CREEP OF METALS UNDER MULTIAXIAL STATES OF STRESS

by

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ABSTRACT

This paper is concerned principally with the basis for the formulation of a creep model that represents creep data of metals sufficiently accurately and that lends itself to numerical stress analysis of creep under multiaxial states of stress. A phenomenological approach is taken. Several forms of creep relations and flow conditions for multiaxial creep are considered. Comparisons of analytical predictions with experimental results are shown. Recommendations for numerical stress analysis of creep in engineering machines and structures subject to multiaxial stress states are given, and a promising technique for collection of the voluminous data that result from numerical stress analysis is indicated. An extensive reference list is included.
CREEP OF METALS UNDER MULTIAXIAL STATES OF STRESS

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A. P. Boresi* and O. M. Sidebottom*

1. INTRODUCTION

Interest in the creep of metals has increased greatly in the recent past as a result of the development of equipment which operates at elevated temperatures (for example, gas turbines, jet propulsion systems), and the higher temperatures used in processes such as nuclear reactor power generation. In this paper, we discuss the application of creep models to the study of time-dependent metal behavior under variable multiaxial states of stress. By creep, we mean all inelastic effects that are included when the relationship between stress and strain is time-dependent. Creep may be a function of the material, stress, strain, temperature, time stress history, and temperature history. This definition of creep includes the phenomenon of relaxation which is characterized by the reduction of stress in a body with time while total strain remains constant, and of recovery, which is characterized by the reduction of inelastic strain with time after the stress has been removed. Unfortunately from the viewpoint of ease of analysis, creep of metals at high temperatures is characterized by the fact that most of the creep deformation is irreversible, since at moderate temperatures only part of the creep strain is recovered after removal of load. Furthermore, the dependence of creep rate upon stress is quite nonlinear. As a consequence linear theories of viscoelasticity (hereditary theories) do not apply to metals. Accordingly, the mechanical theory of creep of metals is patterned after the theory of plasticity whose object is to describe time-independent irreversible deformation. Since at higher temperatures plastic deformation of metals is usually accompanied by creep, in

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real problems the concepts of creep and plasticity intertwine. However, from a convenience viewpoint, in studying creep, creep deformation is separated from plastic deformation and likewise from elastic deformation (Appendix A).

Since phenomenological creep theory is patterned after plasticity theory, many of the fundamental concepts of plasticity enter. For example, analogous to the deformation theory and the incremental theory of plasticity, there are total strain theories and incremental theories of creep. Furthermore, plasticity concepts such as isotropic work-hardening and anisotropic work-hardening have their counterpart in creep concepts of isotropic strain-hardening and anisotropic strain hardening. In addition, however, creep is further complicated by time-dependent behavior. The description of the time-dependent deformation of creep under multiaxial states of stress has historically developed from a phenomenological description of creep under uniaxial tension (Appendix A), the basis of which is the creep curve (Figure A-1, Appendix A). Plots of creep strain versus time for uniaxial tension tests at constant load and temperature normally exhibit three stages of creep rate, the primary stage in which the creep rate decreases, a secondary stage in which the creep rate attains a minimum constant value and a tertiary stage in which the creep rate increases. Since these creep curves are often expensive and time consuming to attain, they are obtained for given generally short periods of time. Hence, there is a continual need to extrapolate to longer times. For steady-state creep (constant stress and temperature) numerous approximations of the secondary stage have been proposed. For example, Norton (Eq. A-22, Appendix A) proposed that

\[ \dot{\varepsilon}_c = B \sigma^n \]  

(1)

where for metals, experiments indicate that \( n \) varies from 3 to 8 and higher. This highly nonlinear dependence on stress level is typical of metals. An alternate form of Eq. (1) which allows a change in scale is
\[
\dot{\varepsilon}_c = \dot{\varepsilon}_n \left( \frac{\sigma}{\sigma_n} \right)^n
\]

(2)

Ludwik (Eq. A-20, Appendix A) proposed

\[
\dot{\varepsilon}_c = \dot{\varepsilon}_s \exp \left( \frac{\sigma}{\sigma_s} \right)
\]

(3)

which in some respects is more convenient than power relations (Eqs. 1 and 2). However Eq. (3) gives a nonzero creep rate when \( \sigma = 0 \). To correct this defect, Nadai (Eq. A-23, Appendix A) proposed the equation

\[
\dot{\varepsilon}_c = 2\dot{\varepsilon}_s \sinh \left( \frac{\sigma}{\sigma_s} \right)
\]

(4)

and Söderberg proposed (Eq. A-21, Appendix A)

\[
\dot{\varepsilon}_c = \dot{\varepsilon}_s \exp \left[ \left( \frac{\sigma}{\sigma_s} \right) - 1 \right]
\]

(5)

For values of \( \sigma \) sufficiently large compared to \( \sigma_s \), Eqs. (4) and (5) practically coincide with Eq. (3). Also for sufficiently small \( \sigma \), Eqs. (4) and (5) differ little from a linear function.

Numerous attempts have been made to represent the creep curve as a whole.

The first systematic study by Andrade included representation of the primary and secondary stages of creep deformation by the form (A-8, Appendix A):

\[
\varepsilon_c = a \left( 1 + \beta t^{1/3} \right) e^{k t} - a
\]

(6)

where \( \beta, k \) are considered functions of stress and temperature. For stresses at which \( \beta \) and \( k \) are sufficiently small the creep strain is

\[
\varepsilon_c = a \left( \beta t^{1/3} + k t \right)
\]

(7)

Andrade and others, mainly British authors, have made numerous studies of Eq. (6), \[1\]. The following more general form has been suggested by Rabotnov \[2\] for the first two stages of creep

\[
\varepsilon_c = S (\sigma) t^m + v (\sigma) t
\]

(8)

where \( v \) is the constant creep rate in the secondary stage. The constant \( m \) is ordinarily close to 1/3 but may be as large as 1/2 or more, and it is generally a function of stress.
and temperature. The function \( S(\sigma) \) is approximated by the same form as \( v(\sigma) \) (for example, by forms such as Eqs. 1 - 5). Thus \( S(\sigma) \) is proportional to \( v(\sigma) \). For short-time creep, the creep deformation is approximated by (See Eq. A-6).

\[
\epsilon_c = S(\sigma) t^m
\]  \( (9) \)

The concept of similarity of creep curves has been used by many authors \([2]\) concerned with the development of phenomenological creep theories. This means that the creep deformation is representable as

\[
\epsilon_c = S(\sigma) \tau(t)
\]  \( (10) \)

Equation (10) is fairly well justified in the first part of creep curves of metals. Indeed Eq. (9) is valid over a certain range and satisfies Eq. (10). Since the functions \( S(\sigma) \) and \( v(\sigma) \) differ by a proportionality constant, creep curves described in the form of Eq. (8) are also similar. The concept of similarity is discussed lucidly by Rabotnov \([2]\).

A series of creep curves may be considered as a graphical representation of the functional equation

\[
\epsilon = \epsilon(\sigma, t)
\]  \( (11) \)

with one \((\epsilon, t)\) curve for each value of \(\sigma\) (Fig. 1). Alternatively, the functional relation between \(\epsilon, \sigma,\) and \(t\) may be expressed by plotting \(\sigma\) versus \(\epsilon\) for given times \(t_1, t_2, t_3 \ldots\) (Fig. 2).

Curves of \((\sigma, \epsilon)\) for given time \(t\) are called isochronous creep curves. For some materials, isochronous creep curves are similar. Thus, by analogy to Eq. (10), they may be represented by the relation

\[
\sigma = F(\epsilon) G(t)
\]  \( (12) \)

However, as Rabotnov has noted, the conditions of similarity of isochronous creep curves are very different from those of ordinary creep curves (Eq. 10). Only when elastic strain can be neglected are Eqs. (10) and (12) equivalent.
If we set \( G(0) = 1 \), then \( \sigma = F(\epsilon) \) is the instantaneous deformation relation. In published experimental data on creep, data for the primary stage of creep is often unreliable, since the instantaneous deformation that occurs upon applying load is not usually accurately recorded. Hence, creep curves for \( t = 0 \) or for very small values of \( t \) frequently are not available. Of course, by selecting \( G(t) \) appropriately we can obtain analytically the instantaneous curve \( \sigma = F(\epsilon) \) by extrapolation, but we have no guarantee that the actual instantaneous deformation curve is obtained. Rabotnov has represented the function \( G(t) \) by the formula
\[
G(t) = \frac{1}{1 + a t^b}
\] (13)
and from experimental data has determined \( b \approx 0.3 \).

In comparing calculated and experimental creep results, very small changes in stress produce a large influence on creep rate, and consequently upon the time required to achieve a particular deformation. Small differences in microstructure or chemical composition of material specimens also greatly affect the creep rate. Consequently, in experiments to determine time required to attain a particular creep strain at a fixed stress level, the scatter of results for individual specimens is ordinarily quite large. However, if we determine experimentally the stress at which a particular deformation is reached in a given time, the scatter is very small. On this basis, Rabotnov notes that there is some experimental evidence of accurately predicted creep deformation which confirms Eq. (12). However, all available experimental evidence does not provide full verification of the rule of similarity of isochronous creep curves. In some cases, similarity is clearly not valid. However, if it exists, it can be used to simplify calculations.

The question of temperature dependence of creep parameters may be summarized by noting that an increase in temperature results in an increase in creep rate. Hence, at a given stress level, a given deformation is obtained more quickly at high temperatures than at low temperatures. In general, all the creep parameters are affected by temperature. Thus, a correlation of the temperature effects on the parameters that occur in the
numerous creep equations that have been proposed is not feasible. Consequently, physical theories are usually employed to predict the temperature dependence of creep curves. The results have been reasonably accurate. A number of formulas that reflect the temperature dependence of creep curves of metals are summarized in Appendix A (Eqs. A-10 through A-18). Ordinarily in practice, the temperature dependency of creep curves is required mainly for interpolation over a fairly narrow temperature range. Hence, the problem of extrapolation to temperatures far outside this range is not particularly important to the engineer. Hence, in general, it is reasonable to use a simple formula for temperature dependency even though the formula may have no physical relationship to a more complicated physically based formula that involves calculation difficulties.

In elementary unidimensional creep theories, a fundamental objective is that of predicting the relaxation process from creep properties obtained from creep tests. To achieve this objective, we require a one-dimensional theory capable of describing creep under variable (decreasing) loads. To test analytical predictions of relaxation we require experimental results of relaxation tests. However, since such tests are somewhat more difficult to perform than classical creep tests, there is not much data available. In themselves, relaxation tests are not particularly important since relaxation properties are not ordinarily used to determine parameters of creep models. Rather, relaxation tests are primarily of interest as a means of checking the predictions of particular creep models.

In the following, we discuss certain procedures employed to extend the applicability of one-dimensional phenomenological creep models to variable load and temperature effects. We then indicate techniques that have been employed to treat creep under states of multiaxial stress. In particular, numerical predictions of various creep models are given and compared with experimental results. A novel computer technique for collection of the voluminous data required to numerically describe creep deformations
in multiaxial states is described briefly. A brief discussion of the effects of radiation on creep is given. Finally recommendations for the study of creep under multiaxial states of stress are listed.
2. ONE-DIMENSIONAL ELEMENTARY CREEP MODELS FOR VARIABLE LOAD AND TEMPERATURE

The objective of an elementary creep theory based upon a mathematical (phenomenological) model is to determine the strain as a function of time, when given temperature and stress as functions of time. By elementary, we mean theories that include all the results of one-dimensional creep tests at constant stress and temperature. Alternatively, this objective is to develop an equation or system of equations to relate measured values of stress, strain, temperature and time. Even though these equations may include parameters which are influenced by internal structural (metallurgical) properties, in the phenomenological model we assume that all the creep properties may be related to mechanically measured quantities such as force and displacement. The form of the stress-strain-temperature-time relation of creep may be chosen to fit only certain parts of the creep curve. For example, in the steady-state creep stage the strain rate is approximately constant for constant stress and temperature. If the creep test is run for a different (higher) stress, the creep rate increases. Thus, for constant temperature creep tests, the creep strain rate $\dot{\varepsilon}_c$ may be expressed as a function of stress level $\sigma$:

$$\dot{\varepsilon}_c = \dot{\varepsilon}_{sc} (\sigma)$$

(14)

where $\dot{\varepsilon}_{sc}$ is the steady-state creep rate. Equation (14) is similar to the equation for quasi-viscous flow of fluids, which, unlike ordinary viscous flow depends nonlinearly upon stress. Equation (14) is sometimes applied to creep of metals which undergo long time use in which most of the strain occurs at constant rate or to short-time creep at very high temperature and high stress. The use of Eq. (14) ignores the primary and tertiary stages of creep. Hence, the creep deformation $\varepsilon_c$ is approximated by straight lines (Fig. 3). Creep models (theories) which employ Eq. (14) are called steady-state creep models (theories). Steady-state creep models do not describe the relaxation
process, since they do not include unloading effects properly. There have been a number of proposed modifications of steady-state models to include approximations of relaxation as well as the effects of the primary stage on long time creep deformation.

For example, Söderberg \[3\] proposed the equation

\[
\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \dot{\varepsilon}_{\text{sc}} (\sigma)
\]  

(15)

where \(\dot{\varepsilon}\) is the total strain rate, \(\dot{\varepsilon}_{\text{sc}}\) is the steady-state creep rate, \(\dot{\sigma}\) is stress rate and \(E\) is the modulus of elasticity, to allow for initial elastic deformation, but ignored the primary creep. However, agreement between experiments and the prediction of Eq. (15) do not ordinarily agree well since the primary stage creep often is as large or larger than the elastic strain. Odqvist \[4\] has proposed an equation for steady state creep that approximates the effects of elastic deformation, the instantaneous plastic deformation and the primary stage creep on the steady-state. It is

\[
\dot{\varepsilon} = \left[ \varepsilon'_{\text{o}} (\sigma) \right] \dot{\sigma} + \dot{\varepsilon}_{\text{sc}} (\sigma)
\]  

(16)

where \(\varepsilon'_{\text{o}} (\sigma)\) = elastic strain + instantaneous plastic strain + primary stage creep, and the prime denotes derivative with respect to \(\sigma\). Equation (16) generally can be made to agree well with experimental results. The creep model curve associated with Eq. (16) starts at \(\varepsilon_{\text{0}}\) at time \(t = 0\) and is a straight line asymptotic to the steady-state stage of creep (Fig. 4).

One may represent the creep curve for constant stress as a function relation of the type

\[
\varepsilon = f (\sigma , t)
\]  

(17)

It is attractive to assume that this relation holds when the stress varies with time. However, on the basis of invariance relative to time, Eq. (17) leads to contradictions \[2\] for structurally stable materials (that is, materials whose structure and properties, e.g., affecting creep mechanisms, do not change under prolonged exposure to test temperature without load). Models (theories) in which stress, strain, temperature,
and time are related functionally are called aging models (theories). Since these models are applied chiefly to the primary and secondary stages of creep and since in these regions the creep rate decreases (that is, resistance to creep increases or hardens), they are also called time-hardening models or theories. In a modified form of aging (time hardening), the creep strain \( \varepsilon_c \), is taken as a function of stress and time. In particular, in a form analogous to Eq. (15), the creep rate is taken as

\[
\dot{\varepsilon}_c = \dot{\varepsilon} - \frac{\dot{\sigma}}{E} = f(\sigma, t) \tag{18}
\]

The time-hardening model of Eq. (18) is more logically acceptable than is that of Eq. (17), since an instantaneous change in stress does not produce an instantaneous change in creep strain, but rather a change in creep strain rate only. The behavior predicted by Eq. (18) is analogous to that of a non-linear viscous liquid with time-dependent viscosity. Generally speaking, relaxation curves predicted by Eq. (18) agree well with experiments [2, 5, 6, 7]. The predictions of Eq. (18), time-hardening theory, also agree well with experiments for small changes in stress levels.

Time-hardening theory is particularly easy to apply for similar creep curves (Eq. 10). Then,

\[
\dot{\varepsilon}_c = \dot{\varepsilon} - \frac{\dot{\sigma}}{E} = S(\sigma) \tau(t) \tag{19}
\]

If we change time scale [2] and take \( \tau(t) \) as the independent variable rather than \( t \), we may write Eq. (19) in the form

\[
\frac{d\varepsilon_c}{d\tau} = \frac{d\varepsilon}{d\tau} - \frac{1}{E} \frac{d\sigma}{d\tau} = S(\sigma) \tag{20}
\]

In particular, then the relaxation problem is solved readily, since then \( \varepsilon = \) constant and integration of Eq. (20) yields

\[
\tau(t) = \frac{1}{E} \int \frac{d\sigma}{S(\sigma)} \tag{21}
\]

a result analogous to that of "viscous" flow theory.
More generally, for structurally stable materials, it is natural to assume the existence of an equation of state which relates the creep rate to the applied stress and the amount of accumulated creep strain. Such an assumption relates the degree of strain hardening uniquely to the amount of plastic deformation in a manner analogous to the way in which work-hardening in the theory of plasticity is related to plastic deformation. Accordingly, for a strain-hardening hypothesis the equation of state is represented functionally as

$$ F ( \dot{\varepsilon}, \sigma, \varepsilon, T ) = 0 $$ (22)

Equations of state were assumed quite early in the study of plasticity [8, 9]. However, Davenport [6] introduced the terminology of strain-hardening. Strain-hardening theories can be checked most simply by creep tests in which stepwise changes of load are made [10]. Generally speaking, stepwise changes of load are easy to make. Also, since the theoretical predictions of various hardening theories show wider differences for stepwise loading than they do for relaxations tests, stepwise loading tests serve as a better check of hardening models than do relaxation tests.

For example, let us examine creep tests with a stepwise change in load. To illustrate the difference between predictions of a time-hardening model (Eq. 18) and a strain-hardening model (Eq. 22), we consider two creep curves for a given metal (creep deformation-time diagrams) for two stresses $\sigma_1, \sigma_2 > \sigma_1$, Fig. 5. Let us subject a specimen of the metal first to stress $\sigma_1$. The creep is predicted by the lower curve. At a given time, say $t_1$, we increased the stress level instantaneously to $\sigma_2$. Deleting the instantaneous increase in elastic strain (since we plot creep deformation $\varepsilon$), the new creep curve leaves the $\sigma_1$ creep curve at point P. Depending upon the hardening theory used, predictions for the new curve differ. For example, if the time hardening theory is employed (Eq. 18), then the creep at point A depends solely upon the time $t_1$ and the stress level $\sigma_2$. Hence, by this time-hardening theory the new creep curve
would start from point $P$ with the creep rate of curve $\sigma_2$ at time $t_1$ (point $B$) and continue parallel to $BC$ of curve $\sigma_2$. On the other hand, by the strain-hardening theory of Eq. (22), the new creep rate depends upon the stress level $\sigma_2$ and the accumulated creep $\epsilon_{CA}$ (assuming $T$ constant). Hence, by this strain-hardening model, the new creep curve would start from point $P$ with the creep rate of curve $\sigma_2$ at point $A$, and continue parallel to $ABC$ of curve $\sigma_2$. Thus, as observed in Fig. 5, appreciable differences in the predictions of the two theories is evident.

Quite often, the strain-hardening theory gives a more accurate prediction of experimental results of stepwise changes of load $[10]$. Unfortunately, however, this strain-hardening model does not always yield accurate predictions, particularly when several step changes in load occur in the same test $[2]$. Furthermore, strain-hardening theory is unable to predict accurately results that arise due to structural instabilities $[10]$. Nevertheless, for structural stable materials, the prediction of strain-hardening generally are fairly reliable and relatively easy to calculate. Finally, although many authors have assumed that $\epsilon_c$ is the total plastic deformation, this procedure in general is incorrect $[2]$. Accordingly, the equation of state approach to hardening (Eq. 22) is best suited to those situations involving structurally stable material in which a predominate creep mechanism exists and in which only elastic deformation $\sigma/E$ exists. Then, the total strain is $\epsilon = \sigma/E + \epsilon_c$.

A more general strain-hardening hypothesis is that the creep rate $\epsilon_c$ depends upon stress, temperature, and a number of structural parameters $s_1, s_2, s_3, \ldots, s_{N_s}$ which characterize the creep process $[11]$. Thus, Eq. (22) is generalized to

$$F(\epsilon_c, \sigma, T, s_i) = 0, \quad i = 1, 2, \ldots, N_s$$  \hspace{1cm} (23)

The time-hardening hypothesis is then characterized by $N_s = 1, s_1 = t$, and the strain-hardening hypothesis is given by $N_s = 1, s_1 = \epsilon_c$. By selecting $N_s > 1$ and different $s_i$, other forms of hardening models may be obtained $[12, 13]$.
In applications, many special forms of Eq. (22) or Eq. (23) have been used. Thus, analytical forms have been chosen to represent Eq. (22), with the objective of including the principal creep mechanism. Often the condition of similarity of creep curves is assumed. Then, for constant temperature, Eq. (22) is represented by (See Eq. 10)

$$\dot{\varepsilon}_c = S(\sigma) \cdot f\left(\frac{\varepsilon_c}{S(\sigma)}\right)$$

(24)

If $f$ is a power function of $\varepsilon_c/S$, we may write

$$\varepsilon_c = \varepsilon_c^{-\beta} \cdot h(\sigma)$$

(25)

Integration of Eq. (25) yields

$$\varepsilon_c = g(\sigma) \cdot t^k$$

(26)

where $k = 1/(1 + \beta)$. Various forms for $g(\sigma)$ have been used; for example, a power function (Appendix A)

$$g(\sigma) = A \sigma^n; \quad n > 1/k$$

(27)

or an exponential function

$$g(\sigma) = B \exp\left(\frac{\sigma}{\varepsilon_c}\right)$$

(28)

This last expression is unsuitable for small values of stress, since $g(\sigma)$ must tend to zero as $\sigma^n$. ($n > 1/k$). Accordingly, an improvement

$$g(\sigma) = C \left(2 \sinh \frac{\sigma}{B}\right)^n$$

(29)

has been suggested by Garofalo [14], but this expression is somewhat more complicated to use in practice.

More generally, to describe adequately both the primary and secondary stages of creep of metals, it is necessary to broaden Eq. (25) to the form

$$\dot{\varepsilon}_c = H(\varepsilon_c) \cdot f(\sigma)$$

(30)
where \( H(\epsilon_c) \) behaves like \( \epsilon_c^{-\beta} \) for small values of \( \epsilon_c \) and tends to a constant for steady-state creep. Rabotnov [2] suggests the form

\[
H(\epsilon_c) = \epsilon_c^{-\beta} + C_1
\]  
(31)

where \( C_1 \) is a constant. For \( \beta = 2 \), Eqs. (30) and (31) are equivalent to Andrade's law in the primary stage, since creep deformation \( \epsilon_c \) is then proportional to the 1/3 power (Appendix A). More generally, techniques for experimental determination of the constants in creep equations such as Eqs. (26), (27), (28), (29), (30), and (31) are discussed extensively by Conway [15]. The creep parameters \( A \) and \( n \) in Eq. (27) or \( B \) and \( c \) in Eq. (28) generally depend on temperature \( T \).

Furthermore, if the temperature range is sufficiently narrow, \( n \) or \( c \) are practically constant. However, \( A \) or \( B \) vary considerably with \( T \). To express the effects of temperature \( T \), the parameters \( A \) and \( B \) are often taken in the form

\[
A = A_o \exp \left(-\frac{U}{RT}\right); \quad B = B_o \exp \left(-\frac{U}{RT}\right)
\]  
(32)

where \( U \) is activation energy [16, 17] and \( R \) is the gas constant (See Appendix A).

Thus, a simple general form of the strain hardening law that includes temperature effects is

\[
\dot{\epsilon}_c = \epsilon_c^{-\beta} f(\sigma) \exp \left(-\frac{U}{RT}\right)
\]  
(33)

where the function \( f(\sigma) \) and the parameter \( \beta \) are considered independent of temperature.
3. CREEP UNDER MULTIAXIAL STATES OF STRESS

Practically all engineering systems (machines, structures, etc.) operate under multiaxial stress conditions. Unfortunately, creep tests of parts of a machine or a structure are very difficult and expensive to perform. In addition, a tremendous amount of experimental and analytical data must be accumulated to make meaningful comparisons between experimental results and analytical predictions. Recently, pioneering work in digital computer representation of analytical predictions in the form of field maps may overcome some of this difficulty by displaying the entire analytic calculation in a single diagram or map. (Article 7, Computer Methods in Creep).

Physical theories of multiaxial stress creep problems are nonexistent, and hence, problems of multiaxial creep are mainly phenomenological in form. These phenomenological theories are based principally upon concepts of the theory of plasticity of metals at relatively normal (room) temperatures (where time effects are negligible or absent). The theory of plasticity predicts reasonably accurate results for proportional loading \[18\]. However, for arbitrary loading paths, considerable differences between experimental results and theoretical predictions may occur. In addition, experimental data does not clearly indicate the validity of any theory of plasticity. Since the extension of plasticity theory to creep may proceed in several ways, the number of possibilities of creep theories is much larger than for plasticity theories. Fortunately, in a number of engineering creep problems, the stressed state ordinarily varies slowly with time. Consequently, different creep theories may predict rather similar results.

The complexity of the creep problem under multiaxial states of stress is compounded greatly compared to the uniaxial case because of the fact that one dimensional quantities (scalars) must now be replaced by tensor quantities. Hence, instead of a single creep deformation \(\varepsilon_c\) and a deformation rate \(\dot{\varepsilon}_c\), now a creep deformation tensor \(\varepsilon_{ij}^c\) and a creep deformation rate tensor \(\dot{\varepsilon}_{ij}^c\), \(i, j = 1, 2, 3\), enter. Hence,
the equation of state becomes a relationship between the creep rate tensor $\epsilon^{c}_{ij}$, the stress tensor $\sigma_{ij}$ and hardening parameters which may be scalars or more generally tensors of any rank. In case the hardening parameters are scalars, the hardening is said to be isotropic. To date, engineering phenomenological theories of multiaxial creep have been based primarily upon isotropic hardening assumptions, even though theoretical predictions so obtained may in some cases be shown to disagree with experiments.

The simplest case of multiaxial creep is that in which the stressed state is homogeneous and constant with time. As expected, most available experimental results are for this case. From this basis, a number of methods are used to extend the analysis to nonhomogeneous stress states that vary with time. As noted above, however, the number of possibilities is much larger than that which exists in plasticity theory. Indeed, it is possible to describe a large number of creep models treating special effects such as steady-state creep (creep deformation rate constant with constant stress), creep with isotropic hardening, creep with anisotropic hardening, and so on. As noted in the unidimensional creep theory (Article 2), it is impossible to determine the dividing line between the end of instantaneous deformation (elastic and plastic) and the beginning of creep deformation in a creep test. However, the error introduced into the theory by this unknown is generally small. Likewise it is not possible to distinguish precisely between nonsteady creep (primary stage) and steady-state creep (secondary stage). The problem of transition from nonsteady to steady-state creep can be approached in several ways. For example, in the one dimensional case, it is often assumed that as creep deformation $\epsilon_c$ increases, the function $H(\epsilon_c)$ in Eq. (30) tends to a definite limit, which is attained either for a particular value of $\epsilon_c$ or as $\epsilon_c \to \infty$. Another method based upon the metallurgical level, may be to assume that transient creep and steady-state creep are controlled by different micromechanisms which coexist simultaneously, but independently. In this case, the total deformation at any time consists
of the instantaneous strain (elastic or elastoplastic) $\varepsilon_0$, the transient creep strain $\varepsilon_{Ic}$ and the strain of steady-state creep $\varepsilon_{IIc}$. The transient creep is described by an equation of state similar to that of Eq. (22), namely $\dot{\varepsilon}_{Ic} = F_I(\sigma, \varepsilon_{Ic}, T)$, but which damps out with time such that $\dot{\varepsilon}_{Ic} \rightarrow 0$. Other more elaborate schemes may be devised. For example, recovery may be also accounted for. Depending on the scheme employed to make the transition from transient to steady-state creep, the analysis may proceed by widely differing paths. However, in engineering problems, questions of simplicity and convenience often dictate the scheme of transition. For example, on this basis, the first method noted above is used most frequently. However, a really rational basis for selection still awaits more definitive experimental data.

Two cases of steady-state creep are prevalent. Under some conditions, strain-hardening may be negligible from the initial time of loading. In other conditions, the creep rate becomes constant only after some time as the material becomes fully strain-hardened and can undergo no further strain-hardening as creep continues. For example, if the temperature and stress levels are sufficiently high (short-term creep tests), strain-hardening is nearly absent. However, at relatively low temperatures the rate may become constant only after long times, steady-state creep then being preceded by a period of strain-hardening. The different response in the two cases is clear with variable loading. If the material does not strain-harden the instantaneous creep rate depends only upon the instantaneous stress and is independent of previous loading history. If steady-state creep is preceded by a transient period, when a step change in load is applied in the steady-state region, the creep rate does not instantaneously change to a new value, but rather the transient period of the creep curve is more or less repeated until finally after some time the creep rate is again steady.

For creep under multiaxial states of stress, the presence or absence of strain-hardening may be even more important, since then strain-hardening which results from creep or short-term plastic deformation may create anisotropy in the material relative
to subsequent creep under conditions of changing loading paths. Hence, in describing steady-state creep (and nonsteady creep) under multiaxial stress conditions, we must distinguish between conditions of isotropic and anisotropic behavior. These conditions are influenced by the initial state of the material, the strain hardening or structural changes caused by instantaneous plastic creep (which may occur due to sufficiently large instantaneous loading or due to creep if the stresses are nonuniform and become redistributed), and by strain hardening that occurs in the transient creep phase. In theory, anisotropy may also arise during steady-state creep. This implies that the creep deformation in the steady-state may also cause structural changes in the material which are not noticed as a strain-hardening effect in simple one-dimensional tests, but nevertheless influence the relationship between the components of the creep deformation rate tensor $\dot{\epsilon}_{ij}^c$. This latter effect is relatively unexplored to date.

As noted above, time dependent inelastic theories for multiaxial states of stress are based on idealized models of material behavior. The models are similar to those used in the theory of plasticity (time independent inelastic behavior). Accordingly, we give a brief review of the simple plasticity models, not only because creep theories use the same models but because more generally part of the inelastic deformation may be time independent and part time dependent.

**Time Independent Inelastic Deformation.** Ziegler [19] notes that the mathematical theory of plasticity is based upon the following four postulates (assumptions).

1. All yield surfaces are independent of hydrostatic states of stress.
2. There exists an initial yield state defined by a point on a yield surface specifying the states of stress for which plastic flow begins.
3. There exists a rule relating the increment of the plastic state of strain to a specified increment in the state of stress. (The so-called flow rule).
4. There exists a rule specifying the modification of the yield surface during the course of the plastic flow. (The so-called hardening rule).
Postulate 1 is an empirical result which has been experimentally verified for a large number of materials. Postulate 2 is based on the fact that most metals at room temperature exhibit a linear relation between stress and strain for states of stress less than that required to initiate yielding. The shape of the initial yield surface can be empirically determined but is usually assumed to have either a circular cross section or a hexagonal cross section. If a material yield is defined best by the circular cross section, it is said to be a von Mises material; if the yield is best defined by the hexagonal cross section, it is said to be a Tresca material.

All plasticity theories require that stress-strain relations be specified to give increments of each component of the state of strain in terms of the increment of the state of stress. These stress-strain relations require a flow rule as indicated in Postulate 3 and a hardening rule as indicated in Postulate 4. The flow rule in general depends upon the idealized model of material behavior chosen to represent the hardening rule. Most plasticity theories are based on the isotropic hardening model. The Prandtl-Reuss stress-strain relations for von Mises materials and stress-strain relations for Tresca materials are based on this model. The model requires that the material be both isotropic and even (have the same true stress versus true strain relationship for tension and compression). The flow rule for this model is given by a plot of true stress versus true plastic strain from a tension specimen of the material, Fig. 6. The flow rule is usually expressed by the relation (see the analogous time dependent (creep) rule of Eq. 17).

\[ \varepsilon_p = f(\sigma_e) \quad (34) \]

where \( \varepsilon_p \) is the effective true plastic strain and \( \sigma_e \) is the effective true stress. The definitions of \( \varepsilon_p \) and \( \sigma_e \) are different for von Mises and Tresca materials.

For von Mises materials, the effective true stress \( \sigma_e \) is defined by the relation

\[ \sigma_e = \frac{1}{\sqrt{2}} \sqrt{\left( \sigma_x - \sigma_y \right)^2 + \left( \sigma_y - \sigma_z \right)^2 + \left( \sigma_z - \sigma_x \right)^2 + 6 \left( \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right)} \quad (35) \]
and the effective true plastic strain $\varepsilon_p$ is defined by the relation

$$\varepsilon_p = \frac{\sqrt{2}}{3} \sqrt{\left(\varepsilon_p^x - \varepsilon_p^y\right)^2 + \left(\varepsilon_p^y - \varepsilon_p^z\right)^2 + \left(\varepsilon_p^x - \varepsilon_p^z\right)^2 + \frac{3}{2} \left[(\gamma_p^x)^2 + (\gamma_p^y)^2 + (\gamma_p^z)^2\right]}$$  \(36\)

These definitions of $\sigma_e$ and $\varepsilon_p$ reduce to the true stress $\sigma$ and true plastic strain $\varepsilon_p$ for the tension specimen, Fig. 6. The function $f$ in Eq. (34) is represented by the diagram in Fig. 6.

For Tresca materials $\sigma_e$ and $\varepsilon_p$ are defined in terms of principal components of true stress and true plastic strain. For convenience, we may assume that $\sigma_1$ and $\sigma_3$ are the maximum and minimum principal true stresses while $\sigma_2$ is the intermediate true stress. Furthermore, it is usually assumed that the plastic strain in the direction of the intermediate stress remains zero. For these conditions, $\sigma_e$ for a Tresca material is given by the relation

$$\sigma_e = \sigma_1 - \sigma_3$$  \(37\)

For a Tresca material, three different relations are used in the current literature to define $\varepsilon_p$. They are

$$\varepsilon_p = \frac{\sqrt{2}}{\sqrt{3}} \sqrt{\left(\varepsilon_p^1\right)^2 + \left(\varepsilon_p^2\right)^2 + \left(\varepsilon_p^3\right)^2}$$  \(38\)

$$\varepsilon_p = \frac{1}{2} \left(\varepsilon_p^1 - \varepsilon_p^3\right)$$  \(39\)

$$\varepsilon_p = \frac{2}{3} \left(\varepsilon_p^1 - \varepsilon_p^3\right)$$  \(40\)

Equation (36) reduces to Eq. (38) for principal true plastic strains since $\varepsilon_p^1 + \varepsilon_p^2 + \varepsilon_p^3 = 0$.

The authors recommend that Eq. (38) be used to define $\varepsilon_p$, since a recent investigation \[20\] indicates that it gives the best correlation between theory and experiment. For the tension test (Fig. 6), the definitions of $\sigma_e$ and $\varepsilon_p$ in Eqs. (37) and (38) reduce to true stress $\sigma$ and true plastic strain $\varepsilon_p$. The function $f$ in Eq. (34) is represented by the diagram in Fig. 6.
We note that for certain loads, it is impossible for the plastic strain in the direction of the intermediate principal stress to remain zero. For example, the intermediate stress direction for a solid circular torsion-tension member coincides with the radial direction, and it is impossible to have plastic strain in the axial direction without having plastic strain in the radial direction.

Plasticity theories based on an isotropic hardening model accurately predict load-deformation relations for structural members subjected to monotonically increasing proportionate loading. Good agreement between theory and experiment may be obtained also for monotonically increasing nonproportionate loading [18]. In general, experimental data either agrees with the von Mises prediction, with the Tresca prediction, or falls between the two predictions.

These theories are not generally valid for reversed loading or loading in which a member is plastically deformed by loading along one path, then unloaded, and reloaded along another path. Once the member has been plastically deformed, the material becomes anisotropic. To predict the behavior of a structural member that is unloaded and either loaded in the opposite sense or loaded along another path, a flow rule and stress-strain relations for the anisotropic material must be included in the theory.

**Time Dependent Inelastic Deformation.** In the case of time dependent inelastic deformation, the concept of yield surfaces does not have any meaning. However, Drucker [21] has pointed out that one may speak of surfaces \( \phi(\sigma_{ij}) = \text{constant} \) in stress space for both time dependent and time independent inelastic behavior. For von Mises and Tresca materials these surfaces correspond to the condition \( \phi = \sigma_e = \text{constant} \). For time independent inelastic deformation, the surface \( \sigma_e = \text{constant} \) is a yield surface. For time dependent inelastic deformation, the surface \( \sigma_e = \text{constant} \) is interpreted by Drucker as a surface along which the rate of dissipation of energy is a constant, i.e., \( \sigma_e \dot{\varepsilon}_c \) is a constant, where \( \dot{\varepsilon}_c = d\varepsilon_c/dt \) is the effective creep strain defined by the relation
\[
\dot{\varepsilon}_c = \frac{\sqrt{2}}{3} \sqrt{\left( \dot{\varepsilon}_x^c - \dot{\varepsilon}_y^c \right)^2 + \left( \dot{\varepsilon}_y^c - \dot{\varepsilon}_z^c \right)^2 + \left( \dot{\varepsilon}_z^c - \dot{\varepsilon}_x^c \right)^2 + \frac{3}{2} \left[ (\dot{\gamma}_{xy}^c)^2 + (\dot{\gamma}_{yz}^c)^2 + (\dot{\gamma}_{zx}^c)^2 \right]} \tag{41}
\]

The mathematical theory of inelasticity for time dependent inelastic deformations is based upon the following three postulates:

1. Surfaces for which \( \phi = \text{constant} \) are independent of hydrostatic states of stress.

2. There exists a rule relating \( \dot{\varepsilon}_c \) to \( \sigma_c \), temperature, and stress histories. (The creep flow rule.)

3. There exists a rule specifying the modification of the surface \( \phi = \text{constant} \) during the process of inelastic deformation. (The creep hardening rule.)

We assume Postulate 1 to be valid. To satisfy Postulate 3, we assume an isotropic hardening model.

Thus there remains the important problem of including Postulate 2 in creep theories. In the case of time independent inelastic deformations, the flow rule for uniaxial state of stress is given by the results of a tension test. However, in the case of time dependent inelastic deformation, even for the uniaxial case, the flow rule is not known precisely, since the strain rate is generally a function not only of stress and temperature, but also of stress and temperature histories (Article 2). We return to the question of selection of flow rules for time dependent inelastic deformation and for multiaxial states of stress below.

Of particular importance to us here then is the question "Is there experimental evidence that creep theories based on the above three postulates may not be valid in some cases"? The answer to this question is yes. For example, as noted above, for time independent inelastic deformation, experimental data for structural members subjected to proportionate loading are found to agree either with the von Mises theory, with the Tresca theory, or lie between the predictions of the two theories. However,
for time dependent inelastic deformation, experimental creep data for structural members subjected to constant loads may fall far outside the region bounded by the curves for von Mises and Tresca theories.

A comparison between the two theories and experimental data for solid circular torsion members is indicated in Fig. 7 for copper alloy 360 (free-cutting brass) at $650^\circ$ \[10\] and in Fig. 8 for 7075-T6 aluminum alloy at $375^\circ$F \[5\]. Free cutting brass was found to be well behaved (structural stable) in \[22\], for a test duration of one hour; excellent agreement was found between the predictions of the von Mises theory and the experimental data. Good agreement between theory and experiment is also indicated in Fig. 7 for the first two hours. Whereas, after several hours, the experimental data for free cutting brass lies below the region bounded by the two theories, the data for 7075-T6 aluminum alloy falls considerably above predictions of the two theories. Other test data for 7075-T6 aluminum alloy subjected to nonproportionate torsion-tension \[18\] at room temperature, was in excellent agreement with the von Mises theory.

The authors have no explanation for the anomalous behavior of the two metals in Figs. 7 and 8. However, it is suspected that metallurgical changes in the metals occurred at the elevated temperatures. If these changes in properties are independent of state of stress, they may be included in the flow rule and good agreement should be expected between theory and experiment. If the time and temperature dependent metallurgical changes are not independent of state of stress, one is led to suspect that Postulate 1 is not valid for certain metals at elevated temperatures.

Many metals are structurally stable at elevated temperatures and good agreement can be expected between theory and experiment. However, further research is needed to determine if other metals exhibit anomalous behavior similar to that exhibited in Figs. 7 and 8. In particular, there may be ranges of time and temperature for which metals exhibit anomalous behavior. For instance, at $800^\circ$F, titanium 6.4 was found to be structurally stable \[5\], but at $1000^\circ$F it exhibited behavior similar to that of 7075-T6 aluminum alloy.
Multiaxial Flow Rule for Time Dependent Inelastic Deformation. The flow rule for multiaxial states of stress generally is based upon the existence of a flow rule for the uniaxial state of stress (for tension specimens). Often, the multiaxial flow rule is obtained from the flow rule for tension specimens by replacing the tensile true stress $\sigma$ by the effective true stress $\sigma_e$ (see Eqs. 35 and 37) and the tensile true creep strain $\epsilon^c$ by the effective true creep strain $\epsilon_c$ which is given by replacing subscript and superscript $p$ in Eqs. (36) and (38) by $c$.

Since an equation of state does not truly exist for metals that creep, the flow rule is not known for the uniaxial state of stress. However, as discussed in Article 2, a number of approximate flow rules have been proposed. All of them are based on a family of constant stress creep curves obtained from tension specimens tested at the temperature of interest. If a temperature gradient is to be included in the analysis, a family of constant stress creep curves is obtained at each of two or more temperatures in the range of interest. The family of constant stress creep curves are incorporated into the multiaxial flow rule in a number of different ways.

Multiaxial Flow Rule for Steady State Creep. Consider a structural member subjected to constant loads at constant temperature for a long period of time. If the deformations are sufficiently small to preclude tertiary creep and geometry changes are negligible, the deformation-time diagram for the member corresponds approximately to the solid curve in Fig. 9. The instantaneous deformation OA may be entirely elastic or may be partly elastic and partly plastic. The primary creep range AB is followed by the secondary creep range BC. The designer may estimate that a sufficiently accurate approximation of the steady-state creep can be obtained by replacing the curve ABC by the dashed straight line OS. In derivations of a steady-state creep theory based upon the straight line OS approximation of Fig. 9, the temperature and the stress components of each volume element are usually assumed to remain constant with time. Since the primary creep effect is neglected, the flow rule is given by a relation that approximates
the creep rate for steady state conditions. Since for constant stress and constant temperature conditions, the creep rate is a function of stress only, the flow rule takes the functional form (see also Eq. 14 and discussion of Appendix A).

$$\dot{\varepsilon}_c = F(\sigma_e)$$

(42)

Several relations have been proposed for the function $F$ (see Eqs. A-19 through A-25). A widely-used formula proposed by Bailey $^{[23]}$ is (see also Eq. 1).

$$\dot{\varepsilon}_c = B \sigma_e^n$$

(43)

Multiaxial creep theories based on Eq. (43) are simple to derive and to use.

Unfortunately creep theories based on Eq. (43) have several limitations. For example, predicted inelastic deformations based upon Eq. (43) are generally nonconservative, since the effects of primary creep are underestimated. The effect of primary creep may be small if the design time is long, but it may be excessive for short design times. Creep theories based on Eq. (43) are not usually used in problems where the member is subjected to either variable load history, variable temperature history, or nonuniform temperature distribution.

For members subjected to constant loads at constant temperature, the nonconservative nature of theories based on Eq. (43) can be overcome by a procedure proposed by Odqvist $^{[4]}$ and used by Marin and Pao $^{[24]}$. The procedure may be explained as follows. Construct tangents to each of the constant stress creep curves in the steady state regions of the curves. The intercept $\varepsilon_0$ (see Fig. 4 and discussion of Eq. 16) of the tangent for a given creep curve with the strain axis determines a hypothetical strain for the given stress which approximates the effects upon the steady state of an elastic part $\varepsilon^e = \sigma/E$, and an inelastic part: a plastic part $\varepsilon^P$, and a primary stage creep $\varepsilon_{IC}$. The inelastic part when plotted versus stress gives a curve similar to the curve in Fig. 6. Assuming that the inelastic part is plastic, a plasticity theory for the member of Fig. 9 may be used to predict OD at zero time. Then, a creep
model based upon this concept and Eq. (43) predicts a deformation given by straight line DC in Fig. 9. In principle, the resulting theory is conservative.

**Multiaxial Flow Rule for Nonsteady Creep.** Multiaxial creep models have been postulated to predict loads or deformations of structural members subjected to prescribed load or deformation histories and subjected to prescribed temperature history including a temperature gradient. Ordinarily, the flow rule for these problems is based upon an equation which approximates the family of constant stress creep curves if only one temperature is considered or approximates families of constant stress creep curves, which cover the temperature range of interest. As a generalization of Eq. (33), the equation takes the form (see also Eqs. A-26 through A-32).

\[ \epsilon_c = F(\sigma_e) G(T_a) \phi(t) \]  

(44)

where \( F(\sigma_e) \), and \( G(T_a) \), and \( \phi(t) \) are functions of effective stress \( \sigma_e \), absolute temperature \( T_a \), and time \( t \), respectively (see Appendix A). A large number of investigations have employed the form (Eq. A-28).

\[ \epsilon_c = B \sigma_e^n (e^{-A/T_a}) t^k \]  

(45)

where \( B, n, A, \) and \( k \) are material constants. However, frequently, better accuracy is obtained by replacing \( \sigma_e^n \) by \( (\sinh b \sigma_e)^n \), where \( b \) is an additional material constant (Eq. A-30).

Forms, such as Eq. (45), are ordinarily postulated for constant stress and constant temperature conditions. The effect of varying either stress or temperature on the effective strain rate \( \dot{\epsilon}_c \) is included in the creep model by the introduction of a hardening rule. Two hardening rules that are commonly used are the time hardening rule and the strain hardening rule (Article 2). The time hardening rule postulates that the creep rate \( \dot{\epsilon}_c \) depends on stress, temperature, and time. For the time hardening rule, the creep rate is obtained by taking the time derivative of Eq. (44). For the specific form of Eq. (45), we obtain
\[ \dot{\epsilon}_c = k B \sigma_e^n e^{-A/T_a} t^{k-1} \]  

(46)

The strain hardening rule states that \( \dot{\epsilon}_c \) depends upon stress, temperature, and strain. The strain hardening form for Eq. (45) is obtained by eliminating time between Eqs. (45) and (46). Thus,

\[ \dot{\epsilon}_c = k ( B \sigma_e^n e^{-A/T_a} )^{1/k} \epsilon_c^{(k-1)/k} \]  

(47)

A comparison of the prediction of creep based upon Eqs. (46) and (47) is given below.

**Multiaxial Flow Rule for Treating Creep Problems as an Equivalent Plasticity Problem.** Creep models based upon either Eqs. (46) or (47) incorporate the stress history of every volume element of a structural member. These creep models predict that redistribution of stress with time occurs very rapidly after initial loading so that after a short time the stress distributions remain constant for constant load and constant temperature. Accordingly, creep properties for these members are accurately approximated by a family of constant stress creep curves. These data can be approximated also by a family of isochronous stress-strain diagrams each diagram of which gives the relation between stress and strain for a specified time. For a given isochronous stress-strain diagram (i.e., for a given time; see Fig. 2) the plastic strain can be plotted versus stress; the resulting diagram determines the flow rule given by Eq. (34). Plasticity theory is then used to predict the deformations of the member for the specified time. This technique can be repeated for other times (other isochronous stress-strain curves Fig. 2).

Creep models based on isochronous stress-strain diagrams are applicable to members subjected to load and temperature histories \([25]\). However, it is not advisable to use these kinds of theories if the member has a temperature gradient or if the member is subjected to deformation histories. The accuracy of isochronous stress-strain
creep models is comparable to those models based upon Eq. (47). A comparison of
the predictions of various creep models is presented below.

Comparison of Multiaxial Flow Rules by the Method of Successive Elastic
Solutions. To compare creep flow rules for multiaxial states of stress, it is desirable
to develop the creep model for the particular structural member being considered. The
method of successive elastic solutions is applicable to general states of stress and mate-
rial behavior. Hence, in the following comparisons, this method is used. It has been
employed extensively by Mendelson [26] and his co-workers [27]. Mendelson attributes
the method to Ince [28] and Ilyushin [29]. The method is particularly applicable to
problems for which elasticity solutions can be obtained readily, either in closed form
or in approximate numerical form. The method of successive elastic solutions as applied
to creep is essentially one of seeking solutions to a system of nonlinear equations by suc-
cessive approximations, the initial approximation being the solution of the elasticity
problem. The nonlinear effects are introduced through the stress-strain relations,
where now strain is no longer proportional to stress as in the linearly elastic problem,
but contains plastic-creep effects that are nonlinearly related to stress. In the general
procedure of solving nonlinear equations by the method of successive approximations,
the nonlinear terms are initially assumed (often set to zero), then the remaining system
of linear equations is solved. The solution so obtained is then used to modify the initial
assumptions for the nonlinear terms and the resulting linear equations are resolved.
The procedure is repeated until no further effects on the solution is observed, i.e., the
method converges.

With reference to the multiaxial creep problem, the method of successive elas-
tic solutions may be described in an equivalent manner. For the purpose here, a brief
description of the technique is given for an incremental creep calculation. For a detailed
description of the method see [26]. However, the procedure is perfectly general and
may be applied equally well to a deformational approach to creep [26]. Thus, the
strain-stress-temperature relationships for the inelastic creep problem are written in the form

\[
\begin{align*}
\varepsilon_x &= \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} + aT + \varepsilon_x^p + d\varepsilon_x^c \\
\varepsilon_y &= \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E} + aT + \varepsilon_y^p + d\varepsilon_y^c \\
\varepsilon_z &= \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + aT + \varepsilon_z^p + d\varepsilon_z^c \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} + \gamma_{xy}^p + d\gamma_{xy}^c \\
\gamma_{yz} &= \frac{\tau_{yz}}{G} + \gamma_{yz}^p + d\gamma_{yz}^c \\
\gamma_{zx} &= \frac{\tau_{zx}}{G} + \gamma_{zx}^p + d\gamma_{zx}^c
\end{align*}
\]

(48)

where the following equivalent notations are used: stress components \(\sigma_{ij} = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})\), total strain components \(\varepsilon_{ij} = (\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})\), plastic strain components \(\varepsilon_{ij}^p = (\varepsilon_{ij}^p, \varepsilon_{ij}^p, \varepsilon_{ij}^p, \gamma_{ij}^p, \gamma_{ij}^p, \gamma_{ij}^p)\), associated plastic strain differentials \(d\varepsilon_{ij}^p\), creep strain components \(\varepsilon_{ij}^c\), and associated creep strain differentials \(d\varepsilon_{ij}^c\), \(i, j = 1, 2, 3\).

Although the method of successive elastic solutions can be applied to a structural member subjected to a known temperature history, to a known load history, and to a known deformation history, for purposes of explanation, the simpler problem of constant load for a given time period is considered. However, the load may be changed instantaneously from one value to another, after which instant it again remains constant. For simplicity, all plastic strains due to the load are assumed to have occurred before creep begins. Furthermore, the initial load or changes in load are applied in increments so that an incremental plasticity theory can be used to compute plastic strains. Also, the differential strains \(d\varepsilon_{ij}^p, d\varepsilon_{ij}^c\) in Eq. (48) are approximated by the incremental quantities
\( \Delta \epsilon^P_{ij} \), \( \Delta \epsilon^c_{ij} \) respectively. In the computation of the plastic strains due to initial loading or due to load changes, the creep strains \( \epsilon^c_{ij} \) and the incremental creep strains \( \Delta \epsilon^c_{ij} \) are set equal to zero. In turn, under conditions of periods of constant load, the plastic strain components \( \epsilon^P_{ij} \) are assumed to remain constant (\( \Delta \epsilon^P_{ij} = 0 \)), additional plastic strains due to stress redistribution during creep at constant load being assumed small. Generally speaking, in many engineering problems, the effects of redistribution of stress with time are small. Furthermore, the effects on the total load carrying capacity of redistribution of stress in regions of higher stress level tend to be compensated for by the effects in regions of lower stress levels. Accordingly, the method proceeds as follows: The region occupied by the structural member is divided into a finite number of volume elements, if possible, each element being selected to admit an approximation of constant stress. For example, in torsion of a circular cross section bar, the elements are selected as concentric rings with respect to the axis of the bar. To compute the plastic strain due to initial load, \( \epsilon^c_{ij} \) and \( \Delta \epsilon^c_{ij} \) are set equal to zero in Eqs. (48). Next an increment \( P_1 \) of load \( P \) is assumed to act, and nonzero initial guesses are made for \( \Delta \epsilon^P_{ij} \sim \Delta \epsilon^P_{ij} \), with \( \epsilon^P_{ij} = 0 \). Hence, Eqs. (48) become linear strain-stress relations. Accordingly, with these relations, the solution of the stress components \( \sigma_{ij} \) may be obtained by elasticity theory. In addition, the corresponding increment \( \Delta \epsilon^P \) of the effective plastic strain \( \epsilon^P \) (initially zero) is obtained for each volume element by replacing \( \epsilon^P_{ij} \) (i.e., \( \epsilon^P_{x}, \epsilon^P_{y}, \epsilon^P_{z}, \gamma^P_{xy}, \gamma^P_{xz}, \gamma^P_{yz} \)) by the assumed values of \( \Delta \epsilon^P_{ij} \) in Eq. (36) for a von Mises material (or in Eq. (38) for a Tresca material). Then, by the flow rule (Eq. 34), the effective stress \( \sigma^e \) for each volume element is calculated. This completes one step of the calculation. To obtain new (better) estimates of the increments \( \Delta \epsilon^P_{ij} \), incremental plasticity theories ordinarily employ the Prandtl-Reuss stress-strain relations. Mendelson [26] has proposed a modification of the usual form of the Prandtl-Reuss relations which allows greater freedom in the choice of initial values of \( \Delta \epsilon^P_{ij} \), and furthermore, results in more rapid convergence.
of the method. Mendelson recommends the modification for all plasticity problems.

With the calculated values of $d\varepsilon_p^e$, $\sigma_e^e$, and the stress components $\sigma_{ij}$, the modified Prandtl-Reuss relations yield new estimates of $\Delta\varepsilon_{ij}^p$; the calculation is redone repeatedly until the new estimates of $\Delta\varepsilon_{ij}^p$ are same as the previous values (to within specified numerical tolerances). Then, the load is increased by another increment $P_2$ of $P$ and again values of $\Delta\varepsilon_{ij}^p$ are selected for the new level of load and the calculation is repeated. In Eq. (48), the plastic strain components $\varepsilon_{ij}^P$ of each volume element for each load level is taken as the sum of the incremental quantities $\Delta\varepsilon_{ij}^P$ for all previous load increments; that is $\varepsilon_{ij}^P = \sum_{k=1}^{P} \Delta\varepsilon_{ij}^P$. This process is continued for $n$ increments of load until the total load $P$ is reached. ($P = P_1 + P_2 + \cdots + P_n$).

The result of this calculation yields an estimate of the plastic strain components $\varepsilon_{ij}^p$ at time $t = 0$. As noted above, these components are assumed to remain constant under creep. However, if the loads on the member are changed (increased) at future times $t = t_1$, $t = t_2$, \ldots, changes (increases) in plastic strains must be computed for all volume elements for which $\sigma_e^e$ increases in magnitude, provided that the new magnitude exceeds the yield stress required by Eq. (34). Hence, in the numerical computation, it is necessary to retain the magnitude of the effective stress $\sigma_e^e$ for each volume element at times $t = 0$, $t_1$, $t_2$, \ldots.

With plastic strain components $\varepsilon_{ij}^p$ and stress components $\sigma_{ij}$ determined for time $t = 0$, creep strain starting at $t = 0$ may be computed. On the basis of incremental creep theory, stress components $\sigma_{ij}$ (and hence effective stress $\sigma_e^e$) are assumed to remain essentially constant for sufficiently small time change $\Delta t_1$. The creep computation proceeds as follows: Assume that the stress components $\sigma_{ij}$ and the creep strain components $\varepsilon_{ij}^c$ are known at time $t$. To obtain the creep strain components $\varepsilon_{ij}^c$ at time $t + \Delta t_1$, the stress components $\sigma_{ij}$ and creep strain increments $d\varepsilon_{ij}^c (\approx \Delta\varepsilon_{ij}^c)$ must be known at time $t + \Delta t_1$. Since the effective stress $\sigma_e^e$ for each volume element is assumed approximately constant from time $t$ to time $t + \Delta t_1$, the increment in effective creep strain $d\varepsilon_c$ for each volume element is
\[ d\epsilon_c \simeq \Delta \epsilon_c = \dot{\epsilon}_c \Delta t_1 \]  \hspace{1cm} (49)

where \( \dot{\epsilon}_c \) is the creep rate (Eq. 46 for the time hardening rule and Eq. 47 for the strain hardening rule). Analogous to plastic theory, the components of creep strain increments \( \Delta \epsilon_{ij}^c \) for each volume element are assumed given by Prandtl-Reuss stress-strain relations as follows:

\[
\Delta \epsilon_x^c = \frac{\Delta \epsilon_c}{2\sigma_c} \left( 2\sigma_x - \sigma_y - \sigma_z \right)
\]
\[
\Delta \epsilon_y^c = \frac{\Delta \epsilon_c}{2\sigma_c} \left( 2\sigma_y - \sigma_x - \sigma_z \right)
\]
\[
\Delta \epsilon_z^c = \frac{\Delta \epsilon_c}{2\sigma_c} \left( 2\sigma_z - \sigma_x - \sigma_y \right) = -\Delta \epsilon_x^c - \Delta \epsilon_y^c
\]
\[
\Delta \gamma_{xy}^c = \frac{3\Delta \epsilon_c}{\sigma_c} \tau_{xy} \quad \Delta \gamma_{yz}^c = \frac{3\Delta \epsilon_c}{\sigma_c} \tau_{yz} \quad \Delta \gamma_{zx}^c = \frac{3\Delta \epsilon_c}{\sigma_c} \tau_{zx}
\]  \hspace{1cm} (50)

The third of Eqs. (50) includes the assumption of incompressibility of creep deformation.

With the estimated creep strain increments of Eq. (50), \( \Delta \epsilon_{ij}^c \simeq d\epsilon_{ij}^c \), the stress components \( \sigma_{ij} \) are computed for time \( t + \Delta t_1 \), using Eqs. (48). If new values of \( \sigma_{ij} \) (at \( t + \Delta t_1 \)) differ by more than a prescribed numerical tolerance from values of \( \sigma_{ij} \) at \( t \), new estimates of \( \Delta \epsilon_{ij}^c \) are computed by Eqs. (50) with the new values \( \sigma_{ij} \). This process is repeated until convergence of \( \sigma_{ij} \) is obtained. Then, the creep strain components \( \epsilon_{ij}^c \) at time \( t \) are augmented by the creep strain increments \( \Delta \epsilon_{ij}^c \) that occur during time \( \Delta t_1 \) (Eq. 50) to obtain values of \( \epsilon_{ij}^c \) for time \( t + \Delta t_2 \). The numerical calculation is repeated for \( n \) time increments \( \Delta t_2, \Delta t_3, \ldots, \Delta t_n \) until the computation is achieved for the design time \( t_d = t + \sum_{i=1}^{n} \Delta t_i \). In the numerical process, the time increments \( \Delta t_i \) may be increased generally with time.

Unfortunately, the method of successive elastic solutions may not converge. However, usually, convergence problems may be overcome by decreasing the size \( P_i \) of load
increments in time independent deformation problems and by decreasing the size $\Delta t_i$ of time increments in time dependent problems.

The choice of a multiaxial flow rule for time dependent inelastic deformations is not particularly critical for many creep and relaxation problems. The flow rule depends on a creep relation, to approximate sufficiently accurately either a family of constant stress creep curves for a specified temperature or a sufficiently large number of families of constant stress creep curves to cover the design temperature range, and on a hardening rule.

In many engineering applications structural members are designed for constant temperature. Often the deformations of such a member and its load carrying capacity are influenced mainly by the portion of the member subjected to higher stress levels. Then only a limited range (covering the higher stress levels) of the constant stress creep curves need be accurately approximated by the creep relation. Any one of several creep relations proposed in the literature may be used for these types of problems. Furthermore, the choice of hardening rule used in the analysis is not extremely critical since the redistribution of stress with time has little influence on the creep curves for the member if the loads and temperature remain constant.

If a structural member is subjected to variable loads at constant temperature, the choice of a creep relation becomes more important since then the creep relation must accurately approximate the constant stress creep curves over a greater range of stress. In addition, the choice of hardening rule becomes a sensitive matter particularly if loads are increasing with time.

The most severe test of applicability of a creep relation arises when both variable loads and variable temperature are considered. Then the creep relation must accurately approximate large number of families of constant stress creep curves over the design temperature range.
In a recent investigation [10] creep relations and hardening rules were evaluated for two metals for a temperature range of 500°F. One of the metals considered was annealed SAE 1035 steel for a temperature range from 900°F to 950°F. The steel was elastic at zero time. Three creep relations were considered for the SAE 1035 steel as follows:

\[ \varepsilon_c = 9.042 \times 10^{-15} e^{-\frac{4.768 \times 10^4}{T_a}} (\sigma_e)^{6.253} t^{0.6671} \]  \hspace{1cm} (51)

\[ \varepsilon_c = 4.664 \times 10^4 \left[ \sinh (3.620 \times 10^{-4} \sigma_e) \right] t^{0.6703} \]  \hspace{1cm} (52)

\[ \varepsilon_c = 1.680 \times 10^{11} e^{-\frac{4.708 \times 10^4}{T_a}} \left[ \sinh (7.808 \times 10^{-5} \sigma_e) \right] 4.051 t^{0.6697} \]  \hspace{1cm} (53)

where \( T_a \) is the absolute temperature in degrees Rankine. The experimental constants for each of the three relations were obtained using a generalized least-squares computer program. Equation (53) contains five experimental constants whereas the other two relations contain four. Hence, it was used to evaluate the experimental data. Good correlation between theory and experiment was found by means of Eq. (53) and the strain hardening rule. A comparison of the predictions given by Eqs. (51), (52), and (53) for stepped loading of both a torsion member and a closed ended cylinder is presented below.

A solid circular torsion member of diameter 0.499 in. and of SAE 1035 steel was subjected to stepped torsional loads of 220, 250, and 280 in. lb at a temperature of 925°F. Experimental data for the member are indicated by the open circles in Fig. 10. Three of the theoretical curves in Fig. 10 are based on Eq. (53); the solid, dotted, and dashed curves represent results for the von Mises strain hardening, the von Mises time hardening, and the Tresca strain hardening theories, respectively. In developing the theories, the circular cross section was divided into 20 concentric ring volume elements of constant thickness. Each theory followed the stress history of each volume element.
A comparison of theoretical predictions and experimental results shows that the material closely approximates a von Mises material. The strain hardening and time hardening theories are nearly identical for the first constant load period. However, the time hardening theory deviates considerably from experimental results after a stepped increase in load.

The von Mises strain hardening theory based on Eq. (51) is indicated by the dash-dot curve in Fig. 10. It predicts lower creep values than the theory based on Eq. (53). The von Mises strain hardening theory based on Eq. (52) is indicated by the dash-double-dot curve. It predicts larger creep values than those based on Eq. (53). The principal reason for the difference between the three theories lies in the fact that Eqs. (51), (52), and (53) predict constant stress creep curves that differ from each other particularly in the lower stress levels. At the end of the three constant torque time intervals, Fig. 10 (t = 22 hrs., 44 hrs., 66 hrs.), the maximum effective stress in the torsion member is approximately 12,500 psi, 14,000 psi, and 15,500 psi, respectively. All three creep relations give nearly identical constant stress creep curves at 925°F for stress levels in the neighborhood of 15,000 psi. However, at a stress level of 12,000 psi, the constant stress creep curve predicted by Eq. (51) gives creep strains 19 percent less than those predicted by Eq. (53); whereas, the constant stress creep curve predicted by Eq. (52) gives creep strains 15 percent greater than those predicted by Eq. (53). This trend is indicated in Fig. 10 for the first 22 hours. All three theories predicted nearly identical changes in creep strains during the third 22 hour period, in which the effective stress in the outer fibers is in the neighborhood of 15,000 psi.

The difference between the three theories in Fig. 10 is not particularly large. Although the creep shearing strain predicted by Eq. (52) at 22 hrs. is more than 45 percent greater than that predicted by Eq. (51), this difference corresponds to an equivalent change in torsional load of only a 7 percent. The differences at 66 hrs. between the three theories corresponds to less than a 2 percent equivalent change in torsional load.
Predicted creep curves for a closed ended thick walled cylinder having an outer radius twice the inner radius are shown in Fig. 11. The recorded strain is the circumferential strain for the outer radius of the cylinder. The cylinder is made of SAE 1035 steel. The cylinder was subjected to stepped internal pressures of 9, 290 psi, 8,170 psi, and 10,400 psi at 925°F. At 22 hours, the pressure was decreased in magnitude from 9290 psi to 8170 psi. The time hardening theory predicts greater deformations than the strain hardening theory after this decrease in load, which is opposite to the result obtained for an increase in load (Fig. 10). The reason for the excellent agreement for the von Mises strain hardening theories based on Eqs. (51), (52), and (53) is that the experimental constants for Eqs. (51) and (52) are adjusted so that all three creep relations give a good fit to the constant stress creep curves for the stress range of 10,000 to 15,000 psi. The maximum effective stress in the cylinder at 66 hours is 15,000 psi. If the experimental constants indicated in Eqs. (51) and (52) are used, the discrepancy between the three theories is slightly greater than that for the torsion member in Fig. 10.

Creep theory predictions based on isochronous stress-strain diagrams are also indicated in Fig. 10 and 11. Sufficient constant stress creep data was not available at 925°F to construct the needed isochronous stress-strain diagrams. Accordingly, the diagrams were constructed using Eq. (53) to predict the constant stress creep curves. The von Mises strain hardening creep theory based on isochronous stress-strain diagrams predicted creep deformations slightly greater than the creep theory based on Eq. (53) for the torsion member (Fig. 10), but predicted creep deformations almost identical with those predicted by the theory based on Eq. (53) for the closed ended thick walled cylinder (Fig. 11).
4. SUMMARY OF SELECTED MULTIAXIAL CREEP EXPERIMENTS

A large number of experimental creep investigations have been undertaken to evaluate multiaxial creep theories since Bailey's [23] early work in 1935. With few exceptions the multiaxial creep theories have been based on the assumption that the material was an isotropic hardening von Mises material or an isotropic hardening Tresca material. Generally the agreement between theory and experiment has not been as good for time dependent inelastic deformations as for time independent inelastic deformations. There are at least four reasons for the poor agreement between theory and experiment.

One reason for discrepancy between theory and experiment is that isotropic hardening theories have been applied to members made of anisotropic metals. To complicate matters further, poor agreement has been found [38] for members made of essentially isotropic materials.

A second reason for the discrepancy between theory and experiment for time dependent inelastic deformations is that the isotropic material may not obey the isotropic hardening model used to idealize the material. It is generally desirable that creep properties be obtained using both tension specimens and hollow torsion specimens to determine if the material obeys the isotropic hardening model. When the inelastic deformation is time independent, experimental data for members made of isotropic materials fall either on the von Mises theory, on the Tresca theory, or between the two theories, at least for monotonically increasing loads. When the inelastic deformation is time dependent, Figs. 7 and 8, experimental data may fall either appreciably above or below the band between von Mises and Tresca creep theories. No explanation has been advanced to explain these anomalies.

A third reason for the discrepancy between theory and experiment for time dependent inelastic deformations is experimental scatter. Because of the time required
to conduct a single creep test, generally sufficient test data are not obtained to determine the reliable average data needed to make a meaningful comparison between theory and experiment.

A fourth reason for the discrepancy between theory and experiment for time dependent inelastic deformation is possible error in the flow rule. The flow rule is never known for any metal but is generally approximated by a creep relation and a hardening rule. The shape of a constant stress creep curve may be affected strongly by time and temperature dependent metallurgical changes and may not be accurately approximated by the creep relation used. Furthermore, discrepancies may be due to an inappropriate hardening rule. The difference between theory and experiment due to an error in the flow rule is generally small compared to the other reasons noted above.

It should be pointed out that the same discrepancy between theory and experiment for time independent (plasticity) and time dependent (creep) inelastic deformations are generally reported so that the discrepancy for creep problems appears to be about one order of magnitude greater than the discrepancy for plasticity problems. An increase in plastic deformations for a member requires an increase in loads, therefore, theoretical and experimental curves are plotted as load versus deformation as indicated in Fig. 12. If the theoretical curve is 5 percent below the experimental curve as indicated in Fig. 12, the theory is said to be conservative by 5 percent. This discrepancy is based on load and not on deformation. On the other hand creep deformations can occur under conditions of constant loads. Theoretical and experimental creep curves are plotted as deformation versus time as indicated in Fig. 13 for a member subjected to constant loads. Suppose that the theoretical and experimental creep curves in Fig. 13 coincide if the loads used in the theoretical calculations are decreased by 5 percent. The theory is said to be conservative by 5 percent, as in Fig. 12. However, the
theoretical deformation at the maximum time shown in Fig. 13 is 48 percent greater than the experimental deformation. The discrepancy is usually reported as 48 percent rather than the 5 percent which is the more significant value. The ratio of the discrepancies based on deformation and on load depends upon the material; the ratio ranges from 3 to 10 for most metals.

Most of the early experimental multiaxial creep investigations consisted of tests on thin walled members subjected to either torsion, torsion-tension, internal pressure, or pressure-tension. In general, experimentally determined creep rates for steady state conditions were compared with rates predicted by methods using tension creep data and the von Mises theory.

Bailey [23] conducted tension, thin wall torsion-tension, and thin wall pressure-tension tests on members made of lead at room temperature, low carbon steel at 475°C, and SAE 1045 steel at 480°C. Marin [30] analyzed Bailey's data and other unpublished data using both Bailey's theory and the von Mises theory. He concluded that the two theories were nearly identical and were in good agreement with test data.

Norton [31, 32] conducted tension and thin wall closed ended cylinder tests of a carbon-molybdenum steel at 900°F and 1050°F for up to 2700 hours. Söderberg [33] compared the experimental steady state creep rates from Norton's tests with the von Mises theory. The data fell between the Tresca and von Mises theories.

A. E. Johnson and his colleagues [34-38] conducted tension and thin wall torsion and torsion-tension tests on members made of 0.17 percent carbon steel at 662°F, 842°F, and 1022°F, 0.20 percent carbon steel at 842°F, cast magnesium alloy at 68°F and 122°F, cast aluminum alloy (RR.59) at 302°F and 392°F, and nickel-chromium alloy (Nimonic 75) at 1022°F and 1202°F. Although much of the data exhibited appreciable scatter, the experimental steady state creep rates in most cases were in good agreement with that predicted by the von Mises theory. Some data was in better agreement with the Tresca theory.
Nishihara, Tanaka, and Shima [39] conducted torsion-tension tests on thin walled members made of 0.10 percent carbon steel at 450°C. Marin, Faupel, and Hu [40] conducted torsion-tension tests on thin walled members of aluminum alloy 25-0 at room temperature. Kennedy, Harms, and Douglas [41] conducted pressure-tension tests on thin walled cylinders made of Miconel at 1500°F. Oding and Tulakov [42] conducted torsion-tension tests on thin walled members made of an austenitic steel at 600°C for test durations up to 2000 hours. Namestnikov [43] conducted torsion-tension tests on thin walled members made of an austenitic steel at 600°C and 500°C. Much of the experimental data in these investigations were in good agreement with the von Mises theory.

Taira, Ohtani, and Ishisaka [44] conducted pressure-tension tests on thin walled members made of 0.14 percent carbon steel at 500°C. Tests were conducted for ratios of circumferential stress to axial stress of 0, 1/4, 1/2, 7/8, 1, 4/3, 8/5, and 2. Good agreement was obtained between a finite strain von Mises theory and test data up to the fracture strain. Some of the data fell between von Mises and Tresca theories. Ohtani [45] conducted similar tests for test members made of 18-8 Nb stainless steel at 650°C. The data fell between the von Mises and Tresca theories but nearer to the von Mises theory. This material did not exhibit tertiary creep. Because of the reduced ductility, the maximum principal stress theory more accurately predicted the fracture of these cylinders.

Other experimental investigations have considered the problem of creep of members having solid or thick sections for which stress distributions change with time as the result of time dependent inelastic deformations. Gubser, Sidebottom, and Shamammy [46] conducted short time stepped loading tests (105 minutes) on solid square and rectangular section torsion members made of untreated hot rolled SAE 1020 steel at 972°F. Excellent agreement was found between theory and experiment when material properties
were obtained using thin wall torsion specimens. The von Mises theory based on tension and compression data was conservative. Dharmarajan and Sidebottom [47] conducted short time constant load torsion-tension tests on solid circular members made of 17-7PH stainless steel at 972°F. Again the von Mises theory was found to be conservative. Solid circular torsion tests [5] on torsion members made of untreated hot rolled bars of mild steel and of SAE 1045 steel at 950°F gave conservative results when compared to a von Mises theory. The discrepancy between theory and experiment was less than 5 percent based on load. Data [5] for solid circular torsion members made of annealed OFHC copper and titanium alloy 6.4 at 800°F fell midway between the von Mises and Tresca theories; the Tresca theory was found to be nonconservative by about 15 percent based on load for the titanium alloy at 1000°F. Stepped loading tests [10] of solid circular torsion members made of annealed SAE 1035 steel at 925°F were in excellent agreement with a von Mises theory. Other tests [5, 10] on torsion members made of aluminum alloy 7075-T6 at 375°F and copper alloy 360 at 650°F gave the results shown in Figs. 7 and 8.

Chu and Sidebottom [22] conducted short time (60 minutes) nonproportionate loading tests on solid circular torsion-tension members made of either annealed SAE 1035 steel at 975°F or copper alloy 360 at 700°F. Good agreement was obtained between a von Mises strain hardening theory and test results. The nonproportionate loading was accomplished in most cases by keeping one load (either axial load or torsion) constant for 60 min. and applying the other load at 30 min. In a few tests the axial load was decreased somewhat when the torque was applied. Johnson, Henderson, and Mathur [38] conducted nonproportionate loading tests in which the loads were always increasing in magnitude on several metals. Their data was in fair agreement with a von Mises theory but the large scatter made interpretation difficult. In these investigations the anisotropy caused by creep for the first load did not have a measurable effect on creep deformations after the nonproportionate load change. Namestikov [48] conducted nonproportionate
load tests on thin walled torsion-tension members on an austenitic steel at 500$^\circ$C and 600$^\circ$C in which the member was allowed to creep under a tension load for 50 hr., the tension load was removed, and a torsion-tension load which produced the same effective stress was applied. The anisotropy caused by the first load had a pronounced effect on the creep deformations after the nonproportionate load change.

Ohnami and his colleagues [49, 50] conducted nonproportionate loading tests on thin walled test members made of an annealed 0.15 percent carbon steel at 450$^\circ$F. The test apparatus used was capable of a combination of axial load, internal pressure, and reversed torsion. A comparison of data from tension specimens and hollow torsion specimens indicated that the material was a von Mises material. In one test, the effective stress was held constant, but the test member was subjected to alternating axial and circumferential stresses for six steps. In other tests torsion-tension members were subjected to six steps of loading such that the principal axes of stress rotated through 180$^\circ$. A plot of effective strain versus time was in good agreement with the tension constant stress creep curve for those tests in which the effective stress was held constant. However, a plot of each strain component versus time was not in good agreement with the theory. The theory based on the time hardening rule gave the best correlation.

Wahl, Sankey, Manjoine, and Shoemaker [51] conducted tests of rotating disks made of 12 percent chrome steel at 1000$^\circ$F for up to 1000 hr. The Tresca theory was found to be at about the center of the scatter band. They concluded that the poor agreement with the von Mises theory was due to anisotropy of the material. Other rotating disk tests by Sakata and Seo [52] have indicated good agreement between a von Mises theory and test data; the latter conducted tests on disks made of 12 percent chrome steel at 550$^\circ$C.

Taira and his colleagues [53-55] have conducted closed ended cylinder tests on cylinders made of 0.19 percent carbon steel at 450$^\circ$C, 470$^\circ$C, and 500$^\circ$C and 2.25Cr-1Mo
steel at $550^\circ$C. Two carbon steel cylinders were tested with a temperature gradient with the outside temperature either $25^\circ$C or $20^\circ$C greater than that for the inner radius. These data are also analyzed by Ohtani \[45\]. Von Mises and Tresca theories were derived for the assumption that the axial component of creep strain remained zero. Test data fell between the two theories. The authors attributed the discrepancy between the von Mises theory and experimental data on anisotropy of the steel. Ohtani \[45\] also subjected closed ended cylinders made of 0.19 percent carbon steel at $500^\circ$C and 2.25 Cr-1Mo steel at $550^\circ$C to stepwise varying loads in which the pressure was applied for 24 hours and released for 24 hours repeatedly for up to six cycles. The cyclic loading increased the creep strain when plotted versus total time under load. The increase was small.

A large amount of experimental data has been obtained \[45, 56-59\] on thick walled cylinders subjected to a combination of axial load and internal pressure. The theories that have been proposed are not adequate to compare with the experimental results. The most comprehensive theory is that proposed by Ohtani \[45\].

Ohtani \[45\] has compared von Mises small deflection and large deflection theories with test data from thin circular plates with clamped edges of 0.17 percent carbon steel at $450^\circ$C. He obtained good correlation between experimental and theoretical creep curves.
5. SOME EFFECTS OF NUCLEAR RADIATION ON CREEP

A nuclear reactor is a strong source of high-energy radiation and heat. Whereas, high temperatures affect creep in a manner that may be treated reasonably accurately for structurally stable materials (Article 3), high-energy radiation can drastically alter physical, chemical, and mechanical properties of materials in rather unpredictable ways. In particular, it can affect strongly micromechanisms related to creep. In other words, radiation generally disrupts the internal structure of materials.

Radiation in reactors may generally be classified as either electromagnetic radiation or energetic particles. Both types of radiation can produce changes in solids, liquids, or gases. Energetic particles, for example, fission fragments and fast neutrons, are of importance in reactor structural materials. The effects of radiation on the properties of materials are due principally to defects produced by radiations. For example, neutron radiation may produce interstitials, phase changes, precipitation-hardening, order-disorder reactions, nucleation of helium bubbles at grain boundaries in certain metals, etc. Many changes in metals are due to a combination of such effects. Some of these effects produce lattice changes which tend to move a system toward thermodynamic equilibrium, while others tend to move it away from equilibrium. In general the net effect is difficult to predict.

Mechanical properties of structural materials can be altered considerably by irradiation. However, at moderate temperatures, generally the main changes are increased yield strength, tensile strength, and hardness, with a loss of ductility and toughness. In some metals these changes are similar to those produced by cold-working. In practice the effects of radiation on metals may be minimized by several methods. For example, as with cold-working, if excessive hardening occurs from radiation, an increase in temperature may reduce the hardening. On the other hand, if diffusion is important, as in metastable solid solutions, a decrease in temperature may insure
metastable response for a longer period of time by decreasing the mobility of radiation induced defects. If temperature changes are impractical from an engineering viewpoint, use of a different material may be necessary.

In general, isotropic materials with simple binding or at worse with a minimum of anisotropy, appear most radiation resistant. Materials possessing unstable phases at or near exposure temperatures should usually be avoided. Since the addition of small percentages of chemical elements or compounds may significantly reduce phase instabilities and anisotropy effects, many nuclear structural materials are alloys, metallurgically designed to resist grain growth, improve ductility, reduce detrimental surface effects, etc. In the phenomenological description of creep of metals in nuclear reactors, these effects are accounted for by empirically determined material parameters as in the case of thermal creep. However, since the creep process is significantly altered by the reactor environment, it is desirable to determine creep parameters under operating conditions.

Some studies have examined the effect on creep of reactor structural metals of various additives and treatments \[60-63\], environment and neutron dosage \[64-68\], in-reactor exposure \[69-74\], and surface effects \[75-77\]. Swindeman and Douglas \[60\] note that depending upon material and service conditions, improvements in strength of reactor materials after fabrication may be achieved by (1) annealing or aging, (2) carburization, and (3) environmental control during service. It is shown that carburization produces the greatest increase in creep strength. Oxidizing and nitriding also increase creep life. A material that has been aged gives a lower creep rate, but only up to a certain annealing temperature.

Creep tests were conducted on Iconel, packed carburization to approximately 1 percent by weight of carbon, at temperatures of 1300, 1500, 1650, and 1850°F. At 1300°F, carburization of Iconel resulted in a reduction of creep rate by a factor of 10 and an increase in rupture life by more than a factor of 3. Creep curves for course-
grained Iconel tested in nitrogen showed that the reaction of nitrogen on Iconel is to increase rupture life and to reduce the creep rate. Air also produced strengthening due to the formation of oxides on the surface. Surface effects are treated more explicitly in [75-77].

Magnesium-zirconium alloys (ZA) have been studied by several investigators [61, 62, 63, 69]. Walker and Fisher [61] show that the creep ductility of magnesium-zirconium alloy (Mg.-0.6 percent by weight of Zr) at 200°C is improved considerably by heat treatment in hydrogen at 550°C prior to working. This treatment produces a metallurgically stable alloy with at least as good creep resistance at 400°C as conventional ZA, with increased resistance to grain growth. However, similar treatment after working, results in course-grained alloys with poor creep ductility at 200°C. Creep behavior of ZA was studied by Kent and Wells [62] in tests of up to 5600 hours duration at 400°C and up to 12,600 hours duration at 450°C in an atmosphere of CO₂. ZA is stronger at 450°C than at 400°C in creep tests. In addition, further strengthening is obtained by prestraining at 250°C prior to creep testing. The increase of creep strength at 450°C and subsequent loss of ductility are attributed mainly to the precipitation of a zirconium-rich phase, tentatively identified as ε-zirconium hydride, which forms both intergranularly (as ribbons and thin hexagonal plates) and as intergranular particles. Similar information is obtained by Brown [63]. Again hardening occurs above 400°C. Here, the author attempts to determine whether reducing the zirconium content improves ductility at high temperatures while still retaining the desirable properties of ZA. In an attempt to explain the difference in properties at 400°C and 450°C, it is noted that the passage of hydrogen across the gas-metal interface is the important mechanism in the pick-up of hydrogen by magnesium-zirconium alloys. In addition, where moisture is the source of hydrogen, the rate of oxidation determines the rate of hydrogen formation. Thus, the differences in the behavior between specimens tested at 400°C and 450°C can probably
be explained by the differences in the rates of hydrogen formation possibly coupled with
the difference in the properties of oxide films formed at the two temperatures.

The effects of radiation on stress relaxation of springs made of Nimonic 80A is
studied by Taylor and Jeffs [65]. The springs were irradiated with fast neutrons under
constant strain conditions in the temperature range 325-525°C. Much larger stress re-
lexation occurs under radiation than under the equivalent thermal tests, thus indicating
a significant effect of radiation. The chief mechanism is believed to be one of dislocation
climb by the absorption of irradiation-induced vacancies. The irradiation may affect
creep or relaxation in two ways as follows: (a) by accelerating or retarding thermal
creep and (b) by producing additional creep independent of thermal creep. The creep
data agrees well with a modified form of the Nutting equation [99].

\[
\log \left( \frac{\sigma_0}{\sigma} \right) = t z(\phi, T) \exp \left( B - \frac{Q}{RT} \right) \tag{54}
\]

where \( z(\phi, T) \) is a function of neutron flux, temperature and possibly other variables.

The influence of boron content and carbon content and a combined mechanical-
heat-treatment on high temperature embrittlement of austenitic 16/13 Cr-Ni-steel after
neutron irradiation is investigated by Böhm, Hanck, and Hess [67] and the effect of liquid
sodium on creep and stress rupture of two austenitic steels at 700°C is examined by Böhm
and Schneider [68]. It was found that boron had little effect on ductility (in the concen-
tration range used), whereas increasing carbon content improved post radiation ductility.
Cold working prior to annealing also improved post irradiation ductility. The secondary
creep of austenitic steels was not affected by sodium. However, the tertiary creep stage
is shortened, resulting in a reduction of fracture strain. In NiCr 3020, the time to rup-
ture is decreased by the sodium environment.

A number of in-reactor creep tests have been conducted [69-74]. Ross-Ross
and Hunt [69] measured the inside diameter of reactor pressure tubes during reactor
shutdown. The measurements show a significant increase in creep rate which can be
directly related to the fast neutron flux. The diametral creep rate (per hour) of a cold-worked Zircaloy-2 pressure tube operating at 270°\textdegree C, 3 \times 10^{13} \text{n/cm}^2\text{-s} (> 1 \text{ Mev}), and 14,000 psi under a transverse to longitudinal stress ratio of 2:1 (closed end tube) is about 1.8 \times 10^{-7}, which is about ten times the creep rate for the tubes out of reactor. The effects of stress and temperature are not as well defined as the effect of flux. Results indicate that for tubes operating between 10,000 psi to 14,000 psi stress, the exponent n in the relation $\dot{\varepsilon}_c = B\sigma^n$ is closer to 1 than to 3 found in the in-reactor uniaxial tests. Results also indicate that the in-reactor creep rate under biaxial stressing is less temperature dependent than under uniaxial stressing. As a first approximation, the hoop creep rate $\dot{\varepsilon}_t$ (per hour) for cold-worked Zircaloy-2 in the temperature (T) range of 250°-300°\textdegree C, hoop stress $\sigma_t$ up to 20,000 psi, and fast neutron flux $\phi$ up to $3.5 \times 10^{13} \text{n/cm}^2\text{-s} (> 1 \text{ Mev})$ is

$$\dot{\varepsilon}_t = 4 \times 10^{-27} (T - 160°\textdegree C) \phi \sigma_t$$ (55)

There is no attempt to attach any significance to Eq. (55), with regard to the mechanism of creep. The creep of cold-worked Zirconium-2.5 percent Niobium has a rate one-third that of Zircaloy-2. Since Zirconium and its alloys are anisotropic, creep properties are different for specimens cut from tubes in the longitudinal and radial directions. However, the isotropic hardening rule was nevertheless employed, since no good estimate of the anisotropic behavior was available. Probably, this explains the poor agreement obtained between the biaxially stressed tube results and the uniaxial creep specimen results with regard to the effects of temperature.

A comprehensive treatment of the subject of radiation induced creep is presented by Piercy [70], and is worthy of detailed study. Piercy considers a number of different models in attempting to explain observed creep rates, including the effects of dislocation loops and their preferred orientation relative to the applied stress, mobile point defects (interstitials and vacancies), new slip dislocation caused by irradiation, and decreased
mobility of dislocations by irradiation. The predominant factor is judged to be dislocation climb caused by vacancies or by these point defects diffusing to the obstacles that impede the dislocation motion. If this diffusion reduces the interaction between the obstacle and the dislocation, then the dislocation will move more rapidly. Thus, all other mechanisms are eliminated from further consideration, except dislocation climb by supersaturation of vacancies, since they disagree with results. On this basis, Piercy explains creep rate at high stresses, high activation energy for in-reactor creep, and transient change in creep rate when a pre-crept sample is put in the reactor or when the reactor is shut down. Piercy also notes that Niobium and Molybdenum solute atoms stabilize the small regions of radiation damage in Zirconium. This effect should reduce the in-reactor creep rate since these damage regions are also sinks for point defects. As a caution, it is noted that different mechanisms can occur at the same time, and hence, should not be ruled out. For example, at high stress, dislocation-climb (a stress-dependent phenomenon) predominates, but at low stresses the mechanisms may be that of radiation growth.

In [71] Fidleris notes the possible existence of three mechanisms of creep, namely, irradiation growth, growth induced yielding, and dislocation climb. However, no single mechanism properly explains the results obtained. Whereas, some of the results suggest a linear dependence exists between in-reactor creep rate and flux intensity (Eq. 55), other results indicate a higher order dependence. Hesketh [72] cites tremendous uncertainties in post irradiation and pre-irradiation values of creep rates as handicaps in testing the hypothesis of yielding creep, that is, a creep rate component (say irradiation creep) that has no relation to thermal creep rate, but is additional to it, and is a direct consequence of growth of a ductile polycrystal. One direct measure of growth that has been made seems to confirm the yielding creep hypothesis. In [78]
Hesketh reexamines diffusion creep under neutron radiation and concludes, in contrast to [72], that steady-state creep is affected by simple diffusion of vacancies and interstitials and by the presence of fixed sinks between dislocations (e.g., voids). In [73], Kreyns and Burkart summarize the results of stress relaxation experiments conducted on Zircaloy-4 and Zr/2.5 percent by weight-Nb/0.5 percent by weight-Cu to exposures of $2 \times 10^{21} \text{nvt} (>1 \text{Mev})$. It was found, in part, that in-reactor relaxation of Zircaloy-4 was temperature sensitive (decreasing with lower radiation temperature) and structure sensitive (increasing with increased amounts of prior plastic deformation and for finer grain size), and consistent with predictions of an accelerated (radiation) thermal creep model.

No significant differences in creep rates of in-reactor and unirradiated conditions were noted at temperatures of $350^\circ$, $375^\circ$, and $400^\circ \text{C}$ by Gilbert [74]. However, in-reactor creep rates were faster than ordinary thermal creep rates at $300^\circ \text{C}$ and stresses of 25.6, 31.6, and 38.7 kg/mm$^2$ and at $320^\circ \text{C}$ and 38.7 kg/mm$^2$. As in [72], the creep rate is separated into thermally induced and irradiationally induced components. The faster creep rates during irradiation at $300^\circ \text{C}$ and $320^\circ \text{C}$ are tentatively related to growth of stress oriented dislocation loops limited by the generation rate of point defects, since little if any dependence on stress is noted. The author explains critical mechanisms under different conditions in terms of simple interesting ideas. Some additional noteworthy papers on the effects of radiation on creep are [79-82]. In particular, Nichols [81] reviews literature concerning creep of in-reactor Zircaloy. He proposes three creep models, each being predominant in a certain stress range. (a) At low stresses, in-pile softening is due to a combination of the growth-directed Roberts-Cottrell yielding creep, and the formation of point defect loops preferentially on certain planes in response to the applied stress, with the second process apparently more important. (b) At high stresses, in-pile hardening is due to cutting by dislocations of depleted zones.
(c) At intermediate stresses radiation-enhanced climb is rate-controlled. Böhm et al. [82] present a wealth of experimental data.

The effects of surface phenomena on creep of structure materials is relatively unexplored. Surface effects on creep of crystals have been studied more extensively [75-77].
6. CREEP RUPTURE

Failure due to rupture is always an important design consideration and has received considerably study in the creep literature. Most of the experimental investigations included in references \[31-59\] treat some aspects of creep rupture. Creep rupture studies are also included in several investigations \[60, 62, 64, 67, 68, 71\] which considered the effect of nuclear radiation on creep. Many of these investigations attempt to determine a creep-rupture criterion, which will predict time to rupture in members having multiaxial states of stress, using time to rupture data obtained from tension tests. Since most members are not subjected to either constant loads or constant temperature, a creep-damage rule is needed to assess the effect of varying stresses and temperature.

The extensive work of Johnson, Henderson, and Kahn \[7\] and of the Japanese investigators reviewed by Ohtani \[45\] indicate that the creep-rupture criterion depends upon material behavior. They conclude that the effective stress \(\sigma_e\) serves as a creep-rupture criterion for materials that show little cracking until final rupture is imminent, while the maximum principal stress \(\sigma_{\text{max}}\) serves as a creep-rupture criterion for materials that show appreciable cracking throughout the life of the member.

Two of the creep damage rules in use are the life fraction rule and the strain fraction rule. The former is expressed by the relation

\[
\sum_i \frac{t_i}{t_{ri}} = 1
\]  
(56)

where \(t_i\) is the period of time spent under a specified stress and temperature and \(t_{ri}\) is the rupture life corresponding to this stress and temperature. The strain fraction rule is expressed by the relation

\[
\sum_i \frac{\varepsilon_i}{\varepsilon_{ri}} = 1
\]  
(57)
where $\epsilon_i$ is the strain for a specified stress and temperature and $\epsilon_{ri}$ is the strain at rupture corresponding to this stress and temperature.

Johnson, Henderson, and Kahn [7] found that the life fraction rule fit test results if the material obeyed the effective stress creep rupture criterion and that the strain fraction rule fit test results if the material obeyed the maximum principal stress creep rupture criterion. Freeman and Voorhees [83] have made a literature survey of creep damage in metals. Abo El Ata and Finnie [84] have made a study of creep damage rules and have presented a creep damage rule that contains both the life fraction and strain fraction rules as special cases.
7. COMPUTER METHODS IN CREEP

Most unidimensional problems of creep are solved by iteration methods. In addition, multiaxial problems that are essentially one dimensional in concept such as torsion of circular bars, pressurization of circular tubes, etc., may be solved directly by iteration techniques \cite{10, 51, 52}. As noted in Article 3, the iteration process of successive elastic solutions is widely employed in numerical solution of creep problems. Two and three dimensional boundary value problems of creep are usually treated numerically by a combination of direct numerical methods, such as finite element methods \cite{85}, and iteration schemes, such as successive elastic solutions \cite{26}. The method of successive elastic solutions, used in conjunction with finite element methods, is readily adapted to anisotropic problems with little more difficulty than in the treatment of isotropic problems. For example, it may be adapted to treat creep of uniaxial metal matrix composites subjected to axial and normal lateral loads \cite{86}. Ford \cite{87} has employed the method to treat composites with nonaxially directed fiber reinforced matrix.

A number of finite element computer programs have been developed for general purpose treatment of creep problems for plane stress, plane strain, and axisymmetrical bodies of revolution \cite{88, 89}. The results of such programs ordinarily lead to voluminous data output, if a complete picture of the stress state or creep state is required for the member. Accordingly, results are usually obtained only for certain regions or points which are thought to be critical, e.g., regions that are highly stressed or strained \cite{88}. Indeed, one of the limitations of any numerical procedure (e.g., direct finite element methods) in multidimensional problems is the large amount of data output required for determination of stress or strain distributions, particularly for time dependent problems, such as creep, in which distributions change continuously with time. Great promise of overcoming this difficulty exists in a recent innovation \cite{87} which allows the presentation of the data in field form that gives in a single diagram a complete picture of the creep at each instant in time. This technique is discussed in more detail below.
In \[90\], the use of finite element methods in conjunction with the incremental procedure of \[90\] is outlined for creep laws of the form

\[
\Delta \varepsilon_c = F(\sigma_e, \varepsilon_c, T, t, \text{strain history})
\]  \hspace{1cm} (58)

where \(\Delta \varepsilon_c\) is the increment of effective creep strain, \(\varepsilon_c\) is the total effective creep strain, \(\sigma_e\) is the effective stress, \(T\) is the temperature, and \(t\) is the time. Results are presented for the creep behavior of a typical Fast Flux Test Facility hexagonal flow duct, a simply-supported beam, and a long thick-walled cylinder with open ends and subjected to internal pressure. The results for the cylinder agree well with those of \[90\]. As in \[88\], the amount of data that can be presented is severely restricted, with results for particular locations being cited as typical.

**Finite Element Method.** A large number of papers on finite element applications to boundary value problems of engineering is presented in \[91\]. However, little of this effort is devoted to the creep problem. Since the usual finite element method is well documented \[85, 91\], here we consider briefly the formulation of the plane stress creep problem. As noted, the problems of plane stress, plane strain, and axial symmetry have been presented in some detail in \[88, 89\].

For the triangular plane element with six nodal points (Fig. 14), the \((x, y)\) plane displacement functions \((u, v)\) are taken in the form

\[
u(x, y) = A_1 + A_2x + A_3y + A_4x^2 + A_5xy + A_6y^2
\]

\[
v(x, y) = A_7 + A_8x + A_9y + A_{10}x^2 + A_{11}xy + A_{12}y^2
\]  \hspace{1cm} (59)

where \((A_1, A_2, \ldots, A_{12})\) are constants. For small strains, the strain-displacement relations are \[92\]

\[
\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\]  \hspace{1cm} (60)
where the strain components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ consist of elastic components $\varepsilon_x^E, \varepsilon_y^E, \gamma_{xy}^E$ and inelastic (here creep) components $\varepsilon_x^c, \varepsilon_y^c, \gamma_{xy}^c$. Thus, we have

$$\begin{align*}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
&= 
\begin{bmatrix}
\varepsilon_x^E + \varepsilon_x^c \\
\varepsilon_y^E + \varepsilon_y^c \\
\gamma_{xy}^E + \gamma_{xy}^c
\end{bmatrix}
\end{align*}$$

(61)

By Eq. (61), $\varepsilon_x^E = \varepsilon_x - \varepsilon_x^c$, etc. Hence, by Hooke's law,

$$\begin{align*}
\{\sigma\} &= \begin{bmatrix} D \end{bmatrix} \{\varepsilon\} - 2G \{\varepsilon^c\}
\end{align*}$$

(62)

where $\{\sigma\} = \begin{bmatrix} \sigma_x, \sigma_y, \tau_{xy} \end{bmatrix}^T$ is the stress vector, $\{\varepsilon\} = \begin{bmatrix} \varepsilon_x, \varepsilon_y, \frac{1}{2} \gamma_{xy} \end{bmatrix}^T$ is the strain vector, and $\{\varepsilon^c\} = \begin{bmatrix} \varepsilon_x^c, \varepsilon_y^c, \frac{1}{2} \gamma_{xy}^c \end{bmatrix}^T$ is the creep strain vector, $G$ is the shear modulus, and $D$ is the plane stress elasticity matrix

$$\begin{bmatrix} D \end{bmatrix} = \frac{E}{1 - \nu^2} 
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1 - \nu^2 \\
1 & 0 & \nu \\
\end{bmatrix}$$

(63)

where $E, \nu$ denote Young's modulus and Poisson's ratio, respectively. In Eq. (62) we have employed the incompressibility condition $(\varepsilon_{ii}^c = 0)$.

To be consistent with Eqs. (59), the creep strains must vary linearly in $(x, y)$.

Thus,

$$\begin{align*}
\{\varepsilon^c\} &= \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} 1 \\
x \\
y \end{bmatrix}
\end{align*}$$

(64)

where the elements of matrix $[a]$ are defined when a set of creep strains is selected for the corners of the finite element $[93]$.

With Eqs. (59) and (60), the strains may be related to the nodal displacements for an element in the form
\[ \{ \epsilon \} = [B] \{ u \} \]  
\[ \text{(65)} \]

where \([B]\) is a known matrix and \(\{ u \}\) is the nodal displacement vector \([u_1, v_1, u_2, v_2, \ldots, u_6, v_6]^T\), Fig. (14). The equilibrium nodal displacements which result from a given set of initial creep strains and a set of known applied forces on the element are determined by the principle of virtual work. The result is

\[ \{ f \} = \int \limits_V [B]^T [D] [B] \; dV \{ u \} - 2G \int \limits_V [B]^T [a] \{ x \} \; dV \]  
\[ \text{(66)} \]

Integration over the element yields

\[ [k] \{ u \} = \{ f \} + \{ f^C \} \]  
\[ \text{(67)} \]

where \([k]\) is the element stiffness matrix, and the vector \(\{ f^C \}\) represents pseudo forces which when applied to the element give the creep strains selected for Eq. (64). The pseudo force vector \(\{ f^C \}\) is called the "creep forces" vector. By superposing element stiffnesses, we obtain the total stiffness matrix of the system \([85]\). The equations among all nodal displacements of the grid are expressed by the usual relation

\[ [K] \{ U \} = \{ F \} + \{ F^C \} \]  
\[ \text{(68)} \]

where the structural stiffness matrix \([K]\) is symmetric. The terms \(\{ F \}, \{ F^C \}\) are the total force and "creep force" vectors, respectively.

The creep force vector \(\{ F^C \}\) is obtained by appropriate substitution of the element creep force vector \(\{ f^C \}\). The element creep force vector is given by \([93; Branca]\)

\[ \{ f^C \} = 2G \left[ [M]^{-1} \right]^T \int \limits_V [B]^T [a] \{ \epsilon \} \; dV \]  
\[ \text{(69)} \]

where the matrix \([M]\) is given by the relation \(\{ u \} = [M] \{ A \}\), \(\{ u \} = [u_1, u_2, \ldots, u_6, v_1, v_2, \ldots, v_6]^T\), and \(\{ A \} = [A_1, A_2, \ldots, A_{12}]^T\). The matrix \([a]\) is obtained from Eqs. (59) and (60) with the relation \(\{ \epsilon \} = [a] \{ A \}\). Accordingly, the creep properties of the material are introduced into the finite element solution through the creep force vector \(\{ F^C \}\) and hence through \(\{ f^C \}\) of Eqs. (68) and (67). To predict the increments
of creep strain necessary to calculate \( R^c_j \), the theories of Article 3 may be employed. Thus, for example, we may write

\[
\Delta \epsilon_i^c - \frac{3}{2} \frac{\Delta \epsilon_c}{\sigma_e} S_{ij}
\]

(70)

where the effective creep strain increment \( \Delta \epsilon_c \) is defined by Eq. (41), the effective stress is given by Eq. (35), and \( \Delta \epsilon_i^c \) and \( S_{ij} \) are the tensor of creep strain increments and the stress deviator tensor (Eq. 50). In addition, a creep strain-stress-time relation is required (see Eqs. 51-53).

To develop the time dependent solution of creep, the method of successive elastic solutions may then be employed \([86-90, 93]\).

A Field Method of Data Presentation (Computer Holography). Ford \([87]\) has successfully employed the high speed digital computer in combination with a high speed printer to reproduce theoretical hologram patterns \([94]\) of displacement fields in structural members. This technique shows great promise as a practical method of presentation of the enormous data required to illustrate the numerical solution of problems of time dependent material behavior. The method also has potential as an aid to the experimenter with the problem of optimum strain gage placement and material selection \([87]\). Here, we briefly indicate the technique of producing a physical hologram and give a few results obtained by the computer. A comparison of a physical hologram and a theoretical hologram is given for the clamped cantilever beam.

The physical system used to produce a hologram is shown schematically in Fig. 15. A deformable body \( B \), initially undeformed, is subjected to an appropriate light source \( L \) (e.g., a helium neon gas laser). Light is reflected from a typical point \( P \) on the surface of \( B \) to a point \( H \) on the hologram plane (sensitive film). Similarly, light from all other points (e.g., \( P_1 \), Fig. 15) on the surface of \( B \) is reflected to every point (e.g., \( H \)) on the hologram plane. The body is then subjected to loads and deformed.
Hence, each point in \( B \) (e.g., points on the surface) undergo displacement components \((u, v, w)\) in the \((x, y, z)\) directions, respectively. Then the film is processed. Upon reconstruction the intensity of the virtual (or real) image is given by \( [\text{Ranson, 95}] \)

\[
I = 0.5 + 0.5 \cos \left( \frac{2\pi}{\lambda} \left[ \left( \ell_1 + n_1 \right) u + \left( \ell_2 + n_2 \right) v + \left( \ell_3 + n_3 \right) w \right] \right)^2
\]

\[
= 0.5 + 0.5 \cos \left( \frac{2\pi}{\lambda} \left( \mathbf{PL} + \mathbf{PH} \right) \cdot \mathbf{D} \right)
\]

(71)

where

\[
\mathbf{D} = \hat{i}u + \hat{j}v + \hat{k}w
\]

(72)

and \( \mathbf{PL} = (\ell_1, \ell_2, \ell_3) \) is the unit vector from any point \( P \) on \( B \) to the point light source \( L \), \( \mathbf{PH} = (n_1, n_2, n_3) \) is the unit vector from point \( P \) to the corresponding point \( H \) on the hologram plane, and \( \hat{i}, \hat{j}, \hat{k} \) are unit vectors associated with axes \((x, y, z)\), Fig. 15. The factor \( \lambda \) is the wave length of light used \((\lambda = 2.49 \times 10^{-5} \text{ in.})\). Thus, the light intensity for every point on the surface of \( B \) is recorded on the hologram plane. The intensity \( I \) (Eq. 71) varies from maximum values to minimum values, producing corresponding dark and light bands or fringes. (Actually, \( I \) varies continuously producing varying shades of white and black, but the resulting pattern appears as a group of dark and light bands or fringes.) The order of the fringe is related to the magnitude of displacement. Hence, a reference displacement (ordinarily zero) is required. As an example, for purposes of verification of the utility of the computer holography technique, Fig. 16 is an actual hologram of a clamped beam (3 in. long, 0.5 in. deep and 0.25 in. thick), subjected to end displacement of 0.0005 in. The physical parameters for the hologram are \( \ell_1 = \ell_3 = 0, \ell_2 = 1, n_1 = n_2 = -0.866, n_3 = 0.5 \) (Eq. 71). Hence,

\[
I = 0.5 + 0.5 \cos \left( \frac{2\pi}{\lambda} \left( -1.866 v + 0.5 w \right) \right)
\]

(73)

where axes \((x, y, z)\) are along the horizontal beam axis, in the vertical direction and perpendicular to the beam plane \((x, y, \text{plane})\), respectively. The outline of the beam is obtained in the deformed state, and the fringe pattern is due to the imposed end
displacement. At the clamped end, the displacement is zero. Hence, with zero displacement as reference (zero order fringe), the hologram shows four dark fringes above the neutral axis and four below. Along a given fringe (white or black), the displacement is constant. The higher the order of fringe, the larger the displacement. In Fig. 17, a theoretical hologram is shown, for the same problem solved by elasticity theory \[96\]. The fringe lines were obtained by substitution of the elasticity solution for the displacement into the equation for light intensity \( I \) (Eq. 73). To obtain a clear distinction between regions of high and low light intensity, a black or dark band (fringe) was printed for \( I < .35 \); thus, a white or light band corresponds to \( I \geq .35 \). The printing of the theoretical fringe pattern was done by means of a high speed printer (GOULD\( \text{®} \)). In the actual physical hologram (Fig. 16) there is a gray transition between white and black regions. Comparison of Figs. 16 and 17 show excellent agreement between the physical and theoretical holograms, except at the clamped boundary, since the theoretical elasticity solution admits zero displacement and zero slope only at the neutral axis, but not along the entire clamped edge. Hence, in the theoretical hologram, displacements (fringes) occur at the clamped boundary away from the neutral axis.

As an application of computer holography to creep, Ford \[87\] has solved an anisotropic plane stress problem of a flat bar consisting of a matrix with fiber reinforcement at an angle \( \theta \) to the direction of loading (Fig. 18) by the method of successive elastic solutions in conjunction with the method of finite elements. For \( L/h = 16 \), \( \theta = 45^0 \), and with material properties defined by the elastic and creep compliance matrices in units of \((\text{psi})^{-1}{\text{in.}}^2/\text{sec}\) \[97\],

\[
\begin{bmatrix}
  C_{11}^E & C_{12}^E & C_{13}^E \\
  C_{21}^E & C_{22}^E & C_{23}^E \\
  C_{31}^E & C_{32}^E & C_{33}^E
\end{bmatrix} = \begin{bmatrix}
  0.601 & -0.259 & -0.215 \\
  -0.259 & 0.001 & -0.215 \\
  -0.215 & -0.215 & 0.912
\end{bmatrix} \times 10^{-6} \tag{74}
\]

*Designed by Automation Technology Incorporated, 2 Henson Place, Champaign, Illinois, 61820, U.S.A.
\[
C_{ij}^c = \begin{bmatrix}
0.970 & -0.970 & 0 \\
-0.970 & 0.970 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \times 10^{-7}
\]

\[
\begin{bmatrix}
-0.001 & 0.421 & -0.420 \\
0.421 & -0.001 & -0.420 \\
-0.420 & -0.420 & 0.840 \\
\end{bmatrix} \times 10^{-7}
\]

where \( t \) is time in hours, the holograms of Figs. 19 and 20 were obtained for the \( y \)-displacement component \( v \), by taking \( \ell_1 = -\frac{1}{\sqrt{3}} \), \( \ell_2 = \ell_3 = n_1 = n_2 = n_3 = \frac{1}{\sqrt{3}} \).

Thus, Eq. (71) becomes

\[
I = 0.5 + 0.5 \cos \left( \frac{2\pi}{\lambda} \cdot \frac{2}{\sqrt{3}} \right) v
\]

\[
\lambda = 2.49 \times 10^{-5} \text{ in.}
\]

Only the holograms for the \( y \) displacement component are shown, since the \( x \) displacement is not particularly interesting. Again, white and black bands correspond to \( I \geq 0.35 \) and \( I < 0.35 \), respectively. In Fig. 19, the hologram corresponds to the elastic \( v \) displacement (time \( t = 0 \)). In Fig. 20, the \( v \) displacement is given after 8 minutes of creep, that is, for time \( t = 2/15 \) hr. To note the change in \( v \), the patterns for \( t = 0 \) and \( t = 2/15 \) hr are superposed in Fig. 21. For simplicity, only one half of the length of the specimen is shown in Figs. 19, 20 and 21, since the \( v \) displacement is antisymmetric with respect to \( x = L/2 \). A difference between the two holograms of about 1 fringe order exists. Since the stress level is low (\( \sigma_x \) maximum is less than 500 psi), it is apparent that the theoretical hologram is a very sensitive method of detecting changes in displacement, with the parameters employed one fringe order being equivalent to a change in \( v \) of \( 1.1 \times 10^{-5} \) in. In addition to its sensitivity, the method has the advantage of displaying in a single diagram the full field of displacement. In other words the method
is capable of documenting a tremendous amount of data output (finite element output) in a single picture. In addition, employing techniques developed by Holloway [98], it appears feasible to produce theoretical holograms of the stress field.
8. DISCUSSION

The phenomenological theory of creep of metals under multiaxial states of stress is patterned after the theory of plasticity. Both plasticity theories and creep theories assume that hydrostatic states of stress do not influence inelastic strains. Plasticity theories require a flow rule and a model of material behavior which specifies the modification of the yield surfaces during the course of plastic flow. The well known isotropic hardening model specifies that the cross section of the yield surfaces are expanding circles for von Mises theories and expanding hexagons for Tresca theories. The flow rule associated with the isotropic hardening model for plasticity theories is a plot of true stress versus true strain for a tension specimen of the material being considered.

Creep theories also require a creep flow rule and a creep model of material behavior. The concept of yield surfaces has no meaning for time dependent inelastic deformations. Hence, Drucker [21] suggests that we consider surfaces \( \phi = \text{constant} \) each of which represents a surface for which the rate of dissipation of energy is a constant. The isotropic hardening model for time dependent inelastic deformation specifies that the cross section of surfaces for \( \phi = \text{constant} \) are expanding circles for von Mises theories and expanding hexagons for Tresca theories. Other creep models of material behavior have been considered; however, most of the multiaxial creep theories in the literature are based on the isotropic hardening model. Creep models have been proposed for isotropic materials; the resulting theories generally reduce to von Mises theories when applied to isotropic materials. The creep flow rule is based on a phenomenological description of creep under uniaxial tension, the basis of which is the constant stress creep curve. This curve is characterized by three stages (primary, secondary, and tertiary) although only the primary and secondary stages are considered in most cases.

The creep flow rule for creep theories based on the isotropic hardening model is expressed in two parts. The first part is a creep relation which approximates a
family of constant stress creep curves if only one temperature is being considered or represents families of constant stress creep curves over the temperature range being considered. The creep relation is a strain-stress-temperature-time relation, which in a certain sense attempts to incorporate the effects of various metallurgical creep mechanisms (Article 2 and Appendix A). The second part of the creep flow rule is a hardening rule (usually either the time hardening rule or the strain hardening rule) to specify the effect of a stepped change in stress and temperature on the creep rate.

A comparison of multiaxial creep theories with experimental data indicates good correlation for some metals and poor correlation for others. Whereas experimental data for members made of isotropic metals fall between the predictions of von Mises and of Tresca theories for time independent plasticity, elevated temperature creep data for members made of the same metals may fall far outside of the band between the von Mises and Tresca theories (see Figs. 7 and 8). Most metals are well behaved (structurally stable) and show good correlation between theory and experiment. At present the causes of anomalous behavior for some metals are not known. Most steels are well behaved; the von Mises theory for members made of steels at elevated temperatures is found to range in general from about 5 percent conservative to 5 percent non-conservative when the discrepancy between theory and experiment is based on load and not deformation.

Before the development of high speed computers most experimental multiaxial creep investigations were concerned with comparing experimental and theoretical creep rates for members subjected to constant load and constant temperature. Only the steady state creep rates from constant stress creep curves were used in developing the theories. Today creep theories can predict behavior for specified load, temperature, or deformation histories. These theories require that the creep relation accurately approximate each constant stress constant temperature creep curve in the range of stress and temperature considered. Many creep relations have been proposed.
Any of the relations can be used if they accurately approximate the constant stress creep curves with reasonable accuracy over the stress and temperature range of interest.

Either the time hardening rule or the strain hardening rule is used in conjunction with the creep relation to develop creep theories that follow changes of stress distribution and temperature distribution with time. The difference between the theories based on the two hardening rules becomes significant only when the member is subjected to variable loads or variable temperatures. Either hardening rule can be used when either loads or temperatures decrease with time; the time hardening theories are more conservative in this case. Only the strain hardening theories should be considered for members subjected to increasing loads or temperatures since the time hardening theories may exhibit appreciable error and the error is nonconservative.

Creep theories based on isochronous stress-strain diagrams can be applied to members subjected to known load histories or to known temperature histories (without temperature gradient). The theories predict deformations at the time specified for the isochronous stress-strain diagram. The accuracy of isochronous creep theories is comparable with that of other creep theories for small deformations.

The effects of radiation upon the stress-strain-temperature-time relation for reactor structural materials have been studied by a number of investigators [60-74; see also Additional Selected Reference, Part b]. Most of these studies have been concerned with the effect of radiation on various creep mechanisms, and the corresponding effects upon the stress-strain-temperature-time creep relation. A number of creep relations which incorporate the effects of radiation flux have been proposed. Some of these models predict reasonably accurately the creep behavior of certain reactor structural materials (e.g., zirconium-base alloys) [80]. Generally, these relations are similar in form to those of Appendix A for the thermal creep, with additive effects due to neutron flux. Hence, radiation effects that are described by such forms may be incorporated directly into the numerical treatment of creep problems (Article 7).
The field method (computer holography) of numerical data presentation is a feasible way of displaying the voluminous data obtainable from finite element methods of creep analysis. Computer developed holograms of displacement and stress fields are presently feasible. Computer holograms of other field quantities are readily conceivable.

ACKNOWLEDGMENT

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APPENDIX A

UNIDIMENSIONAL CREEP FORMULAS FOR METALS

We define the term creep as the time-dependent inelastic deformation of materials. For one-dimensional states, the classical creep curve is a plot of creep strain versus time under constant tension stress $\sigma$ and temperature $T$. The creep curve for metals usually exhibits three regions in which the creep deformation takes on different character (curve $C_1$, Figure A-1). With reference to curve $C_1$, section OA represents the instantaneous deformation that occurs immediately after the load is applied. This strain is denoted by $\varepsilon_0 (\sigma, T)$. Depending on the stress level $\sigma$ and temperature $T$, $\varepsilon_0 (\sigma, T)$ may contain both elastic and inelastic parts. The section AB represents a portion of creep deformation in which the creep is changing (transient) at a decreasing rate. This region of the creep curve is called the primary creep stage. The section BC represents the second stage of creep in which the creep rate reaches a minimum value and remains constant or steady (the steady-state or secondary stage creep). In this region, the creep time rate $\varepsilon_c = f (\sigma, T)$ is a function of stress level $\sigma$ and temperature $T$. The section CD represents the third stage of creep (tertiary creep stage) in which the creep rate increases rapidly and fracture soon occurs. Although the division of the creep curve into three sections is rather conventional for metals, depending upon the stress and temperature levels, a variety of creep curves is a consequence of the complexity of the physical-metallurgical processes involved in the creep of metals. Additionally, it is difficult to define the instantaneous deformation $\varepsilon_0$ precisely, since it depends on the method of applying load. Much of the published creep data ignore this fact, and simply present creep curves as plots of $\varepsilon_c$ versus time (Part ABCD, curve $C_1$, Fig. A-1).

Because of the extreme complexity of creep, its study has rested heavily upon empirical representation of creep curves. These representations generally attempt
to include the effects of temperature $T$, stress $\sigma$, and time $t$ on the creep strain $\epsilon_c$. Such representations are referred to as strain-stress-temperature-time relations. These equations have usually been developed in one of three ways; namely, Method (1) by deriving empirical formulas that agree with the experimental data [10], Method (2) by deriving equations based upon assumptions of the physical-metallurgical creep mechanisms [17] and checking the equations against experimental data, or Method (3) combinations of Methods (1) and (2) [99]. In these methods, attempts have been made to separate the various influences for each group. For example, in the first method one may represent any one of the three creep stages separately by empirical formulas or one may represent two stages, say the primary and secondary, or all three stages, by a single formula. Alternatively in Method (2), one may consider the effect of a particular creep producing mechanism, say dislocations, on the various stages of creep, and attempt to relate various parameters in the strain-stress-temperature-time relations to the properties of the dislocations. In the following, we list a number of formulas that have been proposed for representing creep curves. Many of these formulas are discussed in detail in reference works such as [2, 14, 99, 100, 101]. Following [99], we separate the equations into time, temperature, and stress dependent parts, and we indicate suggested combinations of time, stress, and temperature components. The time dependence formulas are usually of the form $\epsilon = \epsilon_0 + \epsilon_c$, where $\epsilon_c = \epsilon_{Ic} + \epsilon_{IIc} + \epsilon_{IIIc}$, where $\epsilon$ is the total strain (deformation), $\epsilon_c$ secondary stage creep (steady-state) and $\epsilon_{IIIc}$ tertiary stage creep, where the various parts are selected to fit creep data at constant stress and temperature. Often, since practical interest does not extend to the tertiary stage, $\epsilon_{IIIc}$ is not included in some of the formulas. Generally, $\epsilon_0$ is the initial strain, $\epsilon_{Ic}$ is a monotonically decreasing function of time, $\epsilon_{IIc}$ is a linear function of time and $\epsilon_{IIIc}$ is a monotonically increasing function of time.
The temperature dependency of creep is often related to thermodynamics and rate processes of solid state physics \[17\]. Hence, it is often of exponential form. Generally, experimental evidence indicates that the steady-state rate of creep increases more rapidly with temperature increase than does the transient rate.

Finally, since stress is a tensor quantity whereas temperature and time are scalar quantities, the introduction of stress dependency into the creep formulas is more difficult. In general, more than one function of stress has to be employed, if the creep behavior over a wide range of stress is to be fitted closely.

In the following tabulation \( \varepsilon_c \) denotes creep strain (deformation), \( \sigma \) stress, \( T \) temperature, \( t \) time, and \( a, b, c, \ldots, m, n, p, \ldots, A, B, C, \ldots, M, N, P, \ldots \) denote parameters that may be functions of \( \sigma, T, t \) or they may be constants. Time derivatives are denoted by dots over a symbol (e.g., \( \dot{\varepsilon}_c \)). The notation \( f(\ ) \) denotes a function of \( (\ ) \).

TIME DEPENDENCE

<table>
<thead>
<tr>
<th>EQUATION FORM</th>
<th>REFERENCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rational Functions</td>
<td></td>
</tr>
<tr>
<td>( \varepsilon_c = a t / (1 + b t) )</td>
<td>[102]</td>
</tr>
<tr>
<td>Logarithmic Functions</td>
<td></td>
</tr>
<tr>
<td>( \varepsilon = a + b \log t )</td>
<td>[103]</td>
</tr>
<tr>
<td>Exponential Functions</td>
<td></td>
</tr>
<tr>
<td>( \varepsilon_c = a + b t - c \exp (-dt) )</td>
<td>[104]</td>
</tr>
<tr>
<td>( \varepsilon_c = a t + b \left[ 1 - \exp (-ct) \right] )</td>
<td>[3]</td>
</tr>
<tr>
<td>( \varepsilon_c = a \left[ 1 - \exp (-bt) \right] + c \left[ 1 - \exp (-dt) \right] )</td>
<td>[105]</td>
</tr>
<tr>
<td>Power Functions</td>
<td></td>
</tr>
<tr>
<td>( \varepsilon_c = b t^n )</td>
<td>[106]</td>
</tr>
</tbody>
</table>
Power Series
\[ \epsilon_c = a t^m + b t^n + c t^p + \ldots \]  \[107\]  \[(A-7)\]

Combined Exponential-Power Functions
\[ \epsilon_c = a (1 + b t^{1/3}) \exp kt - a \]  \[1\]  \[(A-8)\]

Combined Logarithmic-Power Function
\[ \epsilon_c = a \log t + b t^n + c t \]  \[108\]  \[(A-9)\]

These equations are not generally applicable. Any one of them may, however, hold for a certain metal and for certain test conditions. However, many of these equations are not conveniently adapted to include effects of temperature and stress.

TEMPERATURE DEPENDENCE

**EQUATION FORM**

<table>
<thead>
<tr>
<th>Exponential Functions</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{\epsilon}_c = a \exp (-Q/RT) )</td>
<td>[109] ( (A-10))</td>
</tr>
<tr>
<td>( \epsilon_c = a \left[ t \exp (-Q/RT) \right] )</td>
<td>[17] ( (A-11))</td>
</tr>
<tr>
<td>( \dot{\epsilon}_c = a T \exp (-Q/RT) )</td>
<td>[110] ( (A-12))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rational Functions</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon_c = a T^{2/3} f(t) )</td>
<td>[111] ( (A-13))</td>
</tr>
<tr>
<td>( \epsilon_c = a T f(t) )</td>
<td>[112] ( (A-14))</td>
</tr>
<tr>
<td>( \epsilon_c = f \left( T (a + \log t) \right) )</td>
<td>[113] ( (A-15))</td>
</tr>
<tr>
<td>( \epsilon_c = f \left( \frac{T - a}{\log (t - b)} \right) )</td>
<td>[114] ( (A-16))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hyperbolic Exponential Functions</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon_c = a \exp (-Q/RT) \sinh \left( \frac{b}{RT} \right) )</td>
<td>[115] ( (A-17))</td>
</tr>
</tbody>
</table>
\[ \varepsilon_c = c f ( t ( T - T')^{-B} ) \]  
\[ \text{[107]} \]  
\[ (A-18) \]

**STRESS DEPENDENCE**

**EQUATION FORM**

**Exponential Function**

\[ \varepsilon_c = a f ( t ) \exp ( b \sigma ) \]  
\[ \text{[17]} \]  
\[ (A-19) \]

\[ \dot{\varepsilon}_c = a \exp ( b + c \sigma ) \]  
\[ [9, 116] \]  
\[ (A-20) \]

\[ \ddot{\varepsilon}_c = a \left[ \exp ( b \sigma ) - 1 \right] \]  
\[ [3] \]  
\[ (A-21) \]

**Power Functions**

\[ \varepsilon_c = a f ( t ) \sigma^b \]  
\[ [17, 23, 117] \]  
\[ (A-22) \]

**Hyperbolic Functions**

\[ \dot{\varepsilon}_c = a \sinh ( b \sigma ) \]  
\[ [8] \]  
\[ (A-23) \]

\[ \dot{\varepsilon}_c = a \sinh ( b \sigma / RT ) \]  
\[ [115] \]  
\[ (A-24) \]

**Other**

\[ \dot{\varepsilon}_c = a \sigma \exp ( f ( \sigma ) ) \]  
\[ [118] \]  
\[ (A-25) \]

By combining the time, temperature, stress representations, the entire functional behavior of creep may be approximated by a single equation. Equations of this type include the following:

**COMBINED TIME-TEMPERATURE-STRESS EFFECTS**

**EQUATION FORM**

\[ \varepsilon_c = a \left[ t \exp ( - Q / RT ) \right]^{1/3} \sinh ( b \sigma / RT ) \]  
\[ [1, 17, 115] \]  
\[ (A-26) \]
\[
\begin{align*}
\dot{\varepsilon}_c &= T \exp \left(- \frac{a}{T} - b + c\sigma \right) \\
\varepsilon_c &= a \exp \left(- \frac{A}{T}\right) \sigma^n t^k \\
\varepsilon_c &= a \exp \left(- \frac{A}{T}\right) \sinh \left(ab\sigma\right) t^k \\
\varepsilon_c &= a \exp \left(- \frac{A}{T}\right) \sinh \left(ab\sigma\right)^m t^k \\
\varepsilon_c &= a \exp \left(- \frac{A}{T}\right) \left(\frac{\sigma}{D}\right)^c + \left(\frac{\sigma}{d}\right)^e t \\
\varepsilon_c &= \sum_{i=1}^{n} C_i \sigma^a_i \phi^b_i
\end{align*}
\]

where \( \phi = t \left( T' - T \right)^{-A} \)

An extensive discussion of the application of Eq. (A-32) is given by Kennedy [99].

Kennedy also discusses at length strain-stress-temperature-time relations based upon quasi-empirical methods and upon physical-metallurgical (microstructure) observations. The distinguishing feature of microstructure formulations is that an attempt is made to relate the parameters in the creep relation to creep producing microstructure mechanisms such as grain boundary displacement, slip, sub-grain size, etc.

Since our chief concern here is with creep models that lend themselves to the solution of complex engineering problems, in the main body of this paper, we restrict ourselves to certain formulas that describe experimental results reasonably well, and to techniques that are best suited to treating engineering creep problems.
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5. O. M. Sidebottom, Unpublished: Creep and Relaxation Data for Solid Circular Torsion Members made of Mild Steel at 940°F, SAE 1045 Steel at 950°F, OFHC Copper at 800°F, Titanium Alloy 6.4 at 800°F and 1000°F, and 7075-T6 Aluminium Alloy at 375°F.


ADDITIONAL SELECTED REFERENCES

a. General


b. Radiation Effects in Creep


c. Creep Rupture


d. Computer Methods in Creep


Figure 1. Constant Stress Creep-Time Curves

Figure 2. Isochronous Stress-Strain Diagram
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    Ignoring OA
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\( t = 8 \) min. (Creep)
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