DIFFUSIVITY OF ULTRASOUND IN POLYCRYSTALS

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Abstract

The diffusivity of ultrasound in an untextured aggregate of cubic crystallites is studied theoretically with a view towards nondestructive characterization of microstructures. Multiple scattering formalisms for the mean Green's Dyadic and for the covariance of the Green's Dyadic (and therefore for the energy density) based upon the method of smoothing are presented. The first-order smoothing approximation used is accurate to leading order in the anisotropy of the constituent crystallites. A further, Born, approximation is invoked which limits the validity of the calculation to frequencies below the geometrical optics regime. Known results for the mean field attenuations are recovered. The covariance is found to obey an equation of radiative transfer for which a diffusion limit is taken. The resulting diffusivity is found to vary inversely with the fourth power of frequency in the Rayleigh, long wavelength, regime, and inversely with the logarithm of frequency on the short wavelength, stochastic, asymptote. The results are found to fit the experimental data.

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I. Introduction

The microstructure of polycrystalline aggregates has been probed by means of ultrasonic scattering for at least 40 years, since the work of Mason and McSkimm [1]. That ultrasound with frequencies of the order of 10 MHz and inverse wave numbers of order 50 microns would be sensitive to microstructural variations on the length scales of relevance in typical polycrystals has long been appreciated. The names of Papadakis [2,3], Bhatia [4], Adler [5], Goebbels [6,7,8] and Vary [9] are amongst those associated with laboratory efforts in the pursuit of the ability to assess microstructure, and potentially therefore, mechanical properties, using nondestructive ultrasound. In spite, though, of considerable effort over many years, the most common technique, that of ultrasonic attenuation, though proven in the laboratory, has not seen much application in the field.

That technique calls for the measurement of the exponential rate of spatial decay of an ensemble averaged plane wave. Due to a host of potential systematic errors, the unambigous measurement of that rate is, however, quite difficult [10]. The finite beam width to wavelength ratio of ultrasonic transducers is responsible for geometric, non exponential, decays as the beam transmits from near to far field conditions. Most studies are carried out in slab-like geometries and the wave amplitude monitored as it reflects between opposite faces. Imperfections in the reflection coefficients at these faces can artifically enhance the apparent attenuation. Faces which are insufficiently parallel can distort the apparent pulse amplitudes. Internal friction, or absorption, which is more properly viewed as a temporal decay than a spatial one, contributes in such a way as to be indistinguishable, in these configurations, from scattering based decay. High attenuation rates are unmeasurable except over distances so short as to bring the ergodic hypothesis, crucial to comparisons between theory and experiment, into severe question.

In more recent years there has been an increasing effort to measure microstructure from the energy which is incoherently and singly backscattered from a beam [6,11,12,13]. The potential for systematic error in these configurations is perhaps less severe than in the conventional parallel wall slab configurations. The technique is also attractive in that it makes fewer demands on specimen geometry and can potentially give information on microstructure as a function of depth. The chief difficulties appear to be traceable to the need for extensive spatial averaging of signal powers in order to minimize the effect of the inevitable meaningless fluctuations of an incoherent wave field, the need for minimum slab thicknesses in order to resolve backwall echoes and distinguish them from the important incoherent grass, and the absence of a complete theory relating backscattered power to microstructure.

It is proposed here that measurement of the evolution of the incoherent and multiply scattered, fully diffuse, ultrasonic wave field in a material with a scattering microstructure may provide a more robust measure of that microstructure. The published literature addressed to such wave fields is meagre, and seems to be confined to the paper by Guo, Holler and Goebbels [8] and to one by Weaver [14]. Weaver's work considered sub MHz range ultrasound in a specimen with an artificial centimeter scale microstructure. While not directly applicable to the usual materials of interest in NDE, it did demonstrate, over several orders of magnitude, the applicability of the concept of diffusion to the evolution of these fields. That is, it was shown that the ultrasonic energy density evolved in accordance with a diffusion, or heat, equation, modified by the inclusion of an extra term representing temporal decay. Weaver has also shown, in work not yet prepared for publication, strong correlations between measured diffusivities in steels and their heat treatments and fracture toughnesses. Guo *et al.* [8] after a study of such fields in a range of materials, also report a good fit to the predictions of a diffusion model. Guo *et al.* [8] and Weaver [14] have both emphasized that the diffuse-field technique promises to be able to distinguish between temporal decay, or absorption, and scattering. Each is potentially a nondestructive testing parameter of interest, the former having received little attention in the past due to the difficulty of its in situ measurement, e.g. [22].

No theory for the diffusivity of an ultrasonic field in a polycrystal is immediately obvious. Guo *et al.* [8] suggest, in analogy to a random-walk model, that the diffusivity D should be one third of the product of a mean free ray path and the wave speed. They then identify that mean free path with the inverse of the attenuation and suggest that the diffusion constant D is simply an inverse measure of attenuation. The argument is attractive due to its simplicity, but it must be modified to account for the presence of different wave modes, the transverse (T) and longitudinal (L), each with its own wave speed c and attenuation α. We must further recognize that the appropriate mean free path should be half of the inverse of the attenuation, (as it is the energy which is diffusing and the attenuation rate of the coherent energy is twice that, α, of coherent amplitude). With these identifications we write, as a conjecture,

$$D = f_T(c_T/6\alpha_T) + f_L(c_L/6\alpha_L). \tag{1.1}$$

where f_T and $f_L = 1 - f_T$ are the mixing weights for the transverse and longitudinal contributions to the diffusivity. Their values are not obvious. The only a-priori likely possibility is

$$f_T/f_L = 2c_L^3/c_T^3 \tag{1.2}$$

corresponding to the equilibrium partition of the energy of a fully developed diffuse wave field [15].

Equation 1.1 has the virtue of being dependent only on quantities available from standard theory and/or experiment. As Guo *et al.* indicate it also implies that the diffusivity should, for frequencies below the geometric optics regime, scale inversely with frequency with a power between two and four. This follows from the standard theory (for which the reader is directed to the recent work by Stanke and Kino [16] or Hirsekorn [17-20] or Karal & Keller [21] or the general discussions of references 1-6) for the attenuation of mean waves in polycrystals. That theory indicates that attenuation scales with the fourth power of frequency in a domain for which wavelength is much larger than microscale length (the "Rayleigh" domain) and with the square of frequency in the opposite limit (the "stochastic asymptote") but is ultimately bounded by the geometrical optics limit where α^{-1} is of the order of the microscale length.

The frequency dependence of the diffusivity observed by Guo *et al.* was however very clearly an inverse *first* power. In consequence, equation 1.1 cannot be correct. We will see in this communication that that error is due to the identification of that "mean free ray path" required by the random walk model with $1/2\alpha$. The model requires a mean free path which corresponds to the distance travelled by a typical ray before it is significantly scattered away from its original direction. At low frequencies, in the Rayleigh regime, grain scattering is isotropic; equal energies are scattered in all directions; hence $1/2\alpha$ does correspond to a mean free path. At higher frequencies the scattering is increasingly biased towards the forward direction; typical rays are scattered through small angles; hence the appropriate mean free path is somewhat greater than $1/2\alpha$.

The present communication is addressed towards the derivation of a more accurate expression for D. The derivation is confined, like the attenuation calculation of Stanke and Kino [16], to the case of an untextured aggregate of cubic-symmetry crystallites. This is the simplest polycrystal and includes iron as a special case, allowing comparison with the results from one of the specimens studied by Guo *et al.* After discussion of the assumed statistics of the polycrystal in section II, section III presents, by means of consideration of the ensemble average Green's function the standard result for the attenuations, explicit expressions for which are given in section VII. Sections IV and V present the analogous theory for the mean square Green's functions, and therefore for the energy. Section VI solves the resulting equations of radiative transfer and concludes with an expression for the diffusivity. Upon choosing a form for the microstructural spatial correlation function, that diffusivity is evaluated in section VII. It is shown that the frequency dependence of D spans a range from strong to weak. The low frequency dependence is with the inverse fourth power of frequency. At high frequencies but below the geometric optics limit D^{-1} scales only logarithmically with frequency. Section VII concludes with recommendation for furtherwork. An appendix contains a list of symbols.

II. Mathematical Preliminaries

The Green's function (dyadic actually) for the response of a heterogeneous anisotropic passive elastodynamic medium is given by the causal solution to

$$[-\delta_{li}\partial_t^2 + \partial_k C_{klij}(\mathbf{x})\partial_j] G_{i\alpha}(\mathbf{x},\mathbf{x}';t) = \delta_{l\alpha}\delta^3(\mathbf{x}-\mathbf{x}')\delta(t) \tag{2.1}$$

In (2.1) units have been used such that the material density is equal to one. $G_{ij}(\mathbf{x},\mathbf{x}')$ is the displacement response in the i direction at position $\mathbf{x}$ due to a unit impulse applied in the j direction at position $\mathbf{x}'$ at time zero.

The usual fourth rank stiffness tensor is given by

$$C_{klrj}(\mathbf{x}) = C_{klrj}^{\circ} + \gamma_{klrj}(\mathbf{x}) \tag{2.2}$$

with

$$<\gamma(\mathbf{x})> = 0 \tag{2.3}$$

Angle brackets represent ensemble averages, hence $\mathbf{C}$ has a mean value of $\mathbf{C}^{\circ}$ and γ represents the modulus fluctuations. We shall be considering the system (2.1) for the case γ small, and discussing mean responses and mean square responses to leading order in the magnitude of γ. As such, all relevant statistical information regarding the material heterogeneity is contained in the covariance function

$$<\gamma_{klrj}(\mathbf{x})\gamma_{\alpha\beta\gamma\delta}(\mathbf{x}')> = \Xi_{klrj}^{\alpha\beta\gamma\delta}\,\eta(|\,\mathbf{x}-\mathbf{x}'\,|). \tag{2.4}$$

Any other statistical information about the medium is necessarily of order γ^n with $n > 2$, and therefore negligible.

Two assumptions regarding the statistics of the polycrystal are implicit in the form of equation (2.4). The first assumption is that the tensorial character of the covariance function decouples from the spatial dependence. This is equivalent to an assertion that there are no orientation correlations between different crystallites. The assumption is a standard one. We have also assumed that the polycrystal's second order statistics are homogeneous (a function of $\mathbf{x}-\mathbf{x}'$ rather than $\mathbf{x}$ and $\mathbf{x}'$separately) and isotropic (independent of the direction of $\mathbf{x}-\mathbf{x}'$).

If v is taken to be a dimensional measure of γ then we define $\eta(r)$ as v^2 times the probability that two points separated by a distance r lie within the same crystallite. Hence the eighth rank tensor Ξ is dimensionless and η is a well defined physical quantity. Like Stanke and Kino [16], we shall choose an η of the form

$$\eta(r) = v^2 e^{-\beta r} \tag{2.5}$$

The present work will not, however, call for any specification of η until necessary, in section VII. We will occasionally cite β^{-1}, though, as a measure of microscale length.

With respect to crystal axes, the moduli of a cubic crystallite are

$$\begin{aligned} C_{ijkl} &= \lambda^I \delta_{ij}\delta_{kl} + \mu^I(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu\delta_{ijkl} \\ &= C^I_{ijkl} + \nu\delta_{ijkl} \end{aligned} \tag{2.6}$$

Where the last term in each line is not a tensor, and is defined as vanishing unless all indices are equal, and being unity otherwise.

With respect to laboratory axes the crystallite modulus tensor is given by

$$\begin{aligned} C_{ijkl} &= C^I_{ijkl} + \nu \sum_{n=1}^{3} a_i^n a_j^n a_k^n a_l^n \\ &= C^I_{ijkl} + t_{ijkl} \end{aligned} \tag{2.7}$$

a_i^n is an element of the rotation matrix representing the transformation between the crystallite axes and the laboratory axes.

If the rotation between crystal and laboratory axes is represented by the three Euler angles ϕ, Θ and ζ, the rotation matrix elements are given by

$$\begin{aligned} a_1^1 &= \cos\Theta\sin\phi\sin\zeta + \cos\phi\cos\zeta; \quad a_2^1 = \cos\Theta\cos\phi\sin\zeta + \sin\phi\cos\zeta; \quad a_3^1 = \sin\zeta\sin\Theta \\ a_1^2 &= -\cos\Theta\sin\phi\cos\zeta - \cos\phi\sin\zeta; \quad a_2^2 = \cos\Theta\cos\phi\cos\zeta - \sin\phi\sin\zeta; \quad a_3^2 = \cos\zeta\sin\Theta \\ a_1^3 &= \sin\phi\sin\Theta; \quad a_2^3 = -\cos\phi\cos\Theta; \quad a_3^3 = \cos\Theta \end{aligned} \tag{2.8}$$

as found in Bunge's treatise, eqn. (2.50) [23].

We now further assume that all crystallite orientations occur in the ensemble with equal weight. This corresponds, when taking averages of quantities which depend on a, to the use of the "uniform measure" [23]

$$\sin\Theta/8\pi^2\, d\Theta\, d\phi\, d\zeta \qquad [0 \le \Theta \le \pi, \quad 0 \le \phi < 2\pi, \quad 0 \le \zeta < 2\pi]$$

when integrating over crystallite orientations.

The ensemble average modulus of our aggregate then becomes

$$\begin{aligned} \langle C_{ijkl} \rangle \equiv C_{ijkl}^{\circ} &= C^I + \nu \sum_n \int a_i^n a_j^n a_k^n a_l^n \frac{\sin\Theta d\Theta d\phi d\zeta}{8\pi^2} \\ &\equiv C^I_{ijkl} + \langle t_{ijkl} \rangle \end{aligned} \tag{2.9}$$

$\langle \mathbf{t} \rangle$ is manifestly an isotropic fourth rank tensor which is invariant under any permutation of its indices. There is only one such tensor

$$< t_{ijkl} > \quad = \quad (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \mid \bar{t} \mid \tag{2.10}$$

where the magnitude $\mid \bar{t} \mid$ may be determined by evaluating any one component of (2.9). For example $< t_{3333} >$ is, according to 2.10, equal to $3 \mid \bar{t} \mid$. It is also, according to (2.9) and (2.8), equal to $3v/5$. Hence $\mid \bar{t} \mid = v/5$ and

$$< t_{ijkl} > \quad = \quad (v/5)\,[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{ik}] \tag{2.11}$$

Therefore the Voight-average, and isotropic, stiffness C° is given by

$$\begin{aligned} C_{ijkl}{}^{\circ} \quad &= \quad (\lambda^{I} + v/5)\delta_{ij}\delta_{kl} + (\mu^{I} + v/5)\,[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] \\ &\equiv \quad (c_L^2 - 2c_T^2)\delta_{ij}\delta_{kl} + c_T^2[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] \end{aligned} \tag{2.12}$$

c_L and c_T represent the longitudinal and transverse wavespeeds in the Voight-average medium.

The variance, equation 2.4, is an eighth rank tensor:

$$\begin{aligned} v^2\,\Xi_{ijkl}^{\alpha\beta\gamma\delta} \quad &\equiv \quad < C_{\alpha\beta\gamma\delta}C_{ijkl} > - < C_{\alpha\beta\gamma\delta} >< C_{ijkl} > \\ &= \quad < t_{\alpha\beta\gamma\delta}t_{ijkl} > - < t_{\alpha\beta\gamma\delta} >< t_{ijkl} > \end{aligned} \tag{2.13}$$

where each of the factors of C is evaluated at the same point, in accordance with equation (2.4) evaluated at $\mathbf{x} = \mathbf{x}'$. This tensor is manifestly invariant under any permutation of greek or of latin indices. It is also invariant under exchange of all the greek for all the latin indices. It is isotropic and not a pseudo-tensor; it is therefore constructable entirely from Kronecker deltas. Correspondingly, each index of any nonvanishing component must occur an even number of times. There are three independent tensors with these properties. The general linear combination is

$$\begin{aligned} < tt > - < t >< t > \quad = \quad & bv^2\{\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\}\,\{\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\gamma} + \delta_{\alpha\delta}\delta_{\beta\gamma}\} \\ & + dv^2\{\delta_{\alpha i}\delta_{\beta j}\delta_{\gamma\delta}\delta_{kl} \text{ plus all permutations consisting} \\ & \qquad \text{of exactly two pairings between greek} \\ & \qquad \text{and latin indices, one pairing between} \\ & \qquad \text{greek and greek indices, and one pairing} \\ & \qquad \text{between latin and latin indices, 72 terms in all}\} \\ & + hv^2\{\delta_{\alpha i}\delta_{\beta j}\delta_{\gamma\delta}\delta_{kl} \text{ plus all permutations consisting} \\ & \qquad \text{of exactly four pairings between greek} \\ & \qquad \text{and latin indices, 24 terms in all}\} \end{aligned} \tag{2.14}$$

The numerical coefficients b, d and h may be determined by evaluating three independent components of Ξ. For example

$$\begin{aligned} v^2\,\Xi^{3222}_{2333} &= <t_{3222}t_{2333}> - <t_{3222}><t_{2333}> \\ &= 9dv^2 \\ &= 6v^2<(a_3^3)(a_2^3)^3(a_2^2)(a_3^2)^3> + 3v^2<(a_3^3)^4(a_2^3)^4> \\ &= -v^2/70. \end{aligned}$$

Hence

$$d = -1/630$$

The other coefficients may be found by similar evaluations. They are

$$b = 13/315 - 1/25$$

and

$$h = 1/180$$

Certain inner products on Ξ will be required for the present calculation. They are tabulated here. In terms of two unit vectors $\hat{p}$ and $\hat{s}$, with angle θ_{ps} between them, they are

$$\begin{aligned} L(\theta_{ps}) \equiv\ & \Xi^{\alpha\beta\gamma\delta}_{ijkl}\hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_\delta\hat{s}_l \\ =\ & \Xi^{\cdots\hat{p}\hat{p}\hat{s}\hat{s}}_{\cdots\hat{p}\hat{p}\hat{s}\hat{s}} \\ =\ & (6/105-1/25)+(\hat{p}\cdot\hat{s})^2(18/105-4/25)+(\hat{p}\cdot\hat{s})^4(17/105-4/25) \\ =\ & \frac{9}{525}+\frac{6}{525}\cos^2\theta_{ps}+\frac{1}{525}\cos^4\theta_{ps} \\ M(\theta_{ps}) \equiv\ & \hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\delta_{\delta l} \\ =\ & \Xi^{\cdots\hat{p}\hat{p}\hat{s}}_{\cdots\hat{p}\hat{p}\hat{s}}I \;=\; \Xi^{\cdots}_{\cdots}I^{\hat{p}\hat{s}\hat{s}}_{\hat{p}\hat{s}\hat{s}} \\ =\ & \frac{24}{525}+\frac{12}{525}\cos^2\theta_{ps} \\ N(\theta_{ps}) \equiv\ & \Xi^{\alpha\beta\gamma\delta}_{ijkl}\delta_{\alpha i}\hat{p}_\beta\hat{p}_j\hat{s}_\gamma\hat{s}_k\delta_{\delta l} \;=\; \Xi^{\cdots}_{\cdots}I^{\hat{p}\hat{s}}_{\hat{p}\hat{s}}I \\ =\ & \frac{63}{525}+\frac{21}{525}\cos^2\theta_{ps}. \end{aligned} \tag{2.15}$$

III. Ensemble Average Responses

The bulk of the theoretical work on waves in polycrystalline media has focussed on attempts to describe an ensemble average plane wave, or more specifically, the effective wavenumber of the average field. This wavenumber is normally taken to indicate the speed and the attenuation of waves observed in the laboratory. The present section presents a derivation equivalent to the usual discussions, but in a language more suited to the requirements of later sections. It concludes with closed form expressions for the average Green's dyadic in the frequency domain and expressions for the attenutations which are equivalent to those derived elsewhere. The focus is on the mean response, which is calculated within an approximation, variously called the first-order-smoothing approximation or Keller approximation, valid to lowest order in the magnitude, ν, of the moduli fluctuations. The final results are expressed in closed form valid for the frequency domain below the geometrical optics limit.

The starting point is equation (2.1) together with the (temporal) Fourier transform pair

$$G_{\omega i\alpha} \equiv \int_{-\infty}^{\infty} G_{i\alpha}(t) e^{i\omega t} dt$$

$$G_{i\alpha}(t) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} G_{\omega i\alpha} e^{-i\omega t} d\omega. \tag{3.1}$$

The causality condition or alternatively, the quiescent initial conditions, together with the boundedness of the energy, assures that $\mathbf{G}_\omega$ is analytic for all ω in the upper half complex plane and in particular for all ω with infinitesimal positive imaginary part.

In the ω-domain the governing PDE for the Green's Dyadic becomes

$$\{(\omega + i\varepsilon)^2 \delta_{li} + C_{klij} \circ \partial_k \partial_j + \partial_k \gamma_{klij}(\mathbf{x}) \partial_j\} G_{i\alpha}(\mathbf{x},\mathbf{x}')$$

$$= \delta_{l\alpha} \delta^3(\mathbf{x} - \mathbf{x}') \tag{3.2}$$

where an $i\varepsilon$ has been included to emphasize the infinitesimal positive imaginary part of ω. The ω subscript on $\mathbf{G}$ has been suppressed as well. It will be re-included at occasional future points in the present exposition, but only when necessary for clarity.

Equation (3.2) is of a form considered in some generality by Karal and Keller [21] and by Frisch [24] and McCoy [25] and others. The last term in brackets is a stochastic operator. The assumed smallness of γ suggests the use of an expansion in powers of γ. But as Frisch [24] emphasizes, any simple perturbative expansion of G in powers of γ will fail at large $|\mathbf{x} - \mathbf{x}'|$. The difficulty is overcome in most expositions by the method of "smoothing" described in Frisch's case very elegantly using diagram methods. The result is an integral equation for $< G >$, the Dyson equation

$$< G_{i\alpha}(\mathbf{x},\mathbf{x}') > \;=\; G_{i\alpha}^{\circ}(\mathbf{x},\mathbf{x}') \tag{3.3}$$

$$+\int\int G_{i\beta}^{\circ}(\mathbf{x},\mathbf{y}) m_{\beta j}(\mathbf{y},\mathbf{z}) < G_{j\alpha}(\mathbf{z},\mathbf{x}') > d^3y d^3z$$

m is the "mass", or "self-energy" operator and G° is the known Green's Dyadic for the bare, Voight-average, medium, i.e., the solution to (3.2) when $\gamma = 0$. It is given below.

The Dyson equation is exact. An approximation is, however, necessary for the evaluation of **m**. The "First-Order-Smoothing Approximation" (FOSA) [24] approximates m by its lowest order term in an expansion in powers of the fluctuating moduli. It is generally argued that FOSA is exact in the limit of small γ. This corresponds in the present case to sufficiently small values of a non-dimensionalized $|\nu|^{*}$. The FOSA expression for m is

$$m_{\beta j}(\mathbf{y},\mathbf{z}) \approx < \frac{\partial}{\partial y_\alpha} \gamma_{\alpha\beta\gamma\delta}(\mathbf{y}) \frac{\partial}{\partial y_\delta} G_{\gamma k}^{\circ}(\mathbf{y},\mathbf{z}) \frac{\partial}{\partial z_i} \gamma_{ikjl}(\mathbf{z}) \frac{\partial}{\partial z_l} > \tag{3.4}$$

The Dyson equation (3.3) may be solved with ease in the domain of spatial Fourier transforms. We define the spatial Fourier transforms of $\mathbf{G}^{\circ}$, **m**, and $< \mathbf{G} >$.

$$G_{i\alpha}^{\circ}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{q}) \equiv \frac{1}{(2\pi)^3} \int\int d^3x d^3x' e^{-i\mathbf{p}\cdot\mathbf{x}} G_{i\alpha}^{\circ}(\mathbf{x},\mathbf{x}') e^{i\mathbf{q}\cdot\mathbf{x}'} \tag{3.5}$$

Where the diagonal, or delta-function, character of G° in transform space follows necessarily from the homogeneity of the bare medium. Equation (3.5) has an inverse

$$G_{i\alpha}^{\circ}(\mathbf{x},\mathbf{x}') \;=\; \frac{1}{(2\pi)^3} \int\int d^3p d^3q e^{i\mathbf{p}\cdot\mathbf{x}} G_{i\alpha}^{\circ}(\mathbf{p}) \delta^3(\mathbf{p}-\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{x}'} \tag{3.6}$$

$\mathbf{G}^{\circ}(\mathbf{p})$ is readily constructed by means of the Fourier transform of its governing equation, (3.2) with γ set to zero.

$$\{(\omega + i\varepsilon)^2 \delta_{li} - p_k p_j C_{klij}^{\circ}\} G_{i\alpha}^{\circ}(\mathbf{p}) \;=\; \delta_{l\alpha} \tag{3.7}$$

In direct notation the equation for $\mathbf{G}^{\circ}(\mathbf{p})$ becomes

$$(\hat{\mathbf{p}}\hat{\mathbf{p}}\{(\omega + i\varepsilon)^2 - p^2 c_L^2\} + (\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}})\{(\omega + i\varepsilon)^2 - p^2 c_T^2\}) \cdot \mathbf{G}^0 = \mathbf{I} \tag{3.8}$$

The solution to (3.8) is available by inspection:

$$\mathbf{G}^{\circ}(\mathbf{p}) \;=\; \hat{\mathbf{p}}\hat{\mathbf{p}} g^{oL}(p) + (\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}}) g^{oT}(p) \tag{3.9}$$

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*The important issue as to how small must $|\nu|$ be for FOSA to be accurate will not be completely addressed here. In the case of iron, $|\nu|$, when measured in the most obvious nondimensional manner, $|\nu|/c_T^2$, has a value of 1.66. On the other hand, Stanke and Kino [16] argue for a nondimensional $|\nu|$ in the case of iron of order 0.14 or less.

where g^{oL} and g^{oT} may be termed the bare longitudinal and transverse propagators

$$g^{oL}(p) = [(\omega+i\varepsilon)^2 - p^2c_L^2]^{-1}; \quad g^{oT}(p) = [(\omega+i\varepsilon)^2 - p^2c_T^2]^{-1} \tag{3.10}$$

It is of particular interest to note that, for infinitesimal positive ε, the imaginary parts of these propagators are given by

$$Img^{oL}(p) = -\pi\, sgn(\omega)\, \delta(\omega^2 - p^2c_L^2)$$

and

$$Img^{oT}(p) = -\pi\, sgn(\omega)\, \delta(\omega^2 - p^2c_T^2) \tag{3.11}$$

The Fourier transform of $< G >$ is also diagonal by reason of the statistical homogeneity of the medium.

$$< G_{i\alpha}(\mathbf{p}) > \delta^3(\mathbf{p}-\mathbf{q}) \equiv \frac{1}{(2\pi)^3}\int\int d^3x d^3x' e^{-i\mathbf{p}\cdot\mathbf{x}} < G_{i\alpha}(\mathbf{x},\mathbf{x}') > e^{i\mathbf{q}\cdot\mathbf{x}'} \tag{3.12}$$

$$< G_{i\alpha}(\mathbf{x},\mathbf{x}') > = \frac{1}{(2\pi)^3}\int\int d^3p d^3x' e^{i\mathbf{p}\cdot\mathbf{x}} < G_{i\alpha}(\mathbf{p}) > \delta^3(\mathbf{p}-\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{x}'} \tag{3.13}$$

The Fourier transform of m is of the form

$$-\sigma_{\beta j}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{q}) \equiv \frac{1}{(2\pi)^3}\int\int d^3x d^3x' e^{-i\mathbf{p}\cdot\mathbf{x}} m_{\beta j}(\mathbf{x},\mathbf{x}') e^{i\mathbf{q}\cdot\mathbf{x}'} \tag{3.14}$$

Like $< G >$, m is diagonal in the Fourier domain in consequence of the statistical homogeneity of the medium.

Substitution of the FOSA approximation for m (3.4) into (3.14) yields

$$\sigma_{\beta j}(\mathbf{p}) = p_\alpha p_l \int d^3r\; G_{\delta k}{}^{\circ}(\mathbf{r})\, \Xi_{ijkl}^{\alpha\beta\gamma\delta}\, \partial_\gamma\partial_i\{\eta(r)e^{-i\mathbf{p}\cdot\mathbf{r}}\} \tag{3.15}$$

It is more convenient to have σ expressed in terms of G° evaluated in the Fourier domain. Hence we define a spatial Fourier transform of the two point correlation function

$$\tilde{\eta}(\mathbf{p}) \equiv \frac{1}{(2\pi)^3}\int d^3r \quad \eta(\mathbf{r})e^{-i\mathbf{p}\cdot\mathbf{r}} \tag{3.16}$$

and find that (3.15) may be re-expressed as a convolution between G° and $\tilde{\eta}$

$$\sigma_{\beta j}(\mathbf{p}) = -p_\alpha p_l \int d^3s\; G_{\delta k}{}^{\circ}(\mathbf{s})\, \tilde{\eta}(\mathbf{p}-\mathbf{s})\, s_\gamma s_i \Xi_{ijkl}^{\alpha\beta\gamma\delta} \tag{3.17}$$

In direct notation

$$\sigma(\mathbf{p}) = -\int d^3s\; {}^{ps}_{ps}\mathbf{G}^{0\cdots}_{\cdots}\Xi\, \tilde{\eta}(\mathbf{p}-\mathbf{s}) \tag{3.18}$$

By general symmetry considerations σ must be, like G°, and any second rank tensor function of a single vector **p**, of the form

$$\sigma(\mathbf{p}) = \sigma^L(p)\hat{\mathbf{p}}\hat{\mathbf{p}} + \sigma^T(p)(\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}}) \tag{3.19}$$

Comparison of (3.18) and (3.19) with substitution of (3.9) gives expression for σ^L and σ^T

$$\sigma^L(\mathbf{p}) = -\int d^3s \, \hat{\mathbf{p}}\mathbf{p}\mathbf{s}\cdots\Xi\cdots\hat{\mathbf{p}}\mathbf{p}\mathbf{s}\cdots \{\hat{\mathbf{s}}\hat{\mathbf{s}} g^{oL}(\mathbf{s}) + (\mathbf{I} - \hat{\mathbf{s}}\hat{\mathbf{s}}) g^{oT}(\mathbf{s})\} \tilde{\eta}(\mathbf{p} - \mathbf{s}) \tag{3.20}$$

and

$$\sigma^T(\mathbf{p}) = -\frac{1}{2}\int d^3s \, (\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}})^{\mathbf{ps}\cdots}_{\mathbf{ps}\cdots}\Xi\cdots \{\hat{\mathbf{s}}\hat{\mathbf{s}} g^{oL}(\mathbf{s}) + (\mathbf{I} - \hat{\mathbf{s}}\hat{\mathbf{s}}) g^{oT}(\mathbf{s})\} \tilde{\eta}(\mathbf{p} - \mathbf{s}) \tag{3.21}$$

A spatial Fourier transform of the Dyson equation and an invocation of the definitions (3.5), (3.12) and (3.14) reduces the Dyson equation to an algebraic equation for the Fourier transform of $< G >$. The solution is

$$< \mathbf{G}(\mathbf{p}) > = g^L(p)\hat{\mathbf{p}}\hat{\mathbf{p}} + g^T(p)(\mathbf{I} - \hat{\mathbf{p}}\hat{\mathbf{p}}) \tag{3.22}$$

with

$$g^L(p) = (\omega^2 - p^2 c_L^2 + \sigma^L(p))^{-1} \tag{3.23}$$

and

$$g^T(p) = (\omega^2 - p^2 c_T^2 + \sigma^T(p))^{-1}. \tag{3.24}$$

These are the propagators for the mean field. They reduce to the bare propagators g^{oL} and g^{oT} when ν, and therefore σ, vanishes. The term in epsilon has been dropped as the small but nevertheless finite imaginary parts of the σ's will dominate the infinitesimal ε.

The wavenumbers of the mean field are extractable directly from equations (3.23) and (3.24) by seeking that p at which the g are singular. Were one to take the inverse Fourier transform (3.13) of the expression (3.22), the major contribution would arise from the pole near $| p | = \omega/c$. The effective wave numbers are therefore the solutions, p, to the algebraic conditions

$$\omega^2 - p^2 c_L^2 + \sigma^L(p) = 0 \tag{3.25}$$

and

$$\omega^2 - p^2 c_T^2 + \sigma^T(p) = 0 \tag{3.26}$$

Evaluation of the effective wavenumbers requires the analytic or numerical solutions to these equations where the σ's are given by equations (3.20) and (3.21). The solutions simplify, however, in what is sometimes called the Born approximation [16]. That approximation is motivated by a recognition that the σ's scale with ν^2 and are therefore small. The solutions, p, to

equations 3.25 and 3.26 are therefore close to the bare solutions $p = \omega/c$. We are thus led to approximate $\sigma(p)$ with $\sigma(\omega/c)$ and to write closed form expressions for the effective wavenumbers

$$p^{Teff} = (\omega^2/c_T^2 + \sigma^T(\omega/c_T)/c_T^2)^{1/2} \tag{3.27}$$

$$p^{Leff} = (\omega^2/c_L^2 + \sigma^L(\omega/c_L)/c_L^2)^{1/2}. \tag{3.28}$$

As Karal and Keller discuss, this approximation fails in the very high frequency, geometrical optics, regime. The approximation is by no means a necessary one and Stanke and Kino [16] give numerical solutions to the more general equations (3.25) and (3.26). We, though, retain the approximation because it simplifies the later analysis and is in any case valid except at very high frequencies.

The attenuations are the imaginary parts of the effective wavenumbers and are given, to the same approximation, by

$$\alpha^T = Im\sigma^T(\omega/c_T)/2\omega c_T \tag{3.29}$$

and

$$\alpha^L = Im\sigma^L(\omega/c_L)/2\omega c_L \tag{3.30}$$

The imaginary parts of the σ's are in turn determinable from equations (3.20) and (3.21) together with the identification (3.11).

$$\begin{aligned} Im\sigma^L(\omega/c_L) &= \frac{\pi}{2}\int \hat{\mathbf{p}}\hat{\mathbf{p}}\hat{\mathbf{s}}\hat{\mathbf{s}}\cdots\Xi \frac{\omega^5}{c_L^7}\tilde{\eta}(\hat{\mathbf{p}}\frac{\omega}{c_L} - \hat{\mathbf{s}}\frac{\omega}{c_L})d^2\hat{s} \\ &+ \frac{\pi}{2}\int \hat{\mathbf{p}}\hat{\mathbf{p}}\hat{\mathbf{s}}(\mathbf{I} - \hat{\mathbf{s}})\cdots\Xi \frac{\omega^5}{c_T^5 c_L^2}\tilde{\eta}(\hat{\mathbf{p}}\frac{\omega}{c_L} - \hat{\mathbf{s}}\frac{\omega}{c_T})\, d^2\hat{s} \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} Im\sigma^T(\omega/c_T) &= \frac{\pi}{4}\int (\mathbf{I} - \hat{\mathbf{p}})\hat{\mathbf{p}}\hat{\mathbf{s}}\hat{\mathbf{s}}\cdots\Xi \frac{\omega^5}{c_L^5 c_T^2}\tilde{\eta}(\hat{\mathbf{p}}\frac{\omega}{c_T} - \hat{\mathbf{s}}\frac{\omega}{c_L})\, d^2\hat{\mathbf{s}} + \\ &\frac{\pi}{4}\int_{4\pi} (\mathbf{I} - \hat{\mathbf{p}})\hat{\mathbf{p}}\hat{\mathbf{s}}(\mathbf{I} - \hat{\mathbf{s}})\cdots\Xi \frac{\omega^5}{c_T^7}\tilde{\eta}(\hat{\mathbf{p}}\frac{\omega}{c_T} - \hat{\mathbf{s}}\frac{\omega}{c_T})d^2\hat{s} \end{aligned} \tag{3.32}$$

where the integrals over the magnitude of the vector $\mathbf{s}$ have been absorbed by the delta-functions in equations (3.11), leaving integrals over the unit sphere $\hat{s}$.

In terms of the three functions of the angle θ_{ps} between the unit vectors $\hat{p}$ and $\hat{s}$ which are defined by

$$\eta^{LL}(\theta_{ps}) \equiv \tilde{\eta}(\hat{\mathbf{p}}\omega/c_L - \hat{\mathbf{s}}\omega/c_L)$$

$$\eta^{LT}(\theta_{ps}) \equiv \eta^{TL}(\theta_{ps}) \equiv \tilde{\eta}(\hat{\mathbf{p}}\omega/c_L - \hat{\mathbf{s}}\omega/c_T) \tag{3.33}$$

$$\eta^{TT}(\theta_{ps}) \equiv \tilde{\eta}(\hat{\mathbf{p}}\omega/c_T - \hat{\mathbf{s}}\omega/c_T)$$

we find

$$\alpha^L = \alpha_{LL} + \alpha_{LT}; \qquad \alpha^T = \alpha_{TT} + \alpha_{TL} \tag{3.34}$$

with

$$\alpha_{LL} \equiv \frac{\pi\omega^4}{4c_L^8}\int d^2\hat{s}\,\eta^{LL}(\theta_{sp})L(\theta_{sp}) = \frac{\pi^2\omega^4}{2c_L^8}\int_{-1}^{+1}\eta^{LL}(\theta)L(\theta)d\cos\theta$$

$$\alpha_{LT} \equiv \frac{\pi}{4}\frac{\omega^4}{c_L^3c_T^5}\int d^2\hat{s}\,\eta^{LT}(\theta_{sp})\{M(\theta_{sp}) - L(\theta_{sp})\}$$

$$= \frac{\pi^2\omega^4}{2c_L^3c_T^5}\int_{-1}^{1}\eta^{LT}(\theta)\,(M(\theta) - L(\theta))d\cos\theta \tag{3.35}$$

$$\alpha_{TL} \equiv \frac{1}{2}(c_T/c_L)^2\alpha_{LT}$$

$$\alpha_{TT} \equiv \frac{\pi\omega^4}{8c_T^8}\int d^2\hat{s}\,\eta^{TT}(\theta_{sp})\{N - 2M + L\} = \frac{\pi^2\omega^4}{4c_T^8}\int_{-1}^{+1}\eta^{TT}(\theta)\,(N(\theta) - 2M(\theta) + L(\theta))d\cos\theta$$

Four additional attenuation-like quantities may also be defined, α_{LL}', α_{LT}', α_{TL}' and α_{TT}', by expressions identical to those of (3.35) except that one extra factor of $\cos\theta$ is inserted in each integrand. The primed quantities appear to represent the preference, if any, of the scattering for the forward direction. It is worth noting that in the Rayleigh limit $\omega \ll \beta c$, all the η'*s* become independent of their arguments. In consequence, all attenuations scale with ω^4, and all primed attenuations vanish.

Upon specification of $\eta(r)$, the above integrals can be evaluated. For the case of an exponential $\eta(r)$ (equation 2.5) this is done in section VII.

The major qualitative difference between the bare medium and the effective medium is the presence of this attenuation in the effective medium. It should be clearly understood, though, that this attenuation is not due to a dissipative process. In a sense it is entirely an artefact of having asked for the ensemble average response. Phase differences accumulating between the disturbances in different realizations lead to destructive interferences upon averaging over the ensemble. Viewed alternatively, the attenuation described above can be thought of as due to the scattering of energy out of a beam. In any case, there is no true dissipation present. Therefore

the energy which is associated with the attenuating mean field, itself attenuating also, is a very poor representation of the actual energy in a typical member of the ensemble. This latter quantity is probably more accurately described by the mean of the energy, not the energy of the mean. In order to describe experiments such as that of Guo, *et al* [8] or Weaver [14] or backscatter studies [6,11,12,13] all of which scrutinize mean square responses, a different theory is required. The next section defines a few energy-related quantities of interest and the following section presents the multiple-scattering theory for their evaluation.

IV. Elastodynamic Energy, Flux and Diffusivity

The energy density at position $\mathbf{x}$ and time t due to a unit impulsive source at $\mathbf{x}'$ polarized in the j direction, and for simplicity summed over that direction, is given by the sum of kinetic and strain energy terms

$$e \;=\; 1/2\dot{G}_{ij}(\mathbf{x},\mathbf{x}',t)\dot{G}_{ij}(\mathbf{x},\mathbf{x}',t) + \text{strain energy density} \tag{4.1}$$

By means of the virial theorem we may recognize that the total energy density is equal, on suitable temporal or spatial averaging, to twice the kinetic energy density. Thus we are led to define a quantity which appears to be a temporal Fourier transform of the energy associated with a band passed process.

$$\tilde{e}(\Omega,\mathbf{x},\mathbf{x}') \equiv \sum_{ij} \int_{-\infty}^{\infty} e^{-i\Omega t} \mid B \otimes \dot{G}_{ij}(\mathbf{x},\mathbf{x}',t) \mid^2 dt \tag{4.2}$$

where B represents a moderately narrow and moderately well damped band-pass process, centered, typically in polycrystalline applications at a frequency of order 5 to 50 MHz, at central frequency ω with width $\Delta\omega$. Ω is of the order of the inverse of the time scale upon which the relatively slow evolution of the diffuse energy is to be studied; $\Omega \ll \Delta\omega \ll \omega$; $\Omega \ll \alpha c \ll \omega$. $\otimes$ represents a temporal convolution.

By employing the Fourier representation (3.1) for $G(t)$, $\tilde{e}$ may be re-expressed as

$$\tilde{e} \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dt\,d\omega\,d\omega'}{(2\pi)^2} e^{-i\Omega t - i\omega t + i\omega' t} B(\omega)B^*(\omega')\omega\omega' G_{\omega ij}(\mathbf{x},\mathbf{x}')G^*_{\omega' ij}(\mathbf{x},\mathbf{x}') \tag{4.3}$$

where an asterisk denotes a complex conjugate. The time integration yields a delta function which then allows ready evaluation of the ω' integration. $\tilde{e}$ becomes, in direct notation,

$$\tilde{e} \;=\; \int \frac{d\omega}{(2\pi)} B(\omega)B^*(\omega+\Omega)\,\omega(\omega+\Omega)\,\mathbf{G}_{\omega}(\mathbf{x},\mathbf{x}'):\mathbf{G}^*_{\omega+\Omega}(\mathbf{x},\mathbf{x}') \tag{4.4}$$

As $B(\omega)$ is a smooth function with width $\Delta\omega \gg \Omega$, and as $\omega \gg \Omega$,

$$\tilde{e} \approx \int \frac{d\omega}{(2\pi)} \mid B(\omega) \mid^2 \omega^2\, \mathbf{G}_{\omega}(\mathbf{x},\mathbf{x}'):\mathbf{G}^*_{\omega+\Omega}(\mathbf{x},\mathbf{x}')$$

The above equation indicates that $\bar{e}$ is a frequency weighted average of the product of the Green's function at two different frequencies. By an ergodic hypothesis this frequency average is equivalent to an ensemble average and we may conclude

$$\bar{e} \approx \frac{|B|^2 \omega^2 \Delta\omega}{2\pi} < \mathbf{G}_{\omega}(\mathbf{x},\mathbf{x}') : \mathbf{G}_{\omega+\Omega}{}^{*}(\mathbf{x},\mathbf{x}') >$$

We now drop the uninteresting pre-factors and take a spatial Fourier transform to define a quantity which shall play the role of a diffuse-field energy density.

$$E(\Omega,\Delta) \equiv \int d^3x e^{i\Delta\cdot(\mathbf{x}-\mathbf{x}')} < \mathbf{G}_{\omega}(\mathbf{x},\mathbf{x}') : \mathbf{G}^{*}_{\omega+\Omega}(\mathbf{x},\mathbf{x}') > \tag{4.5}$$

E should, by reason of statistical homogeneity, be independent of source position $\mathbf{x}'$.

On sufficiently long time scales ($\Omega \ll \alpha c$) and long length scales $|\Delta| \ll \alpha$, E according to our main hypothesis, should obey a diffusion equation. It is therefore expected, for small Ω and Δ, that E should have an Ω and Δ dependence of the form

$$E = \frac{\text{constant}}{i\Omega + D\Delta^2} \tag{4.6}$$

characteristic of the Fourier-domain solution of a heat equation. D is the desired diffusivity.

The constant in the numerator of (4.6) may be evaluated by means of a Ward identity [26,27]. Note that

$$E(\Omega,0) = < \int d^3x \mathbf{G}_{\omega}(\mathbf{x}',\mathbf{x}) : \mathbf{G}^{*}_{\omega+\Omega}(\mathbf{x},\mathbf{x}'') >_{\mathbf{x}'=\mathbf{x}''}$$

If G is expanded in terms of its unknown, but real, normal modes $\mathbf{u}^{(n)}$ each with natural frequency ω_n.

$$G_{\omega ij}(\mathbf{x}',\mathbf{x}) = \sum_n u_i^n(\mathbf{x}') u_j^n(\mathbf{x}) \frac{1}{(\omega + i\varepsilon)^2 - \omega_n^2} \tag{4.7}$$

with orthonormality condition

$$\int d^3x \, \mathbf{u}^n(\mathbf{x}) \cdot \mathbf{u}^m(\mathbf{x}) = \delta_{nm} \tag{4.8}$$

then

$$E(\Omega,0) = <\sum_n \sum_m \int d^3x \frac{u_i^n(\mathbf{x}')u_j^n(\mathbf{x})u_j^m(\mathbf{x})u_i^m(\mathbf{x}'')}{[(\omega+i\varepsilon)^2-\omega_n^2]\,[(\omega-i\varepsilon+\Omega)^2-\omega_m^2]}>_{\mathbf{x}'=\mathbf{x}''}$$

$$= <\sum_n \frac{u_i^n(\mathbf{x}')u_i^n(\mathbf{x}'')}{2\omega\Omega+\Omega^2-2i\varepsilon\omega}\left[\frac{1}{(\omega+i\varepsilon)^2-\omega_n^2}-\frac{1}{(\omega-i\varepsilon+\Omega)^2-\omega_n^2}\right]>_{\mathbf{x}=\mathbf{x}''}$$

$$= \frac{1}{2\omega\Omega+\Omega^2}<\mathbf{G}_{\omega ii}(\mathbf{x}',\mathbf{x}'')-\mathbf{G}^*_{\omega+\Omega ii}(\mathbf{x}',\mathbf{x}'')>_{\mathbf{x}'=\mathbf{x}''} \qquad (4.9)$$

The $\Delta = 0$ value of the mean square response E has been found to be related to a simple mean response. The numerator in the last line of (4.9) can be recognized, as $\Omega \to 0$, as the imaginary part of the trace of $< G >$, divided by the total system volume. As such it is related to the modal density in the effective medium [26]. As the modal density will be perturbed from its bare value by terms only of order v^2, that modal density can be well approximated here by its bare value. More formally, equation (4.9) may be evaluated by replacing $< G(x) >$ with the inverse Fourier transform (3.13) of the expression derived for $< G(p) >$ in the preceeding section.

$$E(\Omega,0) = \frac{1}{2\omega\Omega+\Omega^2}\frac{1}{(2\pi)^3}\int d^3p\{g_\omega^L - g_{\omega+\Omega}^{*L} + 2g_\omega^T - 2g_{\omega+\Omega}^{*T}\}$$

$$= \frac{1}{2\omega\Omega}\frac{1}{2\pi^2}\int p^2dp\left\{\frac{2\omega\Omega+\sigma_L^*-\sigma_L}{(\omega^2-p^2c_L^2+\sigma_L)(\omega^2-p^2c_L^2+\sigma_L^*)}+2\frac{2\omega\Omega+\sigma_T^*-\sigma_T}{(\omega^2-p^2c_T^2+\sigma_T)(\omega^2-p^2c_T^2+\sigma_T^*)}\right\} \qquad (4.10)$$

where Ω^2 has been neglected in comparison with $2\omega\Omega$. We may further neglect $2\omega\Omega$ relative to $\sigma^* - \sigma = \omega\alpha c$; that is, Ω, the rate scale for study of diffusive field evolution, is slow compared to the rate of multiple scattering. We also approximate $\sigma \ll \omega^2$, which is valid at all frequencies if the modulus fluctuation scale v is small. (4.10) is then easily evaluated;

$$E(\Omega,0) = \frac{1}{4i\pi\Omega}[1/c_L^3+2/c_T^3]. \qquad (4.11)$$

E has been found to be inversely proportional to $i\Omega$, corresponding to a step function temporal behaviour of the total energy. The relative contribution of longitudinal and transverse waves in 4.11 is consistent with the standard mode counting methods [15] used in the conjecture (1.2). Equation (4.11) can serve as a check on any theory of mean square responses.

Comparison of (4.11) and (4.6) allows one to conclude with an expression for the leading behaviour of the diffusion propagator

$$E(\Omega,\Delta) = \frac{1}{4\pi}[1/c_L^3+2/c_T^3]/(i\Omega+D\Delta^2) \qquad (4.12)$$

Of importance equal to that of the energy density is the energy flux. That flux may be written in terms similar to those used for the energy by considering minus the product of material velocity and stress.

$$j_k = -\dot{G}_{ij}(\mathbf{x},\mathbf{x}',t)G_{vj,\mu}(\mathbf{x},\mathbf{x}',t)C_{ik\mu\nu}(\mathbf{x}) \tag{4.13}$$

The energy flux is summed over source polarization as was the energy. The modulus $\mathbf{C}(\mathbf{x})$ may be taken to be the Voight average modulus $\mathbf{C}^\circ$ to leading order in the magnitude of the fluctuations.

Energy conservation takes the form

$$\dot{e} = -\partial_k j_k + \text{source} \tag{4.14}$$

which may be corroborated by appeal to the governing equations (2.1). As above we construct a temporal Fourier transform of the energy flux associated with the band-passed response.

$$\tilde{j}_k \equiv C^\circ{}_{ik\mu\nu} \int\int\int \frac{dt\,d\omega\,d\omega'}{(2\pi)^2} e^{-i\Omega t - i\omega t + i\omega' t} B(\omega)B^*(\omega')(-i\omega)G_{\omega ij}(\mathbf{x},\mathbf{x}')G^*{}_{\omega' vj,\mu}(\mathbf{x},\mathbf{x}')$$

$$= \frac{i\omega}{(2\pi)} C^\circ{}_{ik\mu\nu} \mid B \mid^2 \Delta\omega < G_{\omega ij}(\mathbf{x},\mathbf{x}')G^*{}_{\omega+\Omega vj,\mu}(\mathbf{x},\mathbf{x}') > \tag{4.15}$$

We drop the same prefactors as was done following eqn. (4.5) and take a spatial Fourier transform to define

$$J_k(\Omega,\Delta) \equiv \frac{i}{\omega} C^\circ{}_{ik\mu\nu} \int d^3x e^{i\Delta\cdot(\mathbf{x}-\mathbf{x}')} < G_{\omega ij}(\mathbf{x},\mathbf{x}')G^*{}_{\omega+\Omega vj,\mu}(\mathbf{x},\mathbf{x}') > \tag{4.16}$$

which shall serve as our definition of diffuse field energy flux. Energy conservation (4.14) then takes the form

$$i\Omega E = i\Delta_k J_k + \text{source} \tag{4.17}$$

While the diffusivity D may be defined by means of equation (4.12), it will prove more convenient to define it by means of a constitutive-like relation between energy flux and energy gradient:

$$J_k = iD\Delta_k E \tag{4.18}$$

hypothesized to be valid at small Ω and Δ, and hypothesized to follow from a study of the multiple scattering. It is readily seen that equations (4.18) and (4.17) together imply (4.6) and therefore (4.12). Thus the two definitions of D are equivalent.

V. Multiple Scattering Theory for Mean Square Responses

Both the energy and the flux may be formed by contractions on the Green's dyadic covariance

$$< G_{\alpha\beta}(\mathbf{x},\mathbf{x}')G^{*}_{ij}(\mathbf{y},\mathbf{y}') >$$

which is a dyadic generalization of a scalar quantity for which there exists a multiple-scattering formalism [24,27]. The covariance is fourth rank tensor function of four independent spatial variables. We again suppress the ω and $\omega+\Omega$ subscripts, henceforth taking the asterisk to indicate a complex conjugate and also to indicate a subscript $\omega+\Omega$ rather than ω. It proves convenient to study the covariance in a spatial transform domain, hence a fourth rank tensor function of four wave vectors is defined.

$$ {}^{\mathbf{p}\alpha}_{\mathbf{p}'i}H^{\beta\mathbf{q}}_{j\mathbf{q}'}\,\delta^3(\mathbf{p}+\mathbf{q}'-\mathbf{q}-\mathbf{p}') \equiv$$

$$(2\pi)^{-6}\int\int\int\int d^3x\,d^3x'\,d^3y\,d^3y' < G_{\alpha\beta}(\mathbf{x},\mathbf{x}')G^{*}_{ij}(\mathbf{y},\mathbf{y}') > e^{-i\mathbf{p}\cdot x+i\mathbf{q}\cdot\mathbf{x}'+i\mathbf{p}'\cdot\mathbf{y}-i\mathbf{q}'\cdot\mathbf{y}'} \tag{5.1}$$

The energy density may be expressed in terms of H by

$$E \;=\; \frac{1}{(2\pi)^3}\int\int d^3q\,d^3p\; {}^{\mathbf{p}}_{\mathbf{p}+\Delta\,i}{}^{\alpha}H^{\beta\mathbf{q}}_{j\mathbf{q}+\Delta}\,\delta_{i\alpha}\delta_{j\beta} \tag{5.2}$$

which follows from (5.1) and the definition of E. Similarly the energy flux may be expressed

$$J_\gamma \;=\; \frac{1}{\omega}\frac{1}{(2\pi)^3}C^{0}_{\gamma\alpha ki}\int\int d^3q\,d^3p\;[p_k+\Delta_k]\;{}^{\mathbf{p}}_{\mathbf{p}+\Delta\,i}{}^{\alpha}H^{\beta\mathbf{q}}_{i\mathbf{q}+\Delta}\delta_{j\beta} \tag{5.3}$$

The Dyson equation governed the mean Green's Dyadic. The Bethe-Salpeter equation [24] governs the covariance. In the Fourier domain it is

$$ {}^{\mathbf{p}}_{\mathbf{p}+\Delta\,i}{}^{\alpha}H^{\beta\mathbf{q}}_{j\mathbf{q}+\Delta} \;=\; {}^{\alpha}_{i}\Gamma^{\beta\mathbf{p}}_{j\mathbf{p}+\Delta}\,\delta^3(\mathbf{p}-\mathbf{q})+\int d^3s\;{}^{\alpha}_{i}\Gamma^{\gamma\mathbf{p}}_{k\mathbf{p}+\Delta}\;\;{}^{\mathbf{p}}_{\mathbf{p}+\Delta\,k}{}^{\gamma}K^{\delta\mathbf{s}}_{ls+\Delta}\;\;{}^{\mathbf{s}}_{\mathbf{s}+\Delta\,i}{}^{\delta}H^{\beta\mathbf{q}}_{j\mathbf{q}+\Delta} \tag{5.4}$$

It is written in terms of the double-mean-field Green's function

$$ {}^{\alpha}_{i}\Gamma^{\gamma\mathbf{p}}_{k\mathbf{p}+\Delta} \equiv < G_{\alpha\gamma}(\mathbf{p}) >< G^{*}_{ik}(\mathbf{p}+\Delta) > \tag{5.5}$$

where $< G >$ is as given in the previous section.

The Bethe-Salpeter equation is formally exact, but approximations are required for the intensity operator **K**. In what is often called the ladder approximation because of the shapes of the associated diagrams **K** is expanded to leading order in the perturbations, in this case, to leading order in v. The approximation is similar to the FOSA and a dyadic version of Frisch's [24] equation (4.36).

$$ {}^{\mathbf{p}}_{\mathbf{p}+\Delta\,k}{}^{\gamma}K^{\delta\mathbf{s}}_{ls+\Delta} \approx \tilde{\eta}(\mathbf{p}-\mathbf{s})\;p_\beta s_\alpha(p_i+\Delta_i)\,(s_j+\Delta_j)\;\Xi^{\alpha\beta\gamma\delta}_{ijkl} \tag{5.6}$$

The specifications (5.5) and (5.6) complete the prescription for the calculation of the covariance H. The Bethe-Salpeter equation (5.4) is however very complex and further simplifications and approximations are required before it becomes tractable.

The first set of simplifications may be obtained by focussing on a different dependent variable, S, instead of H. S is a contracted version of the source of the covariance and is defined by

$$S^{\mu}_{m}(\mathbf{p},\Delta) \equiv \int d^3q \; {}^{\mu}_{m}(\Gamma^{-1})^{\alpha\, \mathbf{p}}_{i\mathbf{p}+\Delta} \; {}^{\mathbf{p}\alpha}_{\mathbf{p}+\Delta i} H^{\beta\mathbf{q}}_{j\mathbf{q}+\Delta} \delta_{\beta j} \tag{5.7}$$

where

$${}^{\mu}_{m}(\Gamma^{-1})^{\alpha\mathbf{p}}_{i\mathbf{p}+\Delta} \equiv < G(\mathbf{p}) >^{-1}_{\mu\alpha} < G^*(\mathbf{p}+\Delta) >^{-1}_{mi} \tag{5.8}$$

By performing suitable contractions upon eqn. (5.4), the Bethe-Salpeter equation becomes

$$S^{\mu}_{m}(\mathbf{p},\Delta) \;=\; \delta_{\mu m} + \int d^3s \; {}^{\mathbf{p}\,\mu}_{\mathbf{p}+\Delta m} K^{\alpha s}_{js+\Delta} \; {}^{\alpha}_{j}\Gamma^{\delta s}_{ls+\Delta} \;\; S^{\delta}_{l}(\mathbf{s},\Delta) \tag{5.9}$$

S contains all information required for the present purposes. In particular, we find that the energy density and flux are given in terms of S by

$$E \;=\; \frac{1}{(2\pi)^3} \int d^3p \, \delta_{\alpha i} \; {}^{\alpha}_{i}\Gamma^{\mu\mathbf{p}}_{m\mathbf{p}+\Delta} \, S^{\mu}_{m}(\mathbf{p},\Delta) \tag{5.10}$$

and

$$J_{\mu} \;=\; \frac{1}{(2\pi)^3} \int d^3p \, \frac{[p_k + \Delta_k]}{\omega} C_{\mu\alpha k i} \circ \; {}^{\alpha}_{i}\Gamma^{\beta\mathbf{p}}_{m\mathbf{p}+\Delta} \, S^{\beta}_{m}(\mathbf{p},\Delta) \tag{5.11}$$

In order to obtain the diffusivity D by means of equation (4.18) it will be sufficient to obtain E and $\mathbf{J}$ as expansions in powers of Δ through the linear terms. This will allow the derivation of an expression for D and also allow verification of the constraint (4.11) and the constraint that $\mathbf{J}$ vanish at zeroth order in Δ. To obtain this expansion it will be sufficient to work with expansions of Γ and K and S to similar, linear order. This will provide a further simplification of the Bethe-Salpeter equation. While doing this expansion it is wise to recognize that the system provides two different wavenumber scales, ω/c and the attenuation α. In the limit $\alpha \ll \omega/c$, a constraint invoked in previous sections, and valid throughout the entire frequency range if ν is small, we recognize that our expansion through terms linear in Δ should retain terms like $\Delta/\alpha \sim \Delta\omega c/Im\sigma$ in comparison to unity, but may neglect terms like $\Delta c/\omega$.

The assumptions $\alpha \ll \omega/c$ and $\Delta \ll \omega/c$ allow the Bethe-Salpeter equation to be re-cast as an equation of radiative transfer. The further approximation $\Delta \ll \alpha$ provides the diffusion limit. This two step procedure is the business of the next section.

VI. Radiative Transfer

In the above three expressions the integrals over the magnitudes of the wave vectors are all highly singular, as can be seen from the form of Γ, eqns. (5.5) and (3.22) while recalling that the

self energies σ, and in particular, their imaginary parts, are small. This observation leads one to attempt to approximate the integral by the contributions from the vicinity of the poles. Such an approximation is equivalent to replacing the factor $p^2\Gamma$ with

$$p^2{}_i{}^{\alpha}\Gamma^{\beta p}_{mp+\Delta} \Rightarrow R^T(\hat{\mathbf{p}},\Delta)(\delta_{\alpha\beta}-\hat{p}_\alpha\hat{p}_\beta)(\delta_{im}-\hat{p}_i\hat{p}_m)\delta(|\,\mathbf{p}|-|\,\omega\,|/c_T)$$

$$+R^L(\hat{\mathbf{p}},\Delta)(\hat{p}_\alpha\hat{p}_\beta\hat{p}_i\hat{p}_m)\delta(|\,\mathbf{p}|-|\,\omega\,|/c_L) \qquad (6.1)$$

This approximation is valid only if $Im\sigma \ll \omega^2$, or equivalently, $\alpha \ll \omega/c$. Note that the unit vectors in the direction of $\mathbf{p}+\Delta$ have been approximated by $\hat{\mathbf{p}}$, in accordance with the assumed limit $\Delta \ll \omega/c$.

If we were to further seek the diffusion limit $\Delta \ll \alpha$ and $\Omega \ll \alpha/\omega c$ we could approximate the energy propagators R by

$$R^T(\hat{\mathbf{p}},\Delta) = \frac{\omega\pi}{2c_T^3 Im\sigma_T}\left[1-\frac{i\omega\Omega}{Im\sigma^T}+i\omega\Delta\cdot\hat{\mathbf{p}}\,c_T/Im\sigma_T+\ldots\right]$$

$$R^L(\hat{\mathbf{p}},\Delta) = \frac{\omega\pi}{2c_L^3 Im\sigma_L}\left[1-\frac{i\omega\Omega}{Im\sigma_L}+i\omega\Delta\cdot\hat{\mathbf{p}}\,c_L/Im\sigma_L+\ldots\right] \qquad (6.2)$$

We now recognize that, as the wavevectors $\mathbf{p}$ and $\mathbf{s}$ will have magnitudes of order ω/c, the terms in Δ in (5.6) and (5.11) may be neglected. Use of (6.1) and the neglect of the Δ's leads to great simplification of the Bethe-Salpeter equation. It is now defined on the unit sphere and all Δ dependence in the kernal is by means of the factors R. In direct notation it is

$$\mathbf{S}(\mathbf{p}) = \mathbf{I}+\int d^3\hat{s}\left[{}^{\mathbf{p}}_{\mathbf{p}}\mathbf{K}^{\hat{\mathbf{s}}\omega/c_L\cdot\hat{\mathbf{s}}\hat{\mathbf{s}}}_{\hat{\mathbf{s}}\omega/c_L\cdot\hat{\mathbf{s}}\hat{\mathbf{s}}}\cdot\mathbf{S}(\hat{\mathbf{s}}\omega/c_L)R^L(\hat{\mathbf{s}})+{}^{\mathbf{p}}_{\mathbf{p}}\mathbf{K}^{\hat{\mathbf{s}}\omega/c_T\cdot\mathbf{I}-\hat{\mathbf{s}}\hat{\mathbf{s}}}_{\hat{\mathbf{s}}\omega/c_T\cdot\mathbf{I}-\hat{\mathbf{s}}\hat{\mathbf{s}}}\cdot\mathbf{S}(\hat{\mathbf{s}}\omega/c_T)R^T(\hat{\mathbf{s}})\right] \qquad (6.3)$$

The Δ and Ω dependencies in $\mathbf{S}$ and R have been suppressed. After the approximation (6.1), the expression for the energy density becomes

$$E = (2\pi)^{-3}\int d^2\hat{s}\,[\mathbf{I}^{\hat{\mathbf{s}}\hat{\mathbf{s}}}_{\hat{\mathbf{s}}\hat{\mathbf{s}}}\cdot\mathbf{S}(\hat{\mathbf{s}}\omega/c_L)R^L(\hat{\mathbf{s}})+\mathbf{I}^{\mathbf{I}-\hat{\mathbf{s}}\hat{\mathbf{s}}}_{\mathbf{I}-\hat{\mathbf{s}}\hat{\mathbf{s}}}\cdot\mathbf{S}(\hat{\mathbf{s}}\omega/c_T)R^T(\hat{\mathbf{s}})] \qquad (6.4)$$

The expression for the energy flux also reduces to an integral over the unit sphere.

$$J_\mu = (2\pi)^{-3}C^0_{\mu\alpha ki}\int d^2\hat{s}\,[(\hat{s}_k/c_L)\hat{s}_\alpha\hat{s}_\beta\hat{s}_i\hat{s}_m\,S^\beta_m(\hat{\mathbf{s}}\omega/c_L)R^L(\hat{\mathbf{s}})$$

$$+(\hat{s}_k/c_T)(\delta_{\alpha\beta}-\hat{s}_\alpha\hat{s}_\beta)(\delta_{im}-\hat{s}_i\hat{s}_m)S^\beta_m(\hat{\mathbf{s}}\omega/c_T)R^T(\hat{\mathbf{s}})] \qquad (6.5)$$

At this point the equations can be made to admit of a simple physical interpretation. $\mathbf{S}(\hat{\mathbf{p}})$ represents a source of radiant intensity in the $\hat{\mathbf{p}}$ direction. It includes both primary and secondary contributions. R represents a propagation of that intensity. Equation (6.3) indicates that the

radiant intensity has two sources. **I** represents the primary source, the original impulsive force. The integral sums the secondary contributions due to scattering of intensity into the $\hat{\mathbf{p}}$ direction from all other directions $\hat{\mathbf{s}}$. Thus equation (6.3) is an equation of radiative transfer.

There is a considerable body of literature on the subject of electromagnetic radiative transfer [28,29,30]. The bulk of it is confined to a parameter range of little relevance for the present purposes. In that literature the effects of radiation polarization (the tensor character of the equations) are usually neglected, as is the time dependence Ω. Any polarization which that literature does consider is, furthermore, confined to transversely polarizable waves and does not include the longitudinal modes implicit in (6.3-6.5). On the other hand, that literature is generally not confined to the diffusion limit $\Delta \ll \alpha$, nor to infinite statistically homogeneous media. It considers secondary sources and sinks other than those due to simple scattering and allows frequencies, ω, to change upon scattering. Thus our present problem is both somewhat more complex and substantially less complex than the usual problems considered in the literature of radiative transfer.

Equation (6.3) is a set of nine coupled integral equations for the nine components of the tensor **S**. As such it appears to remain somewhat intractable. However, our purposes are such that only two of those components will prove to be relevant. One might write, in general

$$\mathbf{S}(\mathbf{p},\Delta) = \sum_{\Lambda M=1}^{3} \hat{\mathbf{p}}^{\Lambda}\hat{\mathbf{p}}^{M} S_{M}^{\Lambda} \tag{6.6}$$

where $\hat{\mathbf{p}}^1 \equiv \hat{\mathbf{p}}$ and $\hat{\mathbf{p}}^2$ and $\hat{\mathbf{p}}^3$ form an orthonormal triad. Substitution of this form for **S** into equations 6.3-6.5 shows that the four components S_2^1, S_1^2, S_1^3 and S_1^3 do not contribute to J or E or to the secondary source of radiation intensity. They may or may not vanish, but they are in any case decoupled and irrelevant. The irrelevancy of these four components may be summarized by requiring that $\mathbf{S}(\hat{\mathbf{p}})$ have as an eigenvector $\hat{\mathbf{p}}$ itself on both the right and the left. The decoupling is due to the vanishing of the cross terms $g^L g^{*T}$ and $g^T g^{*L}$ in the approximation (6.1). That vanishing is in turn traceable to the different speeds with which transverse and longitudinal waves propagate. There is no coherent propagation of interference between waves of different velocity. Transverse modes can, however, interfere coherently and S_3^2 and S_2^3 do not in general decouple from the radiative transfer equations. It is at first somewhat surprising that one would need four components to describe the intensity of a transverse wave. One should, though, recognize that an elliptically polarized transverse wave requires specification of four independent quantities: the two principle intensities, the orientation of the ellipse, and also the relative phase. These four quantities are equivalent to the four "Stokes" parameters used in the treatment of

electromagnetic polarization in the theory of radiative transfer [28]. Thus the present system may be said to require a fifth Stokes parameter as well, to represent the intensity of the longitudinal waves.

S may be reduced further, from five independent components to three if it is recognized that in an infinite statistically homogeneous and isotropic medium any second rank tensor function of two vectors $\hat{\mathbf{p}}$ and Δ must be a linear combination of the five dyadics $\hat{\mathbf{p}}\hat{\mathbf{p}}$, **I**, $\Delta\Delta$, $\hat{\mathbf{p}}\Delta$ and $\Delta\hat{\mathbf{p}}$. The condition that **S** also take $\hat{\mathbf{p}}$ as a right and a left eigenvector then reduces this to three dyadics. The most convenient representation is of the form

$$S(\mathbf{p},\Delta) = \hat{\mathbf{p}}\hat{\mathbf{p}}\frac{A(\mathbf{p},\Delta)}{R^L} + (\mathbf{I}-\hat{\mathbf{p}}\hat{\mathbf{p}})B\frac{(\mathbf{p},\Delta)}{R^T} \tag{6.7}$$

$$+[(\mathbf{I}-\hat{\mathbf{p}}\hat{\mathbf{p}})\cdot\Delta]\,[(\mathbf{I}-\hat{\mathbf{p}}\hat{\mathbf{p}})\cdot\Delta]\frac{C(\mathbf{p},\Delta)}{R^T}$$

where A, B and C are isotropic scalar functions of $\mathbf{p}$ and Δ, and therefore of $\mathbf{p}^2, \mathbf{p}\cdot\Delta$ and Δ^2 (as well as Ω). Because, as discussed above, the product RS represents radiation intensity we see that A, B and C represent radiation intensity. B and C represent transverse wave intensities; A represents longitudinal wave intensity.

The energy density and energy flux take very simple forms in terms of A, B and C:

$$E = \frac{1}{(2\pi)^3}\int d^2\hat{\mathbf{s}}\{A(\hat{\mathbf{s}}\omega/c_L) + 2B(\hat{\mathbf{s}}\omega/c_T) + (\Delta^2 - (\hat{\mathbf{s}}\cdot\Delta)^2)C(\hat{\mathbf{s}}\omega/c_T)\} \tag{6.8}$$

and

$$\mathbf{J} = \frac{1}{(2\pi)^3}\int d^2\hat{s}\;\hat{\mathbf{s}}\{c_L A(\hat{\mathbf{s}}\omega/c_L) + 2c_T B(\hat{\mathbf{s}}\omega/c_T) + c_T(\Delta^2 - (\hat{\mathbf{s}}\cdot\Delta^2)C(\hat{\mathbf{s}}\omega/c_T)\} \tag{6.9}$$

where the Δ and Ω dependences in the Stokes-like parameters A, B and C have been suppressed.

When the form (6.7) is employed on the right-hand side of the equation, (6.3), of radiative transfer, it becomes, in direct notation,

$$\mathbf{S}(\mathbf{p}) = \mathbf{I} + \int d^2\hat{s}\;\tilde{\eta}(\mathbf{p}-\hat{\mathbf{s}}\omega/c_L)\omega^2/c_L^2\,\Xi^{\cdots\mathbf{p}\hat{\mathbf{s}}\hat{\mathbf{s}}}_{\cdots\mathbf{p}\hat{\mathbf{s}}\hat{\mathbf{s}}}A(\hat{\mathbf{s}}\omega/c_L) \tag{6.10}$$

$$+\int d^2\hat{s}\;\tilde{\eta}(\mathbf{p}-\hat{\mathbf{s}}\omega/c_T)\omega^2/c_T^2\,\Xi^{\cdots\mathbf{p}\hat{\mathbf{s}}}_{\cdots\mathbf{p}\hat{\mathbf{s}}}\left[(\mathbf{I}-\hat{\mathbf{s}}\hat{\mathbf{s}})B(\hat{\mathbf{s}}\omega/c_T) + \frac{(\mathbf{I}-\hat{\mathbf{s}}\hat{\mathbf{s}})\cdot\Delta}{(\mathbf{I}-\hat{\mathbf{s}}\hat{\mathbf{s}})\cdot\Delta}C(\hat{\mathbf{s}}\omega/c_T)\right]$$

(6.10) is equivalent to three coupled scalar integral equations defined on the unit sphere. They are obtained by contracting (6.10) on the three different dyadics $\hat{\mathbf{p}}\hat{\mathbf{p}}, \mathbf{I}-\hat{\mathbf{p}}\hat{\mathbf{p}}$ and $(\Delta\cdot(\mathbf{I}-\hat{\mathbf{p}}\hat{\mathbf{p}}))(\Delta\cdot(\mathbf{I}-\hat{\mathbf{p}}\hat{\mathbf{p}}))$ respectively. The first two are, with terms of explicit order Δ^2 neglected,

$$A(\hat{\mathbf{p}}\omega/c_L)/R^L = 1+\int d^2\hat{s}\,\eta^{LL}(\theta_{ps})\omega^4/c_L^4(^{\hat{p}\hat{p}\hat{s}\hat{s}\cdots}_{\hat{p}\hat{p}\hat{s}\hat{s}\cdots}\Xi)A(\hat{\mathbf{s}}\omega/c_L)$$

$$+\int d^2\hat{s}\,\eta^{LT}(\theta_{ps})\frac{\omega^4}{c_L^2c_T^2}[^{\hat{p}\hat{p}\hat{s}}_{\hat{p}\hat{p}\hat{s}}(\mathbf{I}-^{\hat{s}}_{\hat{s}})^{\cdots}\Xi\,B(\hat{\mathbf{s}}\omega/c_T)] \tag{6.11}$$

and

$$2B(\hat{\mathbf{p}}\omega/c_T)/R^T = 2+\int d^2\hat{s}\,\eta^{TL}(\theta_{ps})\frac{\omega^4}{c_L^2c_T^2}\left[(\mathbf{I}-^{\hat{p}}_{\hat{p}})^{\hat{p}\hat{s}\hat{s}\cdots}_{\hat{p}\hat{s}\hat{s}\cdots}\Xi\,A(\hat{\mathbf{s}}\omega/c_L)\right]$$

$$+\int d^2\hat{s}\,\eta^{TT}(\theta_{ps})\frac{\omega^4}{c_T^4}[(\mathbf{I}-^{\hat{p}}_{\hat{p}})^{\hat{p}\hat{s}}_{\hat{p}\hat{s}}(\mathbf{I}-^{\hat{s}}_{\hat{s}})^{\cdots}\Xi\,B(\hat{\mathbf{s}}\omega/c_T)] \tag{6.12}$$

The third equation is not required for the present purposes and is therefore not included here. θ_{ps} is the angle between the $\hat{\mathbf{p}}$ and $\hat{\mathbf{s}}$ directions and the $\eta(\theta)$ were defined in equations (3.33). Except for the factors of A and B in the above integrands, these integrals are equivalent to those of equations (3.31)-(3.35).

In the diffusion limit, where $\Delta \ll \alpha$, we seek a regular expansion of the Stokes parameters A and B in powers of Δ/α. As discussed above, retention of Δ dependence through the linear terms will suffice. Hence we write

$$A(\hat{\mathbf{p}}\omega/c_L,\Delta) = A_0+A_1(\Delta\cdot\hat{\mathbf{p}}) \tag{6.13}$$

and

$$B(\hat{\mathbf{p}}\omega/c_T,\Delta) = B_0+B_1(\Delta\cdot\hat{\mathbf{p}}) \tag{6.14}$$

where the coefficients A_0, A_1, B_0 and B_1, are independent of Δ. As they are isotropic scalars they must also be independent of $\hat{\mathbf{p}}$. They may, however, retain Ω dependence.

The energy propagators R must also be expanded;

$$R^T(\hat{\mathbf{p}}) = \rho_0^T+\rho_1^T(\Delta\cdot\hat{\mathbf{p}})$$

$$R^L(\hat{\mathbf{p}}) = \rho_0^L+\rho_1^L(\Delta\cdot\hat{\mathbf{p}}) \tag{6.15}$$

whence, upon reference to (6.2) we identify

$$\rho_0^L = \frac{\omega\pi}{2c_L^3 Im\sigma^L}\left(1-\frac{i\omega\Omega}{Im\sigma^L}\right) = \frac{\pi}{4c_L^4\alpha_L}\left(1-\frac{i\omega\Omega}{2c_L\alpha_L}\right)$$

$$\rho_0^T = \frac{\omega\pi}{2c_T^3 Im\sigma^T}\left(1-\frac{i\omega\Omega}{Im\sigma^T}\right) = \frac{\pi}{4c_T^4\alpha_T}\left(1-\frac{i\omega\Omega}{2c_T\alpha_T}\right) \tag{6.16}$$

and

$$\rho_1^L = \frac{i\omega^2\pi}{2c_L^2(Im\sigma_L)^2} = \frac{i\pi}{8c_L^4\alpha_L^2}$$

$$\rho_1^T = \frac{i\omega^2\pi}{2c_T^2(Im\sigma_L)^2} = \frac{i\pi}{8c_T^4\alpha_T^2} \tag{6.17}$$

The diffuse field energy density (6.8) is given through first order in Δ in terms of the expansion coefficients of the field by

$$E = \frac{1}{2\pi^2}(A_0 + 2B_0) \tag{6.18}$$

The energy flux (6.9) is given through order Δ^2 by

$$\mathbf{J} = (\Delta/6\pi^2)[A_1 c_L + 2B_1 c_T] \tag{6.19}$$

By reference to equation (4.18) the diffusivity is then given by

$$D = -\frac{i}{3}\frac{A_1 c_L + 2B_1 c_T}{A_0 + 2B_0} \tag{6.20}$$

Were we to seek the diffusivity by comparison of E with the expected form of the diffusion propagator (4.12) instead of by means of comparison of the flux with (4.18), E would have been required through second order in Δ. Correspondingly, A and B would have been required through that same order, and C to zeroth order. By considering energy flux and equation (4.18) instead of (4.12) an energy conservation constraint (4.17) is implicitly being employed to aid in the solution to the equations of radiative transfer

The integrals in 6.11 and 6.12 over solid angle $\hat{\mathbf{s}}$ will be carried out using conventional spherical co-ordinates θ_{sp} and ϕ_{sp} referred to a polar axis in the direction $\hat{\mathbf{p}}$, and to an arbitrary zero of longitude. The factor $\hat{\mathbf{s}} \cdot \Delta = \Delta\cos\theta_{s\Delta}$ appears in the integrand and must be re-expressed in terms of the chosen spherical co-ordinates. This expression is available from spherical trigonometry, or alternatively and in greater generality, from the addition theorem of spherical harmonics. It is

$$\hat{\mathbf{s}} \cdot \Delta = \Delta\cos\theta_{s\Delta} =$$

$$\Delta[\cos\theta_{sp}\cos\theta_{p\Delta} + \sin\theta_{sp}\sin\theta_{p\Delta}\cos[\phi_{sp} - \phi_{\Delta p}]]$$

$$= (\hat{\mathbf{p}} \cdot \Delta)\cos\theta_{sp} + \text{azimuthal terms} \tag{6.21}$$

The azimuthal terms will vanish upon integration over ϕ_{sp} in any integrand such as those of (6.11) and (6.12) which are otherwise independent of ϕ_{sp}. This was to be expected. There are no terms on the left hand side depending on $\sin\theta_{p\Delta}$, hence such terms should not appear on the right hand side.

The expansions (6.13) through (6.15) are now inserted in the integral equations (6.11) and (6.12). Like powers of $\hat{\mathbf{p}}\cdot\Delta$ are collected and the integrations carried out in terms of the definitions (3.35 ff) of the attenuations and their angle-weighted primed versions. From the Δ-independent terms one deduces

$$A_0 = \rho_0^L + (4\rho_0^L c_L^3/\pi)\,[\alpha_{LL}c_L A_0 + 2\alpha_{TL}c_T B_0]$$

and

$$2B_0 = 2\rho_0^T + (4\rho_0^T c_T^3/\pi)\,[\alpha_{LT}c_L A_0 + 2\alpha_{TT}c_T B_0] \tag{6.22}$$

From the terms linear in $\hat{\mathbf{p}}\cdot\Delta$ one finds

$$\begin{aligned} A_1 = {} & \rho_1^L + (4\rho_1^L c_L^3/\pi)\,[\alpha_{LL}c_L A_0 + 2\alpha_{TL}c_T B_0] \\ & + (4\rho_0^L c_L^3/\pi)\,[\alpha_{LL}'c_L A_1 + 2\alpha_{TL}'c_T B_1] \end{aligned} \tag{6.23}$$

and

$$\begin{aligned} 2B_1 = {} & 2\rho_1^T + (4\rho_1^T c_T^3/\pi)\,[\alpha_{LT}c_L A_0 + 2\alpha_{TT}c_T B_0] \\ & + (4\rho_0^T c_T^3/\pi)\,[\alpha_{LT}'c_L A_1 + 2\alpha_{TT}'c_T B_1]. \end{aligned} \tag{6.23}$$

To leading order in Ω the solution to (6.22) for the lower order terms is

$$A_0 = \frac{\pi}{2i\Omega}\frac{1}{c_L^3} \qquad B_0 = \frac{\pi}{2i\Omega}\frac{1}{c_T^3} \tag{6.24}$$

This result together with (6.18) corroborates the expected total energy constraint (4.11). It serves as a valuable verification of the extensive algebra of the present section and also of the consistency of the ladder approximation.

At the next order in Δ and again at leading order in Ω, we find

$$A_1 = \frac{\pi}{4\Omega}\frac{(\alpha_T - \alpha_{TT}' + \alpha_{LT}')/c_L^3}{(\alpha_L - \alpha_{LL}')(\alpha_T - \alpha_{TT}') - \alpha_{LT}'\alpha_{TL}'}$$

and

$$B_1 = \frac{\pi}{4\Omega}\frac{(\alpha_L - \alpha_{LL}' + \alpha_{TL}')/c_T^3}{(\alpha_L - \alpha_{LL}')(\alpha_T - \alpha_{TT}') - \alpha_{LT}'\alpha_{TL}'} \tag{6.25}$$

Hence, using (6.20),

$$D = \frac{1}{6\left(\frac{1}{c_L^3}+\frac{2}{c_T^3}\right)} \frac{(\alpha_T - \alpha_{TT}' + \alpha_{LT}')/c_L^2 + 2(\alpha_L - \alpha_{LL}' + \alpha_{TL}')/c_T^2}{(\alpha_L - \alpha_{LL}')(\alpha_T - \alpha_{TT}') - \alpha_{LT}'\alpha_{TL}'} \tag{6.26}$$

This is the desired result. The diffusivity of the multiply-scattered wave field is expressed in terms of the wave speeds, the attenuations, and the scattering-angle-weighted attenuations. It is valid for statistically isotropic and homogeneous media in the limit of weak attenuation per inverse wavenumber, $\alpha \ll \omega/c$, and equivalently, weak fluctuations, v, in moduli. It has been established for frequencies below the very high frequency, geometric optics regime, that is, within the "Born" approximation.

In the Rayleigh limit where all primed attenuations vanish, the expression for the diffusivity simplifies.

$$D_{\text{Rayleigh}} = \left(\frac{1}{c_L^3}\frac{c_L}{6\alpha_L}+\frac{2}{c_T^3}\frac{c_T}{6\alpha_T}\right)\Big/\left(\frac{1}{c_L^3}+\frac{2}{c_T^3}\right) \tag{6.27}$$

which is precisely that diffusivity argued as plausible in the first section. It is striking that the naive rule of mixtures used there was correct.

VII. Numerical Results and Discussion

Explicit expressions for the attenuations, and therefore for the diffusivity, can be obtained only upon specification of a two point correlation function $\eta(r)$. The choice (2.5) has the virtue of simplicity while adhering to the requirement that η vanish at large r and reach v^2 at $r = 0$. That specification has a single length scale β^{-1}. In order to make comparisons with real materials one must either determine the actual $\eta(r)$ of the specimen at hand [31] (and by implication verify the decoupling assumed in (2.4)) or make a best estimate for an effective β. At low frequencies it is well understood that the key microstructural quantity is mean grain volume. Hence one identifies an effective β by the equation

$$\overline{V} = \int d^3 r e^{-\beta r} = (2\pi)^3 \tilde{\eta}(0)/v^2 = 8\pi/\beta^3 \tag{7.1}$$

where $\overline{V}$ is the mean grain volume. At frequencies of order $\omega \sim \beta c$ or higher, though, it is probably not the grain volume which is critical, but rather the short distance behaviour of η, or the volume density of grain boundary. This leads to the identification of β with half that density. Such a β is something like half the β determined from (7.1), depending on the assumed grain shapes. This uncertainty in the choice for an effective β underlines the naivete of the simple form (2.5).

We shall, though, in order to generate explicit expressions for the diffusivity, make the choice (2.5). It should be kept in mind, however, that in any comparison between the results to be presented and diffusivities in real materials, the best choice of an effective β is uncertain, and should, probably, even vary somewhat with frequency. With choice (2.5) the spatial Fourier transform of η is obtained from equation (3.10).

$$\tilde{\eta}(\mathbf{q}) = (\nu^2\beta/\pi^2)(\beta^2+q^2)^{-2} \tag{7.2}$$

The $\eta(\theta)$ are then found from equations (3.33)

$$\eta^{LL}(\theta) = (\nu^2\beta/\pi^2)\left[\beta^2+\frac{2\omega^2}{c_L^2}(1-\cos\theta)\right]^{-2}$$

$$\eta^{TT}(\theta) = (\nu^2\beta/\pi^2)\left[\beta^2+\frac{2\omega^2}{c_T^2}(1-\cos\theta)\right]^{-2} \tag{7.3}$$

$$\eta^{TL}(\theta) = \eta^{LT}(\theta) = (\nu^2\beta/\pi^2)\left[\beta^2+\frac{\omega^2}{c_T^2}+\frac{\omega^2}{c_L^2}-\frac{2\omega^2}{c_Tc_L}\cos\theta\right]^{-2}$$

which in turn allows the attenuations, (3.35) to be written as

$$\alpha_{LL} = \frac{\beta}{1050}\frac{\nu^2}{c_L^4}x_L^4\int_{-1}^{1}d\mu\,(3+\mu^2)/[1+2x_L^2(1-\mu)]^2$$

$$\alpha_{TT} = \frac{\beta}{2100}\frac{\nu^2}{c_T^4}x_T^4\int_{-1}^{1}d\mu\,(24+3\mu^2+\mu^4)/[1+2x_T^2(1-\mu)]^2$$

$$\alpha_{LT} = \frac{\beta}{1050}\frac{\nu^2}{c_Lc_T^3}x_L^2x_T^2\int_{-1}^{1}d\mu\,(15+6\mu^2-\mu^4)/[1+x_L^2+x_T^2-2x_Lx_T\mu)]^2$$

$$\alpha_{TL} = \alpha_{LT}c_T^2/2c_L^2 \tag{7.4}$$

$x_L \equiv \omega/c_L\beta$ and $x_T \equiv \omega/c_T\beta > x_L$ are dimensionless measures of frequency. μ has been used for $\cos\theta$. The primed attenuations are given by expressions identical to those above, but with one extra factor of μ in each integrand.

The integrals are easily done in the Rayleigh limit where $x_L < x_T \ll 1$. The primed attenuations vanish in comparison to the unprimed attenuations, the unprimed attenuations are

$$\begin{aligned}
\alpha_{LL}^{\text{Rayleigh}} &= (8/375)\beta v^2 x_L^4/c_L^4 \\
\alpha_{TT}^{\text{Rayleigh}} &= (9/375)\beta v^2 x_T^4/c_T^4 \\
\alpha_{LT}^{\text{Rayleigh}} &= (12/375)\beta v^2 x_L^2 x_T^2/c_L c_T^3 \\
\alpha_{TL}^{\text{Rayleigh}} &= (6/375)\beta v^2 x_L^2 x_T^2/c_T c_L^3
\end{aligned} \tag{7.5}$$

and the attenuations proper are (equation 3.34)

$$\begin{aligned}
\alpha_{\text{Rayleigh}}^{L} &= (1/375)\beta(v^2/c_L^4)x_L^4(8+12K^5) \\
\alpha_{\text{Rayleigh}}^{T} &= (1/375)\beta(v^2/c_T^4)x_T^4(9+6K^{-5})
\end{aligned} \tag{7.6}$$

where K is the ratio of Voight wave speeds; $K \equiv c_L/c_T$. These attenuations are in agreement with expressions found in the literature, e.g., Stanke and Kino [16] equations 24 and 25.

The Rayleigh limit diffusivity is then given by (7.6) and (6.27)

$$D_{\text{Rayleigh}} = \frac{c_T}{6\beta} x_T^{-4}(v^2/c_T^4)^{-1} \frac{125}{(2+K^{-3})\left(1+\frac{2}{3}K^{-5}\right)}\left(\frac{2}{3}+\frac{K}{4}\right) \tag{7.7}$$

The result is nearly insensitive to K in K's physical range. For K about 2, D_{Rayleigh} is given fairly accurately by

$$D_{\text{Rayleigh}} \approx 11.2(c_T/\beta)x_T^{-4}(v^2/c_T^4)^{-1} \tag{7.8}$$

It is noteworthy that the transverse and longitudinal contributions to the energy flux are roughly equal.

Outside the Rayleigh limit the integrals (7.4) are more difficult. They are all of the form

$$I_n \equiv \int_{-1}^{1} d\mu \frac{\mu^n}{(a-b\mu)^2} \tag{7.9}$$

for $n = 0, 1...5$ and $|\, b \,| < |\, a \,|$. Defining $\phi \equiv a/b$ the integrals are given by

$$I_0 = b^{-2}\left\{\frac{2}{\phi^2-1}\right\}$$

$$I_1 = b^{-2}\left\{\frac{2\phi}{\phi^2-1}-\log\frac{\phi+1}{\phi-1}\right\}$$

$$I_2 = b^{-2}\left\{\frac{2\phi^2}{\phi^2-1}-2\phi\log\frac{\phi+1}{\phi-1}+2\right\}$$

$$I_3 = b^{-2}\left\{\frac{2\phi^3}{\phi^2-1}-3\phi^2\log\frac{\phi+1}{\phi-1}+4\phi\right\}$$

$$I_4 = b^{-2}\left\{\frac{2\phi^4}{\phi^2-1}-4\phi^3\log\frac{\phi+1}{\phi-1}+6\phi^2+\frac{2}{3}\right\}$$

and

$$I_5 = b^{-2}\left\{\frac{2\phi^5}{\phi^2-1}-5\phi^4\log\frac{\phi+1}{\phi-1}+8\phi^3+\frac{4}{3}\phi\right\} \tag{7.10}$$

In terms of $\phi_L \equiv 1 + 1/2x_L^2$ the longitudinal attenuations are

$$\alpha_{LL} = \frac{\beta}{4200}\frac{v^2}{c_L^4}\left[\frac{2[3+\phi_L^2]^2}{\phi_L^2-1}-4\phi_L(3+\phi_L^2)\log\frac{\phi_L+1}{\phi_L-1}+6\phi_L^2+38/3\right]$$

and

$$\alpha'_{LL} = \frac{\beta}{4200}\frac{v^2}{c_L^4}\left[\frac{2\phi_L[3+\phi_L^2]^2}{\phi_L^2-1}-(9+18\phi_L^2+5\phi_L^4)\log\frac{\phi_L+1}{\phi_L-1}+8\phi_L^3+76\phi_L/3\right] \tag{7.11}$$

The transverse attenuations are, in terms of $\phi_T \equiv 1 + 1/2x_T^2$,

$$\alpha_{TT} = \frac{\beta}{8400}\frac{v^2}{c_T^4}\left[\frac{2[24+3\phi_T^2+\phi_T^4]}{\phi_T^2-1}-2\phi_T(3+2\phi_T^2)\log\frac{\phi_T+1}{\phi_T-1}+6\phi_T^2+20/3\right]$$

and

$$\alpha'_{TT} = \frac{\beta}{8400}\frac{v^2}{c_T^4}\left[\frac{2\phi_T(24+3\phi_T^2+\phi_T^4)}{\phi_T^2-1}-(24+9\phi_T^2+5\phi_T^4)\log\frac{\phi_T+1}{\phi_T-1}+8\phi_T^3+40\phi_T/3\right]$$

The longitudinal-to-transverse mode conversion attentuations are, in terms of $\phi_{TL} \equiv (1+x_T^2+x_L^2)/2x_Lx_T$,

$$\alpha_{LT} = \frac{\beta}{4200}\frac{v^2}{c_Lc_T^3}\left[\frac{2(15+6\phi_{TL}^2-\phi_{TL}^4)}{\phi_{TL}^2-1}-4\phi_{TL}(3-\phi_{TL}^2)\log\frac{\phi_{TL}+1}{\phi_{TL}-1}-6\phi_{TL}^2+34/3\right]$$

and

$$\alpha'_{LT} = \frac{\beta}{4200}\frac{v^2}{c_L c_T^3}\left[\frac{2\phi_{TL}(15+6\phi_{TL}^2-\phi_{TL}^4)}{\phi_{TL}^2-1} - (15+18\phi_{TL}^2-5\phi_{TL}^4)\log\frac{\phi_{TL}+1}{\phi_{TL}-1} - 8\phi_{TL}^3 + 68\frac{\phi_{TL}}{3}\right] \quad (7.13)$$

The other mode conversion attenuations may be found from

$$\alpha_{TL} = \alpha_{LT}(c_T^2/2c_L^2)$$

and

$$\alpha'_{TL} = \alpha'_{LT}(c_T^2/2c_L^2) \quad (7.14)$$

As $x \to \infty$ the resulting expressions for the attenuations $\alpha^L = \alpha_{LL} + \alpha_{LT}$ and $\alpha^T = \alpha_{TT} + \alpha_{TL}$ approach the "stochastic asymptote" and agree with expressions given by Stanke and Kino [16] in their equations 33 and 34.

The diffusivity is obtained upon substitution of (7.11) through (7.14) into (6.26). The result is plotted in Figure 1 in scaled dimensionless units of diffusivity against dimensionless frequency for a choice of Poisson ratio 0.287 corresponding to a wave speed ratio $K = 1.827$ appropriate to untextured polycrystalline iron. The diffusivity is given in units of $4200c_T^5/v^2\beta$; frequency is measured by $x_T \equiv \omega/c_T\beta$. The inverse fourth power dependence is apparent in the Rayleigh regime $x_T \ll 1$. An inverse logarithmic dependence is seen in the opposite limit. The diffusivity must certainly become frequency independent in the very large x, geometrical optics regime. Thus the observed inverse logarithmic (and therefore weak) dependence closely, but not precisely, captures the actual very large x limit.

The approximate point at which geometrical optics becomes significant and the "Born" approximation fails can be determined from the attenuation curves published by Stanke and Kino. For the case of iron, with a moderate value of v that point is found to be at $x_T = 4.5$; for aluminum, with a small value of v one find that geometrical optics obtains for $x_T > 20$. One speculates that the diffusivities should become frequency independent for frequencies in excess of these limiting points. It is noteworthy that the calculated diffusivities are nearly frequency independent there anyway, in spite of being based on attenuations calculated using the Born approximation.

The analogous situation for the attenuations was very different and should be contrasted with that for the diffusivity. The stochastic asymptote has attenuations increasing with the square of frequency while the geometrical optics very high frequency limit has frequency independent attenuation. Thus the stochastic asymptote is a very bad approximation in the geometrical optics limit if the interest is in attenuation. The error is apparently much less severe in the case of diffusivity.

The four diffusivities measured by Guo, Holler and Goebbels [8] are also plotted, as isolated points. In order to do so, their measurements had to be nondimensionalized. A shear wave speed of $c_T = 3.23mm/\mu sec$ was chosen corresponding to untextured iron. ν/c_T^2 was set to -1.66 in accordance with the moduli of single crystal iron. A value of β was chosen equal to $10/mm$. The one-parameter fit represented by our free choice of a value of β is seen to agree fairly well with the limited laboratory data.

$\beta^{-1} \approx 100$ microns appears, from the optical micrograph published by Guo *et al.* [8], to correspond to the radii of the regions of similar shading. It is reasonable to conjecture that the grains clump in groups with similar crystallographic orientations and thus with similar shadings in the micrograph. The ultrasonics would of course, be most sensitive to the large angle grain boundaries between the clumps, and hence respond to an effective β^{-1} of the order of 100 microns. It is, unfortunately, not possible to make detailed comparisons without more information on the actual sample microstructure and also on error bounds on the diffusivities. The agreement is, however, promising.

VIII. Conclusions

The present theory has presented a formula for the diffusivity of a multiply scattered diffuse wavefield in an untextured cubic polycrystal. It appears to agree with laboratory measurements. The theory is valid in the limit of weakly anisotropic crystallites and is derived within the context of the "Born" approximation which thus confines confident application to the so called Rayleigh and "stochastic" regimes where the frequency lies below the geometrical optics limit. The diffusivity was derived in a two step process. Radiative transfer equations were obtained from the multiple scattering equations in the ladder approximation by invoking a weak attenuation and long length scale assumption. The diffusion limit was then obtained from the radiative transfer equations by considering still longer length scales.

There is room for further study on a variety of issues. Perhaps the most obvious need is for definitive laboratory studies comparing the diffusivities in known microstructures with the predictions of a theory such as the present one. Inasmuch as it is as yet unclear how small the numerical quantity $|\nu/c_T^2|$ must be for the theory to be valid, it would be desirable to study diffusivities in a range of materials with different values for that parameter. Such experiments would also tend towards the development of efficient techniques for the measurements of diffusivities, and by implication, local internal friction as well. The possible consequences for NDE are exciting.

It is conceivable that the formula for diffusivity could have been derived from a starting point within the theory of radiative transfer without any direct reference to the formalism of multiple scattering, with a consequent saving of effort, and with a consequent loss of grounding in the fundamental mechanics. Diffusivity in systems with non-trivial geometries or with statistical anisotropy might be best treated in that manner.

The present calculation has been carried out within the "Born" approximation and is therefore not safely applied to the geometrical optics very high frequency regime. While attenuations have been calculated in that regime [16,21] they are of little practical relevance, being of order β itself and too large for meaningful laboratory measurement. Laboratory diffusivity in such a regime, though, is not inaccessible ; indeed the geometric regime has even been termed the "diffusion regime" by some authors. Such an appellation is somewhat of a misnomer in that diffusion is operational at all frequencies on sufficiently long length scales. It is appropriate though in that no other ultrasonic microstructural assessment method is likely to be feasible there. Thus it is unfortunate that the present calculation cannot be applied there. There is a need therefore for a diffusivity calculation in the geometrical optics limit.

In the limit of strong crystallite anisotropy, FOSA and the ladder approximation will fail, as will the radiative transfer approximation $\alpha \ll \omega/c$. The condition $\alpha \sim \omega/c$ is precisely the Ioffe-Regel criterion for the absence of diffusion and the onset of Anderson localization [14,26,27,32]. While an analysis of the high crystallite anisotropy regime is presumably beyond any exact treatment, experiments there would not be difficult and should certainly be attempted.

With a theory now available it is possible to make the case that ultrasonic diffusivity measurements should take their place with attenuation and backscatter measurements for purposes of nondestructive characterization. While the technical aspects of making diffusivity measurements are non-trivial, and will be discussed in another communication, it seems that the severe systematic errors which can sometimes plague attenuation measurements do not trouble diffusivity measurements. The ease with which quantitative assessment of microscale length was effected at the end of the preceeding section is encouraging. The diffusivity measurements also promise to deliver assessments of the local internal friction, something which it is difficult to imagine measuring in any other non-destructive way and which could conceivably be an interesting nondestructive materials evaluation parameter in its own right.

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X. Appendix - List of Symbols

A, B, C	Stokes-like parameters for the radiation intensity (6.7)
A_0, B_0, A_1, B_1	expansion coefficients for A and B (6.13) (6.14)
a_i^n	rotation matrix element (2.7)
C, C_{ijkl}	stiffness tensor (2.1)
C°_{ijkl}	ensemble averaged stiffness, also the Voight estimate of effective stiffness (2.2)
c_L, c_T	effective (Voight average) longitudinal and transverse wave speeds
c	generic wavespeed
D	diffusivity (1.1)
E	diffuse field energy density in transform domain (4.5)
G_{ij}	unaveraged Green's function (2.1) (3.2)
G°_{ij}	Green's function of the bare, Voight average, medium (3.3),(3.7)
$< G_{ij} >$	ensemble average Green's function (3.3)
g^{oL}, g^{oT}	bare longitudinal and transverse propagators (3.10)
g^L, g^T	mean field propagators (3.22) (3.23)
H	covariance of the Green's function (5.1)
ijklmn	latin Cartesian indices
I	unit dyadic
J	diffuse field energy flux in transform domain (4.16)
K	ratio c_L/c_T of effective medium wave speeds (7.6)
K	intensity operator (5.4)
$L(\theta), M(\theta), N(\theta)$	(2.15)
m_{ij} **m**	self energy operator (3.3)
p,q,s,p′,q′	wave vectors

$\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{s}}$	unit wave vectors
R^L, R^T	mean field energy propagators (6.1)
S_i^α	covariance source (5.7)
x_L, x_T	dimensionless measures of frequency (7.4)
$\mathbf{x}, \mathbf{x}', \mathbf{x}'', \mathbf{y}, \mathbf{y}'$	configuration space position vectors
α	generic attenuation
α_L, α_T	longitudinal and transverse wave attenuations (3.29) (3.30) (1.1)
$\alpha_{LL}, \alpha_{LT}, \alpha_{TL}, \alpha_{TT}$	partial attenuations (3.35)
$\alpha_{LL}', \alpha_{LT}', \alpha_{TL}', \alpha_{TT}'$	angle weighted partial attenuations (3.35)
$\alpha\beta\gamma\delta\gamma\mu\nu$	Greek Cartesian indices
β	microscale inverse length (2.5)
$\gamma(\mathbf{x})$	fluctuating modulus (2.2)
Γ	double mean field propagator (5.5)
Δ	spatial Fourier transform parameter for mean square quantities (4.5)
δ_{nm}	Kronecker delta
$\delta^3()$	three dimensional Dirac delta function
ε	infinitesimal positive constant (3.2)
$\eta(r)$	microstructural two point correlation function (2.4)
$\tilde{\eta}(\mathbf{q})$	Fourier transform of $\eta(r)$ (3.16)
$\eta^{LL}, \eta^{LT}, \eta^{TL}, \eta^{TT}$	(3.33)
ν	the third independent modulus of a cubic crystal (2.6)
$\rho_o^L, \rho_o^T, \rho_1^L, \rho_1^T$	expansion coefficients for R^L and R^T (6.15)
$\sigma^L, \sigma^T,$	self energies for the mean field propagators (3.14) (3.19)
σ	generic self energy
ω	wave frequency
Ω	temporal Fourier transform parameter for mean square quantities (4.2)
$\Xi_{ijkl}^{\alpha\beta\gamma\delta}$	constant eighth rank tensor, the dimensionless modulus variance (2.4)

∂_k	$\partial/\partial x_k$, the partial spatial derivative
*	complex conjugate (4.3), also $\omega \rightarrow \omega + \Omega$ (5.1)
< >	ensemble average

XI. References

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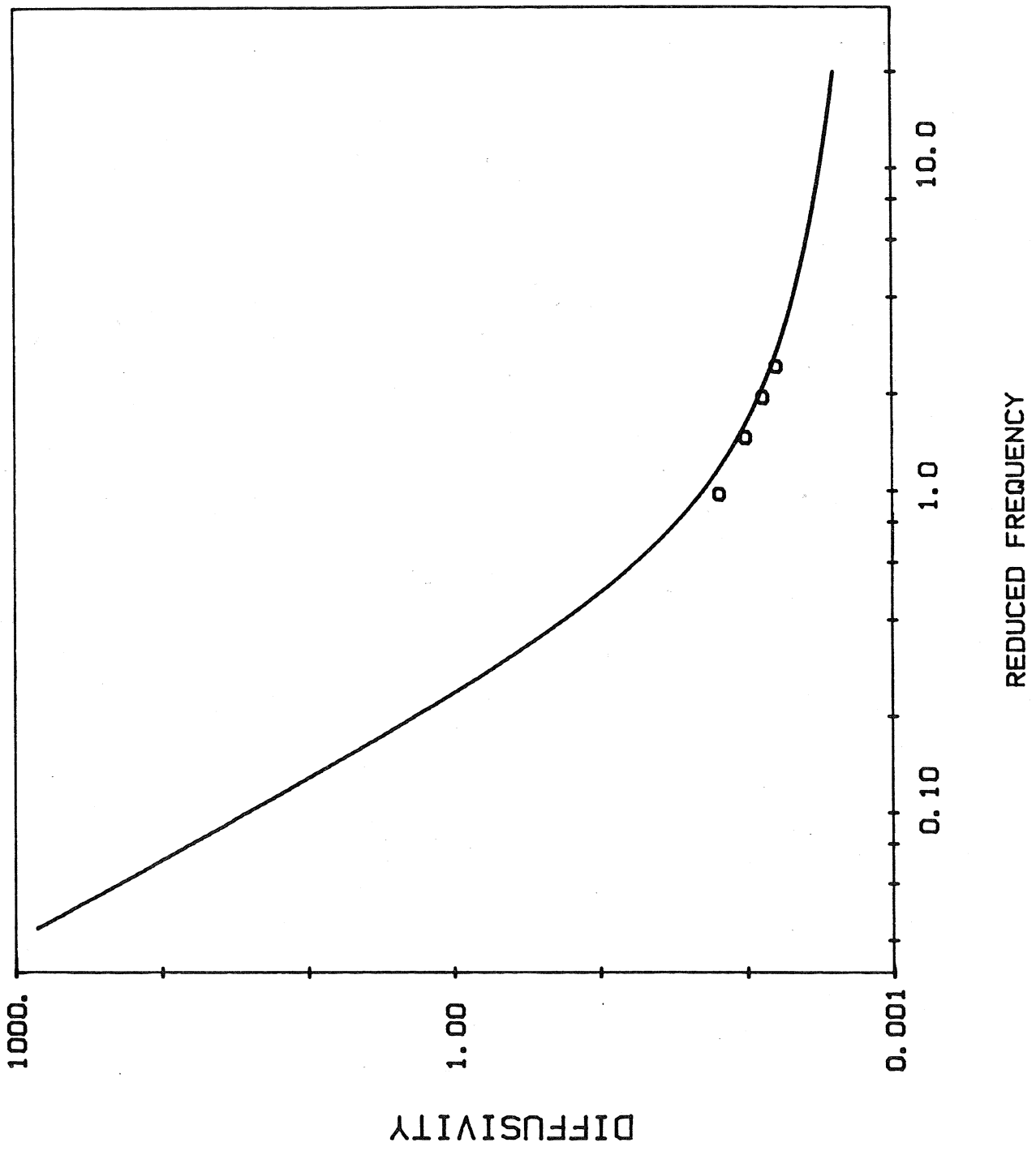

Figure 1. Dimensionless diffusivity is plotted versus dimensionless frequency for the case of an untextured aggregate of cubic crystallites of weak anisotropy. The low frequency Rayleigh regime is characterized by an inverse scaling with the fourth power of frequency. The high frequency stochastic regime is characterized by an inverse logarithmic dependence. The four data points from the experimental work of Guo, Holler and Goebbels are also shown.