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CONSERVATION LAWS IN ELASTICITY OF THE J-INTEGRAL TYPE

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## ABSTRACT

Conservation laws which are expressible as functionals linear in the strain energy and its derivatives are laws of the same type as the J-integral. For finite elastic deformations of homogeneous bodies, relations between the conservation laws are shown through the use of inverse deformation results. Completeness of the laws are established for homogeneous materials and for materials whose strain energies satisfy objectivity, isotropy, or are homogeneous functions. Laws for a class of membranes inflated by pressure are derived and applied to a cylindrical membrane. For infinitesimal deformations of linear elastic bodies, new laws which relate two independent equilibrium states are presented and applied to the problem of a line crack in a plate under mixed-mode loading conditions. A relation is shown to exist between the J-integral and the reciprocal work theorem of Betti.

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## 1. Introduction

Recent applications (e.g. [1, 2, 3])<sup>1</sup> of a two-dimensional conservation law often referred to as the J-integral to the important field of fracture mechanics have generated interest in the theoretical foundations of conservation laws in elasticity. The J-integral was obtained by Rice [4] in the course of approximate analyses of singular stress fields associated with cracks and voids in homogeneous elastic bodies. Earlier work which provides a three-dimensional version of the J-integral is to be found in papers by Eshelby [5] and by Günther [6]. The J-integral of [4], while applicable to nonlinear materials, holds only for infinitesimal deformations.

More recently, Knowles and Sternberg [7] have shown that an additional conservation law holds if the material is linear, while another law results if the material is isotropic (and perhaps nonlinear). In [7], and in a later paper by Green [8], it was shown that analogous laws exist for finite deformations of homogeneous elastic materials. The derivation of conservation laws in [7] was based on a restricted form of Noether's theorem [9] on invariant variational principles together with the principle of stationary potential energy. On the other hand, in [8] a direct approach was used to demonstrate that the conservation laws follow from certain symmetries which are satisfied by the strain energy density function.

Except for the last two sections of this paper, our discussion concerns conservation laws in finite elasticity. In Section 2, we summarize results of [10] which show that an equilibrium deformation of a homogeneous elastic body  $B$  into a body  $B^*$  implies the existence of an inverse equilibrium deformation of a body  $B^*$  into  $B$  for another (homogeneous)

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<sup>1</sup>Numbers in brackets designate references at the end of the paper.

elastic material. The J-integral is then shown to be a consequence of vanishing total force on arbitrary portions of the body for the inverse deformation.<sup>2</sup> When the strain energy function depends only on the Cauchy strains, and so satisfies objectivity, the resultant surface moment on arbitrary portions of the body vanishes; and this result, through the inverse deformation, is shown to lead to the conservation law [7, 8] for isotropic materials.

For infinitesimal deformations, the completeness of the conservation laws within the framework of Noether's theorem was proved in [7]. In Sections 3, 4 and 5, we investigate completeness of the conservation laws for finite elastic deformations among the class of laws expressible in the form (3.1) of Section 3 as functionals linear in the strain energy  $W$  and its first derivatives with respect to the deformation gradients. This approach allows for laws which do not necessarily follow from an infinitesimal invariance of a (strain energy) functional,<sup>3</sup> as required by Noether's theorem (see [7]), and the procedure provides a means for deriving new laws.

If the form of the strain energy function for a homogeneous elastic body of grade-1 material<sup>4</sup> is not restricted in any way (for example, no assumptions of objectivity or material symmetry), we show in Section 3 that the total force integral and the J-integral are the only two conservation laws belonging to the class of laws described earlier. Section 4 establishes

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<sup>2</sup>This result is implicit in the articles of Chadwick [11] and Eshelby [12].

<sup>3</sup>See eq. (7.11) and ensuing remarks.

<sup>4</sup>For conservation laws for materials of higher grades, the reader is referred to a recent paper by Eshelby [12]. Also, higher order theories lead to laws for plates and shells [6].



completeness for objective and/or isotropic materials with the assumption that the coefficients  $a_m$  and  $b_{ikm}$  in (3.1) depend only on the initial and final coordinates  $x_r$  and  $y_s$  and the derivatives  $\partial y_p / \partial x_q$ ,<sup>5</sup> and no new laws result. In Section 5, a similar proof is sketched for the completeness of conservation laws for materials whose strain energy is a homogeneous function of degree  $N$  in the deformation gradients. In this case a new law results which, for  $N = 2$ , is an extension to finite deformations of a law [7] for infinitesimal deformations. At the end of Section 3, we briefly discuss laws which arise when a linear combination of the second derivatives of the strain energy is included. We show that only one additional law results which, in contrast to the other laws, requires equilibrium to be satisfied only on the surface and not within the region enclosed.

While conservation laws do not hold in general for membranes finitely deformed under an applied pressure or body forces, we show in Section 6 that if the homogeneous membrane is initially a plane or developable surface, conservation laws can also be derived. When one of the resulting laws is applied to cylindrical membranes inflated by pressure, it is shown to reduce to a first integral of the meridional equilibrium equation.

The well-known reciprocal work theorem of Betti is an example of a conservation law for two independent equilibrium displacement fields in linear elasticity. In Section 7, we obtain additional conservation laws for two equilibrium deformations in linear elasticity. Although these laws could be derived with the use of Noether's theorem, we derive them directly from the conservation laws for a single deformation and the principle

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<sup>5</sup>This restriction is not imposed in Section 3.



of superposition. We note that it does not appear possible to derive Betti's theorem from Noether's theorem. However, the J-integral for linear elasticity is shown to be related to a particular application of Betti's theorem. Section 8 discusses application of the results of Section 7 to the determination of the stress intensity factors for a line crack under mixed-mode loading conditions.



## 2. Conservation Laws for Elastic Bodies

We consider a body composed of elastic material (of grade 1) and use coordinates referred to a fixed rectangular Cartesian coordinate system to describe particle locations. During a deformation of the unstrained body  $B$ , a particle at the point  $x_i$  ( $i = 1, 2, 3$ ) in  $B$  is displaced to the point  $y_i$  in the deformed body  $B^*$ , with

$$y_i = f_i(x_r). \quad (2.1)$$

The functions  $y_i$  are assumed to have as many continuous derivatives as the analysis requires. Here, and subsequently, Latin indices assume the values 1, 2, 3 and we adopt the notations

$$\frac{\partial y_r}{\partial x_s} = y_{r,s}, \quad \frac{\partial x_r}{\partial y_s} = x_{r;s}. \quad (2.2)$$

It follows that

$$y_{i,r} x_{r;j} = x_{i;r} y_{r,j} = \delta_{ij}, \quad (2.3)$$

where a repeated index implies summation over its range. Conservation of mass requires

$$\det(y_{r,s}) = |y_{r,s}| = \frac{\rho}{\rho^*}, \quad (2.4)$$

where  $\rho$  and  $\rho^*$  are the densities for  $B$  and  $B^*$  respectively. Throughout this paper, an asterisk (\*) is used to denote a quantity which is referred to the deformed body  $B^*$ .

We denote by  $W$  the strain energy per unit volume of the undeformed body  $B$ . For an inhomogeneous material,  $W$  depends explicitly on  $x_i$  as well as on the deformation gradients  $y_{r,s}$ . We shall consider only initially homogeneous materials so that we have

$$W = W(y_{r,s}). \quad (2.5)$$



We emphasize that, at this stage, no restriction is placed on the functional form of  $W$  through material objectivity or material symmetry.

When body forces are absent, the well-known equations of equilibrium are

$$\frac{\partial}{\partial x_r} \left( \frac{\partial W}{\partial y_{i,r}} \right) = 0 . \quad (2.6)$$

These equations are the Euler differential equations for the functional  $F$  defined by

$$F \{y_i\} = \int_V W \, dV, \quad (2.7)$$

where  $V$  is the region in space occupied by the body in the reference state.

Alternatively, we can write

$$W = \frac{\rho}{\rho^*} W^* = |y_{r,s}| W^* , \quad (2.8)$$

where  $W^*$  is the strain energy per unit volume of the deformed body  $B^*$ . By transforming the integral in (2.7) to an integral over the deformed body, we have

$$F = \int_{V^*} W^* \, dV^* . \quad (2.9)$$

Because we can express  $W^*(y_{i,r})$  as a function  $G^*(x_{i;r})$ , we may treat  $F$  as a functional  $F^*\{x_i\}$  of the functions  $x_i(y_r)$ . The Euler equations for  $F^*\{x_i\}$  are

$$\frac{\partial}{\partial y_r} \left( \frac{\partial W^*}{\partial x_{i;r}} \right) = 0 , \quad (2.10)$$

and they are alternative forms of the equilibrium equations. Equations (2.10) were derived in [10] (see also [13]), and they apply only for materials homogeneous in the reference state.



As indicated in [10], comparison of (2.6) and (2.10) provides a result concerning the inverse deformation which would deform the body  $B^*$  into the body  $B$ . If the body  $B^*$  is treated as the reference state, eq. (2.10) shows that the deformation  $y_i \rightarrow x_i$  (of the body  $B^*$  into the body  $B$ ), defined implicitly by eq. (2.1), would provide an equilibrium state for a material with strain energy  $W^* = G^*(x_{i;r})$  per unit volume of the (homogeneous) body  $B^*$ .

We turn now to the various conservation laws applicable to finite elastostatics. Let  $V$  denote an arbitrary portion of the region  $V$ , and let  $S$  be its closed boundary with unit outward normals  $n_i$ . Under the deformation (2.1),  $V$  is deformed to  $V^*$  and  $S$  to  $S^*$  with unit outward normals  $n_i^*$ . For corresponding surface elements  $dS$  and  $dS^*$ , we have

$$n_i^* dS^* = |y_{j,k}| x_{r;i} n_r dS, \quad n_i dS = |x_{j;k}| y_{r,i} n_r^* dS^*. \quad (2.11)$$

When the equilibrium equations (2.6) hold throughout the region  $V$ , then for each portion  $V$  of the body, use of the divergence theorem leads to

$$\int_S \frac{\partial W}{\partial y_{i,r}} n_r dS = 0, \quad (2.12)$$

which states that the total force acting on the surface  $S$  in the deformed position vanishes. For homogeneous materials, we also have the conservation law [7]

$$\int_S \left\{ W n_i - \frac{\partial W}{\partial y_{r,s}} y_{r,i} n_s \right\} dS = 0. \quad (2.13)$$

The result (2.13) is the three-dimensional counterpart for finite elastic deformations of the two-dimensional J-integral of [4] (see also [8, 12]).

For the deformation  $y_i \rightarrow x_i$  of the body  $B^*$  into  $B$  with strain energy  $W^*$ , the total force on a portion of  $B^*$  with boundary surface  $S^*$  must vanish so that, as implied by (2.10),

$$\int_{S^*} \frac{\partial W^*}{\partial x_{i;r}} n_r^* dS^* = 0 . \quad (2.14)$$

The integral in (2.14) can be transformed directly to an integral over  $S$  by using (2.8) and (2.11). A straight-forward calculation shows that the statement (2.14) is identical to the conservation law (2.13).

Thus we can interpret (2.13) as a consequence of vanishing total surface force on an arbitrary portion of the body  $B^*$  with strain energy  $W^* = G^*(x_{i;r})$  undergoing the inverse deformation  $y_i \rightarrow x_i$ , defined implicitly by (2.1).

On the other hand, (2.12) can be transformed into

$$\int_{S^*} \left\{ W^* n_i^* - \frac{\partial W^*}{\partial x_{r;s}} x_{r;i} n_s^* \right\} dS^* = 0 , \quad (2.15)$$

and the integral in (2.15) is the three-dimensional J-integral for the deformation  $y_i \rightarrow x_i$  of  $B^*$  into  $B$ . The duality between the integrands of (2.13) and (2.15) has been noted in [11] and the equivalence of (2.13) and (2.14) in [12].

Additional conservation laws hold if restrictions are placed on the functional form for  $W$ . If  $W$  depends only on the Cauchy strains,

$$W = W(C_{rs}) , \quad C_{rs} = y_{i,r} y_{i,s} , \quad (2.16)$$

so that material objectivity is satisfied but not necessarily any material symmetry conditions, then in addition to (2.12) and (2.13), we have the conservation law



$$\int_S e_{rim} y_r \frac{\partial W}{\partial y_{i,s}} n_s dS = 0 . \quad (2.17)$$

This is a statement that the total moment of the surface tractions on  $S$  vanishes, a consequence of the symmetry of the Eulerian stress  $t_{ij}$  (see [8]),

$$t_{ij} = |x_{r;s}| \frac{\partial W}{\partial y_{j,k}} y_{i,k} = |x_{r;s}| \frac{\partial W}{\partial y_{i,k}} y_{j,k} = t_{ji} , \quad (2.18)$$

which follows directly from (2.16).

If the material is initially isotropic, but does not necessarily satisfy material objectivity, then

$$W = W(B_{rs}) , \quad B_{rs} = y_{r,i} y_{s,i} , \quad (2.19)$$

where  $B_{rs}$  represents the Finger strains. We then have the symmetry relation

$$\frac{\partial W}{\partial y_{k,j}} y_{k,i} = \frac{\partial W}{\partial y_{k,i}} y_{k,j} \quad (2.20)$$

and with this relation it can be shown that [8]

$$\int_S e_{srm} x_r \left\{ W n_s - \frac{\partial W}{\partial y_{i,k}} y_{i,s} n_k \right\} dS = 0 . \quad (2.21)$$

As pointed out in [8], (2.17) and (2.21) together provide the conservation law derived by Knowles and Sternberg for isotropic and objective materials (eq. (4.27) of [7]).

For a material transversely isotropic about the  $x_3$  axis, (2.20) is replaced by

$$\frac{\partial W}{\partial y_{k,\beta}} y_{k,\alpha} = \frac{\partial W}{\partial y_{k,\alpha}} y_{k,\beta} \quad (\alpha, \beta = 1, 2) . \quad (2.22)$$

This relation implies that the conservation law (2.21) holds with the subscript  $m$  set equal to 3 (also see [12]).

When the material of  $B$  obeys material objectivity, condition (2.16), the strain energy  $W^*$  is invariant under rigid rotations of  $B^*$  so that, with  $B^*$  viewed as the reference state,  $W^*$  is isotropic. Conversely, when  $W$  for  $B$  is an isotropic function,  $W^*$  for  $B^*$  satisfies material objectivity. In the latter case, for zero resultant moment for a portion of  $B^*$  with boundary surface  $S^*$ , we must have

$$\int_{S^*} e_{rim}^x \frac{\partial W^*}{\partial x_{i;s}} n_s^* dS^* = 0. \quad (2.23)$$

Direct transformation of the integral in (2.23) to an integral over  $S$  leads to the conservation law (2.21) for isotropic  $B$ . Alternatively, the law corresponding to (2.21) for an isotropic body  $B^*$  with reference state strain energy  $W^*$  transforms directly into the statement (2.17) of vanishing resultant moment for the body  $B$ .

We therefore see that because of the existence of the inverse equilibrium deformation, through the alternative form of the equilibrium equations (2.10) for homogeneous bodies, the conservation laws (2.13) and (2.21) can be deduced from the statements (2.12) and (2.17), respectively, of vanishing resultant force and moment.

We now introduce a new conservation law which holds for a particular class of materials undergoing finite deformations. When  $W(y_{i,k})$  is a homogeneous function of degree  $N$ , we have the relation

$$\frac{\partial W}{\partial y_{i,k}} y_{i,k} = N W. \quad (2.24)$$

For a body composed of a homogeneous material whose strain energy  $W$  satisfies (2.24), then in addition to the conservation laws (2.12) and (2.13), it is easily shown that we have, for  $N \neq 0$ ,

$$\int_S \left\{ W x_m n_m - \frac{\partial W}{\partial y_{i,k}} \left[ x_m y_{i,m} + \left( \frac{3}{N} - 1 \right) y_i \right] n_k \right\} dS = 0. \quad (2.25)$$

(The number 3 comes from the term  $\delta_{ii}$  and is to be replaced by 2 for deformations in a two-dimensional space.) When  $N = 0$ , we have

$$\int_S \frac{\partial W}{\partial y_{i,k}} y_i n_k dS = 0. \quad (2.26)$$

The law (2.25) is the immediate counterpart in finite elasticity of the law (3.19) of [7] for linear elasticity ( $N = 2$ ).<sup>6</sup>

When  $W(y_{i,k})$  is a homogeneous function of degree  $N$  in  $y_{i,k}$ , the strain energy  $W^*(x_{r;s})$ , for the inverse deformation  $y_i \rightarrow x_i$ , will be a homogeneous function of degree  $3-N$  in  $x_{r;s}$ . The law for  $B^*$  corresponding to (2.25) for  $B$  is

$$\int_{S^*} \left\{ W^* y_m n_m^* - \frac{\partial W^*}{\partial x_{i,k}} \left[ y_m x_{i,m} + \left( \frac{3}{3-N} - 1 \right) x_i \right] n_k^* \right\} dS^* = 0. \quad (2.27)$$

However, no new law is generated since a direct transformation of (2.27) leads back to (2.25).

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<sup>6</sup>See also a comment in [14]; the term  $(m-2)T_i u_i$  should be replaced by  $[(m-2)/m]T_i u_i$ .

### 3. Completeness for Arbitrary $W(y_{i,k})$

If the strain energy function  $W(y_{i,k})$  of a homogeneous elastic material is not restricted in form either by material symmetry or by material objectivity, then the two conservation laws (2.12) and (2.13) hold. For solutions  $y_i(x_r)$  of the equilibrium equations (2.6) with  $W(y_{i,k})$  arbitrary, we now show that (2.12) and (2.13) are the only conservation laws which belong to the class of laws

$$\int_S \left\{ W a_m + \frac{\partial W}{\partial y_{i,k}} b_{ikm} \right\} n_m dS = 0, \quad (3.1)$$

where  $a_m$ ,  $b_{ikm}$  are functions of  $x_r$  and of  $y_s$  and its derivatives with respect to  $x_r$ , but are independent of the form of  $W$ . It will be sufficient to assume that  $a_m$  and  $b_{ikm}$  are  $C^2$  in their arguments. Equation (3.1) includes all conservation laws with integrands which are linear in the strain energy  $W$  and its first derivatives  $\partial W / \partial y_{i,k}$ . At the end of this section we discuss the laws which arise if a term linear in the second derivatives of  $W$  is also included in the integrand in (3.1).

We begin by applying the divergence theorem to the integral in (3.1).

The result is

$$\int_V \left\{ W a_{m,m} + \frac{\partial}{\partial x_m} \left( \frac{\partial W}{\partial y_{i,k}} \right) b_{ikm} + \frac{\partial W}{\partial y_{i,k}} (a_m y_{i,km} + b_{ikm,m}) \right\} dV = 0. \quad (3.2)$$

We recall that  $V$  is an arbitrary portion of the region of space  $V$  occupied by the undeformed body  $B$ , and therefore (3.2) implies that the integrand must vanish identically in  $V$ .

At any point  $x_i$  of  $V$ , the values of  $y_i$ ,  $y_{i,k}$  may be chosen arbitrarily and independently of the values of higher derivatives of  $y_i$ . Also because the function  $W(y_{i,k})$  is arbitrary, the values of  $W$  and

$\partial W / \partial y_{i,k}$  at a point may be chosen independently of the values of higher derivatives of  $W$ . Thus the integrand in (3.2) will vanish, for arbitrary  $W$ , only if

$$a_{m,m} = 0, \quad (3.3)$$

and

$$a_m y_{i,km} + b_{ikm,m} = 0, \quad (3.4)$$

and then we must also have

$$b_{ikm} \frac{\partial}{\partial x_m} \left( \frac{\partial W}{\partial y_{i,k}} \right) = 0. \quad (3.5)$$

Now the values of the second derivatives  $y_{r,sk}$  and  $\partial^2 W / \partial y_{i,k} \partial y_{r,s}$  are restricted by the equilibrium equations (2.6), which are

$$\frac{\partial^2 W}{\partial y_{i,k} \partial y_{r,s}} y_{r,sk} = 0, \quad (3.6)$$

while (3.5) can be written

$$\frac{\partial^2 W}{\partial y_{i,k} \partial y_{r,s}} b_{ikm} y_{r,sm} = 0. \quad (3.7)$$

With  $b_{ikm}$  independent of the form of the arbitrary functions  $W$ , (3.7) will be an identity for all functions  $y_i$  satisfying (3.6) if and only if

$$b_{ikm} = \lambda_i \delta_{km}, \quad (3.8)$$

where  $\lambda_i$  depends only on  $x_r$ ,  $y_s$  and the derivatives of  $y_s$ .

Substitution of (3.8) into (3.4) implies that

$$a_m y_{i,km} + \lambda_{i,k} = 0, \quad (3.9)$$

and eliminating  $\lambda_i$ , we obtain

$$a_{m,r} y_{i,km} - a_{m,k} y_{i,rm} = 0. \quad (3.10)$$

If we now write

$$a_M = (a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}, a_{3,1}, a_{3,2}, a_{3,3}) ,$$

equations (3.10) can be written in the form

$$\sum_N A_{MN} a_N = 0 , \quad (3.11)$$

where  $M, N$  range from 1 to 9. The first, fifth and ninth columns of the coefficient matrix  $A_{MN}$  are found to be linearly dependent and so  $\det(A_{MN})$  is zero. However,  $A_{MN}$  is of rank 8 in general because the cofactors of the elements  $A_{M1}$  (which are identical to the cofactors of  $A_{M5}$  and  $A_{M9}$ ) do not vanish for arbitrary  $y_{i,km}$ . Note that for arbitrary values of  $y_{i,km}$  at a point  $x_i$ , we can choose  $W$  so that (3.6) is satisfied at that point. Thus the only possible solutions of (3.10) for general  $y_i$  are given by the one-parameter family

$$a_{m,r} = c \delta_{mr} , \quad (3.12)$$

with  $c$  arbitrary. If we return to the condition (3.3), it immediately requires that  $c$  be identically zero. Therefore the derivatives  $a_{m,r}$  all vanish and

$$a_m = \alpha_m , \quad (3.13)$$

where  $\alpha_m$  are arbitrary constants.

With  $a_m$  known, (3.9) yields

$$\lambda_i = -\alpha_m y_{i,m} + \beta_i , \quad (3.14)$$

where  $\beta_i$  are arbitrary constants.

From (3.8), (3.13) and (3.14), we conclude that the only conservation laws of the form (3.1) are given by

$$\int_S \left\{ \alpha_m (W n_m - \frac{\partial W}{\partial y_{i,k}} y_{i,m} n_k) + \beta_i \frac{\partial W}{\partial y_{i,k}} n_k \right\} dS = 0 . \quad (3.15)$$

By taking all of the arbitrary constants  $\alpha_m$  and  $\beta_i$  zero except one in turn, we obtain the conservation laws (2.12) and (2.13).

If a term linear in the second derivatives of  $W$  with respect to  $y_{i,k}$  is included in the integrand in (3.1), it can be shown that, in addition to the laws (3.15), the only other law which arises is

$$\int_S \left\{ \frac{\partial W}{\partial y_{i,k}} \phi_{ir,r} - \frac{\partial W}{\partial y_{i,r}} \phi_{ik,r} + \frac{\partial^2 W}{\partial y_{i,k} \partial y_{p,q}} y_{p,qr} \phi_{ir} \right\} n_k dS = 0, \quad (3.16)$$

where  $\phi_{ir}$  are arbitrary functions. However, we note that the identity

$$\frac{\partial^2}{\partial x_r \partial x_k} \left( \phi_{ir} \frac{\partial W}{\partial z_{i,k}} - \phi_{ik} \frac{\partial W}{\partial z_{i,r}} \right) = 0 \quad (3.17)$$

holds for all functions  $\phi_{ir}(x_s)$ ,  $z_i(x_s)$  and  $W(z_{i,r})$ . Use of the divergence theorem then leads to

$$\int_S \left\{ \frac{\partial W}{\partial z_{i,k}} \phi_{ir,r} - \frac{\partial W}{\partial z_{i,r}} \phi_{ik,r} + \frac{\partial}{\partial x_r} \left( \frac{\partial W}{\partial z_{i,k}} \right) \phi_{ir} - \frac{\partial}{\partial x_r} \left( \frac{\partial W}{\partial z_{i,r}} \right) \phi_{ik} \right\} n_k dS = 0. \quad (3.18)$$

Substituting  $y_i$  for  $z_i$  in (3.18), we arrive at the law (3.16) if the equilibrium equations (2.6) hold on  $S$ . Thus, in contrast to the other laws, the law (3.16) requires the equilibrium equations to hold only on the surface  $S$  and not in the region enclosed by  $S$ .

An infinite number of higher order laws can be derived if successively higher order derivatives of  $W$  are included (see e.g., pp. 79-84 of [15]).

#### 4. Completeness for Arbitrary Objective $W(C_{rs})$ and Isotropic $W(B_{rs})$

If we impose the condition that the form of the strain energy function  $W(y_{i,k})$  must satisfy material objectivity but otherwise remains arbitrary, we then have the conservation laws (2.12), (2.13) and (2.17). Here we again show completeness in the sense that these laws are the only conservation laws of the form (3.1), but, in contrast to the previous Section we restrict  $a_m$  and  $b_{ikm}$  to depend on  $x_r$ ,  $y_s$  and  $y_{p,q}$  only.<sup>7</sup>

In this and the following section, the notation  $[\partial\phi/\partial x_i]_e$  will be used to denote the explicit derivative of a function  $\phi$  with respect to an explicit argument  $x_r$ . Thus we write the gradient  $\partial\phi/\partial x_r$  of a function  $\phi(x_r, y_s, y_{p,q})$  as

$$\frac{\partial\phi}{\partial x_r} = \phi_{,r} = \left[ \frac{\partial\phi}{\partial x_r} \right]_e + \frac{\partial\phi}{\partial y_s} y_{s,r} + \frac{\partial\phi}{\partial y_{p,q}} y_{p,q,r} . \quad (4.1)$$

When  $W$  satisfies material objectivity, (2.16) holds so that we have

$$\frac{\partial W}{\partial y_{i,k}} = \frac{\partial W}{\partial C_{pq}} (\delta_{kp} y_{i,q} + \delta_{kq} y_{i,p}) . \quad (4.2)$$

If the integrand in (3.2) is to vanish at all points of  $V$  for arbitrary values of  $W$  and  $\partial W/\partial C_{pq}$ , then with (4.2), it follows that

$$a_{m,m} = 0 , \quad (4.3)$$

$$(a_m y_{i,pm} + b_{ipm,m}) y_{i,q} + (a_m y_{i,qm} + b_{iqm,m}) y_{i,p} = 0 . \quad (4.4)$$

These in turn will imply that

$$b_{ikm} \frac{\partial}{\partial x_m} \left( \frac{\partial W}{\partial y_{i,k}} \right) = 0 . \quad (4.5)$$

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<sup>7</sup>The result also follows if the dependence of  $b_{ikm}$  on  $y_i$  and its derivatives is unrestricted.



In an analogous way as in the previous section, for (4.5) to be an identity for all  $y_i$  which satisfy the equilibrium equations (2.6) for arbitrary functions  $W(C_{rs})$ , we deduce that

$$b_{ikm} = \lambda_i \delta_{km} . \quad (4.6)$$

In (4.6),  $\lambda_i$  is again a function of  $x_r$ ,  $y_s$  and  $y_{p,q}$ . Substitution of (4.6) into (4.4) leads to

$$(a_m y_{i,pm} + \lambda_{i,p}) y_{i,q} + (a_m y_{i,qm} + \lambda_{i,q}) y_{i,p} = 0 . \quad (4.7)$$

With the assumption that  $a_m$  and  $b_{ikm}$  depend on  $x_r$ ,  $y_s$  and  $y_{p,q}$  only, (4.7) can be written

$$\begin{aligned} & \left[ (a_m y_{i,pm} + \frac{\partial \lambda_i}{\partial y_{r,m}} y_{r,mp}) y_{i,q} + (a_m y_{i,qm} + \frac{\partial \lambda_i}{\partial y_{r,m}} y_{r,mq}) y_{i,p} \right] \\ & + \left[ \left( \left[ \frac{\partial \lambda_i}{\partial x_p} \right]_e + \frac{\partial \lambda_i}{\partial y_m} y_{m,p} \right) y_{i,q} + \left( \left[ \frac{\partial \lambda_i}{\partial x_q} \right]_e + \frac{\partial \lambda_i}{\partial y_m} y_{m,q} \right) y_{i,p} \right] = 0 . \end{aligned} \quad (4.8)$$

Because second derivatives of  $y_s$  appear only in the first square bracket, it must vanish independently. This leads to

$$(a_m \delta_{ir} + \frac{\partial \lambda_i}{\partial y_{r,m}}) (y_{r,mp} y_{i,q} + y_{r,mq} y_{i,p}) = 0 . \quad (4.9)$$

By taking  $p = q$  and equal to the values 1,2,3 in turn, we must have

$$(a_m \delta_{ir} + \frac{\partial \lambda_i}{\partial y_{r,m}}) y_{i,k} = 0 \quad (4.10)$$

which follows because the values of  $y_{r,mp}$  can be chosen arbitrarily at any point  $x_r$ . Since  $y_{i,k}$  always has an inverse  $x_{k,r}$ , (4.10) still has to hold with  $y_{i,k}$  omitted and then we have

$$\frac{\partial \lambda_i}{\partial y_{r,m}} = 0 \quad (i \neq r) , \quad \frac{\partial \lambda_i}{\partial y_{r,m}} = -a_m \quad (i = r) . \quad (4.11)$$

From (4.11), it can be deduced that  $a_m$  cannot depend on  $y_{p,q}$  so that (4.11) can be integrated to yield

$$\lambda_i = -a_m(x_r, y_s) y_{i,m} + c_i(x_r, y_s), \quad (4.12)$$

where  $c_i$  is an arbitrary function of  $x_r$  and  $y_s$ .

Now the second square bracket in (4.8) must also vanish and, using (4.12), we have

$$\begin{aligned} & \left[ \left[ \frac{\partial c_i}{\partial x_p} \right]_e y_{i,q} + \left[ \frac{\partial c_i}{\partial x_q} \right]_e y_{i,p} \right] \\ & + \left[ \left( \frac{\partial c_i}{\partial y_m} y_{m,p} - \left[ \frac{\partial a_m}{\partial x_p} \right]_e y_{i,m} \right) y_{i,q} + \left( \frac{\partial c_i}{\partial y_m} y_{m,q} - \left[ \frac{\partial a_m}{\partial x_q} \right]_e y_{i,m} \right) y_{i,p} \right] \\ & - \left[ \frac{\partial a_m}{\partial y_r} y_{i,m} (y_{r,q} y_{i,p} + y_{r,p} y_{i,q}) \right] = 0. \end{aligned} \quad (4.13)$$

Since the left hand side of (4.13) is a sum of linear, quadratic and cubic terms in  $y_{p,q}$ , each of the three square brackets must vanish separately. The vanishing of the first square bracket requires, after a differentiation with respect to  $y_{r,s}$ , that

$$\left[ \frac{\partial c_r}{\partial x_p} \right]_e \delta_{qs} + \left[ \frac{\partial c_r}{\partial x_q} \right]_e \delta_{ps} = 0. \quad (4.14)$$

Choosing  $q = s \neq p$  in (4.14), we conclude that

$$\left[ \frac{\partial c_r}{\partial x_p} \right]_e = 0, \quad (4.15)$$

so that  $c_i$  now depends on  $y_s$  only.

The vanishing of the cubic terms in  $y_{p,q}$  in (4.13) requires

$$\frac{\partial a_m}{\partial y_r} y_{i,m} (y_{r,q} y_{i,p} + y_{r,p} y_{i,q}) = 0. \quad (4.16)$$



Multiplying (4.16) by the inverses  $x_{p,u}x_{q,v}$  leads to

$$\frac{\partial a_m}{\partial y_v} y_{u,m} + \frac{\partial a_m}{\partial y_u} y_{v,m} = 0, \quad (4.17)$$

and then upon differentiation with respect to  $y_{r,s}$ , we have

$$\frac{\partial a_s}{\partial y_v} \delta_{ru} + \frac{\partial a_s}{\partial y_u} \delta_{rv} = 0. \quad (4.18)$$

With the choice  $r = u \neq v$ , (4.18) implies that

$$\frac{\partial a_m}{\partial y_v} = 0. \quad (4.19)$$

Hence  $a_m$  is a function of  $x_r$  only.

On the other hand, the vanishing of the quadratic terms in  $y_{p,q}$  in (4.13) implies, after differentiating with respect to  $y_{r,s}$  and  $y_{u,v}$  in turn, that

$$\begin{aligned} \left( \frac{\partial c_u}{\partial y_r} + \frac{\partial c_r}{\partial y_u} \right) (\delta_{ps} \delta_{qv} + \delta_{pv} \delta_{qs}) &= \delta_{ru} \left( \frac{\partial a_s}{\partial x_p} \delta_{qv} + \frac{\partial a_s}{\partial x_q} \delta_{pv} \right. \\ &\quad \left. + \frac{\partial a_v}{\partial x_p} \delta_{qs} + \frac{\partial a_v}{\partial x_q} \delta_{ps} \right). \end{aligned} \quad (4.20)$$

Because the left hand side of (4.20) is independent of  $x_r$  while the right hand side is independent of  $y_s$ , they are both equal to a constant  $\eta_{pqrsuv}$  which must be of the form

$$\eta_{pqrsuv} = \eta \delta_{ru} (\delta_{ps} \delta_{qv} + \delta_{pv} \delta_{qs}), \quad (4.21)$$

where  $\eta$  is again a constant. If we take  $p = v = q = s$  and  $r = u$ , we find that

$$\frac{\partial a_s}{\partial x_s} = \frac{1}{2} \eta \text{ (no sum on } s \text{)}. \quad (4.22)$$



However, from (4.9) we see that we must have  $\eta = 0$ . Therefore both sides of (4.20) vanish and  $\partial a_s / \partial x_p$  is zero for  $s = p$ . If we now take  $q = v$  and  $s \neq p \neq q \neq s$ , then  $\partial a_s / \partial x_p$  is also zero for  $s \neq p$  and so  $\partial a_s / \partial x_p$  vanishes for all  $s, p$ . Hence

$$a_s = \alpha_s, \quad (4.23)$$

with  $\alpha_s$  arbitrary constants. For the left hand side of (4.20) to vanish identically, with  $c_u$  dependent only on  $y_r$ , it is necessary to have

$$c_u = e_{urs} \beta_s y_r + \gamma_u, \quad (4.24)$$

and  $\beta_s, \gamma_u$  are arbitrary constants.

Thus we conclude from (4.24), (4.23), (4.12) and (4.6) that the only conservation laws in the form of (3.1) for a homogeneous material with a strain energy  $W$  satisfying material objectivity but otherwise arbitrary are contained in

$$\int_S \left\{ \alpha_m (W n_m - \frac{\partial W}{\partial y_{i,k}} y_{i,m} n_k) + \beta_s (e_{irs} y_r \frac{\partial W}{\partial y_{i,k}} n_k) + \gamma_i \frac{\partial W}{\partial y_{i,k}} n_k \right\} dS = 0, \quad (4.25)$$

where  $\alpha_m, \beta_s, \gamma_i$  are arbitrary constants.

Because of the existence of the inverse deformation relationships discussed in Section 2, the completeness of the conservation laws (2.12), (2.13), (2.17) for objective but not necessarily isotropic  $W$  will imply that, for isotropic but not necessarily objective  $W$ , laws (2.12), (2.13) and (2.21) are the complete set of laws of the form of (3.1). Alternatively a similar proof to that of this section can be constructed to lead to the conclusion that when  $W$  is of the form  $W(B_{rs})$  then the only conservation



laws of type (3.1) with  $a_m$ ,  $b_{ikm}$  functions of  $x_r$ ,  $y_s$  and  $y_{p,q}$  are given by

$$\int_S \left\{ \alpha_m (W n_m - \frac{\partial W}{\partial y_{i,k}} y_{i,m} n_k) + \beta_m e_{srm} x_r (W n_s - \frac{\partial W}{\partial y_{i,k}} y_{i,s} n_k) + \gamma_i \frac{\partial W}{\partial y_{i,k}} n_k \right\} dS = 0 , \quad (4.26)$$

where  $\alpha_m$ ,  $\beta_m$ ,  $\gamma_i$  are arbitrary constants.

In the case when  $W$  is both objective and isotropic, so that  $W$  is a function only of the three strain invariants, we find that no conservation laws exist in the form of (3.1) other than those which already hold for the two cases separately.

### 5. Completeness for Arbitrary $W(y_{i,k})$ Homogeneous of Degree $N$

When the strain energy  $W(y_{i,k})$  is an arbitrary homogeneous function of degree  $N$ , we can use a similar approach to show that the laws (2.12), (2.13) and (2.25) for  $N \neq 0$  or (2.26) for  $N = 0$  are also complete in the sense of (3.1).

We begin as before by requiring the integrand in (3.2) to vanish. From (2.24), for  $N \neq 0$ , we have

$$W = \frac{1}{N} \frac{\partial W}{\partial y_{i,k}} y_{i,k}, \quad (5.1)$$

and then the integrand in (3.2) vanishes only if

$$b_{ikm} = \lambda_i \delta_{km}, \quad (5.2)$$

$$a_m y_{i,km} + \frac{1}{N} a_{m,m} y_{i,k} + \lambda_{i,k} = 0. \quad (5.3)$$

With  $a_m$  and  $\lambda_i$  assumed to depend on  $x_r$ ,  $y_s$  and on  $y_{p,q}$  only, (5.3) can be written

$$\left[ a_m y_{i,km} + \frac{\partial \lambda_i}{\partial y_{r,m}} y_{r,mk} + \frac{1}{N} \frac{\partial a_m}{\partial y_{r,s}} y_{i,k} y_{r,sm} \right] + \left[ \left[ \frac{\partial \lambda_i}{\partial x_k} \right]_e + \frac{\partial \lambda_i}{\partial y_m} y_{m,k} + \frac{1}{N} \left[ \frac{\partial a_m}{\partial x_m} \right]_e y_{i,k} + \frac{1}{N} \frac{\partial a_m}{\partial y_r} y_{i,k} y_{r,m} \right] = 0. \quad (5.4)$$

Because second derivatives  $y_{r,sm}$  appear only in the first square bracket of (5.4), it must vanish separately, and it can then be deduced that

$$\frac{\partial a_m}{\partial y_{r,s}} = 0, \quad (5.5)$$

which in turn leads to

$$\lambda_i = -a_m(x_r, y_j) y_{i,m} + c_i(x_r, y_j). \quad (5.6)$$

With these results, the vanishing of the second square bracket in (5.4) now implies that

$$\begin{aligned}
& \left[ \frac{\partial c_i}{\partial x_k} \right]_e + \left[ \frac{\partial c_i}{\partial y_m} y_{m,k} - \left[ \frac{\partial a_r}{\partial x_k} \right]_e y_{i,r} + \frac{1}{N} \left[ \frac{\partial a_m}{\partial x_m} \right]_e y_{i,k} \right] \\
& + \left[ \frac{1}{N} \frac{\partial a_m}{\partial y_r} y_{i,k} y_{r,m} - \frac{\partial a_r}{\partial y_m} y_{i,r} y_{m,k} \right] = 0 . \quad (5.7)
\end{aligned}$$

Since  $a_m$ ,  $c_i$  are functions of  $x_r$ ,  $y_s$  only, it follows that the term independent of  $y_{r,s}$ , the term linear in  $y_{r,s}$ , and the term quadratic in  $y_{r,s}$  must vanish separately. We then arrive at

$$a_m = \alpha x_m + \beta_m \quad (5.8)$$

and after solving for  $c_i$ , we find that

$$\lambda_i = -\alpha x_m y_{i,m} - \beta_m y_{i,m} - \alpha \left( \frac{3}{N} - 1 \right) y_i + \gamma_i \quad (5.9)$$

where  $\alpha$ ,  $\beta_m$ , and  $\gamma_i$  are arbitrary constants. This gives us the desired results for  $N \neq 0$ . The corresponding result for the case  $N = 0$  can be treated independently or it can be derived as a limiting case.

## 6. Conservation Laws for Membranes

In this section we limit our attention to initially plane membranes of uniform thickness and composed of homogeneous elastic materials. Rectangular Cartesian coordinates  $x_\alpha$  ( $\alpha = 1, 2$ ) in the plane of the membrane are used to define particle locations in the undeformed state. Here, Greek indices assume the values 1, 2 only. Under a pressure differential across its major surfaces, the plane membrane is finitely deformed to an equilibrium state with particle locations  $y_i$  given by

$$y_i = f_i(x_\alpha), \quad (6.1)$$

where  $y_i$  are referred to a rectangular Cartesian coordinate system. We note that the following analysis applies also to membranes which form developable surfaces in the undeformed state (such as a cylindrical membrane) provided the coordinates  $x_\alpha$  are chosen appropriately.

Denoting the strain energy per unit area of the undeformed membrane by  $w$ , it is in general a function  $w(y_{i,\alpha})$  of the derivatives  $y_{i,\alpha}$  (through the strains  $y_{i,\alpha}y_{i,\beta}$ ). For the moment no material symmetry is assumed.

The equilibrium equations for the deformed membrane can be obtained by application of the principle of virtual work. If we choose the virtual displacements to have the form

$$\delta y_i = y_{i,\alpha} \epsilon_\alpha, \quad (6.2)$$

then the vector  $\delta y_i$  is tangential to the deformed membrane and there is no virtual work done by the applied pressure. The arbitrary functions  $\epsilon_\alpha$  in (6.2) are assumed to vanish outside an arbitrary portion  $A$  of the undeformed membrane. For zero body forces, the principle of virtual work



requires that the total change in strain energy under the virtual displacements (6.2) be zero to first order, so that we have

$$\delta \int_A w(y_{i,\alpha}) dA = \int_A \frac{\partial w}{\partial y_{i,\alpha}} \delta(y_{i,\alpha}) dA = 0, \quad (6.3)$$

and we note that  $dA = dx_1 dx_2$ . An application of the divergence theorem then leads to

$$\int_A \frac{\partial}{\partial x_\alpha} \left( \frac{\partial w}{\partial y_{i,\alpha}} \right) y_{i,\beta} \varepsilon_\beta dA = 0. \quad (6.4)$$

Boundary terms do not appear in (6.4) because  $\varepsilon_\beta$  vanishes on the boundary  $L$  of  $A$ . Since  $A$  and  $\varepsilon_\beta$  are arbitrary, (6.4) provides the tangential equilibrium equations for the membrane

$$\frac{\partial}{\partial x_\alpha} \left( \frac{\partial w}{\partial y_{i,\alpha}} \right) y_{i,\beta} = 0. \quad (6.5)$$

Integration of (6.5) over  $A$  leads immediately to the conservation law

$$\int_L \left\{ w n_\beta - \frac{\partial w}{\partial y_{i,\alpha}} y_{i,\beta} n_\alpha \right\} dL = 0, \quad (6.6)$$

where  $n_\alpha$  are the unit outward normals to the boundary  $L$ . We emphasize that (6.6) holds independently of the pressure loading (and whether it is conservative or not). Also in deriving (6.6), we have not used the fact that  $w$  depends on  $y_{i,\alpha}$  only through the strains  $y_{i,\alpha} y_{i,\beta}$ .

If the membrane is composed of isotropic material, then  $w$  depends on  $y_{i,\alpha}$  only through the strains  $B'_{rs} = y_{r,\alpha} y_{s,\alpha}$  so that we have,

$$\frac{\partial w}{\partial y_{i,\gamma}} y_{i,\beta} = \frac{\partial w}{\partial y_{i,\beta}} y_{i,\gamma}. \quad (6.7)$$

In this case, if we choose  $\epsilon_\beta = (x_2, -x_1) = e_{3\beta\gamma} x_\gamma$  in (6.4), then with (6.7), it follows after an application of the divergence theorem that

$$\int_L e_{3\beta\gamma} x_\gamma \left\{ w n_\beta - \frac{\partial w}{\partial y_{i,\alpha}} y_{i,\beta} n_\alpha \right\} dL = 0, \quad (6.8)$$

which is a conservation law of the same form as (2.21). Laws corresponding to (2.12) and (2.17) do not exist for the membrane in the presence of applied pressure because the pressure contributes to the total force and moment. It also appears that a law corresponding to (2.25) does not apply when applied pressure is present.

If, in addition to pressure loading, the membrane is also acted upon by body forces  $F_i(y_r)$  per unit area of the undeformed membrane, then use of the principle of virtual work with the displacements (6.2) leads to

$$\left\{ \frac{\partial}{\partial x_\alpha} \left( \frac{\partial w}{\partial y_{i,\alpha}} \right) + F_i \right\} y_{i,\beta} = 0, \quad (6.9)$$

for tangential equilibrium. For conservative body force fields, a potential  $H(y_i)$  exists, with

$$F_i = \frac{\partial H}{\partial y_i}, \quad (6.10)$$

and integrating (6.9) over  $A$  provides the conservation law

$$\int_L \left\{ (w - H) n_\beta - \frac{\partial w}{\partial y_{i,\alpha}} y_{i,\beta} n_\alpha \right\} dL = 0. \quad (6.11)$$

Similarly, if the membrane is isotropic, we can show that

$$\int_L e_{3\beta\gamma} x_\gamma \left\{ (w - H) n_\beta - \frac{\partial w}{\partial y_{i,\alpha}} y_{i,\beta} n_\alpha \right\} dL = 0. \quad (6.12)$$

In the case when the membrane is initially cylindrical and undergoes axisymmetric deformations, the deformation is essentially one-dimensional

and the conservation law (6.6) reduces to a first integral, derived by other means previously in [16], of the equilibrium equation in the meridional direction in the absence of tangential loading. Referring to FIG. 1, the boundary  $L$  in (6.6) is the sum of the paths  $L_1$ ,  $L_2$ ,  $L_3$ ,  $L_4$ . Because the contributions to the integrals in (6.6) along  $L_2$  and  $L_3$  cancel, the integration reduces to that along  $L_1$  and  $L_4$ . Then, we find, with  $\beta = 2$  in (6.6),<sup>8</sup> that

$$w - \frac{\partial w}{\partial y_{3,2}} y_{3,2} = \text{constant} . \quad (6.13)$$

Because  $y_{3,2}$  is the extension ratio in the meridional direction, (6.13) agrees with the result derived in [16].

We remark that a similar application of the conservation law (2.13) for elastic bodies to the problem of flexure of an aeolotropic and compressible cuboid also leads to a first integral which corresponds to eq. (2.15.10) of [17].

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<sup>8</sup>The case  $\beta = 1$  leads to a trivial result.



## 7. Conservation Laws for Two Equilibrium States

For infinitesimal deformations of homogeneous and elastic bodies, the applicable conservation laws can be deduced separately [6, 7] or from the corresponding laws for finite deformations. With displacements  $u_i$  referred to a rectangular Cartesian coordinate system, the strain energy per unit volume  $W$  is a function  $W(u_{i,k})$  of the derivatives  $u_{i,k}$ . The law corresponding to (2.13) is the three-dimensional J-integral,

$$J_i \{u\} = \int_S \left\{ W n_i - \frac{\partial W}{\partial u_{r,k}} u_{r,i} n_k \right\} dS = 0, \quad (7.1)$$

for an arbitrary closed surface  $S$  which encloses a portion of the body in equilibrium with no body forces. The law (7.1) holds for linear and nonlinear elastic materials under infinitesimal deformations. For our purposes, we consider only linear elastic materials so that the strain energy is

$$W = \frac{1}{2} c_{ijkl} u_{i,j} u_{k,l}, \quad (7.2)$$

where  $c_{ijkl}$  are elastic constants and, without loss of generality, we assume

$$c_{ijkl} = c_{klij}. \quad (7.3)$$

We note that for linear materials (7.1) can be written in the alternative form [5]

$$\int_S \left\{ W n_i - \frac{\partial}{\partial x_i} \left( \frac{\partial W}{\partial u_{r,k}} \right) u_{r,i} n_k \right\} dS = 0.$$

Suppose we now consider two independent equilibrium states for the linear elastic body and denote the displacements by  $u_i^{(1)}$ ,  $u_i^{(2)}$ . From superposition, the sum of the two states is also an equilibrium state with displacements

$$u_i^{(0)} = u_i^{(1)} + u_i^{(2)} . \quad (7.4)$$

Since (7.1) holds for any equilibrium state,  $J_i \{u^{(0)}\}$  vanishes. But, with (7.2), we have

$$J_i \{u^{(0)}\} = J_i \{u^{(1)}\} + J_i \{u^{(2)}\} + \int_S \left\{ W^{(1,2)} n_i - \left( \frac{\partial W^{(1,2)}}{\partial u_{r,s}^{(1)}} u_{r,i}^{(1)} + \frac{\partial W^{(1,2)}}{\partial u_{r,s}^{(2)}} u_{r,i}^{(2)} \right) n_s \right\} dS , \quad (7.5)$$

where

$$W^{(1,2)} = c_{ijkl} u_{i,j}^{(1)} u_{k,l}^{(2)} = c_{ijkl} u_{i,j}^{(2)} u_{k,l}^{(1)} \quad (7.6)$$

denotes the mutual potential energy of the body [18]. Since  $u_i^{(1)}$ ,  $u_i^{(2)}$  are independent equilibrium displacements,  $J_i \{u^{(1)}\}$ ,  $J_i \{u^{(2)}\}$  in (7.5) vanish and the vanishing of  $J_i \{u^{(0)}\}$  therefore provides the conservation law

$$M_i \{u^{(1)}, u^{(2)}\} = \int_S \left\{ W^{(1,2)} n_i - \left( \frac{\partial W^{(1,2)}}{\partial u_{r,s}^{(1)}} u_{r,i}^{(1)} + \frac{\partial W^{(1,2)}}{\partial u_{r,s}^{(2)}} u_{r,i}^{(2)} \right) n_s \right\} dS = 0 . \quad (7.7)$$

We note that a term  $\sigma_{ik} u_{i,k}$  where  $\sigma_{ik}$  are constants can be added to the strain energy in (7.2) to include initial stresses. For small deformations superposed on a finite homogeneous deformation of a homogeneous elastic body, the equilibrium equations can be written as the Euler differential equations associated with an integrand of the form (7.2) with  $c_{ijkl}$  constants. Thus the laws (7.1) and (7.7) apply also in this case.

We also note that the decomposition

$$J_i \{u^{(0)}\} = J_i \{u^{(1)}\} + J_i \{u^{(2)}\} + M_i \{u^{(1)}, u^{(2)}\} \quad (7.8)$$

holds whether or not  $u_i^{(1)}, u_i^{(2)}$  satisfy equilibrium. For equilibrium displacement fields, different surfaces  $S_1, S_2, S_3$  can be used for the three integrals on the right hand side of (7.8) provided that the surface  $S$  used for the integral  $J_i \{u^{(0)}\}$  can be deformed continuously into each of the other surfaces without passing through a region in which equilibrium is not satisfied. This is a consequence of the so-called "path-independence" of the J-integral.

Since  $W$  in (7.2) is a homogeneous function of degree 2 in the displacement gradients, a conservation law [7] analogous to (2.25) applies, and with superposition, we arrive at the law

$$\int_S \left\{ W^{(1,2)} x_m n_m - \left( \frac{\partial W^{(1,2)}}{\partial u_{i,k}^{(1)}} u_{i,m}^{(1)} + \frac{\partial W^{(1,2)}}{\partial u_{i,k}^{(2)}} u_{i,m}^{(2)} \right) x_m n_k - \frac{1}{2} \left( \frac{\partial W^{(1,2)}}{\partial u_{i,k}^{(1)}} u_i^{(1)} + \frac{\partial W^{(1,2)}}{\partial u_{i,k}^{(2)}} u_i^{(2)} \right) n_k \right\} dS = 0. \quad (7.9)$$

If the material is isotropic, then we also have

$$\int_S e_{srm} \left\{ x_r \left[ W^{(1,2)} n_s - \left( \frac{\partial W^{(1,2)}}{\partial u_{i,k}^{(1)}} u_{i,s}^{(1)} + \frac{\partial W^{(1,2)}}{\partial u_{i,k}^{(2)}} u_{i,s}^{(2)} \right) n_k \right] + \left( \frac{\partial W^{(1,2)}}{\partial u_{s,k}^{(1)}} u_r^{(1)} + \frac{\partial W^{(1,2)}}{\partial u_{s,k}^{(2)}} u_r^{(2)} \right) n_k \right\} dS = 0. \quad (7.10)$$

The laws (7.7), (7.9), (7.10) can alternatively be derived from Noether's theorem on invariant principles [6, 7] if the appropriate functional is taken to depend on the derivatives  $u_{i,k}^{(1)}$  and  $u_{i,k}^{(2)}$ .

The three conservation laws (7.7), (7.9), (7.10) are not complete since at least one other non-trivial law is provided by Betti's reciprocal work theorem. In terms of the mutual potential energy, the reciprocal work theorem takes the form

$$\int_S \left( \frac{\partial W^{(1,2)}}{\partial u_{r,k}^{(1)}} u_r^{(1)} - \frac{\partial W^{(1,2)}}{\partial u_{r,k}^{(2)}} u_r^{(2)} \right) n_k dS = 0 \quad (7.11)$$

for equilibrium displacements  $u_i^{(1)}$ ,  $u_i^{(2)}$ . The reciprocal theorem is a consequence of the invariance of the term  $(\partial W^{(1,2)} / \partial u_{i,k}^{(1)}) u_{i,k}^{(1)}$  when the displacements  $u_i^{(1)}$  and  $u_i^{(2)}$  are interchanged. Such a transformation does not appear to render the mutual potential energy functional infinitesimally invariant and hence is not contained in the restricted version of Noether's theorem stated in [7].

Finally, we note that by use of the divergence theorem and (7.6), the conservation law (7.7) can be written in an alternative form

$$M'_i \{u^{(1)}, u^{(2)}\} = \int_S \left\{ \frac{\partial}{\partial x_i} \left( \frac{\partial W^{(1,2)}}{\partial u_{r,k}^{(1)}} u_r^{(1)} - \frac{\partial W^{(1,2)}}{\partial u_{r,k}^{(2)}} u_{r,i}^{(2)} \right) \right\} n_k dS = 0. \quad (7.12)^9$$

A comparison between (7.11) and (7.12) shows that (7.12) is merely Betti's theorem (7.11) with  $u_r^{(2)}$  replaced by the displacements  $u_{r,i}^{(2)}$  (with  $i$  fixed) which also satisfy equilibrium when the material is homogeneous.

The conservation law (7.1) can similarly be written in the form

$$J'_i \{u\} = \frac{1}{2} \int_S \left\{ \frac{\partial}{\partial x_i} \left( \frac{\partial W}{\partial u_{r,k}} u_r - \frac{\partial W}{\partial u_{r,k}} u_{r,i} \right) \right\} n_k dS = 0, \quad (7.13)$$

---

<sup>9</sup>Compare with eq. (5.6) of [5].

and it corresponds to an application of Betti's theorem (7.11) with the auxiliary displacements  $u_r^{(2)}$  chosen to be  $u_{r,i}^{(1)}$  (with  $i$  fixed). The integral  $J'_i \{u\}$  is therefore "path-independent" for any closed surface  $S$ . It should be noted, however, that although the two equivalent conservation laws (7.1) and (7.13) both lead to "path-independent" integrals, the values  $J_i \{u ; S_1\}$  and  $J'_i \{u ; S_1\}$  of the integrals are not necessarily equal for an arbitrary surface  $S_1$ . The difference between them is given by

$$J_i \{u ; S_1\} - J'_i \{u ; S_1\} = \frac{1}{2} \int_{S_1} e_{pjk} e_{psi} \frac{\partial}{\partial x_j} \left( \frac{\partial W}{\partial u_{r,s}} u_r \right) n_k dS . \quad (7.14)$$

The surface integral in (7.14) can be transformed into a line integral by Stokes' theorem so that we have

$$J_i \{u ; S_1\} - J'_i \{u ; S_1\} = \frac{1}{2} \int_{L_1} e_{isp} \frac{\partial W}{\partial u_{r,s}} u_r m_p dL , \quad (7.15)$$

where  $m_p$  are unit tangent vectors to the boundary curve  $L_1$  of  $S_1$ .



## 8. Application to Fracture Mechanics

The J-integral (i.e.,  $J_1 \{u\}$  in two-dimensions) has been shown [4] to be related to the stress intensity factors associated with a line crack in a homogeneous flat plate under in-plane loading. For two integration paths  $\Gamma$  and  $\bar{\Gamma}$  (FIG. 2) enclosing a crack tip located at the origin, the values of  $J_1 \{u; \Gamma\}$  and  $J_1 \{u; \bar{\Gamma}\}$  are equal because the faces of the crack are free from traction.<sup>10</sup> If we consider the crack to be under mixed-mode loading conditions with displacements

$$u_{\alpha}^I + u_{\alpha}^{II} = u_{\alpha}^I + u_{\alpha}^{II},$$

then in the limit as  $\bar{\Gamma}$  shrinks to zero, we have [4]

$$J_1 \{u^I + u^{II}; \Gamma\} = \alpha^I (K^I)^2 + \alpha^{II} (K^{II})^2, \quad (8.1)$$

where  $K^I$ ,  $K^{II}$  are the stress intensity factors for mode I (opening) and mode II (in-plane shearing) respectively. Here  $\alpha^I$ ,  $\alpha^{II}$  are constants given by

$$\alpha^I = \alpha^{II} = \begin{cases} \frac{1 - \nu^2}{E} & \text{for plane strain} \\ \frac{1}{E} & \text{for plane stress,} \end{cases}$$

where  $E$  is Young's Modulus and  $\nu$  Poisson's ratio. We note that a comparison between (8.1) and (7.8) shows that in this case  $M_1 \{u^I, u^{II}; \Gamma\}$  vanishes identically, and this is a form of orthogonality for the mode I and mode II solutions.

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<sup>10</sup>In this case, the counterpart of the integral in (7.15) for two-dimensions vanishes so that the values  $J_1 \{u; \Gamma\}$  and  $J_1' \{u; \Gamma\}$  are also equal.

When only one of the modes is present, (8.1) provides a means for estimating the stress intensity factor if appropriate boundary data on  $\Gamma$  are known [1, 2]. But in mixed-mode cases, (8.1) alone clearly does not allow  $K^I$  and  $K^{II}$  to be determined separately. Now consider the  $M_1$  integral (in two-dimensions) defined in (7.7) and let  $v_\alpha^I$  be the equilibrium displacements of a known auxiliary solution due to a mode I deformation only. Thus we have

$$\begin{aligned} M_1 \{u^I + II, v^I; \Gamma\} &= M_1 \{u^I + II, v^I; \bar{\Gamma}\} \\ &= M_1 \{u^I, v^I; \bar{\Gamma}\} + M_1 \{u^{II}, v^I; \bar{\Gamma}\}, \end{aligned} \quad (8.2)$$

where the first equality holds because of the path-independence of  $M_1$  and the second because  $M_1$  is linear in each of the displacement fields. If the dominant singularity of the  $v^I$  solution is characterized by a factor  $A^I$ , then in the limit the integral  $M_1 \{u^I, v^I; \bar{\Gamma}\}$  is proportional to  $A^I K^I$  while  $M_1 \{u^{II}, v^I; \bar{\Gamma}\}$  vanishes so that

$$M_1 \{u^I + II, v^I; \Gamma\} = 2\alpha^I A^I K^I. \quad (8.3)$$

If we then use a second known auxiliary equilibrium solution  $v^{II}$  which is due to a mode II deformation only, we have in the same way

$$M_1 \{u^I + II, v^{II}; \Gamma\} = 2\alpha^{II} A^{II} K^{II}, \quad (8.4)$$

where  $A^{II}$  again characterizes the  $v^{II}$  singularity. Thus if sufficient information is known on  $\Gamma$ ,  $K^I$  and  $K^{II}$  can be determined from (8.3) and (8.4).

The fact that (8.1) alone does not supply enough information to determine  $K^I$  and  $K^{II}$  separately was noted by Budiansky and Rice [14]. We remark that their conclusion that the singularity at the crack tip must be



either mode I or mode II under all mixed-mode loadings is incorrect. From superposition, it can be seen that the stress intensity factors  $K^I$  and  $K^{II}$  must change continuously if the loading mode changes continuously.

The argument given in [14] does not apply because the singular part of the stress  $\sigma_{11}$  for mode I loading vanishes identically on the upper and lower crack faces.

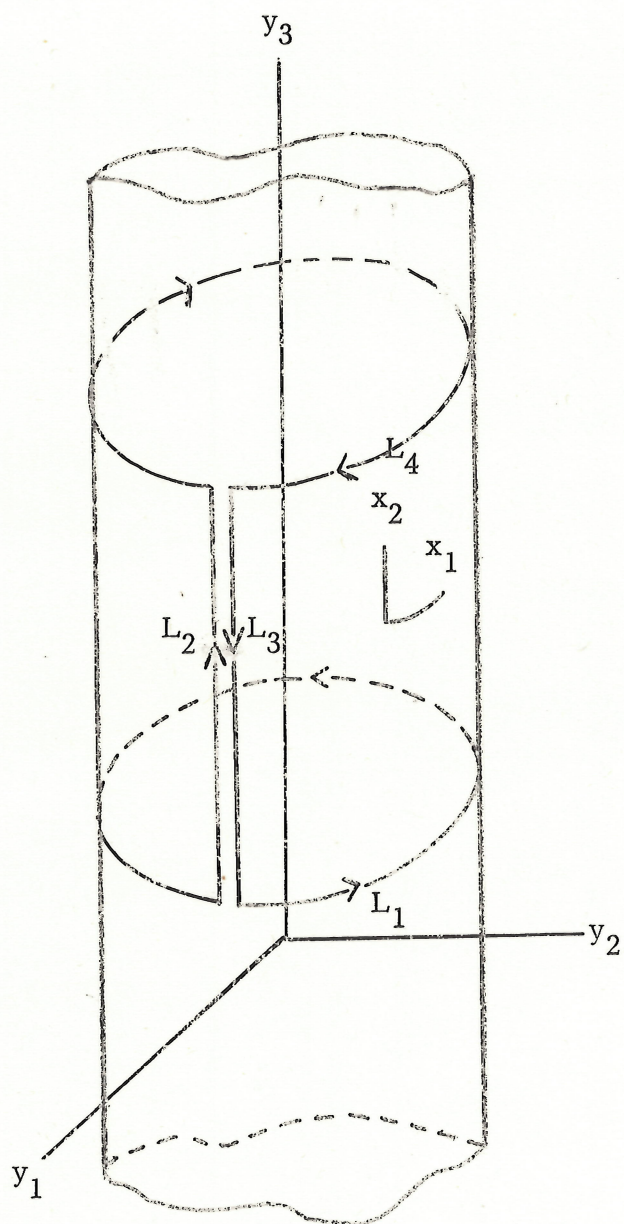


FIG. 1 Integration Paths for Cylindrical Membrane

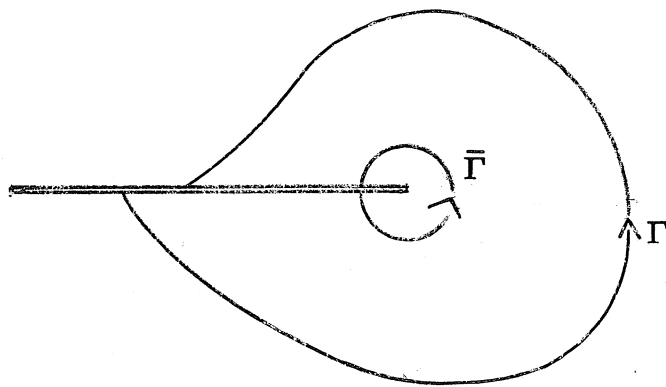


FIG. 2 Integration Paths for Crack



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