for \underline{X} , the displacements of all the nodes in the plate. Since \underline{X} is known, \underline{x} is obviously known, thus the strains in any element can be determined from Eq. (15) and the stresses from Eq. (16). For a more detailed account of finite elements, the reader is referred to [24].

The solution is approximate unless the assumption in Eq. (12) is accurate, in which case the solution is exact. Equations (12) imply by (14) and (16) that the stresses and strains are linear within an element. It is known, however, that the stress components near the crack tip behave according to an inverse square root law. In order to obtain a good approximate solution, the elements are chosen small close to the crack tip, thus the inverse square root relation is approximated by a series of linear relationships.

The plate to be analyzed is divided into triangular elements as shown in Fig. 5.

The geometry is defined by specifying seven lengths:

L_1 = distance from crack to top of plate

L_2 = distance from side of plate to crack tip

$L_3 = L_4$ = half the crack length = a

$L_5 = L_6$ = half the distance to the material interface = $\frac{b}{2}$

L_7 = the width of the second material = W_2

The elements are chosen by dividing each length, L_i , into n_i division in such a way that if the division closest to the crack tip (or material interface for L_6 and L_7) has length α , the next division has length $(1 + S)\alpha$. The next furthest is $(1 + S)^2\alpha$, etc., and the j^{th} division away from the crack tip (or material interface) has length $\alpha(1 + (j - 1)S)$, where S is a variable parameter and can be different for each L . Thus with all values of S equal to zero all the elements are the same size. As S increases, the elements decrease in size near the crack (or interface) and increase in size away from it. Obviously increasing the values of n_i will also decrease the element size.

Figure 5 shows only the top half of the plate, and this is all that is analyzed since the boundary conditions along the line of symmetry are such that the vertical displacements of the nodes* are zero, and the horizontal forces on the nodes are zero. Along the crack face, the vertical force also is zero. Both the nodes and the elements are consecutively numbered from the bottom of a column to the top starting at the lower left corner of the plate, as indicated in Figure 5. Once the elements have been defined and the stress analysis complete, the stress intensity factors can be determined from a work integral.

B. Determination of Stress Intensity Factor

Rice [25] gives a formula for what he calls the J-integral, which is a path independent integral around the crack tip which is equal to the value \mathcal{J} in Eqs. (3) and (4).

* not on crack face

The equation is

$$\mathcal{G} = \mathcal{G}_I + \mathcal{G}_{II} = J = \int_{\Gamma} (W dy - \underline{T} \cdot \frac{\partial \underline{u}}{\partial x} ds) \quad (18)$$

where Γ is a curve surrounding the crack tip

W is the strain energy density defined by

$$W = \frac{1}{2} \sigma_{ij} \epsilon_{ij} \text{ for an elastic body}$$

σ_{ij} is the stress tensor

ϵ_{ij} is the strain tensor

\underline{T} is the traction vector defined according to an outward normal of Γ

\underline{u} is the displacement vector

ds is an element of arc length along Γ

The integral is evaluated in a counter clockwise sense starting at the lower crack surface and continuing along Γ to the upper crack surface.

Since only mode I is investigated, $\mathcal{G}_{II} = 0$ and Eq. (18) reduces to the value \mathcal{G}_I . From now on \mathcal{G}_I will be designated \mathcal{G} , also K_I will be designated K .

In order to program Eq. (18) the symmetry of the plate must be used to determine the value of the integral in the lower half of the plate. The path of integration is a rectangular path which is divided into six sections as shown in Fig. 6. Integration is carried out counter clockwise from section 1 through section 6. If we let the length of sections 1, 3, 4, and 6 be c and sections 2 and 5 be d , \mathcal{G} can be evaluated as follows:

The integral is first separated into two parts

$$\mathcal{G} = J_W - J_T \quad (19)$$

where

$$J_W = \int_{\Gamma} W dy$$

$$J_T = \int_{\Gamma} \underline{T} \cdot \frac{\partial \underline{u}}{\partial x} ds$$

Now J_W can be written as

$$J_W = \int_0^{-c} W_1 dy + \int_{-c}^{-c} W_2 dy + \int_{-c}^0 W_3 dy + \int_0^c W_4 dy + \int_c^c W_5 dy + \int_c^0 W_6 dy \quad (20)$$

where the subscripts on W refer to the corresponding section on Γ .

Since W is symmetric and $\int_{-c}^{-c} W dy = \int_c^c W dy = 0$, Eq. (20) reduces to

$$J_W = 2 \int_c^0 W_6 dy + 2 \int_0^c W_4 dy \quad (21)$$

J_T can be written as

$$J_T = \int_0^c \underline{T} \cdot \frac{\partial u}{\partial x} ds + \int_c^{c+d} \underline{T} \cdot \frac{\partial u}{\partial x} ds + \int_{c+d}^{2c+d} \underline{T} \cdot \frac{\partial u}{\partial x} ds + \int_{2c+d}^{3c+d} \underline{T} \cdot \frac{\partial u}{\partial x} ds \\ + \int_{3c+d}^{3c+2d} \underline{T} \cdot \frac{\partial u}{\partial x} ds + \int_{3c+2d}^{4c+2d} \underline{T} \cdot \frac{\partial u}{\partial x} ds \quad (22)$$

where the six integrals above have the value

$$\int_0^c \underline{T} \cdot \frac{\partial u}{\partial x} ds = \int_0^c (-\sigma_x^{(1)} \epsilon_x^{(1)} - \tau_{xy}^{(1)} \frac{\partial v^{(1)}}{\partial x}) ds \quad (22a)$$

$$\int_c^{c+d} \underline{T} \cdot \frac{\partial u}{\partial x} ds = \int_0^d (-\tau_{xy}^{(2)} \epsilon_x^{(2)} - \sigma_y^{(2)} \frac{\partial v^{(2)}}{\partial x}) ds \quad (22b)$$

$$\int_{c+d}^{2c+d} \underline{T} \cdot \frac{\partial u}{\partial x} ds = \int_0^d (\sigma_x^{(3)} \epsilon_x^{(3)} + \tau_{xy}^{(3)} \frac{\partial v^{(3)}}{\partial x}) ds \quad (22c)$$

$$\int_{2c+d}^{3c+d} \underline{T} \cdot \frac{\partial u}{\partial x} ds = \int_0^c (\sigma_x^{(4)} \epsilon_x^{(4)} + \tau_{xy}^{(4)} \frac{\partial v^{(4)}}{\partial x}) ds \quad (22d)$$

$$\int_{3c+d}^{3c+2d} \underline{T} \cdot \frac{\partial u}{\partial x} ds = \int_0^d (\tau_{xy}^{(5)} \epsilon_x^{(5)} + \sigma_y^{(5)} \frac{\partial v^{(5)}}{\partial x}) ds \quad (22e)$$

$$\int_{3c+2d}^{4c+2d} \underline{T} \cdot \frac{\partial u}{\partial x} ds = \int_0^c (-\sigma_x^{(6)} \epsilon_x^{(6)} - \tau_{xy}^{(6)} \frac{\partial v^{(6)}}{\partial x}) ds \quad (22f)$$

where superscripts in parentheses designate the section on which the stresses and strains are evaluated. From the conditions of symmetry on the plate,

$$\begin{aligned}\sigma_x^{(1)} &= \sigma_x^{(6)}, \tau_{xy}^{(1)} = -\tau_{xy}^{(6)}, \epsilon_x^{(1)} = \epsilon_x^{(6)}, \frac{\partial v^{(1)}}{\partial x} = \frac{\partial v^{(6)}}{\partial x} \\ \tau_{xy}^{(2)} &= -\tau_{xy}^{(5)}, \sigma_y^{(2)} = \sigma_y^{(5)}, \epsilon_x^{(2)} = \epsilon_x^{(5)}, \frac{\partial v^{(2)}}{\partial x} = -\frac{\partial v^{(5)}}{\partial x} \\ \sigma_x^{(3)} &= \sigma_x^{(4)}, \tau_{xy}^{(3)} = -\tau_{xy}^{(4)}, \epsilon_x^{(3)} = \epsilon_x^{(4)}, \frac{\partial v^{(3)}}{\partial x} = -\frac{\partial v^{(4)}}{\partial x}\end{aligned}\quad (23)$$

Using relations (23), Eq. (22) reduces to

$$\begin{aligned}J_T &= 2 \int_0^c (-\sigma_x^{(6)} \epsilon_x^{(6)} - \tau_{xy}^{(6)} - \frac{\partial v^{(6)}}{\partial x}) ds + 2 \int_0^d (\tau_{xy}^{(5)} \epsilon_x^{(5)} + \sigma_y^{(5)} \frac{\partial v^{(5)}}{\partial x}) ds \\ &+ 2 \int_0^c (\sigma_x^{(4)} \epsilon_x^{(4)} + \tau_{xy}^{(4)} \frac{\partial v^{(4)}}{\partial x}) ds\end{aligned}\quad (24)$$

Putting Eq. (21) and Eq. (24) into Eq. (19), \mathcal{G} can be determined knowing only the stresses in the upper half of the plate as determined by the finite element solution. The actual integration of Eqs. (21) and (24) is done numerically using the product of an average value of the stresses over some length of Γ times the corresponding length of Γ .

Thus knowing \mathcal{G} , the correction factors C , of Eq. (1), can be determined by using Eq. (3).

$$C = \frac{1}{\sigma} \left[\frac{E\mathcal{G}}{\pi a} \right]^{1/2} \quad (25)$$

where: σ = average stress on plate, $\frac{P}{BW}$

P = the total load on the plate

B = plate thickness = unity

W = plate width = $W_1 + W_2$

W_1 = width of material containing the crack

W_2 = width of second material

E = elastic modulus of material containing crack

a = the half crack length

\mathcal{G} = value computed by Eq. (18)

The use of the J integral was chosen because it is independent of path. Because the stress gradient is so steep near the crack tip, the finite element solution is least accurate in this region. By integrating around the tip at a distance far enough away, the inaccuracy of the solution near the crack tip is avoided. This method is simple, straight forward and can be applied to most crack problems.

The computation of stresses using the finite element solution, the J-integral, and the correction factor C is done on an IBM systems 360/75 computer. The computer program for doing this is listed in Appendix B.

III. RESULTS

All computer calculations were done in double precision, which means that all numbers were computed to sixteen significant figures. It was found that the value of the parameter S_i which determines the gradient of element sizes had little effect on the results as long as S was greater than 3. A value of S_i equal to 5 was used for all values of S in each program. The number n_i ($i = 1$ thru 7) was kept constant for each program with the values 4, 5, 5, 5, 5, 4, and 3 respectively. This was the largest possible combination of n 's that could be used without exceeding the storage capacity of the computer. The height of the plate (L_1) was maintained at 10 units with W_1 equal to 15.5 and W_2 equal to 4.5 units (see Fig. 5).

The path of integration for Eq. (19) was chosen as follows: Section 6 in Fig. 6 went vertically from the center of the crack to Section 5. Section 4 in Fig. 6 went vertically from a point half way between the closest crack tip to the interface and the interface to Section 5. Section 5 was along the first horizontal line formed by element boundaries below the top of the plate between Sections 4 and 6. The path of integration was latered a few times keeping everything else equal to determine if the results were independent of path as they should be if dictated by Eq. (19). The variation in the results for the alternate paths were found to be less than one per cent. However, for all results reported here the path of integration was kept as described above.

The first cases analyzed were for $E_1 = E_2$ and $\nu_1 = \nu_2$. Subscript 1 refers to the properties of the material containing the crack and subscript 2 refers to the second material, throughout the remainder of the paper. By changing the ratio of crack length to plate width, $\frac{2a}{w}$, called the aspect ratio, λ , the finite width correction factors were obtained. These are compared with those many other investigators [24] and are given in Table I. As can be seen, the results are in good

agreement with other investigators. This indicates that the use of finite elements in conjunction with Eq. (19) is a useful tool to determining stress intensity factors for problems for which analytical solutions are not available.

Results were obtained for ratios for $\frac{E_1}{E_2}$ having the values .01, .1, .3, 1, 2, 10, and 100. The ratio $\frac{\nu_1}{\nu_2}$ was kept at 1. Figure 7 shows the variation of C with $\frac{E_1}{E_2}$ for various values of λ from .1 to .5. According to Fig. 7 the addition of a stiffer second material would cause a drop in the stress intensity factor, while the addition of a less stiff second material would cause the stress intensity factor to increase. However, care must be given to this interpretation. What has happened in the first case is that a section of the material containing the crack has been replaced by a stiffer material. To cause a uniform vertical displacement equal to that of the one material problem, the load on material two must be increased by the ratio $\frac{E_2}{E_1}$, while the load on material one over the distance W_1 must remain the same to obtain the same vertical displacement. Thus the total load which is applied to the plate has been increased by the factor $(W_1 + \frac{E_2}{E_1} W_2) \frac{1}{W}$. Since the total load on the plate is increased and incorporated into Eq. (25), there is an apparent decrease in C . Likewise, if a less stiff second material is used C exhibits an apparent increase.

The correction factors used in Fig. 7 can thus be expressed as

$$C = C_\lambda \left[\frac{W}{(W_1 + \frac{E_2}{E_1} W_2)} \right] C_m \quad (26)$$

where C_λ = the finite width correction factors given in Table I

$\frac{W}{(W_1 + \frac{E_2}{E_1} W_2)}$ = correction due to the change in the loading on the plate

C_m = correction for the effect of the second material on the crack tip stress field.

Since W_2 was a constant for all problems, varying λ also changed the ratio $\frac{b}{a}$. Since λ should not effect C_m , Fig. 8 shows the change in C_m as a function of $\frac{E_1}{E_2}$ for different values of $\frac{b}{a}$. These data are also cross plotted in Fig. 8 to show the variation in C_m with $\frac{b}{a}$ for different values of $\frac{E_1}{E_2}$.

As Fig. 8 shows, the second material has an influence on the crack tip stress field which is not of any great importance until $\frac{b}{a}$ gets lower than about 3. As the crack tip moves closer to the interface, the stress intensity factor is considerably altered by a stiff second material tending to decrease K while a less stiff second material increases K .

The value of Poisson's ratio appeared to have little effect on the correction factor, C . Table 2 gives a few typical values of C where $\frac{\nu_1}{\nu_2}$ is varied from 0.73 to 1.32 for various values of $\frac{E_1}{E_2}$ and λ . The change caused by varying $\frac{\nu_1}{\nu_2}$ is insignificant.

The shear stress along the interface and normal stress perpendicular to the interface did not become significant in magnitude for the range of variables studied in this investigation ($\frac{b}{a} = .05$, $\frac{E_1}{E_2} = 100$).

Once the crack moves to the interface and into the second material, it is hard to determine what would happen. If the second material has a lower elastic modulus, the nominal stresses in the second material would be lower, and thus the crack tip would be entering a smaller stress field which could cause K to decrease. Depending upon the ratio of the elastic moduli and the magnitudes of the fracture toughness for the two materials, this lower stress field could cause the crack to arrest. The opposite would be true for a second material having a higher elastic modulus. This is an area requiring further investigation which can be accomplished employing the method used in this paper.

Applying these results to composites, two cases can be distinguished. First, assume that the high modulus fibers develop cracks which propagate through the fiber

to the interface. Once at the interface the crack encounters the low modulus matrix which frequently has a higher fracture toughness. Since K_{Ic} for the matrix is higher and the nominal stress in the matrix lower than that of the fiber, it is questionable whether the crack will penetrate through the interface into the matrix. Zak and Williams [23] point out in their solution that once the crack is touching the interface, the stresses along the interface are about an order of magnitude larger than those ahead of the crack when material two has the lower elastic modulus. Thus one would expect that the bond holding the two materials together might fail near the fractured fibers.

In the second case, if a crack were to initiate in the matrix and propagate toward a fiber, the stress intensity factor, K , is decreased as the crack approaches the fiber as indicated in Fig. 9 by the curves for $\frac{E_1}{E_2} < 1.0$. If K decreases sufficiently, the crack may be arrested before reaching the material interface. In this case, the crack can be extended further only by an increase in load on the composite.

This discussion of the behavior of cracks either in the fibers or in the matrix of a composite is: (1) in qualitative agreement with observations of crack propagation in composites as well as our intuitive expectations, (2) based on a two-dimensional solution which can be extended to a three-dimensional problem only in a qualitative manner at this time.

The result that the stresses on the interface between material one and material two are small requires further study, particularly for $\frac{E_1}{E_2} > 1.0$ as $\frac{b}{a}$ becomes very small. This conclusion appears to be a direct result of the constant displacement boundary condition on the top and bottom edges of the plate.

The values of C obtained from the computed program are tabulated in Table 3.

IV. CONCLUSIONS

1. The use of a finite element analysis in conjunction with the J-integral is a useful method for the determination of stress intensity factors.
2. The stress intensity factors for a plate of one material containing a crack bonded to a second material with the bond perpendicular to the crack (see Fig. 1) can be determined from

$$K_I = C_\lambda \frac{W}{\left(W_1 + \frac{E_2}{E_1} W_2 \right)} C_m \frac{P}{BW} \sqrt{\pi a} \quad (27)$$

where W_1 = width of the material containing the crack

W_2 = width of the second material

W = $W_1 + W_2$

P = total load on the plate normal to the crack

B = the plate thickness

a = the half crack length

C_m = correction values given in Figure 7

C_λ = correction value given in Table 1

3. If a plate containing a crack is bonded to a second material having a higher elastic modulus, the value of the stress intensity factor will be decreased by an amount dependent on the ratio of the elastic moduli as compared to the same size plate made of only one material.
4. If a plate containing a crack is bonded to a second material having a lower elastic modulus, the value of the stress intensity factor will be increased by an amount dependent on the ratio of the elastic moduli as compared to the same size plate made of only one material.
5. If a plate containing a crack is bonded to a second material having a different Poisson's ratio, the effect of the difference in Poisson's ratio on the stress intensity factors is negligible.

6. The stress intensity factor corrections given in Figures 7 and 8 are valid only for $\frac{W_2}{W_1} = .225$. The value of C_m should most likely depend upon this ratio, and this should be further investigated.
7. For the problems studied in this report there seemed to be no significant increase in any of the stresses along the interfaces due to the presence of the crack. However, the crack tip was never on the interface as in the Zak and Williams report [23] .

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APPENDIX A DETERMINATION OF MATRIX B

If we assume that the displacements are of the form (12) then we may express this as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12} \end{bmatrix} \quad (A1)$$

Let the pairs of nodal coordinates (x_i, y_i) be given by the constants: $x_1 = 0$
 $y_1 = 0$, $x_2 = a$, $y_2 = b$, $x_3 = c$, $y_3 = d$, $x_4 = e$, $y_4 = f$, $x_5 = g$, $y_5 = h$, $x_6 = k$, $y_6 = \ell$.

Using these we can express a set of equations for the twelve displacements

$u_i, v_i, i = 1, \dots, 6$.

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & a & b & a^2 & ab & b^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & b & a^2 & ab & b^2 \\ 1 & c & d & c^2 & cd & d^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & c & d & c^2 & cd & d^2 \\ 1 & e & f & e^2 & ef & f^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & e & f & e^2 & ef & f^2 \\ 1 & g & h & g^2 & gh & h^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & g & h & g^2 & gh & h^2 \\ 1 & k & \ell & k^2 & k\ell & \ell^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & k & \ell & k^2 & k\ell & \ell^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12} \end{bmatrix} \quad (A2)$$

or

$$\underline{\underline{X}} = \underline{\underline{\Delta}} \underline{\underline{A}} \quad (A3)$$

Thus one can solve (A2) for $\underline{\underline{A}}$ and obtains

$$\underline{\underline{A}} = \underline{\underline{\Delta}}^{-1} \underline{\underline{X}} \quad (A4)$$

Applying the strain displacement relations to (A1) yields

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12} \end{bmatrix} \quad (A5)$$

or

$$\underline{\underline{\epsilon}} = \underline{\underline{C}} \underline{\underline{A}} \quad (A6)$$

however using equation (A4) in (A6) gives

$$\underline{\underline{\epsilon}} = \underline{\underline{C}} \underline{\underline{\Delta}}^{-1} \underline{\underline{X}} \quad (A7)$$

Since $\underline{\underline{B}}$ is defined by

$$\underline{\underline{\epsilon}} = \underline{\underline{B}} \underline{\underline{X}} \quad (A8)$$

it is obvious that

$$\underline{\underline{B}} = \underline{\underline{C}} \underline{\underline{\Delta}}^{-1}$$

where $\underline{\underline{C}}$ is given in (A5) and $\underline{\underline{\Delta}}$ is given in (A2). The inverting of $\underline{\underline{\Delta}}$ and multiplying of the matrices is done on the computer.

TABLE 1
Comparison of Finite Width Correction Factors

| λ | Leverenz | Isida | Dixon | Greenspan | Modified Greenspan (Brussman & Kies) | Irwin | Brown & Srawley | |
|-----------|----------|-------|-------|-----------|--|-------|-----------------|---------------|
| | | | | | | | 3rd Degree | 2nd Degree |
| .1 | .992 | 1.006 | 1.005 | 1.005 | 1.005 | 1.001 | 1.012 | 1.00 |
| .2 | 1.012 | 1.025 | 1.021 | 1.029 | 1.029 | 1.017 | 1.026 | 1.02 |
| .3 | 1.046 | 1.058 | 1.037 | 1.051 | 1.052 | 1.040 | 1.053 | 1.06 |
| .4 | 1.097 | 1.109 | 1.091 | 1.101 | 1.107 | 1.076 | 1.103 | 1.12 |
| .5 | 1.166 | 1.187 | 1.155 | 1.185 | 1.205 | 1.130 | 1.183 | 1.20 |

TABLE 2
Comparison fo the Effect of Poisson's Ratio on the Correction Factors

| | $\nu_1/\nu_2 = 1$ | $\nu_1/\nu_2 = .73$ | $\nu_1/\nu_2 = 1.32$ |
|------------------------------|-------------------|---------------------|----------------------|
| $E_1/E_2 = 10, \lambda = .1$ | 1.25231 | 1.25231 | 1.25230 |
| $E_1/E_2 = 10, \lambda = .4$ | 1.56605 | 1.56707 | 1.56527 |
| $E_1/E_2 = .1, \lambda = .1$ | 0.32258 | 0.32246 | 0.32262 |
| $E_1/E_2 = .1, \lambda = .4$ | 0.28950 | 0.28941 | 0.28955 |

TABLE 3
 $\nu_1 = \nu_2 = 0.33$

| Crack Length 2a | Plate Width W | E ₁ | E ₂ | Distance from Crack tip to interface, b | C |
|--------------------|------------------|----------------|----------------|---|---------|
| 2 | 20.0 | 10000 | 100 | 4.5 | 1.28620 |
| 4 | 20.0 | 10000 | 100 | 3.5 | 1.33905 |
| 6 | 20.0 | 10000 | 100 | 2.5 | 1.44086 |
| 8 | 20.0 | 10000 | 100 | 1.5 | 1.64589 |
| 10 | 20.0 | 10000 | 100 | 0.5 | 2.18456 |
| 2 | 20.0 | 10000 | 1000 | 4.5 | 1.25231 |
| 4 | 20.0 | 10000 | 1000 | 3.5 | 1.30020 |
| 6 | 20.0 | 10000 | 1000 | 2.5 | 1.39037 |
| 8 | 20.0 | 10000 | 1000 | 1.5 | 1.56605 |
| 10 | 20.0 | 10000 | 1000 | 0.5 | 1.99288 |
| 2 | 20.0 | 10000 | 5000 | 4.5 | 1.12139 |
| 4 | 20.0 | 10000 | 5000 | 3.5 | 1.15391 |
| 6 | 20.0 | 10000 | 5000 | 2.5 | 1.21028 |
| 8 | 20.0 | 10000 | 5000 | 1.5 | 1.30874 |
| 10 | 20.0 | 10000 | 5000 | 0.5 | 1.49169 |
| 2 | 20.0 | 10000 | 10000 | 4.5 | 0.99202 |
| 4 | 20.0 | 10000 | 10000 | 3.5 | 1.01293 |
| 6 | 20.0 | 10000 | 10000 | 2.5 | 1.04590 |
| 8 | 20.0 | 10000 | 10000 | 1.5 | 1.09745 |
| 10 | 20.0 | 10000 | 10000 | 0.5 | 1.16605 |
| 2 | 20.0 | 3000 | 10000 | 4.5 | 0.64500 |
| 4 | 20.0 | 3000 | 10000 | 3.5 | 0.64617 |
| 6 | 20.0 | 3000 | 10000 | 2.5 | 0.64359 |
| 8 | 20.0 | 3000 | 10000 | 1.5 | 0.63499 |
| 10 | 20.0 | 3000 | 10000 | 0.5 | 0.60105 |
| 2 | 20.0 | 10000 | 100000 | 4.5 | 0.32258 |
| 4 | 20.0 | 10000 | 100000 | 3.5 | 0.31765 |
| 6 | 20.0 | 10000 | 100000 | 2.5 | 0.30697 |
| 8 | 20.0 | 10000 | 100000 | 1.5 | 0.28950 |
| 10 | 20.0 | 10000 | 100000 | 0.5 | 0.25763 |
| 2 | 20.0 | 10000 | 1000000 | 4.5 | 0.04162 |
| 4 | 20.0 | 10000 | 1000000 | 3.5 | 0.04037 |
| 6 | 20.0 | 10000 | 1000000 | 2.5 | 0.03804 |
| 8 | 20.0 | 10000 | 1000000 | 1.5 | 0.03470 |
| 10 | 20.0 | 10000 | 1000000 | 0.5 | 0.02990 |

TABLE 3
(continued)

| Crack Length 2a | Plate Width W | E ₁ | E ₂ | ν_1 | ν_2 | Distance from Crack tip to interface, b | C |
|--------------------|------------------|----------------|----------------|---------|---------|---|---------|
| 2 | 20.0 | 10000 | 1000 | .33 | .45 | 4.5 | 1.25231 |
| 8 | 20.0 | 10000 | 1000 | .33 | .45 | 1.5 | 1.56707 |
| 2 | 20.0 | 10000 | 1000 | .33 | .25 | 4.5 | 1.25230 |
| 8 | 20.0 | 10000 | 1000 | .33 | .25 | 1.5 | 1.56527 |
| 2 | 20.0 | 10000 | 100000 | .33 | .25 | 4.5 | 0.32262 |
| 8 | 20.0 | 10000 | 100000 | .33 | .25 | 1.5 | 0.28955 |
| 2 | 20.0 | 10000 | 100000 | .33 | .45 | 4.5 | 0.32246 |
| 8 | 20.0 | 10000 | 100000 | .33 | .45 | 1.5 | 0.28941 |

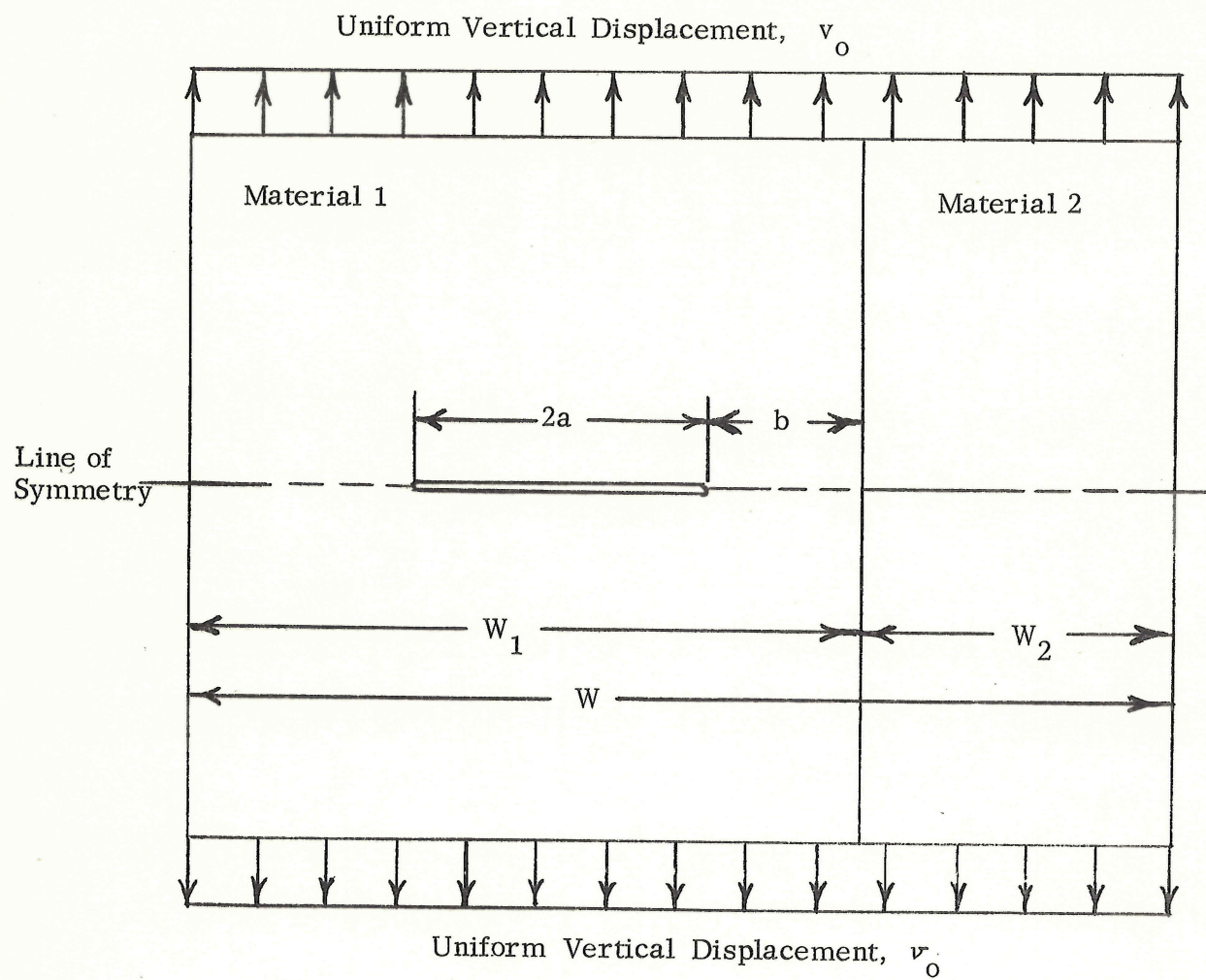


Figure 1. Bimaterial Plate with a Crack Subjected to Uniform Displacement

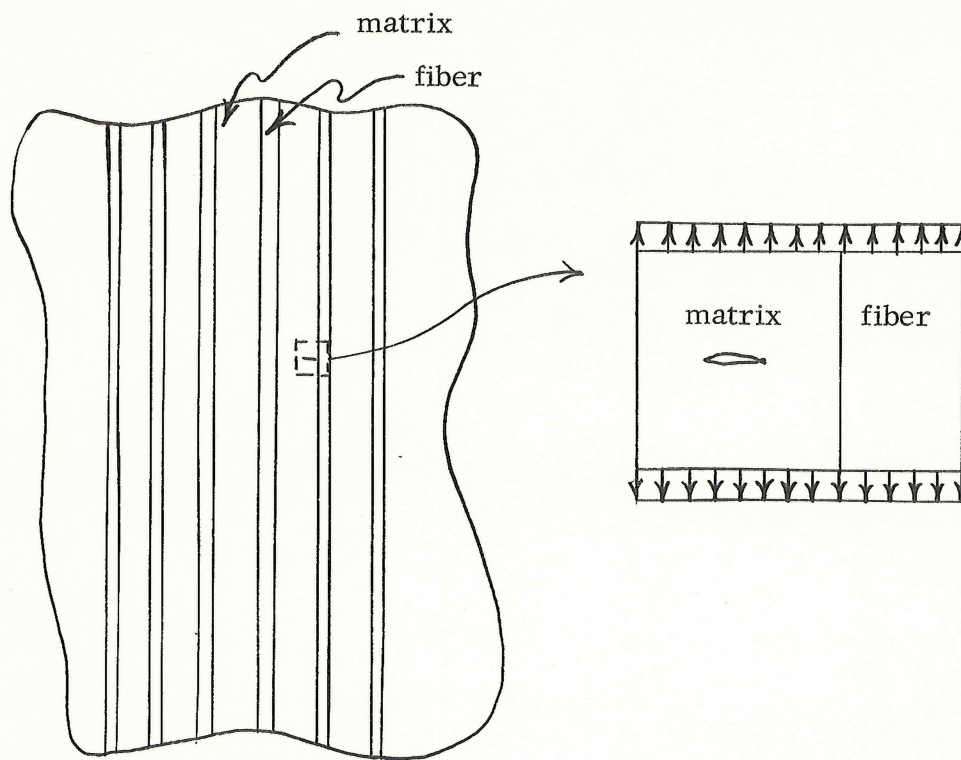
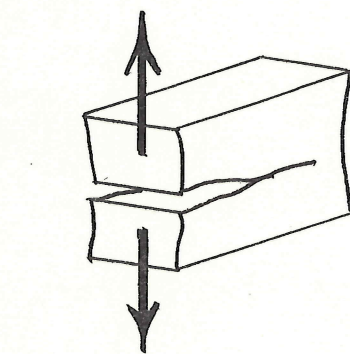
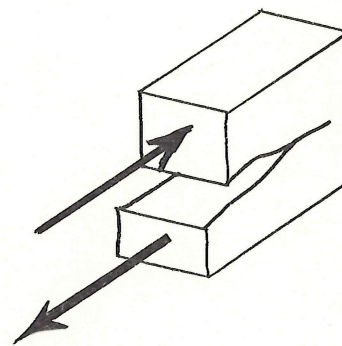


Figure 2. Composite Section Applicable to Analysis



3a opening mode
or mode I



3b in plane shear or
mode II

Figure 3. Different Modes of Crack Propagation

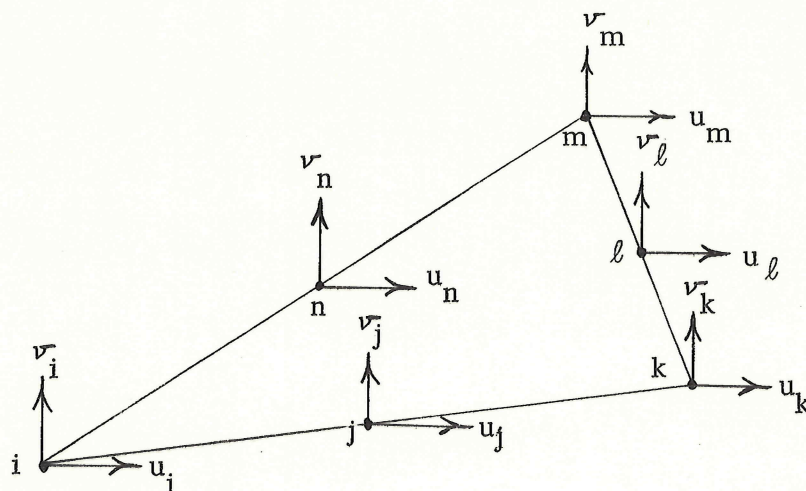


Figure 4. Typical Element Containing Nodes Number i, j, k, l, m and n .

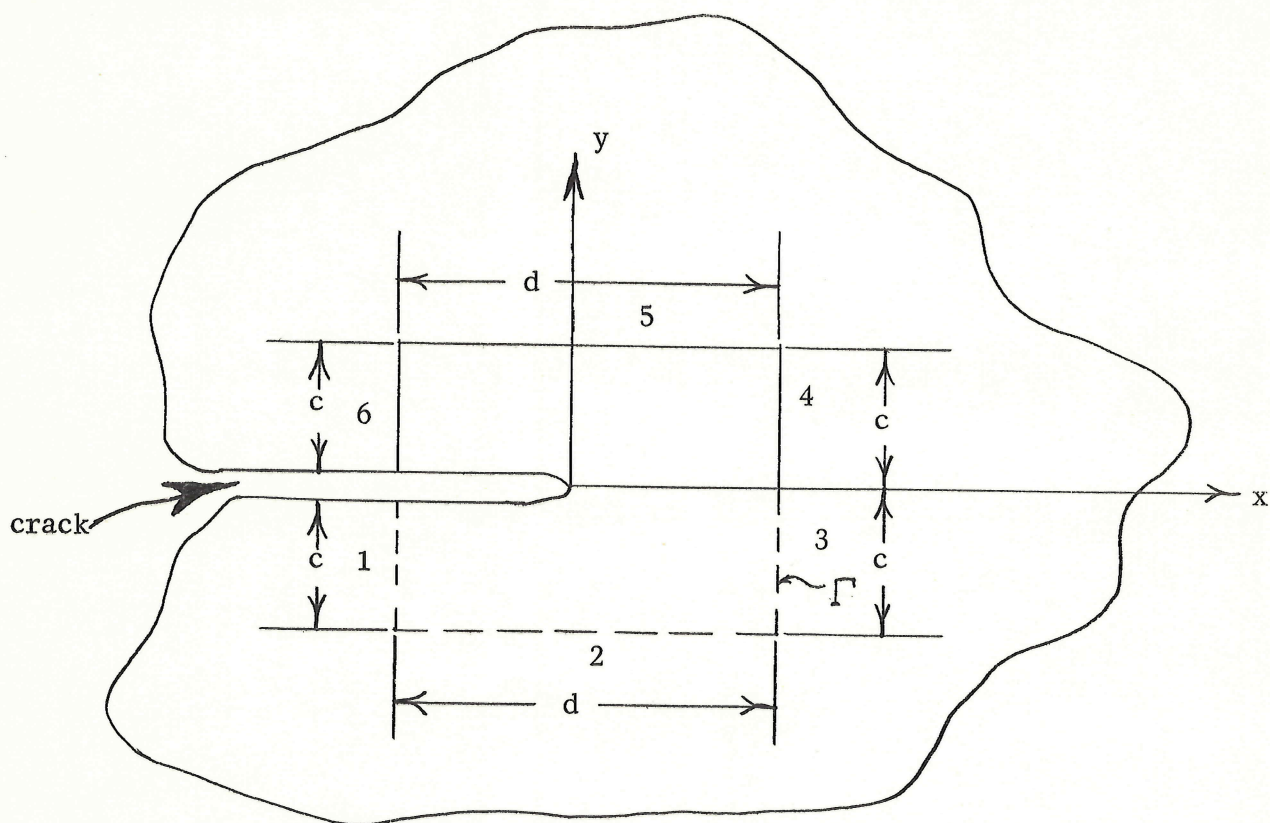


Figure 6. Path of Integration for Determination of Strain Energy Release Rate

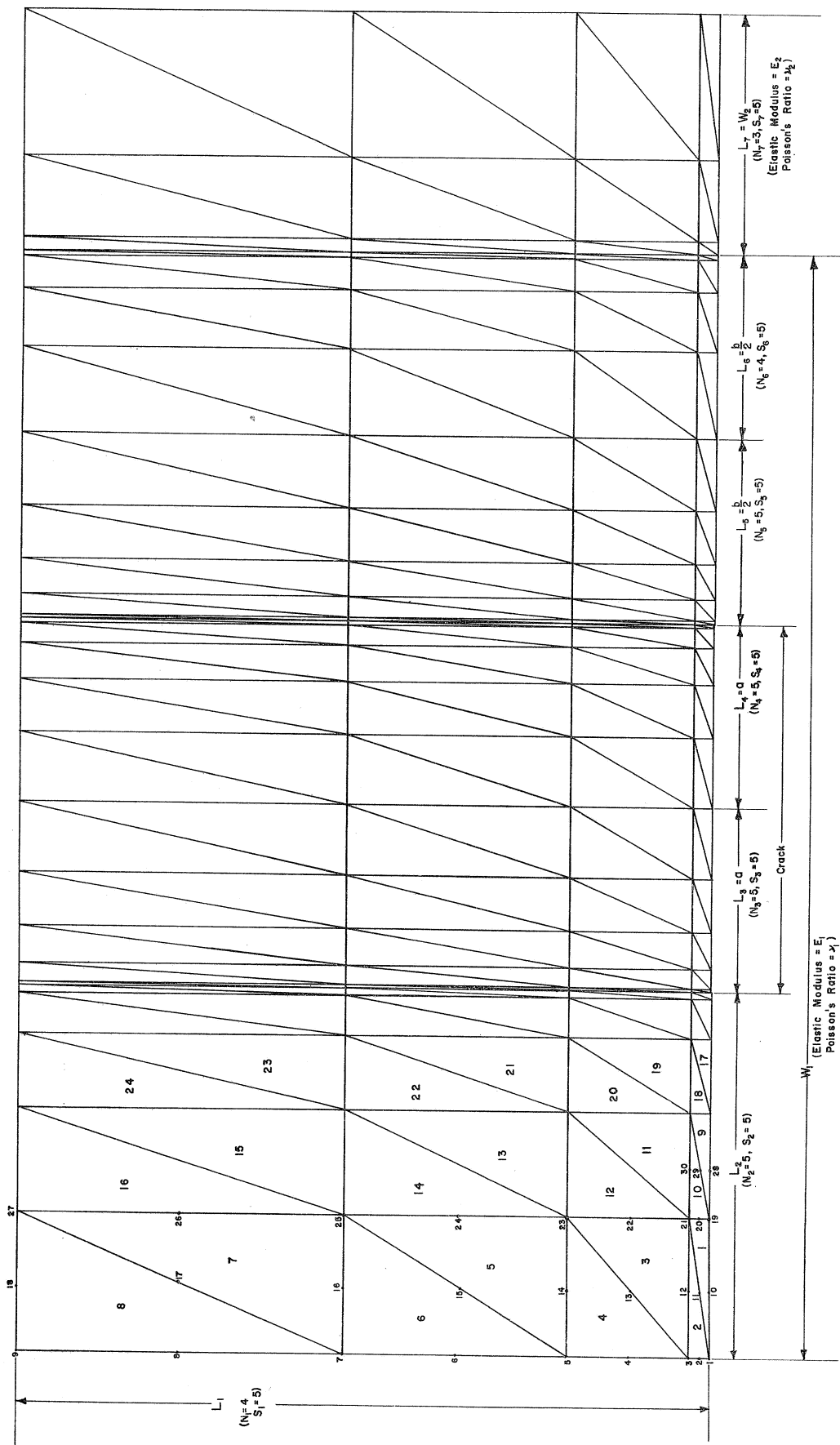


Fig. 5 Finite Element Grid Pattern

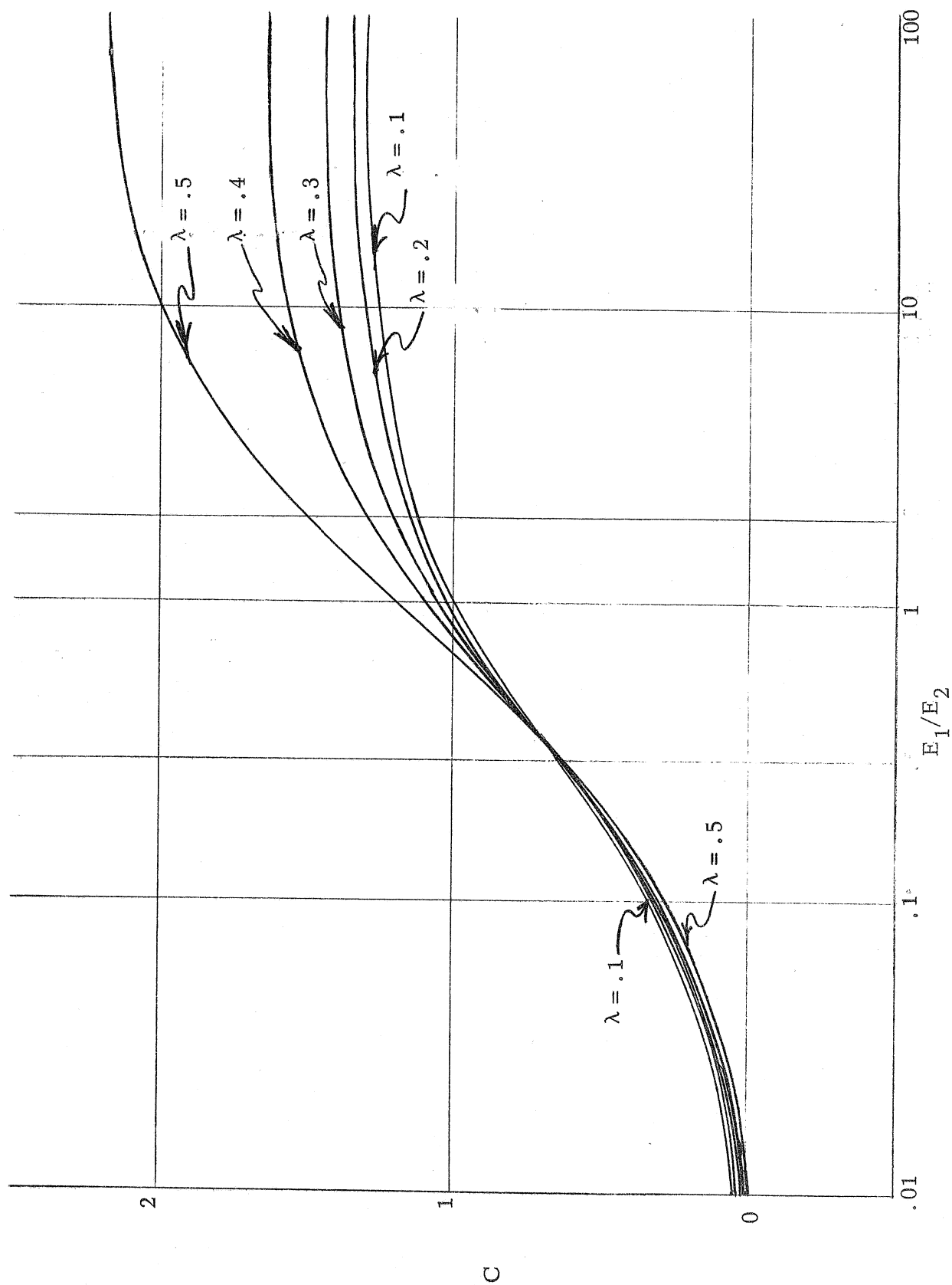


Fig. 7 Stress Intensity Factor Correction for Various Ratios of Elastic Moduli and λ

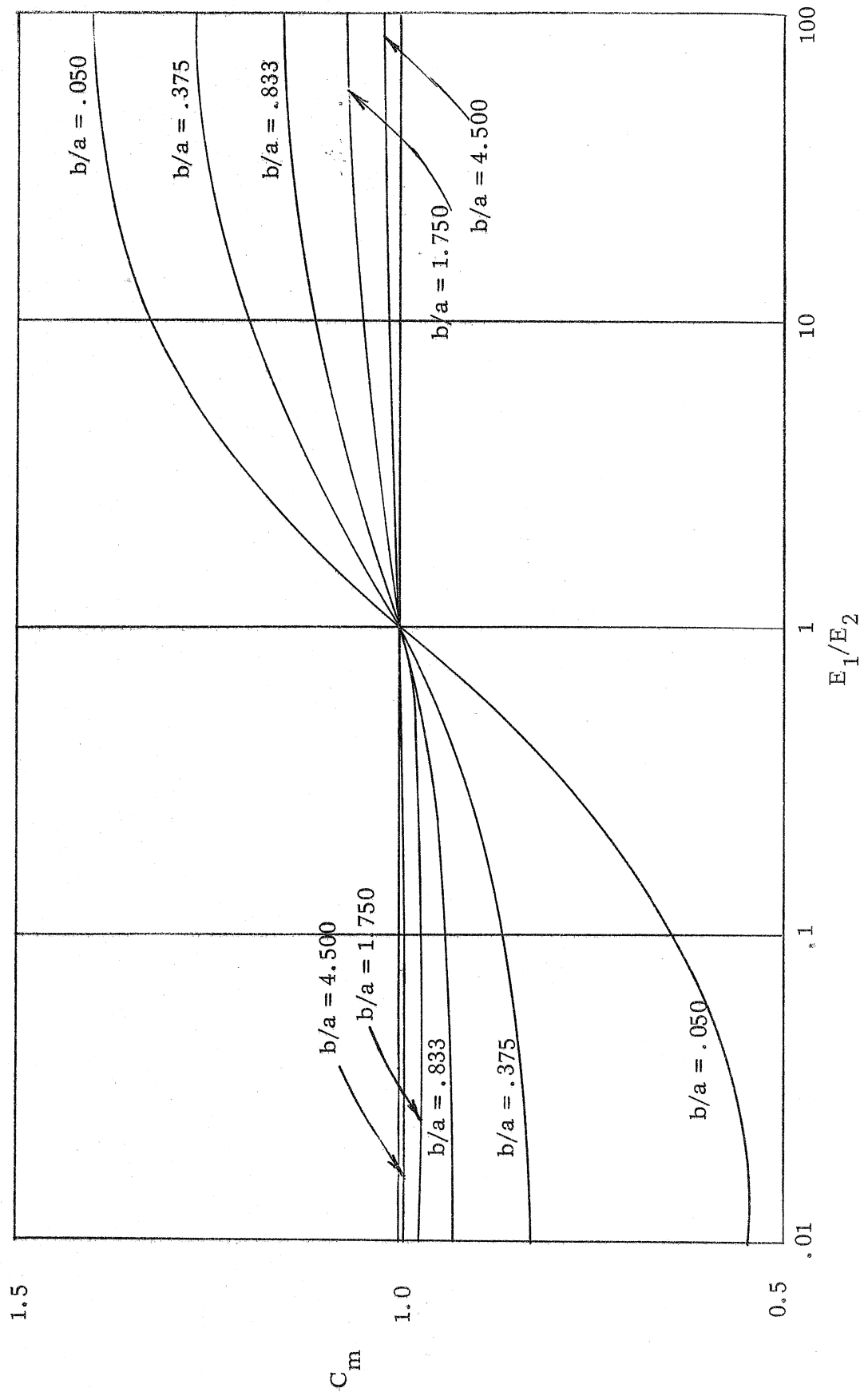


Figure 8 Correction C_m as a function of E_1/E_2

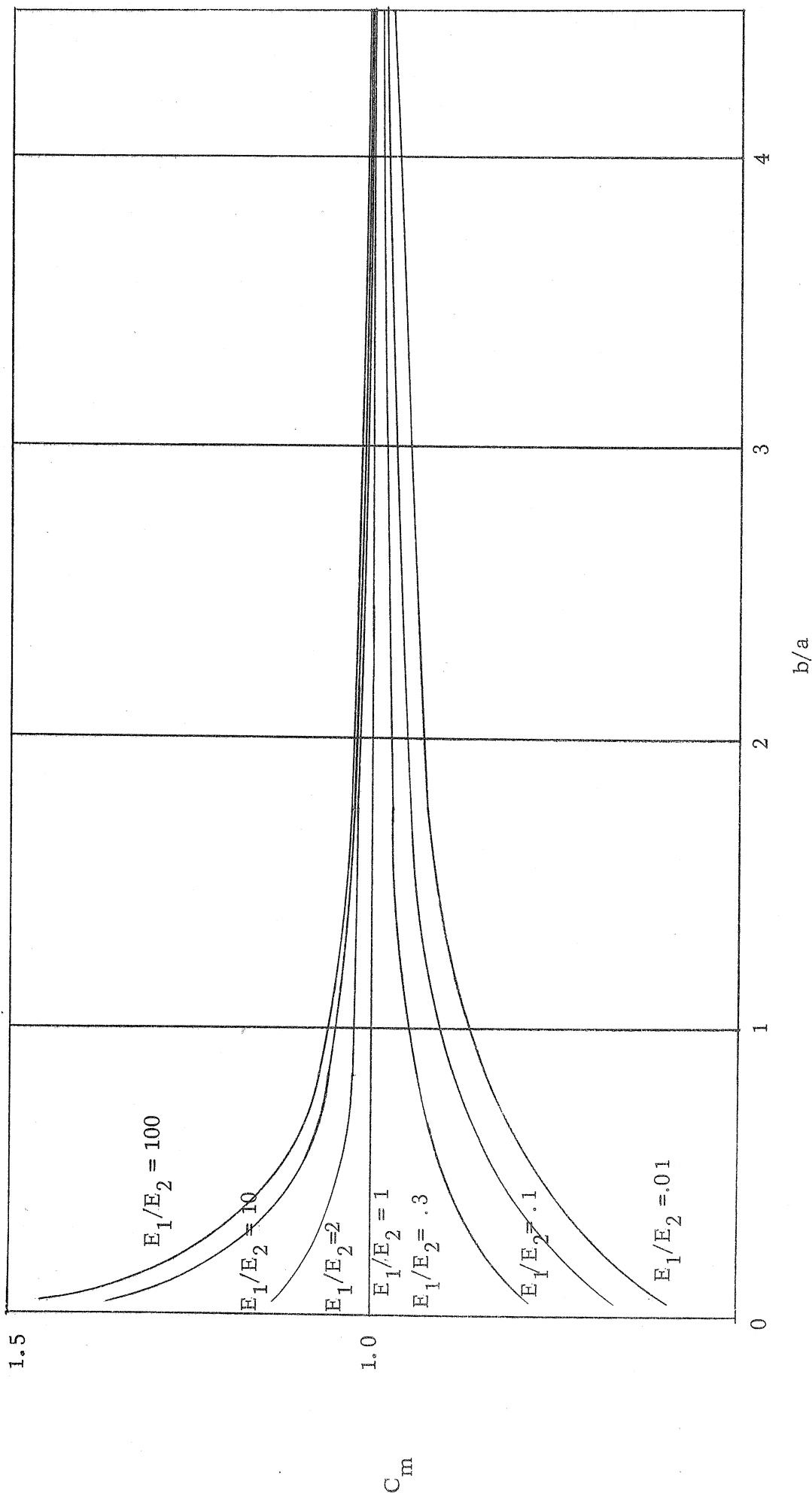


Figure 9 Correction C_m as a Function of b/a

APPENDIX B COMPUTER PROGRAM

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*****
THIS PROGRAM IS A FINITE ELEMENT SOLUTION FOR CRACKS IN DISSIMILAR
MEDIA. THE DEFINITIONS OF THE VARIABLES WHICH ARE USED ARE: AL EQUALS
THE VARIOUS LENGTHS WHICH DEFINE THE PLATE GEOMETRY, N EQUALS THE
NUMBER OF DIVISIONS FOR EACH LENGTH AL, SMALK EQUALS THE STIFFNESS
MATRIX FOR AN ELEMENT, BIGK EQUALS THE STIFFNESS MATRIX FOR THE
PLATE, NNODE EQUALS THE TOTAL NUMBER OF NODES, NELEM EQUALS THE TOTAL
NUMBER OF ELEMENTS, KK EQUALS THE NUMBES OF NODES IN ANY VERTICAL
LINE, E AND PR ARE THE ELASTIC CONSTANTS. THE STRAIN ENERGY RELEASE
RATE IS COMPUTED USING THE J INTEGRAL METHOD. THE ELEMENTS USED ARE
TRIANGLES HAVING SIX NODES.
*****
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION STRESS(3),STRAIN(3),AL(7),S(7),SMALK(12,12),
      1BIGK( 990,42),WORK(2),TRACT(2)
      COMMON X(495),Y(495),E(2),PR(2),FORCES( 990,1),NODES(216,6),N(7)
101  FORMAT(1H1,36H THE PLATE DIMENSIONS ARE AS FOLLOWS: ,/,10X,14H CRACK
      1LENGTH =,F11.5,/,10X,19H THE PLATE WIDTH W =,F11.5,/,10X,34H DISTANC
      2E FROM CRACK TIP TO JOINT =,F11.5,/,10X,27H DISTANCE TO THE CRACK T
      3IP =,F11.5,/,35H THE MATERIAL CONSTANTS ARE; E(1) =,F12.2,7H,E(2)
      4 =,F12.2,8H,NU(1) =,F5.2,8H,NU(2) =,F5.2,///)
102  FORMAT(44H FOR MODE I LOADING, WITH A VERTICLE LOAD OF,F11.5,15H T
      1HE G VALUE IS,F11.5,/,37H WHICH GIVES A Y CORRECTION FACTOR OF,F11
      2.5,///)
103  FORMAT (5X,23H DISTANCE FROM CRACK TIP,5X,6H SIGMAX,10X,6H SIGMAY,10X
      1,5H TAUXY,/)
104  FORMAT (10X,F11.5,9X,F11.5,5X,F11.5,5X,F11.5)
105  FORMAT (5X,24H DISTANCE ALONG INTERFACE,4X,6H SIGMAX,10X,6H SIGMAY,10
      1X,5H TAUXY/)
106  FORMAT (34H FIRST DISPLACEMENT ALONG CRACK IS,F16.8,14H AT A DISTA
      1NCE,F16.8)
107  FORMAT (1H1)
201  FORMAT (7I10)
202  FORMAT (4F20.2)
      1 READ(5,201) (N(I),I=1,7)
      READ(5,202) (AL(I),I=1,7)
      READ(5,202) (S(I),I=1,7)
      READ(5,202) E(1),E(2),PR(1),PR(2)
      NNODE = (2*N(1)+1)*(2*(N(2)+N(3)+N(4)+N(5)+N(6)+N(7))+1)
      NELEM = 2*N(1)*(N(2)+N(3)+N(4)+N(5)+N(6)+N(7))
      K = N(1)+1
      KK = 2*N(1)+1
*****
THIS SECTION OF THE PROGRAM DETERMINES THE VERTICAL COORDINATES,Y(I),
OF THE NODE I FOR ALL NODES IN THE PLATE.
*****
      DIV2=(AL(1)/(N(1)+S(1)*(N(1)**2-N(1))/2.DO))/2.DO
      NUM = NNODE - 2*N(1)
      DO 2 I=1,NUM,KK
2    Y(I) = 0.0
      M = N(1)

```



```

DO 3 I=1,M
  I2 = 2*I
  Y(I2) = (1+(I-1)*S(1))*DIV2 + Y(I2-1)
3 Y(I2+1) = (1+(I-1)*S(1))*DIV2 + Y(I2)
  NUM = NUM - 1
  DO 4 J=KK,NUM,KK
  DO 4 I=2,KK
4 Y(I+J) = Y(I)
*****
THIS SECTION OF THE PROGRAM DETERMINES THE HORIZONTAL COORDINATE,
X(I), OF THE NODE I FOR ALL NODES IN THE PLATE.
*****
DO 5 I=1,KK
5 X(I) = 0.0
  TOTAL = 0.0
  DO 15 II=2,6,2
  IF (N(II).EQ.0) GO TO 11
  M = N(II)
  DIV2=(AL(II)/(N(II)+S(II)*(N(II)**2-N(II))/2.DO))/2.DO
  DO 10 I=1,M
  TOTAL = TOTAL + (1+(N(II)-I)*S(II))*DIV2
  ISUM = 0
  NM = II - 1
  IF (2-NM) 6,8,8
6 DO 7 J=2,NM
7 ISUM = ISUM + N(J)
8 KL = KK*(2*(I+ISUM)-1)
  DO 9 J=1,KK
9 X(KL+J) = TOTAL
  TOTAL = TOTAL + (1+(N(II)-I)*S(II))*DIV2
  KL = KL+KK
  DO 10 J=1,KK
10 X(KL+J) = TOTAL
11 III = II+1
  IF (N(III).EQ.0) GO TO 15
  DIV2=(AL(III)/(N(III)+S(III)*(N(III)**2-N(III))/2.DO))/2.DO
  M = N(III)
  DO 15 I=1,M
  TOTAL = TOTAL + (1+(I-1)*S(III))*DIV2
  ISUM = 0
  DO 12 J=2,II
12 ISUM = ISUM + N(J)
  KL = KK*(2*(I+ISUM)-1)
  DO 13 J=1,KK
13 X(KL+J) = TOTAL
  TOTAL = TOTAL + (1+(I-1)*S(III))*DIV2
  KL = KL+KK
  DO 14 J=1,KK
14 X(KL+J) = TOTAL
15 CONTINUE
*****
THIS SECTION OF THE PROGRAM ASSIGNS THE NODES TO EACH ELEMENT. FOR
NODE I, THE NUMBERS OF THE SIX NODES, TAKEN IN A COUNTERCLOCKWISE
DIRECTION AROUND THE ELEMENT, ARE STORED IN NODES(I,J), J=1 THRU 6.
*****
M = N(1)

```



```

KK2 = KK*2
DO 16 I=1,M
  I2 = 2*I
  I2N = I2-1
  NODES(I2N,1) = I2N
  NODES(I2N,2) = I2N+KK
  NODES(I2N,3) = I2N+KK2
  NODES(I2N,4) = I2+KK2
  NODES(I2N,5) = I2+1+KK2
  NODES(I2N,6) = I2+KK
  NODES(I2,1) = I2N
  NODES(I2,2) = I2+KK
  NODES(I2,3) = I2+1+KK2
  NODES(I2,4) = I2+1+KK
  NODES(I2,5) = I2+1
16 NODES(I2,6) = I2
  KIJ = (NELEM/2) - N(1)+1
  DO 18 LK=K,KIJ,M
    J = LK+N(1)-1
    DO 18 I=LK,J
      I2 = 2*I
      I2N = 2*(I-N(1))
      I2M = I2-1
      I2NM = I2N-1
      DO 17 L=1,6
17 NODES(I2M,L) = NODES(I2NM,L) + KK2
      DO 18 L=1,6
18 NODES(I2,L) = NODES(I2N,L) + KK2
*****
THIS SECTION OF THE PROGRAM ASSEMBLES THE STIFFNESS MATRIX OF THE
PLATE AND STORES IT IN BIGK WHICH IS A BANDED MATRIX HAVING THE
PRINCIPAL DIAGONAL IN THE FIRST COLUMN OF THE MATRIX, AND ONLY THE
PORTION OF THE MATRIX ABOVE THE PRINCIPAL DIAGONAL IS STORED IN THE
REMAINDER SINCE THE MATRIX IS SYMMETRIC. NEQ IS THE TOTAL NUMBER OF
EQUATIONS, OR NODAL DISPLACEMENTS, AND NBW IS THE WIDTH OF THE
BANDED MATRIX.
*****
NEQ = 2*NNODE
NBW = 8*N(1)+10
DO 19 I=1,NEQ
  DO 19 J=1,NBW
19 BIGK(I,J) = 0.0
  DO 23 L=1,NELEM
    IF (2*N(1)*(N(2)+N(3)+N(4)+N(5)+N(6))-L) 21,20,20
20 IND = 1
    GO TO 22
21 IND = 2
22 A = X(NODES(L,3)) - X(NODES(L,1))
    B = Y(NODES(L,3)) - Y(NODES(L,1))
    C = X(NODES(L,5)) - X(NODES(L,1))
    D = Y(NODES(L,5)) - Y(NODES(L,1))
    CALL STIFF(A,B,C,D,E(IND),PR(IND),SMALK)
    DO 23 I=1,6
    DO 23 J=1,6
    NROW = 2*NODES(L,I) - 1
    NCOLM = 2*NODES(L,J)-NROW

```

```

      IF (NCOLM.LE.0) GO TO 23
      BIGK (NROW,NCOLM) = BIGK ( NROW,NCOLM) + SMALK(2*I-1,2*J-1)
      NCOLM = NCOLM+ 1
      BIGK (NROW,NCOLM) = BIGK (NROW,NCOLM) + SMALK(2*I-1,2*J)
      NROW = NROW + 1
      NCOLM = NCOLM - 1
      BIGK(NROW,NCOLM) = BIGK(NROW,NCOLM) + SMALK(2*I,2*J)
      NCOLM = NCOLM - 1
      IF (NCOLM.LE.0) GO TO 23
      BIGK(NROW,NCOLM) = BIGK(NROW,NCOLM) + SMALK(2*I,2*J-1)
23  CONTINUE
*****
THIS SECTION OF THE PROGRAM DEFINES THE BOUNDARY CONDITIONS OF THE
PLATE AND STORES THEM IN THE MATRIX FORCES. THE FIRST SUBSCRIPT OF
FORCES REFERS TO THE NODE AT WHICH THE BOUNDARY CONDITION IS DES-
CRIBED. THE SECOND SUBSCRIPT REFERS TO THE LOADING CONDITION, IN
CASE MORE THAN ONE IS DESIRED.
*****
      BODIS = .001D0*AL(1)
      DO 24 I=1,NEQ
24  FORCES(I,1) = 0.
      DO 27 I=KK,NNODE,KK
      I2 = 2*I
      FORCES(I2,1) = BODIS
      DO 25 J=2,NBW
      IJ = I2+J-1
      IF(IJ.GT.NEQ) GO TO 25
      FORCES(IJ,1) = FORCES(IJ,1) - BIGK(I2,J)*BODIS
25  BIGK(I2,J) = 0.
      DO 26 J=2,NBW
      IJ = I2-J+1
      IF (IJ.EQ.0) GO TO 27
      FORCES(IJ,1) = FORCES(IJ,1) - BIGK(IJ,J)*BODIS
26  BIGK(IJ,J) = 0.
27  BIGK(I2,1) = 1.
      NUM = 2*N(2)*KK+1
      DO 31 I=1,NUM,KK
      I2 = 2*I
      DO 28 J=2,NBW
28  BIGK(I2,J) = 0.
      DO 29 J=2,NBW
      IJ = I2-J+1
      IF (IJ.EQ.0) GO TO 30
29  BIGK(IJ,J) = 0.
30  FORCES(I2,1) = 0.
31  BIGK(I2,1) = 1.
      NUM = KK2*(N(2)+N(3)+N(4))+1
      NUM1 = NUM + KK2*(N(5)+N(6)+N(7))
      DO 35 I=NUM,NUM1,KK
      I2 = 2*I
      DO 32 J=2,NBW
32  BIGK(I2,J) = 0.
      DO 33 J=2,NBW
      IJ = I2-J+1
      IF (IJ.EQ.0) GO TO 34
33  BIGK(IJ,J) = 0.

```

```

34 FORCES(I2,1) = 0.
35 BIGK(I2,1) = 1.
   IJ = NUM*2-1
   IJP = IJ+1
   DO 36 I=2,NBW
36 BIGK(IJ,I) = 0.
   DO 37 J=2,NBW
   IF (IJP-J.EQ.0) GO TO 38
37 BIGK(IJP-J,J) = 0.
38 FORCES(IJ,1) = 0.
   BIGK(IJ,1) = 1.
*****
EXECUTION OF THE NEXT STATEMENT COMPUTES THE NODAL DISPLACEMENTS FOR
ALL THE NODES IN THE PLATE, AND STORES THEM IN FORCES. THIS DESTROYS
THE BOUNDARY CONDITIONS.
*****
   CALL SYBAN (BIGK, FORCES, NEQ, NBW, 1, KO)
*****
THIS SECTION OF THE PROGRAM COMPUTES THE STRAIN ENERGY RELEASE RATE
AND STORES IT IN G. G IS COMPUTED USING A WORK INTEGRAL AROUND THE
CRACK TIP.
*****
   NCOUT2 = 0
   G = 0.
   NUM = 2*N(1)*(N(2)+N(3)-1)+1
   NUM2 = 2*N(1)*(N(4)+N(5))
   NUM1 = NUM + NUM2
   DO 40 I=NUM, NUM1, NUM2
   II = I+4
   NCOUT2 = NCOUT2 + 1
   DO 40 L=I, II, 2
   NCOUT1 = 0
   N2 = 2*N(1)+1
   N1 = N2+L
   DO 39 J=L, N1, N2
   NCOUT1 = NCOUT1 + 1
   CALL STR(J, STRESS, STRAIN, PV)
   WORK(NCOUT1) = STRESS(1)*STRAIN(1) + STRESS(2)*STRAIN(2) + 2*
1STRESS(3)*STRAIN(3)
39 TRACT(NCOUT1) = 2*(STRESS(3)*PV + STRESS(1)*STRAIN(1))
   AWORK = (WORK(1)+WORK(2))/2.
   ATRACT = (TRACT(1)+TRACT(2))/2.
40 G = G + (AWORK-ATRACT)*(-1)**NCOUT2*(Y(NODES(L,5))-Y(NODES(L,3)))
   NUM = 2*N(1)*(N(2)+N(3))+6
   NUM1 = NUM1 + 5
   NUM2 = 2*N(1)
   DO 42 I=NUM, NUM1, NUM2
   II = I+1
   NCOUT1 = 0
   DO 41 J=I, II
   NCOUT1 = NCOUT1 + 1
   CALL STR(J, STRESS, STRAIN, PV)
41 TRACT(NCOUT1) = (STRESS(3)*STRAIN(1)+STRESS(2)*PV)*(X(NODES(I,3))-
1X(NODES(I,5)))
42 G = G - TRACT(1) - TRACT(2)

```

 THIS SECTION OF THE PROGRAM WRITES OUT THE VARIOUS PARAMETERS WHICH
 C DEFINE THE PLATE, AND COMPUTES THE VERTICAL LOAD ON THE PLATE, YLOAD,
 AND THE CORRECTION FACTOR, YFAC, FOR THE STRESS INTENSITY FACTORS.

```

CRACK = 2*AL(4)
WIDTH = AL(2)+AL(3)+AL(4)+AL(5)+AL(6)+AL(7)
TIP = AL(2)+AL(3)+AL(4)
DISTC = AL(5) + AL(6)
WRITE(6,101) CRACK,WIDTH,DISTC,TIP,E(1),E(2),PR(1),PR(2)
YLOAD = 0.
DO 43 I=NUM2,NELEM,NUM2
CALL STR(I,STRESS,STRAIN,PV)
ASTRES = STRESS(2)
II = I-1
CALL STR(II,STRESS,STRAIN,PV)
ASTRES = (ASTRES + STRESS(2))/2.
43 YLOAD = YLOAD + (ASTRES*(X(NODES(I,3))-X(NODES(I,5))))
PI = 3.141592653589793
YFAC = (WIDTH/YLOAD)*(E(1)*G/(PI*(CRACK/2.0)))**.5
WRITE(6,102) YLOAD,G,YFAC

```

 THIS SECTION OF THE PROGRAM COMPUTES THE STRESS AHEAD OF THE CRACK
 AND ALONG THE INTERFACE.

```

WRITE(6,103)
NUM = 2*N(1)*(N(2)+N(3)+N(4))+1
NUM1 = 2*N(1)*(N(5)+N(6)+N(7)-1) + NUM
DO 44 I=NUM,NUM1,NUM2
CALL STR(I,STRESS,STRAIN,PV)
AVG1 = STRESS(1)
AVG2 = STRESS(2)
AVG3 = STRESS(3)
II = I+1
CALL STR(II,STRESS,STRAIN,PV)
AVG1 = (AVG1 + STRESS(1))/2.
AVG2 = (AVG2 + STRESS(2))/2.
AVG3 = (AVG3 + STRESS(3))/2.
DFC = (X(NODES(I,1))+X(NODES(I,3)))/2.00
44 WRITE(6,104) DFC,AVG1,AVG2,AVG3
WRITE(6,105)
NUM = 2*N(1)*(N(2)+N(3)+N(4)+N(5)+N(6)-1)+1
NUM1 = NUM + 2*(N(1)-1)
DO 45 I=NUM,NUM1,2
CALL STR(I,STRESS,STRAIN,PV)
AVG1 = STRESS(1)
AVG2 = STRESS(2)
AVG3 = STRESS(3)
J = I+2*N(1)+1
CALL STR(J,STRESS,STRAIN,PV)
AVG1 = (AVG1 + STRESS(1))/2.00
AVG2 = (AVG2 + STRESS(2))/2.00
AVG3 = (AVG3 + STRESS(3))/2.00
DFC = (Y(NODES(I,3))+Y(NODES(I,5)))/2.00
45 WRITE(6,104) DFC,AVG1,AVG2,AVG3
NUM = 2*N(1)*(N(2)+N(3)+N(4)-1)+1

```

```
DIS = FORCES(2*NODES(NUM,1),1)
DIST = X(NODES(NUM,3))-X(NODES(NUM,1))
WRITE(6,106) DIS,DIST
WRITE(6,107)
GO TO 1
46 STOP
END
```


SUBROUTINE STIFF (A,B,C,D,E,G,S)

THIS SUBROUTINE DETERMINES THE STIFFNESS MATRIX OF A TRIANGULAR ELEMENT HAVING TWELVE DEGREES OF FREEDOM, AND STORES IT IN S. A AND C ARE THE X COORDINATES OF TWO OF THE TRIANGLE CORNERS RELATIVE TO THE OTHER, AND B AND D ARE THE CORRESPONDING Y COORDINATES. E AND G ARE THE MATERIAL CONSTANTS.

IMPLICIT REAL*8 (A-H,O-Z)

DIMENSION DEL(12,12),S(12,12),T(12),AR(3,3)

DO 1 I=1,12

DO 1 J=1,12

S(I,J) = 0.

1 DEL(I,J) = 0.

AC = A+C

BD = B+D

DO 2 I=1,11,2

J = I+1

DEL(I,1) = 1.

2 DEL(J,7) = 1.

DEL(3,2) = A/2.

DEL(3,3) = B/2.

DEL(3,4) = A*A/4.

DEL(3,5) = A*B/4.

DEL(3,6) = B*B/4.

DEL(5,2) = A

DEL(5,3) = B

DEL(5,4) = A*A

DEL(5,5) = A*B

DEL(5,6) = B*B

DEL(7,2) = AC/2.

DEL(7,3) = BD/2.

DEL(7,4) = AC*AC/4.

DEL(7,5) = AC*BD/4.

DEL(7,6) = BD*BD/4.

DEL(9,2) = C

DEL(9,3) = D

DEL(9,4) = C*C

DEL(9,5) = C*D

DEL(9,6) = D*D

DEL(11,2) = C/2.

DEL(11,3) = D/2.

DEL(11,4) = C*C/4.

DEL(11,5) = C*D/4.

DEL(11,6) = D*D/4.

DO 3 I=3,11,2

J = I+1

DO 3 K=2,6

L = K+6

3 DEL(J,L) = DEL(I,K)

CALL INVERT(DEL,12)

AR(1,1) = (A*D - B*C)/2.

AR(2,1) = AR(1,1)*AC/3.

AR(1,2) = AR(1,1)*BD/3.

AR(3,1) = AR(1,1)*(AC*AC+(C-A)**2/3.)/8.

AR(2,2) = AR(1,1)*(AC*BD+(C-A)*(D-B)/3.)/8.

```

AR(1,3) = AR(1,1)*(BD*BD+(D-B)**2/3.)/8.
F = 1.-G
H = F/2.
S(2,2) = AR(1,1)
S(2,4) = 2.*AR(2,1)
S(2,5) = AR(1,2)
S(2,9) = G*AR(1,1)
S(2,11) = G*AR(2,1)
S(2,12) = 2.*G*AR(1,2)
S(3,3) = H*AR(1,1)
S(3,5) = H*AR(2,1)
S(3,6) = F*AR(1,2)
S(3,8) = H*AR(1,1)
S(3,10) = F*AR(2,1)
S(3,11) = H*AR(1,2)
S(4,4) = 4.*AR(3,1)
S(4,5) = 2.*AR(2,2)
S(4,9) = 2.*S(2,11)
S(4,11) = 2.*G*AR(3,1)
S(4,12) = 4.*G*AR(2,2)
S(5,5) = AR(1,3)+H*AR(3,1)
S(5,6) = F*AR(2,2)
S(5,8) = S(3,5)
S(5,9) = G*AR(1,2)
S(5,10) = F*AR(3,1)
S(5,11) = (1.+G)*AR(2,2)/2.
S(5,12) = 2.*G*AR(1,3)
S(6,6) = 2.*F*AR(1,3)
S(6,8) = F*AR(1,2)
S(6,10) = 2.*S(5,6)
S(6,11) = F*AR(1,3)
S(8,8) = S(3,3)
S(8,10) = S(3,10)
S(8,11) = S(3,11)
S(9,9) = AR(1,1)
S(9,11) = AR(2,1)
S(9,12) = 2.*AR(1,2)
S(10,10) = 2.*S(5,10)
S(10,11) = F*AR(2,2)
S(11,11) = AR(3,1)+H*AR(1,3)
S(11,12) = 2.*AR(2,2)
S(12,12) = 4.*AR(1,3)
DO 4 I=1,11
  J = I+1
  DO 4 K=J,12
4 S(K,I) = S(I,K)
  F = E/(1.-G*G)
  DO 6 I=1,12
    DO 5 J=1,12
      T(J) = 0.
      DO 5 K=1,12
5 T(J) = T(J) + S(I,K)*DEL(K,J)
      DO 6 J=1,12
6 S(I,J) = T(J)
      DO 8 I=1,12
      DO 7 J=1,12

```

```
T(J) = 0.  
DO 7 K=1,12  
7 T(J) = T(J) + DEL(K,J)*S(K,I)  
DO 8 J=1,12  
8 S(J,I) = T(J)*F  
RETURN  
END
```


SUBROUTINE STR (K,STRESS,STRAIN,PV)

 THIS SUBROUTINE COMPUTES THE AVERAGE STRESS AND STRAIN IN THE ELEMENT K. THE NORMAL STRESS AND STRAIN IN THE HORIZONTAL DIRECTION ARE STORED IN STRESS(1) AND STRAIN(1), THE NORMAL STRESS AND STRAIN IN THE VERTICAL DIRECTION ARE STORED IN STRESS(2) AND STRAIN(2), THE SHEAR STRESS AND STRAIN ARE STORED IN STRESS(3) AND STRAIN(3). PV IS THE PARTIAL DERIVATIVE OF THE VERTICAL DISPLACEMENT WITH RESPECT TO X.

```

    IMPLICIT REAL*8 (A-H,O-Z)
    DIMENSION STRESS(3),STRAIN(3),DMAT(3,3),DEL(12,12),CMAT(3,12),
    ICDEL(3,12),DISP(12)
    COMMON X(495),Y(495),E(2),PR(2),FORCES( 990,1),NODES(216,6),N(7)
    IF (2*N(1)*(N(2)+N(3)+N(4)+N(5)+N(6))-K) 2,2,1
1  IND = 1
    GO TO 3
2  IND = 2
3  A = X(NODES(K,3)) - X(NODES(K,1))
    B = Y(NODES(K,3)) - Y(NODES(K,1))
    C = X(NODES(K,5)) - X(NODES(K,1))
    D = Y(NODES(K,5)) - Y(NODES(K,1))
    AC = A+C
    BD = B+D
    DO 4 I=1,3
    DO 4 J=1,3
4  DMAT(I,J) = 0.0
    DMAT(1,1) = E(IND)/(1-PR(IND)**2)
    DMAT(2,2) = DMAT(1,1)
    DMAT(2,1) = PR(IND)*E(IND)/(1.-PR(IND)**2)
    DMAT(1,2) = DMAT(2,1)
    DMAT(3,3) = E(IND)/(2*(1.+PR(IND)))
    DO 5 I=1,12
    DO 5 J=1,12
5  DEL(I,J) = 0.
    DO 6 I=1,11,2
    J = I+1
    DEL(I,1) = 1.
6  DEL(J,7) = 1.
    DEL(3,2) = A/2.
    DEL(3,3) = B/2.
    DEL(3,4) = A*A/4.
    DEL(3,5) = A*B/4.
    DEL(3,6) = B*B/4.
    DEL(5,2) = A
    DEL(5,3) = B
    DEL(5,4) = A*A
    DEL(5,5) = A*B
    DEL(5,6) = B*B
    DEL(7,2) = AC/2.
    DEL(7,3) = BD/2.
    DEL(7,4) = AC*AC/4.
    DEL(7,5) = AC*BD/4.
    DEL(7,6) = BD*BD/4.
    DEL(9,2) = C
    DEL(9,3) = D
  
```

```

DEL(9,4) = C*C
DEL(9,5) = C*D
DEL(9,6) = D*D
DEL(11,2) = C/2.
DEL(11,3) = D/2.
DEL(11,4) = C*C/4.
DEL(11,5) = C*D/4.
DEL(11,6) = D*D/4.
DO 7 I=3,11,2
  J = I+1
  DO 7 KL=2,6
    L = KL+6
7  DEL(J,L) = DEL(I,KL)
  CALL INVERT(DEL,12)
  XVAL = A/3. + C/3.
  YVAL = B/3. + D/3.
  DO 8 I=1,3
    DO 8 J=1,12
8  CMAT(I,J) = 0.
    CMAT(1,2) = 1.
    CMAT(1,4) = 2*XVAL
    CMAT(1,5) = YVAL
    CMAT(2,9) = 1.
    CMAT(2,11) = XVAL
    CMAT(2,12) = 2*YVAL
    CMAT(3,3) = 1.
    CMAT(3,5) = XVAL
    CMAT(3,6) = 2*YVAL
    CMAT(3,8) = 1.
    CMAT(3,10) = 2*XVAL
    CMAT(3,11) = YVAL
    DO 9 I=1,3
      DO 9 J=1,12
9  CDEL(I,J) = 0.
      DO 10 I=1,3
        DO 10 J=1,12
          DO 10 L=1,12
10 CDEL(I,J) = CDEL(I,J) + CMAT(I,L)*DEL(L,J)
        DO 11 I=1,6
          DISP(2*I-1) = FORCES(2*NODES(K,I)-1,1)
11 DISP(2*I) = FORCES(2*NODES(K,I),1)
        DO 12 I=1,3
12 STRAIN(I) = 0.
        DO 13 I=1,3
          DO 13 J=1,12
13 STRAIN(I) = STRAIN(I) + CDEL(I,J)*DISP(J)
        DO 14 I=1,3
14 STRESS(I) = 0.
        DO 15 I=1,3
          DO 15 J=1,3
15 STRESS(I) = STRESS(I) + DMAT(I,J)*STRAIN(J)
        AVL8 = 0.
        DO 16 I=1,12
16 AVL8 = AVL8 + DEL(8,I)*DISP(I)
        AVL10 = 0.
        DO 17 I=1,12

```

```
17 AVL10 = AVL10 + DEL(10,I)*DISP(I)
   AVL11 = 0.
   DO 18 I=1,12
18  AVL11 = AVL11 + DEL(11,I)*DISP(I)
   PV = AVL8 + XVAL*AVL10 + YVAL*AVL11
   RETURN
   END
```

SUBROUTINE INVERT (A,NUM)

 THIS SUBROUTINE INVERTS THE MATRIX A AND STORES THE INVERSE IN A.
 NUM IS THE ORDER OF THE MATRIX. AFTER EXECUTION, THE ORIGINAL
 MATRIX A IS DESTROYED.

```

    IMPLICIT REAL*8 (A-H,O-Z)
    DIMENSION A(12,12),IP(12),IN(12,2),PI(12)
    EQUIVALENCE (IR,JR),(IC,JC),(AM,T,SW)
101  FORMAT(20H MATRIX IS SINGULAR)
    DO 1 J=1,NUM
      1  IP(J)=0
      DO 14 I=1,NUM
        AM=0.
        DO 6 J=1,NUM
          IF(IP(J)-1)2,6,2
        2  DO 5 K=1,NUM
          IF(IP(K)-1)3,5,19
        3  IF(DABS(AM)-DABS(A(J,K))) 4,4,5
        4  IR=J
           IC=K
           AM=A(J,K)
        5  CONTINUE
        6  CONTINUE
           IP(IC)=IP(IC)+1
           IF(IR-IC)7,9,7
        7  DO 8 L=1,NUM
           SW=A(IR,L)
           A(IR,L)=A(IC,L)
        8  A(IC,L)=SW
        9  IN(I,1)=IR
           IN(I,2)=IC
           PI(I)=A(IC,IC)
           IF(PI(I))10,18,10
       10  A(IC,IC)=1.
           DO 11 L=1,NUM
       11  A(IC,L)=A(IC,L)/PI(I)
           DO 14 L1=1,NUM
           IF(L1-IC)12,14,12
       12  T=A(L1,IC)
           A(L1,IC)=0.
           DO 13 L=1,NUM
       13  A(L1,L)=A(L1,L)-A(IC,L)*T
       14  CONTINUE
           DO 17 I=1,NUM
           L=NUM+1-I
           IF(IN(L,1)-IN(L,2))15,17,15
       15  JR=IN(L,1)
           JC=IN(L,2)
           DO 16 K=1,NUM
           SW=A(K,JR)
           A(K,JR)=A(K,JC)
           A(K,JC)=SW
       16  CONTINUE
       17  CONTINUE
    RETURN
  
```

```
18 WRITE(6,101)
19 RETURN
END
```

SUBROUTINE SYBAN (A,B,NROW,NCOL,NRHS,KO)

 THIS SUBROUTINE SOLVES SYSTEMS OF SIMULTANEOUS LINEAR EQUATIONS FOR WHICH THE MATRIX OF COEFFICIENTS (CALLED A) IS BOTH SYMMETRIC AND BANDED. THE MATRIX A IS DIMENSIONED A(NROW,NCOL) WHERE NROW IS THE NUMBER OF EQUATIONS AND NCOL IS THE BAND WIDTH. THE PRINCIPAL DIAGONAL IS STORED IN THE FIRST COLUMN OF A AND THE NONZERO BANDS ABOVE THE PRINCIPAL DIAGONAL FORM THE BALANCE OF A. THE RIGHT HAND SIDE COLUMN VECTORS ARE STORED IN B WITH EACH RIGHT HAND SIDE FORMING ONE COLUMN. HENCE B IS DIMENSIONED B(NROW,NRHS) WHERE NRHS IS THE NUMBER OF RIGHT HAND SIDES FOR WHICH THE SOLUTION IS DESIRED. KO IS AN EXECUTION INDICATOR. IF, UPON RETURN TO THE CALLING PROGRAM, KO IS ZERO, THE EXECUTION WAS SUCCESSFUL AND THE ANSWERS ARE STORED IN B. IF KO IS NOT ZERO, THE EXECUTION FAILED BECAUSE THE PRINCIPAL DIAGONAL ELEMENT IN THE KO' TH ROW WAS ZERO. IN THIS EVENT A NOTE IS WRITTEN ON THE OUTPUT PAGE. THIS ROUTINE DESTROYS THE MATRICES A AND B.

IMPLICIT REAL*8 (A-H,O-Z)
 DIMENSION A(990,42),B(990,1)
 101 FORMAT('//64H YOU GOOFED --- THERE IS A ZERO ON THE PRINCIPAL DIAGONAL IN THE,I4,8H TH ROW.//')
 NHALF=NCOL-1
 NROWM=NROW-1
 NROWH=NROW-NHALF
 DO 6 I=1,NROWM
 IF(A(I,1)) 1,15,1
 1 RECIP=1./A(I,1)
 IF(I-NROWH) 2,2,3
 2 LIMIT=NHALF
 GO TO 4
 3 LIMIT=NROW-I
 4 DO 6 J=1,LIMIT
 JROW=I+J
 RATIO=-A(I,J+1)*RECIP
 DO 5 K=J,LIMIT
 JCOL=K+1-J
 5 A(JROW,JCOL)=A(JROW,JCOL)+RATIO*A(I,K+1)
 DO 6 K=1,NRHS
 6 B(JROW,K)=B(JROW,K)+RATIO*B(I,K)
 IF(A(NROW,1)) 7,14,7
 7 RECIP=1./A(NROW,1)
 DO 8 I=1,NRHS
 8 B(NROW,I)=B(NROW,I)*RECIP
 DO 13 I=1,NROWM
 IROW=NROW-I
 RECIP=1./A(IROW,1)
 IF(I-NHALF) 9,10,10
 9 LIMIT=I
 GO TO 11
 10 LIMIT=NHALF
 11 DO 12 J=1,LIMIT
 JROW=IROW+J
 JCOL=J+1

```
DO 12 K=1, NRHS
12 B(IROW,K)=B(IROW,K)-A(IROW,JCOL)*B(JROW,K)
DO 13 J=1, NRHS
13 B(IROW,J)=B(IROW,J)*RECIP
KO=0
RETURN
14 I=NROW
15 KO=I
WRITE(6,101)KO
RETURN
END
```