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THE BEHAVIOR OF ELASTIC HEAT CONDUCTORS WITH SECOND-ORDER RESPONSE FUNCTIONS

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Abstract

Second-order forms for the material response functions of an elastic heat conductor are derived by approximating the response functions by polynomials in the appropriate invariants. Solutions based upon these forms of the response functions are exact for special materials and approximate for general materials.

Several aspects of the behavior of incompressible elastic heat conductors are investigated. The second-order dependence on temperature of isothermal elasticity solutions is found and the results are shown to agree quite well with experimental data taken on rubber. The second-order solutions to the problems of biaxial stretching with transverse heat flow, and simultaneous extension and shear of a cylindrical annulus with radial heat flow are obtained.

If a material which is thermally insulated is deformed, in general, its temperature will change. This phenomenon, known as Gough-Joule heating, is investigated for materials subjected to strain impulses.

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1. Introduction

The solution of most problems in nonlinear thermoelasticity requires the use of the inverse method. In the inverse method, deformation and temperature fields are specified and it is then verified that the state can be supported without mechanical body force or extrinsic heat supply. In general, one must have a knowledge of the response functions of the particular material under consideration to know which states can be supported. States which can be supported in every elastic heat conductor are of particular interest as these states could be used in the design of experiments to measure the response functions. Such states are said to be controllable or universal. Recent work by Petroski and Carlson [1, 2, 3] has determined all deformation and temperature fields that can be supported in every homogeneous, isotropic, elastic heat conductor in the absence of body force and heat supply. It is shown that the class of controllable states is extremely small, being limited to the controllable deformations of nonlinear elasticity coupled with uniform temperature fields.

Another approach to the problem of finding exact solutions in thermoelasticity is to specify the deformation and heat flux fields. Laws [4] has found all such states which can be supported, without body force or heat supply, in every homogeneous, isotropic, incompressible elastic heat conductor. Although it is shown that several classes of these states exist, the material response functions must be known before the conductivity equation can be integrated to obtain the temperature field.

In order to obtain more detailed solutions, it seems appropriate to turn to approximate methods. In nonlinear elasticity, theories appropriate to small but finite deformations have been consistently developed by Murnaghan [5] and Rivlin [6] from the general theory. This approach has also been used by Singh [7] in the study of elastic dielectrics. In the present work, this method is extended and used in the investigation of the behavior of elastic heat conductors. Essentially, the scheme is to approximate the material response functions by polynomials in the appropriate invariants. An elastic heat conductor is then characterized by constants (the coefficients in the polynomials) rather than by functions. The field equations remain exact and are not approximated. Thus, solutions based on this method are exact for special materials and approximate for general materials.

Other theories of second-order thermoelasticity have been proposed by several authors, including Dillon [8], Herrman [9], and Chadwick and Seet [10]. In each of these theories, both constitutive and geometrical nonlinearities are approximated. We also mention the general method of successive approximations, which has been recently applied to nonlinear thermoelasticity by Yarrington and Carlson [11].

After presenting the basic field and constitutive equations of nonlinear thermoelasticity in Sections 2 and 3, approximate constitutive relations are derived in Section 4. First- and second-order approximations for the free energy function are derived, which for an incompressible material represent thermoelastic extensions of the neo-Hookean and Mooney strain energy functions of nonlinear elasticity. In order that the material behave in a reasonable manner, various restrictions may be placed on the thermoelastic constants. Some of these restrictions are given in Section 5.

In Section 6, solutions are obtained for the problems of simple extension and pure torsion of incompressible materials in which the temperature is uniform. A general method of finding the dependence on temperature of any known elasticity solution is given.

Problems involving nonuniform temperature fields are considered in Section 7. For incompressible materials, we obtain second-order solutions to the problems of biaxial stretching with transverse heat flow, and simultaneous extension and shear of a cylindrical annulus with radial heat flow.

If a material which is thermally insulated is deformed, in general, its temperature will change. This phenomenon, known as Gough-Joule heating, is investigated in Section 8. If the motion of the body is a strain impulse, then the change in the temperature can be found as a function of the change in the deformation of the body. In Section 9, this relationship is shown to represent a type of difference equation which, in general, cannot be integrated to yield the temperature as a function of the configuration of the body. This fact is interpreted as implying that, in general, the Gough-Joule heating will depend on the rate of deformation as well as on the amount of deformation. Examples of Gough-Joule heating in simple extension and pure torsion are given in Section 8. In Section 9, it is shown that a temperature buildup occurs in a cyclically twisted cylinder.

Notation. Direct vector and tensor notation is used throughout. We denote the determinate, trace, transpose, and inverse of the second-order tensor $\underset{\sim}{A}$ by det $\underset{\sim}{A}$, tr $\underset{\sim}{A}$ $\underset{\sim}{A}^T$, and $\underset{\sim}{A}^{-1}$. In specific problems, we employ the notation $T^{\langle\cdot\rangle}$ for physical components of stress. In general, second-order tensors will be denoted by upper case boldface letters (e.g. $\underset{\sim}{A}$), while vectors will be denoted by lower case boldface letters (e.g. $\underset{\sim}{a}$). The same notation is used for functions whose values are these quantities. One exception to this rule will be the vector $\underset{\sim}{X}$, which denotes the position of a particle in the reference configuration.

2. Field Equations

If $\mathcal{D}$ represents a subregion of a three-dimensional Euclidean space E_3, a deformation is defined by a one-to-one function $\underset{\sim}{\chi} : \mathcal{D} \rightarrow E_3$ such that $\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X})$, where $\underset{\sim}{X}$ is the position vector of a material point X in the reference configuration and $\underset{\sim}{x}$ is the position vector of X in the deformed configuration. A family of deformations $\underset{\sim}{\chi}(\underset{\sim}{X}, t)$, parameterized by the time t, represents a motion of the body.

The derivative of the function $\underset{\sim}{\chi}$ with respect to $\underset{\sim}{X}$ is the deformation gradient $\underset{\sim}{F}(\underset{\sim}{X}, t)$. The polar decomposition theorem may be used to write

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} = \underset{\sim}{V}\,\underset{\sim}{R},$$

The tensor $\underset{\sim}{R}$ is orthogonal and is known as the rotation tensor, while the tensors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are symmetric positive definite and are known as the right and left stretch tensors, respectively. The tensor $\underset{\sim}{B}$, defined by

$$\underset{\sim}{B} = \underset{\sim}{V}^2$$

is the left Cauchy-Green tensor. The first, second, and third fundamental invariants of the tensor $\underset{\sim}{B}$ will be denoted by I, II, and III, respectively.

In terms of the Cauchy stress tensor $\underset{\sim}{T}$, the conditions of local balance of linear and angular momentum are equivalent to

$$\nabla \cdot \underset{\sim}{T} + \rho\,\underset{\sim}{b} = \rho\,\ddot{\underset{\sim}{x}} \qquad (2.1)$$

and

$$\underset{\sim}{T} = \underset{\sim}{T}^T, \qquad (2.2)$$

where $\nabla \cdot$ represents the spatial divergence, $\underset{\sim}{b}$ is the body force per unit mass, ρ is the mass density, and superposed dots indicate material time differentiation. The existence of an internal energy function ϵ is assumed, and local balance of energy is equivalent to

$$\nabla \cdot \underset{\sim}{q} + \rho\, r + \operatorname{tr}(\underset{\sim}{T}\,\underset{\sim}{D}) = \rho\,\dot{\epsilon}, \qquad (2.3)$$

where $\underset{\sim}{q}$ is the heat flux vector, r is the extrinsic heat supply per unit mass, and $\underset{\sim}{D}$ is the symmetric part of the spatial velocity gradient.

In this work, we will be primarily, but not exclusively, concerned with steady-state heat flow and mechanical equilibrium in the absence of body forces and extrinsic heat supply. Under these conditions, equations (2.1) and (2.3) become

$$\nabla \cdot \underset{\sim}{T} = \underset{\sim}{0} \tag{2.4}$$

and

$$\nabla \cdot \underset{\sim}{q} = 0. \tag{2.5}$$

3. Constitutive Equations

The modern theory of nonlinear thermoelasticity was initiated by the pioneer work of Coleman and Noll [12] in 1963. Their method is to restrict the form of the material response functions so that all processes satisfy the Clausius-Duhem inequality. We summarize their results below.

It is assumed that

$$\begin{aligned} \underset{\sim}{T} &= \underset{\sim}{\overline{T}}\,(\underset{\sim}{F},\ \theta,\ \underset{\sim}{g}), \\ \psi &= \overline{\psi}\,(\underset{\sim}{F},\ \theta,\ \underset{\sim}{g}), \\ \eta &= \overline{\eta}\,(\underset{\sim}{F},\ \theta,\ \underset{\sim}{g}), \\ \underset{\sim}{q} &= \underset{\sim}{\overline{q}}\,(\underset{\sim}{F},\ \theta,\ \underset{\sim}{g}), \end{aligned} \tag{3.1}$$

where θ is the absolute temperature, $\underset{\sim}{g}$ is the spatial temperature gradient, η is the specific entropy, and ψ is the free energy defined by

$$\psi = \epsilon - \eta\,\theta. \tag{3.2}$$

The Clausius-Duhem inequality, which expresses the assumption that the local entropy production is non-negative, is given by

$$-\rho\,(\psi - \eta\,\theta) + \mathrm{tr}\,(\underset{\sim}{T}\,\underset{\sim}{D}) + (\frac{1}{\theta})\ \underset{\sim}{q} \cdot \nabla\theta \geq 0. \tag{3.3}$$

If we now substitute the constitutive assumptions (3.1) into (3.3) and require that the inequality hold for all processes, we obtain the following:

(1) The response functions $\underset{\sim}{\overline{T}}$, $\overline{\psi}$, and $\overline{\eta}$ are independent of $\underset{\sim}{g}$.

(2) The response function $\underset{\sim}{\overline{T}}$ is determined from $\overline{\psi}$ through the stress relation

$$\underset{\sim}{\overline{T}}\,(\underset{\sim}{F},\ \theta) = \rho\,\underset{\sim}{F}\,[\frac{\partial\overline{\psi}}{\partial\underset{\sim}{F}}\,(\underset{\sim}{F},\ \theta)]^{T}. \tag{3.4}$$

(3) The response function $\overline{\eta}$ is determined from $\overline{\psi}$ by the entropy relation

$$\overline{\eta}\,(\underset{\sim}{F},\ \theta) = -\frac{\partial\psi}{\partial\theta}\,(\underset{\sim}{F},\ \theta). \tag{3.5}$$

(4) The response function $\bar{q}$ obeys the heat conduction inequality

$$\bar{q}(F, \theta, g) \cdot g \geq 0. \tag{3.6}$$

If we apply the principle of material frame indifference and assume that the material is isotropic in the reference configuration, the stress relation (3.4) becomes

$$T = 2\rho \left[\left(II \frac{\partial\psi}{\partial II} + III \frac{\partial\psi}{\partial III}\right) 1 + \frac{\partial\psi}{\partial I} B - III \frac{\partial\psi}{\partial II} B^{-1}\right]. \tag{3.7}$$

For an incompressible material, $III = 1$ and (3.7) reduces to

$$T = -p\, 1 + \frac{\partial\psi}{\partial I} B - \frac{\partial\psi}{\partial II} B^{-1}, \tag{3.8}$$

where p denotes an arbitrary hydrostatic pressure.

For an isotropic material, the heat conduction equation $(3.1)_4$ becomes

$$q = (h_0\, 1 + h_1 B + h_2 B^2)\, g, \tag{3.9}$$

where

$$h_i = h_i(I, II, III, \theta, g \cdot g, g \cdot B g, g \cdot B^2 g),$$

$$i = 0, 1, 2.$$

With conditions (1), (2) and (3), it is easily shown that the energy equation (2.3) can be written as

$$\theta \rho \dot{\eta} = \nabla \cdot q + \rho r. \tag{3.10}$$

Thus, in thermoelasticity, locally adiabatic ($\nabla \cdot q + \rho r \equiv 0$) is equivalent to isentropic ($\dot{\eta} \equiv 0$).

4. Particular Forms of the Material Response Functions

An elastic heat conductor is defined once the free energy function ψ and the heat conduction functions h_i are known. Particular forms of these functions will be considered in this section. In Rivlin's terminology [6], we derive the first- and second-order complete forms for the free energy functions of compressible and incompressible materials. The first- and second-order complete forms of the free energy function of an incompressible material represent thermoelastic extensions of the well-known neo-Hookean and Mooney strain energy functions of nonlinear elasticity. Specific forms for the heat conduction functions h_i are also obtained and are used to form a second-order heat conduction equation.

In what follows, the order symbol O will be used. We write

$$\phi = O(a) \quad \text{as} \quad a \longrightarrow 0$$

if there exist constants $k > 0$ and $\epsilon > 0$ such that

$$|\phi| \leq k \; |a| \quad \text{whenever} \quad |a| < \epsilon.$$

We say that a quantity ϕ is of order N and write

$$\text{ord } \phi = N \quad \text{as} \quad a \longrightarrow 0$$

if $N = \max \{n\}$ such that

$$\phi = O(a^n) \quad \text{as} \quad a \longrightarrow 0,$$

where $\{n\}$ denotes the set of all integers. We say that a quantity $\hat{\phi}$ is an Nth-order approximation to a quantity ϕ if

$$\text{ord } (\phi - \hat{\phi}) = N + 1 \quad \text{as} \quad a \longrightarrow 0.$$

When ϕ is a polynomial in a, then ord ϕ is the exponent of the lowest degree term. Of course, for small a, the lowest degree term is the dominant one. The notion of order as defined above is a measure of smallness in the sense that the higher the order, the smaller the quantity. The order of the nonzero constant polynomial is zero, while the order of the zero polynomial is infinity. If $\hat{\phi}$ is an Nth-order polynomial approximation to a polynomial ϕ, then $\phi - \hat{\phi}$ contains no powers of a less than $N + 1$.

As a fundamental assumption, we assume that the principal extension ratios u_i*, the magnitude of the spatial temperature gradient $\underset{\sim}{g}$, and the nondimensional temperature $\frac{\theta - \theta_0}{\theta_0}$ are of first-order in some small parameter.** Here, θ_0 is the uniform temperature of the reference configuration.

Compressible Materials

The free energy function ψ will be expanded in a power series in the invariants. To obtain an Nth-order approximation, we retain the minimum number of terms of the power series which will contain all quantities of order N or less. In order to minimize the number of terms which need to be retained, it is convenient to define a new set of invariants in the following manner.

$$\begin{aligned} J_1 &= I - 3, \\ J_2 &= (II - 3) - 2\,(I - 3), \\ J_3 &= (III - 1) - (II - 3) + (I - 3), \\ J_4 &= \left(\frac{\theta - \theta_0}{\theta_0}\right). \end{aligned} \tag{4.1}$$

It is easily shown that there is a one-to-one correspondence between the old and the new invariants.

In terms of the principal extensions, J_1 can be written as

$$\begin{aligned} J_1 &= (u_1 + 1)^2 + (u_2 + 1)^2 + (u_3 + 1)^2 - 3 \\ &= (u_1^2 + u_2^2 + u_3^2) + 2\,(u_1 + u_2 + u_3) \end{aligned}$$

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*The eigenvalues of the left Cauchy-Green tensor $\underset{\sim}{B}$ are $(1 + u_i)^2$.

**Of course, strictly speaking, the spatial temperature gradient should also be nondimensionalized. For example, we could replace g by $(\ell_0/\theta_0)\,g$, where ℓ_0 is some characteristic length. The final form of the theory is not altered by ignoring this detail.

Thus, J_1 is of first-order.

In a similar manner, it is easily verified that

$$\text{ord } J_2 = 2,$$

$$\text{ord } J_3 = 3,$$

$$\text{ord } J_4 = 1 \text{ (by definition).}$$

Expanding the free energy function ψ in a power series in the new invariants, we obtain

$$\psi = \hat{a}_0 + \hat{a}_1 J_1 + \hat{a}_2 J_4 + \hat{a}_3 J_2 + \hat{a}_4 J_1^2 + \hat{a}_5 J_4^2 + \hat{a}_6 J_1 J_4 + \hat{a}_7 J_3 + \hat{a}_8 J_1^3 + \hat{a}_9 J_1^2 J_4 + \hat{a}_{10} J_1 J_4^2 + \hat{a}_{11} J_2 J_1 + \hat{a}_{12} J_2 J_4 + \dots \quad (4.2)$$

If we require that the free energy, the entropy, and the stress vanish when the body occupies the reference configuration at the reference temperature, then

$$\hat{a}_0 = \hat{a}_1 = \hat{a}_2 = 0. \quad (4.3)$$

If only terms of order two or less are retained in (4.2), we obtain

$$\psi = \hat{a}_3 J_2 + \hat{a}_4 J_1^2 + \hat{a}_5 J_4^2 + \hat{a}_6 J_1 J_4 \quad (4.4)$$

as the second-order approximation to the free energy. In accordance with Rivlin's terminology [6], we will call (4.4) the first-order complete form of ψ for compressible materials.

If, in (4.2) we retain all terms of order three or less, we obtain

$$\psi = \hat{a}_3 J_2 + \hat{a}_4 J_1^2 + \hat{a}_5 J_4^2 + \hat{a}_6 J_1 J_4 + \hat{a}_7 J_3 + \hat{a}_8 J_1^3 + \hat{a}_9 J_1^2 J_4 + \hat{a}_{10} J_1 J_4^2 + \hat{a}_{11} J_2 J_1 + \hat{a}_{12} J_2 J_4 + \hat{a}_{13} J_4^3. \quad (4.5)$$

Equation (4.4) corresponds to the second-order complete form for ψ for compressible materials.

Substitution of (4.4) into the stress relation (3.7) yields the first-order stress relation

$$\underset{\sim}{T} = 2\rho\, [\hat{a}_3\, \mathrm{II}\, \underset{\sim}{1} + (2\hat{a}_4\, \mathrm{I} - 6\hat{a}_4 + \hat{a}_6\, J_4 - 2\,\hat{a}_3)\, \underset{\sim}{B} - \hat{a}_3\, \underset{\sim}{B}^{-1}]. \tag{4.6}$$

Substitution of (4.5) into (3.7) yields the second-order stress relation

$$\begin{aligned} \underset{\sim}{T} = 2\rho \Big\{ & [\,\mathrm{II}\, (\hat{a}_3 + \hat{a}_{11}\, J_1 + \hat{a}_{12}\, J_4 - \hat{a}_7) + \hat{a}_7\, \mathrm{III}\,]\, \underset{\sim}{1} \\ & + [(2\hat{a}_4 - 2\hat{a}_{11})\, J_1 + (\hat{a}_6 - 2\hat{a}_{12})\, J_4 + 3\hat{a}_8\, J_1^2 \\ & + 2\hat{a}_9\, J_1\, J_4 + \hat{a}_{10}\, J_4^2 + \hat{a}_{11}\, J_2 - 2\hat{a}_3 + \hat{a}_7]\, \underset{\sim}{B} \\ & - \mathrm{III}\, (\hat{a}_3 + \hat{a}_{11} + \hat{a}_{12}\, J_4 - \hat{a}_7)\, \underset{\sim}{B}^{-1} \Big\} . \end{aligned} \tag{4.7}$$

From the entropy relation (3.5) and the first-order complete free energy function (4.4) we obtain the first-order entropy relation

$$\eta = \frac{-1}{\theta_0}\, [2a_5\, J_4 + a_6\, J_1]. \tag{4.8}$$

Similarly, substitution of (4.5) into (3.5) yields the second-order entropy relation

$$\begin{aligned} \eta = -\frac{1}{\theta_0}\, [& 2a_5\, J_4 + a_6\, J_1 + a_9\, J_1^2 + 2a_{10}\, J_1\, J_4 \\ & + a_{12}\, J_2 + 3a_{13}\, J_4^2]. \end{aligned} \tag{4.9}$$

Incompressible Materials

The constraint of incompressibility requires the invariant III to equal unity in all deformations. Thus, the free energy function for incompressible materials depends only upon I, II and θ. In addition to this simplification, the equation III $= 1$, which may be written as

$$(u_1 + 1)^2\, (u_2 + 1)^2\, (u_3 + 1)^2 - 1 = 0, \tag{4.10}$$

can be used to increase the order of some of the invariants.

For example, we have already seen that J_1 may be written as

$$J_1 = (u_1^2 + u_2^2 + u_3^2) + 2(u_1 + u_2 + u_3). \tag{4.11}$$

Equations (4.10) and (4.11) imply that

$$\text{ord } J_1 = 2$$

Instead of using J_2 as defined for compressible materials, we will introduce a new invariant J_2', defined by

$$J_2' = II - I.$$

In conjunction with (4.10), it is easily verified that J_2' is of third order.

Expanding ψ in a power series in J_1, J_2', and J_4 yields

$$\psi = a_0 + a_1' J_1 + a_2 J_4 + a_3 J_2' + a_4 J_1 J_4 + a_5 J_4^2 + a_6 J_4^3 + \ldots \tag{4.12}$$

The requirement that the free energy and the entropy vanish in the reference state leads to

$$a_0 = a_2 = 0.$$

If we retain only terms of order two or less in (4.12), we obtain

$$\psi = a_1 (I - 3) + a_5 J_4^2, \tag{4.13}$$

which will be called the first-order complete form for incompressible materials. When the temperature is the reference temperature, (4.13) reduces to the neo-Hookean strain energy function of nonlinear elasticity. This form has been derived from a theory of long-chain molecules by Treloar [13].

If we retain all terms of order three or less in (4.10), we obtain

$$\psi = a_1 (I - 3) + a_3 (II - 3) + a_4 (I - 3) J_4 + a_5 J_4^2 + a_6 J_4^3, \tag{4.14}$$

which will be called the second-order complete form for incompressible materials. Again, when the temperature is the reference temperature, (4.14) reduces to the Mooney strain energy function, which has been experimentally shown to accurately approximate the strain energy function of certain types of rubber [14].

Substitution of (4.13) into the stress relation (3.8) gives the first-order stress relation

$$\underset{\sim}{T} = -p\underset{\sim}{1} + a_1\underset{\sim}{B}. \tag{4.15}$$

Thus, the first-order stress relation is independent of the temperature. Substitution of (4.14) into (3.8) gives

$$\underset{\sim}{T} = -p\underset{\sim}{1} + (a_1 + a_4 J_4)\underset{\sim}{B} - a_3\underset{\sim}{B}^{-1}, \tag{4.16}$$

which corresponds to the second-order stress relation. Therefore, the second-order stress relation involves the temperature through the term $J_4 = \frac{\theta - \theta_0}{\theta_0}$.

From (3.5) and (4.13), we obtain the first-order entropy relation

$$\eta = -\frac{1}{\theta_0}(2a_5 J_4); \tag{4.17}$$

while (3.5) and (4.14) yield the second-order entropy relation

$$\eta = -\frac{1}{\theta_0}[a_4(I - 3) + 2a_5 J_4 + 3a_6 J_4^2]. \tag{4.18}$$

Remark - At this point it should be pointed out that an alternative method for defining the various order stress and entropy relations is available. We could have substituted the complete power series expansion of the free energy function into the stress and entropy relations. Various order stress and entropy relations could then be defined by deleting higher order terms from the exact stress and entropy relations. For incompressible materials, the method just described and the method used in this work give identical results through third-order theories. However, in general, the two methods will give different results for higher-order theories.

Heat Conduction

In addition to the invariants already defined, we will define three new ones.

$$\begin{aligned} J_5 &= \underset{\sim}{g} \cdot \underset{\sim}{g}, \\ J_6 &= \underset{\sim}{g} \cdot \underset{\sim}{B}\,\underset{\sim}{g} - \underset{\sim}{g} \cdot \underset{\sim}{g}, \\ J_7 &= \underset{\sim}{g} \cdot \underset{\sim}{B}^2 \underset{\sim}{g} - 2\,(\underset{\sim}{g} \cdot \underset{\sim}{B}\,\underset{\sim}{g} - \underset{\sim}{g} \cdot \underset{\sim}{g}) - \underset{\sim}{g} \cdot \underset{\sim}{g}. \end{aligned} \tag{4.19}$$

It is not difficult to verify that

$$\begin{aligned} \text{ord } J_5 &= 2, \\ \text{ord } J_6 &= 3, \\ \text{ord } J_7 &= 4. \end{aligned}$$

We observe that the heat conduction equation (3.9) can be written in the form

$$\underset{\sim}{q} = [h_0 \underset{\sim}{1} + h_1 (\underset{\sim}{B} - \underset{\sim}{1}) + h_2 (\underset{\sim}{B} - \underset{\sim}{1})^2]\, \underset{\sim}{g}, \tag{4.20}$$

where

$$h_i = h_i (J_1, J_2, J_3, J_4, J_5, J_6, J_7).$$

We next expand each of the h_i in a power series in the invariants:

$$\begin{aligned} h_0 &= b_0 + b_1 J_1 + b_2 J_2 + \ldots, \\ h_1 &= c_0 + c_1 J_1 + b_2 J_2 + \ldots, \\ h_2 &= d_0 + d_1 J_1 + d_2 J_2 + \ldots . \end{aligned}$$

If we define a vector to be of order N if its magnitude is of order N, the vector $(\underset{\sim}{B} - \underset{\sim}{1})^2\, \underset{\sim}{g}$ is one order higher than $(\underset{\sim}{B} - \underset{\sim}{1})\, \underset{\sim}{g}$, which is one order higher than $\underset{\sim}{g}$. In order to obtain an Nth-order approximation of (4.20) with the least number of terms, we will approximate h_0 one order higher than h_1 and two orders higher than h_2.

We will call the usual Fourier's law

$$\underset{\sim}{q} = b_o\, \underset{\sim}{g} \tag{4.21}$$

the first-order conductivity relation. This corresponds to choosing

$$h_0 = b_0,$$

$$h_1 = h_2 = 0.$$

If we wish to obtain a second-order conductivity relation, we must include first-order terms from the expansion for h_0 and include zeroth-order terms from the expansion for h_1. For compressible materials, this corresponds to choosing

$$h_0 = b'_0 + b_1 J_4 + b_2 J_1,$$

$$h_1 = c_0,$$

$$h_2 = 0.$$

Therefore, the second-order heat conduction equation for compressible materials is given by

$$\underset{\sim}{q} = [(b_0 + b_1 J_4 + b_2 J_1) \underset{\sim}{1} + c_0 \underset{\sim}{B}] \underset{\sim}{g}. \qquad (4.22)$$

For incompressible materials, the invariant J_1 is of second order; and, therefore, the second-order heat conduction equation is given by

$$\underset{\sim}{q} = [(b_0 + b_1 J_4) \underset{\sim}{1} + c_0 \underset{\sim}{B}] \underset{\sim}{g}. \qquad (4.23)$$

5. Restrictions on the Thermoelastic Constants of Incompressible Materials

The heat conduction inequality, the requirement of thermal stability, and the somewhat intuitive requirement that materials respond in a reasonable manner, place certain restrictions upon the response functions of an elastic heat conductor. For the special response functions derived in Section 4, these restrictions limit the range of the thermoelastic constants.

The second-order complete free energy function for incompressible materials (4.14) may be written in the form

$$\psi = [a_1 + a_4 J_4](I - 3) + a_3 (II-3) + a_5 J_4^2 + a_6 J_4^3. \tag{5.1}$$

If the temperature is the reference temperature θ_0, (5.1) reduces to the Mooney strain energy function

$$\psi = a_1 (I - 3) + a_3 (II - 3);$$

and the Baker-Ericksen inequalities [15] (see also Truesdell and Noll [16] and Wang and Truesdell [17]) imply

$$a_1 \geq 0, \quad a_3 \geq 0. \tag{5.2}$$

Since the last two terms of (5.1) do not affect the stress, changing the temperature amounts to changing the coefficient of the term $(I - 3)$. If we require that the Baker-Ericksen inequalities hold for all temperatures in some neighborhood of θ_0, then

$$a_1 + a_4 J_4 \geq 0, \quad a_3 \geq 0 \tag{5.3}$$

We observe that $(5.3)_1$ cannot be satisfied for all temperatures if $a_4 \neq 0$. However, in Section 6, we will present experimental evidence for $a_4 > 0$ for rubber. To alleviate this difficulty, we will adopt the viewpoint that (5.1) is valid for temperatures in a certain neighborhood of θ_0.

The specific heat c of a material is defined by

$$c = \theta \frac{\partial \eta}{\partial \theta} = -\theta \frac{\partial^2 \psi}{\partial \theta^2}. \tag{5.4}$$

If we require that the material be thermally stable in the sense of Coleman and Noll [18], then

$$c > 0. \tag{5.5}$$

For the second-order material defined by (5.1), condition (5.5) becomes

$$-\frac{\theta}{\theta_0^2} (2a_5 + 6a_6 J_4) > 0 \tag{5.6}$$

Since (5.6) cannot hold for arbitrary J_4 if $a_6 \neq 0$, we again restrict the range of J_4 so as to satisfy (5.6). At the reference temperature θ_0, (5.6) becomes

$$a_5 < 0. \tag{5.7}$$

Since the first-order complete free energy function for incompressible materials (4.13) represents a special case of (5.1) in which

$$a_3 = a_4 = a_6 = 0,$$

we have

$$a_1 > 0, \quad a_5 < 0, \tag{5.8}$$

for that form.

If the second-order heat conduction equation (4.23) is substituted into the heat conduction inequality (3.6), we obtain

$$(b_0 + b_1 J_4) \underset{\sim}{g} \cdot \underset{\sim}{g} + c_0 \underset{\sim}{g} \cdot \underset{\sim}{B} \underset{\sim}{g} \geq 0 \quad . \tag{5.9}$$

Again, (5.9) cannot hold for arbitrary J_4 and therefore the range of J_4 must be restricted. When $\underset{\sim}{B} = \underset{\sim}{1}$, (5.9) reduces to

$$(b_0 + b_1 J_4 + c_0) \underset{\sim}{g} \cdot \underset{\sim}{g} \geq 0 \tag{5.10}$$

and therefore

$$b_0 + c_0 + b_1 J_4 \geq 0 \quad \text{when} \quad \underset{\sim}{g} \neq \underset{\sim}{0} \tag{5.11}$$

We note that (5.9) may be interpreted as requiring that the tensor $(b_0 + b_1 J_4)\underset{\sim}{1} + c_0 \underset{\sim}{B}$ be positive semi-definite. Of course, $\underset{\sim}{B}$ is necessarily positive definite.

6. Solutions with Uniform Temperature Fields

As an example of the use of the special stress relations derived in Section 4, we obtain the second-order solution to the problem of simple extension of an incompressible, isotropic, prismatic bar with uniform temperature. The solution reflects the dependence of the surface tractions on the temperature, and it is shown to agree favorably with experimental data taken on rubber.

In addition, a method is outlined whereby the second-order dependence on temperature of any known elastic solution may be found. As an example of this method, we consider simple torsion of an incompressible, isotropic, right-circular cylinder with uniform temperature.

Simple Extension of an Incompressible, Isotropic, Prismatic Bar with Uniform Temperature

We assume that the particle, whose rectangular Cartesian coordinates in the reference configuration are given by (X, Y, Z), moves to the new position (x, y, z) given by

$$x = W^{-\frac{1}{2}} X, \qquad y = W^{-\frac{1}{2}} Y, \qquad z = W Z. \tag{6.1}$$

We then have

$$\underset{\sim}{B} = \begin{bmatrix} W^{-1} & 0 & 0 \\ 0 & W^{-1} & 0 \\ 0 & 0 & W^{2} \end{bmatrix}, \quad \underset{\sim}{B}^{-1} = \begin{bmatrix} W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W^{-2} \end{bmatrix}. \tag{6.2}$$

It is easily seen that det $\underset{\sim}{B} = 1$, and hence the deformation (6.1) is volume preserving.

Substitution of (6.2) into the second-order stress relation (4.16) yields

$$T^{<xx>} = T^{<yy>} = -p + (a_1 + a_4 J_4) W^{-1} - a_3 W,$$

$$T^{<zz>} = -p + (a_1 + a_4 J_4) W^2 - a_3 W^{-2}, \tag{6.3}$$

$$T^{<xy>} = T^{<xz>} = T^{<yz>} = 0.$$

In order that the solution correspond to the simple tension test, we require

$$T^{<zz>} = T^{<yy>} = 0.$$

Hence,

$$p = (a_1 + a_4 J_4) W^{-1} - a_3 W. \tag{6.4}$$

Substitution of (6.4) into $(6.3)_2$ yields

$$T^{<zz>} = T_0^{<zz>} + a_4 J_4 (W^2 - W^{-1}), \tag{6.5}$$

where

$$T_0^{<zz>} = a_1 (W^2 - W^{-1}) - a_3 (W^{-2} - W)$$

denotes the stress which would be obtained if the body were at the reference temperature.

Since the stresses and the temperature are independent of position, the field equations

$$\nabla \cdot \underset{\sim}{T} = \underset{\sim}{0},$$
$$\nabla \cdot \underset{\sim}{q} = 0$$

are automatically satisfied. Thus, the assumed deformation and temperature fields can exist without body force or heat supply.

The general behavior of equation (6.5) is shown in Figures 1 and 2. Figure 3 represents experimental data taken on rubber [18]. The slope of the curves shown in Figures 2 and 3 is known as the stress-temperature coefficient s_t. From (6.5), we may calculate the theoretical stress-temperature coefficient to be given by

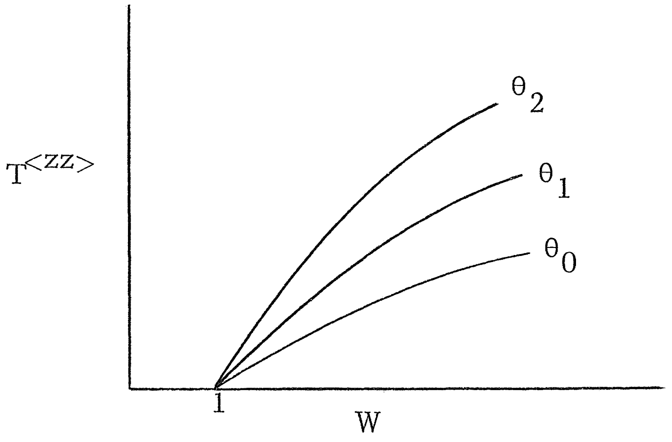

Figure 1. Theoretical stress at constant temperature as a function of the extension with $\theta_0 < \theta_1 < \theta_2$ and $a_4 > 0$.

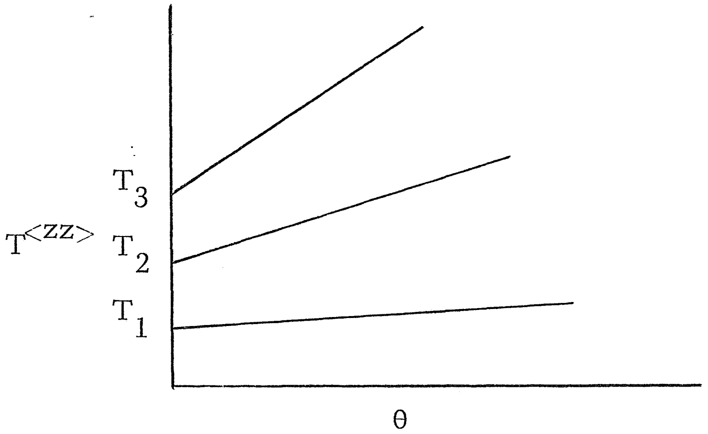

Figure 2. Theoretical stress at constant elongation as a function of the absolute temperature if $a_4 > 0$.

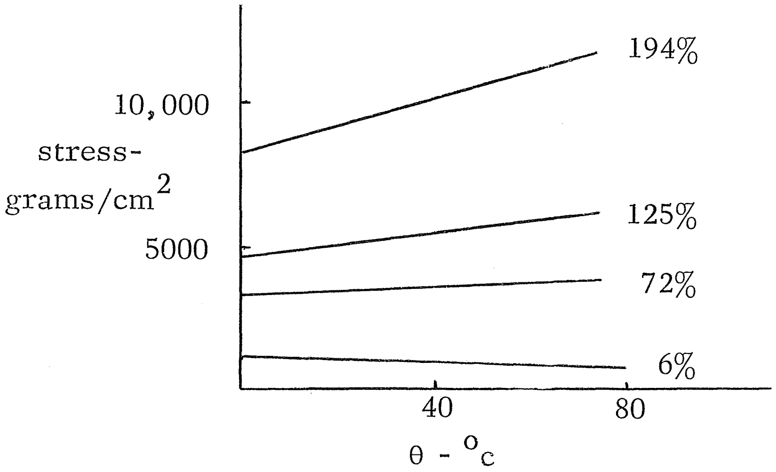

Figure 3. Experimental stress at constant elongation as a function of the temperature [18].

$$s_t = \frac{a_4}{\theta_0}(W^2 - W^{-1}) \quad . \tag{6.6}$$

Except at very small elongations, the stress-temperature coefficient of rubber is positive. Thus, from (6.6), the constant a_4 must be positive for rubber.

From Figure 1, it may be seen that if the stress $T^{<zz>}$ is held constant, an increase in the temperature will produce a decrease in the elongation in a material for which $a_4 > 0$. According to Treloar [13], this effect was first observed in rubber by Gough in 1805.

From (6.6), we observe that when there is no elongation ($W = 1$), the stress-temperature coefficient s_t should be zero. However, Figure 3 shows that, in fact, s_t is slightly negative at zero extension. According to Treloar [13] (see also Treloar [19]), this behavior could be due to the slight compressibility of rubber.

General Method

In the literature of finite elasticity, many problems have been solved, either in terms of the functions ψ_i appearing in the elastic constitutive equation

$$\underset{\sim}{T} = \psi_0 \underset{\sim}{1} + \psi_1 \underset{\sim}{B} + \psi_{-1} \underset{\sim}{B}^{-1}$$

or in terms of the constants C_1 and C_2 appearing in the Mooney strain energy function

$$\psi = C_1 (I - 3) + C_2 (II - 3).$$

For the approximate theories which have been derived here, we have the following correspondences:

First-order complete

1) Compressible

$$\psi_0 = 2\rho\, a_3\, II, \quad \psi_1 = 2\rho\,(2a_4 I - 6a_4 + a_6 J_4 - 2a_3)$$

$$\psi_{-1} = -2\rho\, a_3$$

2) Incompressible

$$\psi_1 = a_1, \ \psi_{-1} = 0$$

$$C_1 = a_1, \ C_2 = 0$$

Second-order complete

1) Compressible

$$\psi_0 = 2\rho \left[\mathrm{II} \left(a_3 + a_{11} J_1 + a_{12} J_4 - a_7 \right) + a_7 \, \mathrm{III} \right]$$

$$\psi_1 = 2\rho \left[(2a_4 - 2a_{11}) J_1 + (a_6 - 2a_{12}) J_4 + 3a_8 J_1^2 + 2a_9 J_1 J_4 + a_{10} J_4^2 + a_{11} J_2 - 2a_3 + a_7 \right]$$

$$\psi_{-1} = 2\rho \left(a_3 + a_{11} J_1 + a_{12} J_4 - a_7 \right)$$

2) Incompressible

$$\psi_1 = a_1 + a_4 J_4, \ \psi_{-1} = -a_3$$

$$C_1 = a_1 + a_4 J_4, \ C_2 = a_3$$

By substituting the above relations into a solution from isothermal elasticity, we may find the effect upon that solution of altering the uniform temperature.

Pure Torsion of an Incompressible, Isotropic, Right-Circular Cylinder

As an example of the method outlined above, we consider simple torsion of an incompressible, isotropic, right-circular cylinder of radius r_0. The particle originally located at the point (R, Φ, Z) in circular cylindrical coordinates moves to the position (r, ϕ, z) given by

$$r = R, \quad \phi = \Phi + BZ, \quad z = Z.$$

The moment T exerted on the ends of the cylinder is known to be (see, e.g., Truesdell and Noll [16])

$$T = 2\pi B \int_0^{r_0} r^3 \left(\psi_1 - \psi_{-1} \right) dr. \tag{6.7}$$

If we substitute the second-order complete forms for ψ_1 and ψ_2 into (6.7), the integration may be carried out and we obtain

$$T = T_0 + \tfrac{1}{2}\pi B r_0^4 a_4 J_4,$$

where

$$T_0 = \tfrac{1}{2}\pi B r_0^4 (a_1 + a_3)$$

is the moment which would exist if the temperature were the reference temperature. For rubber cylinders, the constant a_4 is positive and the moment increases with increasing temperature. If the moment is held constant, the amount of twist will decrease with increasing temperature. This effect represents a torsional analog of Gough's observation for simple extension.

7. Solutions Involving Nonuniform Temperature Fields

In this section we obtain second-order solutions to the problems of biaxial stretching of incompressible materials with transverse heat flow, and simultaneous extension and shear of an incompressible cylindrical annulus with radial heat flow.

Biaxial Stretching of Incompressible Materials with Transverse Heat Flow

We assume that the particle whose rectangular Cartesian coordinates in the reference configuration are given by (X, Y, Z) moves to the position (x, y, z) given by

$$x = UX, \quad y = VY, \quad z = U^{-1} V^{-1} Z. \tag{7.1}$$

In addition, we assume that the nondimensional temperature J_4 is given by

$$J_4 = J_4(z) . \tag{7.2}$$

Laws [4] has shown that this deformation and temperature field represents a controllable state with prescribed heat flux.

Corresponding to the deformation (7.1), we have

$$\underset{\sim}{B} = \begin{bmatrix} U^2 & 0 & 0 \\ 0 & V^2 & 0 \\ 0 & 0 & U^{-2}V^{-2} \end{bmatrix}, \quad \underset{\sim}{B}^{-1} = \begin{bmatrix} U^{-2} & & 0 \\ 0 & V^{-2} & 0 \\ 0 & 0 & U^2V^2 \end{bmatrix}. \tag{7.3}$$

Again, $\det \underset{\sim}{B} = 1$, and the deformation (7.1) is volume preserving.

Substitution of (7.3) into the second-order stress relation (4.16) yields

$$\begin{aligned} T^{<xx>} &= -p + (a_1 + a_4 J_4) U^2 - a_3 U^{-2}, \\ T^{<yy>} &= -p + (a_1 + a_4 J_4) V^2 - a_3 V^{-2}, \\ T^{<zz>} &= -p + (a_1 + a_4 J_4) (U V)^{-2} - a_3 (U V)^2, \\ T^{<xy>} &= T^{<xz>} = T^{<yz>} = 0. \end{aligned} \tag{7.4}$$

Since J_4 depends only on z, substitution of (7.4) into the field equation $\nabla \cdot \underset{\sim}{T} = \underset{\sim}{0}$ yields

$$\begin{aligned} p &= p(z), \\ T^{<zz>} &= T_0^{<zz>} = \text{constant.} \end{aligned} \tag{7.5}$$

Substitution of $(7.5)_2$ into $(7.4)_3$ gives

$$p = (z) = -T_0^{<zz>} + (a_1 + a_4 J_4)(U\,V)^{-2} - a_3 (U\,V)^2. \tag{7.6}$$

With (7.6), the stresses (7.4) become

$$\begin{aligned} T^{<xx>} &= (a_1 + a_4 J_4)[U^2 - (U\,V)^{-2}] - a_3 [U^{-2} - (U\,V)^2] + T_0^{<zz>}, \\ T^{<yy>} &= (a_1 + a_4 J_4)[V^2 - (U\,V)^{-2}] - a_3 [V^{-2} - (U\,V)^2] + T_0^{<zz>}, \\ T^{<zz>} &= T_0^{<zz>} \\ T^{<xy>} &= T^{<xz>} = T^{<yz>} = 0. \end{aligned}$$

To correspond to biaxial stretching, we set $T_0^{<xx>} = 0$ to obtain

$$\begin{aligned} T^{<xx>} &= (a_1 + a_4 J_4)[U^2 - (U\,V)^{-2}] - a_3 [U^{-2} - (U\,V)^2], \\ T^{<yy>} &= (a_1 + a_4 J_4)[V^2 - (U\,V)^{-2}] - a_3 [V^{-2} - (U\,V)^2], \\ T^{<zz>} &= T^{<xy>} = T^{<xz>} = T^{<yz>} = 0. \end{aligned} \tag{7.7}$$

Using the method presented in Section 6, it is easily shown that at each point $\underset{\sim}{x}$, the stresses given by equations (7.7) represent the stresses which would be obtained if the body were at the uniform temperature $J_4(\underset{\sim}{x})$.

It now remains to calculate the temperature distribution. With (7.2), the second-order heat conduction equation (4.23) can be written as

$$\underset{\sim}{q} = \frac{\partial}{\partial z}[(b_0 + c_0 U^{-2} V^{-2}) J_4 + \frac{b_1}{2} J_4^2]\, \underset{\sim}{e}_z. \tag{7.8}$$

Equation (7.8) and the field equation $\nabla \cdot \underset{\sim}{q} = 0$ imply

$$\underset{\sim}{q} = q_0 \underset{\sim}{e}_z = \text{constant}. \tag{7.9}$$

If the temperatures on the faces of a plate originally of thickness t are given by

$$\theta(0) = \theta_0, \quad \theta(U^{-1} V^{-1} t) = \theta_1$$

we may equate (7.9) and (7.8) and integrate to obtain

$$J_4(z) = m_1^{-1} [- m_2 + (m_2^2 + 2m_1 q_0 z)^{\frac{1}{2}}], \tag{7.10}$$

where

$$m_1 = b_1 \theta_0, \quad m_2 = \theta_0 (b_0 + c_0 U^{-2} V^{-2}).$$

The heat flux q_0 is given by

$$q_0 = U V t^{-1} (m_2 J_{4_1} + \tfrac{1}{2} m_1 J_{4_1}^2), \tag{7.11}$$

where

$$J_{4_1} = \frac{\theta_1 - \theta_0}{\theta_0}.$$

The temperature distribution (7.10) may now be substituted into (7.7) to determine the lateral stresses.

This solution may be interpreted as corresponding to a thin stretched membrane whose faces are held at different temperatures. Because the temperature varies in the z-direction, the stresses on the edges of the membrane must also vary in the z-direction.

If we set

$$U = V = W^{-1}$$

in the solution just obtained, we obtain the solution to the problem of simple extension with uniaxial heat flow. Since the lateral stresses vary with temperature, it is not possible, in general, to make these vanish. Thus, the solution cannot be made to correspond to the simple tension test with axial heat flow.

Simultaneous Extension and Shear of an Incompressible, Isotropic Cylindrical Annulus with Radial Heat Flow

We assume deformation and temperature fields of the form

$$r = \lambda R, \quad \phi = \Phi + \hat{\phi}(R), \quad z = \lambda^{-2} Z + \hat{w}(R), \qquad (7.12)$$
$$J_4 = J_4(r),$$

where (r, ϕ, z) represent the circular cylindrical coordinates of the position of a particle whose position in the reference configuration was (R, Φ, Z). The constant λ is assumed to be given while the functions $\hat{\phi}(R)$ and $\hat{w}(R)$ are to be determined. The physical components of $\underset{\sim}{B}$ and $\underset{\sim}{B}^{-1}$ corresponding to (7.12) are given by

$$\underset{\sim}{B} = \begin{bmatrix} \lambda^2 & \lambda \hat{\phi}' r & \lambda \hat{w}' \\ \lambda \hat{\phi}' r & (\lambda^2 + r^2 \hat{\phi}'^2) & \hat{\phi}' \hat{w}' r \\ \lambda \hat{w}' & \hat{\phi}' \hat{w}' r & (\hat{w}'^2 + \lambda^{-4}) \end{bmatrix}, \qquad (7.13)$$

$$\underset{\sim}{B}^{-1} = \begin{bmatrix} (\lambda^2 \hat{w}'^2 + \lambda^{-2} + r^2 \hat{\phi}'^2 \lambda^{-4}) & (-\lambda^{-3} \hat{\phi}' r) & (-\hat{w}' \lambda^3) \\ (-\lambda^{-3} \hat{\phi}' r) & \lambda^{-2} & 0 \\ (-\hat{w}' \lambda^3) & 0 & \lambda^4 \end{bmatrix},$$

where a prime denotes differentiation with respect to R. It is easily verified that the deformation (7.12) is volume preserving.

Substitution of (7.13) into the second-order stress relation (4.16) yields

$$T^{<rr>} = -p + (a_1 + a_4 J_4)\;{}^{2} - a_3 (\lambda^2 \hat{w}'^2 + \lambda^{-2} + r^2 \hat{\phi}'^2 \lambda^{-4})$$

$$T^{<\phi\phi>} = -p + (a_1 + a_4 J_4)(\lambda^2 + r^2 \hat{\phi}'^2) - a_3 \lambda^{-2},$$

$$\begin{aligned}
T^{<zz>} &= -p + (a_1 + a_4 J_4)(\hat{w}'^2 + \lambda^{-4}) - a_3 \lambda^4, \\
T^{<r\phi>} &= (a_1 + a_4 J_4)(\lambda \hat{\phi}' r) + a_3 \lambda^{-3} \hat{\phi}' r, \\
T^{<rz>} &= (a_1 + a_4 J_4)(\lambda \hat{w}') + a_3 \hat{w}' \lambda^3, \\
T^{<\phi z>} &= (a_1 + a_4 J_4)(\hat{\phi}' \hat{w}' r).
\end{aligned} \tag{7.14}$$

If we assume that the pressure p depends only upon r, then all of the stresses depend only upon r; and the field equation $\nabla \cdot \underset{\sim}{T} = \underset{\sim}{0}$ becomes

$$\begin{aligned}
&\frac{\partial}{\partial r}[T^{<rr>}] + \frac{1}{r}[T^{<rr>} - T^{<\phi\phi>}] = 0, \\
&\frac{\partial}{\partial r}[T^{<r\phi>}] + \left(\frac{2}{r}\right) T^{<r\phi>} = 0, \\
&\frac{\partial}{\partial r}[T^{<rz>}] + \left(\frac{1}{r}\right) T^{<rz>} = 0.
\end{aligned} \tag{7.15}$$

Equations $(7.15)_2$ and $(7.15)_3$ imply

$$\begin{aligned}
T^{<r\phi>} &= k_1/r^2, \\
T^{<rz>} &= k_2/r,
\end{aligned} \tag{7.16}$$

where k_1 and k_2 are constants. Substituting (7.14) into (7.16) and integrating yields

$$\begin{aligned}
\hat{\phi}(r) - \hat{\phi}(r_0) &= k_1 I_1, \\
\hat{w}(r) - \hat{w}(r_0) &= k_2 I_2,
\end{aligned} \tag{7.17}$$

where

$$\begin{aligned}
I_1 &= \int_{r_0}^{r} \lambda^4 [(a_0 + a_1 J_4)\lambda^4 + a_3]^{-1} r^{-3}\, dr, \\
I_2 &= \int_{r_0}^{r} [(a_1 + a_4 J_4 + a_3 \lambda^2) \lambda r]^{-1}\, dr.
\end{aligned} \tag{7.18}$$

If we assume the boundary conditions

$$\hat{\phi}(r_0) = \hat{w}(r_0) = 0,$$

$$\hat{\phi}(r_1) = \hat{\phi}_1 \qquad \hat{w}(r_1) = \hat{w}_1,$$

we obtain

$$\begin{aligned} \hat{\phi}(r) &= k_1 I_1, \\ \hat{w}(r) &= k_2 I_2, \end{aligned} \tag{7.19}$$

where

$$k_1 = \hat{\phi}_1 / I_1(r_1), \; k_2 = \hat{w}_1 / I_2(r_1).$$

In order to proceed further, we must find the temperature field. For the assumed deformation, the second-order heat conduction equation (4.23) becomes

$$\begin{aligned} \underset{\sim}{q} = {} & \theta_0 (b_0 + b_1 J_4 + c_0 \lambda^2) \left(\frac{\partial J_4}{\partial r}\right) \underset{\sim r}{e} + \theta_0 \lambda \hat{\phi}' r \left(\frac{\partial J_4}{\partial r}\right) \underset{\sim \phi}{e} \\ & + \theta_0 \lambda \hat{w}' \left(\frac{\partial J_4}{\partial r}\right) \underset{\sim z}{e}. \end{aligned} \tag{7.20}$$

We assume the boundary conditions

$$\theta(r_0) = \theta_0, \quad \theta(r_1) = \theta_1.$$

Substituting (7.20) into the field equation $\nabla \cdot \underset{\sim}{q} = 0$ and integrating yields

$$J_4(r) = \xi_1 [-\xi_2 + (\xi_2^2 + \xi_3 \ln \frac{r}{r_0})^{\frac{1}{2}}], \tag{7.21}$$

where

$$\xi_1 = (b_1 \theta_0)^{-1},$$

$$\xi_2 = \theta_0 (b_0 + c_0 \lambda^2),$$

$$\xi_3 = b_1 \theta_0^2 [2 (b_0 + c_0 \lambda^2) (\frac{\theta_1 - \theta_0}{\theta_0}) + b_1 (\frac{\theta_1 - \theta_0}{\theta_0})^2] [\ln \frac{r_1}{r_0}]^{-1}.$$

Substitution of (7.21) into (7.18) and (7.19) gives $\hat{\phi}$ and $\hat{w}$ as functions of r. However, because of the complexity of the expression, the integration would have to be carried out numerically.

All of the field equations except $(7.15)_1$ have now been satisfied. Substitution of $(7.14)_1$ and $(7.14)_2$ into $(7.15)_1$ yields

$$\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}\left[(a_1 + a_4 J_4)\lambda^2 - a_3(\lambda^2 \hat{w}'^2 + \lambda^{-2} + r^2 \hat{\phi}'^2 \lambda^{-4})\right]$$
$$- r^{-1}\left[(a_1 + a_4 J_4) r^2 \hat{\phi}'^2 - a_3(\lambda^2 \hat{w}'^2 - r^2 \hat{\phi}'^2 \lambda^{-4})\right] \qquad (7.22)$$

Since all of the functions on the right-hand side of (7.22) have been determined, we may, in theory, calculate p to within an additive constant c:

$$p\,(r) = p\,(r) + c.$$

By choosing the constant c properly, we may free either the inner or the outer surface of the cylindrical annulus of normal traction. By choosing p in this manner, we have satisfied the remaining field equation $(7.15)_1$.

We may simplify the problem considerably by choosing

$$\hat{\phi}\,(r) = \hat{w}\,(r) = 0. \qquad (7.23)$$

This corresponds to stretching the cylindrical annulus without shearing. Under these conditions, equation $(7.15)_1$ becomes

$$T^{<rr>} = T_0^{<rr>} = \text{constant} . \qquad (7.24)$$

Substitution of $(7.14)_1$ into (7.24) yields

$$p\,(r) = (a_1 + a_4 J_4)\lambda^2 - a_3 \lambda^{-2} + T_0^{<rr>}. \qquad (7.25)$$

With (7.25), we obtain

$$
\begin{aligned}
T^{<rr>} &= T_0^{<rr>}, \\
T^{<\phi\phi>} &= -\,T_0^{<rr>}, \\
T^{<zz>} &= (a_1 + a_4 J_4)(\lambda^{-4} - \lambda^2) - a_3(\lambda^4 - \lambda^{-2}) - T_0^{<rr>}, \\
T^{<r\phi>} &= T^{<rz>} = T^{<\phi z>} = 0.
\end{aligned}
\qquad (7.26)
$$

The temperature distribution (7.21) remains unchanged. By choosing $T_0^{<rr>}$ equal to zero, we may free the inner and the outer surface of the annulus of stress. The only non-zero stress is the normal stress acting on the ends of the cylindrical annulus. This stress must vary radially with the temperature.

8. Gough-Joule Heating

It is generally observed that if a material which is thermally insulated is deformed, its temperature will change. This phenomenon, known as Gough-Joule heating, will be investigated in this section.

In order to make the concept of Gough-Joule heating more precise, we consider the following situation. We assume that at time $t = 0$, a thermally insulated elastic heat conductor which occupies some configuration $\underset{\sim}{\chi}_1$, is at the uniform temperature θ_1. From time $t = 0$ to $t = \Delta t$, the body is deformed to some new configuration $\underset{\sim}{\chi}_2$. The body is then held in the configuration $\underset{\sim}{\chi}_2$ until the temperature reaches some new uniform temperature θ_2 at time $t = \infty$.* This process of going from one uniform temperature before deformation, to another uniform temperature after deformation, will be called a Gough-Joule process. Our goal is to find the new uniform temperature θ_2.

If we assume that the motion $\underset{\sim}{\chi}(\underset{\sim}{X}, t)$ of an elastic heat conductor may be specified arbitrarily, we may assume the motion to be given by

$$\underset{\sim}{\chi}(\underset{\sim}{X}, \tau) = \begin{cases} \underset{\sim}{\chi}(\underset{\sim}{X}, 0) , & \tau < 0 , \\ \underset{\sim}{\chi}(\underset{\sim}{X}, \tau) , & 0 \leq \tau \leq 1 , \\ \underset{\sim}{\chi}(\underset{\sim}{X}, 1) , & \tau > 1 , \end{cases} \tag{8.1}$$

where τ is a parameter related to time t through

$$\tau = \delta^{-1} t . \tag{8.2}$$

Thus, δ units of time are required for the body to deform from the configuration $\underset{\sim}{\chi}(\underset{\sim}{X}, 0)$ to the configuration $\underset{\sim}{\chi}(\underset{\sim}{X}, 1)$. In the limit as δ goes to zero, the motion becomes discontinuous in time. This motion has been referred to as a strain impulse by Coleman [20].

Basic to all that follows is the assumption that "during" a strain impulse,

---

*Of course, this assumes a certain kind of stability. However, the equilibrium equations and the condition of insulation do support the assumption that θ_2 is uniform.

no heat flows; and hence, by the result at the end of Section 3, the deformation is locally isentropic. This assumption, which leads to a correct final temperature in the linearized case (see Martin and Carlson [21]), may be motivated as follows. If a body is thermally isolated from its surroundings, and if we assume a conductivity equation of the form

$$\underset{\sim}{q} = \underset{\sim}{K}(\underset{\sim}{F}, \theta, \nabla\theta)\, \nabla\theta, \tag{8.3}$$

then the energy equation (3.10) may be written as

$$\theta\, \rho\, \dot{\eta}(\underset{\sim}{F}(\underset{\sim}{X}, t), \theta) = \nabla \cdot [\underset{\sim}{K}(\underset{\sim}{F}(\underset{\sim}{X}, t), \theta, \nabla\theta)\nabla\theta] \tag{8.4}$$

Equation (8.4) represents a partial differential equation in θ, $\underset{\sim}{X}$ and t if the motion and the conductivity functions are known.

Substituting the parameter τ for t (recall (8.2)) in (8.4) yields

$$\theta\, \rho \frac{\partial}{\partial\tau}[\eta(\underset{\sim}{F}(\underset{\sim}{X}, \tau), \theta)] = \nabla \cdot [\underset{\sim}{K}^*(\underset{\sim}{F}(X, \tau), \theta, \nabla\theta)\nabla\theta] \tag{8.5}$$

where

$$\underset{\sim}{K}^* = \delta\, \underset{\sim}{K}.$$

The tensor $\underset{\sim}{K}^*$ will be called the equivalent conductivity.

We observe that, except for the change in the conductivity tensor, equation (8.5) is identical to (8.4). Thus, we conclude that changing the rate of deformation (changing δ) is equivalent to keeping the rate of deformation fixed and changing the conductivity tensor. For a strain impulse, δ goes to zero and therefore the equivalent conductivity tensor goes to zero. Equation (8.5) becomes

$$\frac{\partial}{\partial\tau}[\eta(\underset{\sim}{X}, \tau, \theta)] = 0 \tag{8.6}$$

and therefore we have

$$\eta(\theta, \underset{\sim}{X}, \tau) = \eta_0(\underset{\sim}{X}).$$

Thus, the entropy at each point $\underset{\sim}{X}$ is the same immediately before and after the strain impulse.

We use this approach to consider the Gough-Joule heating associated with strain impulses. For strain impulses, a Gough-Joule process consists of two stages. If we regard a strain impulse as taking place over an infinitesimal time interval δ, the first stage lasts from $t = 0$ to $t = \delta$. During this time interval, the body is deformed isentropically from the initial configuration $\underset{\sim}{\chi}(\underset{\sim}{X}, 0) = \underset{\sim}{\chi}_1(\underset{\sim}{X})$ to the final configuration $\underset{\sim}{\chi}(\underset{\sim}{X}, 1) = \underset{\sim}{\chi}_2(\underset{\sim}{X})$. Employing the notation θ_u whenever a temperature field is to be regarded as uniform, the temperature at $t = 0$ is θ_{u_1}. At $t = \delta$, the temperature will, in general, be nonuniform; and it will be denoted by $\theta_1(\underset{\sim}{X})$.

The second stage lasts from $t \doteq \delta$ to $t \doteq \infty$. During this time interval, the body is held in the configuration $\underset{\sim}{\chi}_2(\underset{\sim}{X})$ while heat flows within the body until the temperature becomes uniform. The final uniform temperature will be denoted by θ_{u_2}.

During the first stage, the entropy at each material point $\underset{\sim}{X}$ is constant. Therefore, from (3.5) we have

$$\eta\,[\underset{\sim}{F}_2(\underset{\sim}{X}),\ \theta_1(\underset{\sim}{X})] = \eta\,[\underset{\sim}{F}_1(\underset{\sim}{X}),\ \theta_{u_1}]\,, \tag{8.7}$$

where $\underset{\sim}{F}_1(\underset{\sim}{X})$ and $\underset{\sim}{F}_2(\underset{\sim}{X})$ are the deformation gradients corresponding to $\underset{\sim}{\chi}_1(\underset{\sim}{X})$ and $\underset{\sim}{\chi}_2(\underset{\sim}{X})$, respectively. If the function $\eta(\underset{\sim}{F}, \theta)$ is known, equation (8.7) may be solved for $\theta_1(\underset{\sim}{X})$ as a function of $\underset{\sim}{F}_1(\underset{\sim}{X})$, $\underset{\sim}{F}_2(\underset{\sim}{X})$, and θ_{u_1}.

In the second stage, the deformation is held fixed while heat is allowed to flow within the body. The temperature distribution at time $t \doteq \delta$ is $\theta_1(\underset{\sim}{X})$. In the absence of external heat supply the energy equation (3.10) is

$$\theta\,\dot{\eta}(\underset{\sim}{F}, \theta) = \nabla \cdot \underset{\sim}{q}\,. \tag{8.8}$$

Since the deformation is fixed, and the boundary of the body is insulated, it is possible to integrate (8.8) to obtain

$$\int_B \epsilon(\underset{\sim}{X}),\ t)\,dm = \int_B \epsilon(\underset{\sim}{X}, \delta)\,dm. \tag{8.9}$$

Of course, this is also evident from overall energy balance. Thus, the total internal energy in the body remains constant during the second stage. If we assume that the temperature reaches some uniform temperature θ_{u_2} at $t = \infty$, (8.9) may be written in terms of the specific internal energy $\epsilon\,(\underset{\sim}{F}, \theta)$ as

$$\int_{\mathcal{B}} \epsilon\,[\underset{\sim}{F}_2\,(\underset{\sim}{X}),\ \theta_{u_2}]\,dm = \int_{\mathcal{B}} \epsilon\,[\underset{\sim}{F}_2\,(\underset{\sim}{X}),\ \theta_1\,(\underset{\sim}{X})]\,dm. \tag{8.10}$$

If the form of the internal energy function is known, the integration may be carried out and θ_{u_2} solved for in terms of $\underset{\sim}{F}_1\,(\underset{\sim}{X})$, $\underset{\sim}{F}_2\,(\underset{\sim}{X})$, and θ_{u_1}.

In order to proceed further, it is necessary to assume a specific form for the free energy function, and hence, specific forms for the corresponding entropy and internal energy functions. We take the form of the free energy function to be given by the second-order complete form (4.14) for incompressible materials. To simplify the calculations, we will assume that the constant $a_6 = 0$. Therefore, we have

$$\psi = a_1\,(I - 3) + a_3\,(II - 3) + a_4\,(I - 3)\,J_4 + a_5\,J_4^2\,. \tag{8.11}$$

At this point, it is convenient to adopt the notation

$\alpha = J_4,$

$\beta = (I - 3),$

$\gamma = (II - 3).$

With this notation, the entropy and internal energy functions corresponding to (8.11) are given by

$$\eta = -\,\theta_0^{-1}\,[a_4\,\beta + 2a_5\,\alpha], \tag{8.12}$$

and

$$\epsilon = (a_1 + a_4)\,\beta + a_3\,\gamma - a_5\,\left(\frac{\theta^2 - \theta_0^2}{\theta_0^2}\right). \tag{8.13}$$

The invariants β and γ corresponding to $\underset{\sim}{\chi}_1(\underset{\sim}{X})$ and $\underset{\sim}{\chi}_2(\underset{\sim}{X})$ will be denoted by β_1, γ_1, and β_2, γ_2, respectively.

Substitution of (8.12) into (8.7) yields

$$\theta_1(\underset{\sim}{X}) = k[\Delta\beta_1(\underset{\sim}{X})] + \theta_{u_1}, \tag{8.14}$$

where

$$\Delta\beta_1(\underset{\sim}{X}) = \beta_2(\underset{\sim}{X}) - \beta_1(\underset{\sim}{X})$$

and

$$k = -\left(\frac{a_4\,\theta_0}{2a_5}\right).$$

With (8.13), the requirement that the total internal energy be constant during the second stage is equivalent to

$$\theta_{u_2}^2 = \frac{1}{m}\int_{\mathcal{B}} [\theta_1(\underset{\sim}{X})]^2\,dm \tag{8.15}$$

where m is the mass of the body. Substitution of (8.14) into (8.15) gives

$$\theta_{u_2}^2 = \frac{1}{m}\int_{\mathcal{B}} [k\,\Delta\beta_1(\underset{\sim}{X}) + \theta_{u_1}]^2\,dm. \tag{8.16}$$

Equation (8.16) allows us to calculate the temperature θ_{u_2} in terms of the temperature θ_{u_1} and the configurations $\underset{\sim}{\chi}_1(\underset{\sim}{X})$ and $\underset{\sim}{\chi}_2(\underset{\sim}{X})$.

If the deformation from $\underset{\sim}{\chi}_1(\underset{\sim}{X})$ to $\underset{\sim}{\chi}_2(\underset{\sim}{X})$ is homogeneous, then (8.16) reduces to

$$\theta_{u_2} = k\,\Delta\beta_1 + \theta_{u_1}. \tag{8.17}$$

Simple Extension

As an example of Gough-Joule heating in a homogeneous deformation, we consider simple extension of an insulated cylinder. From Section 6, we recall that for incompressible materials in simple extension

$$\beta = J_1 = 2\,W^{-1} + W^2 - 3.$$

If we deform the material from the reference state,

$$\Delta\beta_1 = 2\,W^{-1} + W^2 - 3,$$

$$\theta_{u_1} = \theta_0.$$

From (8.17), we obtain

$$\theta_{u_2} = \theta_0 - \left(\frac{a_4\,\theta_0}{2a_5}\right)[2\,W^{-1} + W^2 - 3]. \tag{8.18}$$

For rubber, we have

$$k = -\left(\frac{a_4\,\theta_0}{2a_5}\right) \geq 0.^*$$

Since the function $(2\,W^{-1} + W^2 - 3)$ has a minimum at $W = 1$, an insulated rubber cylinder will increase in temperature whenever it is stretched (or compressed) from the reference configuration.

Pure Torsion of a Right-Circular Cylinder

As an example of Gough-Joule heating in a nonhomogeneous deformation, we consider pure torsion of an insulated right-circular cylinder of radius r_0. If the particle originally located at the point (R, Φ, Z) in circular cylindrical coordinates moves to the position (r, ϕ, z) given by

$$r = R, \qquad \phi = \Phi + B\,Z, \qquad z = Z, \tag{8.19}$$

then

$$\beta(\underset{\sim}{X}) = B^2\,r^2.$$

If the body, initially at the temperature θ_1, is deformed from the reference configuration, we have

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*In Section 6 it was shown that $a_4 > 0$ for rubber. From equation (5.7), $a_5 < 0$.

$$\Delta\beta_1 = B^2 r^2, \quad \theta_{u_1} = \theta_1. \tag{8.20}$$

Substituting (8.20) into (8.16) and integrating gives

$$\theta_{u_2}^2 = \frac{1}{3}\,(y^2 + 3\,y\,\theta_1 + 3\,\theta_1^2), \tag{8.21}$$

where

$$y = k\,B^2\,r_0^2. \tag{8.22}$$

If the cylinder is composed of rubber, $k \geq 0$, and therefore $y \geq 0$. From (8.21), θ_{u_2} is a monotonically increasing function of y if $y \geq 0$. Thus, from (8.22), we see that the temperature increases if the cylinder is twisted in either the positive or negative direction -- a result which should hold due to symmetry.

9. Successive Gough-Joule Processes

In the previous section, it was shown that if the motion of a body is a strain impulse, it is possible to calculate the final uniform temperature of a Gough-Joule process in terms of the initial uniform temperature and the change in configuration. This relation represents a type of difference equation between the uniform temperatures and the configurations of the body in a Gough-Joule process. In the present section, it will be shown that for general motions this difference equation cannot be integrated to yield the uniform temperature as a function of the deformation only, as is possible in linear thermoelasticity [22]. The fact that the temperature cannot be found as a function of the configuration of the body means that the Gough-Joule heating is dependent on the path which the body takes in time-configuration space. For Gough-Joule processes involving motions other than strain impulses, the above results indicate that the Gough-Joule heating will depend upon the rate of deformation as well as upon the amount of deformation.

We assume that we have a sequence of Gough-Joule processes (of the strain impulse type of Section 8) in which the uniform temperature reached at the end of the i^{th} process represents the initial uniform temperature of the $(i+1)^{th}$ process. The uniform temperature at the beginning of the i^{th} process will be denoted by θ_{u_i}. During the i^{th} process, the body is deformed from the i^{th} to the $(i+1)^{th}$ configuration. The invariants β and γ corresponding to the i^{th} configuration will be denoted by β_i and γ_i.

With this notation, equation (8.16) becomes

$$\theta^2_{u_{(i+1)}} = \frac{1}{m} \int_{\mathcal{B}} [k\, \Delta\beta_i (\underset{\sim}{X}) + \theta_{u_i}]^2\, dm, \qquad (9.1)$$

where

$$\Delta\beta_i (\underset{\sim}{X}) = \beta_{(i+1)} (\underset{\sim}{X}) - \beta_i (\underset{\sim}{X}).$$

Equation (9.1) represents what could be called an integral-difference equation of the form

$$y_{i+1} = \mathcal{G}\,[y_i,\ \Delta u_i\,(\underset{\sim}{X})]$$

Conceivably, one might hope to solve (9.1) for θ_u as a function of the scalar field $\beta\,(\underset{\sim}{X})$, as is possible in linear thermoelasticity. That this is, in general, impossible, will now be shown.

We denote the set of all scalar fields by S, the set of all real numbers by R, and establish the following theorem.

Theorem. A necessary condition for a functional-difference equation of the form

$$y_{i+1} = \mathcal{G}\,[y_i,\ \Delta u_i\,(\underset{\sim}{X})] \qquad (9.2)$$

to have a solution $F\colon\ S \rightarrow R$, such that

$$y = F\,[u\,(\underset{\sim}{X})] \qquad (9.3)$$

is that

$$\mathcal{G}\,[y_1,\ \Delta u_1\,(\underset{\sim}{X}) + \Delta u_2(\underset{\sim}{X})] = \mathcal{G}\Big\{\mathcal{G}\,[y_1,\ \Delta u_1\,(\underset{\sim}{X})],\ \Delta u_2\,(\underset{\sim}{X})\Big\} \qquad (9.4)$$

for every scalar y_1 and for all scalar fields $\Delta u_1\,(\underset{\sim}{X})$ and $\Delta u_2\,(\underset{\sim}{X})$.

Proof. Suppose that (9.2) has a solution of the form (9.3). Let $u_1\,(\underset{\sim}{X})$, $u_2\,(\underset{\sim}{X})$, and $u_3\,(\underset{\sim}{X})$ be arbitrary elements of S. Define $\Delta u_1\,(\underset{\sim}{X}), \Delta u_2\,(\underset{\sim}{X})$, and $\Delta u_3\,(\underset{\sim}{X})$ by

$$\Delta u_1\,(\underset{\sim}{X}) = u_2\,(\underset{\sim}{X}) - u_1\,(\underset{\sim}{X}),$$

$$\Delta u_2\,(\underset{\sim}{X}) = u_3\,(\underset{\sim}{X}) - u_2\,(\underset{\sim}{X}),$$

$$\Delta u_3\,(\underset{\sim}{X}) = u_3\,(\underset{\sim}{X}) - u_1\,(\underset{\sim}{X}) = \Delta u_1\,(\underset{\sim}{X}) + \Delta u_2\,(\underset{\sim}{X}).$$

Then, by (9.2) and (9.3),

$$y_2 = F\,[u_2\,(\underset{\sim}{X})] = \mathcal{G}\,[y_1,\ \Delta u_1\,(\underset{\sim}{X})],$$

$$y_3 = F\,[u_3\,(\underset{\sim}{X})] = \mathcal{G}\,[y_2,\ \Delta u_2\,(\underset{\sim}{X})]$$

$$= \mathcal{G}\Big\{\mathcal{G}\,[y_1,\ \Delta u_1\,(\underset{\sim}{X})],\ \Delta u_2\,(\underset{\sim}{X})\Big\}\ .$$

We may also express y_3 by

$$y_3 = \mathcal{G}\,[y_1, \Delta u_3 (\underset{\sim}{X})] = \mathcal{G}\,[y_1, \Delta u_1 (\underset{\sim}{X}) + \Delta u_2 (\underset{\sim}{X})],$$

and therefore, we must have

$$\mathcal{G}\left\{\mathcal{G}\,[y_1, \Delta u_1 (\underset{\sim}{X})]\,,\, \Delta u_2 (\underset{\sim}{X})\right\} = \mathcal{G}\,[y_1, \Delta u_1 (\underset{\sim}{X}) + \Delta u_2 (\underset{\sim}{X})]$$

for every scalar y_1 and for all scalar fields $\Delta u_1 (\underset{\sim}{X})$ and $\Delta u_2 (\underset{\sim}{X})$. This completes the proof.

We now show that (9.1) does not, in general, satisfy the condition (9.4). Let $\beta_1 (\underset{\sim}{X})$, $\beta_2 (\underset{\sim}{X})$, and $\beta_3 (\underset{\sim}{X})$ be arbitrary elements of S. Define $\Delta\beta_2 (\underset{\sim}{X})$ and $\Delta\beta_2 (\underset{\sim}{X})$ by

$$\Delta\beta_1 (\underset{\sim}{X}) = \beta_2 (\underset{\sim}{X}) - \beta_1 (\underset{\sim}{X}), \tag{9.5}$$

$$\Delta\beta_2 (\underset{\sim}{X}) = \beta_3 (\underset{\sim}{X}) - \beta_2 (\underset{\sim}{X}).$$

Substitution of (9.1) and (9.5) into (9.4) yields

$$\int_{\mathcal{B}} \left\{ k\, \Delta\beta_2 (\underset{\sim}{X}) + \left[\frac{1}{m} \int_{\mathcal{B}} (k\, \Delta\beta_1 (\underset{\sim}{X}) + \theta_{u_1})^2\, dm\right]^{\frac{1}{2}} \right\}^2 dm$$

$$= \int_{\mathcal{B}} \left\{ k\, [\Delta\beta_1 (\underset{\sim}{X}) + \Delta\beta_2 (\underset{\sim}{X})] + \theta_{u_1} \right\}^2 dm \tag{9.6}$$

Expanding and collecting terms in (9.6) leads to

$$\int_{\mathcal{B}} \left\{ 2k\, \Delta\beta_2 (\underset{\sim}{X}) \left[\frac{1}{m} \int_{\mathcal{B}} (h (\underset{\sim}{X}))^2\, dm\right]^{\frac{1}{2}} + \frac{1}{m} \int_{\mathcal{B}} (h (\underset{\sim}{X}))^2\, dm \right.$$

$$\left. - 2k\, \Delta\beta_2 (\underset{\sim}{X})\, h (\underset{\sim}{X}) - [h (\underset{\sim}{X})]^2 \right\} dm = 0, \tag{9.7}$$

where

$$h (\underset{\sim}{X}) = k\, \Delta\beta_1 (\underset{\sim}{X}) + \theta_{u_1}. \tag{9.8}$$

If $\Delta\beta_1$ is arbitrary, then (9.8) makes it clear that $h (\underset{\sim}{X})$ is arbitrary.

To show that (9.7) is not valid for arbitrary choices of $\Delta\beta_2 (\underset{\sim}{X})$ and $h (\underset{\sim}{X})$, we take

$$\Delta\beta_2 (\underset{\sim}{X}) = 1 \tag{9.9}$$

and define the quantities I_1 and I_2 by

$$I_1 = \left\{ \frac{1}{m} \int_{\mathcal{B}} [h (\underset{\sim}{X})]^2 \, dm \right\}^{\frac{1}{2}} \tag{9.10}$$

and

$$I_2 = \frac{1}{m} \int_{\mathcal{B}} h (\underset{\sim}{X}) \, dm. \tag{9.11}$$

Substitution of (9.9), (9.10) and (9.11) into (9.7) yields

$$I_1 = I_2. \tag{9.12}$$

Since I_1 does not generally equal I_2, we conclude that the integral-difference equation (9.1) does not satisfy the conditions of the theorem. Hence, we cannot solve (9.1) for θ_u as a function of β.

Thus, we conclude that two rapid deformations, the second occurring only after the body has reached uniform temperature, are not equivalent to a single rapid deformation which is the composition of the two deformations. In particular, if the second deformation is the inverse of the first, the body will not necessarily return to its initial temperature. This result suggests that the Gough-Joule heating depends upon the rate of deformation. Since the free energy function which has been used could, conceivably, correspond to a real material, we conclude that the Gough-Joule heating of a general elastic heat conductor is rate dependent.

If the deformation of the material is homogeneous, equation (9.1) reduces to

$$\theta_{u_{(i+1)}} = k \, \Delta\beta_i + \theta_{u_i} \tag{9.13}$$

Equation (9.13) represents a linear first-order difference equation whose solution is given by [22] as

$$\theta_u = k \beta + \theta_0 . \tag{9.14}$$

Thus, if the deformation is homogeneous and compounded from a sequence of strain impulses, the Gough-Joule heating depends only upon the deformation.

Cyclic Pure Torsion of a Right-Circular Cylinder

We assume that a right-circular cylinder, which is initially in the reference configuration and at the uniform temperature θ_1, is deformed according to the deformation (8.19). In Section 8, it was found that the final uniform temperature θ_{u_2} was given by (8.21) and (8.22).

Now, let us assume that after the body has reached the uniform θ_{u_2}, the body is returned to the reference configuration by another strain impulse. Thus, for this case we have

$$\Delta\beta_2 = - B^2 r^2, \qquad (9.15)$$

$$\theta_{u_2} = [\tfrac{1}{2}(y^2 + 3\, y\, \theta_0 + 3\, \theta_0^2)]^{\frac{1}{2}}. \qquad (9.16)$$

Substitution of (9.15) and (9.16) into (9.1) yields

$$\theta_{u_3}^2 - \theta_1^2 = \frac{2}{3}\, y^2 + y\, \theta_1 - y\, (\frac{1}{3}\, y^2 + y\, \theta_1 + \theta_1^2)^{\frac{1}{2}} \qquad (9.17)$$

If $y > 0$ (which is an appropriate assumption for rubber), it is easily shown that (9.17) implies

$$\theta_{u_3}^2 - \theta_1^2 > 0.$$

Hence, if $y > 0$, a cylinder subjected to cyclic strain impulses will experience a temperature buildup. A similar result has been obtained by Dillon [8] in his theory for a cylinder subjected to infinitesimal sinusoidal twisting.

List of References

1. Petroski, H. J., and D. E. Carlson, Controllable states of rigid heat conductors. Z. angew. Math. Phys. 19, 372-376 (1968).

2. Petroski, H. J., and D. E. Carlson, Controllable states of elastic heat conductors. Arch. Rational Mech. Anal. 31, 127-150 (1968).

3. Petroski, H. J., and D. E. Carlson, Some exact solutions to the equations of non-linear thermoelasticity. J. Appl. Mech. 37, 1151-1154 (1970).

4. Laws, N., Controllable states in thermoelasticity. Quart. Appl. Math. 28, 432-436 (1970).

5. Murnaghan, F. D., Finite deformation of an elastic solid. Amer. J. Math. 59, 235-260 (1937).

6. Rivlin, R. S., Some topics in finite elasticity. Proc. of the First Naval Symp. on Structural Mech., Pergamon Press, New York, 169-198 (1960).

7. Singh, S., Small finite deformations of elastic dielectrics. Quart. Appl. Math. 25, 275-284 (1967).

8. Dillon, O. W., A nonlinear thermoelasticity theory. J. Mech. Phys. Solids 10, 123-131 (1962).

9. Herrmann, G., On second-order thermoelastic effects. Z. angew. Math. Phys.15, 235-262 (1964).

10. Chadwick, P., and L. T. C. Seet, Second-order thermoelasticity theory for isotropic and transversely isotropic materials. In Trends in Elasticity and Thermoelasticity, edited by R. E. Czarnota-Bojarski, M. Sokolowski, and H. Zorski. Gronigen, Walters-Noordhoff (1971).

11. Yarrington, P., and D. E. Carlson, Successive approximations in nonlinear thermoelasticity. (Forthcoming).

12. Coleman, B. D., and W. Noll, The thermodynamics of elastic materials with heat conduction and viscosity. Arch. Rational Mech. Anal. 13, 245-261 (1963).

13. Treloar, L. R. G., The Physics of Rubber Elasticity. 2nd ed., Clarendon Press, Oxford (1958).

14. Rivlin, R. S., and D. W. Saunders, Large elastic deformations of isotropic materials, VII. Experiments on the deformation of rubber. Phil. Trans. Roy. Soc. Lond., A 243, 251-288 (1951).

15. Baker, M., and J. L. Ericksen, Inequalities restricting the form of the stress-deformation relations for isotropic elastic solids and Reiner-Rivlin fluids. J. Wash. Acad. Sci. 44, 33-35 (1954).

16. Truesdell, C. and W. Noll, The non-linear field theories of mechanics. In Vol. III/3 of the Handbuch der Physik, edited by S. Flugge. Berlin-Gottingen-Heidelberg: Springer-Verlag (1965).

17. Wang, C. -C., and C. Truesdell, Introduction to Rational Elasticity, Noordhoff, Leyden (1973).

18. Anthony, R. L., R. H. Caston, and E. Guth, Equations of state for natural and synthetic rubber-like materials. I. J. Phys. Chem. 46, 826-840 (1942).

19. Treloar, L. R. G., The elasticity and related properties of rubbers. Rep. Prog. Phys. 36, 755-826 (1973).

20. Coleman, B. D., On thermodynamics, strain impulses, and viscoelasticity. Arch. Rational Mech. Anal. 17, 230-254 (1964).

21. Martin, S. E., and D. E. Carlson, On the Gough-Joule heating of insulated bodies in linear thermoelasticity. (Forthcoming).

22. Goldberg, S., Introduction to Difference Equations, John Wiley and Sons, Inc., New York (1958).