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A FURTHER LOOK AT RAYLEIGH-TAYLOR AND OTHER SURFACE INSTABILITIES IN SOLIDS

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Summary

An analysis is presented of the instability or out-of-plane displacement h of the surface of a semi-infinite solid of density ρ_s subjected to extremely large in-surface compressive strain rate $\dot{\varepsilon}_L$ or normal deceleration α (inward acceleration) or both. An earlier study of Rayleigh-Taylor instability in a perfectly plastic solid of yield strength σ_o serves as a guide to the selection of useful natural lengths λ_N and natural times. One of the natural lengths is h_o^{TH} = (const) $\sigma_o/\rho_s\alpha$, the threshold amplitude of surface protuberance below which Rayleigh-Taylor growth will not take place for α alone. Another is a screening length in space $\lambda_N = \sqrt{E/\rho_s}/\dot{\varepsilon}_L$ beyond which even elastic or pressure waves can transmit no information so that material motion or surface instability at each instant of time cannot be coordinated.

1. Introduction

The stability of the surface of a solid subjected to extremely large normal deceleration (inward acceleration) α, or overall lateral compressive strain rate $\dot{\varepsilon}_L$, or both (Fig. 1) is central to the feasibility of the class of prospective fusion power generators employing imploding hollow spheres or other geometries. An earlier examination [1] of classical Rayleigh-Taylor instability ($\dot{\varepsilon}_L = 0$) showed that, in contrast to a fluid, an elastic-plastic solid can tolerate a threshold amplitude of surface roughness or protuberance height $h_o = h_o^{TH}$ (Fig. 2) below which there will be no appreciable growth of h.

For $h_o << \lambda_o$, Fig. 2, in a two- or three-dimensional problem

$$\rho_s \alpha h_o^{TH} = (1 + \frac{\pi}{2}) \sigma_o F \qquad (1.1)$$

where ρ_s is the mass density of the solid, σ_o its yield or flow strength for moderate plastic strains, and F is a factor of order unity that takes the shape of the protuberance into account. The mass density ρ_f of the fluid in contact with the solid is negligible under directly observable experimental conditions [2] and is ignored in (1.1), but may be significant in practice.

It was pointed out also that when lateral compressive plastic straining ($\dot{\varepsilon}_L$) takes place simultaneously, as in an imploding sphere or cylinder, the threshold value of h_o goes to zero. Rayleigh-Taylor instability then will begin for the smoothest of solid surfaces.

When $\dot{\varepsilon}_L = 0$ and $h_o > h_o^{TH}$, but $h_o \ll \lambda_o$, the amplitude of a surface wave or protuberance at time t during the early stages of growth is given approximately by

$$h - h_o^{TH} = (h_o - h_o^{TH})\cosh(t\sqrt{\beta\alpha/\lambda_o}) \qquad (1.2)$$

The growth rate $\sqrt{\beta\alpha/\lambda_o}$ appearing in the almost exponential multiplier $\cosh(t\sqrt{\beta\alpha/\lambda_o})$ involves the initial wave length λ_o or twice the in-surface dimension of the protuberance (Fig. 2) and a geometric factor $\beta = \frac{16}{(4+\pi)F}$ that is of order unity but may be as high as 5 or so.

This earlier analysis [1] explained the reported experimental observations [2] but did not address a number of important questions. One is whether or not significant surface instability occurs for large lateral compressive straining in the plastic range in the absence of normal deceleration α. Others involve the identification of critical lengths and more generally the critical variables that control Rayleigh-Taylor instability for combined $\dot{\varepsilon}_L$ and α applied to elastic-plastic, viscous, and elasto-visco-plastic solids.

Constitutive relations and the governing equations can be written for each idealization of material behavior. The boundary conditions are known. Therefore numerical solutions in two and three dimensions can be obtained for a wide variety of initial configurations and conditions. However, such an effort is far from trivial and the abstracting of needed generalizations is very difficult. It is appropriate instead to start with a physical description of likely behavior and follow with a dimensional analysis exploration of the existence and possible significance of the natural lengths and times associated with each idealization and mode of instability or rapid growth of surface perturbations. Such quantities govern the ability of detailed numerical calculations based upon that idealization to predict or match experimental information such as the initiation of surface roughening in a perfectly flat or cylindrical or spherical surface, the existence of preferred wave lengths, the growth rates of surface waves or bumps, and the time for separation of the growing protuberances from the main body of material.

2. A Dimensional Analysis

Unless the underlying mathematics and science are well understood and the solutions to problems are transparent, it is always helpful to begin by listing the variables and parameters to be taken into account in a proposed analytical or experimental approach and perform a dimensional analysis. In a general examination of surface instability, independent variables with the dimension of length or of time (natural lengths and natural times) are of obvious interest along with dimensionless independent variables.

The list of independent variables begins with the imposed kinematic variables α, $\dot{\varepsilon}_L$, the mass densities of the solid and fluid ρ_s, ρ_f, and the appropriate constitutive parameters such as the yield or flow strength σ_o, the elastic moduli typified by a shear modulus or Young's modulus or bulk modulus E, plastic moduli typified by G_p, and viscosity coefficients μ. Atomic, microstructural and additional macroscopic mechanical and thermodynamic material variables such as surface energy may be included. Certainly the geometric description of the initial surface is needed: one or two principal radii R for a smooth curved surface and one or more combinations of irregularity dimensions h_o, λ_o for a nominally flat or smooth curved surface.

Each of the quantities listed may be zero, each may represent one, two, or any number of like independent parameters whose ratios in pairs are dimensionless. In what follows, these dimensionless ratios will not normally be exhibited but they are always to be thought of as included implicitly.

3. Natural Lengths and Times

The now understood classical type of Rayleigh-Taylor instability of an almost flat surface of a perfectly plastic solid undergoing inward acceleration (normal deceleration) α in a vacuum provides a helpful starting point to guide the choice of dimensionless variables likely to prove useful for more general geometries, materials and conditions. To provide this guidance suppose nothing to be known about the instability for the perfectly plastic solid. At any time t the only independent variables other than position in the solid are h_o, λ_o, α, ρ_s, and σ_o. The only natural length that emerges is $\sigma_o/\rho_s\alpha$ but there are a variety of choices of natural time: $(\sqrt{\sigma_o/\rho_s})/\alpha$, $\sqrt{\lambda_o/\alpha}$, $\sqrt{h_o/\alpha}$, $\lambda_o/\sqrt{\sigma_o/\rho_s}$, ...

Is there an instability in this highly idealized model? If so, are there critical or preferred lengths? Dimensional analysis cannot give unequivocal answers to such questions, cannot say whether or not critical dimensions exist. What can be said from dimensional analysis alone is that <u>if</u> a λ_o^{cr} or an h_o^{cr} exists for a wavy surface then

$$\frac{\lambda_o^{cr}}{\sigma_o/\rho_s\alpha} \text{ can be a function only of } \frac{h_o}{\sigma_o/\rho_s\alpha} \qquad (3.1)$$

and

$$\frac{h_o^{cr}}{\sigma_o/\rho_s\alpha} \text{ can be a function only of } \frac{\lambda_o}{\sigma_o/\rho_s\alpha} \qquad (3.2)$$

Of course, a function can be constant and (3.2) reduces to (1.1) with h_o^{TH} for h_o^{cr} and $(1+\frac{\pi}{2})F$ for the constant. However, (3.2) serves as a reminder that λ_o can play an important role in the initial motion. For a rectangular bump of height $h_o>\lambda_o/2$, as indicated schematically in Fig. 3, $h_o^{TH}/(\sigma_o/\rho_s\alpha) = 1$ and not $(1+\frac{\pi}{2})$. Clearly h_o^{TH} does depend strongly on λ_o when the two are comparable in magnitude.

More generally, the initial and intermediate stages of growth of h, when $h_o^{TH}<h<<\lambda_o$, involves plastic deformation below the surface to a depth of order λ_o while the final stage of separation of the growing protuberance involves mainly necking and stretching of the protuberance itself. Until this relatively brief final stage, the driving force for the growth of h is proportional to $\rho_s \alpha h \lambda_o$ in the two-dimensional or plane strain problem as indicated in Fig. 3. The plastic resistance (assumed rate-independent) is proportional to $\sigma_o \lambda_o$, and the inertial mass to be accelerated is proportional to $\rho_s \lambda_o{}^2$. Accelerations are of order and proportional to d^2h/dt^2. This crude analysis leads to an ordinary differential equation with constant coefficients A, B, C

$$A\rho_s \lambda_o^2 \; d^2h/dt^2 = B\rho_s \alpha h \lambda_o - C\sigma_o \lambda_o$$

or

$$\frac{d^2}{dt^2}\left(\frac{h}{h_o}\right) = \frac{B}{A}\,\frac{\alpha}{\lambda_o}\left[\frac{h}{h_o} - \frac{C}{B}\,\frac{\sigma_o}{\rho_s \alpha h_o}\right] \qquad (3.3)$$

In the three-dimensional problem of a bump of height h_o and area $\lambda_o{}^2$ in the plane of the surface, also shown in Fig. 3 the driving force is proportional to $\rho_s \alpha h_o \lambda_o{}^2$, the plastic resistance is proportional to $\sigma_o \lambda_o{}^2$, and the inertial mass is proportional to $\rho_s \lambda_o{}^3$. Therefore the differential equation remains unchanged in form from (3.3). In two or three dimensions the magnitude of λ_o does not enter the quasi-static component of the solution for a semi-infinite solid subjected to a geometrically similar pattern of surface loading.

The result (1.2), given earlier, is the solution of (3.3). Quite apart from the degree of validity of the approximation, the reasoning followed suggests strongly that

$$h = h(h_o,\ \lambda_o,\ \alpha,\ \rho_s,\ \sigma_o,\ t) \qquad (3.4)$$

be written in the particular dimensionless form

$$\frac{h}{h_o} \text{ or } \frac{\rho_s \alpha h}{\sigma_o} \text{ is a function of } \frac{h_o}{\lambda_o}, \frac{\rho_s \alpha h_o}{\sigma_o}, t\sqrt{\alpha/\lambda_o} \tag{3.5}$$

with $\rho_s \alpha h_o/\sigma_o$ and $t\sqrt{\alpha/\lambda_o}$ controlling the initial and intermediate stages (when $h \ll \lambda_o$), supplemented by h_o/λ_o for the later stages. Perhaps even more it suggests that the time to separation of the protuberance might be given closely enough by (1.2) with h set equal to $\lambda_o/2$. Form (3.5) is especially convenient for devising model experiments but all choices of any three independent dimensionless variables must be equivalent.

4. A More Complete Set of Variables ($\dot{\varepsilon}_L = 0$)

After a reasonable understanding has been achieved of the response of a perfectly plastic solid decelerating into a vacuum, greater realism can be introduced into the model. The presence of a fluid of density ρ_f rather than a vacuum adds a dimensionless ratio ρ_f/ρ_s or the Atwood number

$$\frac{\rho_s - \rho_f}{\rho_s + \rho_f} \tag{4.1}$$

Its influence is small under most laboratory conditions when the fluid is a gas but can become significant when the Atwood number departs appreciably from unity, as it may in imploding spheres and cylinders.

If the material is modeled more realistically as elastic-perfectly plastic, σ_o/E is added to the list of independent dimensionless variables. When, as is usual, $\sigma_o \ll E$, the elasticity would not be expected to have much influence.

The addition of work-hardening introduces plastic moduli G_p and dimensionless variables of the form G_p/E or G_p/σ_o. If σ_o is chosen as an initial yield strength and the details of initial growth of h are of interest, the work-hardening of the solid plays a major role. If σ_o is chosen as the flow stress at moderately large plastic strain, say 5%, then $G_p/E \ll 1$ and work hardening has little influence on the gross features of the growth of a given protuberance.

5. The Addition of $\dot{\varepsilon}_L$

With the addition of a high compressive lateral strain rate to the list of independent variables, the situation changes radically. If $\dot{\varepsilon}_L$ were in the usual laboratory range, it would be helpful to modify (3.4) to

$$h = h(h_o, \lambda_o, \alpha, \dot{\varepsilon}_L, \rho_s, \rho_f, \sigma_o, E, G_p, t) \qquad (5.1)$$

and (3.5) could be rewritten in terms of two natural lengths $\sigma_o/\rho_s\alpha$ and $(\sqrt{G_p/\rho_s})/\dot{\varepsilon}_L$ as

$$\frac{h}{h_o} \text{ or } \frac{\rho_s \alpha h}{\sigma_o} \text{ as a function of } \frac{h_o}{\lambda_o}, \frac{\rho_s \alpha h_o}{\sigma_o}, t\sqrt{\alpha/\lambda_o},$$
$$\frac{\rho_s-\rho_f}{\rho_s+\rho_f}, \frac{\sigma_o}{E}, \frac{G_p}{\sigma_o}, \lambda_o\dot{\varepsilon}_L/\sqrt{G_p/\rho_s} \qquad (5.2)$$

Note that for quasi-static lateral straining ($\dot{\varepsilon}_L \to 0$), in the absence of normal acceleration ($\alpha = 0$), there are no natural lengths in the problem and therefore no special critical wave lengths λ_o^{cr} [3].

Under dynamic conditions, however, a region of material away from one protuberance or indentation does not feel its influence until a pressure wave or a plastic shear wave or an elastic unloading wave reaches the region. Shear waves travel at the rather slow speeds of plastic waves when $\dot{\varepsilon}_L$ is high and overwhelms any reversed strain rates. The waves, which are the signals that coordinate the motion of any one point on the surface with another, travel at speeds with respect to the moving solid of the order of $\sqrt{E/\rho_s}$ for pressure waves and $\sqrt{G_p/\rho_s}$ for plastic shear waves. In a time $1/\dot{\varepsilon}_L$ they travel respective distances in the material of

$$\lambda_N^e = \sqrt{E/\rho_s}/\dot{\varepsilon}_L \qquad (5.3)$$

and

$$\lambda_N^p = \sqrt{G_p/\rho_s}/\dot{\varepsilon}_L \qquad (\simeq \lambda_N^e/10) \qquad (5.4)$$

A better physical interpretation of these natural lengths is that they represent the extent of the domain in space over which motions can be coordinated. The material currently at point B in space, a distance λ_N^e from A, Fig. 4, is moving toward A with a relative velocity of $\lambda_N^e\dot{\varepsilon}_L$. This is exactly opposite to the velocity of the fastest wave with respect to the material, $\sqrt{E/\rho_s}$. Therefore, a signal from A cannot reach a material point C until C passes through position B in space.

Indentations and protuberances must develop independently of each other during the time they remain farther apart than a distance of order λ_N^e . In ordinary laboratory tests on solids, even λ_N^p covers the entire specimen because $\sqrt{E/\rho_s}$ is about 5000 m/s, $\sqrt{G_p/\rho_s}$ is about 1/10 as large, and $\dot{\varepsilon}_L$ is far below 100. Independent development still may and does occur but not because of lack of communication. However, at the enormous rates of strain possible in implosions, the screening distances λ_N become microscopic. At $\dot{\varepsilon}_L = 10^6$/s, $\lambda_N^e = 5$mm; at $\dot{\varepsilon}_L = 10^9$/s, $\lambda_N^e = 5\mu$m. At 10^9/s, as long as a protuberance and a dent remain separated by more than 5μm they <u>cannot</u> interact.

If a critical wave length exists at any stage of the deformation it must be of order λ_N or smaller. The amplitudes of Rayleigh-Taylor waves that grow exponentially are the differences h_N in surface level over dimension λ_N, not over λ_o or λ, until λ gets down to a dimension of order λ_N, Fig. 4. Of course, as time goes on, the lateral contraction rapidly brings surface points closer and closer to each other. Quite independently of any Rayleigh-Taylor amplification, surface waves would become steeper and steeper, as illustrated in Fig. 4, and each initial λ_o decreases to become of order λ_N and less. Even at the "low" strain rate $\dot{\varepsilon}_L$ of 10^6/sec, enormous contraction occurs in microseconds. With such large changes in geometry it is better to think in terms of the rate of change of h in the current geometry rather than attempt reference back to the original geometry, although it is the original and not the current geometry that is known or can be specified

$$\frac{dh_N/dt}{h_N/\sqrt{\lambda_N/\alpha}} \text{ is a function of } \frac{h_N}{\lambda_N},\ \frac{\rho_s \alpha h_N}{\sigma_o},\ \frac{\rho_s-\rho_f}{\rho_s+\rho_f},\ \frac{\sigma_o}{E},\ \frac{G_p}{\sigma_o} \tag{5.5}$$

where $\dot{\varepsilon}_L$ appears in the λ_N explicitly and in the h_N implicitly. Also h_N, the surface protuberance in distance λ_N at time t is at least as large as the initial amplitude difference over initial distances of order $\lambda_N \exp(\dot{\varepsilon}_L t) = \lambda_N \exp[(E/\rho_s)t/\lambda_N] >> (\sqrt{E/\rho_s})t$ for small λ_N. Rayleigh-Taylor instability is superposed on this geometrically sharpening picture with a growth rate exponent of order $\sqrt{\beta\alpha/\lambda_N}$.

6. A Side Remark on Linear Viscous Response

If a linear viscous material is postulated instead of an elastic-plastic one, σ_o of the preceding discussion is replaced by a viscosity μ_o with units of stress per unit velocity gradient or stress multiplied by time. The threshold value of h_o always is zero because the resistance of the material to deformation drops to zero as the rate of deformation vanishes. Yet there is a natural length introduced in the problem by viscosity, $(\mu_o{}^2/\rho_s{}^2\alpha)^{1/3}$, as well as a variety of natural times: $(\mu_o/\rho_s\alpha^2)^{1/3}$, $\rho_s\lambda_o{}^2/\mu_o$, ...

For $\dot{\varepsilon}_L = 0$, the approximate ordinary differential equation with constant coefficients that corresponds to (3.3) for an Atwood number (4.1) of unity is

$$A_1 \rho_s \lambda_o{}^2 d^2h/dt^2 + D_1 \mu_o dh/dt = B_1 \rho_s h \alpha \lambda_o$$

or

$$\frac{d^2}{dt^2}\left(\frac{h}{h_o}\right) + \frac{D_1}{A_1}\,\frac{\mu_o}{\rho_s \lambda_o^2}\,\frac{d}{dt}\left(\frac{h}{h_o}\right) - \frac{B_1}{A_1}\,\frac{\alpha}{\lambda_o}\left(\frac{h}{h_o}\right) = 0 \tag{6.1}$$

which suggests that the choice of

$$\frac{h}{h_o} \text{ as a function of } \frac{h_o}{\lambda_o},\ \frac{\mu_o t}{\rho_s \lambda_o{}^2},\ t\sqrt{\alpha/\lambda_o} \tag{6.2}$$

will prove convenient and that h_o/λ_o will not play much of a role in the initial stage of exponential growth governed by a growth rate that combines $\mu_o/\rho_s\lambda_o{}^2$ and $\sqrt{\alpha/\lambda_o}$.

7. Concluding Remark

In Section 4 the statement was made that the dimensionless variable σ_0/E was likely to be small and of little importance. Yet it is worth keeping in mind that, under enormously rapid loading, the compressive stresses in the surface may be very large compared with static values of σ_o. Then a transition from a smooth to a wavy or buckled surface may involve a significant destabilizing elastic followerforce of the type studied extensively by Professor Ziegler [4].

Acknowledgment

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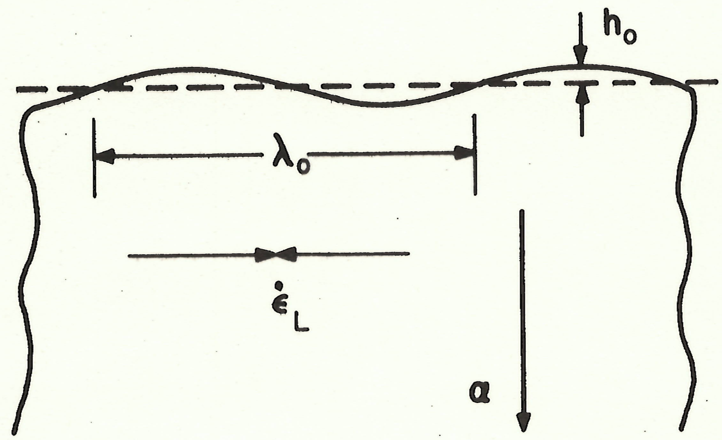

FIG. 1

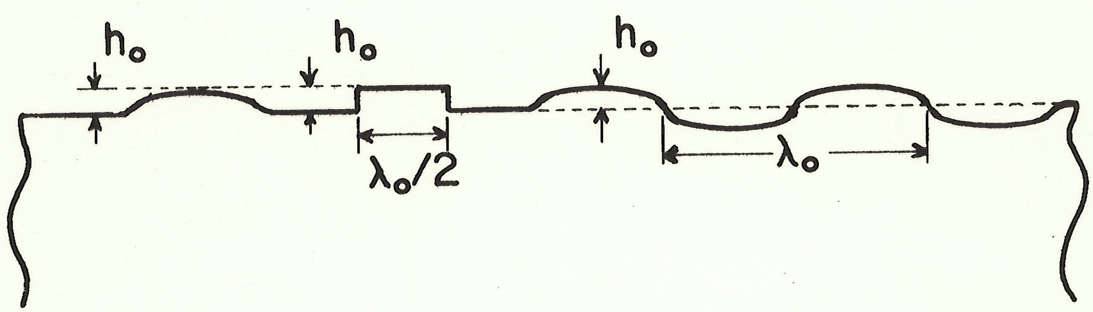

FIG. 2

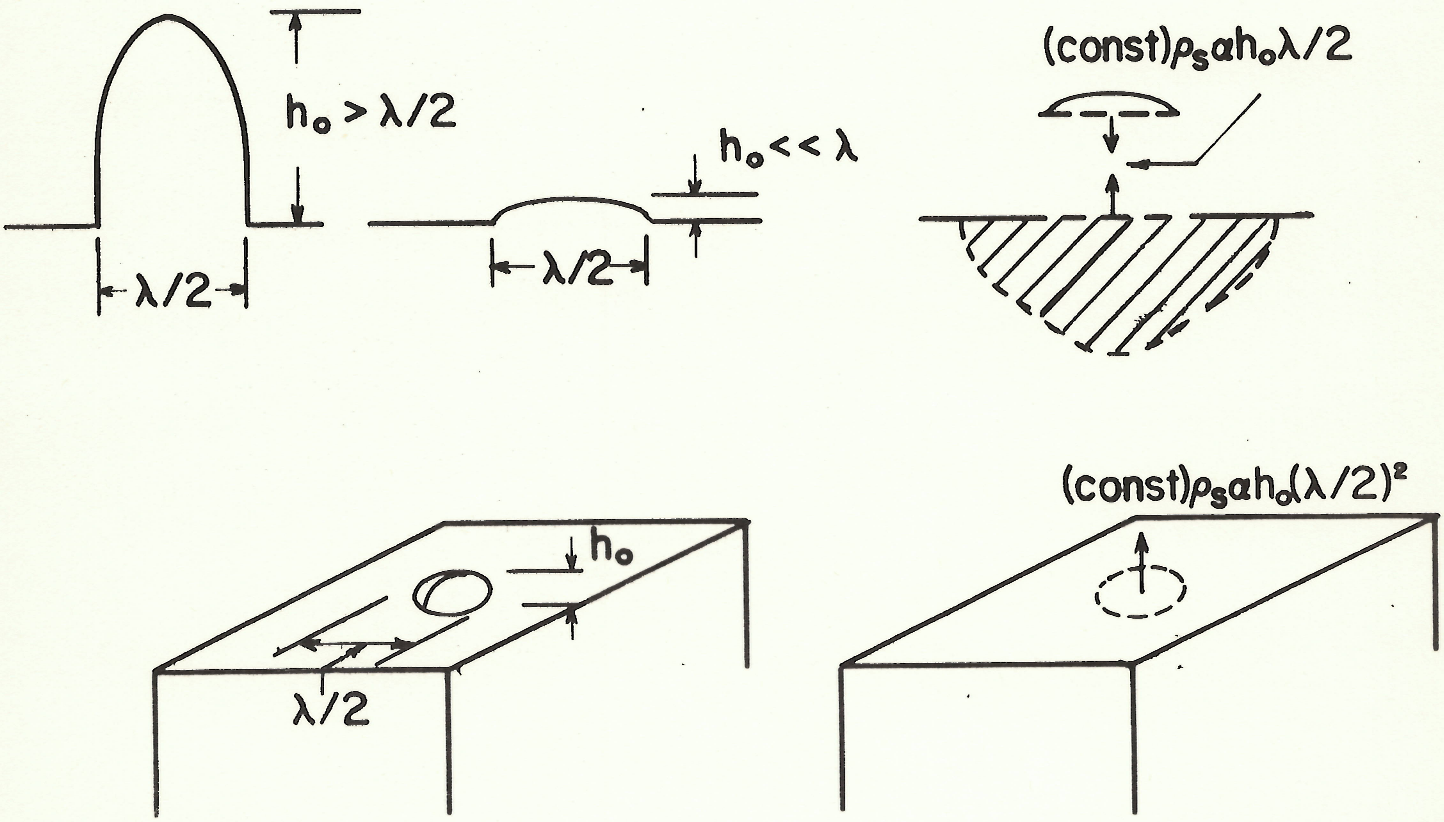
$h_o > \lambda/2$
$\lambda/2$
$h_o << \lambda$
$\lambda/2$
$(const)\rho_s \alpha h_o \lambda/2$
$(const)\rho_s \alpha h_o (\lambda/2)^2$
h_o
$\lambda/2$

FIG. 3

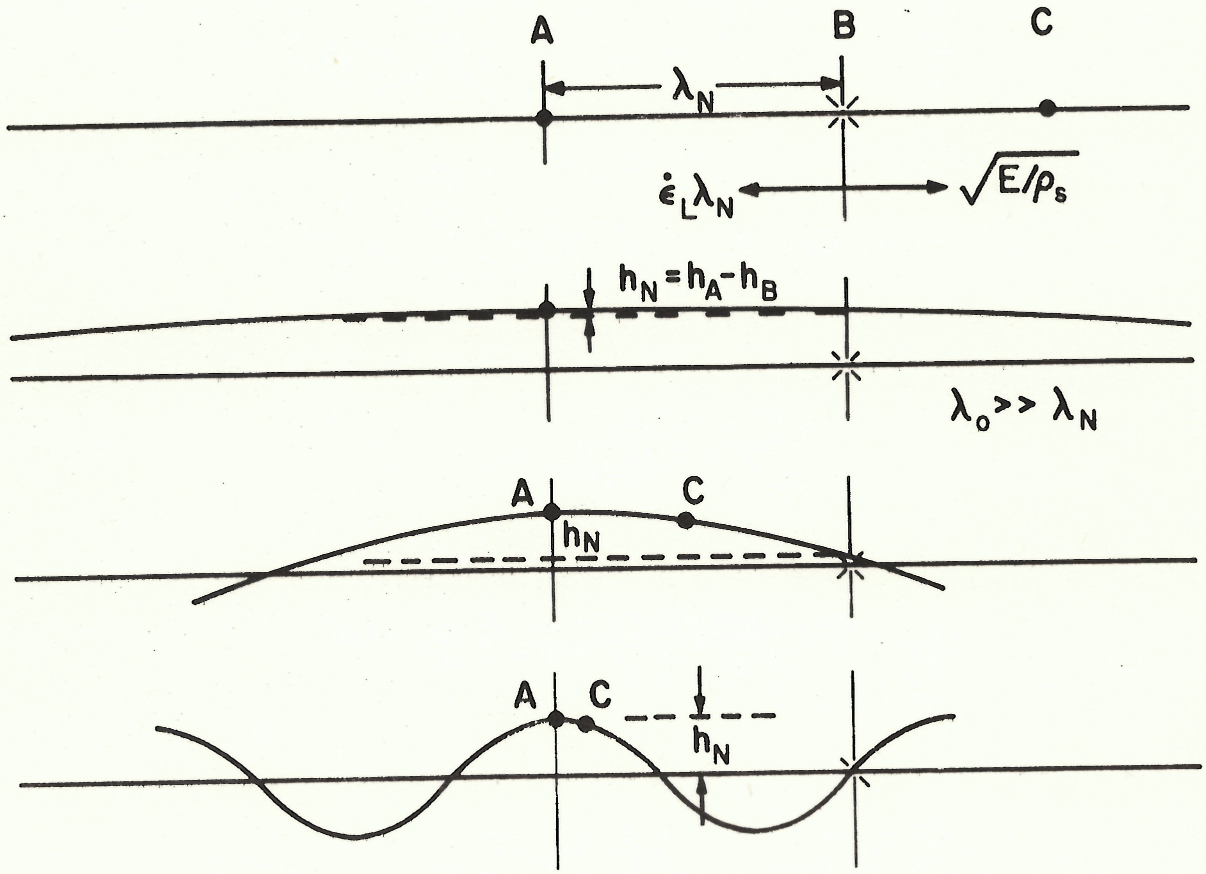
A
B
C
λ_N
$\dot{\epsilon}_L \lambda_N$
$\sqrt{E/\rho_s}$
$h_N = h_A - h_B$
$\lambda_o >> \lambda_N$
A
C
h_N
A
C
h_N

FIG. 4