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SUCCESSIVE APPROXIMATIONS IN NONLINEAR THERMOELASTICITY

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Abstract

Starting with the basic equations of finite thermoelasticity for an incompressible, isotropic, homogeneous body, the usual method of successive approximations is generalized by expanding the relevant fields in terms of two parameters. These parameters separately characterize the mechanical and thermal loading on the body. The successive equations are derived, and it is seen that at each order the basic equations are linear and completely uncoupled. The displacement is determined by the equations of linear isothermal elasticity, while the temperature field is determined by the equation of linear heat conduction. Solutions, through second-order terms, are found for the following problems: annular cylinder with axial mechanical loading and radial thermal loading, torsion of an axial heat conductor, sperical shell with radial mechanical and thermal loading, solid cylinder with axial mechanical and thermal loading, constrained annular cylinder with internal pressure and radial thermal loading.

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1. INTRODUCTION

The coupled, nonlinear character of the basic equations of finite thermoelasticity clearly militates against the exact solution of specific boundary-value problems. It is appropriate, therefore, to turn to approximate procedures. In the present paper we employ the method of successive approximations to investigate thermodynamic processes in an incompressible, isotropic solid. A detailed account of this technique, as it applies to the isothermal case, has been provided by Truesdell and Noll [1].

There appears to be very little in the literature of thermoelasticity on this method. Iesan [2,3] has derived successive equations but has not applied them to any boundary-value problems. Chaudhry [4] has considered successive approximations for plane problems and has given an example that involves an insulated hole in an infinite body with tension at infinity. By following the usual linearization scheme but retaining the appropriate higher-order terms, Chadwick and Seet [5] have derived a consistent (and nonlinear) second-order theory. Starting with the formulation of Chadwick and Seet, Johnson [6] has used successive approximations to study one-dimensional longitudinal wave motion. Actually, Iesan, Chaudhry, and Johnson expand the relevant fields in terms of a single parameter; this would not seem to allow adequately for independence of mechanical and thermal loading.

We generalize the usual procedure of successive approximations and expand all fields in terms of two parameters. These parameters separately characterize the mechanical and thermal loading on the body.

The basic equations of the nonlinear theory are summarized in Section 2. In Section 3, the successive approximation technique is introduced and the equations of the various order theories are derived from the equations of the previous section. Some remarks are made in Section 4 about the uncoupled, linear character of the successive equations. Then, in Sections 5-9, we apply the successive equations to specific boundary-value problems.

We choose to employ direct notation in the development of Sections 2-4, with scalars appearing as light face letters, vectors as bold face lower case letters, and second-order tensors as bold face upper case letters. In Sections 5-9, we select particular coordinate systems appropriate to the geometries of the problems considered.

2. BASIC EQUATIONS

In this section, we summarize the basic equations of nonlinear thermoelasticity for an incompressible, isotropic, homogeneous body.

It is assumed that at time $t = 0$ the body occupies some unstressed reference configuration. The displacement vector $\underset{\sim}{u}$ carries the point $\underset{\sim}{x}$ occupied by a particle in the reference configuration into the corresponding point $\underset{\sim}{y}$ in the deformed configuration according to

$$\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x},t) \ .$$

The deformation gradient is then

$$\underset{\sim}{F} = \underset{\sim}{1} + \nabla\underset{\sim}{u} \ . \qquad (2.1)^{1}$$

We will adopt the convention of placing a caret over the symbol for a field quantity when it is to be regarded as a function of the spatial variable $\underset{\sim}{y}$. The symbol is left free of such marking when it is to be regarded as a function of the referential variable $\underset{\sim}{x}$.[2]

Initially, we choose to present the local form of the basic balance laws in terms of the usual spatial description. Accordingly, conservation of mass is expressed by

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[1] We use the "del" notation, ∇ and $\nabla\cdot$, to indicate differential operations with respect to the referential variable $\underset{\sim}{x}$, while the analogous operations with respect to the spatial variable $\underset{\sim}{y}$ will be indicated by grad and div, respectively.

[2] E.g., $\hat{r} = \hat{r}(\underset{\sim}{y},t)$, while $r = r(\underset{\sim}{x},t)$, where $\hat{r}(\underset{\sim}{y},t) = \hat{r}(\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x},t),t) = \tilde{r}(\underset{\sim}{x},t)$.

$$\dot{\hat{\rho}} + \hat{\rho}\nabla\cdot\hat{\underset{\sim}{v}} = 0 \ , \qquad (2.2)^3$$

where $\hat{\rho}$ is the density and $\hat{\underset{\sim}{v}}$ is the spatial velocity field. Balance of linear and angular momentum are accounted for through the familiar equations of motion

$$\text{div}\ \hat{\underset{\sim}{T}} + \hat{\rho}\hat{\underset{\sim}{b}} = \hat{\rho}\dot{\hat{\underset{\sim}{v}}} \qquad (2.3)$$

and

$$\hat{\underset{\sim}{T}} = \hat{\underset{\sim}{T}}^T \ , \qquad (2.4)$$

where $\hat{\underset{\sim}{T}}$ is the Cauchy stress tensor and $\hat{\underset{\sim}{b}}$ is the body force per unit mass. The equation of energy balance is

$$\text{div}\ \hat{\underset{\sim}{q}} + \hat{\rho}\hat{r} + \text{tr}(\hat{\underset{\sim}{T}}\hat{\underset{\sim}{D}}) = \hat{\rho}\dot{\hat{e}} \ , \qquad (2.5)$$

where $\hat{\underset{\sim}{q}}$ is the heat flux into the body, $\hat{r}$ is the extrinsic heat supply per unit mass, $\hat{e}$ is the internal energy per unit mass, and $\hat{\underset{\sim}{D}}$ is the symmetric part of the spatial velocity gradient. For an elastic body, (2.5) can be written in terms of the absolute temperature $\hat{\theta}$ and the entropy per unit mass $\hat{\eta}$ as

$$\hat{\theta}\hat{\rho}\dot{\hat{\eta}} = \text{div}\ \hat{\underset{\sim}{q}} + \hat{\rho}\hat{r} \ . \qquad (2.6)^4$$

---

[3] Here, and in what follows, a dot above a field quantity indicates the material time derivative.

[4] This form follows from restrictions imposed on elastic bodies by the second law of thermodynamics (see, e.g., [7]).

For the method of successive approximations, we will want the referential description of the field equations. Conservation of mass is conveniently expressed in terms of referential variables by the density equation

$$\rho J = \rho_o \ , \tag{2.7}$$

where ρ_o is the density in the reference configuration and

$$J = \det \underset{\sim}{F} \ . \tag{2.8}$$

Next we introduce the first Piola-Kirchhoff stress tensor

$$\underset{\sim}{S} = J\underset{\sim}{T}(\underset{\sim}{F}^T)^{-1} \ , \tag{2.9}$$

and the referential heat flux vector

$$\underset{\sim}{h} = J\underset{\sim}{F}^{-1}\underset{\sim}{q} \ . \tag{2.10}$$

Then, with (2.7), (2.9), and (2.10), the equations of motion (2.3) and (2.4) and the energy equation (2.5) can be written in terms of the referential variables as

$$\nabla \cdot \underset{\sim}{S} + \rho_o \underset{\sim}{b} = \rho_o \ddot{\underset{\sim}{u}} \ , \tag{2.11}$$

$$\underset{\sim}{S}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{S}^T \ , \tag{2.12}$$

$$\nabla \cdot \underset{\sim}{h} + \rho_o r = \rho_o \theta \dot{\eta} \ . \tag{2.13}$$

The condition of incompressibility is

$$J = \det \underset{\sim}{F} = 1 \ . \tag{2.14}$$

In terms of the left Cauchy-Green deformation tensor

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} \ , \tag{2.15}$$

(2.14) requires

$$\det \underset{\sim}{C} = 1 \ . \tag{2.16}$$

The constitutive equations describing the material response of elastic heat conductors have a rigorous foundation in the modern theory of continuum thermodynamics (see, e.g., [7]). Without going back to first principles, we assume constitutive equations for the stress, entropy, and heat flux that have been previously developed.[5] Accordingly, for an incompressible, isotropic, homogeneous, elastic solid, we take

$$\underset{\sim}{T} = p\underset{\sim}{1} + 2\rho_o \underset{\sim}{F}[(\frac{\partial \psi}{\partial I_1} + I_1 \frac{\partial \psi}{\partial I_2})\underset{\sim}{1} - \frac{\partial \psi}{\partial I_2} \underset{\sim}{C}]\underset{\sim}{F}^T \ , \tag{2.17}$$

$$\eta = - \frac{\partial \psi}{\partial \theta} \ , \tag{2.18}$$

$$\underset{\sim}{q} = \underset{\sim}{F}[\Gamma_o \underset{\sim}{1} + \Gamma_1 \underset{\sim}{C} + \Gamma_{-1} \underset{\sim}{C}^{-1}]\nabla\theta \ , \tag{2.19}$$

where p is an indeterminate pressure and

$$\psi = \psi(I_1, I_2, \theta) \tag{2.20}$$

is the free energy per unit mass, defined by

$$\psi = e - \eta\theta \ . \tag{2.21}$$

[5] Implicit in the form of the constitutive equations assumed here is that the principle of material indifference and the second law of thermodynamics have been satisfied.

The

$$\Gamma_k = \Gamma_k(I_1, I_2, K_1, K_2, K_3, \theta), \; k = 0,1,-1 \tag{2.22}$$

are material response functions, and the invariants I_i, K_j in (2.20) and (2.22) are given by

$$\begin{aligned} I_1 &= \operatorname{tr} \underset{\sim}{C} \;, \\ I_2 &= \tfrac{1}{2}[(\operatorname{tr} \underset{\sim}{C})^2 - \operatorname{tr}(\underset{\sim}{C}^2)] \;, \end{aligned} \tag{2.23}$$

$$\begin{aligned} K_1 &= \nabla\theta \cdot \nabla\theta \;, \\ K_2 &= \nabla\theta \cdot \underset{\sim}{C} \nabla\theta \;, \\ K_3 &= \nabla\theta \cdot \underset{\sim}{C}^{-1} \nabla\theta \;. \end{aligned} \tag{2.24}$$

It follows, from (2.9), (2.14), and (2.17), that

$$\underset{\sim}{S} = p(\underset{\sim}{F}^T)^{-1} + 2\rho_o \underset{\sim}{F}[(\frac{\partial\psi}{\partial I_1} + I_1 \frac{\partial\psi}{\partial I_2})\underset{\sim}{1} - \frac{\partial\psi}{\partial I_2} \underset{\sim}{C}] \;. \tag{2.25}$$

Similarly, from (2.10), (2.14), and (2.19),

$$\underset{\sim}{h} = [\Gamma_o \underset{\sim}{1} + \Gamma_1 \underset{\sim}{C} + \Gamma_{-1} \underset{\sim}{C}^{-1}]\nabla\theta \;. \tag{2.26}$$

It is essential to understand that $\underset{\sim}{S}$ and $\underset{\sim}{h}$ are not the customary Cauchy type stress tensor and heat flux vector. For physical interpretations, one can invert (2.9) and (2.10) to get $\underset{\sim}{T}$ and $\underset{\sim}{q}$ in terms of $\underset{\sim}{S}$ and $\underset{\sim}{h}$, respectively. This is especially important in the applications in Sections 5-9 where traction type boundary conditions are set in terms of the Piola-Kirchhoff stress tensor $\underset{\sim}{S}$.

3. SUCCESSIVE APPROXIMATIONS

The method of successive approximations is a technique used to reduce the basic equations of the nonlinear theory to a sequence of problems in the linear theory (see, e.g., [1]). To allow complete independence of mechanical and thermal loading, we will generalize the usual procedure by expanding the relevant fields in terms of two parameters.

Proceeding formally, we assume the series expansions:

$$\underset{\sim}{u} = \epsilon_1 \underset{\sim}{u}_1 + \epsilon_2 \underset{\sim}{u}_2 + \epsilon_1^2 \underset{\sim}{u}_{11} + \epsilon_2^2 \underset{\sim}{u}_{22} + \epsilon_1 \epsilon_2 \underset{\sim}{u}_{12} + \cdots , \tag{3.1}$$

$$\theta = \theta_o + \epsilon_1 \theta_1 + \epsilon_2 \theta_2 + \epsilon_1^2 \theta_{11} + \epsilon_2^2 \theta_{22} + \epsilon_1 \epsilon_2 \theta_{12} + \cdots , \tag{3.2}$$

$$\underset{\sim}{S} = \epsilon_1 \underset{\sim}{S}_1 + \epsilon_2 \underset{\sim}{S}_2 + \epsilon_1^2 \underset{\sim}{S}_{11} + \epsilon_2^2 \underset{\sim}{S}_{22} + \epsilon_1 \epsilon_2 \underset{\sim}{S}_{12} + \cdots , \tag{3.3}$$

$$\eta = \eta_o + \epsilon_1 \eta_1 + \epsilon_2 \eta_2 + \epsilon_1^2 \eta_{11} + \epsilon_2^2 \eta_{22} + \epsilon_1 \epsilon_2 \eta_{12} + \cdots , \tag{3.4}$$

$$\underset{\sim}{h} = \epsilon_1 \underset{\sim}{h}_1 + \epsilon_2 \underset{\sim}{h}_2 + \epsilon_1^2 \underset{\sim}{h}_{11} + \epsilon_2^2 \underset{\sim}{h}_{22} + \epsilon_1 \epsilon_2 \underset{\sim}{h}_{12} + \cdots \tag{3.5}$$

for the displacement, temperature, stress, entropy, and heat flux fields, respectively. If a body force $\underset{\sim}{b}$ and extrinsic heat supply r are present, they will be assumed to admit expansions of the form

$$\underset{\sim}{b} = \epsilon_1 \underset{\sim}{b}_1 + \epsilon_1^2 \underset{\sim}{b}_{11} + \cdots \tag{3.6}$$

and

$$r = \epsilon_2 r_2 + \epsilon_2^2 r_{22} + \cdots \tag{3.7}$$

so that ϵ_1 and ϵ_2 become measures of the mechanical and thermal loading, respectively. More often it is the case that the parameters ϵ_1 and ϵ_2 arise naturally in the specification of loading conditions on the surface of the body (e.g., as measures of temperature and displacement differences).

Introducing the expansions (3.1)-(3.7) into (2.11) and (2.13), and collecting coefficients, we obtain the equations of motion and energy balance for the various order theories. The first-order equations associated with ϵ_1 are

$$
\begin{aligned}
&\nabla\cdot\underset{\sim}{S}_1 + \rho_o\underset{\sim}{b}_1 = \rho_o\ddot{\underset{\sim}{u}}_1 \ , \\
&\nabla\cdot\underset{\sim}{h}_1 = \rho_o\theta_o\dot{\eta}_1 \ ;
\end{aligned}
\tag{3.8}
$$

while those associated with ϵ_2 are

$$
\begin{aligned}
&\nabla\cdot\underset{\sim}{S}_2 = \rho_o\ddot{\underset{\sim}{u}}_2 \ , \\
&\nabla\cdot\underset{\sim}{h}_2 + \rho_o r_2 = \rho_o\theta_o\dot{\eta}_2 \ .
\end{aligned}
\tag{3.9}
$$

In the second-order theory we have three sets of equations corresponding to ϵ_1^2, ϵ_2^2, and $\epsilon_1\epsilon_2$, respectively:

$$
\begin{aligned}
&\nabla\cdot\underset{\sim}{S}_{11} + \rho_o\underset{\sim}{b}_{11} = \rho_o\ddot{\underset{\sim}{u}}_{11} \ , \\
&\nabla\cdot\underset{\sim}{h}_{11} = \rho_o\theta_o\dot{\eta}_{11} + \rho_o\theta_1\dot{\eta}_1 \ ,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&\nabla\cdot\underset{\sim}{S}_{22} = \rho_o\ddot{\underset{\sim}{u}}_{22} \ , \\
&\nabla\cdot\underset{\sim}{h}_{22} + \rho_o r_{22} = \rho_o\theta_o\dot{\eta}_{22} + \rho_o\theta_2\dot{\eta}_2 \ ,
\end{aligned}
\tag{3.11}
$$

$$\nabla\cdot\mathbf{S}_{12} = \rho_o\ddot{\mathbf{u}}_{12} ,$$
$$\nabla\cdot\mathbf{h}_{12} = \rho_o\theta_o\mathring{\eta}_{12} + \rho_o\theta_1\mathring{\eta}_2 + \rho_o\theta_2\mathring{\eta}_1 . \tag{3.12}$$

To find the first- and second-order fields for the stress, entropy, and heat flux which appear in (3.8)-(3.12), we first expand all fields appearing in the constitutive equations (2.18), (2.25), and (2.26) in power series in ϵ_1 and ϵ_2. From (2.1),

$$\mathbf{F} = \mathbf{1} + \epsilon_1\nabla\mathbf{u}_1 + \epsilon_2\nabla\mathbf{u}_2 + \epsilon_1^2\nabla\mathbf{u}_{11} + \epsilon_2^2\nabla\mathbf{u}_{22} + \epsilon_1\epsilon_2\nabla\mathbf{u}_{12} + \dots , \tag{3.13}$$

and from (2.15) and (3.13),

$$\begin{aligned}\mathbf{C} = \mathbf{1} &+ \epsilon_1(\nabla\mathbf{u}_1^T+\nabla\mathbf{u}_1) + \epsilon_2(\nabla\mathbf{u}_2^T+\nabla\mathbf{u}_2)\\ &+ \epsilon_1^2(\nabla\mathbf{u}_{11}^T + \nabla\mathbf{u}_{11} + \nabla\mathbf{u}_1^T\nabla\mathbf{u}_1) + \epsilon_2^2(\nabla\mathbf{u}_{22}^T + \nabla\mathbf{u}_{22} + \nabla\mathbf{u}_2^T\nabla\mathbf{u}_2)\\ &+ \epsilon_1\epsilon_2(\nabla\mathbf{u}_{12}^T + \nabla\mathbf{u}_{12} + \nabla\mathbf{u}_1^T\nabla\mathbf{u}_2 + \nabla\mathbf{u}_2^T\nabla\mathbf{u}_1) + \dots .\end{aligned} \tag{3.14}$$

Now, the Taylor series expansion[6] of det $\mathbf{C}$ is

$$\begin{aligned}\det\mathbf{C} = 1 &+ \epsilon_1[2\mathrm{tr}\nabla\mathbf{u}_1] + \epsilon_2[2\mathrm{tr}\nabla\mathbf{u}_2]\\ &+ \epsilon_1^2[4\mathrm{tr}\nabla\mathbf{u}_{11} - 2\mathrm{tr}(\nabla\mathbf{u}_1)^2 + 4(\mathrm{tr}\nabla\mathbf{u}_1)^2]\\ &+ \epsilon_2^2[4\mathrm{tr}\nabla\mathbf{u}_{22} - 2\mathrm{tr}(\nabla\mathbf{u}_2)^2 + 4(\mathrm{tr}\nabla\mathbf{u}_2)^2]\\ &+ \epsilon_1\epsilon_2[2\mathrm{tr}\nabla\mathbf{u}_{12} - 2\mathrm{tr}(\nabla\mathbf{u}_1\nabla\mathbf{u}_2) + 4(\mathrm{tr}\nabla\mathbf{u}_1)(\mathrm{tr}\nabla\mathbf{u}_2)] + \dots ,\end{aligned} \tag{3.15}$$

so the incompressibility condition (2.16) requires that

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[6] Here, the Taylor series representation is obtained by expanding the function about $\epsilon_1,\epsilon_2 = 0$, which corresponds to conditions in the reference configuration.

$$\mathrm{tr}\nabla \underset{\sim}{u}_1 = 0 \ ,$$
$$\mathrm{tr}\nabla \underset{\sim}{u}_2 = 0 \ , \tag{3.16}$$

$$\mathrm{tr}\nabla \underset{\sim}{u}_{11} = \frac{1}{2}\,\mathrm{tr}(\nabla \underset{\sim}{u}_1)^2 \ ,$$
$$\mathrm{tr}\nabla \underset{\sim}{u}_{22} = \frac{1}{2}\,\mathrm{tr}(\nabla \underset{\sim}{u}_2)^2 \ , \tag{3.17}$$
$$\mathrm{tr}\nabla \underset{\sim}{u}_{12} = \mathrm{tr}(\nabla \underset{\sim}{u}_1 \nabla \underset{\sim}{u}_2) \ .$$

Taking into account (3.16) and (3.17), we find from (2.23) that, through second-order terms, the expansions of I_1 and I_2 are identical:

$$I_i = 3 + \epsilon_1^2[\frac{1}{2}\,\mathrm{tr}(\nabla \underset{\sim}{u}_1^T + \nabla \underset{\sim}{u}_1)^2] + \epsilon_2^2[\frac{1}{2}\,\mathrm{tr}(\nabla \underset{\sim}{u}_2^T + \nabla \underset{\sim}{u}_2)^2]$$
$$+ \epsilon_1\epsilon_2[\mathrm{tr}[(\nabla \underset{\sim}{u}_1^T + \nabla \underset{\sim}{u}_1)(\nabla \underset{\sim}{u}_2^T + \nabla \underset{\sim}{u}_2)]] + \ldots \ , \quad i = 1,2. \tag{3.18}$$

Similarly, we find from (2.24) that

$$K_j = \epsilon_1^2(\nabla\theta_1 \cdot \nabla\theta_1) + \epsilon_2^2(\nabla\theta_2 \cdot \nabla\theta_2) + \epsilon_1\epsilon_2(\nabla\theta_1 \cdot \nabla\theta_2) + \ldots \ ,$$
$$j = 1,2,3 \ . \tag{3.19}$$

Series expansions for the response functions Γ_k and the partial derivatives of ψ appearing in the constitutive equations are obtained by expanding these functions in Taylor series about the reference values of their arguments, i.e., about $I_i = 3$, $K_j = 0$, and $\theta = \theta_o$. The coefficients in the expansions will be used to define a set of material constants as follows:

$$a_o = 2\rho_o[\frac{\partial\psi}{\partial\theta}]_o \ ,$$

$$a_i = 2\rho_o[\frac{\partial\psi}{\partial I_i}]_o \ , \quad i = 1,2$$

$$b_o = -\left[\frac{\partial^2 \psi}{\partial \theta^2}\right]_o ,$$

$$b_i = 2\rho_o\left[\frac{\partial^2 \psi}{\partial \theta \partial I_i}\right]_o , \quad i = 1,2$$

$$c_o = -\frac{1}{2}\left[\frac{\partial^3 \psi}{\partial \theta^3}\right]_o ,$$

$$\kappa_1 = [\Gamma_o + \Gamma_1 + \Gamma_{-1}]_o ,$$

$$\kappa_2 = [\frac{\partial}{\partial \theta}(\Gamma_o + \Gamma_1 + \Gamma_{-1})]_o ,$$

$$\kappa_3 = [\Gamma_1 - \Gamma_{-1}]_o ,$$

where $[\ ]_o$ denotes evaluation of the enclosed quantity in the reference configuration. After substituting these expansions into (2.18), (2.25), and (2.26), and collecting coefficients, we find that the first-order constitutive equations for the stress, entropy, and heat flux are

$$\begin{aligned} \underset{\sim}{S}_1 &= p_1\underset{\sim}{1} + (a_1 + a_2)(\nabla\underset{\sim}{u}_1^T + \nabla\underset{\sim}{u}_1) , \\ \underset{\sim}{S}_2 &= p_2\underset{\sim}{1} + (a_1 + a_2)(\nabla\underset{\sim}{u}_2^T + \nabla\underset{\sim}{u}_2) , \end{aligned} \tag{3.20}$$

$$\begin{aligned} \eta_1 &= b_o\theta_1 , \\ \eta_2 &= b_o\theta_2 , \end{aligned} \tag{3.21}$$

$$\begin{aligned} \underset{\sim}{h}_1 &= \kappa_1\nabla\theta_1 , \\ \underset{\sim}{h}_2 &= \kappa_1\nabla\theta_2 , \end{aligned} \tag{3.22}$$

while the second-order constitutive equations are

$$
\begin{aligned}
\underset{\sim}{S}_{11} &= p_{11}\underset{\sim}{1} + (a_1+a_2)(\nabla\underset{\sim}{u}_{11}^T+\nabla\underset{\sim}{u}_{11}) + (b_1+b_2)\theta_1(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1) \\
&\quad - p_1\nabla\underset{\sim}{u}_1^T - a_2(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1)^2 - (a_1+a_2)(\nabla\underset{\sim}{u}_1^T)^2 \ , \\
\underset{\sim}{S}_{22} &= p_{22}\underset{\sim}{1} + (a_1+a_2)(\nabla\underset{\sim}{u}_{22}^T+\nabla\underset{\sim}{u}_{22}) + (b_1+b_2)\theta_2(\nabla\underset{\sim}{u}_2^T+\nabla\underset{\sim}{u}_2) \\
&\quad - p_2\nabla\underset{\sim}{u}_2^T - a_2(\nabla\underset{\sim}{u}_2^T+\nabla\underset{\sim}{u}_2)^2 - (a_1+a_2)(\nabla\underset{\sim}{u}_2^T)^2 \ , \\
\underset{\sim}{S}_{12} &= p_{12}\underset{\sim}{1} + (a_1+a_2)(\nabla\underset{\sim}{u}_{12}^T+\nabla\underset{\sim}{u}_{12}) + (b_1+b_2)\theta_2(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1) \\
&\quad + (b_1+b_2)\theta_1(\nabla\underset{\sim}{u}_2^T+\nabla\underset{\sim}{u}_2) - p_1\nabla\underset{\sim}{u}_2^T - p_2\nabla\underset{\sim}{u}_1^T \\
&\quad - a_2(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1)(\nabla\underset{\sim}{u}_2^T+\nabla\underset{\sim}{u}_2) - a_2(\nabla\underset{\sim}{u}_2^T+\nabla\underset{\sim}{u}_2)(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1) \\
&\quad - (a_1+a_2)(\nabla\underset{\sim}{u}_2\nabla\underset{\sim}{u}_1+\nabla\underset{\sim}{u}_1\nabla\underset{\sim}{u}_2)^T \ ,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\eta_{11} &= b_o\theta_{11} + c_o\theta_1^2 - \tfrac{1}{2}(b_1+b_2)\,\mathrm{tr}(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1)^2 \ , \\
\eta_{22} &= b_o\theta_{22} + c_o\theta_2^2 - \tfrac{1}{2}(b_1+b_2)\,\mathrm{tr}(\nabla\underset{\sim}{u}_2^T+\nabla\underset{\sim}{u}_2)^2 \ , \\
\eta_{12} &= b_o\theta_{12} + 2c_o\theta_1\theta_2 \\
&\qquad - (b_1+b_2)\,\mathrm{tr}[(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1)(\nabla\underset{\sim}{u}_2^T+\nabla\underset{\sim}{u}_2)] \ ,
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
\underset{\sim}{h}_{11} &= \kappa_1\nabla\theta_{11} + \kappa_2\theta_1\nabla\theta_1 + \kappa_3(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1)\nabla\theta_1 \, , \\
\underset{\sim}{h}_{22} &= \kappa_1\nabla\theta_{22} + \kappa_2\theta_2\nabla\theta_2 + \kappa_3(\nabla\underset{\sim}{u}_2^T+\nabla\underset{\sim}{u}_2)\nabla\theta_2 \, , \\
\underset{\sim}{h}_{12} &= \kappa_1\nabla\theta_{12} + \kappa_2\nabla(\theta_1\theta_2) + \kappa_3(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1)\nabla\theta_2 \\
&\qquad + \kappa_3(\nabla u_2^T+\nabla u_2)\nabla\theta_1 \ ,
\end{aligned}
\tag{3.25}
$$

where p_1, p_2, and p_{11}, p_{22}, p_{12} are first- and second-order pressures, respectively.

The first-order constitutive equations (3.20)-(3.22) make it clear that $(a_1 + a_2)$, κ_1, and $\rho_o \theta_o b_o$ are the shear modulus, conductivity, and specific heat, respectively, of the classical theory. We also point out that the constants b_1 and b_2 always appear together as the single constant $(b_1 + b_2)$.

Finally, we remark that the assumption of incompressibility has considerably simplified both the first- and second-order constitutive equations.

4. REMARKS ON THE SUCCESSIVE EQUATIONS

It is easily seen from $(3.8)_1$, $(3.9)_1$, and (3.20) that each of the first-order displacement equations of motion is of the form[7]

$$\nabla\cdot(p\mathbf{1}) + (a_1+a_2)(\nabla^2\mathbf{u}+\nabla\nabla\cdot\mathbf{u}) + \rho_o\mathbf{b} = \rho_o\ddot{\mathbf{u}} \ . \tag{4.1}$$

From (3.16), each of the first-order incompressibility conditions requires that

$$\nabla\cdot\mathbf{u} = 0 \ . \tag{4.2}$$

With (3.21) and (3.22), each of the first-order energy equations $(3.8)_2$ and $(3.9)_2$ can be written in the form

$$\kappa_1\nabla^2\theta + \rho_o r = (\rho_o\theta_o b_o)\dot{\theta} \ . \tag{4.3}$$

We note that (4.1), (4.2), and (4.3) are the usual equations of linear thermoelasticity in the incompressible, isotropic, and homogeneous case. Moreover, these equations are completely uncoupled. The displacement is determined by (4.1) and (4.2) as in isothermal elasticity, while the temperature is determined by (4.3), which is the ordinary heat equation. This uncoupling is due to incompressibility and isotropy. It is not surprising at the first-order since it is a familiar feature of the linear theory.

Perhaps unexpected, however, is the fact that the basic equations of each second-order theory are also uncoupled. From (4.23), (4.24),

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[7] Since in this section we will be concerned only with the structure of equations, the field variables entering the equations will be presented without using subscripts to distinguish any specific order theory. Instead, such distinctions will be made in the discussion preceding the equations.

and (4.25), we see that each of the second-order fields for the stress, entropy, and heat flux has the structure

$$\underset{\sim}{S} = p\underset{\sim}{1} + (a_1+a_2)(\nabla\underset{\sim}{u}^T+\nabla\underset{\sim}{u}) + \underset{\sim}{S}^* , \qquad (4.4)$$

$$\eta = b_o\theta + \eta^* , \qquad (4.5)$$

and

$$\underset{\sim}{h} = \kappa_1\nabla\theta + \underset{\sim}{h}^* , \qquad (4.6)$$

respectively, where the terms $\underset{\sim}{S}^*$, η^*, and $\underset{\sim}{h}^*$ are functions of the first-order displacement and temperature fields.

From $(3.10)_1$, $(3.11)_1$, $(3.12)_1$, and (4.4), each of the second-order displacement equations of motion can be written as

$$\nabla\cdot(p\underset{\sim}{1}) + (a_1+a_2)(\nabla^2\underset{\sim}{u}+\nabla\nabla\cdot\underset{\sim}{u}) + \rho_o\underset{\sim}{b}^* = \rho_o\ddot{\underset{\sim}{u}} , \qquad (4.7)$$

where $\underset{\sim}{b}^*$ is the sum of the prescribed body force of that order and a term involving first-order fields. Each of the second-order energy equations $(3.10)_2$, $(3.11)_2$, and $(3.12)_2$ is of the form

$$\nabla\cdot\underset{\sim}{h} + \rho_o r = \rho_o\theta_o\dot{\eta} + \rho_o f^* , \qquad (4.8)$$

where f^* is a function of first-order temperature and entropy fields. Using (4.5) and (4.6), (4.8) becomes

$$\kappa_1\nabla^2\theta + \rho_o r^* = (\rho_o\theta_o b_o)\dot{\theta} , \qquad (4.9)$$

where r^* is the sum of the prescribed extrinsic heat supply r and a term involving first-order fields. Both r^* in (4.9) and $\underset{\sim}{b}^*$ in (4.7) can be considered as known functions in the second-order theory.

We see, then, from (4.7) and (4.9) that the displacement equations of motion and energy equation of each second-order theory consist of a set of uncoupled linear equations, similar in structure to those of the first-order theories. It should be noted that the form of the second-order incompressibility equations (3.17) differs from that of the first-order equations (3.16) in that, the first-order equations are homogeneous, while, in general, those of the second-order are not.

5. ANNULAR CYLINDER WITH AXIAL MECHANICAL LOADING AND RADIAL THERMAL LOADING

In this section, we will find the first- and second-order effects of combined mechanical and thermal loading on an incompressible, isotropic, homogeneous, elastic cylinder in equilibrium. Both body force and extrinsic heat supply are assumed to be absent.

We will use cylindrical coordinates (r,ϕ,z) and work with the physical components of the field quantities.[8] The associated set of orthonormal base vectors is $\{\underset{\sim}{e}_{\langle r\rangle},\underset{\sim}{e}_{\langle \phi\rangle},\underset{\sim}{e}_{\langle z\rangle}\}$. The cylinder is taken to be of annular cross-section, and in the reference configuration it occupies the region: $a \leq r \leq b$, $0 \leq z \leq L$, $0 \leq \phi \leq 2\pi$.

We consider the case in which a first-order axial displacement is specified on the ends of the cylinder, which are assumed to be free of shearing stress, while the lateral surfaces are entirely free of stress. Thus, the mechanical boundary conditions are

$$\left.\underset{\sim}{u}\cdot\underset{\sim}{e}_{\langle z\rangle}\right|_{z=0} = 0\ , \qquad \left.\underset{\sim}{u}\cdot\underset{\sim}{e}_{\langle z\rangle}\right|_{z=L} = \epsilon_1 L\ , \tag{5.1}$$

$$\left.(\underset{\sim}{S}\underset{\sim}{e}_{\langle z\rangle})\cdot\underset{\sim}{e}_{\langle r\rangle}\right|_{z=0,L} = 0\ , \qquad \left.(\underset{\sim}{S}\underset{\sim}{e}_{\langle z\rangle})\cdot\underset{\sim}{e}_{\langle \phi\rangle}\right|_{z=0,L} = 0\ , \tag{5.2}$$

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[8] Component indices will be enclosed in brackets, $\langle\ \rangle$, as an additional reminder that we are working with physical components.

and

$$\underset{\sim\sim}{Se}_{\langle r \rangle} \Big|_{r=a,b} = \underset{\sim}{0} \; . \qquad (5.3)$$

A first-order temperature difference is specified between the lateral surfaces as follows:

$$\begin{aligned} \theta \Big|_{r=a} &= \theta_o \; , \\ \theta \Big|_{r=b} &= \theta_o + \epsilon_2 \theta_o \; , \end{aligned} \qquad (5.4)$$

while the ends of the cylinder are assumed to be insulated, so that

$$\underset{\sim}{h} \cdot \underset{\sim}{e}_{\langle z \rangle} \Big|_{z=0,L} = 0 \; . \qquad (5.5)$$

From these conditions and the axial symmetry of the problem, we seek displacement and temperature fields in the form:

$$\begin{aligned} \underset{\sim}{u} &= u(r)\underset{\sim}{e}_{\langle r \rangle} + w(z)\underset{\sim}{e}_{\langle z \rangle} \; , \\ \theta &= \theta(r) \; . \end{aligned} \qquad (5.6)$$

We note that the boundary conditions (5.2) are automatically satisfied by displacements of the form $(5.6)_1$, and assuming $(5.6)_2$ restricts heat flow to the radial direction so that (5.5) is also met.

First-Order Solutions

Equations (3.20) relate the first-order stress and displacement fields. From them we find that

$$S_i^{\langle rr\rangle} = p_i + 2(a_1+a_2)\frac{\partial}{\partial r}u_i ,$$

$$S_i^{\langle \phi\phi\rangle} = p_i + 2(a_1+a_2)\frac{u_i}{r} , \qquad i = 1,2 \qquad (5.7)$$

$$S_i^{\langle zz\rangle} = p_i + 2(a_1+a_2)\frac{\partial}{\partial z}w_i ,$$

with all other components of the $\underset{\sim}{S}_i$ being identically zero. Then, using (5.7) in $(3.8)_1$ and $(3.9)_1$, we get the displacement equations of equilibrium

$$\frac{\partial}{\partial r}p_i + 2(a_1+a_2)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_i)\right] = 0 ,$$

$$i = 1,2 \qquad (5.8)$$

$$\frac{\partial}{\partial z}p_i + 2(a_1+a_2)\frac{\partial^2}{\partial z^2}w_i = 0 .$$

The incompressibility conditions (3.16) require that

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_i) + \frac{\partial}{\partial z}w_i = 0 , \qquad i = 1,2 . \qquad (5.9)$$

According to (5.1), the first-order displacements must satisfy the boundary conditions:

$$w_1\Big|_{z=0} = 0 ,$$

$$(5.10)$$

$$w_1\Big|_{z=L} = L ,$$

$$w_2\Big|_{z=0,L} = 0 , \qquad (5.11)$$

and, from (5.3) and (5.7),

$$[p_i + 2(a_1+a_2) \frac{\partial}{\partial r} u_i] \Big|_{r=a,b} = 0 \ . \tag{5.12}$$

The mechanical parts of the first-order solutions are found to be[9]

$$\begin{aligned} \underset{\sim}{u}_1 &= -\frac{1}{2} r\underset{\sim}{e}_{\langle r \rangle} + z\underset{\sim}{e}_{\langle z \rangle} \ , \\ p_1 &= (a_1+a_2) \ , \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \underset{\sim}{u}_2 &= \underset{\sim}{0} \ , \\ p_2 &= 0 \ . \end{aligned} \tag{5.14}$$

Using (5.13) in (5.7), we see that the only nonzero component of $\underset{\sim}{S}_1$ is

$$S_1^{\langle zz \rangle} = 3(a_1+a_2) \ , \tag{5.15}$$

while, from (5.14) and (5.6),

$$\underset{\sim}{S}_2 \equiv \underset{\sim}{0} \ . \tag{5.16}$$

For the case of steady-state radial heat flow, we find from (3.21) and (3.22) that the first-order energy equations $(3.8)_2$ and $(3.9)_2$ reduce to

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[9] Since the basic equations of each order theory are of the same form as those of the classical infinitesimal theory, we invoke the uniqueness arguments associated with the classical theory to claim at each order that any sufficiently smooth solution we find is, in fact, the only (sufficiently smooth) solution.

$$\frac{d}{dr}\left(r \frac{d}{dr} \theta_i\right) = 0 , \qquad i = 1,2 . \tag{5.17}$$

From (5.4), the boundary conditions on the θ_i are

$$\theta_1 \Big|_{r=a,b} = 0 \tag{5.18}$$

and

$$\theta_2 \Big|_{r=a} = 0 , \qquad \theta_2 \Big|_{r=b} = \theta_o . \tag{5.19}$$

The differential equations (5.17) subject to the boundary conditions (5.18) and (5.19) yield the first-order temperature fields

$$\theta_1 \equiv 0 \tag{5.20}$$

and

$$\theta_2 = \frac{\theta_o}{\ell n(b/a)} \ell n(r/a) . \tag{5.21}$$

Second-Order Solutions

Placing the first-order solutions (5.13) and (5.20) in $(3.23)_1$, we find that the nonzero components of the stress $\underset{\sim}{S}_{11}$ are

$$\begin{aligned} S_{11}^{\langle rr \rangle} &= p_{11} + 2(a_1+a_2) \frac{\partial}{\partial r} u_{11} + \left(\frac{1}{4} a_1 - \frac{3}{4} a_2\right) , \\ S_{11}^{\langle \phi\phi \rangle} &= p_{11} + 2(a_1+a_2) \frac{u_{11}}{r} + \left(\frac{1}{4} a_1 - \frac{3}{4} a_2\right) , \\ S_{11}^{\langle zz \rangle} &= p_{11} + 2(a_1+a_2) \frac{\partial}{\partial z} w_{11} - (2a_1+6a_2) , \end{aligned} \tag{5.22}$$

so that, from $(3.10)_1$, the associated displacement equations of equilibrium are

$$\frac{\partial}{\partial r} p_{11} + 2(a_1+a_2) \frac{\partial}{\partial r} [\frac{1}{r} \frac{\partial}{\partial r} (ru_{11})] = 0 \ ,$$
$$\frac{\partial}{\partial z} p_{11} + 2(a_1+a_2) \frac{\partial^2}{\partial z^2} w_{11} = 0 \ . \tag{5.23}$$

With (5.13), the incompressibility condition $(3.17)_1$ becomes

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_{11}) + \frac{\partial}{\partial z} w_{11} = \frac{3}{4} \ . \tag{5.24}$$

According to (5.1),

$$w_{11} \Big|_{z=0,L} = 0 \ , \tag{5.25}$$

while, with (5.22), the condition (5.3) requires

$$[p_{11} + 2(a_1+a_2) \frac{\partial}{\partial r} u_{11} + (\frac{1}{4} a_1 - \frac{3}{4} a_2)] \Big|_{r=a,b} = 0. \tag{5.26}$$

The boundary value problem (5.23)-(5.26) has the solution

$$\underset{\sim}{u}_{11} = \frac{3}{8} r \underset{\sim}{e}_{\langle r \rangle} \ ,$$
$$p_{11} = - a_1 \ . \tag{5.27}$$

From (5.22) and (5.27),

$$S_{11}^{\langle zz \rangle} = - 3(a_1+2a_2) \tag{5.28}$$

is the only nonzero component of $\underset{\sim}{S}_{11}$.

Taking into account (5.14), the second-order problem for $\underset{\sim}{u}_{22}$, $\underset{\sim}{S}_{22}$ reduces to the form of the problem for the first-order fields $\underset{\sim}{u}_i$, $\underset{\sim}{S}_i$ with homogeneous data. Therefore,

$$\begin{aligned} \underset{\sim}{u}_{22} &\equiv \underset{\sim}{0} \ , \\ \underset{\sim}{S}_{22} &\equiv \underset{\sim}{0} \ . \end{aligned} \tag{5.29}$$

Placing (5.13), (5.14), (5.20), and (5.21) in $(3.23)_1$ yields

$$\begin{aligned} S_{12}^{\langle rr \rangle} &= p_{12} + 2(a_1+a_2) \frac{\partial}{\partial r} u_{12} \\ &\qquad - (b_1+b_2) \frac{\theta_o}{\ell n(b/a)} \ell n(r/a) \ , \\ S_{12}^{\langle \phi\phi \rangle} &= p_{12} + 2(a_1+a_2) \frac{u_{12}}{r} \\ &\qquad - (b_1+b_2) \frac{\theta_o}{\ell n(b/a)} \ell n(r/a) \ , \\ S_{12}^{\langle zz \rangle} &= p_{12} + 2(a_1+a_2) \frac{\partial}{\partial z} w_{12} \\ &\qquad + 2(b_1+b_2) \frac{\theta_o}{\ell n(b/a)} \ell n(r/a) \ , \end{aligned} \tag{5.30}$$

with all other components of $\underset{\sim}{S}_{12}$ being identically zero. From $(3.12)_1$ and (5.30), we get the displacement equations of equilibrium

$$\frac{\partial}{\partial r}\, p_{12} + 2(a_1+a_2)\, \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\,(ru_{12})\right]$$

$$- (b_1+b_2)\, \frac{\theta_o}{\ell n(b/a)}\, \left(\frac{1}{r}\right) = 0 \;, \tag{5.31}$$

$$\frac{\partial}{\partial z}\, p_{12} + 2(a_1+a_2)\, \frac{\partial^2}{\partial z^2}\, w_{12} = 0 \;.$$

With the first-order result (5.17), the incompressibility condition $(3.17)_3$ becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\,(ru_{12}) + \frac{\partial}{\partial z}\, w_{12} = 0 \;. \tag{5.32}$$

The boundary condition (5.1) requires that

$$w_{12}\Big|_{z=0,L} = 0 \;, \tag{5.33}$$

while, from (5.3) and (5.30),

$$\Big[p_{12} + 2(a_1+a_2)\, \frac{\partial}{\partial r}\, u_{12}$$

$$- (b_1+b_2)\, \frac{\theta_o}{\ell n(b/a)}\, \ell n(r/a)\Big]\Big|_{r=a,b} = 0 \;. \tag{5.34}$$

The solution to the system (5.31)-(5.34) is

$$\begin{aligned} \underset{\sim}{u}_{12} &\equiv \underset{\sim}{0} \;, \\ p_{12} &= (b_1+b_2)\, \frac{\theta_o}{\ell n(b/a)}\, \ell n(r/a) \;. \end{aligned} \tag{5.35}$$

From (5.30) and (5.35), we see that the only nonzero component of $\underset{\sim}{S}_{12}$ is

$$S_{12}^{\langle zz\rangle} = 3(b_1+b_2)\,\frac{\theta_o}{\ell n(b/a)}\,\ell n(r/a)\ . \tag{5.36}$$

With the first-order solutions (5.13), (5.14), (5.19), and (5.20), the second-order problems for the temperature fields θ_{11} and θ_{12} lead to the trivial solutions

$$\theta_{11}, \theta_{12} \equiv 0\ . \tag{5.37}$$

From (5.12), together with $(3.25)_2$ and $(3.11)_2$, we find that the second-order temperature θ_{22} must be a solution to

$$\kappa_1 \frac{d}{dr}\left(r \frac{d}{dr}\theta_{22}\right) + \kappa_2 \left[\frac{\theta_o}{\ell n(b/a)}\right]^2 \frac{1}{r} = 0\ . \tag{5.38}$$

By (5.4), θ_{22} must satisfy

$$\theta_{22}\Big|_{r=a,b} = 0\ . \tag{5.39}$$

The solution to (5.38) which meets the conditions (5.39) is

$$\theta_{22} = \frac{\kappa_2}{2\kappa_1}\left[\frac{\theta_o}{\ell n(b/a)}\right]^2 \ell n\left(\frac{r}{b}\right)\ell n\left(\frac{a}{r}\right)\ . \tag{5.40}$$

Then, through second-order terms, we get, from (5.13), (5.14), (5.27), (5.29), and (5.35), that

$$\underset{\sim}{u} = \epsilon_1\left[-\frac{1}{2}\,r\underset{\sim}{e}_{\langle r\rangle} + z\underset{\sim}{e}_{\langle z\rangle}\right] + \epsilon_1^2\left[\frac{3}{8}\,r\underset{\sim}{e}_{\langle r\rangle}\right], \tag{5.41}$$

while, from (5.20), (5.21), (5.37), and (5.40),

$$\theta = \theta_o + \epsilon_2\left[\frac{\theta_o}{\ln(b/a)} \ln(r/a)\right]$$

$$+ \epsilon_2^2\left[\frac{\kappa_2}{2\kappa_1}\left\{\frac{\theta_o}{\ln(b/a)}\right\}^2 \ln\left(\frac{r}{b}\right)\ln\left(\frac{a}{r}\right)\right] . \tag{5.42}$$

Finally, from (5.15), (5.16), (5.28), (5.29), and (5.36), we see that, through second-order terms, the only component of stress not identically zero is

$$S^{\langle zz\rangle} = \epsilon_1[3(a_1+a_2)] + \epsilon_1^2[-3(a_1+2a_2)]$$

$$+ \epsilon_1\epsilon_2\left[3(b_1+b_2)\frac{\theta_o}{\ln(b/a)} \ln(r/a)\right] . \tag{5.43}$$

We note, from (5.41) and (5.42), that the displacement is independent of thermal effects while the temperature is independent of mechanical effects. However, the $\epsilon_1\epsilon_2$-term in (5.43) represents a second-order cross effect in the stress that appears only in the presence of both mechanical and thermal loading. One could make use of the present solution to experimentally determine the constants a_1, a_2, $b_1 + b_2$, and κ_2. We will leave such considerations to the subsequent sections, where examples that could be more readily realized in the laboratory are discussed.

6. TORSION OF AN AXIAL HEAT CONDUCTOR

We turn now to the problem of torsion of an incompressible, isotropic, homogeneous, elastic cylinder with axial heat conduction. We take the cylinder to be of circular cross section with the generators of the lateral surface parallel to the z-axis. As in the previous section, we will use cylindrical coordinates (r,ϕ,z) and work with the physical components of the field quantities. We let $\{\underset{\sim}{e}_{\langle r\rangle},\underset{\sim}{e}_{\langle\phi\rangle},\underset{\sim}{e}_{\langle z\rangle}\}$ denote the associated set of orthonormal base vectors. In the reference configuration, the body occupies the region defined by: $0 \le r \le a$, $0 \le z \le L$, $0 \le \phi \le 2\pi$. For this problem, both the body force and extrinsic heat supply will be taken to be zero.

We consider the case of equilibrium with the mechanical loading on the cylinder statically equivalent to a first-order torque on the ends, while the lateral surface is free of stress. Accordingly, taking into account the symmetry of the problem, the mechanical boundary conditions are

$$[\int_R rS^{\langle z\phi\rangle}\Big|_{z=0,L}\, dA]\underset{\sim}{e}_{\langle z\rangle} + \int_R \underset{\sim}{u} \times \underset{\sim\sim}{S}\underset{\sim}{e}_{\langle z\rangle}\Big|_{z=0,L}\, dA = \epsilon_1 T_z \underset{\sim}{e}_{\langle z\rangle}\ , \tag{6.1}$$

$$\int_R S^{\langle zz\rangle}\Big|_{z=0,L}\, dA = 0\ , \tag{6.2}$$

$$\underset{\sim\sim}{S}\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=a} = \underset{\sim}{0}\ , \tag{6.3}$$

where in (6.1) and (6.2) R denotes the cross section $0 \leq r \leq a$. We take ϵ_1 to be the classical angle of twist that arises in the linearized isothermal problem (see, e.g., [8]).[10]

The axial thermal loading is achieved by insisting that the lateral surface be insulated, while specifying a first-order temperature difference between the ends of the cylinder:

$$\underset{\sim}{h} \cdot \underset{\sim}{e}_{\langle r \rangle} \Big|_{r=a} = 0 \ , \tag{6.4}$$

$$\begin{aligned} \theta \Big|_{z=0} &= \theta_o \ , \\ \theta \Big|_{z=L} &= \theta_o + \epsilon_2 \theta_o \ . \end{aligned} \tag{6.5}$$

For this problem, we find that it is sufficient to seek displacement and temperature fields in the forms

$$\underset{\sim}{u} = u(r,z)\underset{\sim}{e}_{\langle r \rangle} + v(r,z)\underset{\sim}{e}_{\langle \phi \rangle} + w(r,z)\underset{\sim}{e}_{\langle z \rangle} \tag{6.6}$$

and

$$\theta = \theta(z) \ . \tag{6.7}$$

By assuming (6.7), we see that the condition (6.4) is automatically satisfied through first-order terms.

First-Order Solutions

We note that the second integral in (6.1) contributes no first-order moment. Thus, from (6.1)-(6.3), the mechanical boundary conditions on the first-order problems are

[10] We will set the arbitrary infinitesimal rigid body displacements that arise at each order equal to zero. This does affect the second-order stresses and may be interpreted as fixing the end $z = 0$ in some sense.

$$\int_R rS_1^{\langle z\phi\rangle}\Big|_{z=0,L} dA = T_z \ , \tag{6.8}$$

$$\int_R rS_2^{\langle z\phi\rangle}\Big|_{z=0,L} dA = 0 \ , \tag{6.9}$$

$$\int_R S_i^{\langle zz\rangle}\Big|_{z=0,L} dA = 0, \quad i = 1,2 \tag{6.10}$$

$$\underset{\sim}{S}_i \underset{\sim}{e}_{\langle r\rangle}\Big|_{r=a} = \underset{\sim}{0}, \quad i = 1,2 \ . \tag{6.11}$$

We get the first-order equilibrium equations

$$\nabla\cdot\underset{\sim}{S}_i = \underset{\sim}{0} \ , \qquad i = 1,2 \tag{6.12}$$

from $(3.8)_1$ and $(3.9)_1$, and, from (3.20),

$$\underset{\sim}{S}_i = p_i\underset{\sim}{1} + (a_1+a_2)(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_1) \ , \quad i = 1,2 \ . \tag{6.13}$$

By (3.16), the incompressibility conditions are

$$\nabla\cdot\underset{\sim}{u}_i = \mathrm{tr}\nabla\underset{\sim}{u}_i = 0 \ , \quad i = 1,2 \ . \tag{6.14}$$

Now, each of the first-order mechanical problems is the same as the classical torsion problem (see, e.g., [8]). Thus,

$$\begin{aligned} \underset{\sim}{u}_1 &= \gamma rz\underset{\sim}{e}_{\langle\phi\rangle} \ , \\ p_1 &\equiv 0 \ , \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} \underset{\sim}{u}_2 &\equiv \underset{\sim}{0} \ , \\ p_2 &\equiv 0 \ , \end{aligned} \tag{6.16}$$

where $\epsilon_1\gamma=\epsilon_1/L$ is the twist per unit length. From (6.13) and (6.15),

$$S_1^{\langle z\phi\rangle} = S_1^{\langle\phi z\rangle} = \gamma(a_1+a_2)r \tag{6.17}$$

are the only nonzero components of $\underset{\sim}{S}_1$; while from (6.13) and (6.16),

$$\underset{\sim}{S}_2 \equiv \underset{\sim}{0} \;. \tag{6.18}$$

Using the result (6.17) in (6.8), we find that

$$T_z = \gamma(a_1+a_2)J \;, \tag{6.19}$$

where $J = \pi a^4/2$ is the polar moment of inertia of the cross-section. By (6.19), the classical angle of twist ϵ_1 is related to the torque T_z through

$$\epsilon_1 = \frac{\epsilon_1 T_z L}{(a_1+a_2)J} \;. \tag{6.20}$$

For temperature fields of the form (6.7), we find, from $(3.8)_2$, $(3.9)_2$, (3.21), and (3.22), that the first-order energy equations reduce to

$$\frac{d^2}{dz^2}\,\theta_i = 0 \;, \qquad i = 1,2 \;. \tag{6.21}$$

From (6.5), the θ_i must satisfy

$$\begin{aligned} \theta_1\Big|_{z=0,L} &= 0 \;, \\ \theta_2\Big|_{z=0} &= 0 \;, \\ \theta_2\Big|_{z=L} &= \theta_o \;. \end{aligned} \tag{6.22}$$

Hence,

$$\theta_1 \equiv 0 \ , \tag{6.23}$$

$$\theta_2 = \frac{\theta_o}{L} z \ . \tag{6.24}$$

Second-Order Solutions

From $(6.15)_1$, $(6.16)_1$, and (6.7), we see that (6.4) is automatically satisfied through second-order terms.

With the first-order results (6.13) and (6.17), the second integral in (6.1) vanishes; and the mechanical boundary conditions (6.1)-(6.3) in the second-order theory become

$$\int_R r\Sigma^{\langle z\phi\rangle}\Big|_{z=0,L} dA = 0 \ , \tag{6.25}$$

$$\int_R \Sigma^{\langle zz\rangle}\Big|_{z=0,L} dA = 0 \ , \tag{6.26}$$

$$\underset{\sim}{\Sigma}\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=a} = \underset{\sim}{0} \ , \tag{6.27}$$

where $\underset{\sim}{\Sigma}$ can be $\underset{\sim}{S}_{11}$, $\underset{\sim}{S}_{22}$, or $\underset{\sim}{S}_{12}$.

We first consider the determination of $\underset{\sim}{u}_{11}$, $\underset{\sim}{S}_{11}$. Using the results (6.15) and (6.23) in $(3.23)_1$, the components of $\underset{\sim}{S}_{11}$ are seen to be

$$S_{11}^{\langle rr\rangle} = p_{11} + 2(a_1+a_2)\frac{\partial}{\partial r}u_{11} + (a_1+a_2)\gamma^2 z^2 ,$$

$$S_{11}^{\langle\phi\phi\rangle} = p_{11} + 2(a_1+a_2)\frac{u_{11}}{r} - a_2\gamma^2 r^2 + (a_1+a_2)\gamma^2 z^2 ,$$

$$S_{11}^{\langle zz\rangle} = p_{11} + 2(a_1+a_2)\frac{\partial}{\partial z}w_{11} - a_2\gamma^2 r^2 ,$$

$$S_{11}^{\langle r\phi\rangle} = S_{11}^{\langle\phi r\rangle} = (a_1+a_2)\left(\frac{\partial}{\partial r}v_{11} + \frac{v_{11}}{r}\right) , \qquad (6.28)$$

$$S_{11}^{\langle rz\rangle} = (a_1+a_2)\left(\frac{\partial}{\partial z}u_{11} + \frac{\partial}{\partial r}w_{11}\right) ,$$

$$S_{11}^{\langle zr\rangle} = (a_1+a_2)\left(\frac{\partial}{\partial z}u_{11} + \frac{\partial}{\partial r}w_{11}\right) + (a_1+a_2)\gamma^2 rz ,$$

$$S_{11}^{\langle\phi z\rangle} = S_{11}^{\langle z\phi\rangle} = (a_1+a_2)\frac{\partial}{\partial z}v_{11} .$$

By $(3.10)_1$ and (6.28), the equations of equilibrium are

$$\frac{\partial}{\partial r}p_{11} + 2(a_1+a_2)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_{11})\right]$$

$$+ (a_1+a_2)\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}u_{11} + \frac{\partial}{\partial r}w_{11}\right) + a_2\gamma^2 r = 0 ,$$

$$\frac{\partial^2}{\partial z^2}v_{11} + \frac{\partial^2}{\partial r^2}v_{11} + \frac{1}{r}\frac{\partial}{\partial r}v_{11} - \frac{v_{11}}{r^2} = 0 , \qquad (6.29)$$

$$\frac{\partial}{\partial z}p_{11} + 2(a_1+a_2)\frac{\partial^2}{\partial z^2}w_1$$

$$+ (a_1+a_2)\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial}{\partial z}u_{11} + \frac{\partial}{\partial r}w_{11}\right)\right]$$

$$+ 2(a_1+a_2)\gamma^2 z^2 = 0 ,$$

while, from (6.15), the incompressibility condition $(3.17)_1$ becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_{11}) + \frac{\partial}{\partial z} w_{11} = -\gamma^2 z^2 . \tag{6.30}$$

The problem (6.25)-(6.29) for $\underset{\sim}{u}_{11}$, $\underset{\sim}{S}_{11}$ is a special case of the isothermal problem treated recently by Chan and Carlson [9]. With the results given there, we find that

$$\begin{aligned} \underset{\sim}{u}_{11} &= -\frac{1}{2}\gamma^2 r(z^2+k)\underset{\sim}{e}_{\langle r\rangle} + \gamma^2 kz\underset{\sim}{e}_{\langle z\rangle} , \\ p_{11} &= -\frac{1}{2}\gamma^2 a_1(a^2-r^2) + \gamma^2(a_1+a_2)k , \end{aligned} \tag{6.31}$$

where

$$k = \frac{1}{12}\frac{(a_1+2a_2)}{(a_1+a_2)} a^2 . \tag{6.32}$$

Then, by (6.28) and (6.31), the stresses are

$$\begin{aligned} S_{11}^{\langle rr\rangle} &= -\frac{1}{2}\gamma^2 a_1(a^2-r^2) , \\ S_{11}^{\langle\phi\phi\rangle} &= -\frac{1}{2}\gamma^2 a_1(a^2-r^2) - a_2\gamma^2 r^2 , \\ S_{11}^{\langle zz\rangle} &= -\frac{1}{2}\gamma^2 a_1(a^2-r^2) + \frac{1}{4}\gamma^2(a_1+2a_2)a^2 - a_2\gamma^2 r^2 , \\ S_{11}^{\langle r\phi\rangle} &= S_{11}^{\langle\phi r\rangle} = 0 , \\ S_{11}^{\langle rz\rangle} &= -\gamma^2(a_1+a_2)rz , \\ S_{11}^{\langle zr\rangle} &= 0 , \\ S_{11}^{\langle\phi z\rangle} &= S_{11}^{\langle z\phi\rangle} = 0 . \end{aligned} \tag{6.33}$$

With (6.16), the second-order problem for $\underset{\sim}{u}_{22}$, $\underset{\sim}{S}_{22}$ reduces to the form of the first-order problem for $\underset{\sim}{u}_2$, $\underset{\sim}{S}_2$. Hence, as in (6.16) and (6.18),

$$\underset{\sim}{u}_{22} \equiv \underset{\sim}{0} \ , \tag{6.34}$$

$$\underset{\sim}{S}_{22} \equiv \underset{\sim}{0} \ . \tag{6.35}$$

Next we consider the problem for $\underset{\sim}{u}_{12}$, $\underset{\sim}{S}_{12}$. For the components of the second-order stress, $\underset{\sim}{S}_{12}$, we find from $(3.23)_3$ together with the results (6.15), (6.16), (6.23), and (6.24) that

$$\begin{aligned}
S_{12}^{\langle rr\rangle} &= p_{12} + 2(a_1+a_2)\frac{\partial}{\partial r}u_{12} \ , \\
S_{12}^{\langle \phi\phi\rangle} &= p_{12} + 2(a_1+a_2)\frac{u_{12}}{r} \ , \\
S_{12}^{\langle zz\rangle} &= p_{12} + 2(a_1+a_2)\frac{\partial}{\partial z}w_{12} \ , \\
S_{12}^{\langle r\phi\rangle} &= S_{12}^{\langle \phi r\rangle} = (a_1+a_2)\left(\frac{\partial}{\partial r}v_{12} + \frac{v_{12}}{r}\right) \ , \\
S_{12}^{\langle rz\rangle} &= S_{12}^{\langle zr\rangle} = (a_1+a_2)\left(\frac{\partial}{\partial z}u_{12} + \frac{\partial}{\partial r}w_{12}\right) \ , \\
S_{12}^{\langle \phi z\rangle} &= S_{12}^{\langle z\phi\rangle} = (a_1+a_2)\frac{\partial}{\partial z}v_{12} \\
&\qquad + (b_1+b_2)\frac{\gamma\theta_o}{L}rz \ .
\end{aligned} \tag{6.36}$$

By (6.36) and $(3.12)_1$, the equilibrium equations are

$$\frac{\partial}{\partial r} p_{12} + 2(a_1+a_2) \frac{\partial}{\partial r} [\frac{1}{r} \frac{\partial}{\partial r} (ru_{12})]$$

$$+ (a_1+a_2) \frac{\partial}{\partial z} (\frac{\partial}{\partial z} u_{12} + \frac{\partial}{\partial r} w_{12}) = 0 ,$$

$$\frac{\partial^2}{\partial z^2} v_{12} + \frac{\partial^2}{\partial r^2} v_{12} + \frac{1}{r} \frac{\partial}{\partial r} v_{12} - \frac{v_{12}}{r^2}$$

$$+ (b_1+b_2) \frac{\gamma\theta_o}{L} r = 0 , \tag{6.37}$$

$$\frac{\partial}{\partial z} p_{12} + 2(a_1+a_2) \frac{\partial^2}{\partial z^2} w_{12}$$

$$+ (a_1+a_2) \frac{1}{r} \frac{\partial}{\partial r} [r(\frac{\partial}{\partial z} u_{12} + \frac{\partial}{\partial r} w_{12})] = 0 .$$

Since the first-order displacement $\underset{\sim}{u}_2$ is zero, the incompressibility condition $(3.17)_3$ reduces to

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_{12}) + \frac{\partial}{\partial z} w_{12} = 0 . \tag{6.38}$$

The solution to the system (6.37), (6.38), which through (6.36) satisfies the boundary conditions (6.25)-(6.27), is

$$\underset{\sim}{u}_{12} = - \frac{(b_1+b_2)}{(a_1+a_2)} \frac{\gamma\theta_o}{2L} rz^2 \underset{\sim}{e}_{\langle\phi\rangle} ,$$

$$p_{12} = 0 . \tag{6.39}$$

Using (6.39) in (6.36), we find that

$$\underset{\sim}{S}_{12} \equiv \underset{\sim}{0} . \tag{6.40}$$

By (6.5), the boundary conditions on the second-order temperature fields are homogeneous in all cases. Taking into account the first-order solutions (6.15), (6.16), (6.23), and (6.24), the boundary-value problems for the temperature fields θ_{11} and θ_{12} lead to the trivial solutions:

$$\theta_{11}, \theta_{12} \equiv 0 \; . \tag{6.41}$$

For the second-order temperature field θ_{22}, we find from the first-order results (6.16) and (6.24), together with $(3.25)_2$ and $(3.11)_2$ that

$$\kappa_1 \frac{d^2}{dz^2} \theta_{22} + \kappa_2 \left(\frac{\theta_o}{L}\right)^2 = 0 \; . \tag{6.42}$$

The solution to (6.42) which satisfies homogeneous conditions on the ends, $z = 0$ and $z = L$, is

$$\theta_{22} = \frac{\kappa_2}{2\kappa_1} \left(\frac{\theta_o}{L}\right)^2 z(L-z) \; . \tag{6.43}$$

Through second-order terms, we find from (6.15), (6.16), (6.31), (6.34), and (6.39) that

$$\begin{aligned} \underset{\sim}{u} &= \epsilon_1[\gamma r z \underset{\sim}{e}_{\langle\phi\rangle}] + \epsilon_1^2[-\tfrac{1}{2}\gamma^2 r(z^2+k)\underset{\sim}{e}_{\langle r\rangle} + \gamma^2 k z \underset{\sim}{e}_{\langle z\rangle}] \\ &\quad + \epsilon_1\epsilon_2[-\frac{(b_1+b_2)}{(a_1+a_2)} \frac{\gamma\theta_o}{2L} r z^2 e_{\langle\phi\rangle}] \; , \end{aligned} \tag{6.44}$$

while from (6.23), (6.24), (6.41), and (6.43),

$$\theta = \theta_o + \epsilon_2[\frac{\theta_o}{L} z] + \epsilon_2^2[\frac{\kappa_2}{2\kappa_1} (\frac{\theta_o}{L})^2 z(L-z)] \ . \qquad (6.45)$$

Finally, from (6.17), (6.18), (6.33), (6.35), and (6.40), the components of the stress tensor are given, through second-order terms, by

$$
\begin{aligned}
S^{\langle rr \rangle} &= \epsilon_1^2[- \tfrac{1}{2} \gamma^2 a_1(a^2-r^2)] \ , \\
S^{\langle \phi\phi \rangle} &= \epsilon_1^2[- \tfrac{1}{2} \gamma^2 a_1(a^2-r^2) - a_2\gamma^2 r^2] \ , \\
S^{\langle zz \rangle} &= \epsilon_1^2[- \tfrac{1}{2} \gamma^2 a_1(a^2-r^2) + \tfrac{1}{4} \gamma^2(a_1+2a_2)a^2 \\
&\qquad - a_2\gamma^2 r^2] \ , \\
S^{\langle rz \rangle} &= \epsilon_1^2[- \gamma^2(a_1+a_2)rz] \ , \\
S^{\langle zr \rangle} &= 0 \ , \\
S^{\langle z\phi \rangle} &= S^{\langle \phi z \rangle} = \epsilon_1[\gamma(a_1+a_2)r] \ , \\
S^{\langle r\phi \rangle} &= S^{\langle \phi r \rangle} = 0 \ .
\end{aligned} \qquad (6.46)
$$

In (6.44)-(6.66), k is given by (6.32), and $\gamma = 1/L$.

We see, from (6.45) and (6.46), that the temperature is unaffected by the mechanical loading, while the stress is unaffected by the thermal loading. However, a second-order cross effect does appear in the displacement given by (6.44). This second-order twist provides a means for determining the constant $b_1 + b_2$. Similarly, a measurement of the elongation would give k; and, consequently, a_1 and

a_2 could be found through (6.32). Since this elongation occurs in the ϵ_1^2-term, k could be found without imposing the temperature gradient on the rod. Finally, it is clear from (6.45) that measuring the temperature profile along the rod would give the second-order conductivity κ_2. This could be done without twisting the rod.

Pipkin and Rivlin [10] have also considered axial heat conduction in a twisted and extended rod. They do not resort to approximate methods; however, their assumed deformation and temperature fields cannot be supported without body force, and they assume that the heat flux does not depend explicitly on the temperature.[11] Both of these important points are overlooked in the account given by Truesdell and Noll [1, §96].

[11] This would imply that our κ_2 vanish.

7. SPHERICAL SHELL WITH RADIAL MECHANICAL AND THERMAL LOADING

We consider here an incompressible, isotropic, homogeneous, spherical shell subjected to an internal pressure and a radial temperature gradient. For this problem, we will use spherical coordinates (r,ϕ,ψ) with the associated orthonormal basis being $\{\underset{\sim}{e}_{\langle r\rangle},\underset{\sim}{e}_{\langle\phi\rangle},\underset{\sim}{e}_{\langle\psi\rangle}\}$. In the reference configuration, the shell occupies the region: $a \leq r \leq b$, $0 \leq \phi \leq \pi$, $0 \leq \psi \leq 2\pi$. We assume that the shell is in equilibrium and that both body force and extrinsic heat supply are absent.

The mechanical loading is specified as follows:

$$\underset{\sim\sim}{S}\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=a} = -\sigma\underset{\sim}{e}_{\langle r\rangle} = -\epsilon_1(a_1 + a_2)\underset{\sim}{e}_{\langle r\rangle} \ , \tag{7.1}$$

$$\underset{\sim\sim}{S}\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=b} = \underset{\sim}{0} \ . \tag{7.2}$$

The radial thermal loading is achieved by setting

$$\theta\Big|_{r=a} = \theta_o \tag{7.3}$$

and

$$\theta\Big|_{r=b} = \theta_o + \epsilon_2\theta_o \ . \tag{7.4}$$

From the symmetry of this problem, we find that it is sufficient to consider displacement and temperature fields in the forms

$$\underset{\sim}{u} = u(r)\underset{\sim}{e}_{\langle r\rangle} \tag{7.5}$$

and

$$\theta = \theta(r) \ . \tag{7.6}$$

First-Order Solutions

From (7.5) and (3.20), the nonzero components of $\underset{\sim}{S}_i$ are

$$S_i^{\langle rr \rangle} = p_i + 2(a_1 + a_2) \frac{d}{dr} u_i \ , \qquad i = 1,2 \tag{7.7}$$
$$S_i^{\langle \phi\phi \rangle} = S_i^{\langle \psi\psi \rangle} = p_i + 2(a_1 + a_2) \frac{u_i}{r} \ .$$

Using (7.7), the displacement equations of equilibrium are found, from $(3.8)_1$ and $(3.9)_1$, to be

$$\frac{d}{dr} p_i + 2(a_1 + a_2)\left(\frac{d^2}{dr^2} u_i + \frac{2}{r}\frac{d}{dr} u_i - \frac{2}{r^2} u_i\right) = 0, \qquad i = 1,2 \ . \tag{7.8}$$

The incompressibility conditions (3.16) require

$$\frac{d}{dr} u_i + 2\frac{u_i}{r} = 0, \ i = 1,2 \ . \tag{7.9}$$

By (7.1), (7.2), and (7.7),

$$\left[p_1 + 2(a_1 + a_2)\frac{d}{dr} u_1\right]\Big|_{r=a} = -(a_1 + a_2) \ , \tag{7.10}$$

$$\left[p_2 + 2(a_1 + a_2)\frac{d}{dr} u_2\right]\Big|_{r=a} = 0 \ , \tag{7.11}$$

$$\left[p_i + 2(a_1 + a_2)\frac{d}{dr} u_i\right]\Big|_{r=b} = 0, \ i = 1,2 \ . \tag{7.12}$$

Thus, the first-order displacements[12] and pressures are

$$u_1 = \frac{k}{r^2} , \qquad p_1 = 4(a_1 + a_2)\frac{k}{b^3} , \tag{7.13}$$

and

$$u_2 \equiv 0 , \qquad p_2 \equiv 0 , \tag{7.14}$$

where

$$k = \frac{a^3 b^3}{4(b^3 - a^3)} . \tag{7.15}$$

From (7.5), (7.11), and (7.13),

$$S_1^{\langle rr\rangle} = 4(a_1 + a_2)\frac{k}{b^3}\left(1 - \frac{b^3}{r^3}\right) , \qquad S_1^{\langle\phi\phi\rangle} = S_1^{\langle\psi\psi\rangle} = 4(a_1 + a_2)\frac{k}{b^3}\left(1 + \frac{b^3}{2r^3}\right) , \tag{7.16}$$

and

$$\underset{\sim}{S}_2 \equiv \underset{\sim}{0} . \tag{7.17}$$

---

[12] We are taking the arbitrary infinitesimal rigid body displacements that arise here to be zero. This is evidently consistent with the Signorini compatibility condition (Truesdell and Noll [1]) in that we will be able to find the corresponding second-order solutions.

With (3.21), (3.22), and (7.6), the energy equations $(3.8)_2$ and $(3.9)_2$ reduce to

$$r \frac{d^2}{dr^2} \theta_i + 2 \frac{d}{dr} \theta_i = 0, \; i = 1,2 \; . \tag{7.18}$$

The boundary conditions (7.3) and (7.4) require

$$\begin{aligned} \theta_i \Big|_{r=a} &= 0, \; i = 1,2 \\ \theta_1 \Big|_{r=b} &= 0 \; , \\ \theta_2 \Big|_{r=b} &= \theta_o \; ; \end{aligned} \tag{7.19}$$

thus, we find that

$$\theta_1 \equiv 0 \; , \tag{7.20}$$

$$\theta_2 = \frac{b\theta_o}{(b-a)} \frac{(r-a)}{r} \; . \tag{7.21}$$

Second-Order Solutions

Each of the second-order solutions will be given only to within an infinitesimal rigid body displacement.

From $(3.23)_1$, (7.5), and (7.13),

$$\begin{aligned} S_{11}^{\langle rr \rangle} &= p_{11} + 2(a_1 + a_2) \frac{d}{dr} u_{11} + 8(a_1 + a_2) \frac{k^2}{b^3} \frac{1}{r^3} \\ &\quad - 4(a_1 + 5a_2) \, k^2 \frac{1}{r^6} \; , \\ S_{11}^{\langle \phi\phi \rangle} &= S_{11}^{\langle \psi\psi \rangle} = p_{11} + 2(a_1 + a_2) \frac{u_{11}}{r} \\ &\quad - 4(a_1 + a_2) \frac{k^2}{b^3} \frac{1}{r^3} - (a_1 + 5a_2) \, k^2 \frac{1}{r^6} \; , \end{aligned} \tag{7.22}$$

with all other components being zero. From $(3.10)_1$, and (7.22), the displacement equation of equilibrium, corresponding to ϵ_1^2, is

$$\frac{d}{dr} p_{11} + 2(a_1 + a_2)(\frac{d^2}{dr^2} u_{11} + \frac{2}{r}\frac{d}{dr} u_{11} - \frac{2}{r^2} u_{11})$$

$$+ 18(a_1 + 5a_2)k^2 \frac{1}{r^7} = 0 \ . \qquad (7.23)$$

From (7.13) and the incompressibility condition $(3.17)_1$, u_{11} must satisfy

$$\frac{d}{dr} u_{11} + \frac{2}{r} u_{11} = 3k^2 \frac{1}{r^6} \ . \qquad (7.24)$$

With (7.22), the boundary conditions (7.1) and (7.2) require

$$[p_{11} + 2(a_1 + a_2) \frac{d}{dr} u_{11} + 8(a_1 + a_2) \frac{k^2}{b^3} \frac{1}{r^3}$$

$$- 4(a_1 + 5a_2)k^2 \frac{1}{r^6}] \Big|_{r=a,b} = 0 \ . \qquad (7.25)$$

The system (7.23)-(7.25) has the solution

$$u_{11} = \frac{\alpha}{r^2} - \frac{k^2}{r^5} \ ,$$

$$p_{11} = k^2[\frac{(3a_1-a_2)}{a^3b^3} - 3\,\frac{(a_1-3a_2)}{r^6}] \ , \qquad (7.26)$$

where

$$\alpha = \frac{k^2}{4}\,\frac{(b^3+a^3)}{(a^3b^3)}\,\frac{(3a_1-a_2)}{(a_1+a_2)} + \frac{k^2}{2b^3} \ . \qquad (7.27)$$

The components of $\underset{\sim}{S}_{11}$ can be computed from (7.26) and (7.22).

Taking into account the result (7.14), the problem for $\underset{\sim}{u}_{22}$, $\underset{\sim}{S}_{22}$ leads to

$$\underset{\sim}{u}_{22} \equiv \underset{\sim}{0} \,, \tag{7.28}$$

$$\underset{\sim}{S}_{22} \equiv \underset{\sim}{0} \,. \tag{7.29}$$

Substituting the first-order solutions (7.13), (7.14), (7.20), and (7.21) into $(3.23)_3$, we find, with (7.5), that the nonzero components of $\underset{\sim}{S}_{12}$ are

$$S_{12}^{\langle rr\rangle} = p_{12} + 2(a_1+a_2)\frac{d}{dr}u_{12} - 4k\theta_o(b_1+b_2)\left(\frac{b}{b-a}\right)\left(\frac{1}{r^3} - \frac{a}{r^4}\right) \,, \tag{7.30}$$

$$S_{12}^{\langle\phi\phi\rangle} = S_{12}^{\langle\psi\psi\rangle} = p_{12} + 2(a_1+a_2)\frac{u_{12}}{r} + 2k\theta_o(b_1+b_2)\left(\frac{b}{b-a}\right)\left(\frac{1}{r^3} - \frac{a}{r^4}\right) \,.$$

Then by $(3.12)_1$, the equilibrium equation becomes

$$\frac{d}{dr}p_{12} + 2(a_1+a_2)\left(\frac{d^2}{dr^2}u_{12} + \frac{2}{r}\frac{d}{dr}u_{12} - \frac{2}{r^2}u_{12}\right) - 4k\theta_o(b_1+b_2)\left(\frac{b}{b-a}\right)\frac{a}{r^5} = 0 \,. \tag{7.31}$$

By (7.13) and (7.14), the incompressibility condition $(3.17)_3$ requires

$$\frac{d}{dr}u_{12} + \frac{2}{r}u_{12} = 0 \,. \tag{7.32}$$

With (7.30), the boundary conditions (7.1) and (7.2) become

$$\left[p_{12} + 2(a_1+a_2)\frac{d}{dr}u_{12} - 4k\theta_o(b_1+b_2)\left(\frac{b}{b-a}\right)\left(\frac{1}{r^3} - \frac{a}{r^4}\right)\right]\Big|_{r=a,b} = 0. \tag{7.33}$$

Equations (7.31)-(7.33) have the solution

$$u_{12} = \frac{\beta}{r^2}, \tag{7.34}$$

$$p_{12} = k\theta_o(b_1+b_2)\left(\frac{b}{b-a}\right)\left[\frac{3(b-a)}{b(b^3-a^3)} - \frac{a}{r^4}\right],$$

where

$$\beta = \frac{k\theta_o}{4}\,\frac{(b_1+b_2)}{(a_1+a_2)}\left(\frac{b}{b-a}\right)\left[\frac{3a^3(b-a)}{b(b^3-a^3)} - 1\right]. \tag{7.35}$$

The components of $\underset{\sim}{S}_{12}$ follow from (7.30) and (7.34).

With (7.20), the problem for the second-order temperature θ_{11} leads to

$$\theta_{11} \equiv 0 . \tag{7.36}$$

Using (7.6), (7.14), and (7.22) in $(3.11)_2$, $(3.24)_2$, and $(3.25)_2$, we find that the energy equation for θ_{22} becomes

$$\frac{d^2}{dr^2}\theta_{22} + \frac{2}{r}\frac{d}{dr}\theta_{22} + \frac{a\kappa_2}{\kappa_1}\left(\frac{b\theta_o}{b-a}\right)^2\frac{1}{r^4} = 0 . \tag{7.37}$$

By (7.3) and (7.4),

$$\theta_{22}\Big|_{r=a,b} = 0 ; \tag{7.38}$$

thus,

$$\theta_{22} = \frac{\kappa_2\theta_o^2 ab}{2\kappa_1(b-a)^2}\,\frac{(b-r)(r-a)}{r^2} \; . \tag{7.39}$$

From the first-order solutions (7.13), (7.14), (7.20), and (7.21), together with $(3.12)_2$, $(3.24)_3$, $(3.25)_3$, and (7.6), we find that

$$\frac{d^2}{dr^2}\,\theta_{12} + \frac{2}{r}\,\frac{d}{dr}\,\theta_{12} + \frac{12\kappa_3 ab\theta_o k}{\kappa_1(b-a)}\,\frac{1}{r^6} = 0 \; . \tag{7.40}$$

By (7.3) and (7.4),

$$\theta_{12}\Big|_{r=a,b} = 0 \; . \tag{7.41}$$

Equations (7.40) and (7.41) imply that

$$\theta_{12} = \frac{\kappa_3\theta_o k}{\kappa_1 ab(b-a)}\,\frac{(r^2-a^2)(r^2-b^2)}{r^4} \; . \tag{7.42}$$

Hence, from (7.13), (7.14), (7.26), (7.28), and (7.34), we find that, through second-order terms,

$$u = \epsilon_1\left[\frac{k}{r^2}\right] + \epsilon_1^2\left[\frac{\alpha}{r^2} - \frac{k^2}{r^5}\right] + \epsilon_1\epsilon_2\left[\frac{\beta}{r^2}\right] \; , \tag{7.43}$$

while, from (7.20), (7.21), (7.36), (7.39), and (7.42),

$$\theta = \epsilon_2\left[\frac{b\theta_o}{(b-a)}\,\frac{(r-a)}{r}\right] + \epsilon_2^2\left[\frac{\kappa_2\theta_o^2 ab}{2\kappa_1(b-a)^2}\,\frac{(b-r)(r-a)}{r^2}\right]$$

$$+\ \epsilon_1\epsilon_2\left[\frac{\kappa_3\theta_o k}{\kappa_1 ab(b-a)}\,\frac{(r^2-a^2)(r^2-b^2)}{r^4}\right] \; . \tag{7.44}$$

The constants k, α, and β in (7.43) and (7.44) are given by (7.15), (7.27), and (7.35), respectively.

It should be noted from (7.43) and (7.44) that, in contrast to the problems of Sections 5 and 6, both the displacement and temperature fields include second-order cross effects. Thus, the thermal loading affects the displacement field, while the mechanical loading affects the temperature field.

Of particular interest is the second-order conductivity κ_3 which shows up in (7.44). In principle, κ_3, as well as the other material constants, could be determined from the above solution. However, the geometry of this problem makes it an unlikely candidate for use in experimental work. The example which follows exhibits the same coupling of thermal and mechanical effects, and it is more likely to be amenable to investigation in the laboratory.

From (7.1) and (7.43), the internal pressure per unit area of the deformed body is $\epsilon_1[(a_1+a_2)] + \epsilon_1^2[-\frac{2k}{a^3}(a_1+a_2)]$ through second-order terms. Due to the uncoupling induced by the assumptions of incompressibility and isotropy, there is no chance of ϵ_2 being introduced into the traction boundary conditions as we transform from the Piola-Kirchhoff stress to the Cauchy stress. However, in less symmetrical problems, boundary conditions like (7.1) might not correspond to pressure loadings in the usual sense.

8. SOLID CYLINDER WITH AXIAL MECHANICAL AND THERMAL LOADING

We will work with the physical components on the field quantities relative to a cylindrical coordinate system (r,ϕ,z), where the associated orthonormal basis is $\{\underset{\sim}{e}_{\langle r\rangle},\underset{\sim}{e}_{\langle\phi\rangle},\underset{\sim}{e}_{\langle z\rangle}\}$. In the reference configuration, the cylinder occupies the region: $0 \leq r \leq a$, $0 \leq \phi \leq 2\pi$, $0 \leq z \leq L$. As before, the body is assumed to be incompressible, isotropic, and homogeneous; and both body force and extrinsic heat supply are taken to be zero. Furthermore, we assume that the cylinder is in equilibrium.

On the ends of the cylinder, we prescribe a first-order axial stress while assuming that the shear stress vanishes there. The lateral surface is taken to be free of stress. Thus, the mechanical boundary conditions are

$$\underset{\sim\sim}{S}\underset{\sim}{e}_{\langle z\rangle}\Big|_{z=0,L} = \sigma\underset{\sim}{e}_{\langle z\rangle} = \epsilon_1(a_1 + a_2)\underset{\sim}{e}_{\langle z\rangle}\ , \qquad (8.1)$$

$$\underset{\sim\sim}{S}\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=a} = \underset{\sim}{0}\ . \qquad (8.2)$$

The axial thermal loading is achieved by specifying a first-order temperature difference between the ends of the cylinder and taking the lateral surface to be insulated. Accordingly, we set

$$\theta\Big|_{z=0} = \theta_o\ , \qquad (8.3)$$

$$\theta\Big|_{z=L} = \theta_o + \epsilon_2\theta_o\ , \qquad (8.4)$$

and

$$\underset{\sim}{h}\cdot\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=a} = 0 \ . \tag{8.5}$$

We will find displacement and temperature fields in the forms

$$\underset{\sim}{u} = u(r,z)\underset{\sim}{e}_{\langle r\rangle} + w(r,z)\underset{\sim}{e}_{\langle z\rangle} \tag{8.6}$$

and

$$\theta = \theta(z) \ . \tag{8.7}$$

With (8.7), the boundary condition (8.5) is automatically satisfied through first-order terms.

First-Order Solutions

From (8.1) and (8.2), the mechanical boundary conditions on the first-order mechanical problems are

$$\underset{\sim}{S}_1\underset{\sim}{e}_{\langle z\rangle}\Big|_{z=0,L} = (a_1 + a_2)\underset{\sim}{e}_{\langle z\rangle} \ , \tag{8.8}$$

$$\underset{\sim}{S}_2\underset{\sim}{e}_{\langle z\rangle}\Big|_{z=0,L} = \underset{\sim}{0} \ , \tag{8.9}$$

and

$$\underset{\sim}{S}_i\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=a} = 0,\ i = 1,2 \ . \tag{8.10}$$

The first-order equations of equilibrium,

$$\nabla\cdot\underset{\sim}{S}_i = \underset{\sim}{0},\ i = 1,2 \tag{8.11}$$

follow from $(3.8)_1$ and $(3.9)_1$; and from (3.20),

$$\underset{\sim}{S}_i = p_i\underset{\sim}{1} + (a_1 + a_2)(\nabla\underset{\sim}{u}_i^T + \nabla\underset{\sim}{u}_i),\ i = 1,2 \ . \tag{8.12}$$

The conditions of incompressibility (3.16) can be written as

$$\nabla\cdot\underset{\sim}{u}_i = 0, \; i = 1,2 \; . \tag{8.13}$$

These first-order problems are familiar from the linear isothermal theory; thus,[13]

$$\begin{aligned} \underset{\sim}{u}_1 &= -\frac{1}{6}\, r\underset{\sim}{e}_{\langle r\rangle} + \frac{1}{3}\, z\underset{\sim}{e}_{\langle z\rangle} \; , \\ p_1 &= \frac{1}{3}\,(a_1 + a_2) \; , \end{aligned} \tag{8.14}$$

and

$$\begin{aligned} \underset{\sim}{u}_2 &\equiv \underset{\sim}{0} \; , \\ p_2 &\equiv \underset{\sim}{0} \; . \end{aligned} \tag{8.15}$$

From (8.12) and (8.14), we find that the only nonzero component of $\underset{\sim}{S}_1$ is

$$S_1^{\langle zz\rangle} = (a_1 + a_2) \; , \tag{8.16}$$

while, by (8.12) and (8.15),

$$\underset{\sim}{S}_2 \equiv \underset{\sim}{0} \; . \tag{8.17}$$

The differential equations and boundary conditions which determine the first-order temperatures θ_i are the same as those of the torsion problem in Section 6, viz., (6.21) and (6.22). Thus, from (6.23) and (6.24),

$$\theta_1 \equiv 0 \; , \tag{8.18}$$

$$\theta_2 = \frac{\theta_o}{L}\, z \; . \tag{8.19}$$

[13] Here, as in Section 7, we can take the first-order infinitesimal rigid body displacements to be zero and still solve the second-order problems.

Second-Order Solutions

From $(8.14)_1$, $(8.15)_1$, and (8.7), we see that (8.5) is automatically satisfied through second-order therms. As in Section 7, each second-order solution will be given only to within an infinitesimal rigid body displacement.

With the first-order solutions (8.14) and (8.15), we find from (8.1) and (8.2) that the mechanical boundary conditions on the second-order problems are

$$\underset{\sim}{\Sigma}\underset{\sim}{e}_{\langle z\rangle}\Big|_{z=0,L} = \underset{\sim}{0} , \tag{8.20}$$

$$\underset{\sim}{\Sigma}\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=a} = \underset{\sim}{0} , \tag{8.21}$$

where $\underset{\sim}{\Sigma}$ can be either $\underset{\sim}{S}_{11}$, $\underset{\sim}{S}_{22}$, or $\underset{\sim}{S}_{12}$.

First we will find $\underset{\sim}{u}_{11}$, $\underset{\sim}{S}_{11}$. Taking into account (8.14) and (8.18), $(3.23)_1$ reduces to

$$\begin{aligned}\underset{\sim}{S}_{11} &= p_{11}\underset{\sim}{1} + (a_1+a_2)(\nabla\underset{\sim}{u}_1^T+\nabla\underset{\sim}{u}_{11}) \\ &\qquad - p_1\nabla\underset{\sim}{u}_1 - (a_1+5a_2)(\nabla\underset{\sim}{u}_1)^2 .\end{aligned} \tag{8.22}$$

The last two terms in (8.22) are tensors with constant components; therefore, by $(3.10)_1$, the displacement equations of equilibrium are

$$\nabla\cdot(p_{11}\underset{\sim}{1}) + (a_1+a_2)(\nabla^2\underset{\sim}{u}_{11}+\nabla\nabla\cdot\underset{\sim}{u}_{11}) = \underset{\sim}{0} . \tag{8.23}$$

With (8.14), the equation of incompressibility $(3.17)_1$ becomes

$$\nabla\cdot\underset{\sim}{u}_{11} = \frac{1}{12} . \tag{8.24}$$

The solution to (8.23) and (8.24) which through (8.22) meets the conditions (8.20) and (8.21) is

$$\underset{\sim}{u}_{11} = -\frac{(a_1+5a_2)}{72(a_1+a_2)}\, r\underset{\sim}{e}_{\langle r\rangle} + \frac{(a_1+2a_2)}{9(a_1+a_2)}\, z\underset{\sim}{e}_{\langle z\rangle}\,,$$

$$p_{11} = \frac{2}{9}\, a_2\,. \tag{8.25}$$

Substituting the solutions (8.14) and (8.25) into (8.22), we find that

$$\underset{\sim}{S}_{11} \equiv \underset{\sim}{0}\,. \tag{8.26}$$

Since $\underset{\sim}{u}_2$ and $\underset{\sim}{S}_2$ vanish identically by (8.15) and (8.17), the second-order problem for $\underset{\sim}{u}_{22}$ and $\underset{\sim}{S}_{22}$ leads to

$$\underset{\sim}{u}_{22} \equiv \underset{\sim}{0}\,, \tag{8.27}$$

$$\underset{\sim}{S}_{22} \equiv \underset{\sim}{0}\,. \tag{8.28}$$

Next, we will find $\underset{\sim}{u}_{12}$ and $\underset{\sim}{S}_{12}$. Using (8.14), (8.15), (8.18), and (8.19) in $(3.23)_3$, we get

$$\begin{aligned}
S_{12}^{\langle rr\rangle} &= p_{12} + 2(a_1+a_2)\frac{\partial}{\partial r}u_{12} - (b_1+b_2)\frac{\theta_o}{3L}z\,,\\
S_{12}^{\langle\phi\phi\rangle} &= p_{12} + 2(a_1+a_2)\frac{u_{12}}{r} - (b_1+b_2)\frac{\theta_o}{3L}z\,,\\
S_{12}^{\langle zz\rangle} &= p_{12} + 2(a_1+a_2)\frac{\partial}{\partial z}w_{12} + 2(b_1+b_2)\frac{\theta_o}{3L}z\,,\\
S_{12}^{\langle rz\rangle} &= S_{12}^{\langle zr\rangle} = (a_1+a_2)\left(\frac{\partial}{\partial z}u_{12} + \frac{\partial}{\partial r}w_{12}\right),\\
S_{12}^{\langle r\phi\rangle} &= S_{12}^{\langle\phi r\rangle} = 0\,,\\
S_{12}^{\langle\phi z\rangle} &= S_{12}^{\langle z\phi\rangle} = 0\,.
\end{aligned} \tag{8.29}$$

From $(3.12)_1$ and (8.29), the displacement equations of equilibrium become

$$\frac{\partial}{\partial r} p_{12} + 2(a_1+a_2) \frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial}{\partial r} (ru_{12})\right] + (a_1+a_2) \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) = 0 \ ,$$

$$\frac{\partial}{\partial z} p_{12} + 2(a_1+a_2) \frac{\partial^2}{\partial z^2} w_{12} + (a_1+a_2) \frac{1}{r} \frac{\partial}{\partial r}\left[r\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right] + 2(b_1+b_2) \frac{\theta_o}{3L} = 0 \ . \tag{8.30}$$

By $(3.17)_3$ and (8.15), incompressibility requires

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_{12}) + \frac{\partial}{\partial z} w_{12} = 0 \ . \tag{8.31}$$

Seeking the components of displacement as power series in r and z, we find after a lengthy calculation that

$$\underset{\sim}{u}_{12} = \frac{(b_1+b_2)\theta_o}{6(a_1+a_2)L} rz\underset{\sim}{e}_{\langle r\rangle} - \frac{(b_1+b_2)\theta_o}{6(a_1+a_2)L} \left(z^2 - \frac{1}{2} r^2\right)\underset{\sim}{e}_{\langle z\rangle} \ ,$$

$$p_{12} \equiv 0 \tag{8.32}$$

satisfies (8.30) and (8.31) together with the boundary conditions (8.20) and (8.21). Through (8.29), the solution (8.32) leads to

$$\underset{\sim}{S}_{12} \equiv 0 \ . \tag{8.33}$$

With (8.18), the second-order problem for θ_{11} reduces in form to that of the first-order problem for θ_1. Thus,

$$\theta_{11} \equiv 0 \ . \tag{8.34}$$

Using (8.15) and (8.19) together with $(3.24)_2$ and $(3.25)_2$, the energy equation $(3.11)_2$ becomes

$$\kappa_1 \frac{d^2}{dz^2} \theta_{22} + \kappa_2 \left(\frac{\theta_o}{L}\right)^2 = 0 \ . \tag{8.35}$$

By (8.3) and (8.4), θ_{22} must vanish on the ends $z = 0$ and $z = L$. Thus,

$$\theta_{22} = \frac{\kappa_2}{2\kappa_1} \left(\frac{\theta_o}{L}\right)^2 z(L-z) \ . \tag{8.36}$$

We find, from $(3.12)_2$, $(3.24)_3$, $(3.25)_3$, and the first-order results (8.14), (8.15), (8.18), and (8.19), that

$$\kappa_1 \frac{d^2}{dz^2} \theta_{12} + 2\kappa_3 \left(\frac{\theta_o}{3L}\right) z = 0 \ . \tag{8.37}$$

By (8.3) and (8.4), θ_{12} must vanish at $z = 0$ and $z = L$. Consequently,

$$\theta_{12} = \frac{\kappa_3 \theta_o}{9\kappa_1 L} z(L^2 - z^2) \ . \tag{8.38}$$

In summary, through second-order terms, we have, from (8.14), (8.15), (8.25), (8.27), and (8.32), that

$$\underset{\sim}{u} = \epsilon_1[-\frac{1}{6}\, r\underset{\sim}{e}_{\langle r\rangle} + \frac{1}{3}\, z\underset{\sim}{e}_{\langle z\rangle}]$$

$$+ \epsilon_1^2[-\frac{(a_1+5a_2)}{72(a_1+a_2)}\, r\underset{\sim}{e}_{\langle r\rangle} + \frac{(a_1+2a_2)}{9(a_1+a_2)}\, z\underset{\sim}{e}_{\langle z\rangle}]$$

$$+ \epsilon_1\epsilon_2[\frac{(b_1+b_2)\theta_o}{6(a_1+a_2)L}\, \{rz\underset{\sim}{e}_{\langle r\rangle} + (\frac{1}{2}\, r^2-z^2)\underset{\sim}{e}_{\langle z\rangle}\}] \,, \tag{8.39}$$

while, from (8.18), (8.19), (8.34), (8.36), and (8.38),

$$\theta = \theta_o + \epsilon_2[\frac{\theta_o}{L}\, z] + \epsilon_2^2[\frac{K_2}{2K_1}\, (\frac{\theta_o}{L})^2 z(L-z)]$$

$$+ \epsilon_1\epsilon_2[\frac{K_3\theta_o}{9K_1 L}\, z(L^2-z^2)] \,. \tag{8.40}$$

Finally, we see, from (8.26), (8.28), and (8.33), that the second-order stresses all vanish for this problem. In fact, from (8.16) and (8.17), the only component of stress, through second-order terms, which does not vanish is

$$S^{\langle zz\rangle} = \epsilon_1(a_1+a_2) \,. \tag{8.41}$$

Of course, the Cauchy stress, which can be calculated from (2.9), will not be this simple. In any case, the traction vector vanishes on the lateral surface, and the stress distribution on the ends is statically equivalent to an axial force.

The presence of the $\epsilon_1\epsilon_2$-terms in (8.39) and (8.40) makes it clear that the displacement and temperature fields are each coupled with respect to the effects of thermal and mechanical loading on the body.

Using (8.39), the constants a_1 and a_2 could be determined from an isothermal tension test. Then, imposing an axial temperature gradient would yield (b_1+b_2). The second-order conductivity κ_2 could be found from (8.40) by measuring the temperature profile along the cylinder in the absence of mechanical loading, i.e., when $\epsilon_1 = 0$, $\epsilon_2 \neq 0$. A similar measurement under combined loading would then determine κ_3.

Petroski and Carlson [11] have given an exact solution of the present problem for conductors whose heat flux does not depend explicitly on the temperature. This would imply that the second-order conductivity κ_2 vanish.

9. CONSTRAINED ANNULAR CYLINDER WITH INTERNAL PRESSURE AND RADIAL THERMAL LOADING

Here, we consider a variation of the annular cylinder problem treated in Section 5. In the present problem, both the mechanical and thermal loading are in the radial direction. Again, we will be working with physical components of the field quantities relative to a cylindrical coordinate system (r,ϕ,z). The body is assumed to be in mechanical and thermal equilibrium with both body force and extrinsic heat supply absent. In the reference configuration, the body occupies the region: $a \leq r \leq b$, $0 \leq \phi \leq 2\pi$, $0 \leq z \leq L$.

A first-order pressure is prescribed on the inner lateral surface, while the outer lateral surface is free of stress. We assume that, on the ends of the cylinder, axial motion is prevented and the shear stress is zero. Thus, the mechanical boundary conditions are

$$\underset{\sim\sim}{S}\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=a} = -\sigma\underset{\sim}{e}_{\langle r\rangle} = -\epsilon_1(a_1+a_2)\underset{\sim}{e}_{\langle r\rangle}\ , \tag{9.1}$$

$$\underset{\sim\sim}{S}\underset{\sim}{e}_{\langle r\rangle}\Big|_{r=b} = \underset{\sim}{0}\ , \tag{9.2}$$

$$(\underset{\sim\sim}{S}\underset{\sim}{e}_{\langle z\rangle})\cdot\underset{\sim}{e}_{\langle r\rangle}\Big|_{z=0,L} = 0\ , \tag{9.3}$$

$$(\underset{\sim\sim}{S}\underset{\sim}{e}_{\langle z\rangle})\cdot\underset{\sim}{e}_{\langle\phi\rangle}\Big|_{z=0,L} = 0\ , \tag{9.4}$$

and

$$\underset{\sim}{u}\cdot\underset{\sim}{e}_{\langle z\rangle}\Big|_{z=0,L} = 0\ . \tag{9.5}$$

A first-order temperature difference is specified between the lateral surfaces, and the ends are assumed to be insulated. Accordingly, we set

$$\theta \Big|_{r=a} = \theta_o \ , \tag{9.6}$$

$$\theta \Big|_{r=b} = \theta_o + \epsilon_2 \theta_o \ , \tag{9.7}$$

and

$$\underset{\sim}{h} \cdot \underset{\sim}{e}_{\langle z \rangle} \Big|_{z=0,L} = 0 \ . \tag{9.8}$$

To get terms through second-order, we find that it is sufficient to consider displacement and temperature fields in the forms

$$\underset{\sim}{u} = u(r) \underset{\sim}{e}_{\langle r \rangle} \tag{9.9}$$

and

$$\theta = \theta(r) \ . \tag{9.10}$$

We note that, with (9.9) and (9.10), the boundary conditions (9.3)-(9.5) and (9.8) are automatically satisfied.

First-Order Solutions

From (3.20) and (9.9),

$$\left.\begin{aligned} S_i^{\langle rr \rangle} &= p_i + 2(a_1+a_2) \frac{d}{dr} u_i \ , \\ S_i^{\langle \phi\phi \rangle} &= p_i + 2(a_1+a_2) \frac{u_i}{r} \ , \\ S_i^{\langle zz \rangle} &= p_i \ , \end{aligned}\right\} \quad i = 1,2 \tag{9.11}$$

with all other components being zero. With $(3.8)_1$, $(3.9)_1$, and (9.11), the displacement equations of equilibrium become

$$\frac{d}{dr} p_i + 2(a_1+a_2) \frac{d}{dr} [\frac{1}{r} \frac{d}{dr} (ru_i)] = 0, \; i = 1,2 \; . \tag{9.12}$$

From (3.16), incompressibility requires that

$$\frac{d}{dr} u_i + \frac{u_i}{r} = 0, \; i = 1,2 \; . \tag{9.13}$$

By (9.1), (9.2), and (9.11),

$$\begin{aligned} &[p_1 + 2(a_1+a_2) \frac{d}{dr} u_1] \Big|_{r=a} = -(a_1+a_2) \; , \\ &[p_1 + 2(a_1+a_2) \frac{d}{dr} u_1] \Big|_{r=b} = 0 \; , \\ &[p_i + 2(a_1+a_2) \frac{d}{dr} u_2] \Big|_{r=a,b} = 0 \; . \end{aligned} \tag{9.14}$$

The solutions to (9.12)-(9.13) meeting (9.14) are

$$\begin{aligned} u_1 &= \frac{k}{r} \; , \\ p_1 &= 2(a_1+a_2) \frac{k}{b^2} \; , \end{aligned} \tag{9.15}$$

and

$$\begin{aligned} u_2 &\equiv 0 \; , \\ p_2 &\equiv 0 \; , \end{aligned} \tag{9.16}$$

where

$$k = \frac{a^2 b^2}{2(b^2-a^2)} \; . \tag{9.17}$$

From (9.11), (9.15), and (9.16), the nonzero components of $\underset{\sim}{S}_1$ are

$$\begin{aligned} S_1^{\langle rr\rangle} &= 2(a_1+a_2)\,\frac{k}{b^2}\,\left(\frac{r^2-b^2}{r^2}\right) , \\ S_1^{\langle\phi\phi\rangle} &= 2(a_1+a_2)\,\frac{k}{b^2}\,\left(\frac{r^2+b^2}{r^2}\right) , \\ S_1^{\langle zz\rangle} &= 2(a_1+a_2)\,\frac{k}{b^2} ; \end{aligned} \tag{9.18}$$

and

$$\underset{\sim}{S}_2 \equiv \underset{\sim}{0} . \tag{9.19}$$

The problems for the first-order temperature fields θ_i are the same as the corresponding problems of Section 5. Thus, from (5.20) and (5.21),

$$\theta_1 \equiv 0 , \tag{9.20}$$

$$\theta_2 = \frac{\theta_o}{\ell n(b/a)}\,\ell n(r/a) . \tag{9.21}$$

Second-Order Solutions

When (9.15) and (9.20) are substituted into $(3.23)_1$, we find that

$$\begin{aligned} S_{11}^{\langle rr\rangle} &= p_{11} + 2(a_1+a_2)\,\frac{d}{dr}\,u_{11} + k^2\left[\frac{2(a_1+a_2)}{b^2r^2} - \frac{(a_1+5a_2)}{r^4}\right] , \\ S_{11}^{\langle\phi\phi\rangle} &= p_{11} + 2(a_1+a_2)\,\frac{u_{11}}{r} \\ &\qquad - k^2\left[\frac{2(a_1+a_2)}{b^2r^2} + \frac{(a_1+5a_2)}{r^4}\right] , \\ S_{11}^{\langle zz\rangle} &= p_{11} . \end{aligned} \tag{9.22}$$

All other components of $\underset{\sim}{S}_{11}$ are zero. From $(3.10)_1$ and (9.22), we get the equilibrium equation

$$\frac{d}{dr} p_{11} + 2(a_1+a_2) \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (ru_{11})\right] + 4(a_1+5a_2) \frac{k^2}{r^5} = 0 . \tag{9.23}$$

With (9.15), the incompressibility condition $(3.17)_1$ requires that

$$\frac{1}{r} \frac{d}{dr} (ru_{11}) = \frac{k^2}{r^4} . \tag{9.24}$$

Using (9.22), the boundary conditions (9.1) and (9.2) become

$$\left\{p_{11} + 2(a_1+a_2) \frac{d}{dr} u_{11} + k^2\left[\frac{2(a_1+a_2)}{b^2 r^2} - \frac{(a_1+5a_2)}{r^4}\right]\right\} \Big|_{r=a,b} = 0 . \tag{9.25}$$

The solution to (9.23)-(9.25) is

$$u_{11} = \frac{1}{2} k^2 \left(\frac{b^2+3a^2}{a^2 b^2 r} - \frac{1}{r^3}\right) ,$$
$$p_{11} = (a_1+a_2) \frac{k^2}{a^2 b^2} - (a_1-3a_2) \frac{k^2}{r^4} . \tag{9.26}$$

By (9.22) and (9.26),

$$S_{11}^{\langle rr\rangle} = (a_1+a_2)\left(\frac{k}{ab}\right)^2\left[\frac{r^4-r^2(a^2+b^2)+a^2b^2}{r^4}\right] ,$$
$$S_{11}^{\langle \phi\phi\rangle} = (a_1+a_2)\left(\frac{k}{ab}\right)^2\left[\frac{r^4+r^2(a^2+b^2)-3a^2b^2}{r^4}\right] , \tag{9.27}$$
$$S_{11}^{\langle zz\rangle} = (a_1+a_2)k^2 \frac{1}{a^2b^2} - (a_1-3a_2)k^2 \frac{1}{r^4} .$$

With (9.16), the problem for the second-order fields $\underset{\sim}{u}_{22}$, $\underset{\sim}{S}_{22}$ leads to

$$\underset{\sim}{u}_{22} \equiv \underset{\sim}{0} \,, \qquad (9.28)$$

$$\underset{\sim}{S}_{22} \equiv \underset{\sim}{0} \,. \qquad (9.29)$$

The components of the second-order stress $\underset{\sim}{S}_{12}$ are given by $(3.23)_3$. Substituting from (9.15), (9.16), (9.20), and (9.21), we get

$$S_{12}^{\langle rr \rangle} = p_{12} + 2(a_1+a_2) \frac{d}{dr} u_{12} - 2(b_1+b_2) \frac{k\theta_o}{\ell n(b/a)} \frac{\ell n(r/a)}{r^2} \,,$$

$$S_{12}^{\langle \phi\phi \rangle} = p_{12} + 2(a_1+a_2) \frac{u_{12}}{r} + 2(b_1+b_2) \frac{k\theta_o}{\ell n(b/a)} \frac{\ell n(r/a)}{r^2} \,, \qquad (9.30)$$

$$S_{12}^{\langle zz \rangle} = p_{12}$$

with all other components being zero. From $(3.12)_1$ and (9.30), we have the single equilibrium equation

$$\frac{d}{dr} p_{12} + 2(a_1+a_2) \frac{d}{dr} [\frac{1}{r} \frac{d}{dr} (ru_{12})]$$

$$- 2(b_1+b_2) \frac{k\theta_o}{\ell n(b/a)} \frac{1}{r^3} = 0 \,. \qquad (9.31)$$

With (9.16), the incompressibility condition $(3.17)_3$ reduces to

$$\frac{1}{r} \frac{d}{dr} (ru_{12}) = 0 \,. \qquad (9.32)$$

Using (9.30), the boundary conditions (9.1) and (9.2) require

$$[p_{12} + 2(a_1+a_2) \frac{d}{dr} u_{12} - 2(b_1+b_2) \frac{k\,\theta_o}{\ell n(b/a)} \frac{\ell n(r/a)}{r^2}] \Big|_{r=a,b} = 0 \,.$$

(9.33)

The solution to (9.31)-(9.32) which satisfies (9.33) is

$$u_{12} = \frac{(b_1+b_2)}{(a_1+a_2)} \frac{k\theta_o}{\ell n(b/a)} \left[\left(\frac{a^2}{b^2-a^2}\right) \ell n(b/a) - \frac{1}{4}\right]\frac{1}{r} ,$$

$$p_{12} = (b_1+b_2) \frac{k\theta_o}{\ell n(b/a)} \left[\left(\frac{2}{b^2-a^2}\right) \ell n(b/a) - \frac{1}{2r^2}\right] . \tag{9.34}$$

Through (9.30), (9.34) generates the stresses

$$S_{12}^{\langle rr\rangle} = 2(b_1+b_2) \frac{k\theta_o}{\ell n(b/a)} \left[\frac{\ell n(b/a)}{b^2-a^2}\left(1 - \frac{a^2}{r^2}\right) - \frac{\ell n(r/a)}{r^2}\right],$$

$$S_{12}^{\langle\phi\phi\rangle} = 2(b_1+b_2) \frac{k\theta_o}{\ell n(b/a)} \left[\frac{\ell n(b/a)}{b^2-a^2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\ell n(r/a)}{r^2}\right], \tag{9.35}$$

$$S_{12}^{\langle zz\rangle} = (b_1+b_2) \frac{k\theta_o}{\ell n(b/a)} \left[\left(\frac{2}{b^2-a^2}\right) \ell n(b/a) - \frac{1}{2r^2}\right] .$$

We turn now to the second-order temperature fields. The problems for θ_{11} and θ_{22} are the same as the corresponding problems in Section 5. Thus, from (5.37),

$$\theta_{11} \equiv 0 , \tag{9.36}$$

while, from (5.40),

$$\theta_{22} = \frac{\kappa_2}{2\kappa_1} \left[\frac{\theta_o}{\ell n(b/a)}\right]^2 \ell n\left(\frac{r}{b}\right) \ell n\left(\frac{a}{r}\right) . \tag{9.37}$$

Using (9.15), (9.16), (9.20), and (9.21), together with $(3.24)_3$ and $(3.25)_3$, in $(3.12)_2$ we get the following differential equation for θ_{12}:

$$\frac{d^2}{dr^2}\,\theta_{12} + \frac{1}{r}\frac{d}{dr}\,\theta_{12} + \frac{4\kappa_3}{\kappa_1}\,\frac{k\theta_o}{\ell n(b/a)} = 0\ . \tag{9.38}$$

By (9.6) and (9.7), we must have

$$\theta_{12}\Big|_{r=a,b} = 0\ ; \tag{9.39}$$

thus,

$$\theta_{12} = -\frac{\kappa_3}{\kappa_1}\,\frac{k\theta_o}{\ell n(b/a)}\,\left[\frac{1}{r^2} + \left(\frac{b^2-a^2}{a^2b^2}\right)\frac{\ell n(r/\delta)}{\ell n(b/a)}\right]\ , \tag{9.40}$$

where

$$\delta = b\left(\frac{b}{a}\right)^{\frac{a^2}{b^2-a^2}}\ . \tag{9.41}$$

Finally, we summarize the displacement, temperature, and stress fields, through second-order terms. From (9.15), (9.16), (9.26), (9.28), and (9.34),

$$\begin{aligned} u = {} & \epsilon_1\left[\frac{k}{r}\right] + \epsilon_1^2\left[\frac{1}{2}\,k^2\left(\frac{b^2+3a^2}{a^2b^2r} - \frac{1}{r^3}\right)\right] \\ & + \epsilon_1\epsilon_2\left[\frac{(b_1+b_2)}{(a_1+a_2)}\,\frac{k\theta_o}{\ell n(b/a)}\,\left\{\left(\frac{a^2}{b^2-a^2}\right)\,\ell n(b/a) - \frac{1}{4}\right\}\frac{1}{r}\right]\ . \end{aligned} \tag{9.42}$$

The temperature field

$$\begin{aligned} \theta = {} & \epsilon_2\left[\frac{\theta_o}{\ell n(b/a)}\,\ell n(r/a)\right] + \epsilon_2^2\left[\frac{\kappa_2}{2\kappa_1}\left\{\frac{\theta_o}{\ell n(b/a)}\right\}^2 \ell n\left(\frac{r}{b}\right)\ell n\left(\frac{a}{r}\right)\right] \\ & + \epsilon_1\epsilon_2\left[-\frac{\kappa_3}{\kappa_1}\,\frac{k\ \theta_o}{\ell n(b/a)}\left\{\frac{1}{r^2} + \left(\frac{b^2-a^2}{a^2b^2}\right)\frac{\ell n(r/\delta)}{\ell n(b/a)}\right\}\right] \end{aligned} \tag{9.43}$$

follows from (9.20), (9.21), (9.36), (9.37), and (9.40). Only the diagonal components of stress are nonzero. From (9.18), (9.19), (9.27), (9.29), and (9.35), these are

$$
\begin{aligned}
S^{\langle rr\rangle} &= \epsilon_1[2(a_1+a_2)\frac{k}{b^2}(\frac{r^2-b^2}{r^2})] \\
&+ \epsilon_1^2[(a_1+a_2)(\frac{k}{ab})^2(\frac{r^4-r^2(a^2+b^2)+a^2b^2}{r^4})] \\
&+ \epsilon_1\epsilon_2[2(b_1+b_2)\frac{k\Theta_o}{\ln(b/a)}\{\frac{\ln(b/a)}{b^2-a^2}(1-\frac{a^2}{r^2})-\frac{\ln(r/a)}{r^2}\}], \\
S^{\langle\phi\phi\rangle} &= \epsilon_1[2(a_1+a_2)\frac{k}{b^2}(\frac{r^2+b^2}{r^2})] \\
&+ \epsilon_1^2[(a_1+a_2)(\frac{k}{ab})^2\{\frac{r^4+r^2(a^2+b^2)-3a^2b^2}{r^4}\}] \qquad (9.44) \\
&+ \epsilon_1\epsilon_2[2(b_1+b_2)\frac{k\Theta_o}{\ln(b/a)}\{\frac{\ln(b/a)}{b^2-a^2}(1-\frac{a^2}{r^2})+\frac{\ln(r/a)}{r^2}\}], \\
S^{\langle zz\rangle} &= \epsilon_1[2(a_1+a_2)\frac{k}{b^2}] \\
&+ \epsilon_1^2[(a_1+a_2)(\frac{k}{ab})^2 - (a_1-3a_2)k^2\frac{1}{r^4}] \\
&+ \epsilon_1\epsilon_2[(b_1+b_2)\frac{k\Theta_o}{\ln(b/a)}\{(\frac{2}{b^2-a^2})\ln(b/a)-\frac{1}{2r^2}\}].
\end{aligned}
$$

The constants k and δ in (9.42)-(9.44) are given by (9.17) and (9.41), respectively.

Notice that the solutions (9.42)-(9.44) for the displacement, temperature, and stress each include $\epsilon_1\epsilon_2$-terms. These cross terms make second-order contributions to the corresponding fields only when both mechanical and thermal loads are present.

As in Section 7, special care must be taken in interpreting the pressure boundary condition (9.1). With (9.42), it is easy to see that the internal pressure per unit area of the deformed body is $\epsilon_1[(a_1+a_2)] + \epsilon_1^2[-\frac{k}{a^2}(a_1+a_2)]$.

LIST OF REFERENCES

1. Truesdell, C., and W. Noll, The nonlinear field theories of mechanics. In Vol. III/3 of the Handbuch der Physik, edited by S. Flügge. Berlin-Göttingen-Heidelberg: Springer-Verlag (1965).

2. Iesan, D., Asupra teorie Termoelasticitatii nelineare. Analele Stünt. Univ. "A. I. Cuza" Iasi, Sect. I, Matematica **13**, 161-175 (1967).

3. Iesan, D., On the thermoelastic finite plane strain. Analele Stünt. Univ. "A. I. Cuza" Iasi, Sect. I, Matematica **15**, 195-207 (1969).

4. Chaudhry, H. R., A note on second order effects in plane strain thermoeleasticity, Int. J. Engrng. Sci. **9**, 673-678 (1971).

5. Chadwick, P., and L. T. C. Seet, Second-order thermoelasticity theory for isotropic and transversely isotropic materials. In Trends in Elasticity and Thermoelasticity, edited by R. E. Czarnota-Bojarski, M. Sokolowski, and H. Zorski. Groningen: Walters-Noordhoff (1971).

6. Johnson, A. F., Some second-order effects in thermoelasticity theory. Report DNAM 84, National Physical Laboratory (1970).

7. Carlson, D. E., Linear Thermoelasticity. In Vol. VIa/2 of the Handbuch der Physik, edited by C. Truesdell. Berlin-Heidelberg-New York: Springer-Verlag (1972).

8. Sokolinkoff, I. S., Mathematical Theory of Elasticity, 2nd Ed., New York: McGraw-Hill (1956).

9. Chan, C., and D. E. Carlson, Second-order incompressible elastic torsion, Int. J. Engrng. Sci. **8**, 415-430 (1970).

10. Pipkin, A. C., and R. S. Rivlin, The formulation of constitutive equations in continuum physics. Div. Appl. Math. Brown Univ. Report DA-4531/4 (1958) [A.C. Pipkin, Brown Univ. Ph.D. Thesis, Univ. Microfilms, Ann Arbor, Michigan (1959)].

11. Petroski, H. J., and D. E. Carlson, Some exact solutions to the equations of nonlinear thermoelasticity, J. Appl. Mech. **37**, 1151-1154 (1970).