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SOME NEW BASIC RESULTS FOR SINGULARLY PERTURBED ORDINARY DIFFERENTIAL EQUATIONS*

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ABSTRACT

Asymptotic results are obtained for an initial-value problem for singularly perturbed systems. Existence of bounded solutions to singularly perturbed systems is deduced from the results of a previous paper [9]. These results significantly enlarge the class of limiting asymptotic solutions of singularly perturbed systems in-as-much as the limiting solutions satisfy equations more general than the classical reduced system. These results generalize those of Tikhonov [3] for the initial value problem, Flatto and Levinson [6] for the existence of periodic solutions and Hale and Seifert [7] for the existence of almost-periodic solutions.

1. Introduction

Systems of ordinary differential equations of the form

$$\frac{dx}{d\tau} = f(\tau, x, y, \varepsilon)$$

$$\varepsilon \frac{dy}{d\tau} = g(\tau, x, y, \varepsilon) \qquad (1.1\text{ a,b})$$

where x, y are real vectors or arbitrary but finite dimensionality and ε is a real, positive parameter, are said to be singularly perturbed* [1], [2]. If we formally put $\varepsilon = 0$ in (1.1), we obtain the lower-order system, the Reduced or Degenerate System:

$$\frac{dx}{d\tau} = f(\tau, x, y, 0)$$

$$0 = g(\tau, x, y, 0). \qquad (1.2\text{ a,b})$$

The standard results of singular perturbations theory are concerned with establishing the relation between solutions of (1.1) and (1.2). For the initial valve problem, the classical theorem of Tikhonov [3] established the convergence as $\varepsilon \to 0$, of solutions of (1.1) to those of (1.2) over finite intervals in τ. This result was used to develop asymptotic expansion in powers of ε by Vasileva [4], and extended to infinite intervals by Hoppensteadt [5]. The existence of periodic and

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*More general singularly perturbed systems have been considered in the literature, but this paper is restricted to systems of the form (1.1).

almost periodic solutions to (1. 1) when (1. 2) possesses such solutions has been established by Flatto and Levinson [6] and Hale and Seifert [7] respectively.

In all of this work it becomes necessary to consider a related equation, the Boundary layer Equation:

$$\frac{dz}{dt} = g(\tau, x, z, 0) \qquad (1.3)$$

in which τ and x are treated as parameters. If the equation (1. 2b) is solved for y to obtain $y = \phi(\tau, x)$ we see that $z = \phi(\tau, x)$ defines a constant solution of (1. 3) in terms of parameters τ and x and it is customary to make assumptions about this solution and its behavior when the parameters τ and x are varied.

The results mentioned above may be made more clear by means of the following heuristic explanation. Equation (1. 1) shows that the y-vector varies much more rapidly than the x-vector. The variation of y is approximated by the boundary layer equation (1. 3). If the solution $\phi(\tau, x)$ of this equation is assumed asymptotically stable, it follows that nearby solutions of (1. 3) approach ϕ as $t \to \infty$. Therefore, under suitable conditions, for small ε the behavior of x may be obtained by replacing y by ϕ in (1. 2a). The assumptions on the stability of ϕ and on its dependence on τ and x are crucial in obtaining these results.

The primary result of this paper is to show that it is possible for solutions of (1. 1) to approach asymptotic limits different from the solutions of (1. 2). This happens when (1. 3) possesses a nonconstant solution $\phi(t, \tau, x)$ that is bounded in a sense to be made precise later.

In this case we show that the equation satisfied by the limiting solution is obtained by substituting $\phi(t, \tau, x)$ for y in equation (1. 1a) and then averaging over t. The precise definition of this average is given in section 2. In addition to assumptions of a stability nature, it turns out to be necessary to make some crucial assumptions about the average of f.

In case ϕ is a constant solution of (1. 3), the averaging process becomes trivial and we are left with the equation (1. 2a) for x in the limit. We thus recover earlier results as special cases of our result. Pontryagin [8] has obtained a result for the initial value problem following somewhat similar analysis for the more restricted case where f and g in (1. 1) are independent of τ and ϕ is periodic in t.

In section 4 we show that an earlier result [9] leads to the existence of bounded solutions to (1. 1). This represents a generalization of [7] in the same sense as above, i. e. the asymptotic limits of the bounded solutions of (1. 1) belong to a larger class than in [7] and the equations they satisfy are obtained by an averaging procedure.

2. Formulation of the Initial Value Problem

Consider the singularly perturbed real differential system:

$$\frac{dx}{d\tau} = f(\tau, x, y, \varepsilon) \qquad x(0) = x_0 ,$$

$$\varepsilon \frac{dy}{d\tau} = g(\tau, x, y, \varepsilon) \qquad y(0) = y_0 , \tag{2.1}$$

in which x and f are n-vectors, y and g are m-vectors and ε

is a positive number. Furthermore, f is a once continuously differentiable, and g, a twice continuously differentiable mapping from S_1 to R^n and R^m respectively, S_1 being the set:

$$S_1 = \left\{ (\tau, x, y, \varepsilon) : 0 \leq \tau \leq L, \; x \in \mathcal{G}_1, \; y \in \mathcal{G}_2, \; 0 \leq \varepsilon \leq \varepsilon_o \right\}$$

where $\mathcal{G}_1 \times \mathcal{G}_2$ is a bounded domain in $R^n \times R^m$, containing the origin. Further restrictions are imposed on f and g through assumptions detailed below.

The equation

$$\frac{d\hat{y}}{dt} = g(\tau, x, \hat{y}, 0) \qquad (2.2)$$

in which τ and x are regarded as parameters, is called the <u>Boundary Layer Equation.</u>

<u>Hypothesis H1:</u>

Equation (2.2) has a solution $y = \phi(t, \tau, x)$ that is contained within $\mathcal{G}_2$ for $(t, \tau, x) \in S_2 = \left\{ t \geq 0, \; 0 \leq \tau \leq L, \; x \in \mathcal{G}_1 \right\}$. The partial derivatives $\frac{\partial \phi}{\partial \tau}$ and $\frac{\partial \phi}{\partial x}$ are also bounded on S_2.

<u>Hypothesis H2:</u>

The limits

$$f_o(\tau, x) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} f(\tau, x, \phi(s, \tau, x), 0)\, ds$$

$$\left(\frac{\partial f}{\partial \tau}\right)_o (\tau, x) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \frac{\partial f}{\partial \tau} (\tau, x, \phi(s, \tau, x), 0)\, ds$$

(2.3)

$$\left(\frac{\partial f}{\partial x}\right)_o (\tau, x) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} \frac{\partial f}{\partial x} (\tau, x, \phi(s, \tau, x), 0)\, ds$$

exist uniformly over $S_3 = \{(\tau, x): 0 \leq \tau \leq L, x \in \mathcal{G}_1\}$ and are independent of t. Thus

$$\left(\frac{\partial f}{\partial \tau}\right)_o = \frac{\partial f_o}{\partial \tau} \quad \left(\frac{\partial f}{\partial x}\right)_o = \frac{\partial f_o}{\partial x} .$$

We strengthen these hypotheses somewhat by assuming that there exists an $M > 0$ such that

$$\left| \int_t^{t+T} [f - f_o]\, ds \right| + \left| \int_t^{t+T} \left[\frac{\partial f}{\partial \tau} - \frac{\partial f_o}{\partial \tau} \right] ds \right| +$$

$$\left| \int_t^{t+T} \left[\frac{\partial f}{\partial x} - \frac{\partial f_o}{\partial x} \right] ds \right| \leq M \qquad (2.4)$$

for $(t, \tau, x) \in S_2$ and all $T \geq 0$.

The equation

$$\frac{d\xi}{d\tau} = f_o(\tau, \xi) \qquad (2.5)$$

is called the Averaged Equation. The general solution of (2.5) is denoted by $\zeta(\tau, x_o)$, satisfying $\zeta(0, x_o) = x_o$.

Hypothesis H3:

If $x_o \in \mathcal{G}_1$, the solution $\zeta(\tau, x_o)$ stays inside $\mathcal{G}_1$ for $0 \leq \tau \leq L$.

Define the matrix $C(\tau, \varepsilon)$ by

$$C(\tau, \varepsilon) = \frac{\partial g}{\partial y}(\tau, \zeta(\tau, x_o), \phi(\tau/\varepsilon, \tau, \zeta(\tau, x_o)), 0). \tag{2.6}$$

We note that C depends on x_o.

Hypothesis H4:

The linear equation

$$\frac{d\chi}{d\tau} = \varepsilon^{-1} C(\tau, \varepsilon)\chi \tag{2.7}$$

has a fundamental matrix $\Psi(\tau, \varepsilon)$ satisfying the inequality,

$$|\Psi(\tau, \varepsilon)\Psi^{-1}(s, \varepsilon)| \leq k_1 \exp[-\varepsilon^{-1}\alpha_1(\tau - s)] \quad \text{fo} \quad \tau \geq s, \tag{2.8}$$

where k_1 and α_1 are positive numbers independent of ε and x_o.

Then we have the following result:

Theorem 1:

Under the hypotheses H1 - H4, there exist positive numbers κ and ε^*, such that if

$$\left| y_o - \phi(0, 0, x_o) \right| \leq \kappa \tag{2.9}$$

and $0 < \varepsilon \leq \varepsilon^*$, the solution $x(\tau, \varepsilon)$, $y(\tau, \varepsilon)$ of the Initial Value Problem (2.1) exists for $0 \leq \tau \leq L$. Furthermore, we have:

$$\lim_{\varepsilon \to 0} \left| x(\tau, \varepsilon) - \xi(\tau, x_o) \right| = 0 \quad \lim_{\varepsilon \to 0} \left| y(\tau, \varepsilon) - \phi(\tau/\varepsilon, \tau, \xi(\tau, x_o)) \right| = 0 . \tag{2.10}$$

The first limit in (2.10) is uniform over $0 \leq \tau \leq L$ and the second, over $\tau_1 \leq \tau \leq L$ for any $\tau_1 > 0$.

Remark: If the solution $\phi(t, \tau, x)$ is independent of t, i.e., ϕ is a constant solution of the Boundary Layer Equation, a reference to (2.3) shows that

$$f_o(\tau, x) \equiv f(\tau, x, \phi(\tau, x), 0) \tag{2.11}$$

As a result, the averaged system (2.5) is just (1.2a) of the reduced system with $\phi(\tau, x)$ substituted for y. The problem therefore reduces to Tikhonov's problem when ϕ is a constant solution of (2.2).

The proof of Theorem 1 is given in the next section. The major steps in the proof are as follows:

(i) We use a transformation from (x, y) to new variables z, ψ such that the equations in the new variables have a nonlinear part satisfying some well-defined bounds. Equation (2.4) of Hypothesis H2 is crucial in obtaining these bounds. This transformation is based on a transformation due originally to Krylov and Bogoliubov that is basic in results employing averaging ([10], [11]).

(ii) We show that, provided the solution of the $z - \psi$ system enters a sufficiently small neighborhood of the origin at a time $\tau_1 > 0$, the solution stays within an arbitrarily small neighborhood of the origin for sufficiently small ε, for $\tau_1 \leq \tau \leq L$. This is done using hypothesis H4 and the bounds on the nonlinear parts obtained in (i).

(iii) We show, finally, that the solution of the $z - \psi$ system does indeed enter an arbitrarily small neighborhood of the origin at a time τ_1 that shrinks to zero with ε. This step again depends heavily on hypothesis H4.

3. Proof of Theorem 1

Define $w(t, \tau, x, \varepsilon)$ by

$$w(t, \tau, x, \varepsilon) = \int_0^t e^{-\varepsilon(t-s)} [f(\tau, x, \phi(s, \tau, x), 0) - f_0(\tau, x)]\, ds. \tag{3.1}$$

Then from hypothesis H2, it can be shown that $\frac{\partial w}{\partial \tau}$, $\frac{\partial w}{\partial x}$ and w are bounded with a bound independent of ε on $S_4 = \{(t, \tau, x, \varepsilon)$: $t \in [0, \infty)$, $\tau \in [0, L]$, $x \in \mathcal{G}_1$, $0 < \varepsilon \leqslant \varepsilon_0\}$. (See, for example, [11]). Furthermore, w satisfies the partial differential equation

$$\frac{\partial w}{\partial t} = -\varepsilon w + f(\tau, x, \phi(t, \tau, x), 0) - f_0(\tau, x). \tag{3.2}$$

Put

$$x = \xi + \varepsilon\, w(\tau/\varepsilon, \tau, \xi, \varepsilon),$$

$$y = \phi(\tau/\varepsilon, \tau, \xi) + \psi. \qquad (3.3)$$

Substituting in (2.1) and observing that for sufficiently small ε, $0 < \varepsilon \leq \varepsilon_2 \leq \varepsilon_0$, the matrix $[I + \varepsilon \frac{\partial w}{\partial x}]$ has an inverse of the form $[I + \varepsilon W(\tau, \xi, \varepsilon)]$ where the matrix W is bounded, we obtain the equations in the new variables:

$$\frac{d\xi}{d\tau} = f_o(\tau, \xi) + \hat{f}(\tau, \xi, \psi, \varepsilon) \quad \xi(0) = x_o$$

$$\frac{d\psi}{d\tau} = \varepsilon^{-1} \hat{g}(\tau, \xi, \psi, \varepsilon) \quad \psi(0) = \psi_o = y_o - \phi(o, o, x_o),$$

$$(3.4)$$

where

$$\hat{f}(\tau, \xi, \psi, \varepsilon) \equiv [I + \varepsilon W(\tau, \xi, \varepsilon)] \Big\{ f(\tau, \xi + \varepsilon w(\tau/\varepsilon, \tau, \xi, \varepsilon), \phi(\tau/\varepsilon, \tau, \xi) + \psi, \varepsilon)$$

$$- f(\tau, \xi, \phi(\tau/\varepsilon, \tau, \xi), 0) + f_o(\tau, \xi) \qquad (3.5)$$

$$- \varepsilon \frac{\partial w}{\partial \tau}(\tau/\varepsilon, \tau, \xi, \varepsilon) \Big\} - f_o(\tau, \xi),$$

$$\hat{g}(\tau, \xi, \psi, \varepsilon) \equiv g(\tau, \xi + \varepsilon w(\tau/\varepsilon, \tau, \xi, \varepsilon), \phi(\tau/\varepsilon, \tau, \xi) + \psi, \varepsilon)$$

$$- g(\tau, \xi, \phi(\tau/\varepsilon, \tau, \xi), 0) - \varepsilon \frac{\partial \phi}{\partial \tau}(\tau/\varepsilon, \tau, \xi)$$

$$- \varepsilon \frac{\partial \phi}{\partial x}(\tau/\varepsilon, \tau, \xi) [f_o(\tau, \xi) + \hat{f}(\tau, \xi, \psi, \varepsilon)]. \qquad (3.6)$$

Introduce another transformation through

$$\xi = \zeta(\tau, x_o) + z \tag{3.7}$$

Then substituting in (3.4) and simplifying, we have

$$\frac{dz}{d\tau} = h(\tau, z, \psi, \varepsilon) \qquad z(0) = 0$$

$$\frac{d\psi}{d\tau} = \varepsilon^{-1} C(\tau, \varepsilon)\psi + \varepsilon^{-1} k(\tau, z, \psi, \varepsilon) \quad \psi(0) = \psi_o, \tag{3.8}$$

where,

$$h(\tau, z, \psi, \varepsilon) \equiv f_o(\tau, \zeta(\tau, x_o) + z) - f_o(\tau, \zeta(\tau, x_o)) +$$

$$\hat{f}(\tau, \zeta(\tau, x_o) + z, \psi, \varepsilon), \tag{3.9}$$

$$k(\tau, z, \psi, \varepsilon) \equiv \hat{g}(\tau, \zeta(\tau, x_o) + z, \psi, \varepsilon) - C(\tau, \varepsilon)\psi.$$

This may be written in the following form:

$$\frac{dz}{d\tau} = A(\tau)z + B(\tau, \varepsilon)\psi + \hat{h}(\tau, z, \psi, \varepsilon) \tag{3.10}$$

$$\frac{d\psi}{d\tau} = \varepsilon^{-1} C(\tau, \varepsilon)\psi + \varepsilon^{-1} k(\tau, z, \psi, \varepsilon)$$

where,

$$A(\tau) \equiv \frac{\partial f_o}{\partial x}(\tau, \zeta(\tau, x_o))$$

$$B(\tau, \varepsilon) \equiv \frac{\partial f}{\partial y}(\tau, \xi(\tau, x_o), \phi(\tau/\varepsilon, \tau, \xi(\tau, x_o)), 0)$$

and $\hat{h}(\tau, z, \psi, \varepsilon) = h(\tau, z, \psi, \varepsilon) - Az - B\psi.$ (3.11)

The smoothness and boundedness properties of the functions occuring in (3.10) are crucial for the subsequent analysis. In order to state these precisely, we define the set

$$S_5 = \{(\tau, z, \psi, \varepsilon): 0 \leq \tau \leq L, z \in R^n, \psi \in R^m,$$

$$|z| + |\psi| \leq \nu_o, \ 0 < \varepsilon \leq \varepsilon_o \quad (3.12)$$

$\nu_o > 0$ is such that

$$|z| + |\psi| \leq \nu_o \Longrightarrow (x, y) \in \mathcal{G}_1 \times \mathcal{G}_2 \quad (3.13)$$

on tracing through the transformations (3.3) and (3.7).

Then it is seen that $\hat{h}$, k as well as their partial derivatives with respect to z and ψ tend to zero as $(|z| + |\psi| + \varepsilon) \to 0$, uniformly in τ. In fact, a close examination of the Taylor expansions of $\hat{h}$ and k shows that there exist positive constants λ, μ_1 and μ_2 such that

$$|\hat{h}(\tau, 0, 0, \varepsilon)| \leq \lambda\varepsilon \qquad |k(\tau, 0, 0, \varepsilon)| \leq \lambda\varepsilon$$

$$|\hat{h}(\tau, z_1, \psi_1, \varepsilon) - \hat{h}(\tau, z_2, \psi_2, \varepsilon)| \leq (\mu_1\varepsilon + \mu_2\nu)\left[|z_1 - z_2| + |\psi_1 - \psi_2|\right]$$

$$\left| k(\tau, z_1, \psi_1, \varepsilon) - k(\tau, z_2, \psi_2, \varepsilon) \right| \leqslant (\mu_1 \varepsilon + \mu_2 \nu)$$

$$[\ |z_1 - z_2| + |\psi_1 - \psi_2|\ ]$$

for $0 \leq \tau \leq L$, $|z_i| + |\psi_i| \leq \nu \leq \nu_o$, $i = 1, 2$, $0 \leq \varepsilon \leq \varepsilon_o$. Furthermore $B(\tau, \varepsilon)$ is bounded for $0 \leq \tau \leq L$, $0 < \varepsilon \leq \varepsilon_o$.

This completes step (i) of the proof. In step (ii) we show that given any $\nu > 0$ there exist positive constants $\gamma(\nu)$ and $\varepsilon_1(\nu)$ such that if $\varepsilon < \varepsilon_1$, any solution of the initial-value problem (3.10) that enters the γ-neighborhood of the origin at some time $\tau_1 > 0$ stays inside the ν-neighborhood of the origin for $\tau_1 \leq \tau \leq L$.

For any $\nu \leq \nu_o$, let

$$B^n_\nu = \left\{ f: J \to R^n, \quad \|f\| \leq \nu \right\}, \qquad (3.15)$$

where $J = [0, L] \times (0, \varepsilon_o]$ and the norm $\|f\|$ is defined by

$$\|f\| = \sup_{0 \leqslant \tau \leqslant L,\ 0 < \varepsilon \leqslant \varepsilon_o} |f(\tau, \varepsilon)| \qquad (3.16)$$

and further, for an n-vector x, we use $|x| = \max\limits_{1 \leqslant i \leqslant n} |x_i|$.

Let $u: J \to R^n$, $v: J \to R^m$ be such $\|u\| + \|v\| < \nu$.

Then $(u, v) \in B_{\nu}^{n+m}$.

We shall next define an operator on B_{ν}^{n+m} and use the contraction mapping principle to obtain our result.

Let $\Phi(\tau)$ be an nxn matrix satisfying

$$\frac{d}{d\tau} \Phi(\tau) = A(\tau)\, \Phi(\tau), \tag{3.17}$$

M_1 a positive number such that

$$|B(\tau, \varepsilon)| \leq M_1 \text{ for } (\tau, \varepsilon) \in J \tag{3.18}$$

and $(a, b) \in R^n \times R^m$, constants with $|a|$, $|b|$ to be chosen later. Also, let τ_1 be a positive number less than L.

Then we define the operator $\mathcal{J}$ on B_{ν}^{n+m} by

$$\mathcal{J}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \left| \begin{array}{l} \Phi(\tau)\, \Phi^{-1}(\tau_1)\, a + \int_{\tau_1}^{\tau} \Phi(\tau)\, \Phi^{-1}(s) \\ \Big\{ B(s, \varepsilon)\ \Psi(s, \varepsilon)\, \Psi^{-1}(\tau_1, \varepsilon)\ b \\ \quad + \hat{h}(s, u(s, \varepsilon), v(s, \varepsilon), \varepsilon) \Big\}\, ds \\ \Psi(\tau, \ )\, \Psi^{-1}(\tau_1, \ )\, b \\ \quad + \varepsilon^{-1} \int_{\tau_1}^{\tau} \Psi(\tau, \varepsilon)\, \Psi^{-1}(s, \varepsilon) \\ \quad k(s, u(s, \ ), v(s, \ ), \ )\, ds \end{array} \right| \tag{3.19}$$

We see that a fixed point of this operator corresponds to a solution of (3. 10) passing through the point (a, b) at time τ_1. We shall establish the existence of a fixed point by showing that $\mathcal{J}$ is a contraction.

Let $M_2 > 0$ be such that

$$|\Phi(\tau)\, \Phi^{-1}(s)| \leqslant M_2 \text{ for } 0 \leqslant \tau \leqslant L, \quad 0 \leqslant s \leqslant L \tag{3.20}$$

Then for $\tau_1 \leq \tau \leq L$ we have the following estimates:

$$\begin{aligned} \|\rho_1\| &\leqslant M_2|a| + \int_{\tau_1}^{\tau} M_2 [M_1 k_1 e^{-\alpha_1/\varepsilon\,(s-\tau_1)}|b| + \lambda\varepsilon + \nu(\mu_1\varepsilon + \mu_2\nu)]\, ds \\ &\leqslant M_2|a| + M_2 M_1 L \frac{k_1}{\alpha_1}|b|\varepsilon + M_2 L[\lambda\varepsilon + \nu(\mu_1\varepsilon + \mu_2\nu)] \end{aligned} \tag{3.21}$$

$$\begin{aligned} \|\rho_2\| &\leqslant k_1 e^{-\alpha_1/\varepsilon\,(\tau-\tau_1)}|b| + \varepsilon^{-1}\int_{\tau_1}^{\tau} k_1 e^{-\alpha_1/\varepsilon\,(\tau-s)}[\lambda\varepsilon + \nu(\mu_1\varepsilon + \mu_2\nu)]\, ds \\ &\leqslant k_1|b| + \frac{k_1}{\alpha_1}[\lambda\varepsilon + \nu(\mu_1\varepsilon + \mu_2\nu)] \end{aligned} \tag{3.22}$$

$$\begin{aligned} \|\rho_1' - \rho_1''\| &\leqslant \int_{\tau_1}^{\tau} M_2(\mu_1\varepsilon + \mu_2\nu)[|u'(s) - u''(s)| + |v'(s) - v''(s)|]\, ds \\ &\leqslant M_2 L(\mu_1\varepsilon + \mu_2\nu)[\|u' - u''\| + \|v' - v''\|] \end{aligned} \tag{3.23}$$

$$\left\| \rho_2' - \rho_2'' \right\| \leq \varepsilon^{-1} \int_{\tau_1}^{\tau} k_1 e^{-\alpha_1/\varepsilon (\tau - s)} (\mu_1 \varepsilon + \mu_2 \nu) [\left| u'(s) - u''(s) \right|$$

$$+ \left| v'(s) - v''(s) \right|] \, ds \qquad (3.24)$$

$$\leq M_2 L (\mu_1 \varepsilon + \mu_2 \nu) [\| u' - u'' \| + \| v' - v'' \|]$$

Pick ν_1 and then ε_1 such that

$$M_2 L [\lambda \varepsilon_1 + \nu_1 (\mu_1 \varepsilon_1 + \mu_2 \nu_1)] < \nu_1/8$$

$$\frac{k_1}{\alpha_1} [\lambda \varepsilon_1 + \nu_1 (\mu_1 \varepsilon_1 + \mu_2 \nu_1)] < \nu_1/4 \qquad (3.25)$$

$$M_2 M_1 L \varepsilon_1/\alpha_1 < 1/2$$

The ε_1 chosen here depends on ν_1 and for any $\nu \leq \nu_1$, we can find $\varepsilon_1(\nu)$ to satisfy the above inequalities. Then $\varepsilon_1(\nu) \to 0$ with ν. Now for any $\nu \leq \nu_1$, let

$$\gamma(\nu) = \min (\nu/2M_2, \ \nu/2k_1) \qquad (3.26)$$

and pick a, b such that

$$|a| < \gamma/2, \quad |b| < \gamma/2 \qquad (3.27)$$

Then for any $\nu \leq \nu_1$ we find from (3.21-3.24) that

$$\left\| \mathcal{J} \begin{bmatrix} u \\ v \end{bmatrix} \right\| \leqslant \|\rho_1\| + \|\rho_2\| < \nu, \tag{3.28}$$

$$\left\| \mathcal{J} \begin{bmatrix} u' \\ v' \end{bmatrix} - \mathcal{J} \begin{bmatrix} u'' \\ v'' \end{bmatrix} \right\| \leqslant \theta \left\| \begin{bmatrix} u' \\ v' \end{bmatrix} - \begin{bmatrix} u'' \\ v'' \end{bmatrix} \right\| \text{ for some } \theta < 1 .$$

This shows that $\mathcal{J}$ is a contraction on B_ν^{n+m} and hence possesses a unique fixed point. The fixed point of $\mathcal{J}$, from (3.19) is evidently a solution of (3.10) that is in the $\gamma(\nu)$ neighborhood at $\tau = \tau_1$ and remains inside the ν-neighborhood for $\tau_1 \leq \tau \leq L$.

Step iii: To show that the solution of the IVP(3.10) does indeed enter the $\gamma(\nu)$ neighborhood at some time τ_1, $0 < \tau_1 < L$, we first assume that the solution of (3.10) exists on $[0, L]$ and that

$$\| z \| + \| \psi \| \leqslant \nu_2, \tag{3.29}$$

where $\nu_2 \leqslant \nu_o$ will be specified later.

At this stage we return to the $z - \psi$ equation in the form (3.8) and write it as an integral eqaution

$$z(\tau) = \int_0^\tau h(s, z(s), \psi(s), \varepsilon)\, ds$$

$$\psi(\tau) = \Psi(\tau, \varepsilon)\psi_o + \varepsilon^{-1} \int_0^\tau \Psi(\tau,\varepsilon)\Psi^{-1}(s,\varepsilon) \tag{3.30 a, b}$$

$$k(s, z(s), \psi(s), \varepsilon)$$

Let $M_3 > 0$ be such that

$$\left| h(\tau, z, \psi, \varepsilon) \right| \leq M_3 \text{ for } 0 \leq \tau \leq L, \ |z| + |\psi| \leq \nu_o, \ 0 < \varepsilon \leq \varepsilon_1. \tag{3.31}$$

Then from (3.30a), we have

$$\left| z(\tau) \right| \leq M_3 \tau \tag{3.32}$$

Choose τ_1' such that

$$M_3 \tau_1' < \gamma(\nu_1)/2 \tag{3.33}$$

Then $\left| z(\tau) \right| < \gamma/2$ for $0 \leq \tau \leq \tau_1'$ (3.34)

Now from (3.30b), we have

$$\left| \psi(\tau) \right| \leq k_1 |\psi_o| e^{-\alpha_1 \tau/\varepsilon} + \varepsilon^{-1} \int_o^\tau [\alpha_1 K e^{-\alpha_1/\varepsilon (\tau - s)} +$$

$$k_1 \mu e^{-\alpha_1/\varepsilon (\tau - s)} \left| \psi(s) \right|] ds, \tag{3.35}$$

where we have put

$$K = \frac{k_1}{\alpha_1} \left[\lambda \varepsilon + \gamma/2 (\mu_1 \varepsilon + \mu_2 \nu_2) \right], \ \mu = \mu_1 \varepsilon + \mu_2 \nu_2. \tag{3.36}$$

Therefore the scalar function

$$r(\tau) = e^{\alpha_1 \tau/\varepsilon} \left| \psi(\tau) \right| \tag{3.37}$$

satisfies the inequality

$$r(\tau) \leqslant k_1 |\psi_o| + \varepsilon^{-1} \int_o^{\tau} \alpha_1 K e^{\alpha_1 s/\varepsilon} ds + \varepsilon^{-1} \int_o^{\tau} k_1 \mu r(s) \; ds$$

$$\leqslant k_1 |\psi_o| + K (e^{\alpha_1 \tau/\varepsilon} - 1) + \varepsilon^{-1} k_1 \mu \int_o^{\tau} r(s) \, ds. \tag{3.38}$$

Applying the generalized Gronwall's inequality [12] to (3. 38) and simplifying, we have:

$$r(\tau) \leqslant \frac{K\alpha_1}{\alpha_1 - k_1 \mu} e^{\alpha_1 \tau/\varepsilon} + \left(k_1 | \psi_o| - \frac{K\alpha_1}{\alpha_1 - k_1 \mu} \right) e^{k_1 \mu \tau/\varepsilon}. \tag{3.39}$$

Therefore,

$$|\psi(\tau)| \leqslant \frac{K\alpha_1}{\alpha_1 - k_1 \mu} + \left(k_1 | \psi_o| - \frac{K\alpha_1}{\alpha_1 - k_1 \mu} \right) \exp \left[- \frac{\alpha_1 - k_1 \mu}{\varepsilon} \tau \right] \tag{3.40}$$

Now we choose ν_2 and $\varepsilon_2 \leqslant \varepsilon_1$ such that

$$k_1 (\mu_1 \varepsilon_2 + \mu_2 \nu_2) < \alpha_1/3 \tag{3.41}$$

With this choice of ν_2 it is possible to pick ε^*, $\nu^* \leq \nu_1$ such that

$$\frac{K\alpha_1}{\alpha_1 - k_1 \mu} = \frac{k_1}{\alpha_1} \frac{\lambda \varepsilon^* + \gamma/2 \, (\nu^*) [\mu_1 \varepsilon^* + \mu_2 \nu_2]}{1 - \frac{k_1}{\alpha_1} (\mu_1 \varepsilon^* + \mu_2 \nu_2)} < \min\left(\frac{\gamma(\nu^*)}{4}, k_1 |\psi_o| \right) \tag{3.42}$$

From (3.42) we see that the right side of (3.40) is a monotone decreasing function of τ having values between $k_1|\psi_o|$ and $\frac{K\alpha_1}{\alpha_1 - k_1}\mu$. Thus for each ε, $0 < \varepsilon \leq \varepsilon^*$, we can pick $\tau_1'' = \varepsilon\hat{\tau}$ such that

$$\left|\psi(\tau_1)\right| < \frac{\gamma}{2}(\nu_1) \tag{3.43}$$

Taking

$$\tau_1 = \min(\tau_1', \tau_1'') \tag{3.44}$$

we see from (3.34) and (3.43) that

$$\left|z(\tau_1)\right| + \left|\psi(\tau_1)\right| < \gamma(\nu_1) \tag{3.45}$$

To complete the proof it is only necessary to show that (3.29) is satisfied and the solution of (3.10) does indeed exist on [O, L]. This is immediate since, if ψ_o is chosen that

$$\left|\psi_o\right| < \kappa = \nu_2/k_1 \tag{3.46}$$

we see from (3.40) that $\left|\psi(\tau)\right|$ is monotone decreasing and this, together with (3.34) shows that the solution of (3.12) cannot get outside the ν_2 neighborhood. Thus Theorem 1 is proved.

4. A Result on the Existence of Bounded Solutions

We show in this section that the main result of [9] can be specialized to singularly perturbed systems of the form (1. 1) to yield sufficient conditions for the existence of bounded solutions which have asymptotic limits as $\varepsilon \to 0$. Results of this nature have been obtained for periodic systems in [6] and for almost periodic systems in [7]. Our results are more general in the same way as in the case of the initial value problem, namely, the class of limiting solutions in our result is much larger than in the above results. However, as noted in [7], the result of [6] does not follow from our results because the special nature of periodic systems makes it possible to utilize less restrictive hypotheses than ours.

We proceed to describe the result of [9] when specialized to the system (1. 1). Let $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$ be as in section 2, except that τ and t may now take any real values, i. e. $-\infty < \tau < \infty$ and $-\infty < t < \infty$. Let f and g satisfy the same smoothness hypotheses as in section 2 on the set $\hat{S}_1$ and further, let hypotheses H1 and H2 hold on the sets $\hat{S}_2$ and $\hat{S}_3$. The hypotheses H3 and H4 are replaced by the hypotheses H5 - H7 below.

<u>Hypothesis H5:</u>

The averaged equation (2. 5) has a bounded solution $\xi^*(\tau)$ which lies in the interior of $\mathcal{G}_1$ for all τ.

<u>Hypothesis H6:</u>

Let

$$A(\tau) \equiv \frac{\partial f_0}{\partial x}(\tau, \xi^*(\tau)). \tag{4. 1}$$

Then the variational equation of (2. 5) with respect to the solution $\xi^*(\tau)$, that is, the linear equation

$$\frac{d\eta}{d\tau} = A(\tau)\,\eta, \tag{4.2}$$

has a fundamental matrix $\Phi(\tau)$ which exhibits exponential dichotomy, i.e., there exist supplementary projections* P_1 Q_1 and positive constants k_1, α_1 such that

$$\left|\Phi(\tau)\,P_1\Phi^{-1}(s)\right| \leq k_1 \exp\left\{-\alpha_1(\tau-s)\right\} \quad \text{for} \quad \tau \geq s,$$

$$\left|\Phi(\tau)\,Q_1\Phi^{-1}(s)\right| \leq k_1 \exp\left\{\alpha_1(\tau-s)\right\} \text{ for } \tau \leq s. \tag{4.3}$$

Hypothesis H7:

Let $C(\tau,\varepsilon)$ be defined by

$$C(\tau,\varepsilon) \equiv \frac{\partial g}{\partial y}(\tau, \xi^*(\tau), \phi(\tau/\varepsilon, \tau, \xi^*(\tau)), 0). \tag{4.4}$$

Then the linear equation

$$\frac{d\chi}{d\tau} = \varepsilon^{-1}C(\tau,\varepsilon)\chi \tag{4.5}$$

has a fundamental matrix $\Psi(\tau,\varepsilon)$ for which there exist supplementary projections P_2, Q_2 and positive constants k_2, α_2 such that

$$\left|\Psi(\tau,\varepsilon)\,P_2\,\Psi^{-1}(s,\varepsilon)\right| \leq k_2 \exp\left\{-\varepsilon^{-1}\alpha_2(\tau-s)\right\} \text{ for } \tau \geq s,$$

---

* A matrix P is called a projection if $P^2 = P$. Two projections P, Q are called Supplementary Projections if $P+Q=I$. (See, for instance, [13]).

$$\left| \Psi(\tau,\varepsilon)\, Q_2\, \Psi^{-1}(s,\varepsilon) \right| \leq k_2 \exp \left\{ \varepsilon^{-1} \alpha_2(\tau - s) \right\} \quad \text{for } \tau \leqslant s. \tag{4.6}$$

A discussion of the hypotheses H6 and H7 and sufficient conditions for the existence of such exponential dichotomy, along with a reference to relevant literature may be found in [9].

The result of [9] as it applies to (1.1) is contained in the following theorem:

Theorem 2

Under the above hypotheses there exists an $\varepsilon^* > 0$, such that for all ε, $0 < \varepsilon \leq \varepsilon^*$, the system (1.1) possesses a unique bounded solution $x^*(\tau,\varepsilon)$, $y^*(\tau,\varepsilon)$ lying in the interior of $\mathcal{G}_1 \times \mathcal{G}_2$ for all τ. Furthermore x^* and y^* are continuous in ε and satisfy

$$\lim_{\varepsilon \to 0} \left[\| x^*(\tau,\varepsilon) - \xi^*(\tau) \| + \| y^*(\tau,\varepsilon) - \phi(\tau/\varepsilon, \tau, \xi^*(\tau)) \| \right] = 0 \tag{4.7}$$

where $\|f\| = \sup |f(\cdot)|$. The solution (x^*, y^*) is uniformly asymptotically stable if the matrices Q_1 or Q_2 are null matrices of order n and m respectively, and unstable if either Q_1 or Q_2 is different from the null matrix.

For a proof of Theorem 2, see [9].

In case the solution $\phi(t,\tau,x)$ is independent of t, i.e., ϕ is a constant solution of (2.2), then as in the case of the initial-value problem, Theorem 2 shows that the solutions x^* and y^* approach asymptotically the postulated bounded solutions of a system which in this case corresponds to the reduced system (1.2). If f, g instead of being merely bounded in τ, are almost-periodic or periodic, this result reduces to those of [7] and [6] respectively except for the reservation noted at the beginning of this section.

As in the case of the initial value problem, our result is considerably more general than previous results because of the fact that the asymptotic limit of the solution of (1. 1) satisfies an equation more general than the reduced system (1. 2).

5. Conclusions

It has been shown that if the Boundary Layer Equation possesses a nonconstant bounded solution, the singularly perturbed system (1. 1) may possess solutions that approach asymptotically suitably defined solutions of a system more general than the classical reduced system (1. 2)

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