

ON STRESS-STRAIN RELATIONS SUITABLE FOR CYCLIC AND OTHER LOADING

by

D. C. Drucker and L. Palgen

Department of Theoretical and Applied Mechanics  
University of Illinois at Urbana-Champaign  
Urbana, Illinois 61801  
July, 1980



# ON STRESS-STRAIN RELATIONS SUITABLE FOR CYCLIC AND OTHER LOADING

Daniel C. Drucker  
Dean, College of Engineering  
University of Illinois, Urbana, IL 61801

Luc Palgen  
Aspirant du Fonds National Belge de la Recherche Scientifique  
Department of Theoretical and Applied Mechanics  
University of Illinois, Urbana, IL 61801

## Summary

The analysis and design of pressure vessels and other structures subjected to cyclic loading and occasional large overloads requires stress-strain relations sufficiently simple to be usable with computer programs and yet adequate to describe the essential aspects of the response of the material. One such form with two quite different options is proposed for the time-independent domain which avoids the difficulties of earlier approaches. It has the kinematic hardening attributes needed for reversal of loading, allows for cyclic hardening or softening, gives zero mean stress as the asymptotic response to cyclic straining between fixed limits of strain, and reduces to a  $J_2$  stress-hardening form for radial or proportional loading so that it can model both cyclic and other loading to a good first approximation.

## Introduction

Many important machines and structures such as pressure vessels, turbines, and railroad wheels are subjected to cycles of load, or to cycles of temperature, or both, that produce significant inelastic response. Their design also must encompass the probabilities of occasional large excursions of load or temperature that may precede, interrupt, or follow this exposure to low cycle fatigue.

Considerable attention has been devoted to the experimental determination



of the behavior of materials subjected to cycles of uniaxial stressing or straining, or subjected to cycles of shear. Cyclic hardening, cyclic softening, and the usual approach to cyclic stability have been demonstrated by the extensive investigations of Dolan, Morrow, their colleagues and students, and many others throughout the world [1, 2]. However, little is known about the response of materials to more complex cycles beyond the preliminary study made by Lamba [3, 4].

Many models have been proposed to fit one or more aspects of the response that has been observed in experiments. Time-independent behavior is of sufficient interest and complexity to have attracted major attention, but with the full recognition that time effects often are significant and may well govern design. Some models aim at a detailed and accurate representation of observed behavior over a wide range of loading paths. Consequently, they are rather elaborate and difficult to incorporate in computer programs for complex structures. Others are addressed to an important but limited aspect of the behavior of the material and are not to be used outside of that range of applicability.

The purpose of this paper is to propose a model for consideration that is simple enough to be used effectively in computer programs and yet matches the essential features of the time-independent inelastic behavior of materials reasonably well for cyclic loading and for occasional overloading separately and in combination. In the Sections that follow, an outline will be given of the features of material behavior perceived as most essential and relevant. A simple model with two options will be proposed for cyclically stable material. One option matches the rounding of cyclic stress-strain curves, the other without the rounding matches the behavior on reloading following almost purely elastic unloading. The difficulties or limitations of earlier models



will be exhibited along with possible physical or mathematical explanations for them. Cyclic hardening or softening then will be introduced into the simple model for each of the two options and the ability to match experimental information will be demonstrated. Finally, more elaborate forms of such a model, that include time-dependent behavior and other ignored aspects of the real world, will be touched upon briefly. Their development seems premature in the absence of an accepted, broadly useful, elementary form for the time-independent idealization.

#### Material Behavior Perceived as Most Essential and Relevant

The complexity of all the details of the inelastic behavior of material is infinitely great even when all time effects are ignored. Obviously, therefore, the selection of just a few key aspects as the most essential and relevant for the purpose is a debatable matter of judgment and definition of essential. The choice is strongly dependent upon the perception of purpose and relevance. Our purpose here is to write a simple usable form that will include as a minimum both large excursions of loading well out into the plastic range and the cyclic loading that gives plastic hysteresis loops and can result in low cycle fatigue. Our short list of the most essential and relevant aspects of material behavior prior to significant material damage is:

A) Load excursions well out into the plastic range overwhelm or wipe out many of the effects of the history of plastic deformation prior to such large overloads. It is relevant but not essential that, for such excursions, a Mises or  $J_2$  stress-hardening form is usually a satisfactory approximation for those metals and alloys that are fairly isotropic in their initial state.

B) Under symmetric cycles of stress or strain, metals and alloys in a soft or annealed state to start will harden cyclically and tend to a stable



limit cycle, Figs. 1, 2; those in a very hard or cold-worked condition to start will soften to the stable cycle; and those already in the stable condition neither harden nor soften but simply go through the stable cycle so evocative of kinematic hardening [5].

C) Unsymmetric cycles of stress in the plastic range will cause progressive "creep" or "ratcheting" in the "direction" of the mean stress, right hand side of Fig. 3.

D) Unsymmetric cycles of strain in the plastic range will cause progressive relaxation to zero of the mean stress in the cycle, left hand side of Fig. 3.

A model will be presented here with two options, each of which satisfies our four requirements (A) - (D). The first option gives full rounding of the stress-strain curve on each load reversal following appreciable plastic deformation, a condition often encountered in practice. The second option gives a sharp transition when going from purely elastic to elastic-plastic response, which is the correct picture for reloading following almost purely elastic unloading. Neither of these model options is as appropriate for both types of loading as are the models of Mroz [6] and the other assemblages of many simple models in parallel that correspond to a large set of closely nested yield or loading surfaces [7]. However, the broader match by such assemblages is at the sacrifice of simplicity and some important elements of reality.

#### A Simple Model for Cyclically Stable Material

The expression for the increment or rate of plastic strain  $\dot{\epsilon}_{ij}^p$  of a time-independent material with a smooth yield surface  $f = 0$  is:

$$\dot{\epsilon}_{ij}^p = G \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{mn}} \dot{\sigma}_{mn} \quad (1)$$



where  $G$  is a scalar multiplier,  $\sigma_{ij}$  is the current stress,  $\dot{\sigma}_{mn}$  its increment or rate, and repeated subscripts denote summation.

Cyclic creep or ratcheting and stress relaxation can occur whether the material hardens, softens, or is stable. Therefore, it is reasonable to postpone the examination of hardening or softening and consider a cyclically stable material first.

A combination of the Mises stress-hardening form and the kinematic hardening proposed by Prager [5] and modified by Shield and Ziegler [8, 9] includes the key aspects (A) and (B) of cyclically stable material behavior. The simplest permissible and yet appropriate choices appear to be:

$$G = A J_2^N \quad (2)$$

and

$$f = \frac{1}{2}(s_{ij} - s_{ij}^c)(s_{ij} - s_{ij}^c) - k^2 = 0 \quad (3)$$

where  $A$  and  $N$  and  $k$  are constants,  $J_2 = \frac{1}{2} s_{ij} s_{ij}$ ,  $s_{ij}$  is the stress deviator  $\sigma_{ij} - \frac{1}{3} \sigma_{qq} \delta_{ij}$ ,  $s_{ij}^c$  is the center of the spherical yield domain in stress deviator space, and  $k$  is the yield stress in simple shear when the yield domain is centered at the origin,  $s_{ij}^c = 0$ . The center moves in the direction of  $s_{ij} - s_{ij}^c$  at a rate equal to the projection of  $\dot{s}_{mn}$  on that direction in accordance with a Ziegler type of rule that satisfies the consistency condition of the stress point  $s_{ij}$  remaining on the current yield surface, or  $\dot{f} = 0$ :

$$\dot{s}_{ij}^c = (s_{ij} - s_{ij}^c)(s_{mn} - s_{mn}^c) \dot{s}_{mn} / 2k^2 \quad (4)$$

and

$$\dot{\epsilon}_{ij}^p = A J_2^N (s_{ij} - s_{ij}^c)(s_{mn} - s_{mn}^c) \dot{s}_{mn} \quad (5)$$



for this simplest of analytic models of an initially isotropic and symmetric cyclically stable elastic-plastic material.

Simple tension or simple shear or any radial (proportional) loading is represented by

$$\sigma_{ij} = R \sigma_{ij}^o, \quad s_{ij} = R s_{ij}^o, \quad \dot{s}_{ij} = \dot{R} s_{ij}^o \quad (6)$$

where the fixed state of stress  $\sigma_{ij}^o$  or stress deviator  $s_{ij}^o$  is on the (initial) yield surface for  $s_{ij}^c = 0 = \sigma_{ij}^c$ . The response of the material is purely elastic up to  $R = 1$  and then is elastic-plastic in accord with Eqs. (4) and (5) as  $R$  increases. When plastic deformation takes place in this forward direction or in the reverse direction as  $R$  decreases and becomes negative,

$$\dot{s}_{ij}^c = \dot{s}_{ij}; \quad s_{ij} - s_{ij}^c = \pm s_{ij}^o; \quad \sigma_{ij} - \sigma_{ij}^c = \pm \sigma_{ij}^o \quad (7)$$

The stress-strain relation (5) can be written in terms of  $\dot{s}_{ij}^c$  from (4) and then with  $\dot{s}_{ij}$  from (7)

$$\begin{aligned} \dot{\epsilon}_{ij}^p &= 2k^2 A J_2^N \dot{s}_{ij}^c = 2k^2 A J_2^N \dot{s}_{ij} = 2A k^{2N+2} |R|^{2N} \dot{s}_{ij} \\ &= 2A k^{2N+2} |R|^{2N} \dot{R} s_{ij}^o \end{aligned} \quad (8)$$

For monotonically increasing  $R$ , direct integration of (8) gives the total plastic strain

$$\epsilon_{ij}^p = \frac{2A k^{2N+2}}{2N+1} (R^{2N+1} - 1) s_{ij}^o \quad (9)$$

but the incremental form (8) will normally be more useful here.



When the radial loading path is simple tension  $\sigma = R \sigma_0$ , where  $\sigma_0 = \sqrt{3} k$  is the initial tensile (or compressive) yield stress, the plastic tensile strain

$$\dot{\epsilon}^p = 2A \left(\frac{1}{3} \sigma_0^2\right)^{N+1} (|\sigma|/\sigma_0)^{2N} (2/3) \dot{\sigma} \quad (10)$$

When the path is simple shear  $\tau = R \tau_0$ , where  $\tau_0 = k$  is the initial yield stress in shear and  $\dot{\epsilon}_{12}^p = \frac{1}{2} \dot{\gamma}^p$

$$\frac{1}{2} \dot{\gamma}^p = 2A (\tau_0^2)^{N+1} (|\tau|/\tau_0)^{2N} \dot{\tau} \quad (11)$$

The introduction of a stress-dependent plastic modulus inversely proportional to  $J_2^N$  gives the correct qualitative picture of cyclic creep and stress relaxation in simple tension or simple shear. This is shown in Fig. 3, where the elastic-plastic stress-strain curves are identical except for a translation along the strain axis.

The rounding option, illustrated in Fig. 3, uses a yield surface of small diameter. Unfortunately, on reloading following appreciable unloading or reverse loading with small plastic deformation, the stress-strain curve exhibits full rounding well before reaching the stress level from which unloading began, Fig. 4. This is the price paid to obtain proper rounding of the hysteresis loops. There is also a somewhat more subtle problem with this option. When reverse plastic deformation occurs on unloading (before reaching zero stress) the reciprocal of the plastic tangent modulus starts off with a positive value but then decreases to zero as the stress goes to zero before it begins to increase again. This clearly is an incorrect representation. However, it occurs only in the region of small stress for a reasonable choice of  $k$  and, although not aesthetic, can be ignored because the plots obtained will differ only very little from purely elastic response.



The apparent elastic range is extended. For all practical purposes, no significant plastic deformation is computed until the sign of the stress reverses and the magnitude of the reversed stress is a significant fraction of the initial yield stress.

Corresponding errors of representation appear for loading paths and cycles that do not include the origin, paths for which the  $J_2$  form itself is a less satisfactory approximation.

The sharp corner option or representation, appropriate for a sequence of unloading and reloading, is obtained by introducing a large diameter yield surface, so that only the flat portion of the elastic-plastic stress-strain curves of Fig. 3 is used. This rules out reverse plastic deformation on unloading.

The choice of a small or large diameter yield surface will be governed by the aspects of behavior most important for the application at hand.

The parameters of the model are found from the upper or lower half of a stable hysteresis loop of the material.  $2N$  comes from the slope of a plot of  $\log \frac{\dot{\epsilon}^p}{\dot{\sigma}}$  or  $\log \frac{\dot{\gamma}^p}{\dot{\tau}}$ , the logarithm of the reciprocal of the plastic tangent modulus, vs  $\log \sigma$  or  $\log \tau$ , in the range of plastic strains that is of greatest interest to the user of the model. Elastic strain increments must, of course, be taken into account in reducing the data;  $\dot{\epsilon}^e = \dot{\sigma}/E$ ,  $\dot{\gamma}^e = \dot{\tau}/\mu$  etc. add to the plastic strain increments or rates to give total increments or rates,  $\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p$ , etc.

In the rounding option  $k$  should be taken as large as possible consistent with the desired rounding of the loop, in order to suppress or minimize plastic deformation on unloading and to reduce improper rounding on direct reloading. Except for these two and closely related loading cases,



the model will be insensitive to the particular choice of  $k$ , as can be seen from Fig. 3.

In the sharp corner approach, however, the size of the yield surface has a dominant effect because it determines the stress level of the plastic response. The elastic range  $2\sqrt{3} k$  is comparable to the total height of the hysteresis loop and is taken as the vertical distance at zero total strain, Fig. 5. Because of their flatness, the computed stress-strain curves (Fig. 7) will be affected little by large variations of  $N$ .

Finally,  $A$  should be chosen to match the value of stress at some intermediate value of strain within the range of plastic strains of greatest interest.

A wide variety of more complex stress-strain relations are available for time-independent behavior [10, 11, 12, 13, 14] that can model one or more aspects of material behavior more closely than the simple three constant form proposed here. Before considering them or the next step of modifying the present form to include cyclic hardening or softening, it is worth examining other time-independent models that have been proposed and used for cyclic loading.

#### Scope and Limitations of Some Earlier Models

One or another aspect of reversed or of cyclic loading has attracted attention in the past and led to suggestions of mathematical models. A set of bars in parallel can model a simple tension curve as accurately as desired. If each bar is elastic - perfectly plastic with the same properties in tension and compression, the assemblage is immediately cyclically stable for an unsymmetric cycle of stress or strain [7], Fig. 6. The assemblage does not "creep" or relax as it should in accord with the requirements (C) and (D) listed under essential material behavior.



The more general assemblages of elastic-perfectly plastic homogeneous elements or states in parallel with their nested yield surfaces have the same ability to model stress-strain behavior accurately for generally outward loading paths and to exhibit a significant realistic Bauschinger effect for a single reversal of loading. However, they too suffer from the defect of not creeping as they should in the direction of the mean stress in an unsymmetric plastic cycle of stress and not settling down to zero mean stress in an unsymmetric plastic cycle of strain.

Permitting one or more elements of the assemblage to harden, as each strains plastically, produces a model of a cyclically hardening material but does not overcome the basic difficulty of an inappropriate response to unsymmetric plastic cycles of stress or of strain. The models of Caulk and Naghdi [13] and of Popov [12] have this basic drawback. So also to a far lesser extent does the model of Mroz [6] with two or more nested yield or loading surfaces. However, each model was devised for its own special set of requirements for matching particular aspects of real world behavior. None began with all the requirements (A) through (D) that have been chosen here as essential.

Differences in principle of the degree of thermodynamic reversibility between assemblages of states in parallel or series and dislocation structure were pointed out still earlier by Drucker [7] for both conventionally cyclically stable models with nested loading surfaces and cyclically hardening models with intertwined loading surfaces. Useful and physically appealing as such assemblages may be for a variety of problems, whether they are in the forms just described or in the form of parallel layers for beams, plates, and shells [15], they cannot represent unsymmetric cyclic behavior properly.

Alternative approaches have been proposed to give proper rounding of reversed loading curves as well as proper cyclic response [11]. They, as

well as several of the earlier suggestions, seem more elaborate than can be handled economically on computers today for boundary value problems with pointwise varying multiaxial states of stress in which large load excursions are superposed occasionally on low-cycle fatigue loading. Some of the complexity appears to be caused by the manner in which the consistency condition is employed.

It is necessary for the stress point to remain on the yield surface(s) and for the center of each surface to move appropriately as plastic deformation continues. However, when this is built into the model automatically through a specification in stress space as in Eq. (4), the choice of a reasonable stress-strain relation is quite free. The reverse approach, which relates the motion of the stress point and the center of the yield surface through an incremental stress-strain relation that is chosen in advance, can lead to a consistency relation that may not be satisfied conveniently. This more difficult approach so common today for nonlinear plastic hardening has not given good results except for the model of Arutyunyan and Vakulenko [14]. It may have developed as a consequence of generalizing Prager's illustrative example of linear kinematic hardening, in which the motion of the yield surface in stress space is proportional to plastic strain. Any linking of the location of the yield surface in stress space to the current components of plastic strain causes a severe difficulty. This class of models can match observed Bauschinger effects for a single reversal of loading extremely well, as demonstrated in a paper with Edelman [10]. However, such models are not appropriate for cyclic loading because unsymmetric strain cycles give unsymmetric stress cycles.

The temptation to think of the special form  $f(\sigma_{ij}, \epsilon_{pq}^p)$  as a good first



approximation to reality must be resisted here. Writing  $\dot{f} = 0 = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \epsilon_{mn}^p} \dot{\epsilon}_{mn}^p$ , and replacing  $\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij}$  by  $-\frac{\partial f}{\partial \epsilon_{mn}^p} \dot{\epsilon}_{mn}^p$  generally leads to undesirable and misleading constraint.

#### Modification of the Proposed Simple Model to Include Hardening or Softening

The term hardening or softening in the cyclic context could refer to the increase or decrease in the diameter of the yield surface or to the increase or decrease in the plastic tangent modulus at a given stress or to both. The first definition is appropriate for the generalization of the sharp corner approach and the second definition for the rounding representation. Pure kinematic hardening with a translating yield surface as given by Eq. (3) and a purely stress-dependent plastic modulus as given by Eq. (5) is neither hardening nor softening in either sense although the stress-strain relation (5) gives rise to the usual work-hardening picture for each radial loading as exhibited by form (10) for simple tension and form (11) for simple shear.

The sharp corner form with cyclic hardening or softening is obtained by replacing  $k^2$  in (3) by  $F$ , a positive scalar function of the path of straining, thereby permitting the diameter of the yield surface to change. Increase in  $F$  gives cyclic hardening, decrease gives cyclic softening. However, the more general expression

$$f = \frac{1}{2} (s_{ij} - s_{ij}^c) (s_{ij} - s_{ij}^c) - F = 0 \quad (12)$$

does not require any alteration in the form of expression (5) of the stress-strain relation

$$\dot{\epsilon}_{ij}^p = A J_2^N (s_{ij} - s_{ij}^c) (s_{mn} - s_{mn}^c) \dot{s}_{mn}$$

and any of the specialized forms such as (10) for simple tension. The



somewhat strange result is that for a hardening material purely in the sense of  $\sigma_0$  increasing, the plastic modulus  $\dot{\sigma}/\dot{\epsilon}^P$  at a given value of stress,  $\sigma$  decreases. The motion of the center of the yield surface  $\dot{s}_{ij}^c$  is affected significantly by the increase or decrease in  $F$ . Equation (4) is replaced by a more general form that reduces to (4) when  $F$  is constant.

$$\dot{f} = 0 = (s_{ij} - s_{ij}^c) (\dot{s}_{ij} - \dot{s}_{ij}^c) - \dot{F} \quad (13)$$

or

$$\dot{s}_{ij}^c (s_{ij} - s_{ij}^c) = (s_{mn} - s_{mn}^c) \dot{s}_{mn} - \dot{F} \quad (14)$$

With the Prager-Ziegler type of assumption that  $\dot{s}_{ij}^c$  is in the direction of  $s_{ij} - s_{ij}^c$

$$\dot{s}_{ij}^c = [(s_{mn} - s_{mn}^c) \dot{s}_{mn} - \dot{F}] (s_{ij} - s_{ij}^c) / 2F \quad (15)$$

Any one of a variety of choices for the functional dependence of  $F$  might be selected on a trial basis.  $F$  as a function of plastic work,  $W^P = \int_t \sigma_{ij} \dot{\epsilon}_{ij}^P dt = \int_t s_{ij} \dot{\epsilon}_{ij}^P dt$ , or of cumulative plastic strain measured by  $\int_t \sqrt{\dot{\epsilon}_{ij}^P \dot{\epsilon}_{ij}^P} dt$ , is not unreasonable. The choice of  $F$  as a function of

$$W^{PC} = \int_t (s_{ij} - s_{ij}^c) \dot{\epsilon}_{ij}^P dt \quad (16)$$

will be made here instead because it also is not unreasonable and it does result in a convenient form for  $\dot{s}_{ij}^c$ . When plastic deformation takes place, (16) may be rewritten using (5) and (12) as

$$W^{PC} = \int_t (2F) A J_2^N (s_{mn} - s_{mn}^c) \dot{s}_{mn} dt \quad (17)$$



It is of some interest to note that  $\int_t \sqrt{\dot{\epsilon}_{ij}^p \dot{\epsilon}_{ij}^p} dt$  is given by the same form (17) except for  $\sqrt{2F}$  instead of  $2F$  and so is proportional to  $W^{pc}$  for cyclically stable material. Substitution of

$$\dot{F} = (dF/dW^{pc}) \dot{W}^{pc} \quad (18)$$

in (15) gives

$$\dot{s}_{ij}^c = [1 - 2F(dF/dW^{pc}) A J_2^N] [(s_{mn} - s_{mn}^c) \dot{s}_{mn}] (s_{ij} - s_{ij}^c)/2F \quad (19)$$

One of many reasonably simple choices for  $F$  that permits an adjustable asymptotic approach to the cyclically stable value  $F = k^2$  is

$$F = k^2 [1 \mp \alpha \exp(-W^{pc}/W_0)]^2 \quad (20)$$

where the upper sign applies for hardening from  $F = k^2(1 - \alpha)^2$  to  $F = k^2$ , with  $\alpha$  restricted to lie between zero and one. The lower sign applies for softening from  $F = k^2(1 + \alpha)^2$  to  $k^2$  for any positive  $\alpha$ . The other disposable constant  $W_0$  in (20) permits adjustment of the rate of hardening or softening.  $W_0$  is the value of  $W^{pc}$  at which  $F = k^2(1 \mp \alpha/e)^2$ . Also, from (20)

$$\begin{aligned} \frac{dF}{dW^{pc}} &= \pm 2k^2 \frac{\alpha}{W_0} \exp(-W^{pc}/W_0) [1 \mp \alpha \exp(-W^{pc}/W_0)] \\ &= \pm \frac{2F(\alpha/W_0) \exp(-W^{pc}/W_0)}{1 \mp \alpha \exp(-W^{pc}/W_0)} \end{aligned} \quad (21)$$

In the rounding representation, expansion or shrinking of the yield surface does not affect the hysteresis loops appreciably. Cyclic hardening or softening must be provided by the alternate definition, i.e., an increase or

decrease in the plastic tangent modulus at each stress point. Perhaps the simplest approach is to maintain a constant (small) diameter yield surface and to give a hardening form by rewriting (5) as

$$\dot{\epsilon}_{ij}^p = B \left( \frac{J_2}{\sigma^2} \right)^N (s_{ij} - s_{ij}^c) (s_{mn} - s_{mn}^c) \dot{s}_{mn}^c \quad (22)$$

Cyclic hardening is given by an increase in the normalizing stress  $\bar{\sigma}$ , cyclic softening by a decrease. Following the same steps which lead to the functional dependence of  $F$  in the previous form, one can write, similarly to Eq. (20),

$$\bar{\sigma} = \sigma^* [1 + \alpha \exp(-W^{pc}/W_0)] \quad (23)$$

where  $\sigma^*$  is a constant stress and the sign before  $\alpha$  is chosen in the same way as for Eq. (20). The stable form of (23) is identical to (4) if one lets

$$A = \frac{B}{(\sigma^*)^{2N}} \quad (24)$$

Because the size of the yield surface remains constant, the motion of its center is still described by (4). However, the specialized forms for radial loading, (8) to (11), must be modified. For example, (10) becomes

$$\dot{\epsilon}^p = \frac{2B\sigma_o^2}{3^{N+1}} \left( \frac{|\sigma|}{\sigma} \right)^{2N} (2/3) \dot{\sigma} \quad (25)$$

No matter what form of incremental stress-strain relation is chosen, the solution to a boundary value problem requires keeping track, at each point of the body, of the state of stress, the location of the current yield surface in stress space or equivalent information, and the increment or rate



of strain accompanying the increment or rate of stress. The iterative process to be followed for each increment of load and displacement or temperature change applied to the body, and the direct updating of  $s_{ij}^c$  to give the state and response of the material at each point, can be done sequentially within the accuracy of representation of the model. Equation (19) for the motion of the center of the yield surface in the sharp corner option and Eq. (22) for plastic strain rates in the rounding option are more complicated than the corresponding Eqs. (4) and (5) for the cyclically stable material, but their use and the calculation of  $\dot{W}^{pc}$  are straightforward given  $\dot{s}_{ij}$  and current values of  $W^{pc}$ ,  $F$  or  $\bar{\sigma}$ ,  $s_{ij}$ , and  $s_{ij}^c$ .

The parameters of the model for both forms are conveniently chosen from a fully reversed, strain-controlled test, in tension or shear. Material constants  $A$  or  $\frac{B}{(\sigma^*)^{2N}}$ ,  $k$ ,  $N$  are determined in the manner already described for the stable loop. However, in the sharp corner form, and for a cyclically hardening material, the log-log plot of the reciprocal of the plastic tangent modulus versus stress, which provides  $N$ , is now a plot for the initial loading curve, where  $N$  has the most influence on model predictions, Fig. 7. Similarly, in the rounding form,  $k$  is taken as large as possible, consistent with the stress level of the initial loading curve.

Quite independently of the values chosen for  $A$  or  $\frac{B}{(\sigma^*)^{2N}}$ ,  $k$ ,  $N$ , the initial response and the approach to the stable cyclic response determine the remaining constants  $\alpha$  and  $\frac{W_0}{k}$ . Let  $\sigma_1$  and  $\sigma_\infty$  be the stresses at the end of the initial curve and at the tip of the stable loop, Fig. 5. Eqs. (20) for the sharp corner option or (23) for the rounding option suggest that a first approximation to  $\alpha$  is

$$\frac{\sigma_1}{\sigma_\infty} = 1 + \alpha \quad (26)$$

The choice of  $\frac{W_o}{k}$  for the rounding option requires picking two points A and B at the end of any two curves in the course of hardening or softening and measuring the stress differences  $\Delta\sigma_A$  and  $\Delta\sigma_B$  from the tip of the stable loop, Fig. 5. In view of the exponential rate of approach to the stable cycle described by (23), an approximate value of  $\frac{W_o}{k}$  is given by

$$\frac{\Delta\sigma_B}{\Delta\sigma_A} = \exp \left[ - \frac{\sqrt{3} k (2n) \Delta\epsilon^P}{W_o} \right] \quad (27)$$

where  $n$  is the number of cycles between A and B and  $\Delta\epsilon^P$  is the average plastic strain range of a cycle and is measured as indicated in Fig. 5. For the sharp corner option, one must in addition use the stresses  $\sigma_A$  and  $\sigma_B$ , Fig. 5, to get  $\frac{W_o}{k}$  from

$$\frac{\sigma_A}{\sigma_B} \frac{\Delta\sigma_B}{\Delta\sigma_A} = \exp \left[ - \frac{\sqrt{3} k (2n) \Delta\epsilon^P}{W_o} \right] \quad (28)$$

More elaborate methods can determine  $\alpha$  and  $W_o$  more accurately. However, they probably are not worth developing because a close match of a single set of data does not guarantee a correct response to a different, e.g., non-radial, loading path.

Of course, a model with 3 constants available for radial loading and stable cyclic response and two more for cyclic hardening or softening can not aim at a precise description of material behavior for a variety of paths of loading. Far more elaborate forms than the  $J_2$  or Mises form are known to be required for radial loading alone. Quite complicated functions of the history of loading, not constants, are needed to obtain just a moderately good representation in detail of the strain history for more general loading paths.

However, just as the very crude approximation of perfect plasticity and the resulting plastic limit theorems and the shakedown theorems have a meaningful place in analysis and design [16, 17], so also should a simple but essentially valid approximation to both cyclic behavior and response to occasional overloads provide a useful basis for life prediction and safe design. It is too early to tell how well the particular proposed simple form will do, but some comparisons with the experimental results of others will prove encouraging. Surely, much more detailed matching of cyclic stress-strain behavior is not essential for low cycle fatigue prediction. Some scalar measure such as  $W^{pc}$ , or perhaps  $W^p$  or  $\int_t \sqrt{\dot{\epsilon}_{ij}^p \dot{\epsilon}_{ij}^p} dt$  based on the proposed simple model should provide a first approximation of value.

It is obvious that any representation that matches the stable cyclic stress range of a material, and is flexible enough to permit a choice of the initial response and the rate of approach to the stable cycle, will match the data from which it is taken reasonably well.

Figure 7 demonstrates this for the sharp corner form and the test of Fig. 1 on 304 stainless [18] with the choice of  $A k^{2N+3} = 0.3$

$$k = 140 \text{ MPa (20 Ksi)} \quad N = 1 \quad \alpha = 0.5 \quad \frac{W_o}{k} = 0.05 \quad E = 120 \text{ GPa (17 x 10}^3 \text{ Ksi)}$$

The rounding form, Fig. 8, shows better agreement with these data, where the improper representation of Fig. 4 cannot appear. Fig. 9 demonstrates how closely the same form matches the cyclic hardening of 2024-T4 aluminum in a symmetric strain cycle followed by a large excursion which in turn is followed by an unsymmetric strain cycle. Corresponding data, Fig. 2, were provided by J. Morrow and Peter Kurath. Parameters in Figs. 8 and 9 were, respectively,

$$\frac{B}{(\sigma^*)^{2N}} k^{2N+3} = 9 \times 10^3$$



$$k = 50 \text{ MPa}(7.3 \text{ Ksi}) \quad N = 6 \quad \alpha = 0.5 \quad \frac{W_o}{k} = 0.07 \quad E = 120 \text{ GPa}(17 \times 10^3 \text{ Ksi})$$

$$\frac{B}{(\sigma^*)^{2N}} k^{2N+3} = 7 \times 10^{-5}$$

$$k = 140 \text{ MPa}(20 \text{ Ksi}) \quad N = 5 \quad \alpha = 0.3 \quad \frac{W_o}{k} = 0.2 \quad E = 70 \text{ GPa}(10^4 \text{ Ksi})$$

No claim can be made that either choice of constants in the above examples is adequate to describe the stress-strain behavior of the material for more general paths of loading. Nevertheless, in distinction to all other simple models that have been proposed, and many of the complex, the character of the response to unsymmetric plastic cycles of stress or strain is correct in principle as is the general behavior for combinations of large excursions of stress with a dominant pattern of cyclic loading.

#### Concluding Remarks on Rounding, Time Effects, and Other Aspects of Reality

It is possible to break the connection between the desirable rounding of the stress-strain curves for reversed plastic loading and the undesirable rounding for reloading following an almost elastic reversal, Fig. 4, by keeping track of the unloading-reloading paths and introducing the physically correct transition from elastic to elastic-plastic response. Whether this added degree of reality is worth the complexity for general paths of loading is doubtful. The inclusion of time and temperature effects certainly is of far greater practical importance for many materials, such as the stainless steels, under operating or emergency conditions in pressure vessels and piping or other engineering structures and devices. Although much is known about time-dependent creep and relaxation at a variety of temperatures [19], very little experimental information exists on the time effects occurring in conjunction

with cyclic loading interspersed with large excursions of load, along with temperature variation. Bodner [20] and Onat [21] have suggested forms on the basis of the limited available experimental information. When time and temperature effects are primary, entirely different models of material behavior are required from that proposed here. However, when elastic-plastic response dominates but time and temperature effects are significant, a modified form of the proposed simple model should be appropriate, one which adds a linear or nonlinear viscous response and employs a temperature modified stress [22] along with a time at temperature modified stress for plastic response.

This assumes the simple model proposed is an adequate model or can be made adequate with minor revision, an assumption that requires further experimental exploration and study. The generalization from isotropic to anisotropic cyclic hardening or softening suggested by the data of Lamba [3] poses no difficulty in principle within the mathematical theory of plasticity. However, the functional form for the yield surface  $f = 0$  will be far more complicated and could hardly be classed as a minor revision.

In concluding it is worth returning to the two somewhat related classes of models that have long been popular because they can exhibit a proper Bauschinger effect for a reversal of loading. One includes the components of plastic strain explicitly in the yield function  $f$ . The other assembles well-defined time-independent simple elements in parallel to produce a model that has great physical appeal because it can actually be constructed and its mechanical behavior is easily visualized. Unfortunately, neither of these classes of models is basically appropriate for cyclic loading. Neither exhibits a proper response for repeated unsymmetric plastic cycles of stress or of strain.



## References

- [1] Dolan, T. J., "Nonlinear Response under Cyclic Loading Conditions," Proceedings of the Ninth Midwestern Mechanics Conference (Madison, Wisconsin), 1965, pp. 3-21.
- [2] Morrow, JoDean and Sinclair, G. M., "Cycle-Dependent Stress Relaxation," Symposium on Basic Mechanisms of Fatigue, ASTM STP 237, American Society for Testing and Materials, 1958, pp. 83-109.
- [3] Lamba, H. S. and Sidebottom, O. M., "Cyclic Plasticity for Nonproportional Paths: Part 1 - Cyclic Hardening, Erasure of Memory, and Subsequent Strain Hardening Experiments," Journal of Eng. Mat. Techn., Vol. 100, 1978, pp. 96-103.
- [4] Lamba, H. S. and Sidebottom, O. M., "Cyclic Plasticity for Nonproportional Paths: Part 2 - Comparison with Predictions of Three Incremental Plasticity Models," Journal of Eng. Mat. Techn., Vol. 100, 1978, pp. 104-111.
- [5] Prager, W., "The Theory of Plasticity - A Survey of Recent Achievements," Proceedings of the Institution of Mechanical Engineers, London, Vol. 169, 1955, pp. 41-57.
- [6] Mroz, Z., "On the Description of Anisotropic Workhardening," Journal of the Mechanics and Physics of Solids, Vol. 15, 1967, pp. 163-175.
- [7] Drucker, D. C., "On the Continuum as an Assemblage of Homogeneous Elements or States," Proceedings of the IUTAM Symposia 1966, Springer-Verlag, Wien-New York, 1968, pp. 77-93.
- [8] Shield, R. T. and Ziegler, H., "On Prager's Hardening Rule," Zeitschrift für angewandte Mathematik und Physik, Vol. 9a, 1958, pp. 260-276.
- [9] Ziegler, H., "A Modification of Prager's Hardening Rule," Quarterly of Applied Mathematics, Vol. 17, 1959, pp. 55-65.
- [10] Edelman, F. and Drucker, D. C., "Some Extensions of Elementary Plasticity Theory," Journal of the Franklin Institute, Vol. 251, 1951, pp. 581-605.
- [11] Eisenberg, M. A., "A Generalization of Plastic Flow Theory with Application to Cyclic Hardening and Softening Phenomena," Journal of Eng. Mat. Techn., Vol. 98, 1976, pp. 221-228.
- [12] Dafalias, Y. T. and Popov, E. P., "Plastic Internal Variables Formalism of Cyclic Plasticity," ASME Journal of Applied Mechanics, Vol. 43, 1976, pp. 645-651.
- [13] Caulk, D. A. and Naghdi, P. M., "On the Hardening Response in Small Deformation of Metals," ASME Journal of Applied Mechanics, Vol. 45, 1978, pp. 755-764.

- [14] Arutyunyan, R. A. and Vakulenko, A. A., "On Repeated Loading of an Elastoplastic Medium," *Izv. Akad. Nauk SSSR, Mekh.*, No. 4, 1965, pp. 53-61.
- [15] Zienkiewicz, O. C., Nayak, G. C. and Owen, D. R. J., 'Composite and "Overlay" Models in Numerical Analysis of Elasto-Plastic Continua', *Foundations of Plasticity*, Edited by Sawczuk, A., Noordhoff, 1972, pp. 107-123.
- [16] Massonnet, Ch. E. and Save, M. A., "Plastic Analysis and Design - Volume One - Beams and Frames," Blaisdell Publishing Company, 1965.
- [17] Save, M. A. and Massonnet, Ch. E., "Plastic Analysis and Design of Plates, Shells and Disks," North-Holland Publishing Company, 1972.
- [18] Pugh, C. E., Liu, K. C., Corum, J. M. and Greenstreet, W. L., "Currently Recommended Constitutive Equations for Inelastic Design Analysis of FFTF Components," ORNL-TM-3602, 1972.
- [19] Leckie, F. A., "A Review of Bounding Techniques in Shakedown and Ratcheting at Elevated Temperature," *Welding Research Council Bulletin*, No. 195, 1974, pp. 1-32.
- [20] Bodner, S. R., Partom, I. and Partom, Y., "Uniaxial Cyclic Loading of Elastic-Viscoplastic Materials," *ASME Journal of Applied Mechanics*, Vol. 46, 1979, pp. 805-810.
- [21] Onat, E. T., "Representation of Inelastic Behavior," Dept. of Eng. App. Sci., Yale University, ORNL-SUB-3863-3, 1976.
- [22] Drucker, D. C., "On Time-Independent Plasticity and Metals under Combined Stress at Elevated Temperature," *Recent Progress in Applied Mechanics*, The Folke Odqvist Volume, Edited by Bertram Broberg, Jan Hult and Frithiof Niordson, Almqvist & Wiksell, Stockholm, John Wiley & Sons, New York, London, Sydney, 1967.



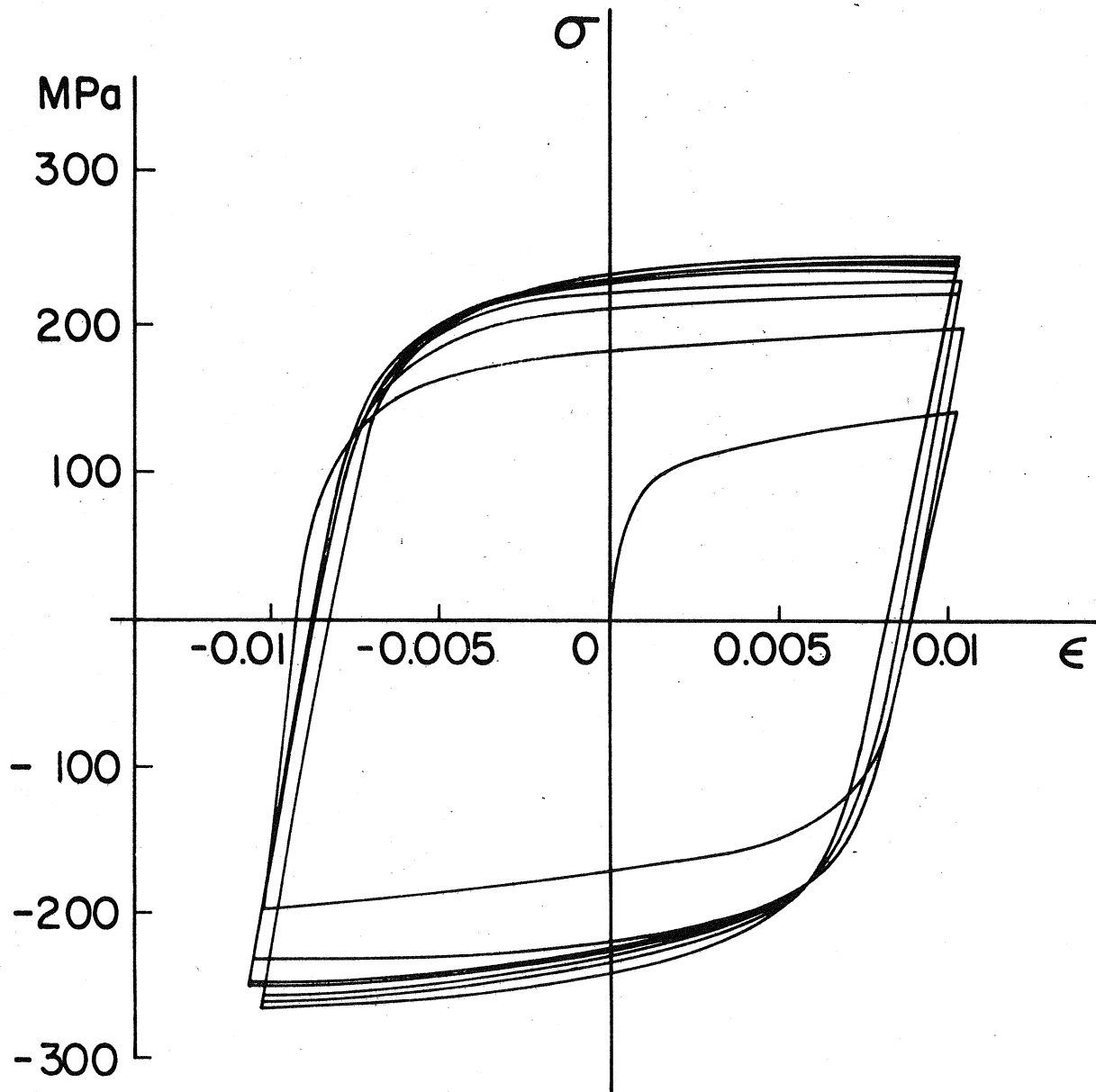


Figure 1. Data on 304 Stainless Steel [18]

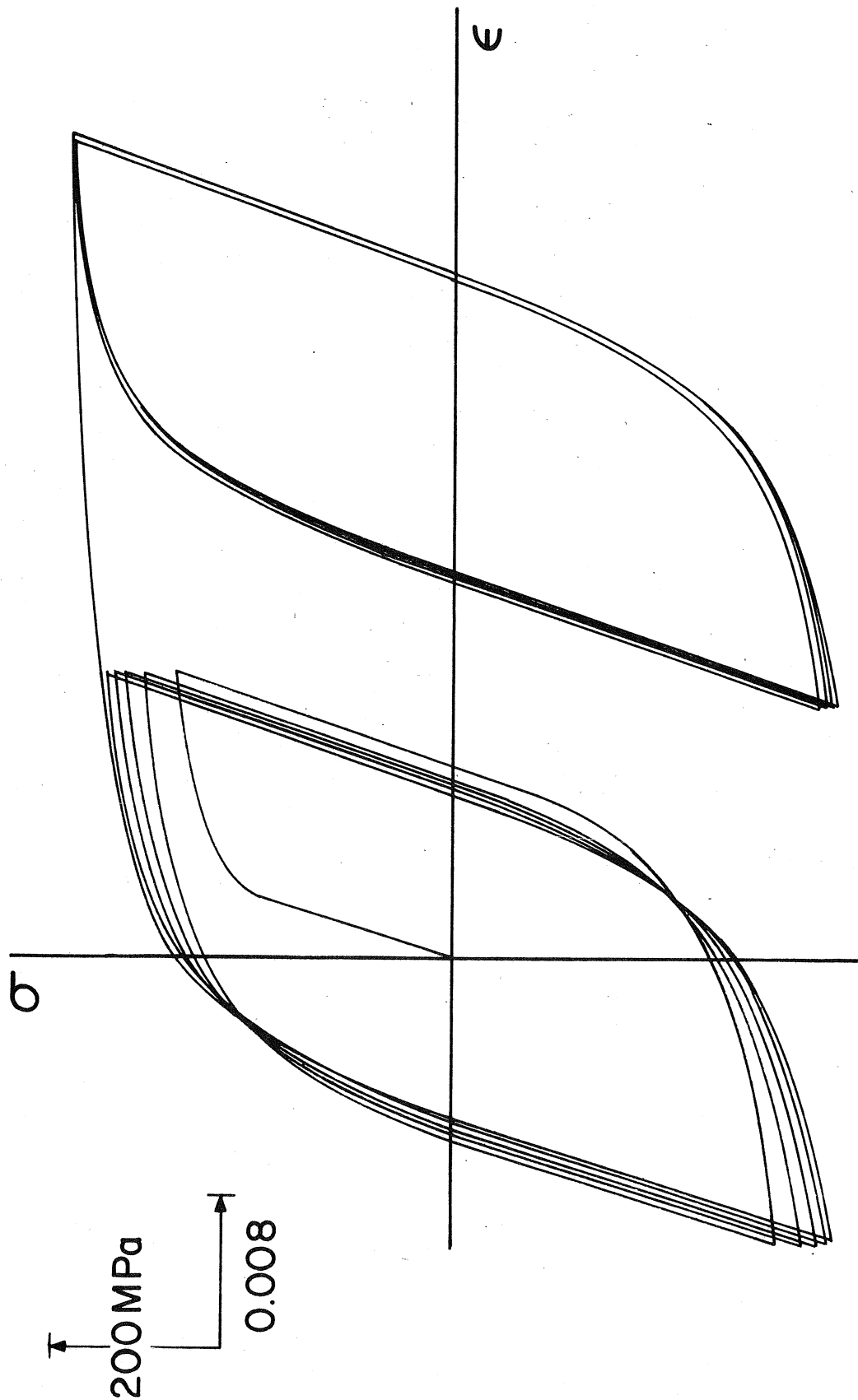


Figure 2. Data on 2024-T4 Aluminum



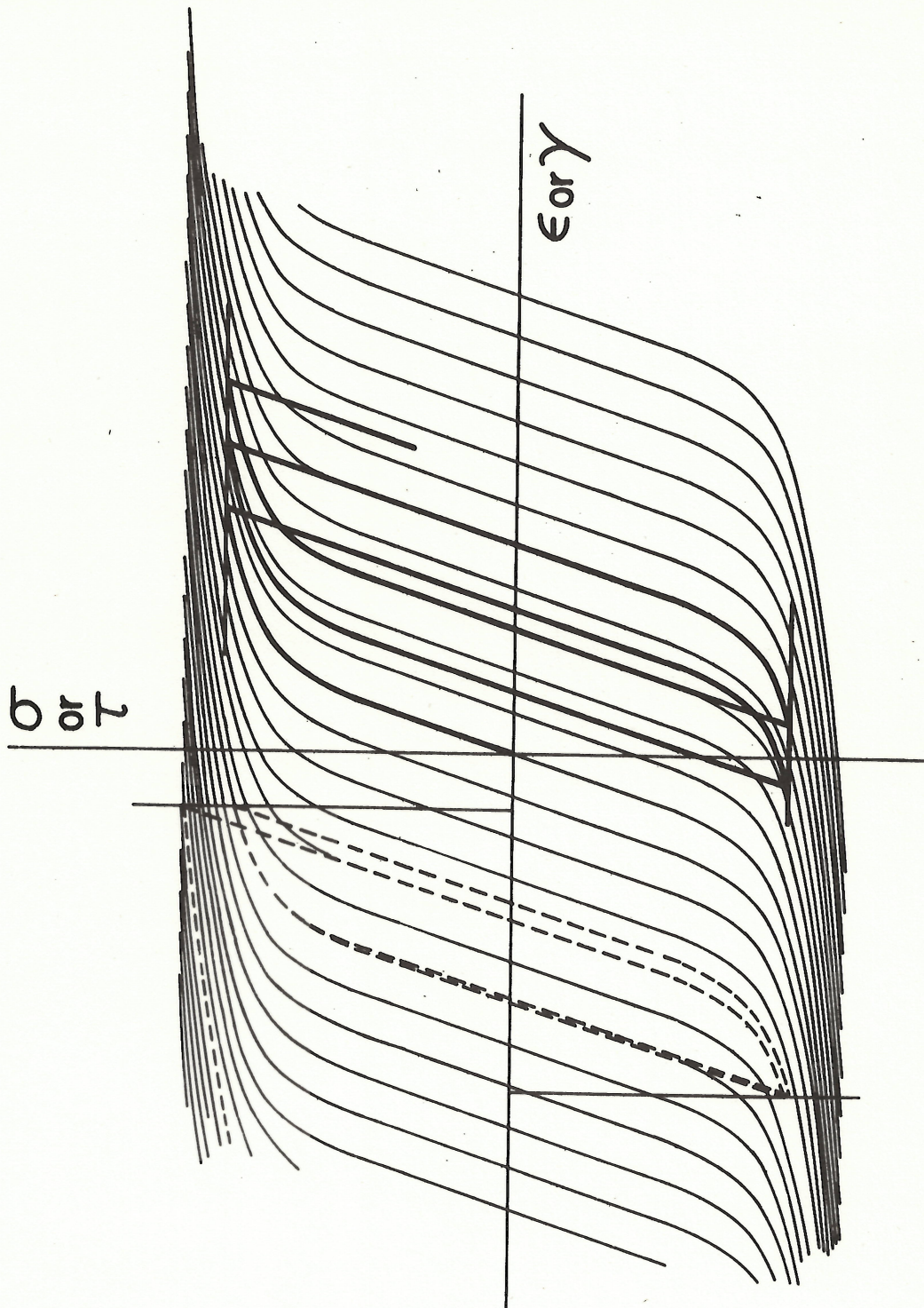


Figure 3. Unsymmetric Cycles of Strain and of Stress for Cyclically Stable Material



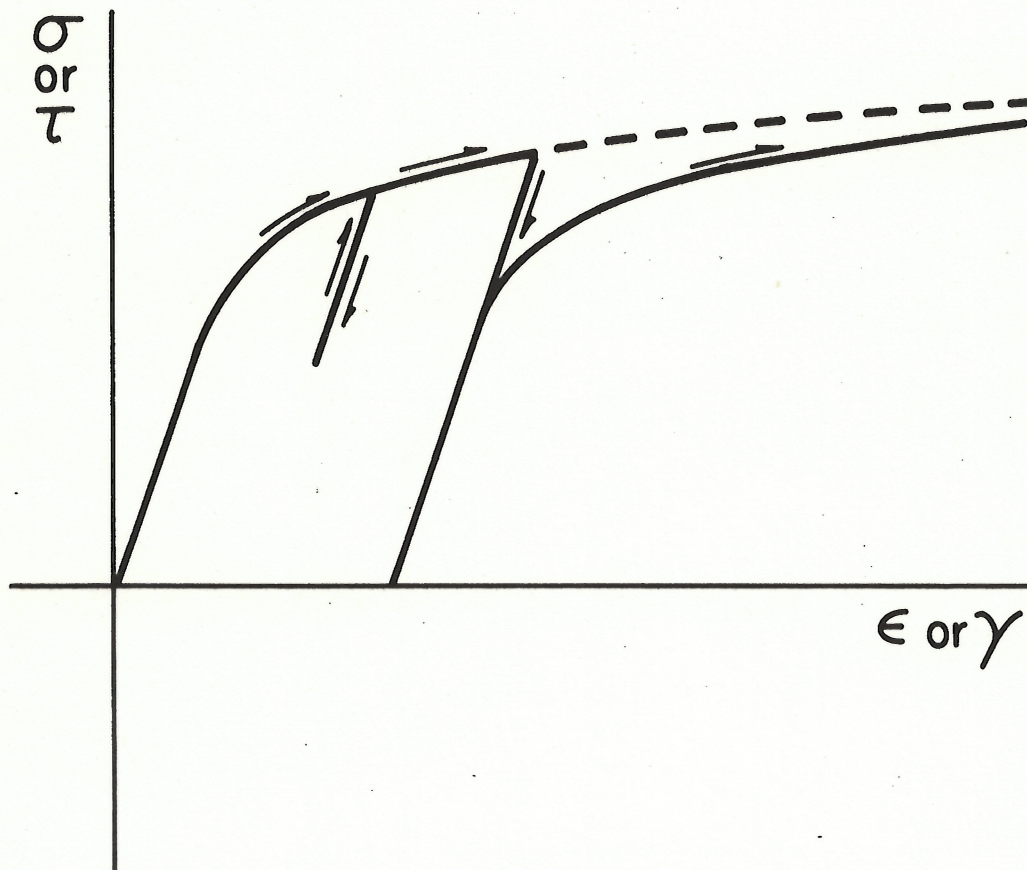


Figure 4. A Poor Representation by the Rounding Option When Plastic Deformation Occurs on Unloading to Zero Stress and on Subsequent Reloading



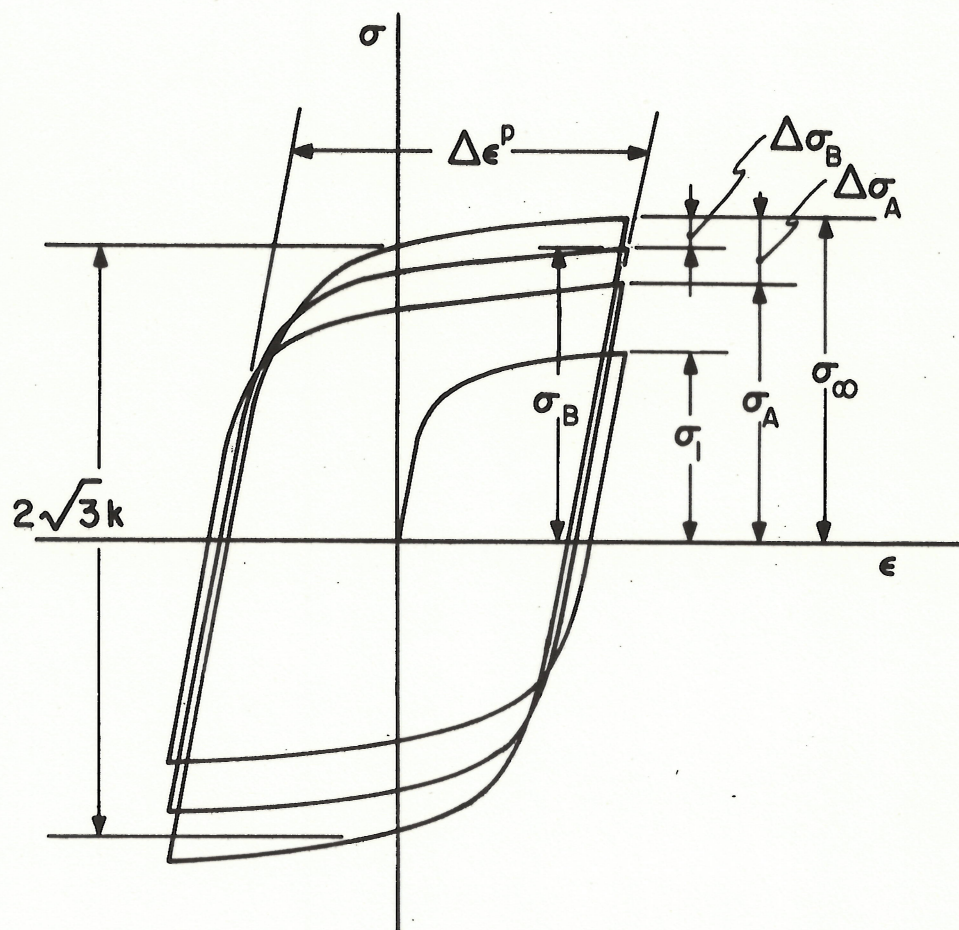


Figure 5. Elastic Range for the Sharp Corner Option and Measurements for the Determination of  $\alpha$  and  $W_0$

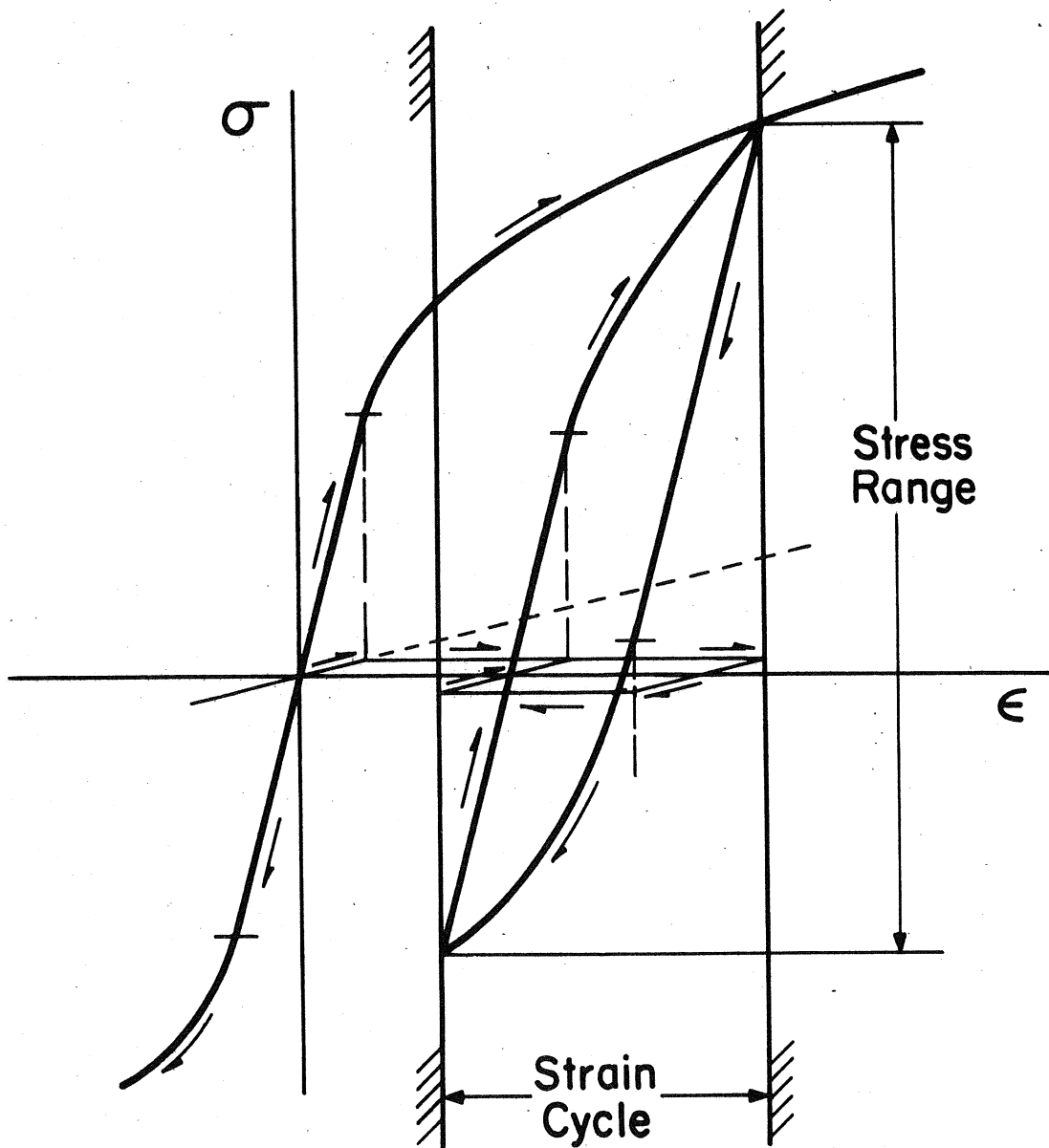


Figure 6. An Immediate and Inappropriate Unsymmetric Stable Cycle



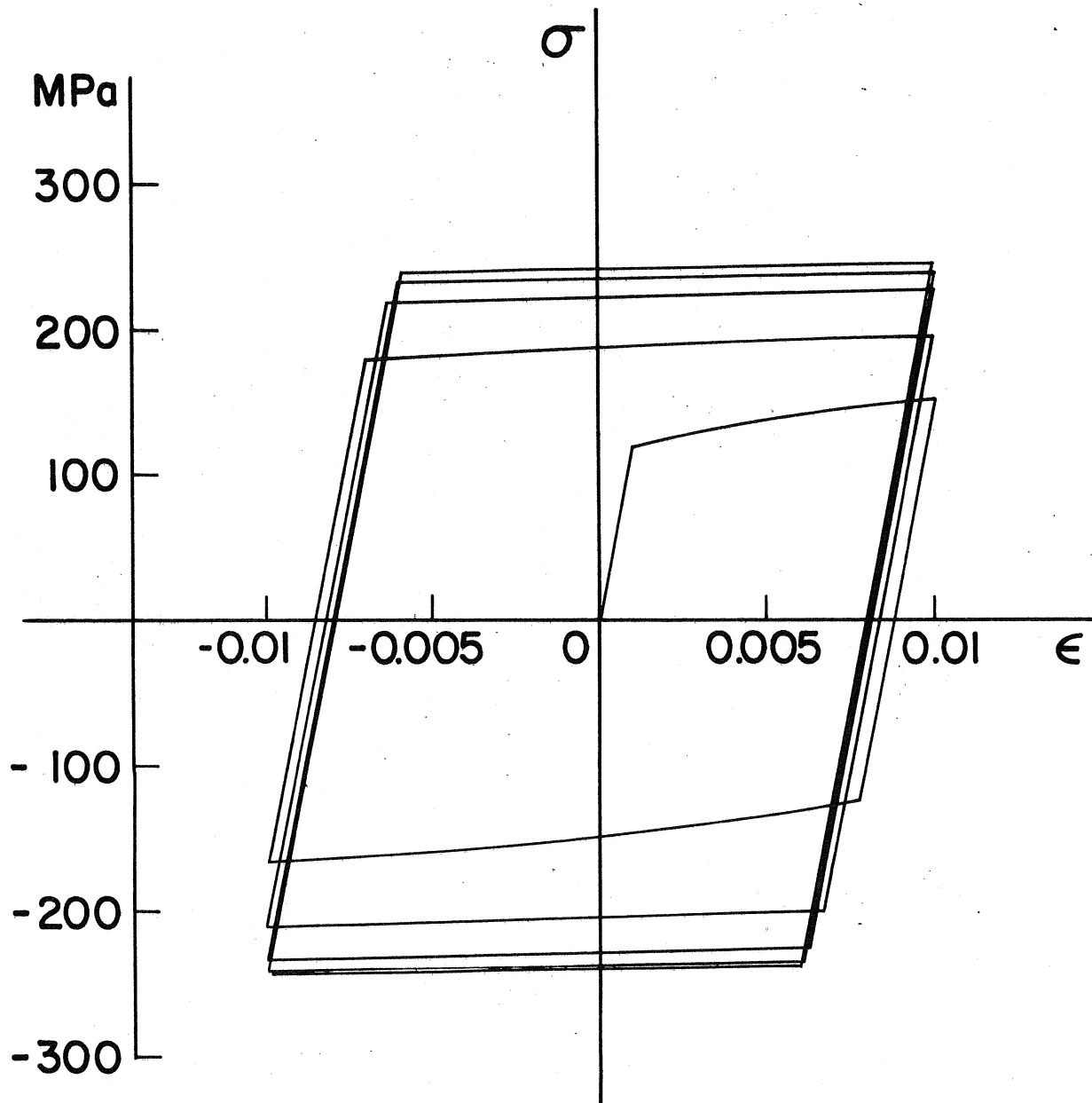


Figure 7. Model Simulation by the Sharp Corner Option Computed with Data of Fig. 1

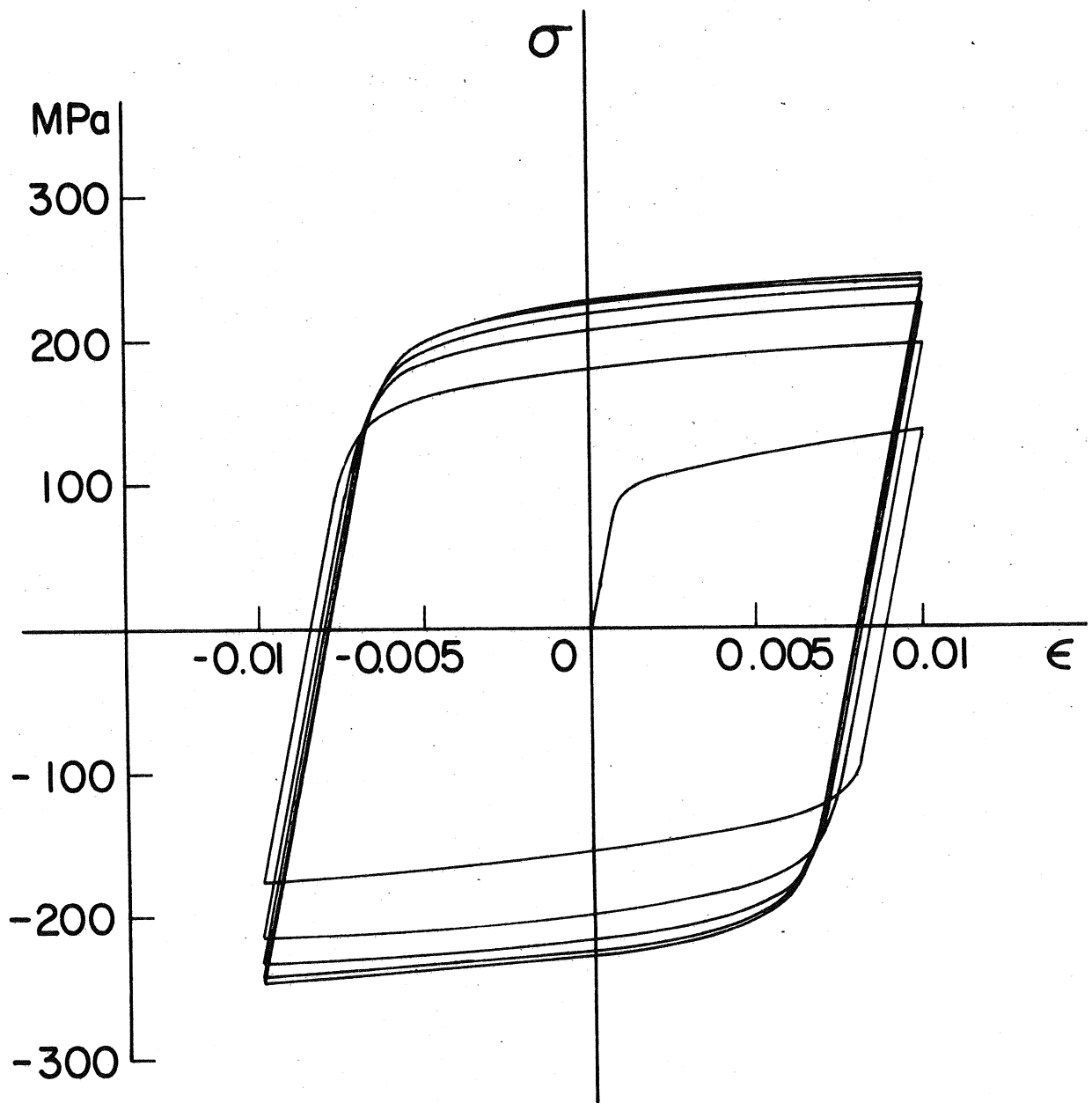


Figure 8. Model Simulation by the Rounding Option Computed with Data of Fig. 1

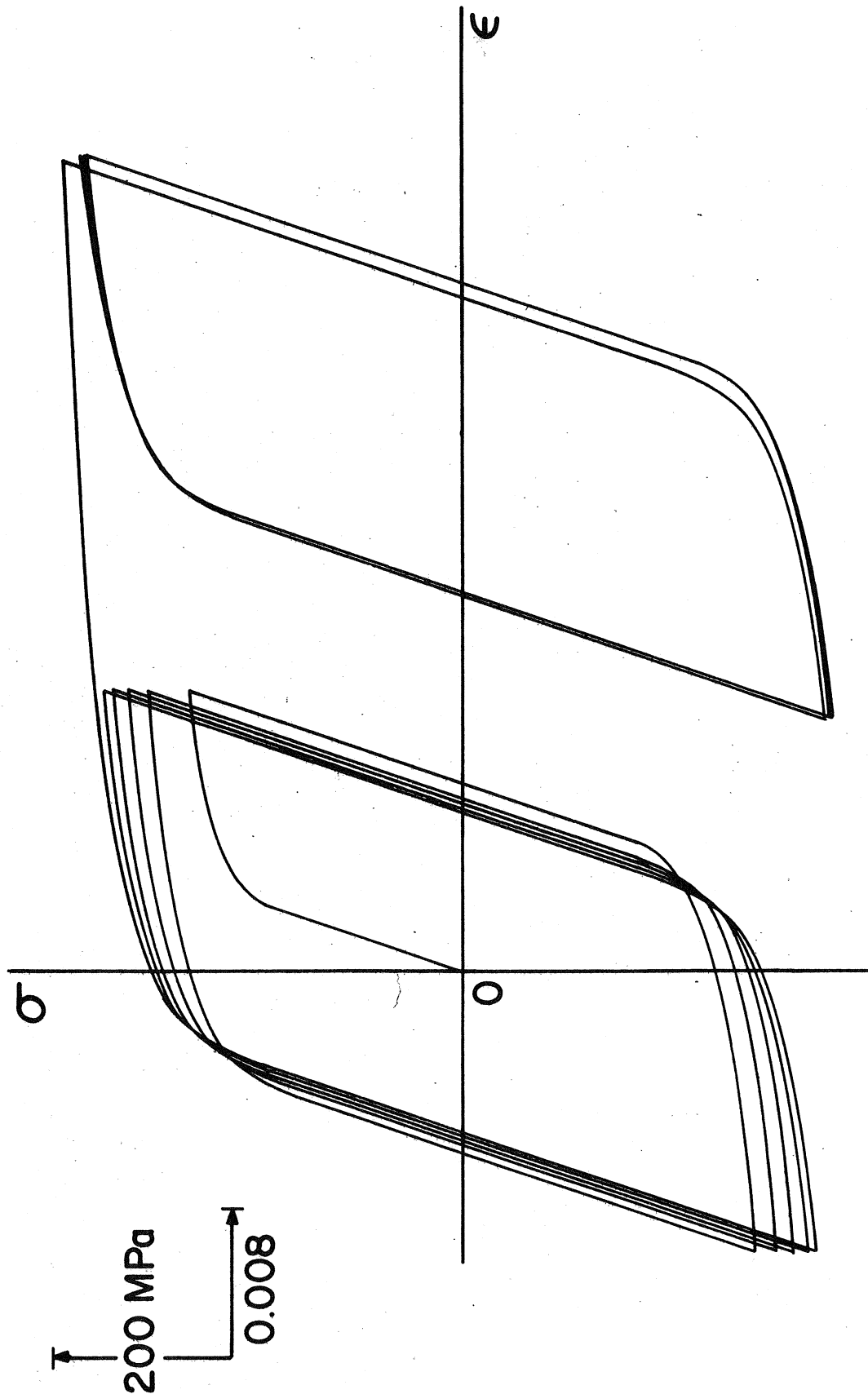


Figure 9. Model Simulation by the Rounding Option Computed with Data of Fig. 2