Correction

Page 14, after equation (52)

Delete the sentence:

"In the above, δ is the tangent of the streamline deflection angle at the interface."

The Shock Dynamics of Stable Multidimensional Detonation

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Abstract

The paper develops a description for the propagation of an unsupported, unsteady, multidimensional detonation wave for an explosive with a fully resolved reaction-zone and a polytropic equation of state. The main features of the detonation are determined once the leading shock surface is known. The principal result is that the detonation velocity in the direction along the normal to the shock is the Chapman-Jouget velocity plus a correction proportional to the local total curvature of the shock. A specific example of unsteady is propogation discussed and the stability of the two-dimensional steady solution is examined.

1. Introduction

This article concerns an extension of the two-dimensional theory of steady detonation developed by Bdzil [1] which describes the steady propagation of an unsupported detonation into a rate stick (a pipe filled with explosive used in many standard explosive tests). We have extended Bdzil's theory to include unsteady effects. The steady theory is an asymptotic theory in which the reaction-zone thickness is assumed to be thin compared to a relevant flow dimension that is imposed geometrically, for example by the radius of the stick.

Bdzil found that the steady detonation is composed of a curved shock followed by a reaction zone. For the class of reaction rates he considered, the end of the reaction zone and the sonic locus are coincident. The curved shock extends across the tube in a symmetric manner, attached at the walls by a condition that is determined by the properties of the inert confining tube. In particular the leading order properties of the detonation are determined by solving a suitable ordinary differential equation that determines the shock locus of the detonation. Having found the shock shape, the gasdynamic state in the reaction zone behind the shock is determined and is parameterized by the shock locus.

The conclusions found here show that an unsteady theory of detonation, with a fully resolved reaction-zone, follows essentially in the same way as Bdzil's original theory. Instead of an ordinary differential equation for the shock locus, we find that a partial differential equation governs the evolution of the shock shape and its steady solution corresponds to the one found by Bdzil.

The plan of this article is as follows: In Section 2., we write down the form of the unsteady reactive Euler equations and the shock relations. In Section 3., we set forth the assumptions of the asymptotic analysis and derive the equation for the shock locus and for the flow in the reaction zone behind the shock. In Section 4., we discuss the boundary conditions that the shock locus must satisfy at the explosive-inert interface. In Section 5., we give the steady solution for the propagation of a 2-D detonation in a rate stick and give the linear stability results. We also present a representative numerical solution of the shock evolution equation. Finally in Section 6., we summarize our conclusions and discuss the limitations of the theory, and directions for further research.

2. The unsteady equations and shock conditions

The notation used here will be the same as Bdzil's [1] in most respects. The coordinates that we will use will have z along the tube (or in the direction of the propagation of the steady detonation) and r and θ in the tangent plane to z. The variable t denotes time. Our coordinates are appropriate for both planar and axisymmetric geometries. The coordinate frame is assumed to move at the steady detonation velocity of the multidimensional wave which we refer to as D. Then the lab coordinate z^{ℓ} is related to z by

$$z = z^{\ell} - Dt. \tag{1}$$

Since our theory is for unsupported detonation, D is generally less than D_{CJ}, the one-dimensional steady Chapman-Jouget velocity. The velocity deficit, $D_{CJ} - D$, is generally small. If $\underset{\sim}{u}$ denotes the particle measured in our moving frame then it is related to the lab frame particle velocity $\underset{\sim}{u}^{\ell}$ by

$$\underset{\sim}{u} = \underset{\sim}{u}^{\ell} - D\hat{e}_z . \tag{2}$$

The unsteady reactive Euler equations in our frame can be written as

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) = 0 , \tag{3}$$

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}(\underset{\sim}{u}) + \rho^{-1} \underset{\sim}{\nabla} P = 0 , \tag{4}$$

$$\frac{\partial E}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} E - (P/\rho^2)[\frac{\partial \rho}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}\rho)] = 0 , \tag{5}$$

$$\frac{\partial \lambda}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}\lambda = R \equiv kf(\lambda) . \tag{6}$$

The variables ρ, P, E and λ denote respectively density, pressure, specific internal energy and the reaction progress variable. The progress variable varies between zero and one and is zero when the explosive is unreacted and one when it is completely reacted. The rate is denoted by R and the parameter k is the rate multiplier whose reciprocal is the characteristic reaction time. The function f is as yet unspecified but vanishes identically whenever $\lambda = 1$. We assume that R is zero in the unshocked state. Finally, we will use a polytropic equation of state for an explosive with a fully resolved reaction-zone.

$$E = (\gamma-1)^{-1} P/\rho - q\lambda \ , \tag{7}$$

where γ is the polytropic exponent and q is the specific heat of reaction.

Starting with these basic equations it is a simple matter to write an equivalent set which is more suitable for computational purposes and follows [1]. Thus we replace (3)-(5) by

$$\frac{\partial H}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}(H) = \rho^{-1} \frac{\partial P}{\partial t} \ , \quad H \equiv (\underset{\sim}{u})^2/2 + P/\rho + E \ , \tag{8}$$

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}(\underset{\sim}{\omega}) = \underset{\sim}{\omega} \cdot \underset{\sim}{\nabla}(\underset{\sim}{u}) + \rho^{-2} \underset{\sim}{\nabla}\rho \times \underset{\sim}{\nabla}P \ , \tag{9}$$

$$-\rho^{-1} \frac{\partial P}{\partial t} + \frac{\partial(|\underset{\sim}{u}|^2/2)}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}(|\underset{\sim}{u}|^2/2) - (\gamma P/\rho) \underset{\sim}{\nabla} \cdot \underset{\sim}{u} = -(\gamma - 1)qR \ . \tag{10}$$

Equation (8) reduces to Bernoulli's equation in the steady case; equation (9) is the vorticity equation. Finally (10) is what we call the basic equation. Equations (8) through (10) with (6) form a complete set.

To these equations we must add the boundary condition of the leading shock. The shock position is assumed to be at

$$z = z_s(r,\Theta,t). \tag{11}$$

We will use a plus subscript to refer to the state immediately behind the shock. Also for the remainder of the paper we will adopt the comma notation to denote partial derivatives. The state immediately behind the shock can be written as

$$\rho_+/\rho_0 = (\gamma + 1)/(\gamma - 1) ,$$

$$u_{z+} = - \{D[(\gamma-1)/(\gamma+1)+z^2_{s,r}+z^2_{s,\theta}/r^2] -2z_{s,t}/(\gamma+1)\}/F ,$$

$$u_{r+} = - [2z_{s,r}/(\gamma + 1)] (D + z_{s,t})/F , \qquad (12)$$

$$u_{\theta+} = - [2(z_{s,\theta}/r)/(\gamma + 1)] (D + z_{s,t})/F ,$$

$$P_+ = [2\rho_0 D^2/(\gamma + 1)] (D + z_{s,t})^2/F ,$$

where

$$F \equiv [1 + z^2_{,r} + z^2_{s,\theta}/r^2] .$$

In the above ρ_0 denotes the density immediately ahead of the shock. Also in deriving conditions (12) we have used the strong shock approximation which neglects the pressure of the unshocked state P_0 relative to the dynamic pressure $\rho_0 D^2$.

Finally it will be convenient to quote a result that can be derived from expressions given by Hayes [2] for the vorticity jump across a curved, unsteady shock. Namely, when the flow ahead of the shock is undisturbed, then the vorticity immediately behind the shock is given by

$$\underset{\sim}{\omega}_+ = - [4/(\gamma^2-1)] (\underset{\sim}{n} \times \underset{\sim}{\nabla}_t V) , \qquad (13)$$

where $\underset{\sim}{n}$ is the normal vector to the shock surface, $\underset{\sim}{\nabla}_t$ is the gradient in the tangent plane to the normal and V is the velocity of the shock along the normal.

3. Asymptotic development

The basic perturbation parameter used in the subsequent development is δ, the ratio of the one-dimensional reaction zone length divided by a characteristic flow length, for example, the tube radius. Thus,

$$\delta = r_\ell / r_* \; . \tag{14}$$

The length r_ℓ depends on the form of the reaction rate which can be calculated explicitly for the assumed equation of state and rate law; r_* is a prescribed dimension.

Our immediate concern is finding an appropriate description in the interior of the flow away from the confinement. The lateral dimensions of the flow are much larger than the reaction zone thickness, hence these variations in the radial direction are properly described in terms of the coordinate

$$\zeta = \delta r \; . \tag{15}$$

The shock locus z_s is assumed to depend explicitly on ζ hence we find that

$$z_{s,r} \sim O(\delta) \; . \tag{16}$$

The shock relations (12) immediately suggest the order of the corrections to the plane shock relations made by the presence of a curved shock. For example, from (12b) we see that corrections to u_z due to a nonplanar shock are $O(\delta^2)$. Bdzil's steady theory shows that the equation for the shock locus is determined by retaining terms to $O(\delta^2)$, where the effects of shock curvature are felt. To bring in time dependent effects at the same order at which the shock locus is determined it is apparent that we should introduce a time scale

$$\tau \equiv \delta^2 t \; . \tag{17}$$

If t is the $O(1)$ time scale measured in terms of characteristic reaction times, then τ is a slow scale representing changes in the flow over many reaction times.

The shock relations (12) suggest that we make expansions in the interior of the form

$$u_z = u_z^{(0)} + \delta^2 u_z^{(2)} + \dots , \quad u_r = \delta u_r^{(1)} + \dots , \quad u_\theta = \delta u_\theta^{(1)} + \dots ,$$

$$P = P^{(0)} + \delta^2 P^{(2)} + \dots , \quad \rho = \rho^{(0)} + \delta^2 \rho^{(2)} + \dots , . \tag{18}$$

In addition we should expand the steady velocity D as

$$D = D_{CJ} + \delta^2 D^{(2)} + \dots , \tag{19}$$

and derivatives of the shock locus are suitably represented as

$$z_{s,r} = \delta\, z_{s,\zeta} , \quad z_{s,\theta}/r = \delta z_{s,\theta}/\zeta ,$$

$$z_{s,t} = \delta^2\, z_{s,\tau} . \tag{20}$$

Great analytic simplification is achieved by using the reaction progress variable λ as a basic coordinate through the reaction-zone. Thus in the analysis that follows, we use λ, ζ, θ, τ as the independent variables where $0 \leq \lambda \leq 1$ and $\lambda_s = 0$.

At leading order, (8) shows

$$\frac{\partial H^{(0)}}{\partial \lambda} = 0 . \tag{21}$$

Integrating (21) and evaluating the integration constant at the shock, $\lambda = 0$, shows

$$H^{(0)} = (u_z^{(0)})^2/2 + \gamma(\gamma - 1)^{-1}\, P^{(0)}/\rho^{(0)} - q\lambda = (D_{CJ})^2/2 . \tag{22}$$

At leading order the basic equation (10), becomes

$$\frac{\partial}{\partial \lambda}\, [1/2\, (u_z^{(0)})^2] - \gamma(P^{(0)}/\rho^{(0)})\, (u_z^{(0)})^{-1} \frac{\partial u_z^{(0)}}{\partial \lambda} = -\,(\gamma - 1)q . \tag{23}$$

Using (22) to substitute for $P^{(0)}/\rho^{(0)}$ in terms of $u_z^{(0)}$ in (23), using the fact that the Chapman–Jouget detonation velocity is given by

$$D_{CJ} \equiv [2q(\gamma^2 - 1)]^{1/2} , \tag{24}$$

and applying the shock boundary condition at $\lambda = 0$ shows

$$u_z^{(0)} = [D_{CJ}/(\gamma + 1)] \, [-\gamma + (1 - \lambda)^{1/2}] . \tag{25}$$

Now we turn to the leading order vorticity equation results. The flow ahead of the shock is irrotational (undisturbed) and an analysis of the vorticity jump through the leading shock shows that the components of vorticity that lie in the tangent plane to the z-direction suffer $O(\delta^3)$ jumps across the shock. This conclusion coupled with equation (9) allow us to conclude that the flow behind shock is irrotational up through terms $O(\delta^2)$. Thus we have

$$\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = O(\delta^3) \ , \quad \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} = O(\delta^3) \ . \tag{26}$$

Thus equations (26) at $O(\delta)$ lead to the following equations

$$\frac{\partial}{\partial \lambda} u_r^{(1)} + z_{s,\zeta} \frac{\partial}{\partial \lambda} u_z^{(0)} = 0 \ , \tag{27}$$

$$\frac{\partial}{\partial \lambda} u_\theta^{(1)} + (z_{s,\theta} \, \zeta^{-1}) \frac{\partial}{\partial \lambda} u_z^{(0)} = 0 \ .$$

Solving the equations and using the shock conditions,

$$u_r^{(1)} = - [D_{CJ}(\gamma + 1)^{-1} \, z_{s,\zeta}] \, [1 + (1 - \lambda)^{1/2}] \ , \tag{28}$$

$$u_\theta^{(1)} = - [D_{CJ}(\gamma + 1)^{-1} \, (z_{s,\theta}/\zeta)] \, [1 + (1 - \lambda)^{1/2}], \tag{29}$$

At leading order the rate equation (6) leads us to the conclusion that

$$\lambda = \lambda^{(0)}(z - z_s(\zeta,\theta,\tau)) \ . \tag{30}$$

Thus the flow is determined to leading order, once $z_s(\zeta,\theta,\tau)$ is known. We will now show that z_s must obey a certain partial differential equation and this equation is found (in a standard way) by insisting that the next order does not lead to secularities.

The starting point is the master equation (10), (using λ as one of the independent variables) which at $O(\delta^2)$ becomes

$$z_{s,\tau}\, kf(\rho^{(0)}u_z^{(0)})^{-1}\, P^{(0)}_{,\lambda} - z_{s,\tau}\, kf(u_z^{(0)})^{-1}\, [1/2\, (u_z^{(0)})^2]_{,\lambda} \tag{31}$$

$$+ \{\underset{\sim}{u} \cdot \underset{\sim}{\nabla}\, (1/2\, |\underset{\sim}{u}|^2)\}^{(2)} - \{(\gamma P/\rho)\, (\underset{\sim}{\nabla} \cdot \underset{\sim}{u})\}^{(2)} = 0$$

Equation (31) relates the $O(\delta^2)$ quantities, $u_z^{(2)}$ and $(P/\rho)^{(2)}$ to the $O(1)$ and $O(\delta)$ quantities $\rho^{(0)}$, $P^{(0)}$, $u_z^{(0)}$ and $u_r^{(1)}$, $u_\theta^{(1)}$. Indeed equation (31) becomes a determining equation solely for $u_z^{(2)}$ if we eliminate the dependence on $(P/\rho)^{(2)}$. This is done by examining (8) to $O(\delta^2)$ which shows that

$$H^{(2)}_{,\lambda} = - z_{s,\tau}\, (\rho^{(0)}\, u_z^{(0)})^{-1}\, P^{(0)}_{,\lambda} \; . \tag{32}$$

The term $\rho^{(0)}\, u_z^{(0)}$ is constant from the continuity equation (3) and by integrating (32) and evaluating the integration constant at the shock, we obtain

$$H^{(2)} = z_{s,\tau}\, [P^{(0)}(\rho_0 D_{CJ})^{-1}] + D^{(2)}\, D_{CJ} \; . \tag{33}$$

Equation (8b) gives another definition of $H^{(2)}$ in terms of $(P/\rho)^{(2)}$ and when combined with (33) obtains the result

$$\gamma(P/\rho)^{(2)} = (\gamma - 1)\{z_{s,\tau}[P^{(0)}(\rho_0 D_{CJ})^{-1}] + D^{(2)}D_{CJ} - [u_z^{(0)}u_z^{(2)} + (u_r^{(1)})^2/2$$

$$+ (u_\theta^{(1)})^2/2]\} \; . \tag{34}$$

The result (34) used in (31) leads to an equation for $u_z^{(2)}$ alone. The details of the final simplification of equation (31) are rather tedious and are not given here. However, the analysis is straightforward and one derives the following equation for $u_z^{(2)}$; namely

$$\frac{\partial u_z^{(2)}}{\partial t} = \gamma[(t-\gamma)t]^{-1} u_z^{(2)} + A(t)D^{(2)} - B(t)[(z_{s,\zeta})^2 + (z_{s,\theta}/\zeta)^2]$$

$$- C(t)\ [z_{s,\zeta\zeta} + \alpha\ z_{s,\zeta}/\zeta + z_{s,\theta\theta}/\zeta^2] - z_{s,\tau}/[(t-\gamma)t]\ . \qquad (35)$$

where

$$t \equiv (1-\lambda)^{1/2},\ A \equiv (\gamma - 1)\ [t(t-\gamma)]^{-1},$$

$$B(t) \equiv D_{CJ}(t - 2\gamma - 1)(1 + t)\ [2(\gamma + 1)\ t(t-\gamma)]^{-1}\ ,$$

$$C \equiv 2\gamma(D_{CJ})^2\ (t-\gamma)(1+t)^2 t\ [(\gamma+1)^3\ kf(\lambda \equiv 1 - t^2)]^{-1}, \qquad (36)$$

and t defined above is not to be confused with time.

Note that $\alpha = 0$ and $\partial/\partial\theta = 0$ for planar geometry, $\alpha = 1$ otherwise.

The general solution of (35) can be written as

$$u_z^{(2)} = c(t-\gamma)/t + [(t-\gamma)/t]\ \{\tilde{A}(t)\ D^{(2)} - \tilde{B}(t)[(z_{s,\zeta})^2 + (z_{s,\theta}/\zeta)^2]$$

$$- \tilde{C}(t)\ [z_{s,\zeta\zeta} + \alpha z_{s,\zeta/\zeta} + z_{s,\theta\theta}/\zeta^2] + z_{s,\tau} t/[\gamma(t-\gamma)]\}\ , \qquad (37)$$

where c is an integration constant and

$$\tilde{A}(t) \equiv \int_0^t [A(\bar{t})\ \bar{t}/(\gamma-\bar{t}),\ \tilde{B}(t) \equiv \int_0^t [B(\bar{t})\bar{t}/(\gamma-\bar{t})d\bar{t},$$

$$\tilde{C}(t) = \int_0^t [C(\bar{t})\bar{t}/(\gamma-\bar{t})d\bar{t}\ . \qquad (38)$$

The functions $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ (with appropriate restrictions placed on $f(\lambda)$) are

$O(t)$ as $t \to 0$. Thus the entire second term in the right hand side of (37) is bounded throughout the reaction zone $0 \leq t \leq 1$. It is clear that to ensure boundedness (hence a uniform description) of $u_z^{(2)}$ in the reaction zone, the integration constant c must be chosen to be zero. With the constant so chosen, the shock boundary condition (at $t = 1$) can only be satisfied if there is a relationship between the various derivatives of z_s that appear in (37). Applying the shock boundary condition (12b) for $u_z^{(2)}$ at $t = 1$ shows that

$$u_z^{(2)}(1) = -2D_{CJ}[z_{s,\zeta}^2 + z_{s,\theta}^2/\zeta^2]/(\gamma + 1) \qquad (39)$$
$$- D^{(2)}(\gamma - 1)/(\gamma + 1) + 2z_{s,\tau}/(\gamma + 1) .$$

Comparing (39) with $u_z^{(2)}(1)$ evalulated from (37) yields the evolution equation for z_s,

$$z_{s,\tau} = a[z_{s,\zeta\zeta} + \alpha\, z_{s,\zeta}/\zeta + z_{s,\theta\theta}/\zeta^2] , \qquad (40)$$
$$+ (D_{CJ}/2)\, [z_{s,\zeta}^2 + z_{s,\theta}^2/\zeta^2] - D^{(2)} ,$$

where

$$a \equiv [2\gamma^2\, D_{CJ}/(k(\gamma + 1)^2)] \int_0^1 \frac{t(1 + t)^2}{f(\lambda \equiv 1 - t^2)}\, dt . \qquad (41)$$

Note that (40) is valid for both planar ($\alpha = 0$, $\partial/\partial\theta = 0$) and cylindrical ($\alpha = 1$) geometries. Inspection of (40) shows that we can generalize the result to read

$$z_{s,\tau} = a\, \nabla^2\, z_s + (D_{CJ}/2)\, |\underset{\sim}{\nabla}\, z_s|^2 - D^{(2)} , \qquad (42)$$

where ∇^2 and $\underset{\sim}{\nabla}$ now represent the scaled Laplacian and the gradient operators in the tangent plane to the direction of steady propagation. Indeed

(42) has even a simpler interpretation in terms of coordinate independent quantities. Note that the propagation velocity of the shock surface along its normal, V, and the sum of the principal curvatures of the surface, κ, are given respectively by

$$V = D_{CJ} + \delta^2[z_{s,\tau} + D^{(2)} - (D_{CJ}/2)\, |\underset{\sim}{\nabla}\, z_s|^2] + o(\delta^2) , \qquad (43)$$

$$\kappa = - \delta^2\, \nabla^2\, z_s + o(\delta^2) ,$$

so that (42) shows that

$$D_{CJ} - V = a\, \kappa + o(\delta^2) , \qquad (44)$$

i.e. the velocity deficit of the shock (the difference of the Chapman-Jouguet velocity and the normal shock velocity) is equal to the the total curvature times a constant that depends on the form of the rate and the equation of state.

It is worth pointing out that simulation of the propagation of stable detonation waves in condensed phase explosives is routinely carried out using a Huygens' construction where the detonation is propagated along its normal at exactly the Chapman-Jouguet velocity. Equation (44) indicates that the curvature of the shock plays an important role.

The parameter a is proportional to the square root of the heat release q (through D_{CJ}), inversely proportional to the rate constant k and proportional to the integral shown in (41). This last dependence is made explicit by choosing a simple depletion form for the rate law

$$f\lambda = (1 - \lambda)^{\nu}, \qquad 0 \leqslant \nu < 1 , \qquad (46)$$

where ν is restricted as shown above to ensure convergence of the integral, which becomes

$$\int_0^1 t^{1-2\nu}(1 - t)^2\, dt = [2(1 - \nu)]^{-1} + 2/(3 - 2\nu) + 1/(4 - 2\nu) \,. \qquad (47)$$

In particular as $\nu \to 1$, $a \to \infty$. The physical reason for the last nonuniformity is discovered by examining the progress variable behind the shock which determines the length of the reaction zone. The rate equation (6) shows that

$$\frac{\partial\lambda^{(0)}}{\partial z} = k\, f(\lambda^{(0)})/u_z^{(0)} \,, \qquad (48)$$

which on integration shows

$$z - z_s(\zeta,\theta,\tau) = [(\gamma+1)k/D_{CJ}]\{\gamma(t^{2(1-\nu)}-1)/(1-\nu) - 2(t^{3-2\nu}-1)/(3-2\nu)\} \,. \qquad (49)$$

Note that the length of the reaction zone, $r_\ell \equiv z(\lambda = 1) - z_s$, can be calculated explicitly from (49). Constant λ surfaces are the same as constant $z - z_s$ surfaces. The end of the reaction zone is found when $\lambda = 1$ $(t = 0)$ and as $\nu \to 1$ we find

$$z(\lambda = 1) \sim z_s - D_{CJ}\, \gamma/(1 - \nu) \,. \qquad (50)$$

Thus $\nu \to 1$ corresponds to the reaction zone having infinite length. Consequently the introduction of a new scale leads to the nonuniformity.

Also one can show for this model that the end of the reaction zone and the sonic locus coincide precisely to leading order. This is shown since

$$(C^{(0)2}) - |\underset{\sim}{u}^{(0)}|^2 = \gamma P^{(0)}/\rho^{(0)} - (u_z^{(0)})^2 = 0 \quad \text{at} \quad \lambda = 1 \ (t = 0) \,, \qquad (51)$$

where C in the above refers to the local sound velocity. Since the flow between the shock and the sonic locus is subsonic, it consequently affects the evolution of the shock. The cases where the parameter a becomes unbounded, $\nu \to 1$, $k \to 0$, and $q \to \infty$ all correspond to the length of the reaction zone or equivalently the size of the flow region that influences the shock being infinite. Our analysis cannot be expected to apply in this case.

4. Discussion of the shock boundary conditions

The flow in the explosive near the confining edge, must be determined in order to derive the boundary condition needed for (40). Different types of confinement can occur, for example, the strong confinement provided by a (metal) tube wall of finite thickness or no confinement when the explosive products expand freely into a vacuum. Equation (40) applies in regions where the shock curvature is small. Therefore, one can expect that there is a boundary layer near the explosive-inert interface, of at least one reaction zone thickness for strong confinement case and larger for the case of no confinement, that provides a flow adjustment from the strongly 2-D flow near the interface to the nearly 1-D interior flow. These boundary layer analyses are often difficult research problems in their own right. The simplifications of the interior flow (nearly 1-D) do not necessarily apply in these layers and questions about the general character of the flow near the explosive-inert interface are still open.

However in [1], [3] and [4] some results have been derived that can be applied to our discussion. For the case of strong confinement, Bdzil's analysis [1] for steady detonation applies if one limits the temporal variations in the boundary layer to occur on the slow time scale $\tau = \delta^2 t$. There it is shown that the interior flow shock equation satisfies the boundary condition

$$L = -\,2z_{s,\zeta}/(\gamma + 1)\ , \qquad (52)$$

at the explosive-inert interface. [redacted] The parameter L, which is nearly one, accounts for the adjustment in the streamline curvature between the interior explosive flow and the flow at the interface.

The case of no confinement was treated in [4] for an explosive with a different equation of state. Some of the results derived there also apply to our discussion. In particular one can show, using a local coordinate expansion in the distance from the shock-vacuum interface, that the flow consists of a constant sonic state immediately behind the shock, followed by a radial, Prandtl-Meyer expansion fan, followed by another constant state. Satisfying the sonic condition immediately behind the shock shows that the shock slope is given by

$$(z_{s,r})^2 = (\gamma - 1)/(\gamma + 1) \ . \tag{53}$$

Importantly for the equation of state chosen in this paper, no explicit time dependence is found in (53). This slope is not small and consequently the flow adjustment boundary layer must predict the appropriate large change of slope through the layer. Some of this analysis is indicated in [3], however the problem of no confinement is not completely resolved. We speculate that a boundary condition of the form (52) still applies where L has a value L_c corresponding to a critical confinement.

Finally we add that in [4], the boundary condition derived for no confinement is of the general form

$$bz_{s,\tau} + 4z^2_{s,\zeta}/(\gamma + 1)^2 = L^2 \ , \tag{54}$$

where b is a known value. This boundary condition can be thought of as an empirical generalization of (52).

5. Simple examples and stability results

In this section we illustrate some representative results found from the shock evolution equation (40) and the boundary condition (52) for the simple case of plane, 2-D geometry. Setting α and $\partial/\partial\theta$ zero we find that z_s must satisfy

$$z_{s,\tau} = a\, z_{s,\zeta\zeta} + (D_{CJ}/2)\, z_{s,\zeta}^2 - D^{(2)} , \tag{55}$$

subject to

$$z_{s,\zeta} = \mp (\gamma + 1)L/2 \qquad \text{at} \qquad \zeta = \pm\, \zeta_* , \tag{56}$$

where $\zeta_* \equiv \delta r_*$ is the location of the boundary. The above problem is analyzed conveniently by introducing the new variables

$$Z = z/(2a/D_{CJ}) , \qquad X = \zeta/[a/(-D_{CJ}\, D^{(2)}/2)^{1/2}] ,$$

$$T = \tau/[2a/(-D_{CJ}\, D^{(2)})] , \; W = (\gamma + 1)L/[2(-2D^{(2)}/D_{CJ})^{1/2}] . \tag{57}$$

The problem (55) and (56) becomes

$$Z_{,T} = Z_{,XX} + (Z_{,X})^2 + 1 , \tag{58}$$

$$Z_{,X} = \mp W \qquad \text{at} \qquad X = \pm X_* . \tag{59}$$

The steady solution for Z and $Z_{,X}$ is given by

$$Z = \ln|\cos(X)| + \text{Const}, \; Z_{,X} = -\tan(X) , \tag{60}$$

and satisfying the boundary condition at $X = \pm X_*$ requires the relation

$$\tan X_* = W . \tag{61}$$

Equation (61) gives the expression that determines the velocity deficit (i.e. $D^{(2)}$) as a function of charge size and other parameters. Explicitly written out the last expression gives

$$\tan(\zeta_*/[a(-D_{CJ}\ D^{(2)}/2)^{1/2}]) = (\gamma + 1)L/[2(-2D^{(2)}/D_{CJ})^{1/2}] \ . \qquad (62)$$

Two limiting cases illustrate the behavior predicted by (62), when

$$W \to 0;\ (\zeta^* \to 0) \qquad D^{(2)} \doteq -(\gamma + 1)La/(2\zeta_*) \ ,$$

$$W \to \infty;(\zeta^* \to \infty) \qquad D^{(2)} \doteq -\pi^2\ a^2/(2\zeta_*^2\ D_{CJ}) \ . \qquad (63)$$

In particular the above illustrates the experimentally observed diameter effect where the velocity deficit increases with decreasing charge size. A similar result to (61) was first derived in [1].

The stability of the steady solution is easily examined as well. First note that if we use the shock slope $S \equiv Z,_X$ as the basic variable then (58) becomes Burger's equation

$$S,_T - 2SS,_X = S,_{XX} \ , \qquad (64)$$

subject to

$$S_X = \mp W \qquad \text{at} \qquad X = \pm X_* \ . \qquad (65)$$

Further introduction of the Hopf-Cole variable

$$S = \Psi,_X/\Psi \ , \qquad (66)$$

shows that Ψ satisfies

$$\Psi,_T = \Psi,_{XX} + \Psi \ , \qquad (67)$$

with

$$\Psi_X \pm W\ \Psi = 0 \qquad \text{at} \qquad X = \pm X^* \ , \qquad (68)$$

where the source term in (67) is the consequence of choosing an integration constant consistent with the existence of the steady solution. The steady solution in Ψ is given simply by

$$\Psi = \cos(X) \ , \tag{69}$$

and condition (61) still holds.

Linear stability is examined in the standard way by looking for a solution of the form

$$\Psi = \cos(X) + \Psi' e^{\lambda t} \ ; \qquad \|\Psi'\| << 1 \ .$$

and the following relations for the growth rate λ are derived

$$\tan\left(X_*(\sqrt{1-\lambda})\right) = W/(\sqrt{1-\lambda}) \ , \ \cos\left(X_*(\sqrt{1-\lambda})\right) = -W/(\sqrt{1-\lambda}) \ . \tag{70}$$

The apparent root $\lambda = 0$ is spurious and (70) shows that $\mathrm{Re}\,\lambda < 0$ and hence the solution is stable.

Figure 1 shows the result of a numerical solution of (58), (59) with a flat shock as initial data. This simulation corresponds to an experiment where a booster charge (a plane wave initiator) introduces a flat shock into the explosive sample of sufficient strength to produce a C-J detonation. From simulations such as that shown in Figure 1., it is possible to calculate the time or distance required from initiation to the achievement of a steady detonation wave. Such information can help determine the length of the rate stick required to produce a steady wave.

Some difficult experiments have been carried out by Engelke to check the validity of the steady theory for condensed phase explosives with remarkable qualitative and even quantitative success. These results are recorded in [1] and [5]. The experiments have photographed the shock locus and the recorded shock shapes are compared directly with the steady theory predictions. Determining the relaxation time to steady state, as suggested by Figure 1, would require multiple exposures and to our knowledge no one has attempted to carry out such experiments.

The simplicity of solving (58) and (59) suggests extensions and numerical experiments to rate sticks of varying widths or with other geometric complications. Equation (58) still holds in the interior and (52) or (54) is applied at the boundary with the radial derivative in these conditions being replaced by the normal derivative to the interface. Some numerical experiments of this type have been performed. An important potential application is the simulation of a corner turning experiment that is used in explosive testing and characaterization.

6. Concluding remarks

We have developed a description of the propagation of multidimensional detonation waves for an explosive with a simple (polytropic) equation of state and a simple rate law. The main features of the detonation structure are determined once the shock locus is determined and we have shown that the shock itself will obey an evolution equation for the shock surface. Importantly the analysis in this paper is the unsteady analog of the conclusions found in Bdzil's paper on the structure of the steady 2-D detonation wave. This theory can be expected to hold and is derived under the assumptions that the distance between the shock and the sonic locus is finite and is small compared to a representative flow scale, which can be thought of as the diameter of the charge or a length calculated from insisting on small streamline deflection at the interface and consequently small lateral flow divergence.

For the rate that we have chosen, the sonic locus and the end of the reaction zone coincide. This can be shown generally to be the case for rates that have sufficiently weak dependence on the state. Rates with stronger state dependence may not have this property but still have the sonic locus at a finite location behind the shock. Engelke and Bdzil discuss such a steady case in [5]. However, here we have shown that when the sonic locus falls a large distance behind the leading shock, the current theory no longer applies.

Additional qualifications for the theory presented can be given and call for more research. Namely, our analysis has restricted the time variation to be on the natural slow scale that affects the shock evolution. In [4], for a slightly different equation of state, we have discussed the implications of retaining faster time scales that reflect the basic hyperbolic character of the flow. In that paper it is shown quite clearly for a particular initial value problem, that disturbances on the shock are propagated in accordance with a wave hierarchy. The slow time scale is associated with the long time evolution after the passage of many acoustic waves that reflect in the interior region between the shock, sonic locus and confinement boundaries. While the discussion in [4] gives us confidence that the slow time scale is the important one for describing certain classes of explosives, it is not generally known under what conditions this is the case.

A methodical way to determine the effect of the fast-time scales is to include them at the outset of the analysis and demonstrate that they have the subsidiary effect that they were shown to have in [4], or that they cause instability. Indeed, the question of the importance of the fast time scales is really a question of detonation stability. For example, the result derived here is consistent with some of the known results for detonation stability in that weak state dependence of the rate is generally known to be stabilizing. We have shown here that the limit of stronger state dependence invalidates our theory and indeed one might suspect that in those cases ignoring the faster time scales may not be appropriate. Analytic investigations of detonation stability with stronger state dependence, such as that afforded by the

Arrehnius rate with a large activation energy, will play an important role in determining when the faster time scales must be retained or ignored.

Application of the current theory generally requires a complete discussion of the confinement boundary layer. We point out that only limited results are known about these layers and their analysis is usually difficult. The stability of these layers and the importance of the faster time scales are relevant questions about which little is known.

However, despite these qualifications, the case that we have analyzed is one of the most practically important cases and can be expected to apply to explosives in which stable multidimensional detonations are observed. The case that we have analyzed allows extension to different equations of state and rate laws that will not change many of our main conclusions. Our principal result is that the detonation velocity deficit (i.e., the difference of the Chapman-Jouguet detonation velocity and the detonation velocity along its normal) is proprotional to the total curvature of the surface. Also the theoretical framework presented in [1], [4] and the present paper will allow the investigation of detonation stability by increasing the state dependence of the rate from what we have presently considered. We hope this to be the focus of future work.

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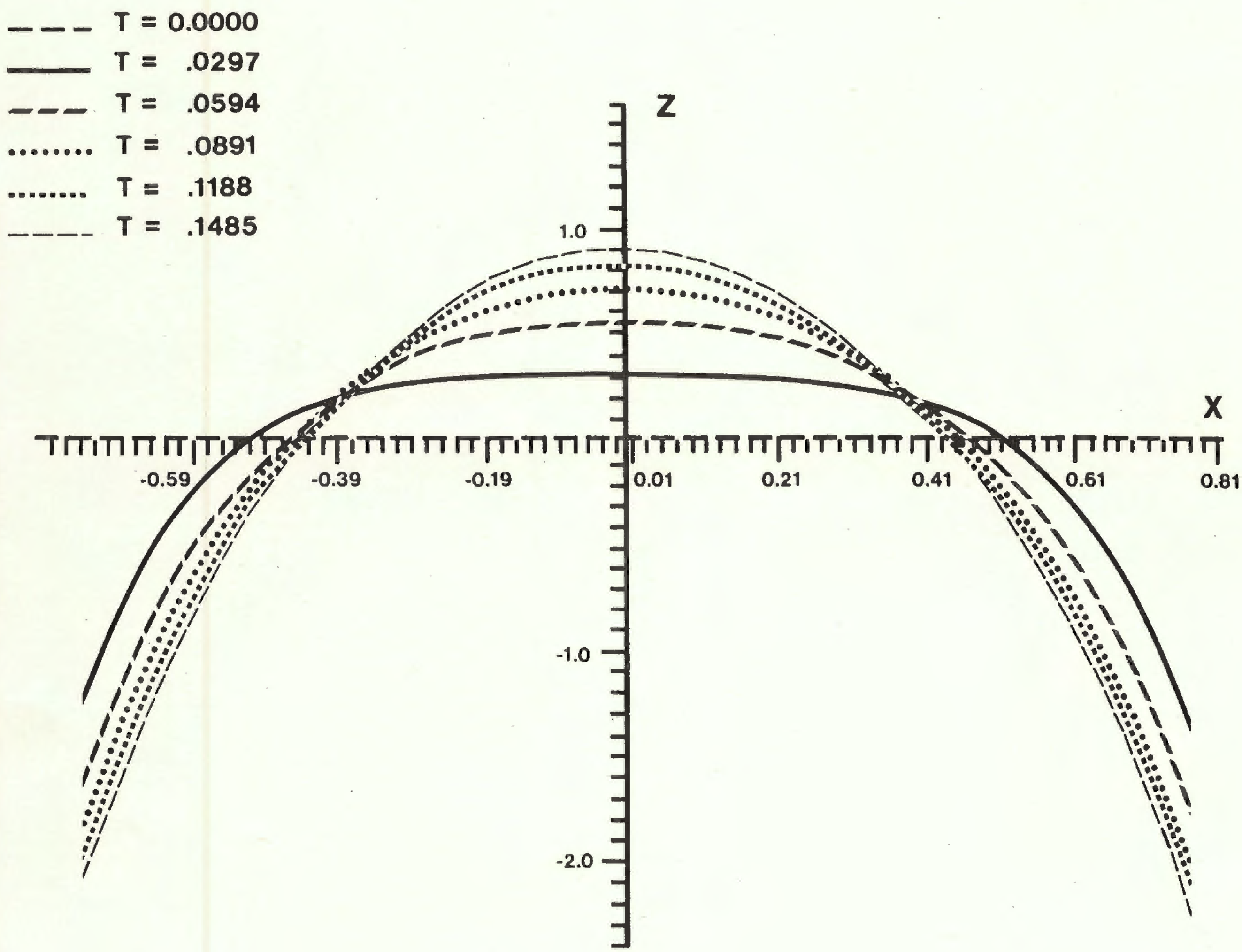

Figure 1. Numerical solution of (58) and (59) with W = 1.0